MA207 Short Notes

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Ref: Prof's slides
All the world's a differential equation,
and the men and women are merely variables!

Differential Equations - 2

September 2021

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Power series

- For real numbers $x_0, a_0, a_1, a_2, \ldots$, an infinite series:

$$\sum_{n\ =\ 0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots$$

  is called a power series in $x - x_0$ with center x_0

- For a real number x_1, if the limit:

$$\lim_{N\ \to\ \infty} \sum_{n\ =\ 0}^{N} a_(x_1 - x_0)^n$$

  exists and is finite, then we say the power series converges at the point $x = x_1$. In this case, the value of the series at x_1 is, by definition, the value of the limit.

- If the series does not converge at x_1, that is, either the limit does not exist, or it is $\pm\infty$, then we say the power series diverges at x_1. Also, a power series always converges at its center $x = x_0$.

- **radius of convergence (R):** for any power series:

$$\sum_{n\ =\ 0}^{\infty} a_n(x-x_0)^n$$

  exactly one of these statements is true:

  - the power series converges ony for $x = x_0$ (here $R = 0$)
  - the power series converges for all values of x (here $R = \infty$)
  - there is a positive number $0 < R < \infty$ such that the power series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$

- **Ratio test:** assume that there is an integer N such that for all $n \geq N$ we have an $a_n \neq 0$ Also assume the following limit exists:

$$\lim_{n\ \to\ \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

  and denote it by L. Then radius of convergence of the power series $\sum_{n\ =\ 0}^{\infty} a_n(x-x_0)^n$ is $R = \frac{1}{L}$.

- **Def:** Suppose we are given a sequence $\{a_n\}_{n\geq 1}$. For every $k \geq 1$ define:

$$b_k = \sup_{n \geq k}\{a_n\}$$

  We know $\{b_k\}_{k\geq 1}$ is a decreasing sequence, and hence we define $\limsup\{a_n\}$ as:

$$\limsup\{a_n\} = \lim_{n\ \to\ \infty} b_n$$

  Similarly, we define $\liminf\{a_n\}$, by replacing sup by inf in the above definition.

- For a sequence $\{a_n\}_{n\geq 1}$, the limit may not exist. However, the lim sup and lim inf always exist (possibly $+\infty$ or $-\infty$)

- **Theorem:** Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then $lim_{n\ \to\ \infty} a_n$ exists if and only if $\limsup a_n = \liminf a_n$. Further, if $lim_{n\ \to\ \infty} a_n$ exists, then

$$\limsup\{a_n\} = \liminf\{a_n\} = \lim_{n\ \to\ \infty} a_n$$

- **Root test:** let $\limsup\{|a_n|^{1/n}\} = L$. Then radius of convergence of the power series $\sum_{n\,=\,0}^{\infty} a_n(x-x_0)^n$ is $R = 1/L$.
- **Theorem:** Let $R > 0$ be the radius of convergence of the power series $\sum_{n\,=\,0}^{\infty} a_n(x-x_0)^n$, then the power series converges (absolutely) for all $x \in (x_0 - R, x_0 + R)$. The open interval $(x_0 - R, x_0 + R)$ is called the interval of convergence of the power series.
- **Theorem:** let R be the radius of convergence of the power series $\sum_{n\,=\,0}^{\infty} a_n(x-x_0)^n$. We assume $R > 0$. We define a function $f : (x_0 - R, x_0 + R) \ \rightarrow\ \mathbb{R}$ by:

$$f(x) = \sum_{n\,=\,0}^{\infty} a_n(x-x_0)^n$$

This function satisfies the following properties:

  - f is infinitely differentiable $\forall x \in (x_0 - R, x_0 + R)$
  - the successive derivatives of f can be computed by differentiating the power series term-by-term, that is:

$$f'(x) = \sum_{n\,=\,0}^{\infty} na_n(x-x_0)^{n-1}$$

  - $f_{(k)}(x) = \sum_{n\,=\,0}^{\infty} n(n-1)\dots(n-k+1)a_n(x-x_0)^{n-k}$
  - the power series representing the derivatives $f_{(n)}(x)$ have same radius of convergence R
  - we can determine the coefficients a_n (in terms of derivatives of f at x_0) as:

$$a_n = \frac{f_{(n)}(x_0)}{n!}$$

  - we can also integrate the function $f(x) = \sum_{n\,=\,0}^{\infty} a_n(x-x_0)^n$ term-wise, that is, if $[a, b] \subset (x_0 - R, x_0 + R)$, then:

$$\int_a^b f(x)dx = \sum_{n\,=\,0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}$$

  - power series representation of f in an open interval I containing x_0 is unique, that is, if:

$$f(x) = \sum_{n\,=\,0}^{\infty} a_n(x-x_0)^n = \sum_{n\,=\,0}^{\infty} b_n(x-x_0)^n$$

for all $x \in I$,then $a_n = b_n$ for all n

  - if:

$$\sum_{n\,=\,0}^{\infty} a_n(x-x_0)^n = 0$$

for all $x \in I$, then $a_n = 0$ for all n

- Power series representation of some familiar functions:

$$e^x = \sum_0^{\infty} \frac{x^n}{n!}\ , \quad -\infty < x < \infty$$

$$sin(x) = \sum_0^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\ , \quad -\infty < x < \infty$$

$$(1-x)^{-1} = \sum_{0}^{\infty} x^n \ , \quad -1 < x < 1$$

$$cos(x) = \sum_{0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \ , \quad -\infty < x < \infty$$

$$\sinh(x) = \sum_{0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \ , \quad -\infty < x < \infty$$

$$\cosh(x) = \sum_{0}^{\infty} \frac{x^{2n}}{(2n)!} \ , \quad -\infty < x < \infty$$

