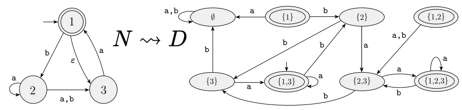
# (1) Reg / DFA / NFA

| | $\overline{\text{REG}}$ | REG | CFL | Turing DECID. | Turing RECOG. | P | NP | NPC |
|---|---|---|---|---|---|---|---|---|
| $L1 \cup L2$ | **no** | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | **no** |
| $L1 \cap L2$ | **no** | ✓ | **no** | ✓ | ✓ | ✓ | ✓ | **no** |
| $\overline{L}$ | ✓ | ✓ | **no** | ✓ | **no** | ✓ | ? | ? |
| $L1 \cdot L2$ | **no** | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | **no** |
| $L^*$ | **no** | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | **no** |
| $L^{\mathcal{R}}$ | | ✓ | ✓ | ✓ | ✓ | ✓ | | |
| $L \cap R$ | | ✓ | ✓ | ✓ | ✓ | ✓ | | |
| $L1 \setminus L2$ | | ✓ | **no** | ✓ | **no** | ✓ | ? | |

- (**DFA**) $M = (Q, \Sigma, \delta, q_0, F)$, $\delta : Q \times \Sigma \to Q$
- (**NFA**) $M = (Q, \Sigma, \delta, q_0, F)$, $\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$

- (**GNFA**) $(Q, \Sigma, \delta, q_0, q_{\text{a}})$, $\delta : (Q \setminus \{q_{\text{a}}\}) \times (Q \setminus \{q_{\text{start}}\} \longrightarrow \mathcal{R}$ (where $\mathcal{R} = \{\text{all regex over } \Sigma\}$)
- GNFA accepts $w \in \Sigma^*$ if $w = w_1 \cdots w_k$, where $w_i \in \Sigma^*$ and there exists a sequence of states $q_0, q_1, \ldots, q_k$ s.t. $q_0 = q_{\text{start}}$, $q_k = q_{\text{a}}$ and for each $i$, we have $w_i \in L(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$.
- (**DFA** $\rightsquigarrow$ **GNFA**) $G = (Q', \Sigma, \delta', s, a)$, $Q' = Q \cup \{s, a\}$, $\delta'(s, \varepsilon) = q_0$, For each $q \in F$, $\delta'(q, \varepsilon) = a$, ((TODO...))
- (**P.L.**) If $A$ is a regular lang., then $\exists p$ s.t. every string $s \in A$, $|s| \geq p$, can be written as $s = xyz$, satisfying: (**i**) $\forall i \geq 0, xy^iz \in A$, (**ii**) $|y| > 0$ and (**iii**) $|xy| \leq p$.
- Every NFA can be converted to an equivalent one that has a single accept state.

- (**reg. grammar**) $G = (V, \Sigma, R, S)$. Rules: $A \to aB$, $A \to a$ or $S \to \varepsilon$. ($A, B, S \in V$; $a \in \Sigma$).
- (**NFA** $\rightsquigarrow$ **DFA**)




  - $N = (Q, \Sigma, \delta, q_0, F)$
  - $D = (Q' = \mathcal{P}(Q), \Sigma, \delta', q_0' = E(\{q_0\}), F')$
  - $F' = \{q \in Q' \mid \exists p \in F : p \in q\}$
  - $E(\{q\}) := \{q\} \cup \{\text{states reachable from } q \text{ via } \varepsilon\text{-arrows}\}$
  - $\forall R \subseteq Q, \forall a \in \Sigma, \delta'(R, a) = E\left(\bigcup_{r \in R} \delta(r, a)\right)$
- $L(\varepsilon \cup 0\Sigma^*0 \cup 1\Sigma^*1) = \{w \mid \#_w(01) = \#_w(10)\}$,

