Inverted Pendulum

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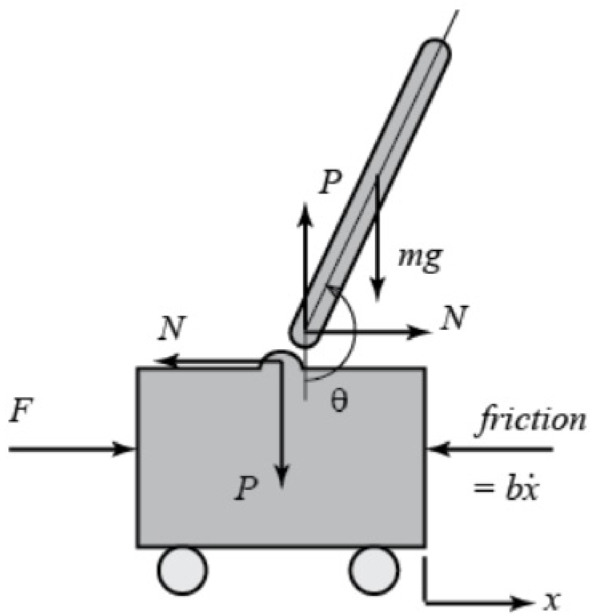
1 Mathematical Modelling

Let the mass of the cart be M and that of the rod be m. The friction coefficient of the road (influencing the cart wheels) is b and the external force on the cart is F.
The free-body diagram of the cart can be drawn as follows:



The EOM (Equations of motion) are:

$$M\ddot{x} + b\dot{x} + N = F \tag{1}$$

$$N = m\ddot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta \tag{2}$$

$$Psin\theta + Ncos\theta - mgsin\theta = ml\ddot{\theta} + m\ddot{x}cos\theta \tag{3}$$

$$-Plsin\theta - Nlcos\theta = I\ddot{\theta} \tag{4}$$

Using equations 1 and 2 we get:

$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\theta}cos\theta - ml\dot{\theta}^2 sin\theta = F \tag{5}$$

Using equations 3 and 4 we get:

$$(I+ml^2)\ddot{\theta} + mglsin\theta = -ml\ddot{x}cos\theta \tag{6}$$

The dynamics of this system are non-linear and we need to linearize the system in-order for the subsequent control design techniques to be applicable. More about linearization of dynamical systems can be found in this lecture slide: *System Analysis and Control*.
The equilibrium point we choose to linearize the system about, is where $\theta = \pi$.

Let the angle of disturbance be ϕ. Thus $\theta = \pi + \phi$.
It follows that $cos\theta = -1$ and $sin\theta = -\theta$.
Also $\dot{\theta}^2 = \dot{\phi}^2 \approx 0$
Note: The angle of disturbance is with regard to the equilibrium position ($\theta = \pi$).

With these substitutions, we get the following equations:

$$(I+ml^2)\ddot{\phi} - mgl\phi = ml\ddot{x} \tag{7}$$

$$(M+m)\ddot{x} + b\dot{x} - ml\ddot{\phi} = F \tag{8}$$

This completes the mathematical analysis of the system.