- If $f(x) = \sum_{n\,=\,0}^{\infty} a_n(x-x_0)^n, \ g(x) = \sum_{n\,=\,0}^{\infty} b_n(x-x_0)^n$ have radii of convergence R_1 and R_2 respectively, then:
$$c_1 f(x) + c_2 g(x) = \sum_{0}^{\infty} (c_1 a_n + c_2 b_n)(x-x_0)^n$$
has radius of convergence $R \geq \min\{R_1, R_2\}$ for $c_1, c_2 \in \mathbb{R}$. Further, we can multiply the series as if they are polynomials, that is:
$$f(x)g(x) = \sum_{0}^{\infty} c_n(x-x_0)^n \ ; \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$
it also has radius of convergence $R \geq \min\{R_1, R_2\}$.

Taylor series and analytic functions

- Let $f(x)$ be infinitely differentiable at x_0. The Taylor series of f at x_0 is defined as the power series:
$$TS\ f|_{x_0} = \sum_{0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$
- Suppose $f(x)$ is infinitely differentiable at x_0 and Taylor series of f at x_0 converges to $f(x)$ for all x in some open interval around x_0, then f is called analytic at x_0. Thus if f is analytic, then there is an interval I around x_0 and f is given by a power series in I.
- Polynomials e^x, $sin(x)$ and $cos(x)$ are analytic at all $x \in \mathbb{R}$. $f(x) = tan(x)$ is analytic at all x except $x = (2n+1)\pi/2$, where $n = \pm 1, \pm 2, \ldots$
- If $f(x)$ and $g(x)$ are analytic at x_0, then $f(x) \pm g(x)$, $f(x)g(x)$ and $f(x)/g(x)$ (if $g(x_0) \neq 0$) are analytic at x_0
- If $f(x)$ is analytic at x_0 and $g(x)$ is analytic at $f(x_0)$, then $g(f(x)) = (g \circ f)(x)$ is analytic at x_0
- If a power series $\sum_{0}^{\infty} a_n(x-x_0)^n$ has radius of convergence $R > 0$, then the function $f(x) = \sum_{0}^{\infty} a_n(x-x_0)^n$ is analytic at all points $x \in (x_0 - R, x_0 + R)$
- **Theorem:** let:
$$F(x) = \frac{N(x)}{D(x)}$$
be a rational function, where $N(x)$ and $D(x)$ are polynomials without any common factors, that is they do not have any common (complex) zeros. Let $\alpha_1, \ldots, \alpha_r$ be distinct complex zeros of $D(x)$.

Then $F(x)$ is analytic at all x except at $x \in \{\alpha_1, \dots, \alpha_r\}$.If x_0 is different from $\{\alpha_1, \dots, \alpha_r\}$, then the radius of convergence R of the Taylor series of F at x_0:

$$TS\ F|_{x_0} = \sum_0^\infty \frac{F^{(n)}(x_0)}{n!}(x - x_0)^n$$

is given by:

$$R = \min\{|x_0 - \alpha_1|, |x_0 - \alpha_2|, \dots, |x_0 - \alpha_r|\}$$

- **Existence theorem:** if $p(x)$ and $q(x)$ are analytic functions at x_0, then every solution of:

$$y'' + p(x)y' + q(x)y = 0$$

is also analytic at x_0, and therefore any solution can be expressed as:

$$y(x) = \sum_0^\infty a_n(x - x_0)^n$$

If $R1$ is the radius of convergence of Taylor series of $p(x)$ at x_0, R_2 is the radius of convergence of Taylor series of $q(x)$ at x_0, then radius of convergence of $y(x)$ is at least $\min(R_1, R_2) > 0$

- The standard form of an ordinary differential equation (ODE) is:

$$y'' + p(x)y' + q(x)y = 0$$

- Steps for series solution of linear ODE:
  - write ODE in the standard form $y'' + p(x)y' + q(x)y = 0$
  - choose x_0 at which $p(x)$ and $q(x)$ are analytic. If boundary conditions at x_0 are given, choose the center of the power series as x_0.
  - find the minimum of radii of convergence of Taylor series of $p(x)$ and $q(x)$ at x_0
  - let $y(x) = \sum_0^\infty a_n(x-x_0)^n$, compute the power series for $y'(x)$ and $y''(x)$ at x_0 and substitute these onto the ODE
  - set the coefficients of $(x - x_0)^n$ to zero and find recursion formula
  - from the recursion formula, obtain (linearly independent) solutions $y_1(x)$ and $y_2(x)$. The general solution then looks like $y(x) = a_1y_1(x) + a_2y_2(x)$

- **initial value problem (IVP) -** is an ordinary differential equation together with an initial condition which specifies the value of the unknown function at a given point in the domain

- **Bessel's equation:**

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$

Legendre polynomials

- **Legendre equation:**

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0, \text{ where } p \text{ is a real number}$$

- The two independent solutions of the Legendre equation are:

$$y_1(x) = a_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p+1)(p-2)(p+3)}{4!}x^4 + \dots\right]$$

$$y_2(x) = a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!}x^5 + \dots\right]$$

If $p \in \{0, 2, 4, \dots\} \cup \{-1, -3, -5, \dots\}$ then $y_1(x)$ is a polynomial function. $y_2(x)$ is an odd function. If $p \in \{1, 3, 5, \dots\} \cup \{-2, -4, -6, \dots\}$ then $y_2(x)$ is a polynomial function. Thus, if p is an integer then exactly one solution is a polynomial and the other is an infinite power series.

