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Numerical Optimization with Python (2024B)

Dry HW 01 – Solutions

Part I: Modeling optimization problems:

<u>Solution 1:</u>

A car manufacturer produces three models: standard cars, pickup trucks, and vans. The goal is to maximize daily profit from manufacturing these vehicles.

Let $x_1,\ x_2,\ and\ x_3$ represent the number of standard cars, pickup trucks, and vans manufactured per day, respectively.

Objective function:

Maximize the total profit per day, calculated based on profit per unit for each type of vehicle:

Maximize $\quad 1000x_1\ +\ 1200x_2 + 1500x_3$

Constraints:

1. **Total Production Capacity Constraint:** The total production should not exceed the assembly line's capacity of 25 vehicles per day.
$$x_1 +\ x_2 +\ x_3 \leq 25$$
2. **Individual Vehicle Production Constraints:**
   - The number of vans produced cannot exceed 8 per day.
$$x_3 \leq 8$$
   - The number of pickup trucks cannot exceed 10 per day.
$$x_2 \leq 10$$
3. **Metal Workshop Time Constraint:** The total metal workshop time should not exceed 8 hours per day. The workshop times are 1 hour per car and 1.5 hours per pickup truck and van.
$$x_1 +\ 1.5x_2 +\ 1.5x_3 \leq 8$$
4. **Minimum Production Requirements:** The production must meet certain minimum sales projections.
   - At least 2 cars per day.
$$x_1 \geq 2$$
   - At least 2 pickup trucks per day.
$$x_2 \geq 2$$
   - At least 1 van per day.
$$x_3 \geq 1$$

Non-Negativity Constraints: Ensure that the production quantities are non-negative:

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$$

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Solution 2:

Problem Definition: Given the vector $x = [-1,\ 0.5,\ 8,\ 12]$,find the probability distribution vector $p = [p_1, p_2, p_3, p_4]^T$ such that the expected value of x is 2.5 and the entropy H of p is maximized.

Variables:

p_1 , p_2 , p_3 , p_4 the probabilities associated with x_1 , x_2 , x_3 and x_4 respectively, which are non-negative and sum to 1.

Objective Function:

Given that n equals 4 in the given problem description, the unknown vector becomes $[p_1, p_2, p_3, p_4]^T$. Optimal entropy is achieved by minimizing the negated objective. Therefore, we express our objective for minimization as:

$$\sum_{i=1}^{4} p_i \log p_i$$

Constraints:

1. Expected Value Constraint: The expected value $E(x)$ should be equal to 2.5:
$$-1p_1 + 0.5p_2 + 8p_3 + 12p_4 = 2.5$$
2. Probability Vector Constraint: The values of p_i must form a valid probability distribution:

- $p_1 + p_2 + p_3 + p_4 = 1$ (The sum of all probabilities must be 1 to form a valid probability distribution)
- $p_i \geq 0\ for\ i = 1,2,3,4$ (Each probability must be non-negative)

Formulation Summary:

The optimization problem to find a probability distribution p for the vector x that maximizes entropy while achieving an expected value of 2.5 is formulated as follows:

$$\min \left[\sum_{i=1}^{4} p_i \log p_i \right]$$

Subject to:

$$-1p_1 + 0.5p_2 + 8p_3 + 12p_4 = 2.5$$
$$p_1 + p_2 + p_3 + p_4 = 1$$
$$p_i \geq 0, \quad i = 1,2,3,4$$

Solution 3:

a. The goal is to find a vector $\hat{x}$ in the column space of A that is closest to b . This can be achieved by minimizing the Euclidean distance between Ax and b , where x are the scalar coefficients of the columns of A .

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Variables:

- $x \in \mathbb{R}^n$: The vector of scalar coefficients for the columns of A .

Objective Function:

- Minimize the squared Euclidean distance between Ax and b (squared to avoid dealing with square roots and ensure differentiability):

$$\text{minimize} \quad \| Ax - b \|^2$$

This can be expanded to:

minimize $(Ax - b)^T(Ax - b)$

minimize $x^T A^T Ax - 2x^T A^T b + b^T b$

Here, the term $b^T b$ can be dropped from optimization since it does not depend on x , simplifying the objective to:
minimize $x^T A^T Ax - 2x^T A^T b$

Constraints:

There are no explicit constraints on x beyond the natural requirement that x must be in $\mathbb{R}^n$.
Summary:
The optimization problem involves minimizing $x^T A^T Ax - 2x^T A^T b$ to find the vector in the column space of A closest to b .

b. Special Conditions:

- $m = n$: Matrix A is square.
- A Is non-singular (invertible).

Unknown Coefficients:

In this special case, since A is square and non-singular, the exact solution for x that minimizes the distance between Ax and b can be directly found using:

$$x = A^{-1}b$$

Realized Distance to b :

To find the realized distance to b , we compute the distance between Ax and b using $x = A^{-1}b$:
distance = $\| A(A^{-1}b) - b \|$
distance = $\| b - b \| = 0$

Thus, when A is square and non-singular, the vector Ax coincides exactly with b , and the realized distance is zero.
Summary:
If A is square and non-singular, the solution is $x = A^{-1}b$ with zero distance to b .

