Homework 1

Problem 2

1. **derive the three step adam-bashforth method's explicit scheme :**

we need to interpolate the polynomial $f(t, y(t))$ and then integrate the interpolated polynomial. for this question $s = 3$ hence we will consider a interpolating polynomial of order $= 2$ passing through (t_{n-2}, f_{n-2}), (t_{n-1}, f_{n-1}), (t_n, f_n)

$$P_3(\tau) = f_{n-2}L_{n-2}(\tau) + f_{n-1}L_{n-1}(\tau) + f_nL_n(\tau)$$

we need to find the L_i's using the following relations :

- $L_{n-2} = \frac{(\tau-t_{n-1})(\tau-t_n)}{(t_{n-2}-t_{n-1})(t_{n-2}-t_n)} = \frac{1}{2h^2}(\tau - t_{n-1})(\tau - t_n)$
- $L_{n-1} = \frac{(\tau-t_n)(\tau-t_{n-2})}{(t_{n-1}-t_{n-2})(t_{n-1}-t_n)} = \frac{-1}{h^2}(\tau - t_n)(\tau - t_{n-2})$
- $L_n = \frac{(\tau-t_{n-2})(\tau-t_{n-1})}{(t_n-t_{n-1})(t_n-t_{n-2})} = \frac{1}{2h^2}(\tau - t_{n-1})(\tau - t_{n-2})$

integrating the expression $y_{n+1} = y_n + \int P_3(\tau)d\tau$; which can be written as the following:

$$y_{n+3} = y_n + f_{n-2}\int_{t_{n+1}}^{t_n} L_{n-2}(\tau)d\tau + f_{n-1}\int_{t_{n+1}}^{t_n} L_{n-1}(\tau)d\tau + f_n\int_{t_{n+1}}^{t_n} L_n(\tau)d\tau$$

we need to substitute the variable τ by a new variable $u = \frac{\tau-t_n}{h}$ such that $0 \leq u \leq 1$

- $L_{n-2} = \frac{u(u+1)}{2}$
- $L_{n-1} = -u(u + 2)$
- $L_n = \frac{(u+1)(u+2)}{2}$

on integrating we can obtain the following values for all the integrating terms on the right hand side:

- $\int_{t_{n+1}}^{t_n} L_{n-2}(\tau)d\tau = \frac{h}{2}\int_0^1 \frac{u(u+1)}{2}du = \frac{5}{12}h$
- $\int_{t_{n+1}}^{t_n} L_{n-1}(\tau)d\tau = -h\int_0^1 u(u + 2)du = \frac{-4}{3}h$
- $\int_{t_{n+1}}^{t_n} L_n(\tau)d\tau = \frac{h}{2}\int_0^1 \frac{(u+2)(u+1)}{2}du = \frac{23}{12}h$

hence we have the update rule for the three step method given by the following relation :

$$y_{n+3} = y_n + \frac{h}{12}\left(5f_{n-2} - 16f_{n-1} + 23f_n\right)$$

2. **find the order of convergence of the three-eight scheme given in the problem two part (b):**

$$y_{n+3} - y_n = h\left(\frac{3}{8}f_{n+3} + \frac{9}{8}f_{n+2} + \frac{9}{8}f_{n+1} + \frac{3}{8}f_n\right)$$

the characteristic polynomial for the method is given by :

$\rho(w) = w^3 - 1$ hence the coefficients are $\alpha_0 = -1$, $\alpha_3 = 1$; the normalising constants β_i's are as follows : $\beta_0 = \frac{3}{8}$, $\beta_1 = \frac{9}{8}$, $\beta_2 = \frac{9}{8}$, $\beta_3 = \frac{3}{8}$

the order of the method is equal to 4 as $c_0 = c_1 = c_2 = c_3 = c_4 = 0$ and $c_5 \neq 0$. we have assumed that the method converges but it can also be shown here as for convergence we need consistence and zero stability which are guaranteed to us using c_0 and c_1 and the fact that the characteristic polynomial has no roots greater than one and the multiplicity of the root equal to one is also one.

$$c_0 = \sum \alpha_j = -1 + 1 = 0 \checkmark$$

$$c_1 = \sum j\alpha_j - \beta_j = 0(-1) + 3(1) - \left(\frac{24}{8}\right) = \underline{0}$$

$$c_2 = \sum \frac{j^2}{2}\alpha_j - j\beta_j = \frac{9}{2}(+1) - \left\{ \frac{3\times 3}{8} + \frac{2\times 9}{8} + \frac{1\times 9}{8} + 0 \right\}$$
$$36/8 = 9/2$$
$$= 0$$

$$c_3 = \sum \frac{j^3}{6}\alpha_j - \frac{j^2}{2}\beta_j = \frac{27}{6}(+1) - \left\{ \frac{9\times 3}{2\ 8} + \frac{4}{2}\times\frac{9}{8} + \frac{1}{2}\times\frac{9}{8} + 0 \right\}$$
$$\frac{27}{6} - \frac{72}{16}$$
$$= 0$$

$$c_4 = \sum \frac{j^4}{24}\alpha_j - \frac{j^3}{6}\beta_j = \frac{81}{24}(+1) - \frac{1}{6}\left\{ 3^3\times\frac{3}{8} + 2^3\times\frac{9}{8} + 1^3\times 9/8 + 0 \right\}$$
$$3.375 - 3.375$$
$$= 0$$

$$c_5 = \sum \frac{j^5}{120}\alpha_j - \frac{j^4}{24}\beta_j$$
$$= 3^5/120\,(1) - \frac{1}{24}\left\{ 3^4\times 3/8 + 2^4\times 9/8 + 1^4\times 9/8 \right\}$$
$$= 2.025 - 2.0625 \neq 0$$

hence order of the method $\underline{= 4}$.