- The general solution (of the Legendre equation):

  $$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

  is called a Legendre function. If $p = m$ is an integer, then precisely one of y_1 or y_2 is a polynomial, and it is called the m^{th} Legendre polynomial $P_m(x)$. For $m \geq 0$ note that $P_m(x)$ is a polynomial of degree m. It is an even function if m is even and an odd function if m is odd.

- A vector space (V) is a set equipped with two operations:
  - addition:
  $$v + w, \quad v, w \in V$$
  - scalar multiplication:
  $$cv, \quad c \in \mathbb{R}, \ v \in V$$

  A vector space V has a dimension, which may not be finite

- Let V be a vector space over $\mathbb{R}$ (not necessarily finite-dimensional). A bilinear form on V is a map:

  $$\langle , \rangle : V \times V \to \mathbb{R}$$

  which is linear in both coordinates, that is:

  $$\langle au + v, w\rangle = a\langle u, v\rangle + \langle v, w\rangle$$

  $$\langle u, av + w\rangle = a\langle u, v\rangle + \langle u, w\rangle$$

  for $a \in \mathbb{R}$ and $u, v \in V$

- An inner product on V is a bilinear form on V which is:
  - symmetric: $\langle v, w\rangle = \langle w, v\rangle$
  - positive definite: $\langle v, v\rangle \geq 0$ for all v and $\langle v, v\rangle = 0$ iff $v = 0$

  A vector space with an inner product is called an inner product space.

- In an inner product space V, two vectors u and v are orthogonal if $\langle v, v\rangle = 0$. More generally, a set of vectors forms an orthogonal system if they are mutually orthogonal.

- A set $\{v_i\}_{i \in I} \subset V$ is called a basis if the vectors in it are:
  - linearly independent i.e., $\sum_{j\,=\,1}^{m} a_j v_{i_j} = 0 \implies a_j = 0$
  - they span V, i.e., every w can be written as $w = \sum_{j\,=\,1}^{m} a_j v_{i_j}$

  An orthogonal basis is an orthogonal system which is also a basis.

- Consider the vector space $\mathbb{R}^n$ with coordinate-wise addition and scalar multiplication. The rule:

  $$\langle (a_1, \dots, a_n), (b_1, \dots, b_n)\rangle = \sum_{i\,=\,1}^{n} a_i b_i$$

  defines an inner product on $\mathbb{R}^n$. The standard basis $\{e_1, \dots, e_n\}$ is an orthogonal basis of $\mathbb{R}^n$.

- **Lemma:** suppose V is a finite dimensional inner product space, and $e_1, \ldots, e_n$ is an orthogonal basis. Then for any $v \in V$:
$$v = \sum_{i\,=\,1}^{n} \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$
- **Lemma:** In a finite-dimensional inner product space, there always exists an orthogonal basis. This result is not necessarily true in infinite-dimensional inner product spaces. For infinite dimensional vector spaces, we can only talk of a maximal orthogonal set. A subset $\{e_1, e_2, \ldots\}$ is called a maximal orthogonal set for V if:
  - $\langle e_i, e_j \rangle = \delta_{ij}$
  - $\langle v, e_i \rangle = 0$ for all i iff $v = 0$
- tbDef: for a vector v in an inner product space, we define the norm or length of the vector v as:
$$||v|| = \langle v, v \rangle^{1/2}$$
It satisfies the following three properties:
  - $||0|| = 0$ and $||v|| > 0$ if $v \neq 0$
  - $||v + w|| \leq ||v|| + ||w||$
  - $||av|| = |a|\,||v||$

  for all $v, w \in V$ and $a \in \mathbb{R}$
- **Pythagoras theorem:** for orthogonal vectors v and w in any inner product space V:
$$||v + w||^2 = ||v||^2 + ||w||^2$$
More generally, for any orthogonal system $\{v_1, \ldots, v_n\}$:
$$||v_1 + \ldots + v_n||^2 = ||v_1||^2 + \ldots + ||v_n||^2$$
- The set of all polynomials in the variable x is a vector space denoted by $\mathscr{P}(x)$. The set $\{1, x, x^2, \ldots\}$ is an infinite basis of the vector space $\mathscr{P}(x)$. $\mathscr{P}(x)$ carries an inner product defined by:
$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$
We are integrating over finite interval $[-1, 1]$ which ensures that the integral is finite. The norm of a polynomial is by definition $\langle f, f \rangle$:
$$||f|| = \left(\int_{-1}^{1} f(x)f(x)dx \right)^{1/2}$$
- **Derivative-transfer:** if $f(1)g(1) = f(-1)g(-1)$, then:
$$\int_{-1}^{1} g\frac{df}{dx} = -\int_{-1}^{1} f\frac{dg}{dx}$$
- **Theorem:** since $P_m(x)$ is a polynomial of degree m,, it follows that:
$$\{P_0(x), P_1(x), P_2(x), \ldots\}$$
is a basis of the vector space of polynomials $\mathscr{P}(x)$. We have:
$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, & if\ m \neq n \\ \frac{2}{2n+1}, & if\ m = n \end{cases}$$

i.e., Legendre polynomials form an orthogonal basis for the vector space $\mathscr{P}(x)$ and:

$$||P_n(x)||^2 = \frac{2}{2n+1}$$

- **Rodrigues' formula for Legendre polynomials P_n:**

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

- Let $f_i(x)$ (for $i \geq 0$) be a collection of non-zero polynomials. Assume that $f_i(x)$ has degree i. Then $\{f_0(x), f_1(x), \ldots, f_n(x)\}$ is a basis for the vector space consisting of polynomials of degree $\leq n$.

