

# MA207 Short Notes

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**Ref:** Prof's slides

All the world's a differential equation,  
and the men and women are merely variables!

Differential Equations - 2

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## Power series

- For real numbers  $x_0, a_0, a_1, a_2, \dots$ , an infinite series:

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$$

is called a **power series in  $x - x_0$  with center  $x_0$**

- For a real number  $x_1$ , if the limit:

$$\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n(x_1 - x_0)^n$$

exists and is finite, then we say the power series **converges** at the point  $x = x_1$ . In this case, the value of the series at  $x_1$  is, by definition, the value of the limit.

- If the series does not converge at  $x_1$ , that is, either the limit does not exist, or it is  $\pm\infty$ , then we say the power series **diverges** at  $x_1$ . Also, a power series always converges at its center  $x = x_0$ .
- **radius of convergence (R):** for any power series:

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n$$

exactly one of these statements is true:

- the power series converges only for  $x = x_0$  (here  $R = 0$ )
- the power series converges for all values of  $x$  (here  $R = \infty$ )
- there is a positive number  $0 < R < \infty$  such that the power series converges if  $|x - x_0| < R$  and diverges if  $|x - x_0| > R$
- **Ratio test:** assume that there is an integer  $N$  such that for all  $n \geq N$  we have an  $a_n \neq 0$ . Also assume the following limit exists:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

and denote it by  $L$ . Then radius of convergence of the power series  $\sum_{n=0}^{\infty} a_n(x-x_0)^n$  is  $R = \frac{1}{L}$ .

- **Def:** Suppose we are given a sequence  $\{a_n\}_{n \geq 1}$ . For every  $k \geq 1$  define:

$$b_k = \sup_{n \geq k} \{a_n\}$$

We know  $\{b_k\}_{k \geq 1}$  is a decreasing sequence, and hence we define  **$\limsup\{a_n\}$**  as:

$$\limsup\{a_n\} = \lim_{n \rightarrow \infty} b_n$$

Similarly, we define  **$\liminf\{a_n\}$** , by replacing sup by inf in the above definition.

- For a sequence  $\{a_n\}_{n \geq 1}$ , the limit may not exist. However, the  $\limsup$  and  $\liminf$  always exist (possibly  $+\infty$  or  $-\infty$ )
- **Theorem:** Let  $\{a_n\}_{n \geq 1}$  be a sequence of real numbers. Then  $\lim_{n \rightarrow \infty} a_n$  exists if and only if  $\limsup a_n = \liminf a_n$ . Further, if  $\lim_{n \rightarrow \infty} a_n$  exists, then

$$\limsup\{a_n\} = \liminf\{a_n\} = \lim_{n \rightarrow \infty} a_n$$

- **Root test:** let  $\limsup\{|a_n|^{1/n}\} = L$ . Then radius of convergence of the power series  $\sum_{n=0}^{\infty} a_n(x-x_0)^n$  is  $R = 1/L$ .
- **Theorem:** Let  $R > 0$  be the radius of convergence of the power series  $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ , then the power series converges (absolutely) for all  $x \in (x_0 - R, x_0 + R)$ . The open interval  $(x_0 - R, x_0 + R)$  is called the **interval of convergence** of the power series.
- **Theorem:** let  $R$  be the radius of convergence of the power series  $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ . We assume  $R > 0$ . We define a function  $f : (x_0 - R, x_0 + R) \rightarrow \mathbb{R}$  by:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

This function satisfies the following properties:

- $f$  is infinitely differentiable  $\forall x \in (x_0 - R, x_0 + R)$
- the successive derivatives of  $f$  can be computed by differentiating the power series term-by-term, that is:

$$f'(x) = \sum_{n=0}^{\infty} n a_n(x-x_0)^{n-1}$$

- $f_{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)\dots(n-k+1)a_n(x-x_0)^{n-k}$
- the power series representing the derivatives  $f_{(n)}(x)$  have same radius of convergence  $R$
- we can determine the coefficients  $a_n$  (in terms of derivatives of  $f$  at  $x_0$ ) as:

$$a_n = \frac{f_{(n)}(x_0)}{n!}$$

- we can also integrate the function  $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$  term-wise, that is, if  $[a, b] \subset (x_0 - R, x_0 + R)$ , then:

$$\int_a^b f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}$$

- power series representation of  $f$  in an open interval  $I$  containing  $x_0$  is unique, that is, if:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

for all  $x \in I$ , then  $a_n = b_n$  for all  $n$

- if:

