# All Pair Shortest Path

1. Bellman-Ford $n$ times $= O(n \cdot nm) = O(n^2 m) = O(n^4)$ for dense.
   (→ adj. list representation)

2. Dijkstra $O(n \cdot n^2) = O(n^3)$ for dense
   (→ priority queue of edges)
   $O(n\,(n+m)\log n) = O((n^2 + mn)\log n)$
   $= O(n^2 \log n)$ for sparse

- We will do 2 dynamic prog. algorithms with tight codes (small coefficients) on $O(n^3 \log n)$ and another $O(n^3)$

**goal** create $n \times n$ matrix of shortest-path distances $\delta(u,v)$

graph Representation (adj matrix)

$W_{n\times n} = (w_{ij})$ of edge weights.

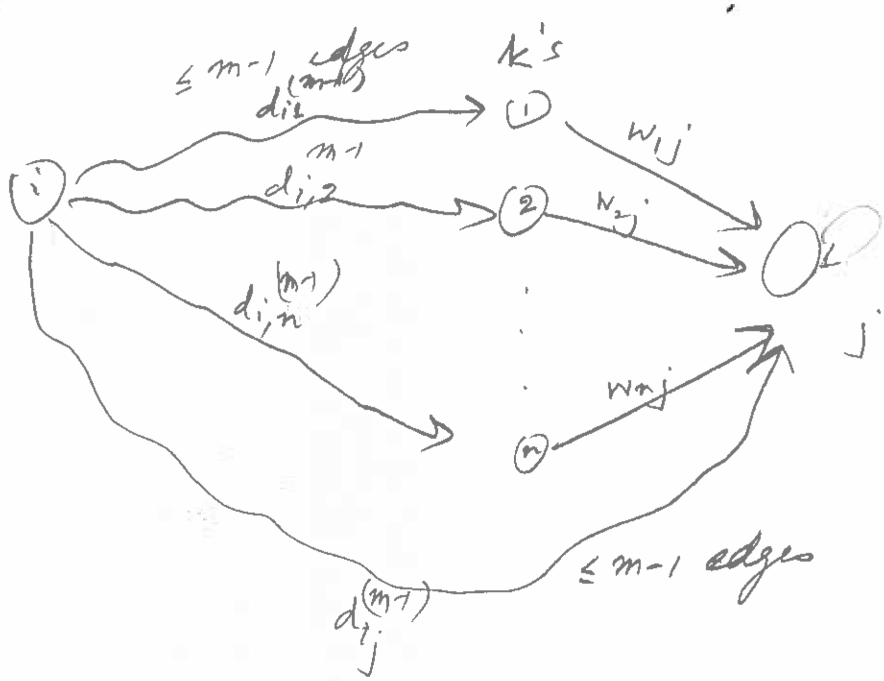
$w_{ii} = 0$

#E

Let $d_{ij}^{(m)}$ = weight of shortest-path from $i$ to $j$ that uses at most $m$ edges

$$d_{ij}^{(0)} = \begin{cases} 0 & i = j \\ \infty & i \neq j \end{cases}$$

$d_{ij}^{(m)} =$




$$d_{ij}^{(m)} = \min\left\{ d_{ij}^{(m-1)}, \min_{k}\left\{d_{ik}^{(m-1)} + w_{kj}\right\}\right\}$$

$$= \min\left\{ d_{ij}^{m-1}, \left\{ d_{ij}^{(m-1)} + w_{jj}\right\}_{k=j}, \min_{k \neq j}\left\{d_{ik}^{(m-1)} + w_{kj}\right\}\right\}$$

$$d_{ij}^{(m)} = \min_{1 \le k \le n}\left\{d_{ik}^{(m-1)} + w_{kj}\right\} \qquad \text{—— (1)}$$

<u>Answer</u>: $\delta(i,j) = d_{ij}^{(n-1)} = d_{ij}^{(n)} = d_{ij}^{(n+1)} = \ldots$

Pseudocode for 'relaxation step'

```
for k ← 1 to n
    if dij > dik + wkj
    then dij ← dik + wkj
```

