

EE229 Short Notes

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Ref: Prof's slides, Signals and Systems by Alan V. Oppenheim

The untuned mind receives no signal from the universe!

Signal Processing - 1

September 2021

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Introduction

- All the concepts described apply to both continuous-time and discrete-time signals unless otherwise mentioned
- **continuous-time signals** - the independent variable is continuous, and thus these signals are defined for a continuum of values of the independent variable
- **discrete-time signals** - defined only at discrete times, i.e. for these signals, the independent variable takes on only a discrete set of values
- The total energy over the time interval $t_1 \leq t \leq t_2$ in a continuous-time signal $x(t)$ is defined as

$$||x||^2 = \int_{t_1}^{t_2} |x(t)|^2 dt$$

- The total energy over the time interval $n_1 \leq n \leq n_2$ in a discrete-time signal $x[n]$ is defined as

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

- Some definitions:
 - **amplitude scaling of signals** : $y(t) = \alpha x(t)$, $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$
 - **time-scaling** : $y(t) = x(\alpha t)$, $\alpha \in \mathbb{R}$
 - **time-shift** : $y(t) = x(t - \tau)$, $\tau \in \mathbb{R}$
 - **time-reversal** : $y(t) = x(-t)$
 - **DC offset** : $y(t) = \alpha + x(t)$

Different types of signals

- **periodic signal** - a signal $x(t)$ having the property that there is a positive value of T for which

$$x(t) = x(t + T)$$

The fundamental period is the smallest positive value of T

- **even and odd signals** - signals satisfying the equations $x(-t) = x(t)$ and $x(-t) = -x(t)$ respectively
- **discrete-time unit impulse or unit sample** is defined as:

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

- **discrete-time unit step** is defined as:

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

- **continuous-time unit impulse (Dirac measure)** is defined as:

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

- **continuous-time unit step** is defined as:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

- Relation between the 2 above signals:

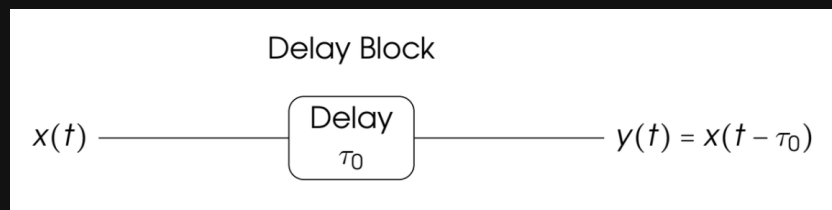
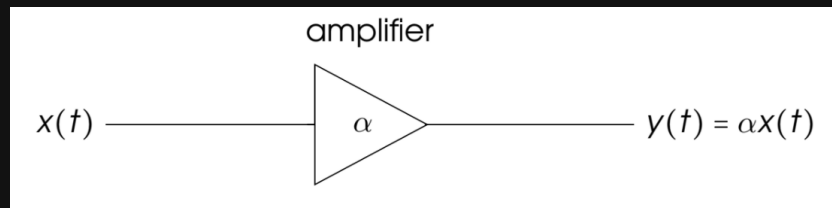
$$\delta(t) = \frac{du(t)}{dt}$$

- **energy and power signals** - energy signal is a signal whose energy is finite and power is zero whereas power signal is a signal whose power is finite and energy is infinite