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$$

for all  $x \in I$ , then  $a_n = 0$  for all  $n$

- Power series representation of some familiar functions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$$

$$(1-x)^{-1} = \sum_0^{\infty} x^n, \quad -1 < x < 1$$

$$\cos(x) = \sum_0^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$$

$$\sinh(x) = \sum_0^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty$$

$$\cosh(x) = \sum_0^{\infty} \frac{x^{2n}}{(2n)!}, \quad -\infty < x < \infty$$

- If  $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ ,  $g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$  have radii of convergence  $R_1$  and  $R_2$  respectively, then:

$$c_1 f(x) + c_2 g(x) = \sum_0^{\infty} (c_1 a_n + c_2 b_n)(x-x_0)^n$$

has radius of convergence  $R \geq \min\{R_1, R_2\}$  for  $c_1, c_2 \in \mathbb{R}$ . Further, we can multiply the series as if they are polynomials, that is:

$$f(x)g(x) = \sum_0^{\infty} c_n(x-x_0)^n; \quad c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$$

it also has radius of convergence  $R \geq \min\{R_1, R_2\}$ .

## Taylor series and analytic functions

- Let  $f(x)$  be infinitely differentiable at  $x_0$ . The Taylor series of  $f$  at  $x_0$  is defined as the power series:

$$TS f|_{x_0} = \sum_0^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

- Suppose  $f(x)$  is infinitely differentiable at  $x_0$  and Taylor series of  $f$  at  $x_0$  converges to  $f(x)$  for all  $x$  in some open interval around  $x_0$ , then  $f$  is called **analytic** at  $x_0$ . Thus if  $f$  is analytic, then there is an interval  $I$  around  $x_0$  and  $f$  is given by a power series in  $I$ .
- Polynomials  $e^x$ ,  $\sin(x)$  and  $\cos(x)$  are analytic at all  $x \in \mathbb{R}$ .  $f(x) = \tan(x)$  is analytic at all  $x$  except  $x = (2n+1)\pi/2$ , where  $n = \pm 1, \pm 2, \dots$
- If  $f(x)$  and  $g(x)$  are analytic at  $x_0$ , then  $f(x) \pm g(x)$ ,  $f(x)g(x)$  and  $f(x)/g(x)$  (if  $g(x_0) \neq 0$ ) are analytic at  $x_0$
- If  $f(x)$  is analytic at  $x_0$  and  $g(x)$  is analytic at  $f(x_0)$ , then  $g(f(x)) = (g \circ f)(x)$  is analytic at  $x_0$
- If a power series  $\sum_0^{\infty} a_n(x-x_0)^n$  has radius of convergence  $R > 0$ , then the function  $f(x) = \sum_0^{\infty} a_n(x-x_0)^n$  is analytic at all points  $x \in (x_0 - R, x_0 + R)$
- **Theorem:** let:

$$F(x) = \frac{N(x)}{D(x)}$$

be a rational function, where  $N(x)$  and  $D(x)$  are polynomials without any common factors, that is they do not have any common (complex) zeros. Let  $\alpha_1, \dots, \alpha_r$  be distinct complex zeros of  $D(x)$ .

Then  $F(x)$  is analytic at all  $x$  except at  $x \in \{\alpha_1, \dots, \alpha_r\}$ . If  $x_0$  is different from  $\{\alpha_1, \dots, \alpha_r\}$ , then the radius of convergence  $R$  of the Taylor series of  $F$  at  $x_0$ :

$$TS F|_{x_0} = \sum_0^{\infty} \frac{F^{(n)}(x_0)}{n!} (x - x_0)^n$$

is given by:

$$R = \min\{|x_0 - \alpha_1|, |x_0 - \alpha_2|, \dots, |x_0 - \alpha_r|\}$$

- **Existence theorem:** if  $p(x)$  and  $q(x)$  are analytic functions at  $x_0$ , then every solution of:

$$y'' + p(x)y' + q(x)y = 0$$

is also analytic at  $x_0$ , and therefore any solution can be expressed as:

$$y(x) = \sum_0^{\infty} a_n (x - x_0)^n$$

If  $R_1$  is the radius of convergence of Taylor series of  $p(x)$  at  $x_0$ ,  $R_2$  is the radius of convergence of Taylor series of  $q(x)$  at  $x_0$ , then radius of convergence of  $y(x)$  is at least  $\min(R_1, R_2) > 0$