# (2) CFL / CFG / PDA

- (**CFG**) $G = (\underset{\text{n.t.}}{V}, \underset{\text{ter.}}{\Sigma}, R, S)$. Rules: $A \to w$. (where $A \in V$ and $w \in (V \cup \Sigma)^*$).
- A derivation of $w$ is a **leftmost derivation** if at every step the leftmost remaining variable is the one replaced.
- $w$ is derived **ambiguously** in $G$ if it has at least two different l.m. derivations. $G$ is **ambiguous** if it generates at least one string ambiguously. A CFG is ambiguous iff it generates some string with two different parse trees. A CFL is **inherently ambiguous** if all CFGs that generate it are ambiguous.
- (**P.L.**) If $L$ is a CFL, then $\exists p$ s.t. any string $s \in L$ with $|s| \geq p$ can be written as $s = uvxyz$, satisfying: (**i**) $\forall i \geq 0, uv^ixy^iz \in L$, (**ii**) $|vxy| \leq p$, and (**iii**) $|vy| > 0$.
- (**CNF**) $A \to BC$, $A \to a$, or $S \to \varepsilon$, (where $A, B, C \in V$, $a \in \Sigma$, and $B, C \neq S$).
- (**CFG** $\rightsquigarrow$ **CNF**) (**1.**) Add a new start variable $S_0$ and a rule $S_0 \to S$. (**2.**) Remove $\varepsilon$-rules of the form $A \to \varepsilon$ (except for $S_0 \to \varepsilon$). and remove $A$'s occurrences on the RH of a rule (e.g.: $R \to uAvAw$ becomes $R \to uAvAw \mid uAvw \mid uvAw \mid uvw$. where $u, v, w \in (V \cup \Sigma)^*$). (**3.**) Remove unit rules $A \to B$ then whenever $B \to u$ appears, add $A \to u$, unless this was a unit rule previously removed. ($u \in (V \cup \Sigma)^*$). (**4.**) Replace each rule $A \to u_1u_2 \cdots u_k$ where $k \geq 3$ and $u_i \in (V \cup \Sigma)$, with the rules $A \to u_1A_1$, $A_1 \to u_2A_2$, ..., $A_{k-2} \to u_{k-1}u_k$, where $A_i$ are new variables. Replace terminals $u_i$ with $U_i \to u_i$.
- If $G \in$ CNF, and $w \in L(G)$, then $|w| \leq 2^{|h|} - 1$, where $h$ is the height of the parse tree for $w$.
- $L \in \textbf{CFL} \Leftrightarrow \exists \underset{\text{CFG}}{G} : L = L(G) \Leftrightarrow \exists \underset{\text{PDA}}{M} : L = L(M)$
- $\forall L \in$ CFL, $\exists G \in$ CNF : $L = L(G)$.
- (**derivation**) $S \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_n = w$, where each $u_i$ is in $(V \cup \Sigma)^*$. (in this case, $G$ **generates** $w$ (or $S$ **derives** $w$), $S \overset{*}{\Rightarrow} w$)

- (**PDA**) $M = (Q, \underset{\text{input}}{\Sigma}, \underset{\text{stack}}{\Gamma}, \delta, q_0 \in Q, \underset{\text{accepts}}{F} \subseteq Q)$. (where $Q$, $\Sigma$, $\Gamma$, $F$ finite). $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$.
- $M$ **accepts** $w \in \Sigma^*$ if there is a seq. $r_0, r_1, \ldots, r_m \in Q$ and $s_0, , s_1, \ldots, s_m \in \Gamma^*$ s.t.:
  - $r_0 = q_0$ and $s_0 = \varepsilon$
  - For $i = 0, 1, \ldots, m-1$, we have $(r_i, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$.
  - $r_m \in F$
- A PDA can be represented by a state diagram, where each transition is labeled by the notation "$a, b \to c$" to denote that the PDA: **Reads** $a$ from the input (or read nothing if $a = \varepsilon$). **Pops** $b$ from the stack (or pops nothing if $b = \varepsilon$). **Pushes** $c$ onto the stack (or pushes nothing if $c = \varepsilon$)
- (**CSG**) $G = (V, \Sigma, R, S)$. Rules: $S \to \varepsilon$ or $\alpha A \beta \to \alpha \gamma \beta$ where: $\alpha, \beta \in (V \cup \Sigma \setminus \{S\})^*$; $\gamma \in (V \cup \Sigma \setminus \{S\})^+$; $A \in V$.

