MA207 Short Notes

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Ref: Prof's slides
All the world's a differential equation,
and the men and women are merely variables!

Differential Equations - 2

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Contents

Power series

- For real numbers $x_0, a_0, a_1, a_2, \ldots$, an infinite series:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots$$

  is called a power series in $x - x_0$ with center x_0

- For a real number x_1, if the limit:

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_{(}x_1 - x_0)^n$$

  exists and is finite, then we say the power series converges at the point $x = x_1$. In this case, the value of the series at x_1 is, by definition, the value of the limit.

- If the series does not converge at x_1, that is, either the limit does not exist, or it is $\pm\infty$, then we say the power series diverges at x_1. Also, a power series always converges at its center $x = x_0$.

- **radius of convergence (R):** for any power series:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n$$

  exactly one of these statements is true:

  - the power series converges ony for $x = x_0$ (here $R = 0$)
  - the power series converges for all values of x (here $R = \infty$)
  - there is a positive number $0 < R < \infty$ such that the power series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$

- **Ratio test:** assume that there is an integer N such that for all $n \geq N$ we have an $a_n \neq 0$ Also assume the following limit exists:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

  and denote it by L. Then radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is $R = \frac{1}{L}$.

- **Def:** Suppose we are given a sequence $\{a_n\}_{n \geq 1}$. For every $k \geq 1$ define:

$$b_k = \sup_{n \geq k}\{a_n\}$$

  We know $\{b_k\}_{k \geq 1}$ is a decreasing sequence, and hence we define $\limsup\{a_n\}$ as:

$$\limsup\{a_n\} = \lim_{n \to \infty} b_n$$

  Similarly, we define $\liminf\{a_n\}$, by replacing sup by inf in the above definition.

- For a sequence $\{a_n\}_{n \geq 1}$, the limit may not exist. However, the lim sup and lim inf always exist (possibly $+\infty$ or $-\infty$)

- **Theorem:** Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. Then $lim_{n \to \infty} a_n$ exists if and only if $\limsup a_n = \liminf a_n$. Further, if $lim_{n \to \infty} a_n$ exists, then

$$\limsup\{a_n\} = \liminf\{a_n\} = \lim_{n \to \infty} a_n$$

3

- **Root test:** let $\limsup\{|a_n|^{1/n}\} = L$. Then radius of convergence of the power series $\sum_{n\,=\,0}^{\infty} a_n(x-x_0)^n$ is $R = 1/L$.

- **Theorem:** Let $R > 0$ be the radius of convergence of the power series $\sum_{n\,=\,0}^{\infty} a_n(x - x_0)^n$, then the power series converges (absolutely) for all $x \in (x_0 - R, x_0 + R)$. The open interval $(x_0 - R, x_0 + R)$ is called the interval of convergence of the power series.

- **Theorem:** let R be the radius of convergence of the power series $\sum_{n\,=\,0}^{\infty} a_n(x - x_0)^n$. We assume $R > 0$. We define a function $f : (x_0 - R, x_0 + R) \to \mathbb{R}$ by:

$$f(x) = \sum_{n\,=\,0}^{\infty} a_n(x - x_0)^n$$

This function satisfies the following properties:

  - f is infinitely differentiable $\forall x \in (x_0 - R, x_0 + R)$
  - the successive derivatives of f can be computed by differentiating the power series term-by-term, that is:

$$f'(x) = \sum_{n\,=\,0}^{\infty} na_n(x - x_0)^{n-1}$$

  - $f_{(k)}(x) = \sum_{n\,=\,0}^{\infty} n(n-1)\ldots(n-k+1)a_n(x - x_0)^{n-k}$
  - the power series representing the derivatives $f_{(n)}(x)$ have same radius of convergence R
  - we can determine the coefficients a_n (in terms of derivatives of f at x_0) as:

$$a_n = \frac{f_{(n)}(x_0)}{n!}$$

  - we can also integrate the function $f(x) = \sum_{n\,=\,0}^{\infty} a_n(x - x_0)^n$ term-wise, that is, if $[a, b] \subset (x_0 - R, x_0 + R)$, then:

$$\int_a^b f(x)dx = \sum_{n\,=\,0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}$$

