Q1

$$X_t = \underbrace{B_1}_{\sim N(0,1)} - \sigma \underbrace{B_{1-t}}_{\sim N(0,z)} \quad z = 1-t \quad \forall t \in [0,1]$$

→ we know that for $\alpha \in \mathbb{R}$, $X \sim N(\mu, \sigma^2)$: $\alpha X \sim N(\alpha\mu, \alpha^2\sigma^2)$

→ using this we can say that $-\sigma B_{1-t} \sim N(0, \sigma^2(1-t))$

also we can say that if $\{X_i\}_{i=1}^{n}$ is a series of randomly distributed normal variables then $\sum_{i=1}^{n} x_i$ also follows normal distribution : $\sum_i x_i \sim N(\sum_i \mu_i, \sum_i \sigma_i^2)$

→ $X_t \sim N(0, 1 + \sigma^2(1-t))$

(I) hence the mean will be $= 0$

(II) Variance $=$ $1 + \sigma^2(1-t)$.

Q2

let us define the auto covariance function as the covariance of B_s, B_t given that $s, t \in \mathbb{R}$

↳ $s \neq t$ is the case of interest for us

↳ define $\tilde{B}_s, \tilde{B}_t = (B_s - sB_1), (B_t - tB_1)$

$$E[\tilde{B}_s, \tilde{B}_t] = E[(B_s - sB_1), (B_t - tB_1)]$$

$$= E[B_t B_s] - tE[B_1 B_s] - stE[B_t B_1] + tsE[B_1^2]$$

$$= s - ts - \cancel{st} + \cancel{ts}$$

$$= s(1-t)$$

hence the auto covariance function for the given brownian bridge can be given by

$$cov(s,t) = s(1-t). \quad \text{——} (i).$$

note → we have used the result that

$$E[B_s, B_t] = \min\{s, t\}$$

↳ we have chosen $s < t$ hence $E[B_s B_t] = s$

↳ as we have $0 \leq s, t \leq 1$ we can write $\min\{1, s \text{ or } t\} = s \text{ or } t$.

Q3

we know that $\rightarrow$

$$d(B_t^2) = 2B_t\,dB_t + dt$$

$$B_t^2 = 2\int_0^t B_\tau\,dB_\tau + t$$

$$\frac{B_t^2 - t}{2} = \int_0^t B_\tau\,dB_\tau \quad \text{—— (I)} \quad \text{(result)}$$

$$E\left[\int_0^t B_\tau\,dB_\tau\right] = E\left[\frac{B_t^2 - t}{2}\right] = \frac{1}{2}\left(E\left[B_t^2\right] - t\right)$$

$$= \tfrac{1}{2}(t - t) = 0 \quad \text{(Ans)}$$

$$\left\{ B_t^2 \sim \chi^2 \text{ distr.} \right\} \text{—— (II)} \text{ (result}$$

$$\mathrm{Var}\left[\int_0^t B_\tau\,dB_\tau\right] = \frac{1}{4}\mathrm{Var}\left[B_t^2\right]$$

$$= \frac{1}{4}\left(E\left[B_t^4\right] - \left(E\left[B_t^2\right]\right)^2\right)$$

$$= \frac{3t^2}{4} - \frac{t^2}{4} = \left(\frac{t^2}{2}\right).$$

Q4

(a) for this part it is enough to show that the expectation of the quadratic variation $Q_n(W,t) = t$

$$\rightarrow Q_n(w,t) = \sum_{i=1}^{n} (W_{s_i} - W_{s_{i-1}})^2$$

$$\rightarrow E[Q_n(w,t)] = E\left[\sum_{i=1}^{n} (W_{s_i} - W_{s_{i-1}})^2\right]$$

$$= \sum_{i=1}^{n} E[(W_{s_i} - W_{s_{i-1}})^2]$$

$$= \sum_{i=1}^{n} var(W_{s_i} - W_{s_{i-1}})$$

$$= \sum_{i=1}^{n} (s_i - s_{i-1}) = S_n - S_0$$

$$= t$$

(b) if we notice the expression $(W_{s_i} - W_{s_{i-1}})(s_i - s_{i-1})$ carefully we can see resemblance of the covariance function with it.