## (3) TM, (4) Decidability

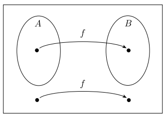
- (**TM**) $M = (Q, \underset{\text{input}}{\Sigma} \subseteq \Gamma, \underset{\text{tape}}{\Gamma}, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where $\sqcup \in \Gamma$ (**blank**), $\sqcup \notin \Sigma$, $q_{\text{reject}} \neq q_{\text{accept}}$, and $\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$
- (**recognizable**) accepts if $w \in L$, rejects/loops if $w \notin L$.
  - $L$ is recognizable $\iff L \leq_{\mathrm{m}} A_{\mathsf{TM}}$.
  - Some languages are unrecognizable.
  - $A$ is **co-recognizable** if $\overline{A}$ is recognizable.
  - Every inf. rec. lang. has an inf. dec. subset.
- (**decidable**) accepts if $w \in L$, rejects if $w \notin L$.
- $L \in \overset{\text{Turing}}{\text{DEC.}} \Leftrightarrow \left( L \in \overset{\text{Turing}}{\text{REC.}} \wedge L \in \overset{\text{Turing}}{\text{co-REC.}} \right) \Leftrightarrow \exists \underset{\text{TM}}{M}$ decides $L$.
- $L \in \overset{\text{Turing}}{\text{DECIDABLE}} \iff L \leq_{\mathrm{m}} \mathtt{0^*1^*}$.
- (**decider**) TM that halts on all inputs.
- (**Rice**) Let $P$ be a lang. of TM descriptions, s.t. (**i**) $P$ is nontrivial (not empty and not all TM desc.) and (**ii**) for each two TM $M_1$ and $M_2$, we have $L(M_1) = L(M_2) \implies (\langle M_1 \rangle \in P \iff \langle M_2 \rangle \in P)$. Then $P$ is undecidable.
- $\{\text{all TMs}\}$ is countable; $\Sigma^*$ is countable (for every finite $\Sigma$); $\{\text{all languages}\}$ is uncountable; $\{\text{all infinite binary sequences}\}$ is uncountable.
- DFA $\equiv$ NFA $\equiv$ GNFA $\equiv$ REG $\subset$ NPDA $\equiv$ CFG $\subset$ DTM $\equiv$ NTM

## FINITE $\subset$ REG $\subset$ CFL $\subset$ CSL $\subset$ $\overset{\text{Turing}}{\text{DECIDABLE}}$ $\subset$ $\overset{\text{Turing}}{\text{RECOGNIZABLE}}$

- (**unrecognizable**) $\overline{A_{TM}}$, $\overline{EQ_{\mathsf{TM}}}$, $EQ_{\mathsf{CFG}}$, $\overline{HALT_{\mathsf{TM}}}$, $\mathrm{REGULAR}_{\mathsf{TM}} = \{M \text{ is a TM and } L(M) \text{ is regular}\}$, $E_{\mathsf{TM}}$, $EQ_{\mathsf{TM}} = \{M_1, M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$
- (**recognizable but undec.**) $A_{TM}$, $HALT_{\mathsf{TM}} = \{\langle M, w \rangle \mid M \text{ is a TM halts on } w\}$, $D = \{p \mid p \text{ is an int. poly. with an int. root}\}$, $\overline{EQ_{\mathsf{CFG}}}$, $\overline{E_{\mathsf{TM}}}$
- (**decidable**) $A_{\mathsf{DFA}}$, $A_{\mathsf{NFA}}$, $A_{\mathsf{REX}}$, $E_{\mathsf{DFA}}$, $EQ_{\mathsf{DFA}}$, $A_{\mathsf{CFG}}$, $E_{\mathsf{CFG}}$, $A_{\mathsf{LBA}}$, $ALL_{\mathsf{DFA}} = \{\langle M \rangle \mid M \text{ is a DFA}, L(A) = \Sigma^*\}$, $A\varepsilon_{\mathsf{CFG}} = \{\langle G \rangle \mid G \text{ is a CFG that generates } \varepsilon\}$, $\mathrm{INFINITE}_{\mathsf{DFA}}$, $\mathrm{INFINITE}_{\mathsf{PDA}}$
- (**CFL but not REG**) $\{w \in \{a,b\}^* \mid w = w^{\mathcal{R}}\}$, $\{ww^{\mathcal{R}} \mid w \in \{a,b\}^*\}$, $\{a^n b^n \mid n \in \mathbb{N}\}$, $\{w \in \{\mathtt{a},\mathtt{b}\}^* \mid \#_{\mathtt{a}}(w) = \#_{\mathtt{b}}(w)\}$
- (**not CFL**) $\{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$, $\{a^n b^n c^n \mid n \in \mathbb{N}\}$, $\{ww \mid w \in \{a,b\}^*\}$, $\{\mathtt{a}^{j^2} \mid j \geq 0\}$, $\{w \in \{\mathtt{a},\mathtt{b},\mathtt{c}\}^* \mid \#_{\mathtt{a}}(w) = \#_{\mathtt{b}}(w) = \#_{\mathtt{c}}(w)\}$

## (5) Mapping Reduction $\leq_{\mathrm{m}}$

- $f : \Sigma^* \to \Sigma^*$ is **computable** if there exists a TM $M$ s.t. for every $w \in \Sigma^*$, $M$ halts on $w$ and outputs $f(w)$ on its tape.




