EE229 Short Notes

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Ref: Prof's slides, Signals and Systems by Alan V. Oppenheim
The untuned mind receives no signal from the universe!

Signal Processing - 1

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Contents

Introduction

- All the concepts described apply to both continuous-time and discrete-time signals unless otherwise mentioned
- **continuous-time signals** - the independent variable is continuous, and thus these signals are defined for a continuum of values of the independent variable
- **discrete-time signals** - defined only at discrete times, i.e. for these signals, the independent variable takes on only a discrete set of values
- The total energy over the time interval $t_1 \leq t \leq t_2$ in a continuous-time signal $x(t)$ is defined as

$$||x||^2 = \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

- The total energy over the time interval $n_1 \leq n \leq n_2$ in a discrete-time signal $x[n]$ is defined as

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

- Some definitions:
  - **amplitude scaling of signals :** $y(t) = \alpha x(t), \ \alpha \in \mathbb{R} \ or \ \alpha \in \mathbb{C}$
  - **time-scaling :** $y(t) = x(\alpha t), \ \alpha \in \mathbb{R}$
  - **time-shift :** $y(t) = x(t - \tau), \ \tau \in \mathbb{R}$
  - **time-reversal :** $y(t) = x(-t)$
  - **DC offset :** $y(t) = \alpha + x(t)$

Different types of signals

- **periodic signal** - a signal $x(t)$ having the property that there is a positive value of T for which

$$x(t) = x(t + T)$$

  The fundamental period is the smallest positive value of T

- **even and odd signals** - signals satisfying the equations $x(-t) = x(t)$ and $x(-t) = -x(t)$ respectively
- **discrete-time unit impulse or unit sample** is defined as:

$$\delta[n] = \begin{cases} 0, \ n \neq 0 \\ 1, \ n = 0 \end{cases}$$

- **discrete-time unit step** is defined as:

$$u[n] = \begin{cases} 0, \ n < 0 \\ 1, \ n \geq 0 \end{cases}$$

- **continuous-time unit impulse (Dirac measure)** is defined as:

$$\delta(t) = \begin{cases} 0, \ t \neq 0 \\ \infty, \ t = 0 \end{cases}$$

- **continuous-time unit step** is defined as:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

- Relation between the 2 above signals:

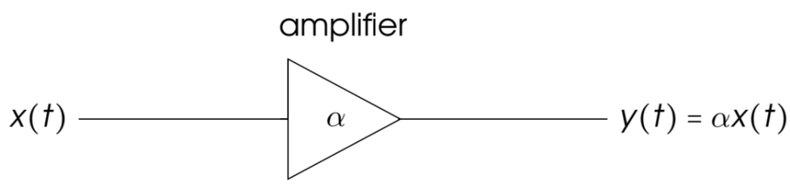
$$\delta(t) = \frac{du(t)}{dt}$$

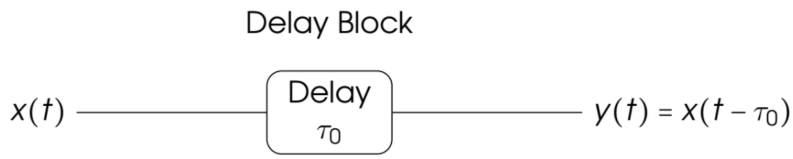
- **energy and power signals** - energy signal is a signal whose energy is finte and power is zero whereas power signal is a signal whose power is finte and energy is infinite

