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Homework 1

Problem 1

for an ode of the form $y'(t) = f(t, y)$, where $f(t, y)$ is a sufficiently well behaved function we say that the non-adaptive numerical integration scheme has order p if :

$$y(t_{n+1}) - y(f, h, y(t_0), ..., y(t_n)) = O(h^{p+1})$$

where h is the time step and the time span is given by $t \in [t_0, t_n]$

1. **prove that the m^{th} taylor expansion has an order of $m \, \forall \, m \in \{1, 2, ...\}$.**

   for this we consider the expansion of the function $y(t)$ around t_n we have the following relation

   $$y(t_{n+1}) = y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2!}y''(t_n) + ... + \frac{h^m}{m!}y^{(m)}(t_n) + \frac{h^{m+1}}{(m+1)!}y^{(m+1)}(\xi)$$

   for some ξ such that $t_n \leq \xi \leq t_{n+1}$; if we transpose all the terms except the residual error term and take the modulus of both sides we have the following equation for the error as :

   $$e_{n+1} = h^{m+1}\left|\frac{y^{(m+1)}(\xi)}{(m+1)!}\right| = O(h^{m+1})$$

   as the $(m+1)^{th}$ order derivative is bounded hence the term in the modulus operator is bounded and the m^{th} order taylor series expansion has the order of m for all $m \in \{1, 2, ...\}$

2. **use the above theorem to prove that euler scheme has an order of 1.**

   we consider the first order expansion for the function $y(t)$ around the point t_n which is given by :

   $$y(t_{n+1}) = y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2!}y''(\xi) = y(t_n) + hf(t_n, y(t_n)) + \frac{h^2}{2!}y''(\xi)$$

   the euler estimate is given by $y_{n+1} = y_n + h \cdot f(t_n, y(t_n))$ if we compute the error term we have :

   $$e_{n+1} = h^2\left|\frac{y''(\xi)}{(2)!}\right| = O(h^2)$$

   hence by the order definition given in the problem we can conclude that the order of the euler method is $= 1$

3. **the implicit mid point method converges to the solution with an order of 2**

   let $t_n + \frac{h}{2} = t_{n+\frac{1}{2}}$ then we have $y(t_{n+1}) = y(t_n) + h \cdot f(t_{n+\frac{1}{2}}, \frac{1}{2}(y(t_n) + y(t_{n+1}) + \eta$, where η is the error term. we can observe that direct substitution of the first derivative to find the error is not possible in this case hence we modify the expression a little bit by introducing η_1

$$y(t_{n+1}) = y(t_n) + h \cdot f\left(t_{n+\frac{1}{2}}, \frac{1}{2}(y(t_n) + y(t_{n+1}))\right) + \eta_1 + \eta$$

   using the lipschitz condition we can impose a bound on η_1 which is given by the following relation :

$$|\eta_1| = h\left|f\left(t_{n+\frac{1}{2}}, \frac{1}{2}(y(t_n) + y(t_{n+1})\right) - f\left(t_{n+\frac{1}{2}}, y(t_{n+\frac{1}{2}})\right)\right| \leq \frac{1}{2}hL\left|y(t_n) + y(t_{n+1}) - 2y(t_{n+\frac{1}{2}})\right|$$

   consider the taylor's expansion at t_n with $y = y(t_n)$ and we will have the following relation:

$$|\eta_1| \leq \frac{1}{2}hL|y + (y + hy') - 2(y + hy') + O(h^2)| = O(h^3)$$

   substitute this bound back into the original expression to get the actual bout for η as follows:

$$|\eta| \leq |\eta_1| + O(h^3) = O(h^3)$$

   hence the method has an order of convergence = 2 same as the explicit mid-point or trapezoidal rule.