Systems

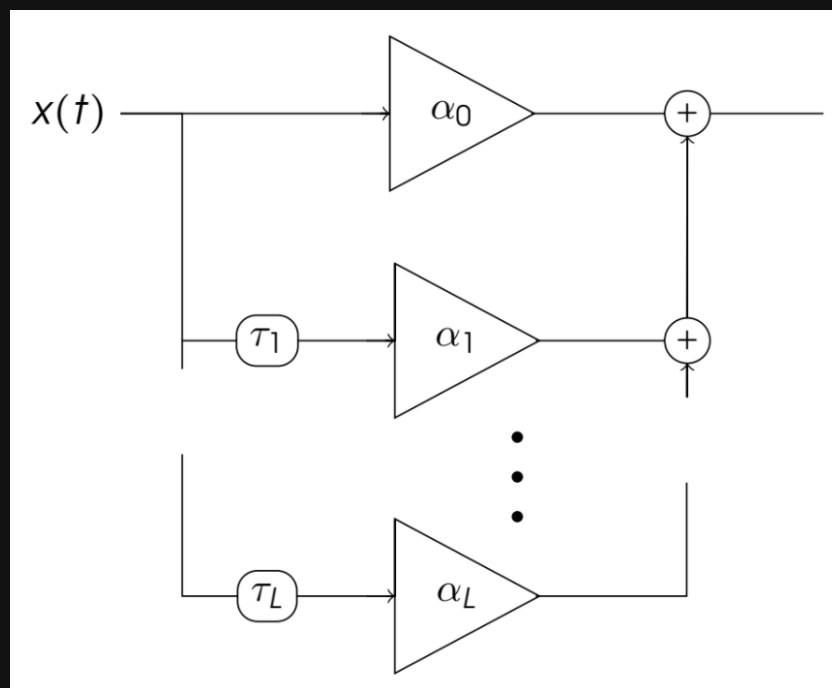
- **system** - any process that produces an output signal in response to an input signal
- **memoryless system** - output for each value of the independent variable at a given time is dependent on the input at only that same time
- The concept of **memory** in a system corresponds to the presence of a mechanism that retains or stores information about input values at times other than the current time. In many physical systems, memory is directly associated with the storage of energy.
- A system is said to be **invertible** if distinct inputs lead to distinct outputs
- If a system is invertible, then an **inverse system** exists that, when cascaded with the original system, yields an output that is equal to the input to the first system
- A system is **causal** if the output at any time depends on values of the input at only the present and past times. Such a system is often referred to as being **nonanticipative**, as the system output does not anticipate future values of the input.
- All memoryless systems are causal, since the output responds only to the current value of the input
- A **stable system** is one in which small inputs lead to responses that do not diverge. Stability of physical systems generally results from the presence of mechanisms that dissipate energy. If the input to a stable system is bounded, then the output must also be bounded and therefore cannot diverge.
- Conceptually, a system is **time-invariant** if the behaviour and characteristics of the system are fixed over time. Specifically, a system is time-invariant if a time shift in the input signal results in an identical time shift in the output signal.
- A **linear system**, in continuous or discrete time, is a system that possesses the important property of **superposition**. Let $y_1(t)$ be the response of a continuous-time system to an input $x_1(t)$, and let $y_2(t)$ be the output corresponding to the input $x_2(t)$. Then the system is linear if:
 - the response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$. (**additivity property**)
 - The response to $ax_1(t)$ is $ay_1(t)$, where a is any complex constant. (**scaling or homogeneity property**)
- A system can be linear without being time-invariant, and it can be time-invariant without being linear
- A direct consequence of the superposition property is that, for linear systems, an input which is zero for all time results in an output which is zero for all time

- Pure all-pass system has impulse response $\delta(t)$
- Some common systems:



- Generalized echo system:

$$y(t) = \sum_{l=0}^L \alpha_l x(t - \tau_l)$$



Linear time-invariant systems

- **linear time-invariant (LTI) system** - systems that are both linear and time-invariant

- Convolution sum or superposition sum of two discrete-time LTI systems $x[n]$ and $h[n]$:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Convolution integral or superposition integral of two continuous-time LTI systems $x(t)$ and $h(t)$:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Convolution is commutative, distributive and associative:

$$x(t) * h(t) = h(t) * x(t)$$

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

- Impulse response of a system ($h(t)$) and its inverse system ($h_1(t)$) satisfy:

$$h(t) * h_1(t) = \delta(t)$$

- A continuous-time LTI system is stable if the impulse response is absolutely integrable, i.e., if

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$$

- A discrete-time LTI system is stable if the impulse response is absolutely summable, i.e., if

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

- Zero-order hold and first-order hold:

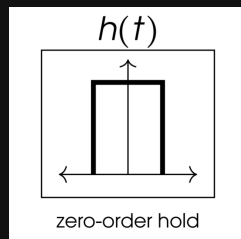


Figure : Zero-order hold

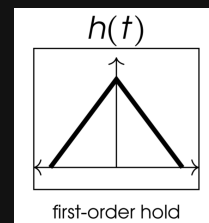


Figure : First-order hold

- **digital interpolation** - upsampling followed by digital convolution
- **cubic interpolation** - this type of interpolation gives rise to a piecewise cubic function that interpolates a set of data points and guarantees smoothness at the data points