Systems

- **system** - any process that produces an output signal in response to an input signal
- **memoryless system** - output for each value of the independent variable at a given time is dependent on the input at only that same time
- The concept of memory in a system corresponds to the presence of a mechanism that retains or stores information about input values at times other than the current time. In many physical systems, memory is directly associated with the storage of energy.
- A system is said to be invertible if distinct inputs lead to distinct outputs
- If a system is invertible, then an colorblueinverse system exists that, when cascaded with the original system, yields an output that is equal to the input to the first system
- A system is causal if the output at any time depends on values of the input at only the present and past times. Such a system is often referred to as being nonanticipative, as the system output does not anticipate future values of the input.
- All memoryless systems are causal, since the output responds only to the current value of the input
- A stable system is one in which small inputs lead to responses that do not diverge. Stability of physical systems generally results from the presence of mechanisms that dissipate energy. If the input to a stable system is bounded, then the output must also be bounded and therefore cannot diverge.
- Conceptually, a system is time-invariant if the behaviour and characteristics of the system are fixed over time. Specifically, a system is time-invariant if a time shift in the input signal results in an identical time shift in the output signal.
- A linear system, in continuous or discrete time, is a system that possesses the important property of superposition. Let $y_1(t)$ be the response of a continuous-time system to an input $x_1(t)$, and let $y_2(t)$ be the output corresponding to the input $x_2(t)$. Then the system is linear if:
  - the response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$. (**additivity property**)
  - The response to $ax_1(t)$ is $ay_1(t)$, where a is any complex constant. (**scaling or homogeneity property**)
- A system can be linear without being time-invariant, and it can be time-invariant without being linear
- A direct consequence of the superposition property is that, for linear systems, an input which is zero for all time results in an output which is zero for all time

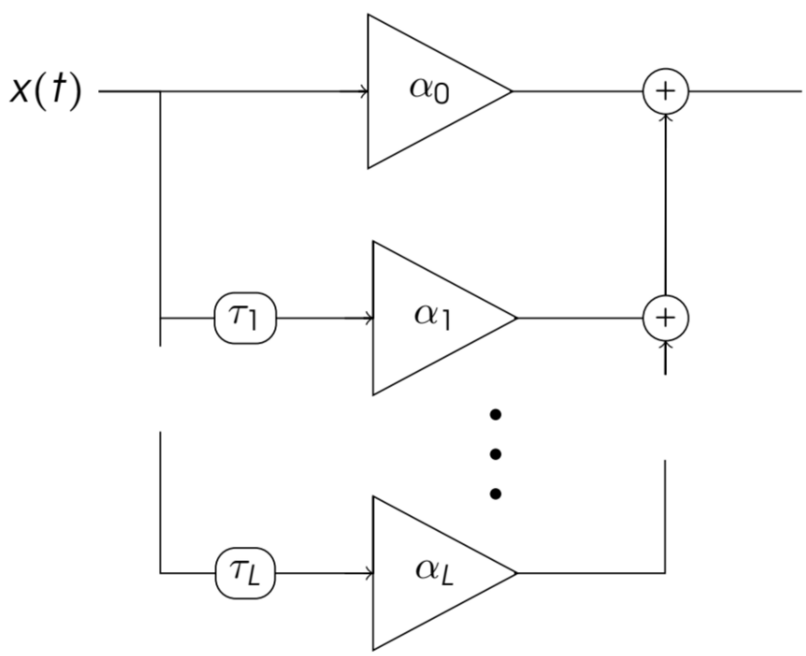
- Pure all-pass system has impulse response $\delta(t)$
- Some common systems:





- Generalized echo system:

$$y(t) = \sum_{l=0}^{L} \alpha_l x(t - \tau_l)$$



Linear time-invariant systems

- **linear time-invariant (LTI) system** - systems that are both linear and time-invariant

- Convolution sum or superposition sum of two discrete-time LTI systems $x[n]$ and $h[n]$:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Convolution integral or superposition integral of two continuous-time LTI systems $x(t)$ and $h(t)$:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- Convolution is commutative, distributive and associative:

$$x(t) * h(t) = h(t) * x(t)$$

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

- Impulse response of a system $(h(t))$ and its inverse system $(h_1(t))$ satisfy:

$$h(t) * h_1(t) = \delta(t)$$

- A continuous-time LTI system is stable if the impulse response is absolutely integrable, i.e., if

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty$$

- A discrete-time LTI system is stable if the impulse response is absolutely summable, i.e., if

$$\sum_{k\ =\ -\infty}^{\infty} |h[k]| < \infty$$

- Zero-order hold and first-order hold:

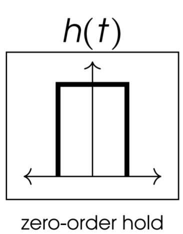


Figure : Zero-order hold

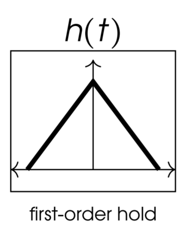


Figure : First-order hold

- **digital interpolation -** upsampling followed by digital convolution
- **cubic interpolation -** this type of interpolation gives rise to a piecewise cubic function that interpolates a set of data points and guarantees smoothness at the data points

Fourier series representation of continuous-time signals

- An even function can be represented as:

$$f(x) = \sum_{m \ \geq \ 0} a_m \cos(\frac{2\pi}{T} mx), \ \text{for} \ \frac{-T}{2} \leq x \leq \frac{-T}{2}$$

- An odd function can be represented as:

$$f(x) = \sum_{m \ \geq \ 1} b_m \sin(\frac{2\pi}{T} mx), \ \text{for} \ \frac{-T}{2} \leq x \leq \frac{-T}{2}$$

- Functions $g1, \ g2$ such that $\langle g1, \ g2 \rangle = 0$ for $t \in T$ is said to be orthogonal in T.

- A representation of a periodic signal as a combination of complex exponentials of discrete frequencies, which are multiples of the fundamental frequency of the signal, is known as the Fourier series representation of the signal

$$x(t) = \alpha_0 + \sum_{n \ \geq \ 1} a_n \cos(\frac{2\pi}{T_d} nT) + \sum_{n \ \geq \ 1} b_n \sin(\frac{2\pi}{T_d} nT)$$

$$= \alpha_0 + \sum_{n \ \geq \ 1} a_n \frac{(e^j \frac{2\pi}{T_d} nt + e^- j \frac{2\pi}{T_d} nt)}{2} + \sum_{n \ \geq \ 1} b_n \frac{(e^j \frac{2\pi}{T_d} nt - e^- j \frac{2\pi}{T_d} nt)}{2j}$$

$$= \sum_{m \ \in \ \mathbb{Z}} \alpha_m e^{j \frac{2\pi}{T_d} mt}$$

$$where \ \alpha_m = \frac{a_m - b_m}{2}, \ m \in \mathbb{Z}^+ \ and \ \alpha_m = \frac{a_{|m|} + b_{|m|}}{2}, m \in \mathbb{Z}^-$$

$$\alpha_m = \frac{\langle \ x(t), \ e^{j \frac{2\pi}{T_d} mt} \ \rangle}{T_d} = \frac{1}{T_d} \int_{-\frac{T_d}{2}}^{\frac{T_d}{2}} x(t) e^{-j \frac{2\pi}{T_d} mt} dt$$

- **Theorem:** No two continuous functions in $[-\frac{T}{2}, \ \frac{T}{2}]$ will have all the Fourier series coefficients the same

- **Lemma:** Let $f(t)$ be a signal locally integrable in $[-\frac{T}{2}, \ \frac{T}{2}]$, if the Fourier series coefficients $\alpha_m = 0$, identically $\forall m \in Z$, then $f(t) = 0$, whenever it is continuous at t

- **Properties of continuous-time Fourier series**

  If the Fourier series coefficients of $x(t)$ are denoted by a_k, we will use the notation:

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

  - **Linearity:**

$$\text{If} \ \ x(t) \xleftrightarrow{\mathcal{FS}} a_k \ \ and \ \ y(t) \xleftrightarrow{\mathcal{FS}} b_k, \ \text{then}$$

$$z(t) = Ax(t) + By(t) \xleftrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k$$

  - **Time shifting:**

$$\text{If} \ \ x(t) \xleftrightarrow{\mathcal{FS}} a_k, \ \text{then}$$

$$x(t - t_0) \xleftrightarrow{\mathcal{FS}} e^{-(j \frac{2\pi}{T} k t_0)} a_k$$

- **Time reversal:**

$$\text{If } \; x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k, \; \text{then}$$
$$x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}$$

- **Time scaling:**

$$x(t) = \sum_{k \, = \, -\infty}^{\infty} a_k e^{(j \frac{2\pi}{T} kt)}$$
$$x(\alpha t) = \sum_{k \, = \, -\infty}^{\infty} a_k e^{(j \frac{2\pi}{T} k\alpha t)}$$