Solution 4:

The goal is to find a solution x of the system $Ax = b$ that minimizes the Euclidean distance to p_0.

Variables:

$x \in \mathbb{R}^n$: The vector of unknowns we are trying to determine.

Objective Function:

Minimize the squared Euclidean distance between x and p_0:

minimize $\| x - p_0 \|^2$

This can be expanded to:

minimize $(x - p_0)^T(x - p_0)$

minimize $x^T x - 2x^T p_0 + p_0^T p_0$

The term $p_0^T p_0$ can be omitted from the optimization as it is constant with respect to x, simplifying the objective to:

minimize $x^T x - 2x^T p_0$

Constraints:

The primary constraint for this problem is the original system of linear equations:

$$Ax = b$$

This ensures that x is indeed a solution to the system.

Problem Summary:

To find the solution x closest to p_0 from among all solutions to $Ax = b$, the optimization model is formulated as follows:

minimize $x^T x - 2x^T p_0$

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Subject to:

$$Ax = b$$

This problem can be addressed by applying methods from linear algebra and optimization. Specifically:

1. **Solution of the Linear System**: First, find a particular solution x_p of the equation

   $Ax = b$.

2. **General Solution Form**: The general solution to $Ax = b$ can be expressed as

   $x = x_p + y$, where y is in the null space of A.

3. **Minimization Problem**: Convert the objective of minimizing $\| x - p_0 \|^2$ into a problem involving y :

minimize $\| x_p + y - p_0 \|^2$

equiv. Minimize $y^T y - 2y^T(p_0 - x_p)$

This minimization problem involves finding y such that $Ay = 0$ (since y is in the null space) and minimizing the revised objective.

Part II: General preview material – geometry representation in $\mathbb{R}^n$:

<u>**Solution 1:**</u>

a. Straight Line:

Given a point $P = [1,1,3,6]^T$ and a direction $v = [2,3,1,-1]$, a parametric equation for a line can be represented as:

$$x(t) = p + tv$$

Where t is a scalar parameter, giving:

$$x(t) = [1 + 2t,\ 1 + 3t,\ 3 + t,\ 6 - t]^T$$

b. Line Segment:

For the line segment between $p\ and\ p - v$, we use a similar representation but restrict t to the interval $[0,1]$:

$$x(t) = p + t(p - v - p) = p - tv$$

Where:

$$x(t) = [1 + 2t,\ 1 + 3t,\ 3 + t,\ 6 - t]^T\ ,\ t \in [0,1]$$

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c. Helix:

A helix around the x_3 axis in $\mathbb{R}^3$ with radius r can be represented as:

$$x(t) = [r \cos t\,,\ r \sin t\,,\ t]^T$$

Here, t is the parameter which effectively measures the height along the x_3 axis, ensuring the helix climbs 1 unit per cycle $(2\pi\ in\ t)$

d. Spiral:

A spiral expanding linearly with the angle in $\mathbb{R}^2$ can be described as:

$$x(t) = [t \cos t\,,\ t \sin t]^T$$

Where t is the polar angle and the radius grows linearly with t .

e. Torus:

A torus centered at the origin with a major radius R and a minor radius r can be parametrized using two angles, θ and ϕ, as:

$$x(\theta, \phi) = [(R + r \cos \phi) cos\theta,\ (R + r \cos \phi) sin\theta,\ r \sin \phi]^T$$

Where θ and ϕ range from 0 to 2π. This representation considers θ as the angle along the large circle and ϕ as the angle along the cross-sectional circle of the torus.

<u>Solution 2:</u>

a. A hyperplane in three dimensions can be defined implicitly by the equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Where (x_0, y_0, z_0) is a point on the hyperplane and (a, b, c) is the normal vector to the hyperplane.

For the hyperplane given:

- Point $[1{,}0{,}1]^T$
- Normal vector $[1{,}2{,}1]^T$

The implicit representation of your hyperplane is:

$$1(x - 1) + 2(y - 0) + 1(z - 1) = 0$$

which simplifies to:

$$x - 1 + 2y + z - 1 = 0$$

Thus, the implicit representation of the hyperplane is:

$$x + 2y + z = 2$$

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b. For any point $x = (x_1 + x_2 + \cdots + x_n)$ in $\mathbb{R}^n$, the condition is:

$$x_1 + x_2 + \cdots + x_n < 1$$

This linear inequality is the implicit representation of the half-space. It specifies that the sum of all components (or entries) of the vector x must be less than one. Here's how this applies:

- Geometric Interpretation: This half-space forms an infinite region in $\mathbb{R}^n$ where all points lie on one side of the hyperplane defined by the equation $x_1 + x_2 + \cdots + x_n = 1$. The region does not include the hyperplane itself since the inequality is strict.