Fourier series representation of continuous-time signals

- An even function can be represented as:

$$f(x) = \sum_{m \geq 0} a_m \cos\left(\frac{2\pi}{T} mx\right), \text{ for } \frac{-T}{2} \leq x \leq \frac{T}{2}$$

- An odd function can be represented as:

$$f(x) = \sum_{m \geq 1} b_m \sin\left(\frac{2\pi}{T} mx\right), \text{ for } \frac{-T}{2} \leq x \leq \frac{T}{2}$$

- Functions g_1, g_2 such that $\langle g_1, g_2 \rangle = 0$ for $t \in T$ is said to be orthogonal in T .
- A representation of a periodic signal as a combination of complex exponentials of discrete frequencies, which are multiples of the fundamental frequency of the signal, is known as the Fourier series representation of the signal

$$\begin{aligned} x(t) &= \alpha_0 + \sum_{n \geq 1} a_n \cos\left(\frac{2\pi}{T_d} nT\right) + \sum_{n \geq 1} b_n \sin\left(\frac{2\pi}{T_d} nT\right) \\ &= \alpha_0 + \sum_{n \geq 1} a_n \frac{(e^{j\frac{2\pi}{T_d} nt} + e^{-j\frac{2\pi}{T_d} nt})}{2} + \sum_{n \geq 1} b_n \frac{(e^{j\frac{2\pi}{T_d} nt} - e^{-j\frac{2\pi}{T_d} nt})}{2j} \\ &= \sum_{m \in \mathbb{Z}} \alpha_m e^{j\frac{2\pi}{T_d} mt} \end{aligned}$$

$$\text{where } \alpha_m = \frac{a_m - b_m}{2}, m \in \mathbb{Z}^+ \text{ and } \alpha_m = \frac{a_{|m|} + b_{|m|}}{2}, m \in \mathbb{Z}^-$$

$$\alpha_m = \frac{\langle x(t), e^{j\frac{2\pi}{T_d} mt} \rangle}{T_d} = \frac{1}{T_d} \int_{-\frac{T_d}{2}}^{\frac{T_d}{2}} x(t) e^{-j\frac{2\pi}{T_d} mt} dt$$

- **Theorem:** No two continuous functions in $[-\frac{T}{2}, \frac{T}{2}]$ will have all the Fourier series coefficients the same
- **Lemma:** Let $f(t)$ be a signal locally integrable in $[-\frac{T}{2}, \frac{T}{2}]$, if the Fourier series coefficients $\alpha_m = 0$, identically $\forall m \in \mathbb{Z}$, then $f(t) = 0$, whenever it is continuous at t
- **Properties of continuous-time Fourier series**

If the Fourier series coefficients of $x(t)$ are denoted by a_k , we will use the notation:

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

– Linearity:

If $x(t) \xleftrightarrow{\mathcal{FS}} a_k$ and $y(t) \xleftrightarrow{\mathcal{FS}} b_k$, then

$$z(t) = Ax(t) + By(t) \xleftrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k$$

– Time shifting:

If $x(t) \xleftrightarrow{\mathcal{FS}} a_k$, then

$$x(t - t_0) \xleftrightarrow{\mathcal{FS}} e^{-j\frac{2\pi}{T} kt_0} a_k$$

– **Time reversal:**

$$\text{If } x(t) \xleftrightarrow{\mathcal{FS}} a_k, \text{ then}$$

$$x(-t) \xleftrightarrow{\mathcal{FS}} a_{-k}$$

– **Time scaling:**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{(j\frac{2\pi}{T}kt)}$$

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{(j\frac{2\pi}{T}k\alpha t)}$$

– **Multiplication:**

$$\text{If } x(t) \xleftrightarrow{\mathcal{FS}} a_k \text{ and } y(t) \xleftrightarrow{\mathcal{FS}} b_k, \text{ then}$$

$$z(t) = x(t)y(t) \xleftrightarrow{\mathcal{FS}} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

– **Conjugation:**

$$\text{If } x(t) \xleftrightarrow{\mathcal{FS}} a_k, \text{ then}$$

$$x^*(t) \xleftrightarrow{\mathcal{FS}} a_{-k}^*$$

• **Parseval's relation for continuous-time periodic signals:**

– If $x(t) = \sum_{m \in \mathbb{Z}} \beta_m \phi_m t$ with $\langle \phi_m, \phi_n \rangle = \delta[m - n]$ in $-\frac{T}{2} \leq x \leq \frac{T}{2}$, then