- A function $f(x)$ on $[-1,1]$ is square-integrable if:

$$\int_{-1}^{1} f(x)g(x)dx < \infty$$

For instance, polynomials, continuous functions, piecewise continuous functions are square-integrable. The set of all square-integrable functions on $[-1,1]$ is a vector space and is denoted by $L^2([-1,1])$. For square-integrable functions f and g, we define their inner product by:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$

- Legendre polynomials form a maximal orthogonal set in $L^2([-1,1])$. This means that a square-integrable function which is orthogonal to all Legendre polynomials is necessarily the constant function "0". We can expand any square-integrable function $f(x)$ on $[11,1]$ in a series of Legendre polynomials:

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)dx$$

This is called the Fourier-Legendre series (or simply the Legendre series) of $f(x)$.

- **Theorem:** The Fourier-Legendre series of $f(x) \in L^2([-1,1])$ given by:

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)dx$$

converges in L^2 norm to f(x), that is:

$$||f(x) - \sum_{n=0}^{m} c_n P_n(x)|| \;\rightarrow\; 0 \;\; as \;\; m \;\rightarrow\; \infty$$

- **Legendre expansion theorem:** if both $f(x)$ and $f'(x)$ have at most a finite number of jump discontinuities in the interval $[-1,1]$, then the Legendre series converges to:

  - $$\frac{1}{2}(f(x_-) + f(x_+)), \quad for \; -1 < x < 1$$

  - $$f(-1_+), \quad for \; x = -1$$

  - $$f(1_-), \quad for \; x = 1$$

In particular, the series converges to $f(x)$ at every point of continuity x

- **Least square approximation theorem:** Suppose we want to approximate $f \in L^2([-1,1])$ in the sense of least square by polynomials $p(x)$ of degree $\leq n$, that is, we want to find a polynomial $p(x)$ which minimizes:

$$I = \int_{-1}^{1} [f(x) - p(x)]^2 dx$$

  Then the minimizing polynomial is precisely the first $n+1$ terms of the Legendre series of $f(x)$, i.e.:

$$c_0 P_0(x) + \ldots + c_n P_n(x) \quad c_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) dx$$

- Steps to solve a second order linear ODE using power series:
  - given an ODE of the type

$$F_0(x)y'' + F_1(x)y' + F_2(x)y = 0 \quad \ldots (1)$$

    first convert it to the standard form

$$y'' + \frac{F_1(x)}{F_0(x)} y' + \frac{F_2(x)}{F_0(x)} y = 0 \quad \ldots (2)$$

    Let

$$p(x) := \frac{F_1(x)}{F_0(x)} \quad q(x) := \frac{F_2(x)}{F_0(x)}$$

  - now find the set:

$$U := \{x_0 \in \mathbb{R} \mid p(x), q(x) \text{ are analytic at } x_0\}$$

  - By the existence theorem, for every $x_0 \in U$, there will exist two independent solutions to the above ODE, call them $y_1(x)$ and $y_2(x)$, such that both of them will be analytic in an interval I around x_0.
  - To find the solutions in a neighborhood of x_0, set $y(x) = \sum_{n \geq 0} a_n (x - x_0)^n$ into the ODE (1) or (2) and get recursive relations involving the a_n. Note that when you do this, the coefficient functions $(p(x), q(x), F_0(x), ..)$ have to be written as power series in $x - x_0$. Note that the recursive relation you get, will be same, irrespective of whether you choose equation (1) or (2)
  - thus, depending on the situation, you may want to choose 1 or 2. For example, for the Legendre equation, in the open interval $(-1, 1)$ around $x_0 = 0$, the equation (1) looks like

$$\left(1 - x^2\right) y'' - 2xy' + p(p+1)y = 0$$

    while (2) looks like

$$y'' - 2\left(\sum_{n \geq 0} x^{2n+1}\right) y' + p(p+1)\left(\sum_{n \geq 0} x^{2n}\right) y = 0$$

    In this case it is clear that, we should choose 1, as it will be easier to work with.

More complicated ODE's

- **Def:** consider the second-order linear ODE in standard form

$$y'' + p(x)y' + q(x)y = 0 \quad \ldots (1)$$

Then:

  - $x_0 \in \mathbb{R}$ is called an ordinary point of (1) if $p(x)$ and $q(x)$ are analytic at x_0
  - $x_0 \in \mathbb{R}$ is called regular singular point if x_0 is not an ordinary point and both $(x - x_0)\, p(x)$ and $(x - x_0)^2\, q(x)$ are analytic at x_0 If x_0 is regular singular then there are functions $b(x)$ and $c(x)$ which are analytic at x_0 such that

$$p(x) = \frac{b(x)}{(x - x_0)} \qquad q(x) = \frac{c(x)}{(x - x_0)^2}$$

  - If $x_0 \in \mathbb{R}$ is not ordinary or regular singular, then we call it irregular singular

- **Cauchy-Euler equation:**

$$x^2 y'' + b_0 x y' + c_0 y = 0 \qquad b_0, c_0 \in \mathbb{R}$$

$x = 0$ is a regular singular point, since we can write the ODE as:

$$y'' + \frac{b_0}{x} y' + \frac{c_0}{x^2} y = 0$$

All $x \neq 0$ are ordinary points. Assume $x > 0$. Note that $y = x^r$ solves the equation iff:

$$r(r-1) + b_0 r + c_0 = 0$$
$$\iff r^2 + (b_0 - 1)\, r + c_0 = 0$$

Let r_1 and r_2 denote the roots of this quadratic equation. Then:

  - if the roots $r_1 \neq r_2$ are real, then x^{r_1} and x^{r_2} are two independent solutions
  - if the roots $r_1 = r_2$ are real, then x^{r_1} and $(\log x) x^{r_1}$ are two independent solutions
  - if the roots are complex (written as $a \pm ib$), then $x^a \cos(b \log x)$ and $x^a \sin(b \log x)$ are two independent solutions

- **Theorem:** consider the ODE:

$$x^2 y'' + x b(x) y' + c(x) y = 0 \quad \ldots (1)$$

where $b(x)$ and $c(x)$ are analytic at 0. Then $x = 0$ is a regular singular point of the ODE. Then (1) has a solution of the form:

$$y(x) = x_1^r \sum_{n \geq 0} a_n x^n \quad a_0 \neq 0, \quad r \in \mathbb{C} \quad \ldots (2)$$

The solution (2) is called Frobenius solution or fractional power series solution. The power series $\sum_{n \geq 0} a_n x^n$ converges on $(-\rho, \rho)$, where ρ is the minimum of the radius of convergence of $b(x)$ and $c(x)$. We will consider the solution $y(x)$ in the open interval $(0, \rho)$.

- **Indicial equation:** An indicial equation, also called a characteristic equation, is a recurrence equation obtained during application of the Frobenius method of solving a second-order ordinary differential equation.

- While solving an ODE around a regular singular point by the Frobenius method, the cases encountered are:
  - roots not differing by an integer
  - repeated roots
  - roots differing by a positive integer

  The larger root always yields a fractional power series solution. In the first case, the smaller root also yields a fractional power series solution. In the second and third cases, the second solution may involve a log term.

Some classical ODE's and their solutions

- The classical ODE's are:
  - Euler equation: $\alpha x^2 y'' + \beta x y' + \gamma y = 0$
  - Bessel equation: $x^2 y'' + x y' + (x^2 - v^2) y = 0$
  - Laguerre equation: $x y'' + (1 - x) y' + \lambda y = 0$
- For all $p \geq 1$, the Gamma function is defined as:

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt$$

- $$\Gamma(p+1) = p\Gamma(p) \quad \Rightarrow \quad \Gamma(p) = \frac{\Gamma(p+1)}{p}$$

$$\lim_{p \to 0} \Gamma(p) = \lim_{p \to 0} \frac{\Gamma(p+1)}{p} = \pm\infty$$

$$\Gamma(1/2) = \sqrt{\pi} \approx 1.772$$

Bessel equation

- Bessel equation is the second-order linear ODE:

$$x^2 y'' + x y' + (x^2 - v^2) y = 0, \quad p \ \geq \ 0 \qquad \ldots 1$$

  its solutions are called Bessel functions. Since $x = 0$ is a regular singular point of (1), we get a Frobenius solution, called Bessel function of first kind. The second linearly independent solution of (1) is called Bessel function of second kind.

- Bessel function of first kind of order p:

$$J_p(x) = \sum_{n \ \geq \ 0} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}, \quad x \ > \ 0$$

- Second solution of the Bessel equation linearly independent of $J_p(x)$:

$$J_{-p}(x) = \left(\frac{x}{2}\right)^{-p} \sum_{n \ \geq \ 0} \frac{(-1)^n}{n!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n}, \quad x \ > \ 0$$

- If $p \notin \{0, 1, 2, \ldots\}$, $J_p(x)$ and $J_{-p}(x)$ are the two independent solutions of the Bessel equation. If $p \in \{0, 1, 2, \ldots\}$, then $J_{-p}(x) = (-1)^p J_p(x)$. Thus in this case the second solution is not $J_{-p}(x)$.

- **Bessel's identities:**
  - $$\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$$
  - $$\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$$
  - $$J_p'(x) + \frac{p}{x} J_p(x) = J_{p-1}(x)$$
  - $$J_p'(x) - \frac{p}{x} J_p(x) = -J_{p+1}(x)$$
  - $$J_{p-1}(x) - J_{p+1}(x) = 2J_p'(x)$$
  - $$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$$
- Spherical Bessel functions arise in solving wave equations in spherical coordinates
- An algebraic function is any function $y = f(x)$ that satisfies an equation of the form:

  $$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \ldots + P_1(x)y + P_0(x) = 0$$

  for some n, where each $P_i(x)$ is a polynomial. Any function which can be constructed using algebraic functions is called an elementary function.
- **Liouville theorem:** $J_{m+\frac{1}{2}}(x)$'s are the only Bessel functions which are elementary functions
- **Sturm separation theorem:** if $y_1(x)$ and $y_2(x)$ are linearly independent solutions of:

  $$y'' + P(x)y' + Q(x)y = 0$$

  P, Q continuous on (a, b). Then:
  - $y_1(x)$ and $y_2(x)$ have no common zero on (a, b)
  - between any two successive zeros of $y_1(x)$, there is exactly one zero of $y_2(x)$ and vice versa
- **Theorem:** let $q(x)$ be continuous on the interval (α, β). Let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$ on finite interval $[a, b] \subset (\alpha, \beta)$. Then $u(x)$ has at most finite number of zeros in $[a, b]$. Hence if $u(x)$ has infinitely many zeros on $(0, \infty)$, then the set of zeros of $u(x)$ are not bounded.
- **Theorem:** let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$ If $q(x) < 0$ in (a, b) and continuous then $u(x)$ has atmost one zero in (a, b)
- **Theorem:** let $u(x)$ be a non-trivial solution of $u'' + q(x)u = 0$. Let $q(x)$ be continuous and $q(x) > 0$ for all $x > x_0 > 0$. If $\int_{x_0}^{\infty} q(x)dx = \infty$, then $u(x)$ has infinitely many zeroes on $(0, \infty)$.
- **Theorem:** any Bessel function has infinitely many zeros on $(0, \infty)$
- **Corollary:** let $Z^{(p)}$ be the set of zeros of Bessel function $J_p(x)$ on $(0, \infty)$. Since $Z^{(p)}$ is an infinite set, it is not bounded