- The **standard form** of an ordinary differential equation (ODE) is:

$$y'' + p(x)y' + q(x)y = 0$$

- Steps for series solution of linear ODE:

- write ODE in the standard form  $y'' + p(x)y' + q(x)y = 0$
- choose  $x_0$  at which  $p(x)$  and  $q(x)$  are analytic. If boundary conditions at  $x_0$  are given, choose the center of the power series as  $x_0$ .
- find the minimum of radii of convergence of Taylor series of  $p(x)$  and  $q(x)$  at  $x_0$
- let  $y(x) = \sum_0^{\infty} a_n (x - x_0)^n$ , compute the power series for  $y'(x)$  and  $y''(x)$  at  $x_0$  and substitute these onto the ODE
- set the coefficients of  $(x - x_0)^n$  to zero and find recursion formula
- from the recursion formula, obtain (linearly independent) solutions  $y_1(x)$  and  $y_2(x)$ . The general solution then looks like  $y(x) = a_1 y_1(x) + a_2 y_2(x)$

- **initial value problem (IVP)** - is an ordinary differential equation together with an initial condition which specifies the value of the unknown function at a given point in the domain

- **Bessel's equation:**

$$x^2 y'' + xy' + (x^2 - v^2)y = 0$$

## Legendre polynomials

- **Legendre equation:**

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0, \text{ where } p \text{ is a real number}$$

- The two independent solutions of the Legendre equation are:

$$y_1(x) = a_0 \left[ 1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p+1)(p-2)(p+3)}{4!} x^4 + \dots \right]$$

$$y_2(x) = a_1 \left[ x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!} x^5 + \dots \right]$$

If  $p \in \{0, 2, 4, \dots\} \cup \{-1, -3, -5, \dots\}$  then  $y_1(x)$  is a polynomial function.  $y_2(x)$  is an odd function. If  $p \in \{1, 3, 5, \dots\} \cup \{-2, -4, -6, \dots\}$  then  $y_2(x)$  is a polynomial function. Thus, if  $p$  is an integer then exactly one solution is a polynomial and the other is an infinite power series.

- The general solution (of the Legendre equation):

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

is called a **Legendre function**. If  $p = m$  is an integer, then precisely one of  $y_1$  or  $y_2$  is a polynomial, and it is called the  $m^{\text{th}}$  Legendre polynomial  $P_m(x)$ . For  $m \geq 0$  note that  $P_m(x)$  is a polynomial of degree  $m$ . It is an even function if  $m$  is even and an odd function if  $m$  is odd.

- A **vector space** ( $V$ ) is a set equipped with two operations:

– addition:

$$v + w, \quad v, w \in V$$

– scalar multiplication:

$$cv, \quad c \in \mathbb{R}, \quad v \in V$$

A vector space  $V$  has a dimension, which may not be finite

- Let  $V$  be a vector space over  $\mathbb{R}$  (not necessarily finite-dimensional). A **bilinear form** on  $V$  is a map:

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$$

which is linear in both coordinates, that is:

$$\langle au + v, w \rangle = a \langle u, w \rangle + \langle v, w \rangle$$

$$\langle u, av + w \rangle = a \langle u, v \rangle + \langle u, w \rangle$$

for  $a \in \mathbb{R}$  and  $u, v \in V$

- An **inner product** on  $V$  is a bilinear form on  $V$  which is:

– symmetric:  $\langle v, w \rangle = \langle w, v \rangle$

– positive definite:  $\langle v, v \rangle \geq 0$  for all  $v$  and  $\langle v, v \rangle = 0$  iff  $v = 0$

A vector space with an inner product is called an **inner product space**.