  - power series representation of f in an open interval I containing x_0 is unique, that is, if:

$$f(x) = \sum_{n\,=\,0}^{\infty} a_n(x - x_0)^n = \sum_{n\,=\,0}^{\infty} b_n(x - x_0)^n$$

for all $x \in I$, then $a_n = b_n$ for all n
  - if:

$$\sum_{n\,=\,0}^{\infty} a_n(x - x_0)^n = 0$$

for all $x \in I$, then $a_n = 0$ for all n

- Power series representation of some familiar functions:

  —

$$e^x = \sum_0^{\infty} \frac{x^n}{n!}\ , \quad -\infty < x < \infty$$

  —

$$sin(x) = \sum_0^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\ , \quad -\infty < x < \infty$$

$$(1-x)^{-1} = \sum_0^\infty x^n \ , \quad -1 < x < 1$$

$$\cos(x) = \sum_0^\infty (-1)^n \frac{x^{2n}}{(2n)!} \ , \quad -\infty < x < \infty$$

$$\sinh(x) = \sum_0^\infty \frac{x^{2n+1}}{(2n+1)!} \ , \quad -\infty < x < \infty$$

$$\cosh(x) = \sum_0^\infty \frac{x^{2n}}{(2n)!} \ , \quad -\infty < x < \infty$$

- If $f(x) = \sum_{n=0}^\infty a_n(x-x_0)^n$, $g(x) = \sum_{n=0}^\infty b_n(x-x_0)^n$ have radii of convergence R_1 and R_2 respectively, then:

$$c_1 f(x) + c_2 g(x) = \sum_0^\infty (c_1 a_n + c_2 b_n)(x-x_0)^n$$

has radius of convergence $R \geq \min\{R_1, R_2\}$ for $c_1, c_2 \in \mathbb{R}$. Further, we can multiply the series as if they are polynomials, that is:

$$f(x)g(x) = \sum_0^\infty c_n(x-x_0)^n \ ; \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$$

it also has radius of convergence $R \geq \min\{R_1, R_2\}$.

Taylor series and analytic functions

- Let $f(x)$ be infinitely differentiable at x_0. The Taylor series of f at x_0 is defined as the power series:

$$TS \ f|_{x_0} = \sum_0^\infty \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

- Suppose $f(x)$ is infinitely differentiable at x_0 and Taylor series of f at x_0 converges to $f(x)$ for all x in some open interval around x_0, then f is called analytic at x_0. Thus if f is analytic, then there is an interval I around x_0 and f is given by a power series in I.

- Polynomials e^x, $sin(x)$ and $cos(x)$ are analytic at all $x \in \mathbb{R}$. $f(x) = tan(x)$ is analytic at all x except $x = (2n+1)\pi/2$, where $n = \pm 1, \pm 2, \ldots$

- If $f(x)$ and $g(x)$ are analytic at x_0, then $f(x) \pm g(x)$, $f(x)g(x)$ and $f(x)/g(x)$ (if $g(x_0) \neq 0$) are analytic at x_0

- If $f(x)$ is analytic at x_0 and $g(x)$ is analytic at $f(x_0)$, then $g(f(x)) = (g \circ f)(x)$ is analytic at x_0

- If a power series $\sum_0^\infty a_n(x-x_0)^n$ has radius of convergence $R > 0$, then the function $f(x) = \sum_0^\infty a_n(x-x_0)^n$ is analytic at all points $x \in (x_0 - R, x_0 + R)$

- **Theorem:** let:

$$F(x) = \frac{N(x)}{D(x)}$$

be a rational function, where $N(x)$ and $D(x)$ are polynomials without any common factors, that is they do not have any common (complex) zeros. Let $\alpha_1, \ldots, \alpha_r$ be distinct complex zeros of $D(x)$.