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{m \in \mathbb{Z}} |\beta_m|^2$$

• The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{m \in \mathbb{Z}} |\alpha_m|^2$$

• A function $h(t)$ is integrable if $\int_{\mathbb{R}} |h(t)| dt < \infty$

•

$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

Property	Periodic Signal	Fourier Series Coefficients
	$\left. \begin{array}{l} x(t) \\ y(t) \end{array} \right\} \begin{array}{l} \text{Periodic with period } T \text{ and} \\ \text{fundamental frequency } \omega_0 = 2\pi/T \end{array}$	$\begin{array}{l} a_k \\ b_k \end{array}$
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting	$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a_{-k}^*
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau)y(t - \tau)d\tau$	$Ta_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk \frac{2\pi}{T} a_k$
Integration	$\int_{-\infty}^t x(\tau) d\tau$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a_{-k}^* \\ \Re\{a_k\} = \Re\{a_{-k}\} \\ \Im\{a_k\} = -\Im\{a_{-k}\} \\ a_k = a_{-k} \\ \angle a_k = -\angle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathcal{E}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\begin{array}{l} \Re\{a_k\} \\ j\Im\{a_k\} \end{array}$
Parseval's Relation for Periodic Signals		
$\frac{1}{T} \int_T x(t) ^2 dt = \sum_{k=-\infty}^{+\infty} a_k ^2$		

Figure : Properties of continuous-time Fourier series

Fourier transforms

- The **Fourier transform** is a mathematical function that decomposes a waveform, which is a function of time, into the frequencies that make it up. The equation of Fourier transform $H(f)$ of a signal $x(t)$ is

$$H(f) = \int_{\mathbb{R}} x(t) e^{-j2\pi f t} dt$$

- The properties of the Fourier transform are:

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(f)$
	$y(t)$	$Y(f)$
<hr/>		
Linearity	$ax(t) + by(t)$	$aX(f) + bY(f)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(f)$
Frequency Shifting	$e^{j\omega_0 t} x(t)$	$X(f - f_0)$
Conjugation	$x^*(t)$	$X^*(-f)$
Time Reversal	$x(-t)$	$X(-f)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{f}{a}\right)$
Convolution	$x(t) * y(t)$	$X(f)Y(f)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
Differentiation in Time	$\frac{d}{dt} x(t)$	$j\omega X(f)$
Integration	$\int_{-\infty}^t x(t)dt$	$\frac{1}{j\omega} X(f) + \pi X(0)\delta(f)$
Differentiation in Frequency	$tx(t)$	$j \frac{d}{d\omega} X(f)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(f) = X^*(-f) \\ \Re\{X(f)\} = \Re\{X(-f)\} \\ \Im\{X(f)\} = -\Im\{X(-f)\} \\ X(f) = X(-f) \\ \angle X(f) = -\angle X(-f) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(f)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(f)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}\{x(t)\}$ [$x(t)$ real]	$\Re\{X(f)\}$
	$x_o(t) = \mathcal{O}\{x(t)\}$ [$x(t)$ real]	$j\Im\{X(f)\}$
<hr/>		
Parseval's Relation for Aperiodic Signals		
$\int_{-\infty}^{+\infty} x(t) ^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) ^2 d\omega$		

Figure : Properties of the Fourier transform

- **Poisson summation formula:** For an integrable $x(t)$ with Fourier transform $H(f)$ -

$$\sum_{n \in \mathbb{Z}} x(t - nT) = \sum_{m \in \mathbb{Z}} \alpha_m e^{(\frac{2\pi}{T} mt)},$$

at points of continuity of the LHS, where

$$\alpha_m = \frac{1}{T} H\left(\frac{m}{T}\right), \quad m \in \mathbb{Z}$$

-

$$e^{(-\pi t^2)} \xleftrightarrow{\mathcal{FT}} e^{(-\pi f^2)}$$

-

$$g_\delta(t) = \frac{1}{\sqrt{\delta}} e^{(-\pi \frac{t^2}{\delta})} \xleftrightarrow{\mathcal{FT}} e^{(-\pi f^2 \delta)} = G_\delta(f), \quad \delta > 0$$

$$\lim_{\delta \downarrow 0} g_\delta(t) := \delta(t) \text{ "Impulse or Dirac Measure"}$$

$$\lim_{\delta \downarrow 0} G_\delta(f) := \mathbb{I}_{\{f \in \mathbb{R}\}}, \text{ "DC value"}$$