- In an inner product space  $V$ , two vectors  $u$  and  $v$  are **orthogonal** if  $\langle v, v \rangle = 0$ . More generally, a set of vectors forms an **orthogonal system** if they are mutually orthogonal.

- A set  $\{v_i\}_{i \in I} \subset V$  is called a **basis** if the vectors in it are:

– linearly independent i.e.,  $\sum_{j=1}^m a_j v_{i_j} = 0 \implies a_j = 0$

– they span  $V$ , i.e., every  $w$  can be written as  $w = \sum_{j=1}^m a_j v_{i_j}$

An **orthogonal basis** is an orthogonal system which is also a basis.

- Consider the vector space  $\mathbb{R}^n$  with coordinate-wise addition and scalar multiplication. The rule:

$$\langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle = \sum_{i=1}^n a_i b_i$$

defines an inner product on  $\mathbb{R}^n$ . The standard basis  $\{e_1, \dots, e_n\}$  is an orthogonal basis of  $\mathbb{R}^n$ .

- **Lemma:** suppose  $V$  is a finite dimensional inner product space, and  $e_1, \dots, e_n$  is an orthogonal basis. Then for any  $v \in V$ :

$$v = \sum_{i=1}^n \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

- **Lemma:** In a finite-dimensional inner product space, there always exists an orthogonal basis. This result is not necessarily true in infinite-dimensional inner product spaces. For infinite dimensional vector spaces, we can only talk of a maximal orthogonal set. A subset  $\{e_1, e_2, \dots\}$  is called a **maximal orthogonal set** for  $V$  if:

- $\langle e_i, e_j \rangle = \delta_{ij}$
- $\langle v, e_i \rangle = 0$  for all  $i$  iff  $v = 0$

- **Def:** for a vector  $v$  in an inner product space, we define the **norm** or **length** of the vector  $v$  as:

$$\|v\| = \langle v, v \rangle^{1/2}$$

It satisfies the following three properties:

- $\|0\| = 0$  and  $\|v\| > 0$  if  $v \neq 0$
- $\|v + w\| \leq \|v\| + \|w\|$
- $\|av\| = |a| \|v\|$

for all  $v, w \in V$  and  $a \in \mathbb{R}$

- **Pythagoras theorem:** for orthogonal vectors  $v$  and  $w$  in any inner product space  $V$ :

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

More generally, for any orthogonal system  $\{v_1, \dots, v_n\}$ :

$$\|v_1 + \dots + v_n\|^2 = \|v_1\|^2 + \dots + \|v_n\|^2$$

- The set of all polynomials in the variable  $x$  is a vector space denoted by  $\mathcal{P}(x)$ . The set  $\{1, x, x^2, \dots\}$  is an infinite basis of the vector space  $\mathcal{P}(x)$ .  $\mathcal{P}(x)$  carries an inner product defined by:

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$$

We are integrating over finite interval  $[-1, 1]$  which ensures that the integral is finite. The norm of a polynomial is by definition  $\langle f, f \rangle$ :

$$\|f\| = \left( \int_{-1}^1 f(x)f(x)dx \right)^{1/2}$$

- **Derivative-transfer:** if  $f(1)g(1) = f(-1)g(-1)$ , then:

$$\int_{-1}^1 g \frac{df}{dx} = - \int_{-1}^1 f \frac{dg}{dx}$$

- **Theorem:** since  $P_m(x)$  is a polynomial of degree  $m$ , it follows that:

$$\{P_0(x), P_1(x), P_2(x), \dots\}$$

is a basis of the vector space of polynomials  $\mathcal{P}(x)$ . We have:

$$\langle P_m, P_n \rangle = \int_{-1}^1 P_m(x)P_n(x)dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$$



i.e., Legendre polynomials form an orthogonal basis for the vector space  $\mathcal{P}(x)$  and:

$$\|P_n(x)\|^2 = \frac{2}{2n+1}$$

- **Rodrigues' formula for Legendre polynomials  $P_n$ :**

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

- Let  $f_i(x)$  (for  $i \geq 0$ ) be a collection of non-zero polynomials. Assume that  $f_i(x)$  has degree  $i$ . Then  $\{f_0(x), f_1(x), \dots, f_n(x)\}$  is a basis for the vector space consisting of polynomials of degree  $\leq n$ .
- A function  $f(x)$  on  $[-1, 1]$  is **square-integrable** if:

$$\int_{-1}^1 f(x)g(x)dx < \infty$$

For instance, polynomials, continuous functions, piecewise continuous functions are square-integrable. The set of all square-integrable functions on  $[-1, 1]$  is a vector space and is denoted by  $L^2([-1, 1])$ . For square-integrable functions  $f$  and  $g$ , we define their inner product by:

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$$

- Legendre polynomials form a **maximal orthogonal set** in  $L^2([-1, 1])$ . This means that a square-integrable function which is orthogonal to all Legendre polynomials is necessarily the constant function "0". We can expand any square-integrable function  $f(x)$  on  $[-1, 1]$  in a series of Legendre polynomials:

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2n+1}{2} \int_{-1}^1 f(x)P_n(x)dx$$

This is called the **Fourier-Legendre series** (or simply the **Legendre series**) of  $f(x)$ .

- **Theorem:** The Fourier-Legendre series of  $f(x) \in L^2([-1, 1])$  given by:

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2n+1}{2} \int_{-1}^1 f(x)P_n(x)dx$$

converges in  $L^2$  norm to  $f(x)$ , that is:

$$\|f(x) - \sum_{n=0}^m c_n P_n(x)\| \rightarrow 0 \text{ as } m \rightarrow \infty$$

- **Legendre expansion theorem:** if both  $f(x)$  and  $f'(x)$  have at most a finite number of jump discontinuities in the interval  $[-1, 1]$ , then the Legendre series converges to:

$$\frac{1}{2}(f(x_-) + f(x_+)), \quad \text{for } -1 < x < 1$$

$$f(-1_+), \quad \text{for } x = -1$$

$$f(1_-), \quad \text{for } x = 1$$

In particular, the series converges to  $f(x)$  at every point of continuity  $x$

- **Least square approximation theorem:** Suppose we want to approximate  $f \in L^2([-1, 1])$  in the sense of least square by polynomials  $p(x)$  of degree  $\leq n$ , that is, we want to find a polynomial  $p(x)$  which minimizes:

$$I = \int_{-1}^1 [f(x) - p(x)]^2 dx$$

Then the minimizing polynomial is precisely the first  $n + 1$  terms of the Legendre series of  $f(x)$ , i.e.:

$$c_0 P_0(x) + \dots + c_n P_n(x) \quad c_k = \frac{2k+1}{2} \int_{-1}^1 f(x) P_k(x) dx$$

- Steps to solve a second order linear ODE using power series:
  - given an ODE of the type

$$F_0(x)y'' + F_1(x)y' + F_2(x)y = 0 \quad \dots (1)$$

first convert it to the standard form

$$y'' + \frac{F_1(x)}{F_0(x)}y' + \frac{F_2(x)}{F_0(x)}y = 0 \quad \dots (2)$$

Let

$$p(x) := \frac{F_1(x)}{F_0(x)} \quad q(x) := \frac{F_2(x)}{F_0(x)}$$

- now find the set:

$$U := \{x_0 \in \mathbb{R} \mid p(x), q(x) \text{ are analytic at } x_0\}$$

- By the existence theorem, for every  $x_0 \in U$ , there will exist two independent solutions to the above ODE, call them  $y_1(x)$  and  $y_2(x)$ , such that both of them will be analytic in an interval  $I$  around  $x_0$ .
- To find the solutions in a neighborhood of  $x_0$ , set  $y(x) = \sum_{n \geq 0} a_n (x - x_0)^n$  into the ODE (1) or (2) and get recursive relations involving the  $a_n$ . Note that when you do this, the coefficient functions  $(p(x), q(x), F_0(x), \dots)$  have to be written as power series in  $x - x_0$ . Note that the recursive relation you get, will be same, irrespective of whether you choose equation (1) or (2)
- thus, depending on the situation, you may want to choose 1 or 2. For example, for the Legendre equation, in the open interval  $(-1, 1)$  around  $x_0 = 0$ , the equation (1) looks like

$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0$$

while (2) looks like

$$y'' - 2 \left( \sum_{n \geq 0} x^{2n+1} \right) y' + p(p+1) \left( \sum_{n \geq 0} x^{2n} \right) y = 0$$

In this case it is clear that, we should choose 1, as it will be easier to work with.

