Numerical Optimization with Python (2024B)

Dry HW 01

Part I: Modeling optimization problems:

For each of the following descriptions, formulate a mathematical optimization problem, namely, pose the problem in the following form:

$$f_0(x) \rightarrow min$$

s.t.

$$f_i(x) \leq 0, i = 1, \dots, m$$

$$h_i(x) = 0, i = 1, \dots, p$$

For your proper choice of the objective f_0, the inequality constraints (if applicable) $f_1, \dots, f_m$ and equality constraints (if applicable) $h_1, \dots, h_p$.

Notes:

a. Not all constraint types must appear, if at all.
b. Sets of linear constraints should be grouped together and presented in matrix form.
c. Only problem formulation is required (not solution attempts!)
d. There may be more than one correct model/problem formulation

1. A car manufacturer produces three models: standard cars, pickup trucks and vans. The profit per one sold car \$1,000, per one sold pickup truck is \$1,200 and per one sold van \$1,500.
   There's one assembly line, which is capable of producing no more than 25 vehicles (total of all types) per day. Further, the assembly line cannot handle more than 8 vans daily, and no more than 10 pickup trucks daily.
   Prior to assembly, each car requires 1 hours of metal workshop time, while pickup trucks and vans each require 1.5 hours of metal workshop time. The metal workshop cannot exceed 8 hours of work per day.
   Sales projection demand production of at least 2 cars, 2 trucks and 1 van per day.
   To maximize profit, how many cars, pickup trucks and vans should be manufactured on a daily

average (namely, number of units are allowed to be real numbers, no need to require integers)?

2. Given a vector x of n real values $x_1, \dots, x_n$, a vector $p \in \mathbb{R}^n$ is called a ***probability distribution*** on x, if $p_i \geq 0$ and $p_1 + \dots + p_n = 1$. In other words, the p_i's are non-negative and sum to 1. Given a probability vector p for x, the ***expected value*** (or expectation) of x is the weighted average $\mathrm{E}x = \sum_{i=1}^{n} p_i x_i$. Given a probability distribution p on n values, the ***entropy*** of p is defined to be:

   $\mathrm{H} = -\sum_{i=1}^{n} p_i \log p_i$.

   For the values $x_1 = -1, x_2 = 0.5, x_3 = 8, x_4 = 12$, formulate an optimization problem that finds a probability distribution for x with expected value 2.5 and has maximal entropy.

3. Let $A \in \mathbb{R}^{m \times n}$ with $m > n$ and let $b \in \mathbb{R}^m$.
   - **a.** Formulate an optimization problem that finds a vector in the column space of A, that is closest to b. Your unknowns should be the scalar coefficients of the columns of A.
   - **b.** Only in the special case where $m = n$ and A is non-singular, what are these unknown coefficients, and what is the realized distance to b?

4. Let $A \in \mathbb{R}^{m \times n}$ with $m < n$ and let $b \in \mathbb{R}^m$. The set of linear equations $Ax = b$ typically has infinitely many solutions, since $m < n$ (the system is under-constrained). Formulate a problem that attempts to find, from all available solutions, the one that is closest to a given point $p_0 \in \mathbb{R}^n$.

Part II: General preview material – geometry representation in $\mathbb{R}^n$:

1. Provide a parametric representation for the following curves:
   a. The straight line passing through $p = [1, 1, 3, 6]^T$ in the direction $v = [2, 3, 1, -1]$.
   b. For the same point p and direction v in part (a): the line segment between the points p and $p - v$. Note: this is a line segment of finite length between the given two points.
   c. The helix in $\mathbb{R}^3$ around the x_3 coordinate axis, with radius r, that climbs exactly 1 unit of height per each cycle completed.
   d. The spiral about the origin, turning counter-clockwise, with radius the grows linearly with its rotation angle.

e. The Torus surface (doughnut surface) about the origin with large radius R and small radius r. Hint: we can think of a torus as a small circle swept along a larger circular path. To uniquely describe a point on the torus, we need to specify where it is along the large circular path, and where it is along the small circle. This can be done with two parameters of angles.

2. Provide an implicit representation of the following geometries (note – you may need more than one constraint to describe them):
   a. The hyper-plane through $[1, 0, 1]^T$ with normal vector $[1, 2, 1]^T$.
   b. The half space of all points in $\mathbb{R}^n$ with the sum of their entries less than one.
   c. The *probability simplex*: all points in $\mathbb{R}^n$ with non-negative entries that sum to one.
   For $n = 2$ and $n = 3$ sketch a picture of it.
   d. The sphere in $\mathbb{R}^n$ about the origin with radius 2, but only the part of it with non-negative entries. For $n = 2,3$ sketch a picture of it.
   e. The unit ball in $\mathbb{R}^3$ for the infinity norm: $\|x\|_\infty = \max(|x_1|, \dots, |x_n|)$, but do not use any absolute values or norms in your implicit representation, use only linear constraints. Sketch a picture of this ball.