Then $F(x)$ is analytic at all x except at $x \in \{\alpha_1, \ldots, \alpha_r\}$. If x_0 is different from $\{\alpha_1, \ldots, \alpha_r\}$, then the radius of convergence R of the Taylor series of F at x_0:

$$TS \ F|_{x_0} = \sum_0^\infty \frac{F^{(n)}(x_0)}{n!}(x - x_0)^n$$

is given by:

$$R = \min\{|x_0 - \alpha_1|, |x_0 - \alpha_2|, \ldots, |x_0 - \alpha_r|\}$$

- **Existence theorem:** if $p(x)$ and $q(x)$ are analytic functions at x_0, then every solution of:

$$y'' + p(x)y' + q(x)y = 0$$

is also analytic at x_0, and therefore any solution can be expressed as:

$$y(x) = \sum_0^\infty a_n(x - x_0)^n$$

If $R1$ is the radius of convergence of Taylor series of $p(x)$ at x_0, R_2 is the radius of convergence of Taylor series of $q(x)$ at x_0, then radius of convergence of $y(x)$ is at least $\min(R_1, R_2) > 0$

- The standard form of an ordinary differential equation (ODE) is:

$$y'' + p(x)y' + q(x)y = 0$$

- Steps for series solution of linear ODE:
    - write ODE in the standard form $y'' + p(x)y' + q(x)y = 0$
    - choose x_0 at which $p(x)$ and $q(x)$ are analytic. If boundary conditions at x_0 are given, choose the center of the power series as x_0.
    - find the minimum of radii of convergence of Taylor series of $p(x)$ and $q(x)$ at x_0
    - let $y(x) = \sum_0^\infty a_n(x-x_0)^n$, compute the power series for $y'(x)$ and $y''(x)$ at x_0 and substitute these onto the ODE
    - set the coefficients of $(x - x_0)^n$ to zero and find recursion formula
    - from the recursion formula, obtain (linearly independent) solutions $y_1(x)$ and $y_2(x)$. The general solution then looks like $y(x) = a_1 y_1(x) + a_2 y_2(x)$

- **initial value problem (IVP)** - is an ordinary differential equation together with an initial condition which specifies the value of the unknown function at a given point in the domain

- **Bessel's equation:**

$$x^2 y'' + xy' + (x^2 - v^2)y = 0$$

Legendre polynomials

- **Legendre equation:**

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0, \text{ where } p \text{ is a real number}$$

- The two independent solutions of the Legendre equation are:

$$y_1(x) = a_0\left[1 - \frac{p(p + 1)}{2!}x^2 + \frac{p(p + 1)(p - 2)(p + 3)}{4!}x^4 + \ldots\right]$$

$$y_2(x) = a_1 \left[x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!} x^5 + \ldots \right]$$

If $p \in \{0, 2, 4, \ldots\} \cup \{-1, -3, -5, \ldots\}$ then $y_1(x)$ is a polynomial function. $y_2(x)$ is an odd function. If $p \in \{1, 3, 5, \ldots\} \cup \{-2, -4, -6, \ldots\}$ then $y_2(x)$ is a polynomial function. Thus, if p is an integer then exactly one solution is a polynomial and the other is an infinite power series.

- The general solution (of the Legendre equation):

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

is called a Legendre function. If $p = m$ is an integer, then precisely one of y_1 or y_2 is a polynomial, and it is called the mth Legendre polynomial $P_m(x)$. For $m \geq 0$ note that $P_m(x)$ is a polynomial of degree m. It is an even function if m is even and an odd function if m is odd.