- **Multiplication:**

$$\text{If } \; x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \;\; \textit{and} \;\; y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k, \; \text{then}$$
$$z(t) = x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = \sum_{l \, = \, -\infty}^{\infty} a_l b_{k-l}$$

- **Conjugation:**

$$\text{If } \; x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k, \; \text{then}$$
$$x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} a^*_{-k}$$

- **Parseval's relation for continuous-time periodic signals:**
  - If $x(t) = \sum_{m \, \in \, \mathbb{Z}} \beta_m \phi_m t$ with $\langle \phi_m, \; \phi_n \rangle = \delta[m-n]$ in $\frac{-T}{2} \leq x \leq \frac{-T}{2}$, then

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{m \, \in \, \mathbb{Z}} |\beta_m|^2$$

- The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \sum_{m \, \in \, \mathbb{Z}} |\alpha_m|^2$$

- A function $h(t)$ is integrable if$\int_{\mathbb{R}} |h(t)| dt < \infty$
-

$$sinc(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting	$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time Reversal	$x(-t)$	a_{-k}
Time Scaling	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution	$\int_T x(\tau) y(t - \tau) d\tau$	$T a_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation	$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration	$\int_{-\infty}^{t} x(t)\, dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x(t)$ real and even	a_k real and even
Real and Odd Signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Figure : Properties of continuous-time Fourier series

Fourier transforms

- The Fourier transform is a mathematical function that decomposes a waveform, which is a function of time, into the frequencies that make it up. The equation of Fourier transform $H(f)$ of a signal $x(t)$ is

$$H(f) = \int_{\mathbb{R}} x(t) e^{-(j2\pi f t)} dt$$

- The properties of the Fourier transform are:

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(f)$
	$y(t)$	$Y(f)$
Linearity	$ax(t) + by(t)$	$aX(f) + bY(f)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(f)$
Frequency Shifting	$e^{j\omega_0 t} x(t)$	$X(j(f - f_0))$
Conjugation	$x^*(t)$	$X^*(-f)$
Time Reversal	$x(-t)$	$X(-f)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{f}{a}\right)$
Convolution	$x(t) * y(t)$	$X(f)Y(f)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(f)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(f) + \pi X(0)\delta(f)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(f)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(f) = X^*(-f) \\ \Re e\{X(f)\} = \Re e\{X(-f)\} \\ \Im m\{X(f)\} = -\Im m\{X(-f)\} \\ \lvert X(f)\rvert = \lvert X(-f)\rvert \\ \sphericalangle X(f) = -\sphericalangle X(-f) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(f)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(f)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(f)\}$
	$x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$j\Im m\{X(f)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Figure : Properties of the Fourier transform

- **Poisson summation formula:** For an integrable $x(t)$ with Fourier transform $H(f)$ -

$$\sum_{n\ \in\ \mathbb{Z}} x(t-nT) = \sum_{m\ \in\ \mathbb{Z}} \alpha_m e^{(\frac{2\pi}{T}mt)},$$

at points of continuity of the LHS, where

$$\alpha_m = \frac{1}{T}H(\frac{m}{T}),\ m\ \in\ \mathbb{Z}$$

- $$e^{(-\pi t^2)} \xleftrightarrow{\mathcal{FT}} e^{(-\pi f^2)}$$

- $$g_\delta(t) = \frac{1}{\sqrt{\delta}}e^{(-\pi\frac{t^2}{\delta})} \xleftrightarrow{\mathcal{FT}} e^{(-\pi f^2\delta)} = G_\delta(f),\ \delta > 0$$

$\lim_{\delta\downarrow 0} g_\delta(t) := \delta(t)$ " Impulse or Dirac Measure "

$\lim_{\delta\downarrow 0} G_\delta(f) := \mathbb{I}_{\{f\in\mathbb{R}\}}$. " DC value "

- The Inverse Fourier transform is given by:

$$x(t) = \int_{\mathbb{R}} H(f)e^{(j2\pi ft)}df$$

where $H(f)$ is the Fourier transform of signal $x(t)$

- **Poisson sum:** Fourier transform (Generalized) of an impulse train:

$$\sum_{n\in\mathbb{Z}} \delta(t-nT) \xrightleftharpoons{F.T.} \sum_{m\in\mathbb{Z}} \frac{1}{T}\delta(f-\frac{m}{T})$$