## More complicated ODE's

- **Def:** consider the second-order linear ODE in standard form

$$y'' + p(x)y' + q(x)y = 0 \quad \dots (1)$$

Then:

- $x_0 \in \mathbb{R}$  is called an **ordinary point** of (1) if  $p(x)$  and  $q(x)$  are analytic at  $x_0$
- $x_0 \in \mathbb{R}$  is called **regular singular point** if  $x_0$  is not an ordinary point and both  $(x - x_0)p(x)$  and  $(x - x_0)^2 q(x)$  are analytic at  $x_0$ . If  $x_0$  is **regular singular** then there are functions  $b(x)$  and  $c(x)$  which are analytic at  $x_0$  such that

$$p(x) = \frac{b(x)}{(x - x_0)} \quad q(x) = \frac{c(x)}{(x - x_0)^2}$$

- If  $x_0 \in \mathbb{R}$  is not ordinary or regular singular, then we call it **irregular singular**

- **Cauchy-Euler equation:**

$$x^2 y'' + b_0 x y' + c_0 y = 0 \quad b_0, c_0 \in \mathbb{R}$$

$x = 0$  is a regular singular point, since we can write the ODE as:

$$y'' + \frac{b_0}{x} y' + \frac{c_0}{x^2} y = 0$$

All  $x \neq 0$  are ordinary points. Assume  $x > 0$ . Note that  $y = x^r$  solves the equation iff:

$$\begin{aligned} r(r-1) + b_0 r + c_0 &= 0 \\ \iff r^2 + (b_0 - 1)r + c_0 &= 0 \end{aligned}$$

Let  $r_1$  and  $r_2$  denote the roots of this quadratic equation. Then:

- if the roots  $r_1 \neq r_2$  are real, then  $x^{r_1}$  and  $x^{r_2}$  are two independent solutions
- if the roots  $r_1 = r_2$  are real, then  $x^{r_1}$  and  $(\log x)x^{r_1}$  are two independent solutions
- if the roots are complex (written as  $a \pm ib$ ), then  $x^a \cos(b \log x)$  and  $x^a \sin(b \log x)$  are two independent solutions

- **Theorem:** consider the ODE:

$$x^2 y'' + x b(x) y' + c(x) y = 0 \quad \dots (1)$$

where  $b(x)$  and  $c(x)$  are analytic at 0. Then  $x = 0$  is a regular singular point of the ODE. Then (1) has a solution of the form:

$$y(x) = x_1^r \sum_{n \geq 0} a_n x^n \quad a_0 \neq 0, \quad r \in \mathbb{C} \quad \dots (2)$$

The solution (2) is called **Frobenius solution** or **fractional power series solution**. The power series  $\sum_{n \geq 0} a_n x^n$  converges on  $(-\rho, \rho)$ , where  $\rho$  is the minimum of the radius of convergence of  $b(x)$  and  $c(x)$ . We will consider the solution  $y(x)$  in the open interval  $(0, \rho)$ .

- **Indicial equation:** An indicial equation, also called a characteristic equation, is a recurrence equation obtained during application of the Frobenius method of solving a second-order ordinary differential equation.

- While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are:
  - roots not differing by an integer
  - repeated roots
  - roots differing by a positive integer

The larger root always yields a fractional power series solution. In the first case, the smaller root also yields a fractional power series solution. In the second and third cases, the second solution may involve a log term.

## Some classical ODE's and their solutions

- The classical ODE's are:
  - Euler equation:  $\alpha x^2 y'' + \beta x y' + \gamma y = 0$
  - Bessel equation:  $x^2 y'' + x y' + (x^2 - v^2) y = 0$
  - Laguerre equation:  $x y'' + (1 - x) y' + \lambda y = 0$

- For all  $p \geq 1$ , the Gamma function is defined as:

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt$$

•

$$\Gamma(p+1) = p\Gamma(p) \Rightarrow \Gamma(p) = \frac{\Gamma(p+1)}{p}$$

$$\lim_{p \rightarrow 0} \Gamma(p) = \lim_{p \rightarrow 0} \frac{\Gamma(p+1)}{p} = \pm\infty$$

$$\Gamma(1/2) = \sqrt{\pi} \approx 1.772$$

- Bessel equation is the second-order linear ODE:

$$x^2 y'' + x y' + (x^2 - v^2) y = 0, \quad p \geq 0 \quad \dots 1$$

its solutions are called Bessel functions. Since  $x = 0$  is a regular singular point of (1), we get a Frobenius solution, called Bessel function of first kind. The second linearly independent solution of (1) is called Bessel function of second kind.

