Problem 5

derive the v-stage runge-kutta method that corresponds to the given collocation points

we have the following relations :

$$a_{ji} = \int_0^{c_j} l_i(\tau) d\tau$$

$$b_j = \int_0^{1} l_i(\tau) d\tau$$

we also know that the form of $l_i(\tau)$ can be given by the following expression :

$$l_i(\tau) = \prod_{k=1, k\neq i}^{3} \frac{\tau - c_k}{c_i - c_k}$$

we need to evaluate the integrals ; let us start with $i = 1, j = 1$ and try to compute a_{11} as follows:

$$a_{11} = \int_0^{\frac{1}{4}} \frac{\tau - \frac{1}{2}}{\frac{1}{4} - \frac{1}{2}} \cdot \frac{\tau - \frac{3}{4}}{\frac{1}{4} - \frac{3}{4}} d\tau = 8 \cdot \int_0^{\frac{1}{4}} \left(\tau - \frac{1}{2}\right) \cdot \left(\tau - \frac{3}{4}\right) d\tau = 8 \cdot \frac{23}{384} = \frac{23}{48}$$

$$a_{12} = \int_0^{\frac{1}{4}} \frac{\tau - \frac{1}{4}}{\frac{1}{2} - \frac{1}{4}} \cdot \frac{\tau - \frac{3}{4}}{\frac{1}{2} - \frac{3}{4}} d\tau = -16 \cdot \int_0^{\frac{1}{4}} \left(\tau - \frac{1}{4}\right) \cdot \left(\tau - \frac{3}{4}\right) d\tau = -16 \cdot \frac{1}{48} = -\frac{1}{3}$$

$$a_{13} = \int_0^{\frac{1}{4}} \frac{\tau - \frac{1}{4}}{\frac{3}{4} - \frac{1}{4}} \cdot \frac{\tau - \frac{1}{2}}{\frac{3}{4} - \frac{1}{2}} d\tau = 8 \cdot \int_0^{\frac{1}{4}} \left(\tau - \frac{1}{4}\right) \cdot \left(\tau - \frac{1}{2}\right) d\tau = 8 \cdot \frac{5}{384} = \frac{5}{48}$$

$$a_{21} = \int_0^{\frac{1}{2}} \frac{\tau - \frac{1}{2}}{\frac{1}{4} - \frac{1}{2}} \cdot \frac{\tau - \frac{3}{4}}{\frac{1}{4} - \frac{3}{4}} d\tau = 8 \cdot \int_0^{\frac{1}{2}} \left(\tau - \frac{1}{2}\right) \cdot \left(\tau - \frac{3}{4}\right) d\tau = 8 \cdot \frac{7}{96} = \frac{7}{12}$$

$$a_{22} = \int_0^{\frac{1}{2}} \frac{\tau - \frac{1}{4}}{\frac{1}{2} - \frac{1}{4}} \cdot \frac{\tau - \frac{3}{4}}{\frac{1}{2} - \frac{3}{4}} d\tau = -16 \cdot \int_0^{\frac{1}{2}} \left(\tau - \frac{1}{4}\right) \cdot \left(\tau - \frac{3}{4}\right) d\tau = -16 \cdot \frac{1}{96} = -\frac{1}{6}$$

$$a_{23} = \int_0^{\frac{1}{2}} \frac{\tau - \frac{1}{4}}{\frac{3}{4} - \frac{1}{4}} \cdot \frac{\tau - \frac{1}{2}}{\frac{3}{4} - \frac{1}{2}} d\tau = 8 \cdot \int_0^{\frac{1}{2}} \left(\tau - \frac{1}{4}\right) \cdot \left(\tau - \frac{1}{2}\right) d\tau = 8 \cdot \frac{1}{96} = \frac{1}{12}$$

we can similarly compute a_{31}, a_{32}, a_{33} and that will come out to be : $\frac{9}{16}, 0, \frac{7}{16}$ respectively and then we should compute b_j's as follows :

$$b_1 = 8\int_0^1 \left(\tau - \frac{1}{2}\right)\cdot\left(\tau - \frac{3}{4}\right)d\tau = \frac{2}{3}$$
$$b_2 = -16\int_0^1 \left(\tau - \frac{1}{4}\right)\cdot\left(\tau - \frac{3}{4}\right)d\tau = -\frac{1}{3}$$
$$b_3 = 8\int_0^1 \left(\tau - \frac{1}{4}\right)\cdot\left(\tau - \frac{1}{2}\right)d\tau = \frac{2}{3}$$

hence we have a butcher tableau for the runge-kutta with the given collocation points as shown in the figure below :

1/4	23/48	-1/3	5/48
1/2	7/12	-1/6	1/12
3/4	9/16	0	9/16
	2/3	-1/3	2/3

we can also find the order for the method using the root polynomial defined by : $g(\tau) = \left(\tau - \frac{1}{4}\right)\left(\tau - \frac{1}{2}\right)\left(\tau - \frac{3}{4}\right)$

we need to check for what least value of m does the integral given by :

$$I_m = \int_0^1 t^m g(\tau)d\tau \neq 0$$

this relation is satisfied for $m = 1$ and so the order is given by $v + m = 3 + 1 = 4$