- **Dual FS formula:**
$$\sum_{n\in\mathbb{Z}} X(f+n\beta) \xrightleftharpoons{I.F.T.} \sum_{m\in\mathbb{Z}} \frac{1}{\beta}x(\frac{m}{\beta})\delta(t-\frac{m}{\beta})$$

- **Shannon's reconstruction formula:**

$$x(t) = \sum_{m\in\mathbb{Z}} x(\frac{m}{\beta})\ sinc(\beta t - m)$$

- **Parseval's relation:**
$$\int_{\mathbb{R}} x(t)y^*(t)dt = \int_{\mathbb{R}} X(f)Y^*(f)df$$
$$\int_{\mathbb{R}} |x(t)|^2dt = \int_{\mathbb{R}} |X(f)|^2df$$

- Wireless communication bandwidths:

Application	Bandwidth
AM Radio	10kHz
2G	200kHz - 1MHz
3G	5MHz
4G	10 – 20MHz
5G	≈ 100MHz

Dirac's formalisms

- **Def:** Dirac Delta is defined as a non-negative unit area operator such that:

$$\int_{\mathbb{R}} x(t)\delta(t)dt = x(0), \quad \text{whenever } x(0^+) = x(0^-) = x(0)$$

- Fourier transform and inverse for Diracs:

$$\delta(t) \xrightleftharpoons{F.T.} \mathbb{I}_{\{f\in\mathbb{R}\}}$$

$$\mathbb{I}_{\{t\in\mathbb{R}\}} \xrightleftharpoons{F.T.} \delta(f)$$

Advanced concepts

- **Inverse DTFT:** given samples $x[n],\ n \in \mathbb{Z}$, having a DTFT $\hat{X}(f),\ -\frac{1}{2} \geq f \geq \frac{1}{2}$:

$$x[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{X}(f)e^{(j2\pi fn)}$$

- **Discrete Fourier transform (DFT):**

$$X[k] = \hat{X}\frac{k}{N} = \sum_{n\,=\,0}^{N\,-\,1} x[n]e^{(-j2\pi\frac{k}{N}n)}$$

- **Matrix form of DFT:**

$$\text{DFT:} \quad \overline{X} = F\overline{x}$$

$$\text{where F } = \begin{bmatrix} \alpha_0^0 & \alpha_0^1 & \alpha_0^2 & \dots & \alpha_0^M \\ \alpha_1^0 & \alpha_1^1 & \alpha_1^2 & \dots & \alpha_1^M \\ \alpha_2^0 & \alpha_2^1 & \alpha_2^2 & \dots & \alpha_2^M \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ \alpha_M^0 & \alpha_M^1 & \alpha_M^2 & \dots & \alpha_M^M \end{bmatrix}$$

$$\text{here } \alpha = e^{(-j\frac{2\pi}{N})}, \quad \alpha_i = \alpha^i \ (for 0 \geq i \geq N-1), \quad M = N-1$$

Proposition:

$$F^H F = N\mathbb{I}_N$$

- **Circular convolution ($\circledast$):** let $x_c[n] = \sum_{i \in \mathbb{Z}} x[n + iN] \;\; \rightarrow \;\; x_c[i] = x[i + N]$ for $-N \geq i \geq -1$, then:

$$x_c[n] \circledast h[n] = \sum_{n\,=\,0}^{N-1} h[n] x_c[k-n]$$

- Circular convolution is commutative

- **Proposition:**

$$x[n] \circledast h[n] \xrightleftharpoons{DFT} X[k]H[k]$$

- **Nyquist condition:** to get the samples back it should satisfy the following condition:

$$\sum_{m\,\in\,\mathbb{Z}} H(f_m f_s) = 1, \quad \forall f \in \mathbb{R}$$

- In signal processing one addition + one multiplication in matrix multiplication is termed a flop or computation. Matrix multiplication requires $O(N^2)$ computations

- **Butterfly structure of FFT:**