- Bessel function of first kind of order  $p$ :

$$J_p(x) = \sum_{n \geq 0} \frac{(-1)^n}{n! \Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}, \quad x > 0$$

- Second solution of the Bessel equation linearly independent of  $J_p(x)$ :

$$J_{-p}(x) = \left(\frac{x}{2}\right)^{-p} \sum_{n \geq 0} \frac{(-1)^n}{n! \Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n}, \quad x > 0$$

- If  $p \notin \{0, 1, 2, \dots\}$ ,  $J_p(x)$  and  $J_{-p}(x)$  are the two independent solutions of the Bessel equation. If  $p \in \{0, 1, 2, \dots\}$ , then  $J_{-p}(x) = (-1)^p J_p(x)$ . Thus in this case the second solution is not  $J_{-p}(x)$ .
- Bessel's identities:

$$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$$

$$\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$$

$$J'_p(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$$

$$J'_p(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)$$

$$J_{p-1}(x) - J_{p+1}(x) = 2J'_p(x)$$

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$$

- **Spherical Bessel functions** arise in solving wave equations in spherical coordinates
- An **algebraic function** is any function  $y = f(x)$  that satisfies an equation of the form:

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \dots + P_1(x)y + P_0(x) = 0$$

for some  $n$ , where each  $P_i(x)$  is a polynomial. Any function which can be constructed using algebraic functions is called an **elementary function**.

- **Liouville theorem:**  $J_{m+\frac{1}{2}}(x)$ 's are the only Bessel functions which are elementary functions
- **Sturm separation theorem:** if  $y_1(x)$  and  $y_2(x)$  are linearly independent solutions of:

$$y'' + P(x)y' + Q(x)y = 0$$

$P, Q$  continuous on  $(a, b)$ . Then:

- $y_1(x)$  and  $y_2(x)$  have no common zero on  $(a, b)$
- between any two successive zeros of  $y_1(x)$ , there is exactly one zero of  $y_2(x)$  and vice versa
- **Theorem:** let  $q(x)$  be continuous on the interval  $(\alpha, \beta)$ . Let  $u(x)$  be a non-trivial solution of  $u'' + q(x)u = 0$  on finite interval  $[a, b] \subset (\alpha, \beta)$ . Then  $u(x)$  has at most finite number of zeros in  $[a, b]$ . Hence if  $u(x)$  has infinitely many zeros on  $(0, \infty)$ , then the set of zeros of  $u(x)$  are not bounded.
- **Theorem:** let  $u(x)$  be a non-trivial solution of  $u'' + q(x)u = 0$  If  $q(x) < 0$  in  $(a, b)$  and continuous then  $u(x)$  has atmost one zero in  $(a, b)$
- **Theorem:** let  $u(x)$  be a non-trivial solution of  $u'' + q(x)u = 0$ . Let  $q(x)$  be continuous and  $q(x) > 0$  for all  $x > x_0 > 0$ . If  $\int_{x_0}^{\infty} q(x)dx = \infty$ , then  $u(x)$  has infinitely many zeroes on  $(0, \infty)$ .
- **Theorem:** any Bessel function has infinitely many zeros on  $(0, \infty)$
- **Corollary:** let  $Z^{(p)}$  be the set of zeros of Bessel function  $J_p(x)$  on  $(0, \infty)$ . Since  $Z^{(p)}$  is an infinite set, it is not bounded

- **Sturm comparison theorem:** let  $y(x)$  be a non-trivial solution of:

$$y'' + q(x)y = 0$$

and  $z(x)$  be a non-trivial solutions of:

$$z'' + r(x)z = 0$$

where  $q(x) > r(x) > 0$  are continuous, then  $y(x)$  vanishes at least once between any two consecutive zeroes of  $z(x)$

