Problem Set #8

Econ 815

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Part A from PS 7

Sellers of commodities such as hotel rooms, and tickets to concerts and sporting events, face a unique problem in that their commodity has no value after a specific time point. For a hotel room, every unoccupied night translates to wasted inventory. Similarly, a ticket to a concert has no value once the concert is done. This makes pricing of the commodity a problem of utmost importance to the seller. More specifically, the seller needs to set the price such that the product is sold at some point within the finite time horizon, whilst also ensuring that the price at which the the sale occurs is as high as possible. This means that the seller needs to find a balance between selling at a particular time point versus waiting for an additional period and selling at a potentially higher price.

I model a single seller who is trying to sell a single unit of a particular commodity. For example, an individual trying to sell a ticket to a concert on Stubhub with some finite number of days until the concert. I denote the time horizon as T distinct time periods. Thus, the seller decides how to price the commodity at each time period $t = 1, \ldots, T$. The likelihood that the commodity is sold depends on the price and the state of the system. For simplicity, I assume that the system can be in one of two states: "up" or "down". The "up" state could represent weather conditions that favor the purchase of the commodity, while the "down" state represents conditions that impede the purchase. In the context of an open-air concert, a forecast of rain is represented the "down" state, while sunny skies are represented by the "up" state. I assume that the system evolves as a first-order discrete time Markov process with transition probabilities as shown in Equation (1).

$$\Pi = \begin{bmatrix} \pi_{DD} & \pi_{DU} \\ \pi_{UD} & \pi_{UU} \end{bmatrix} \tag{1}$$

where D and U represent the "down" and "up" states respectively.

If the system is in state D, the probability of the commodity being sold is ϵ_D. If it is in state U, the probability is ϵ_U. Further, I assume that the probability of sale is dependent on price through a logistic function, i.e., $\frac{\exp(A-Bp)}{1+\exp(A-Bp)}$, where A and B are constants, and p is the price. I also assume independence between the impact of price and the state of the system on the probability of sale. Thus, the probability of sale at time t is

$$q_t\left(p, x\right) = \frac{\exp\left(A - Bp\right)}{1 + \exp\left(A - Bp\right)} \cdot \epsilon_x, \quad x \in \{D, U\} \tag{2}$$

The transition probabilities given by Π govern the evolution of the system, i.e., if today is sunny/rainy, how likely is it that tomorrow will be sunny/rainy? At the start of each time period (day), the seller and the mass of consumers observe the state of the system (up or down), and the seller sets the price which is then subsequently common knowledge. The sellers choice of p thus depends on the value of ϵ_x and Π. ϵ_x directly affects the likelihood of selling in the current period, while Π defines where the system will be in the following period and hence affects the expected future value the seller could obtain if the sale doesn't occur in the current period. The evolution of the system is governed solely by "nature" i.e., the seller's choice of p does not impact the evolution of the system.

The utility that the seller gains if the sale occurs is the price set in that time period, p. By choosing not to sell, she obtains the expected value of attempting to sell in the next period. The Bellman equation is given in Equation 3.

$$V_t(x) = \max_p pq_t(p,x) + (1 - q_t(p,x))\, \mathbb{E}_{X'|X}(V_{t+1}(x')) \tag{3}$$

If the seller doesn't sell the product over the finite time horizon, she obtains a utility of 0. This defines the boundary condition $V_{T+1}(x) = 0$, $\forall x$.

The first order condition is given below

$$q_t(p,x) + p\frac{\partial q_t(p,x)}{\partial p} - \frac{\partial q_t(p,x)}{\partial p}\mathbb{E}_{X'|X}(V_{t+1}(x')) + [1 - q_t(p,x)]\,\frac{\partial \mathbb{E}_{X'|X}(V_{t+1}(x'))}{\partial p} = 0 \tag{4}$$

where $t = 1, 2, \ldots, T-1$.

Given the scale of the problem, one could just use the boundary condition to solve the optimization problem for period $T-1$, and subsequently use backward induction to obtain the optimal price.

Part B for PS 8

I solve the DPP defined by Equation 3 by first solving the maximization problem for the final period. Since $V_{T+1}(x) = 0$, $\forall x$, this problem reduces to a simple static single variable optimization problem. Further, in Equation 4, the term $\frac{\partial \mathbb{E}_{X'|X}(V_{t+1}(x'))}{\partial p} = 0$ since this opportunity cost is a function of the state of the system alone. I then recursively call a root finder on Equation 4 in order to find the optimal price for each period.

I set the following parameter values in `execute.py`.

