Numerical Optimization with Python (2024B)

<u>Dry HW 02</u>

1. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by: $f(x) = 2x_1^2 + 4x_2^2 + 6x_1x_2$.
    a. Write f as a quadratic function in matrix form, namely find a symmetric matrix Q such that f is given by: $f(x) = \frac{1}{2}x^T Q x$.
    b. Compute ∇f at the point $x = [1,1]^T$
    c. Compute $\nabla^2 f$
    d. Compute $\frac{\partial f}{\partial v}$ at the point $x = [-2,0]^T$ in the direction $v = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^T$
    e. Compute $\frac{\partial^2 f}{\partial v^2}$ for $v = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^T$
2. Prove that the intersection of any number (can be infinite, and need not be countable) of convex sets is a convex set. Does the same claim hold for unions of convex sets? Prove or give a counter example.
3. Consider $S_+^n \subset \mathbb{R}^{n\times n}$: the set of all symmetric, positive semidefinite matrices.
    a. Prove that S_+^n is a convex set.
    b. Explain why in fact S_+^n is a cone: for any $A, B \in S_+^n$, and any $\alpha, \beta \geq 0$ (not necessarily that sum to one, as we require for convex combinations).
4. Let $A \in \mathbb{R}^{m\times n}$ be a matrix and $b \in \mathbb{R}^m$ a vector.
    a. Denote by $S \subset \mathbb{R}^n$ the set of solutions of $Ax = b$. Show that S is a convex set.
    b. Explain why in fact S is an affine set: for any $x_1, x_2 \in S$, the entire line supported by x_1, x_2 is also in S (not just the line segment, as we required in convex sets).
5. A sub-level set of a function $f: \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is defined as follows: $\{x \in \mathcal{D}: f(x) \leq c\}$ where c is a constant scalar.
    a. Sketch the sublevel sets of $f(x,y) = 4x^2 + y^2$ for $c = 1, 2$.
    b. Prove that sub-level sets of convex functions are convex sets
6. Let $f: \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $h: \mathbb{R} \to \mathbb{R}$ be convex and monotone increasing.
    a. Prove that the composition $h \circ g: \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}$ is convex.
    b. Provide a counter example if we drop only the monotonic requirement on h.
    c. Provide another counter example if we drop only the convexity requirement on h.
7. For a number $a \in \mathbb{R}$ we know that $a^2 \geq 0$, and if $a > 0$ then $a^2 > 0$. In this question we discuss the analogue for matrices: let $A \in \mathbb{R}^{m\times n}$ be any matrix.
    a. Show that $A^T A$ is positive semi-definite.

b. Show that if, in addition, we assume $\dim Ker(A) = 0$ then $A^T A$ is positive definite.

8. Consider $f(x_1, x_2) = 2 + 6x_1 + 2x_2 + 3x_1^2 - 1x_2^2$.

   a. Write f in vector form, that is: represent $f(x) = \frac{1}{2}x^T Qx + q^T x + c$ for appropriate matrix, vector and scalar Q, q and c, respectively.

   b. Show that f has only one stationary point, and that it is neither a max nor a min, but a saddle.

   c. Provide a rough sketch of the contour lines of f.

9. Define the Rosenbrock function: $f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$.

   a. Compute $\nabla f(x)$ and $\nabla^2 f(x)$

   b. Show that $x^* = (1,1)^T$ is the only local minimizer of f, and that the Hessian is positive definite at that point.