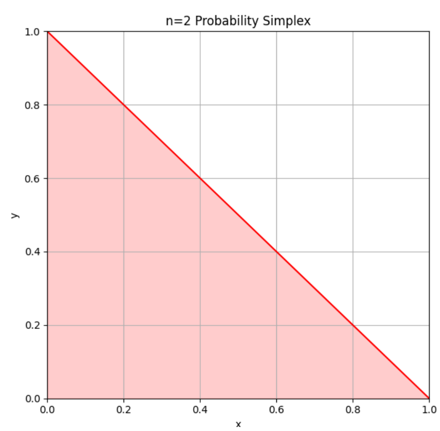
Example in Lower Dimensions:

1. In $\mathbb{R}^2$, this region would be represented as all points (x, y) such that $x + y < 1$. This can be visualized as the area below and to the left of the line $x + y = 1$ in the Cartesian plane.
2. In $\mathbb{R}^3$, it represents the volume below the plane formed by $x + y + z = 1$.

c. For $n = 2$ **:**

Let the coordinates be x and y . The probability simplex in $\mathbb{R}^2$ is defined by:

1. $x \geq 0$ (non-negative)
2. $y \geq 0$ (non-negative)
3. $x + y = 1$ (sums to one)



For $n = 3$:

Let the coordinates be x , y , and z . The probability simplex in $\mathbb{R}^3$ is defined by:

1. $x \geq 0$
2. $y \geq 0$
3. $z \geq 0$
4. $x + y + z = 1$

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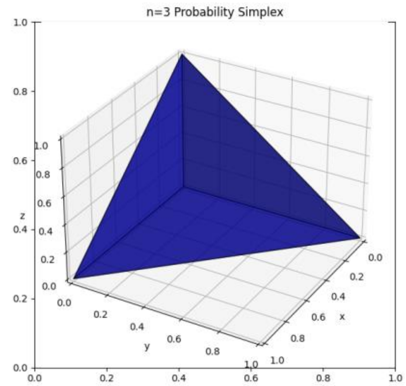


d. Sphere Equation:

The general equation for a sphere centered at the origin with radius r in $\mathbb{R}^n$ is:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = r^2$$

For a sphere with radius 2, this becomes:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 4$$

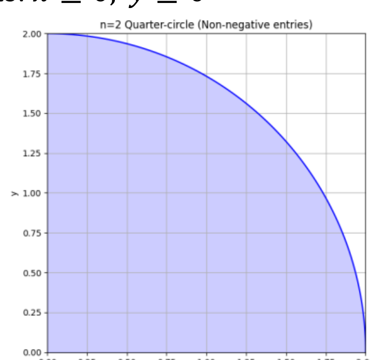
Non-Negativity Constraints:

Each coordinate x_i must be non-negative:

$$x_i \geq 0 \; for \; each \; i$$

For $n = 2$ (A circle):

- Implicit Equation: $x^2 + y^2 = 4$
- Non-Negativity Constraints: $x \geq 0, \; y \geq 0$



For $n = 3$ (A sphere):

- Implicit Equation: $x^2 + y^2 + z^2 = 4$
- Non-Negativity Constraints: $x \geq 0, \; y \geq 0, \; z \geq 0$

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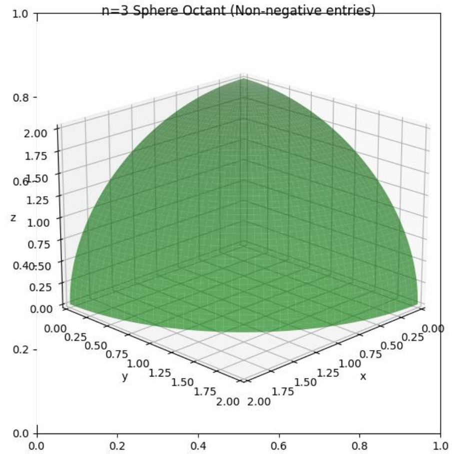

e. The infinity norm of a vector $x = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ is given by the maximum absolute value of its components:

$$\| x \|_\infty = \max(|x_1|, |x_2|, |x_3|)$$

For the unit ball under the infinity norm, we want all vectors x such that:

$$\| x \|_\infty \leq 1$$

This condition translates to each component of x having an absolute value of at most 1. However, since we need to avoid absolute values and represent this with only linear constraints, we can express this condition as:

$$-1 \leq x_i \leq 1 \; for \; i = 1,2,3$$

Linear Constraints:

Thus, the implicit representation of the unit ball in $\mathbb{R}^3$ for the infinity norm, using linear constraints only, is given by:

1. $-1 \leq x_1 \leq 1$
2. $-1 \leq x_2 \leq 1$
3. $-1 \leq x_3 \leq 1$

These constraints ensure that no component of x exceeds 1 in absolute value, which satisfies the definition of the infinity norm being at most 1 for all vectors inside the unit ball. These are linear inequalities and do not use any absolute value terms or norm expressions directly.

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