- **Sturm comparison theorem:** let $y(x)$ be a non-trivial solution of:

$$y'' + q(x)y = 0$$

and $z(x)$ be a non-trivial solutions of:

$$z'' + r(x)z = 0$$

where $q(x) > r(x) > 0$ are continuous, then $y(x)$ vanishes at least once between any two consecutive zeroes of $z(x)$

- **Theorem:** Substituting $u(x) = \sqrt{x}y(x)$ in Bessel equation, we get Bessel equation in normal form ($p \geq 0$):

$$u'' + q(x) = 0, \quad q(x) = 1 + \frac{1 - 4p^2}{4x^2}$$

Now for different values of p:

  - $p < 1/2 \Rightarrow$ between any two roots of $\alpha cos(x) + \beta sin(x)$ there is a root of $y_p(x)$
  - $p = 1/2 \Rightarrow x_2 - x_1 = \pi$
  - $p > 1/2 \Rightarrow$ between any two roots of $y_p(x)$ there is a root of $\alpha cos(x) + \beta sin(x)$

- **Theorem:** if $p < 1/2$ then the sequence of differences of roots of u, $x_{n+1} - x_n$ is increasing and tends to π. Similarly, we can prove that if $p > 1/2$ then the sequence of difference of roots of u is decreasing and tends to π.

- **Def:** for a scalar a, the scaled Bessel functions $J_p(ax)$ are solutions of:

$$x^2y'' + xy' + (a^2x^2 - p^2)y = 0$$

known as scaled Bessel equation

- **Def:** an inner product on functions on $[0, 1]$ by:

$$\langle f, g\rangle = \int_0^1 xf(x)g(x)dx$$

This is similar to the previous inner product except that $f(x)g(x)$ is now multiplied by x and the interval of integration is from 0 to 1. We call a function on $[0, 1]$ square integrable with respect to this inner product if:

$$\int_0^1 xf^2(x)dx < \infty$$

The multiplying factor x is called a weight function.

- **Theorem:** fix $p \geq 0$. Let $Z^{(p)} = \{\lambda_{p,1}, \lambda_{p,2}, \ldots\}$ denote the set of zeroes of $J_p(x)$ on $(0, \infty)$. Then the set of scaled Bessel functions:

$$\{J_p(\lambda_{p,1}), J_p(\lambda_{p,2}), \ldots\}$$

form an orthogonal family with respect to the above inner product, i.e., $\langle J_p(\lambda_{p,k}x), J_p(\lambda_{p,l}x)\rangle =$

$$\int_0^1 xJ_p(\lambda_{p,k}x)J_p(\lambda_{p,l}x)dx = \begin{cases} \frac{1}{2}[J_{p+1}(\lambda_{p,k})]^2, & if\ k\ =\ l \\ 0, & if\ k\ \neq\ l \end{cases}$$

- **Theorem:** fix $p \geq 0$ and $Z_{(p)} = \lambda_{p,1}, \lambda_{p,2}, \ldots\}$ be zeroes of $J_p(x)$ on $(0, \infty)$. Any square-integrable function $f(x)$ on $[0,1]$ can be expanded in a series of scaled Bessel functions $J_p(\lambda_{p,n}x)$ as:
$$f(x) = \sum_{n \geq 1} c_n J_p(\lambda_{p,n}x)$$
where
$$c_n = \frac{2}{[J_{p+1}(\lambda_{p,n})]^2} \int_0^1 x f(x) J_p(\lambda_{p,n} x dx$$
This is Fourier-Bessel series of $f(x)$ for parameter p.

- Fourier-Bessel series converges to $f(x)$ in norm, i.e.:
$$\left|\left| f(x) - \sum_{n=1}^{m} c_n J_p(\lambda_{p,n}x) \right|\right| \text{ converges to 0 as } m \to \infty$$

- **Bessel expansion theorem:** assume f and f' have at most a finite number of jump discontinuities in $[0,1]$, then the Bessel series converges for $0 < x < 1$ to:
$$\frac{f(x_-) + f(x_+)}{2}$$
At $x = 1$, the series always converges to 0 for all f. At $x = 0$, if $p = 0$ then it converges to $f(0_+)$. At $x = 0$, if $p > 0$ then it converges to 0.

Fourier series

- A Boundary value problem (BVP) is a system of ordinary differential equations with solution and derivative values specified at more than one point
- An eigen value is each of a set of values of a parameter for which a differential equation has a non-zero solution (an eigenfunction) under given conditions
- Nonzero solutions for an eigenvalue λ are called λ-eigenfunction, or eigenfunction associated with λ.
- Solving an eigenvalue problem means finding all its eigenvalues and associated eigenfunctions
- **Theorem:** the eigenvalue problem:
$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$
has infinitely many positive eigenvalues:
$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \ldots$$
with associated eigenfunctions:
$$y_n(x) = sin\frac{n\pi x}{L}, \quad n = 1, 2, \ldots$$
there are no other eigenvalues

- **Theorem:** the eigenvalue problem:

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

has an eigenvalue $\lambda_0 = 0$ with eigenfunction $y_0 = 1$, and infinitely many positive eigenvalues:

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \ldots$$

with associated eigenfunctions:

$$y_n(x) = cos\frac{n\pi x}{L}, \quad n = 1, 2, \ldots$$

there are no other eigenvalues

- **Theorem:** the eigenvalue problem:

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$$

has infinitely many positive eigenvalues:

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4L^2}, \quad n = 1, 2, \ldots$$

with associated eigenfunctions:

$$y_n(x) = sin\frac{(2n+1)\pi x}{2L}, \quad n = 1, 2, \ldots$$

there are no other eigenvalues

- **Def:** we say two integrable unctions f and g are orthogonal on an interval $[a, b]$ if:

$$\int_a^b f(x)g(x)dx = 0$$

More generally, we say functions $\phi 1, \phi 2, \ldots, \phi n, \ldots$ (finite or infinitely many) are orthogonal on $[a, b]$ if:

$$\int_a^b \phi_i(x)\phi_j(x)dx = 0 \quad \text{whenever} \quad i \neq j$$

- Considering the vector space of functions on $[a, b]$, the inner product on it is defined as:

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx$$

- $L^2[a, b]$ is the subspace of those functions satisfying $\langle f, g\rangle < \infty$

- **Theorem:** let $f \in L^2[-L, L]$. Consider the series:

$$F_f(x) = a_0 + \sum_{n=1}^{\infty} (a_n cos\frac{n\pi x}{L} + b_n sin\frac{n\pi x}{L})$$

which is called the Fourier series of f on $[-L, L]$. Here:

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

and for $n > 0$:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)dxcos\frac{n\pi x}{L} \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)dxsin\frac{n\pi x}{L}$$

The above series converges to f in the L^2-norm, that is:

$$\lim_{N \to \infty} \left|\left| f - a_0 - \sum_{n=1}^{N}\left(a_n cos\frac{n\pi x}{L} + b_n sin\frac{n\pi x}{L}\right)\right|\right| = 0$$

- **Def:** a function f on $[a, b]$ is said to be piecewise smooth if:
  - f has atmost finitely many points of discontinuity
  - $f'(0)$ exists and has atmost finitely many points of discontinuity
  - $f(x_0^+) = \lim_{x \to x_0^+} f(x)$ and $f'(x_0^+) = \lim_{x \to x_0^+} f'(x)$ exists if $a \geq x_0 < b$
  - $f(x_0^-) = \lim_{x \to x_0^-} f(x)$ and $f'(x_0^-) = \lim_{x \to x_0^-} f'(x)$ exists if $a < x_0 \geq b$
- **Theorem:** let $f(x)$ be a piecewise smooth function on $[-L, L]$. Then the Fourier series:

$$F_f(x) = a_0 + \sum_{n=1}^{\infty}(a_n cos\frac{n\pi x}{L} + b_n sin\frac{n\pi x}{L})$$

  of f converges to:

$$F_f(x) = \begin{cases} \frac{1}{2}[f((-L)^+) + f(L^-)], & x = -L, L \\ \frac{1}{2}[f(x^+) + f(x^-)], & x \in (-L, L) \end{cases}$$

  Therefore, at every point x of continuity of f, the Fourier series converges to $f(x)$. If we re-define $f(x)$ at every point of discontinuity x as $\frac{1}{2}[f(x^+) + f(x^-)]$ then the Fourier series represents the function everywhere. Thus two functions can have same Fourier series.
- Suppose we have an orthogonal set $\{\phi 1, \phi 2, \ldots\}$ which has the following property. For every function f we have a series $\sum_{i \geq 1} a_i \phi_i$ which converges to f, that is:

$$\lim_{n \to \infty} ||f - \sum_{i=1}^{n} a_i \phi_i|| = 0$$

  then we say that the set $\{\phi 1, \phi 2, \ldots\}$ is a normed basis for V. Note that this is different from the notion of basis, where we need that every vector should be written as a finite linear combination of the basis vectors. The the coefficient of ϕ_n in the expansion of f is given by:

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$$

Heat equation

- A partial differential equation (PDE) is an equation involving u and the partial derivatives of u. The order of the PDE is the order of the highest partial derivative of u in the equation.
- Examples of some famous PDEs:
  - $u_t - k^2(u_{xx} + u_{yy}) = 0$: two dimensional heat equation, order 2. Here u is a function of three variables
  - $u_{tt} - c^2(u_{xx} + u_{yy}) = 0$: two dimensional wave equation, order 2. Here u is a function of three variables

- $u_{xx} + u_{yy} = 0$: two dimensional Lapalace equation, order 2. Here u is a function of two variables
- $u_{tt} + u_{xxxx} = 0$: Beam equation, order 4. Here u is a function of two variables

- Let $\mathscr{S}$ denote a space of functions. A differential operator is a map $D : \mathscr{S} \to \mathscr{S}$
- A differential operator is said to be linear if it satisfies the condition:

$$D(u + v) = D(u) + D(v)$$

heat equation, wave equation, Laplace equation and Beam equation are linear PDEs.

- The general form of first order linear differential operator in two variables x, y is:

$$L(u) = A(x, y)u_x + B(x, y)u_y + C(x, y)u$$

The general form of first order linear differential operator in three variables x, y, z is:

$$L(u) = Au_x + Bu_y + Cu_z + Du$$

where coefficients A, B, C, D and f are functions of x, y and z. The general form of second order linear PDE in two variables x, y is:

$$L(u) = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu$$

where coefficients A, B, C, D, E, F and f are functions of x and y.