$O(n)$

$d_{ij}^{(n-1)}$ $\quad O(n^2 \cdot n)$

$$\text{Time} = O(n \cdot n^3) = O(n^4)$$

# Similarity to Matrix Multiplication

$C_{n\times n} = A_{n\times n} \cdot B_{n\times n}$ $n \times n$ matrices

$$c_{ij} = \sum_{1 \le k \le n} a_{ik} \cdot b_{kj} \quad — (2)$$

Compare with

$$d_{ij}^{(m)} = \min_{1 \le k \le n} \{ d_{ik}^{(m-1)} + w_{kj} \}. \quad — (1)$$

Replace $c_{ij} \to d_{ij}^{(m)}$

$a_{ik} \to d_{ik}^{(m-1)}$

$b_{kj} \to w_{kj}$

$+ \to \min$

$\cdot \to +$

$$\Rightarrow \quad D^{(m)} = D^{(m-1)} \text{"}\times\text{"} \; W \quad — (3)$$

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identity of "X"

mat mult

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \\ & & \ddots & & \\ 0 & & & & 1 \end{bmatrix} \to \begin{bmatrix} 0 & \infty & \infty & \cdots & \infty \\ & 0 & & & \\ & & \ddots & & \\ & & & & 0 \end{bmatrix} = D^{(0)}$$

| min | + |
|---|---|
| $\infty \leftarrow 0$ | |

| | |
|---|---|
| $+ \leftarrow \cdot$ | |
| $0 \leftarrow 1$ | |

$D^{(1)} = D^{0} W = W$

$D^{2} = D^{1} W = W^{2}$

$D^{(n-1)} = W^{n-1}$ $\qquad n(\Theta(n^3)) = \Theta(n^4)$

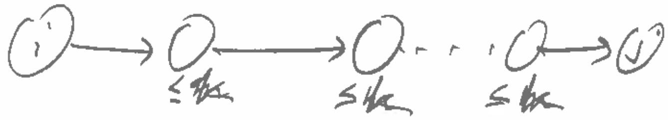
But $W \to W^2 \to W^4 \to \cdots \to W^{2^{\lceil \log_2 n-1 \rceil}}$ overshoot, no problem

$\Theta(n^3 \log n)$

# Floyd-Warshall Algorithm

Dynamic Prog but faster : $\Theta(n^3)$

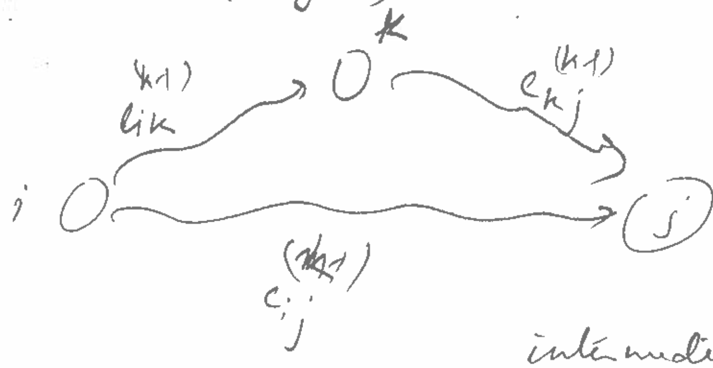
$c_{ij}^{(k)}$ = weight of a shortest path from $i$ to $j$ with intermediate vertices in $\{1, 2, \ldots, k\}$