- A vector space (V) is a set equipped with two operations:

  - addition:
  $$v + w, \quad v, w \in V$$

  - scalar multiplication:
  $$cv, \quad c \in \mathbb{R}, \ v \in V$$

  A vector space V has a dimension, which may not be finite

- Let V be a vector space over $\mathbb{R}$ (not necessarily finite-dimensional). A bilinear form on V is a map:
$$\langle, \rangle : V \times V \to \mathbb{R}$$

  which is linear in both coordinates, that is:

$$\langle au + v, w \rangle = a \langle u, v \rangle + \langle v, w \rangle$$

$$\langle u, av + w \rangle = a \langle u, v \rangle + \langle u, w \rangle$$

  for $a \in \mathbb{R}$ and $u, v \in V$

- An inner product on V is a bilinear form on V which is:

  - symmetric: $\langle v, w \rangle = \langle w, v \rangle$
  - positive definite: $\langle v, v \rangle \geq 0$ for all v and $\langle v, v \rangle = 0$ iff $v = 0$

  A vector space with an inner product is called an inner product space.

- In an inner product space V, two vectors u and v are orthogonal if $\langle v, v \rangle = 0$. More generally, a set of vectors forms an orthogonal system if they are mutually orthogonal.

- A set $\{v_i\}_{i \in I} \subset V$ is called a basis if the vectors in it are:

  - linearly independent i.e., $\sum_{j=1}^{m} a_j v_{i_j} = 0 \implies a_j = 0$
  - they span V, i.e., every w can be written as $w = \sum_{j=1}^{m} a_j v_{i_j}$

  An orthogonal basis is an orthogonal system which is also a basis.

- Consider the vector space $\mathbb{R}^n$ with coordinate-wise addition and scalar multiplication. The rule:

$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle = \sum_{i=1}^{n} a_i b_i$$

defines an inner product on $\mathbb{R}^n$. The standard basis $\{e_1, \ldots, e_n\}$ is an orthogonal basis of $\mathbb{R}^n$.

- **Lemma:** suppose V is a finite dimensional inner product space, and $e_1, \ldots, e_n$ is an orthogonal basis. Then for any $v \in V$:

$$v = \sum_{i=1}^{n} \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

- **Lemma:** In a finite-dimensional inner product space, there always exists an orthogonal basis. This result is not necessarily true in infinite-dimensional inner product spaces. For infinite dimensional vector spaces, we can only talk of a maximal orthogonal set. A subset $\{e_1, e_2, \ldots\}$ is called a maximal orthogonal set for V if:

  - $\langle e_i, e_j \rangle = \delta_{ij}$
  - $\langle v, e_i \rangle = 0$ for all i iff $v = 0$

- tbDef: for a vector v in an inner product space, we define the norm or length of the vector v as:

$$||v|| = \langle v, v \rangle^{1/2}$$

It satisfies the following three properties:

  - $||0|| = 0$ and $||v|| > 0$ if $v \neq 0$
  - $||v + w|| \leq ||v|| + ||w||$
  - $||av|| = |a| \, ||v||$

for all $v, w \in V$ and $a \in \mathbb{R}$

- **Pythagoras theorem:** for orthogonal vectors v and w in any inner product space V:

$$||v + w||^2 = ||v||^2 + ||w||^2$$

More generally, for any orthogonal system $\{v_1, \ldots, v_n\}$:

$$||v_1 + \ldots + v_n||^2 = ||v_1||^2 + \ldots + ||v_n||^2$$

- The set of all polynomials in the variable x is a vector space denoted by $\mathcal{P}(x)$. The set $\{1, x, x^2, \ldots\}$ is an infinite basis of the vector space $\mathcal{P}(x)$. $\mathcal{P}(x)$ carries an inner product defined by:

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$$

We are integrating over finite interval $[-1, 1]$ which ensures that the integral is finite. The norm of a polynomial is by definition $\langle f, f \rangle$:

$$||f|| = \left(\int_{-1}^{1} f(x) f(x) dx \right)^{1/2}$$

- **Derivative-transfer:** if $f(1)g(1) = f(-1)g(-1)$, then:

$$\int_{-1}^{1} g \frac{df}{dx} = - \int_{-1}^{1} f \frac{dg}{dx}$$

- **Theorem:** since $P_m(x)$ is a polynomial of degree m,, it follows that:

$$\{P_0(x), P_1(x), P_2(x), \ldots\}$$

is a basis of the vector space of polynomials $\mathcal{P}(x)$. We have:

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x) P_n(x) dx = \begin{cases} 0, & if \ m \neq n \\ \frac{2}{2n+1}, & if \ m = n \end{cases}$$

i.e., Legendre polynomials form an orthogonal basis for the vector space $\mathcal{P}(x)$ and:

$$||P_n(x)||^2 = \frac{2}{2n+1}$$

- **Rodrigues' formula for Legendre polynomials P_n:**

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

- Let $f_i(x)$ (for $i \geq 0$) be a collection of non-zero polynomials. Assume that $f_i(x)$ has degree i. Then $\{f_0(x), f_1(x), \ldots, f_n(x)\}$ is a basis for the vector space consisting of polynomials of degree $\leq n$.

- A function $f(x)$ on $[-1, 1]$ is square-integrable if:

$$\int_{-1}^{1} f(x)g(x)dx < \infty$$

For instance, polynomials, continuous functions, piecewise continuous functions are square-integrable. The set of all square-integrable functions on $[-1, 1]$ is a vector space and is denoted by $L^2([-1, 1])$. For square-integrable functions f and g, we define their inner product by:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$

- Legendre polynomials form a maximal orthogonal set in $L^2([-1, 1])$. This means that a square-integrable function which is orthogonal to all Legendre polynomials is necessarily the constant function "0". We can expand any square-integrable function $f(x)$ on $[11, 1]$ in a series of Legendre polynomials:

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)dx$$

This is called the Fourier-Legendre series (or simply the Legendre series) of $f(x)$.

- **Theorem:** The Fourier-Legendre series of $f(x) \in L^2([-1, 1])$ given by:

$$\sum_{n=0}^{\infty} c_n P_n(x), \quad c_n = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)dx$$

converges in L^2 norm to f(x), that is:

$$||f(x) - \sum_{n=0}^{m} c_n P_n(x)|| \to 0 \ as \ m \to \infty$$

- **Legandre expansion theorem:** if both $f(x)$ and $f'(x)$ have at most a finite number of jump discontinuities in the interval $[-1, 1]$, then the Legendre series converges to:

$$\frac{1}{2}(f(x_-) + f(x_+)), \ for \ -1 < x < 1$$

$$f(-1_+), \ for \ x = -1$$

$$f(1_-), \ for \ x = 1$$

In particular, the series converges to $f(x)$ at every point of continuity x

- **Least square approximation theorem:** Suppose we want to approximate $f \in L^2([-1, 1])$ in the sense of least square by polynomials $p(x)$ of degree $\leq n$, that is, we want to find a polynomial $p(x)$ which minimizes:

$$I = \int_{-1}^{1} [f(x) - p(x)]^2 dx$$

Then the minimizing polynomial is precisely the first $n + 1$ terms of the Legendre series of $f(x)$, i.e.:

$$c_0 P_0(x) + \ldots + c_n P_n(x) \quad c_k = \frac{2k + 1}{2} \int_{-1}^{1} f(x) P_k(x) dx$$

- Steps to solve a second order linear ODE using power series:

  - given an ODE of the type

    $$F_0(x) y'' + F_1(x) y' + F_2(x) y = 0 \quad \ldots (1)$$

    first convert it to the standard form

    $$y'' + \frac{F_1(x)}{F_0(x)} y' + \frac{F_2(x)}{F_0(x)} y = 0 \quad \ldots (2)$$

    Let

    $$p(x) := \frac{F_1(x)}{F_0(x)} \quad q(x) := \frac{F_2(x)}{F_0(x)}$$

  - now find the set:
    $$U := \{x_0 \in \mathbb{R} \mid p(x), q(x) \text{ are analytic at } x_0\}$$

  - By the existence theorem, for every $x_0 \in U$, there will exist two independent solutions to the above ODE, call them $y_1(x)$ and $y_2(x)$, such that both of them will be analytic in an interval I around x_0.