- **Classification of second order linear PDE:** consider the linear differential operator L on functions in two variables:

$$L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

where $A, \ldots, F$ are functions of x and y. To the operator L we associate the discriminant $\mathbb{D}(x, y)$ given by:

$$\mathbb{D}(x, y) = A(x, y)C(x, y) - B^2(x, y)$$

The operator L is said to be:

  - elliptic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) > 0$
  - parabolic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) = 0$
  - hyperbolic at (x_0, y_0), if $\mathbb{D}(x_0, y_0) < 0$

- Two dimensional Laplace operator, $\delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is elliptic in $\mathbb{R}^2$, since $\mathbb{D} = 1$
- One dimensional heat operator (there are two variables, t and x), $H = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ is parabolic in $\mathbb{R}^2$, since $\mathbb{D} = 0$
- One dimensional wave operator (there are two variables, t and x), $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ is hyperbolic in $\mathbb{R}^2$,since $\mathbb{D} = -1$
- For the Tricomi operator, $T = \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2}$, the discriminant $\mathbb{D} = x$. Hence T is elliptic in the half-plane $x > 0$, parabolic on the y-axis and hyperbolic in the half-plane $x < 0$
- **Def:** let L be a linear differential operator. The PDE $Lu = 0$ is called homogeneous and the PDE $Lu = f$, $(f \neq 0)$ is non-homogeneous.
- **Principle 1:** if $u_1, \ldots, u_N$ are solutions of $Lu = 0$ and $c_1, \ldots, c_N$ are constants, then $\sum_{i=1}^{N} c_i u_i$ is also a solution of $Lu = 0$. In general, space of solutions of $Lu = 0$ contains infinitely many independent solutions and we may need to use infinite linear combinations of them.

- **Principle 2:** Let L be a differentiable operator of order n. Assume:
  - $u_1, u_2, \ldots$ are infinitely many solutions of $Lu = 0$
  - the series $w = \sum_{i \geq 1} c_i u_i$ with $c_1, c_2, \ldots$ constants, converges to a function, which is differentiable n times
  - term by term partial differentiation is valid for the series, that is, $Dw = \sum_{i \geq 1} c_i Du_i$, D is any partial differentiation of order $\geq$ order of L

  Then w is again a solution of $Lu = 0$.

- **Principle 3 (for non-homogenous PDE):** if u_i is a solution of $Lu = f_i$, then:

$$w = \sum_{i=1}^{N} c_i u_i$$

  with constants c_1, is a solution of $Lu = \sum_{i=1}^{N} c_i f_i$

- The formal solution of IBVP:

$$u_t = k^2 u_{xx}, \quad 0 < x < L,\ t > 0$$
$$u(0,t) = 0, \quad t \geq 0$$
$$u(L,t) = 0, \quad t \geq 0$$
$$u(x,0) = f(x), \quad 0 \leq x \leq L$$

  is:

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{(\frac{-n^2\pi^2k^2}{L^2}t)} sin\frac{n\pi x}{L}$$

  where:

$$f(x) = \sum_{n=1}^{\infty} \alpha_n sin\frac{n\pi x}{L} \text{ is the Fourier series of } f \text{ on } [0,L]$$

  that is:

$$\alpha_n = \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx$$

- **Theorem:** let $f(x)$ be continuous and piecewise smooth on $[0,L]$. Let $f(x) = \sum_{n=1}^{\infty} \alpha_n sin\frac{n\pi x}{L}$ with $\alpha_n = \frac{2}{L}\int_0^L f(x) sin\frac{n\pi x}{L} dx$ be the Fourier series of f on $[0,L]$. Then the IBVP:

$$u_t = k^2 u_{xx}, \quad 0 < x < L,\ t > 0$$
$$u(0,t) = 0, \quad t \geq 0$$
$$u(L,t) = 0, \quad t \geq 0$$
$$u(x,0) = f(x), \quad 0 \leq x \leq L$$

  has a solution:

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{(\frac{-n^2\pi^2k^2}{L^2}t)} sin\frac{n\pi x}{L}$$

  Here u_t and u_{xx} can be obtained by term-wise differentiation for $t > 0$

- The formal solution of IBVP:

$$u_t = k^2 u_{xx}, \quad 0 < x < L,\ t > 0$$

$$u_x(0,t) = 0, \quad t > 0$$
$$u_x(L,t) = 0, \quad t > 0$$
$$u(x,0) = f(x), \quad 0 \leq x \leq L$$

is:

$$u(x,t) = \sum_{n\,=\,0}^{\infty} \alpha_n e^{(\frac{-n^2\pi^2k^2}{L^2}t)} cos\frac{n\pi x}{L}$$

where:

$$S(x) = \sum_{n\,=\,0}^{\infty} \alpha_n sin\frac{n\pi x}{L} \text{ is the Fourier series of } f \text{ on } [0, L]$$

that is:

$$\alpha_0 = \frac{1}{L}\int_0^L f(x)dx$$

$$\alpha_n = \frac{2}{L}\int_0^L f(x)cos\frac{n\pi x}{L}dx$$

- **Theorem:** let $f(x)$ be continuous and piecewise smooth on $[0, L]$; $f'(0) = f'(L) = 0$. Let $S(x) = \sum_{n\,=\,0}^{\infty} \alpha_n cos\frac{n\pi x}{L}$ be the Fourier series of f on $[0, L]$. Then the IBVP:

$$u_t = k^2 u_{xx}, \quad 0 < x < L,\ t > 0$$

$$u_x(0,t) = 0, \quad t > 0$$
$$u_x(L,t) = 0, \quad t > 0$$
$$u(x,0) = f(x), \quad 0 \leq x \leq L$$

has a solution:

$$u(x,t) = \sum_{n\,=\,0}^{\infty} \alpha_n e^{(\frac{-n^2\pi^2k^2}{L^2}t)} cos\frac{n\pi x}{L}$$

Here u_t and u_{xx} can be obtained by term-wise differentiation for $t > 0$