so let us define $CV_n = \sum_{i=1}^{n} (W_{s_i} - W_{s_{i-1}})(s_i - s_{i-1})$

to prove that this tends to '0' we can show that the mean, variance also tend to '0'.

$$\hookrightarrow \mu(CV_n, E[CV_n]) \xlongequal{\text{mean square deviation}} \mu(CV_n, 0) \quad \left\{\begin{matrix}\text{expectation} \\ \text{of the latter} \\ \text{is zero}\end{matrix}\right\}$$

$$\hookrightarrow \mu(CV_n, 0) = var(CV_n)$$

$$= \sum_{i=1}^{n} var((W_{s_i} - W_{s_{i-1}}))(s_i - s_{i-1})^2$$

$$= \sum (s_i - s_{i-1}) \cdot (s_i - s_{i-1})^2 = \sum (s_i - s_{i-1})^3$$

$$\sum (s_i - s_{i-1})^3 \leq \max_{1 \leq i \leq n} (s_i - s_{i-1}) \sum_{i=1}^{n} (s_i - s_{i-1})^2$$

$$= \max_{1 \leq i \leq n} (s_i - s_{i-1}) \, \theta_n(\underbrace{I}_{\text{identity function}}, t)$$

$$= \lim_{n \to \infty} \max (s_i - s_{i-1}) \, \theta_n(I, t)$$

$$\longrightarrow 0$$

(c) we have used this identity for the above proof :)

$$\longrightarrow \sum_{i=1}^{n} (s_i - s_{i-1})^2 = \theta_n(I, t)$$

$$\longrightarrow \lim_{n \to \infty} \theta_n(I, t) = \theta(I, t) = 0$$

$$\lim_{n \to \infty} \theta_n(I, t) = \lim \max_{i \leq n} (s_i - s_{i-1}) \sum (s_i - s_{i-1})$$

$$= \lim \max_{i \leq n} (s_i - s_{i-1}) V_n(I, t)$$

$$= \lim \max_{i \leq n} (s_i - s_{i-1}) \, t$$

$$\longrightarrow 0$$

$$\left\{ I(s) = s \right\}$$

Q5 (a)

consider the case of $d\,g(t\ X_t\ Y_t)$; we use the taylor expansion for the given function to write the following :

$$dg(t,x,y) = \frac{\partial g(t,x,y)}{\partial t}dt + \frac{\partial g(t,x,y)}{\partial x}dX_t + \frac{\partial g(t,x,y)}{\partial y}dY_t$$

$$+ \frac{1}{2}\frac{\partial^2 g(t,x,y)}{\partial x^2}(dX_t)^2 + \frac{\partial^2 g(t,x,y)}{\partial x\partial y}\partial X_t \partial Y_t$$

$$+ \frac{1}{2}\frac{\partial^2 g(t\,X_t\,Y_t)}{\partial y^2}(dY_t)^2$$

$$(dX_t)^2 = dX_t \cdot dX_t = d[X,X](t) = \sigma_X^2(t)\,dt$$

$$(dY_t)^2 = dY_t \cdot dY_t = d[Y\,Y](t) = \sigma_Y^2(t)\,dt$$

$$(dX_t)(dY_t) = d[X,Y] = \sigma_X(t)\,\sigma_Y(t)\,dt.$$

replace the $(X,Y) = \overline{Z}_t \longrightarrow$ 2-diffusion.

$$g(t,\overline{Z}) = g(0,\overline{Z}_0) + \sum_i\sum_j \Delta g(t_j\,\overline{Z}_{ji})$$

$$= g(0,\overline{z}_0) + \sum_i\sum_j \frac{\partial g_i}{\partial t}\Delta t_j + \sum_i\sum_j \frac{\partial g_i}{\partial \overline{z}}\Delta\overline{Z}_j$$

$$+ \frac{1}{2}\sum\sum\frac{\partial^2 g_i}{\partial t^2}(\Delta t_j)^2$$

$$+ \sum_i\sum_j \frac{\partial^2 g_i}{\partial t\,\partial \overline{z}}\Delta t_j \Delta\overline{Z}$$

$$+ \frac{1}{2}\sum\sum\frac{\partial^2 g_i}{\partial z}(\Delta\overline{Z}_j)^2 + \sum R_j$$