Then $\delta(i,j) = c_{ij}^{n}$

$c_{ij}^{0} = w_{ij}$

$c_{ij}^{(k)} = \min\left\{ c_{ij}^{k-1},\ c_{ik}^{(k-1)} + c_{kj}^{(k-1)} \right\}$




intermediates are $\{1, 2, \ldots, k-1\}$

pseudocode

for $k \leftarrow 1$ to $n$
  for $i \leftarrow 1$ to $n$
    for $j \leftarrow 1$ to $n$
      $c_{ij}^{k} = \min\left(c_{ij}^{k-1},\ c_{ik}^{k-1} + c_{kj}^{k-1}\right)$

space $O(n^3)$ $O(n^2)$

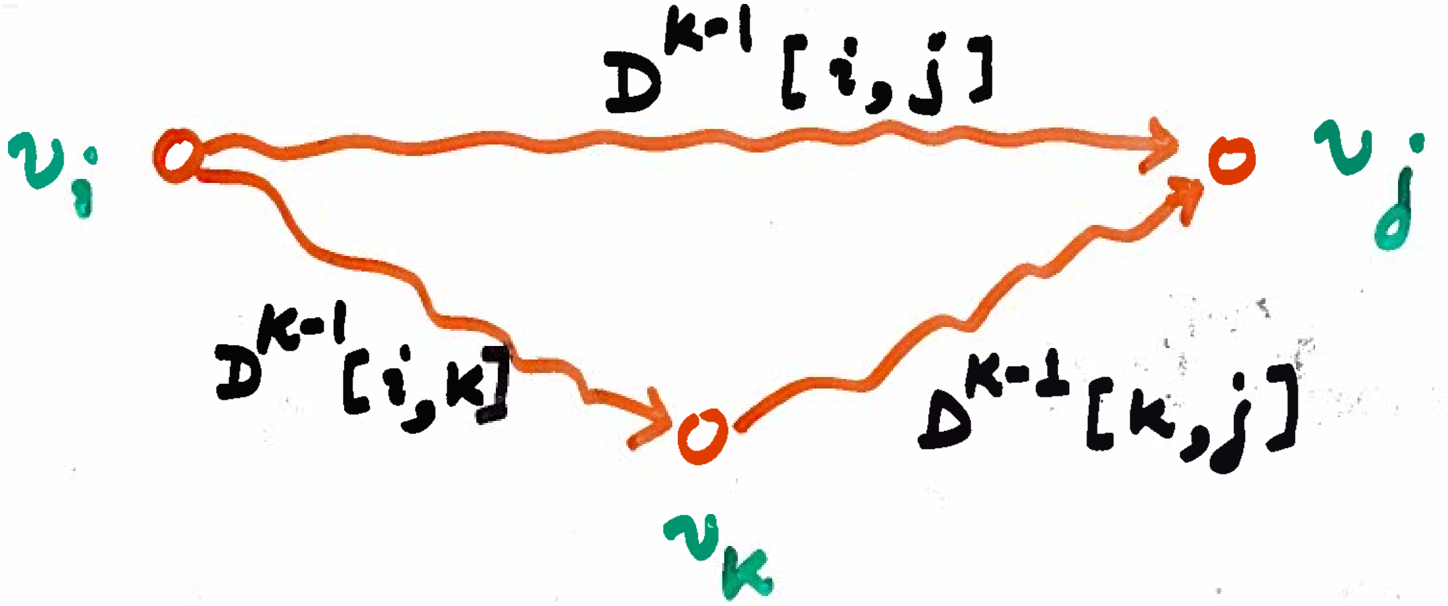
$\Theta(n^3)$.

# ALL-TO-ALL SHORTEST PATHS

$D^{k}[i,j]$ = the weight of a shortest path for $v_i$ to $v_j$ using nodes from $\{v_1, v_2, \dots v_k\}$ as intermediate vertices in the path.

$$D^{(0)} = (w_{ij})_{n \times n} = \begin{cases} 0 & i = j \\ \infty & \{i,j\} \notin E \\ \text{weight of } \{i,j\} & \end{cases}$$