  - To find the solutions in a neighborhood of x_0, set $y(x) = \sum_{n \geq 0} a_n (x - x_0)^n$ into the ODE (1) or (2) and get recursive relations involving the a_n. Note that when you do this, the coefficient functions $(p(x), q(x), F_0(x), ..)$ have to be written as power series in $x - x_0$. Note that the recursive relation you get, will be same, irrespective of whether you choose equation (1) or (2)

  - thus, depending on the situation, you may want to choose 1 or 2. For example, for the Legendre equation, in the open interval $(-1, 1)$ around $x_0 = 0$, the equation (1) looks like

    $$\left(1 - x^2\right) y'' - 2xy' + p(p + 1)y = 0$$

    while (2) looks like

    $$y'' - 2 \left(\sum_{n \geq 0} x^{2n+1}\right) y' + p(p + 1) \left(\sum_{n \geq 0} x^{2n}\right) y = 0$$

    In this case it is clear that, we should choose 1, as it will be easier to work with.

More complicated ODE's

- **Def:** consider the second-order linear ODE in standard form

$$y'' + p(x)y' + q(x)y = 0 \quad \dots (1)$$

Then:

- $x_0 \in \mathbb{R}$ is called an ordinary point of (1) if $p(x)$ and $q(x)$ are analytic at x_0
- $x_0 \in \mathbb{R}$ is called regular singular point if x_0 is not an ordinary point and both $(x - x_0)\, p(x)$ and $(x - x_0)^2\, q(x)$ are analytic at x_0 If x_0 is regular singular then there are functions $b(x)$ and $c(x)$ which are analytic at x_0 such that

$$p(x) = \frac{b(x)}{(x - x_0)} \qquad q(x) = \frac{c(x)}{(x - x_0)^2}$$

- If $x_0 \in \mathbb{R}$ is not ordinary or regular singular, then we call it irregular singular

- **Cauchy-Euler equation:**

$$x^2 y'' + b_0 xy' + c_0 y = 0 \qquad b_0, c_0 \in \mathbb{R}$$

$x = 0$ is a regular singular point, since we can write the ODE as:

$$y'' + \frac{b_0}{x} y' + \frac{c_0}{x^2} y = 0$$

All $x \neq 0$ are ordinary points. Assume $x > 0$. Note that $y = x^r$ solves the equation iff:

$$r(r - 1) + b_0 r + c_0 = 0$$
$$\iff r^2 + (b_0 - 1)\, r + c_0 = 0$$

Let r_1 and r_2 denote the roots of this quadratic equation. Then:

- if the roots $r_1 \neq r_2$ are real, then x^{r_1} and x^{r_2} are two independent solutions
- if the roots $r_1 = r_2$ are real, then x^{r_1} and $(\log x)x^{r_1}$ are two independent solutions
- if the roots are complex (written as $a \pm ib$), then $x^a \cos(b \log x)$ and $x^a \sin(b \log x)$ are two independent solutions

- **Theorem:** consider the ODE:

$$x^2 y'' + xb(x)y' + c(x)y = 0 \quad \dots (1)$$

where $b(x)$ and $c(x)$ are analytic at 0. Then $x = 0$ is a regular singular point of the ODE. Then (1) has a solution of the form:

$$y(x) = x_1^r \sum_{n \geq 0} a_n x^n \quad a_0 \neq 0, \quad r \in \mathbb{C} \quad \dots (2)$$

The solution (2) is called Frobenius solution or fractional power series solution. The power series $\sum_{n \geq 0} a_n x^n$ converges on $(-\rho, \rho)$, where ρ is the minimum of the radius of convergence of $b(x)$ and $c(x)$. We will consider the solution $y(x)$ in the open interval $(0, \rho)$.

- **Indicial equation:** An indicial equation, also called a characteristic equation, is a recurrence equation obtained during application of the Frobenius method of solving a second-order ordinary differential equation.