(b) $$dg(x,y) = \frac{\partial g(x,y)}{\partial x} dX + \frac{\partial g(x,y)}{\partial y} dY + \frac{1}{2}\left(\frac{\partial^2 g(x,y)}{\partial x^2}(dx)^2 + 2\frac{\partial^2 g(x,y)}{\partial x \partial y} dx\, dY + \frac{\partial^2 g(x,y)}{\partial y^2}(dY)^2 \right)$$

#

$$\Rightarrow dg(x,y) = d(xY) = Y dX + X dY + dX dY$$

$$\frac{\partial^2 g(x,y)}{\partial x^2} = 0 = \frac{\partial^2 g(x,y)}{\partial y^2}$$

$$\frac{\partial^2 g(x,y)}{\partial x \partial y} = 1$$

$$\longrightarrow d(XY) = Y dX + X dY + d[X,Y](t)$$

$$\hookrightarrow \int_0^t X dY = \int_0^t d(XY) - Y dX - d[X,Y]$$

$$= (X_t Y_t - X_0 Y_0) - \int_0^t Y dX - \int_0^t dX\, dY$$

$$[x,y](t) = \lim_{\Delta n \to 0} \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j})(Y_{t_{j+1}} - Y_{t_j})$$

$$= \lim_{\Delta n \to 0} \sum_{j=0}^{n-1} (X_{t_{j+1}})(Y_{t_{j+1}}) - X_{t_j} Y_{t_j}$$

$$- \lim_{\Delta n \to 0} \sum_{0}^{n} X(t_j)(Y_{t_{j+1}} - Y_{t_j})$$

$$- \lim \sum_{j=0}^{n} Y(t_j)(X(t_{j+1}) - X_{t_j})$$

$$= (X_t Y_t - X_0 Y_0) - \lim_{\Delta n \to 0} \sum_0^t X_{t_j} (Y_{t_{j+1}} - Y_{t_j})$$

$$- \lim_{\Delta n \to 0} \sum_{j=0}^{n-1} Y_{t_j} (X_{t_{j+1}} - X_{t_j})$$

we can see that as $\Delta n \to 0$ or $n \to \infty$ the integrals on the RHS converge to yield

$$[X, Y](t) = (X_t Y_t - X_0 Y_0) - \int_0^t X \, dY - \int_0^t Y \, dX$$

(c) $X_1(t) = W(t)$ $\quad X_2(t) = e^{-W(t)}$

→ we need to find $d(X_1 X_2)$.

→ we can use the product rule for the same. { we $\sigma^2 = 1$ }

$$\underline{dX_1 = dW} \rightarrow (i)$$

$$dX_2 = d(e^{-W(t)}) = -e^{-W} dW + e^{-W} \frac{dt}{2} \rightarrow (ii)$$

$$d(X_1 X_2) = d(W e^{-W(t)}) = e^{-W} dX_1 + W \, dX_2 - e^{-W} dt$$

$$= e^{-W} dW + W\left(-e^{-W} dW + \frac{e^{-W}}{2} dt\right) - e^{-W} dt$$

$$= e^{-W}\left(\frac{W}{2} - 1\right) dt + e^{-W}(1 - W) dW$$

(d) here we need to consider the use of Ito's lemma for one dependent variable.

$$dX_t = \mu_t\, dt + \sigma_t\, dW_t$$

then $$dg(X_t) = g'(X_t)\, dX_t + \frac{1}{2} g''(X_t)\, \sigma^2(t)\, dt$$

choose
$$\left.\begin{aligned} g(X_c) &= X_c^2 \\ g'(X_c) &= 2X_c \\ g''(X_c) &= 2 \end{aligned}\right\} \longrightarrow (i)$$

$\longrightarrow$ $$dX_c(t) = cX_c(t)\, dt + dW_t$$

$\longrightarrow$ $$dX_c^2(t) = 2X_c(t)\, dX_c(t) + \frac{\cancel{2}^{1}}{\cancel{2}}\, dt$$

$$= (2cX_c^2 + 1)\, dt + 2X_c(t)\, dW_t$$

$\longrightarrow$ we integrate this from $0, t$ using $X_c(0) = 0$

$\longrightarrow$ $$X_c^2(t) - 0 = \int_0^t (2cX_c^2 + 1)\, ds + 2\int_0^t X_c(s)\, dW(s)$$

$$\Rightarrow \int_0^t X_c(s)\, dW(s) = \frac{1}{2}\left\{X_c^2(t) - \int_0^t (2cX_c^2 + 1)\, ds\right\}$$

$$\Rightarrow \int_0^t X_c(s)\, dW(s) = \frac{1}{2}\left(X_c^2 - \int_0^t ds\right) - c\int_0^t X_c^2\, ds$$