1. $A = 3$ – constant in the logistic function.
2. $B = 1$ – coefficient of price in the logistic function.
3. $\epsilon = 0.4$ – component of the probability of sale that depends on the state of the system (specifically when the system is in the "down" state).
4. $T = 10$ – number of time periods
5. $\Pi = \begin{bmatrix} 0.1 & 0.9 \\ 0.4 & 0.6 \end{bmatrix}$ – transition probability matrix

The optimal prices (policy function) for each period are shown in Figure 1. We see that the optimal price is constant for the first eight time periods but drops considerably for the last two time periods. This is expected since the seller's opportunity cost of holding on to the ticket starts to drop. I would presume that if I allowed *epsilon* to vary with time, these monotone structure would not hold.

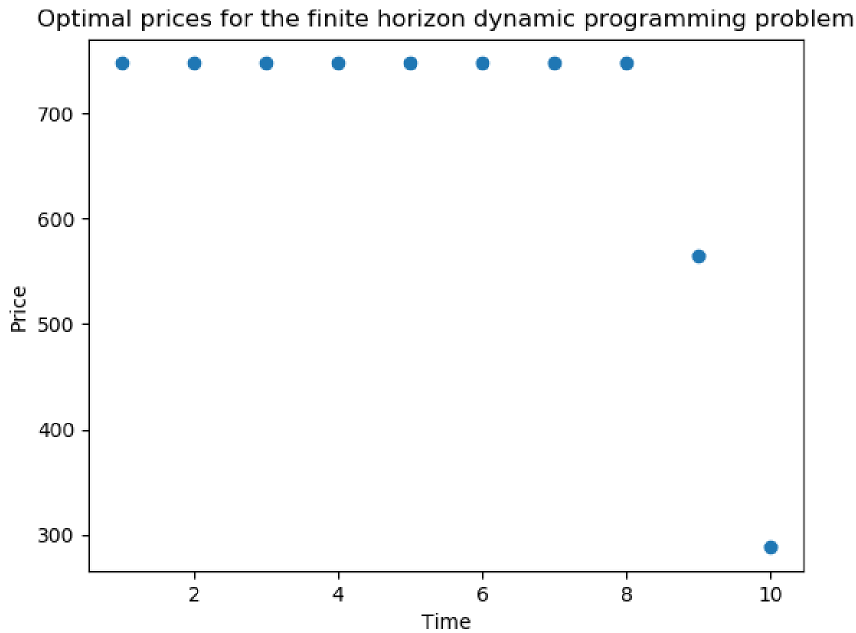

Figure 1

The value functions for the two states (over time) are shown in Figure 2. It is interesting to note that the value obtained by the seller is almost identical for both states that the system can be in for the first 7 time periods. Subsequently, we see that being in the "up" state yields a higher value to the seller (which is expected). For some reason when I attempt to create the plot by running my code through Terminal on my Mac, the plot has the first nine periods' value functions being identical for both states. I have presented the plot from what I generated when I tried the exact same code using a Jupyter notebook since this seems like the more reasonable plot. I was also having some difficulty with the scale of the vertical axis in Figure 2. The value functions take values in the range of $[10^{-26}, 10^{-21}]$ but I was unable to get this to show on the horizontal axis. I used the `py.ylim()` command to set the limits of the vertical axis. This certainly scales the graph better but the values on the axis are still not right.

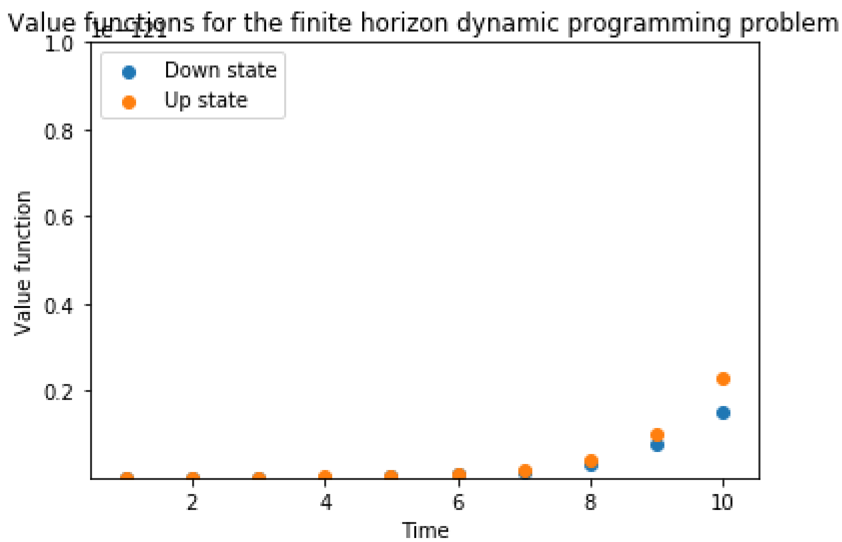
Value functions for the finite horizon dynamic programming problem
1e−121
1.0
0.8
0.6
0.4
0.2
Value function
Down state
Up state
2
4
6
8
10
Time

Figure 2