- **Theorem:** Substituting  $u(x) = \sqrt{x}y(x)$  in Bessel equation, we get Bessel equation in normal form ( $p \geq 0$ ):

$$u'' + q(x)u = 0, \quad q(x) = 1 + \frac{1 - 4p^2}{4x^2}$$

Now for different values of  $p$ :

- $p < 1/2 \Rightarrow$  between any two roots of  $\alpha \cos(x) + \beta \sin(x)$  there is a root of  $y_p(x)$
- $p = 1/2 \Rightarrow x_2 - x_1 = \pi$
- $p > 1/2 \Rightarrow$  between any two roots of  $y_p(x)$  there is a root of  $\alpha \cos(x) + \beta \sin(x)$
- **Theorem:** if  $p < 1/2$  then the sequence of differences of roots of  $u$ ,  $x_{n+1} - x_n$  is increasing and tends to  $\pi$ . Similarly, we can prove that if  $p > 1/2$  then the sequence of difference of roots of  $u$  is decreasing and tends to  $\pi$ .
- **Def:** for a scalar  $a$ , the [scaled Bessel functions](#)  $J_p(ax)$  are solutions of:

$$x^2 y'' + xy' + (a^2 x^2 - p^2)y = 0$$

known as [scaled Bessel equation](#)

- **Def:** an inner product on functions on  $[0, 1]$  by:

$$\langle f, g \rangle = \int_0^1 x f(x) g(x) dx$$

This is similar to the previous inner product except that  $f(x)g(x)$  is now multiplied by  $x$  and the interval of integration is from 0 to 1. We call a function on  $[0, 1]$  square integrable with respect to this inner product if:

$$\int_0^1 x f^2(x) dx < \infty$$

The multiplying factor  $x$  is called a [weight function](#).

- **Theorem:** fix  $p \geq 0$ . Let  $Z^{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \dots\}$  denote the set of zeroes of  $J_p(x)$  on  $(0, \infty)$ . Then the set of scaled Bessel functions:

$$\{J_p(\lambda_{p,1}), J_p(\lambda_{p,2}), \dots\}$$

form an orthogonal family with respect to the above inner product, i.e.,  $\langle J_p(\lambda_{p,k}x), J_p(\lambda_{p,l}x) \rangle =$

$$\int_0^1 x J_p(\lambda_{p,k}x) J_p(\lambda_{p,l}x) dx = \begin{cases} \frac{1}{2} [J_{p+1}(\lambda_{p,k})]^2, & \text{if } k = l \\ 0, & \text{if } k \neq l \end{cases}$$

- **Theorem:** fix  $p \geq 0$  and  $Z_{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \dots\}$  be zeroes of  $J_p(x)$  on  $(0, \infty)$ . Any square-integrable function  $f(x)$  on  $[0, 1]$  can be expanded in a series of scaled Bessel functions  $J_p(\lambda_{p,n}x)$  as:

$$f(x) = \sum_{n=1}^{\infty} c_n J_p(\lambda_{p,n}x)$$

where

$$c_n = \frac{2}{[J_{p+1}(\lambda_{p,n})]^2} \int_0^1 x f(x) J_p(\lambda_{p,n}x) dx$$

This is [Fourier-Bessel series](#) of  $f(x)$  for parameter  $p$ .

- Fourier-Bessel series converges to  $f(x)$  in norm, i.e.:

$$\left\| f(x) - \sum_{n=1}^m c_n J_p(\lambda_{p,n}x) \right\| \text{ converges to 0 as } m \rightarrow \infty$$

- **Bessel expansion theorem:** assume  $f$  and  $f'$  have at most a finite number of jump discontinuities in  $[0, 1]$ , then the Bessel series converges for  $0 < x < 1$  to:

$$\frac{f(x_-) + f(x_+)}{2}$$

At  $x = 1$ , the series always converges to 0 for all  $f$ . At  $x = 0$ , if  $p = 0$  then it converges to  $f(0_+)$ . At  $x = 0$ , if  $p > 0$  then it converges to 0.