Q7 for this we will use the simpler case and hermite polynomials.

$$\int_0^t \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} dW_{t_1}\, dW_{t_2} \cdots dW_{t_n}$$

$$dH_{n+1}(t, W_t) = H_n(t, W_t)\, dW_t \longrightarrow (i)$$

lemma-1 $\quad H_{n+1}(t,x) = \frac{x}{n+1} H_n(t,x) - \frac{t}{n+1} H_{n-1}(t,x)$

Proof $\quad G_n(y) = (-1)^n \left(e^{y^2/2}\right) \frac{d^n}{dy^n} \exp\left(-\frac{y^2}{2}\right)$

$$H_n(t,x) = t^{n/2} \frac{1}{n!} G_n(y) \hookrightarrow y = \left(\frac{x}{\sqrt{t}}\right) = \text{similarity variable.}$$

$$G_{n+1}(y) = y G_n(y) - n G_{n-1}(y)$$

$$\hookrightarrow H_{n+1}(t,x) = t^{\frac{n+1}{2}} \frac{1}{(n+1)!} G_{n+1}\left(\frac{x}{\sqrt{t}}\right)$$

$$= t^{\frac{n+1}{2}} \frac{1}{(n+1)!} \left[\frac{x}{\sqrt{t}} G_n\left(\frac{x}{\sqrt{t}}\right) - n G_{n-1}\left(\frac{x}{\sqrt{t}}\right)\right]$$

$$= \frac{x}{n+1} H_n(t,x) - \frac{t}{n+1} H_{n-1}(t,x)$$

lemma-2 $\quad \frac{\partial}{\partial t} H_n(t,x) = \frac{-1}{2} H_{n-2}(t,x)$

lemma-3 $\quad \frac{\partial}{\partial x}(H_n(t,x)) = H_{n-1}(t,x)$

Proof : $\quad \frac{\partial}{\partial x}(H_n(t,x)) = \frac{(-t)^n}{n!} \exp\left(\left(\frac{x}{2\sqrt{t}}\right)^2\right) \left(\frac{x}{(\sqrt{t})^2} \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2t}\right)\right) + \frac{d^{n+1}}{dx^{n+1}}\left(\exp\left(\frac{-x^2}{2t}\right)\right)\right)$

$$= \frac{x}{t} H_n(t,x) + \frac{n+1}{-t} H_{n+1}(t,x)$$

$$= H_{n-1}(t,x)$$

lemma 4 $\quad d H_{n+1}(t, w_t) = H_n(t, w_t)\, dw_t$

↳ prove from ito's formulae + lemma 2 and lemma 3.

↳ we claim that :

$$I_n = \int\int \cdots \int dw_{t_1}\, dw_{t_2} \cdots dw_{t_n} = H_n(t, w_t)$$

prove using induction.

$n=1$ $\quad I(1) = w_t = H_1(t, w_t)$ true.

$n=k$ $\quad$ assume true.

$n=k+1$ $\quad H_{k+1}(t, w_t) = \int_0^t H_k(s\, w_s)\, dw_s$

$$= \int_0^t \left(\int_0^{s_n} \int_0^{s_{n-1}} \cdots \int_0^{s_2} dw_{s_1} \cdots dw_{s_k} \right) w_s$$

⟶ that is true for $n = k+1$.

hence $\quad I_n = H_n(t, w_t)$

$$I_3 = H_3(t, w_t) = \frac{1}{2}\left(\frac{1}{3} w_t^3 - t\, w_t\right)$$

in the given problem we have shifted the limits a bit and hence

we use $\longrightarrow$ $\Delta B = B_{s_{j+1}} - B_{s_j}$ instead of $W_{t/s}$

$\longrightarrow$ $\Delta t/s = t_{j+1} - t_j$ instead of t

and hence $I_s = \frac{1}{6} \Delta B^3 - \left(\frac{\Delta B \, \Delta t}{2} \right)$.