- The Inverse Fourier transform is given by:

$$x(t) = \int_{\mathbb{R}} H(f) e^{(j2\pi ft)} df$$

where $H(f)$ is the Fourier transform of signal $x(t)$

- **Poisson sum:** Fourier transform (Generalized) of an impulse train:

$$\sum_{n \in \mathbb{Z}} \delta(t - nT) \xleftrightarrow{F.T.} \sum_{m \in \mathbb{Z}} \frac{1}{T} \delta\left(f - \frac{m}{T}\right)$$

- **Dual FS formula:**

$$\sum_{n \in \mathbb{Z}} X(f + n\beta) \xleftrightarrow{I.F.T.} \sum_{m \in \mathbb{Z}} \frac{1}{\beta} x\left(\frac{m}{\beta}\right) \delta\left(t - \frac{m}{\beta}\right)$$

- **Shannon's reconstruction formula:**

$$x(t) = \sum_{m \in \mathbb{Z}} x\left(\frac{m}{\beta}\right) \text{sinc}(\beta t - m)$$

- **Parseval's relation:**

$$\int_{\mathbb{R}} x(t) y^*(t) dt = \int_{\mathbb{R}} X(f) Y^*(f) df$$

$$\int_{\mathbb{R}} |x(t)|^2 dt = \int_{\mathbb{R}} |X(f)|^2 df$$

- Wireless communication bandwidths:

Application	Bandwidth
AM Radio	10kHz
2G	200kHz - 1MHz
3G	5MHz
4G	10 – 20MHz
5G	≈ 100MHz

Dirac's formalisms

- **Def:** Dirac Delta is defined as a non-negative unit area operator such that:

$$\int_{\mathbb{R}} x(t)\delta(t)dt = x(0), \quad \text{whenever } x(0^+) = x(0^-) = x(0)$$

- Fourier transform and inverse for Diracs:

$$\delta(t) \xLeftrightarrow{F.T.} \mathbb{I}_{\{f \in \mathbb{R}\}}$$

$$\mathbb{I}_{\{t \in \mathbb{R}\}} \xLeftrightarrow{F.T.} \delta(f)$$

Advanced concepts

- **Inverse DTFT:** given samples $x[n]$, $n \in \mathbb{Z}$, having a DTFT $\hat{X}(f)$, $-\frac{1}{2} \geq f \geq \frac{1}{2}$:

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(f) e^{j2\pi f n} df$$

- **Discrete Fourier transform (DFT):**

$$X[k] = \hat{X} \frac{k}{N} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}$$

- **Matrix form of DFT:**

$$\text{DFT: } \bar{X} = F \bar{x}$$

$$\text{where } F = \begin{bmatrix} \alpha_0^0 & \alpha_0^1 & \alpha_0^2 & \dots & \alpha_0^M \\ \alpha_1^0 & \alpha_1^1 & \alpha_1^2 & \dots & \alpha_1^M \\ \alpha_2^0 & \alpha_2^1 & \alpha_2^2 & \dots & \alpha_2^M \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \alpha_M^0 & \alpha_M^1 & \alpha_M^2 & \dots & \alpha_M^M \end{bmatrix}$$

here $\alpha = e^{(-j\frac{2\pi}{N})}$, $\alpha_i = \alpha^i$ (for $0 \leq i \leq N-1$), $M = N-1$

Proposition:

$$F^H F = N \mathbb{I}_N$$

- **Circular convolution (\circledast):** let $x_c[n] = \sum_{i \in \mathbb{Z}} x[n + iN] \rightarrow x_c[i] = x[i + N]$ for $-N \geq i \geq -1$, then:

$$x_c[n] \circledast h[n] = \sum_{n=0}^{N-1} h[n]x_c[k-n]$$

- Circular convolution is commutative
- **Proposition:**

$$x[n] \circledast h[n] \xrightarrow{DFT} X[k]H[k]$$

- **Nyquist condition:** to get the samples back it should satisfy the following condition:

$$\sum_{m \in \mathbb{Z}} H(f_m f_s) = 1, \quad \forall f \in \mathbb{R}$$

- In signal processing one addition + one multiplication in matrix multiplication is termed a **flop** or **computation**. Matrix multiplication requires $O(N^2)$ computations
- **Butterfly structure of FFT:**