$D^{(n)}$ = output




$$D^{(k)}[i,j] =$$

$$\min\left[D^{(k-1)}[i,j],\ D^{(k-1)}[i,k] + D^{(k-1)}[k,j]\right]$$

# ALL-TO-ALL

D = Weight Matrix

for $k \leftarrow 1$ to $n$ do

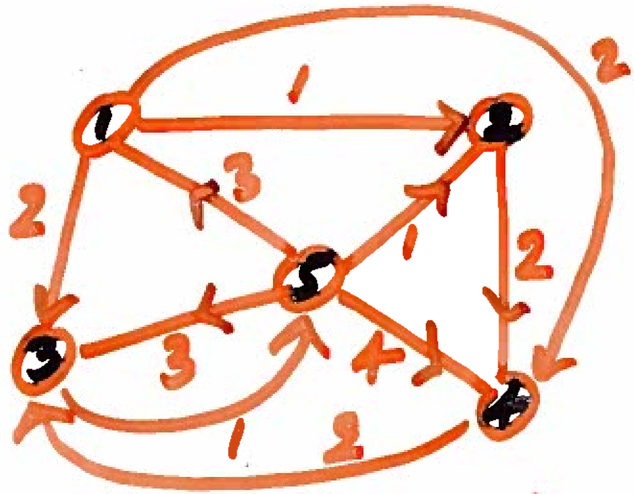
for $i \leftarrow 1$ to $n$ do

for $j \leftarrow 1$ to $n$ do

$D[i,j] = \min\left[D[i,j],\ D[i,k] + D[k,j]\right]$

endfor ; endfor

endfor.




$D^{(0)} =$

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 2 | ∞ |
| 2 | ∞ | 0 | ∞ | 2 | ∞ |
| 3 | ∞ | ∞ | 0 | ∞ | 1 |
| 4 | ∞ | ∞ | 2 | 0 | ∞ |
| 5 | 3 | 1 | 3 | 4 | 0 |

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 2 | ∞ |
| 2 | ∞ | 0 | ∞ | 2 | ∞ |
| 3 | ∞ | ∞ | 0 | ∞ | 1 |
| 4 | ∞ | ∞ | 2 | 0 | ∞ |
| 5 | 3 | 1 | 3 | 4 | 0 |

$D^{(1)}$

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 2 | ∞ |
| 2 | ∞ | 0 | ∞ | 2 | ∞ |
| 3 | ∞ | ∞ | 0 | ∞ | 1 |
| 4 | ∞ | ∞ | 2 | 0 | ∞ |
| 5 | 3 | 1 | 3 | 3 | 0 |

$D^{(2)}$

2.1

$D^{(3)}$

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 2 | 3 |
| ∞ | 0 | ∞ | 2 | ∞ |
| ∞ | ∞ | 0 | ∞ | 1 |
| ∞ | ∞ | 2 | 0 | 3 |
| 3 | 1 | 3 | 3 | 0 |

$D^{(4)}$

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 2 | 3 |
| ∞ | 0 | 4 | 2 | 5 |
| ∞ | ∞ | 0 | ∞ | 1 |
| ∞ | ∞ | 2 | 0 | 3 |
| 3 | 1 | 3 | 3 | 0 |

$D^{(5)}$

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 2 | 3 |
| 8 | 0 | 4 | 2 | 5 |
| 4 | 2 | 0 | 4 | 1 |
| 6 | 4 | 2 | 0 | 3 |
| 3 | 1 | 3 | 3 | 0 |

Q.2

# Transcritive Closure

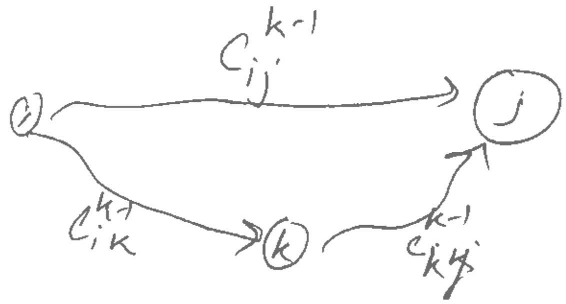
$G^* = (V, E^*)$

$(i,j) \in E^*$ iff $\exists$ path from $i$ to $j$ in $G$.

## Solution

Adj Mat (0,1) matrix

In Floyd Warshal




$$c_{ij}^{k} = c_{ij}^{k-1} \text{ OR } \left(c_{ik}^{k-1} \text{ AND } c_{kj}^{k-1}\right)$$

$O(n^3)$.

Shortest paths prove
Need to be proved