- $A$ is **m. reducible** $B$ (denoted by $A \leq_{\mathrm{m}} B$), if there is a comp. func. $f : \Sigma^* \to \Sigma^*$ s.t. for every $w$, we have $w \in A \iff f(w) \in B$. (Such $f$ is called the **m. reduction** from $A$ to $B$.)
- If $A \leq_{\mathrm{m}} B$ and $B$ is decidable, then $A$ is dec.
- If $A \leq_{\mathrm{m}} B$ and $A$ is undecidable, then $B$ is undec.
- If $A \leq_{\mathrm{m}} B$ and $B$ is recognizable, then $A$ is rec.
- If $A \leq_{\mathrm{m}} B$ and $A$ is unrecognizable, then $B$ is unrec.
- (transitivity) If $A \leq_{\mathrm{m}} B$ and $B \leq_{\mathrm{m}} C$, then $A \leq_{\mathrm{m}} C$.
- If $A$ is recognizable and $A \leq_{\mathrm{m}} \overline{A}$, then $A$ is decidable.
- $A \leq_{\mathrm{m}} B \iff \overline{A} \leq_{\mathrm{m}} \overline{B}$

## (7) Complexity, Polytime Reduction $\leq_{\mathrm{P}}$

- ((**Running time**) decider $M$ is a $f(n)$**-time TM**.) $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the max. num. of steps that DTM (or NTM) $M$ takes on any $n$-length input (and any branch of any $n$-length input. resp.).
- $\mathsf{TIME}(t(n)) = \{L \mid L \text{ is dec. by } O(t(n)) \text{ DTM}\}$.
- $\mathsf{NTIME}(t(n)) = \{L \mid L \text{ is dec. by } O(t(n)) \text{ NTM}\}$.
- $\mathbf{P} = \bigcup_{k \in \mathbb{N}} \mathsf{TIME}(n^k)$
- (**verifier** for $L$) TM $V$ s.t. $L = \{w \mid \exists c : V(\langle w, c \rangle) = \mathsf{accept}\}$.
  - (**certificate** for $w \in L$) str. $c$ s.t. $V(\langle w, c \rangle) = \mathsf{accept}$.
- $\mathbf{NP} = \bigcup_{k \in \mathbb{N}} \mathsf{NTIME}(n^k)$
- $\mathbf{NP} = \{L \mid L \text{ is decidable by a PT verifier}\}$.
- $\mathrm{P} \subseteq \mathrm{NP}$.
- $f : \Sigma^* \to \Sigma^*$ is **PT computable** if there exists a PT TM $M$ s.t. for every $w \in \Sigma^*$, $M$ halts with $f(w)$ on its tape.
- $A$ is **PT (mapping) reducible** to $B$, denoted $A \leq_{\mathrm{P}} B$, if there exists a PT computable func. $f : \Sigma^* \to \Sigma^*$ s.t. for every $w \in \Sigma^*$, $w \in A \iff f(w) \in B$. (in such case $f$ is called the **PT reduction** of $A$ to $B$).
  - If $A \leq_{\mathrm{P}} B$ and $B \in \mathrm{P}$, then $A \in \mathrm{P}$.
  - If $A \leq_{\mathrm{P}} B$ and $B \leq_{\mathrm{P}} A$, then $A$ and $B$ are **PT equivalent**, denoted $A \equiv_P B$. $\equiv_P$ is an equivalence relation on NP. $\mathrm{P} \setminus \{\emptyset, \Sigma^*\}$ is an equivalence class of $\equiv_P$.
- $\textbf{NP-complete} = \{B \mid B \in \mathrm{NP}, \forall A \in \mathrm{NP}, A \leq_{\mathrm{P}} B\}$.
- CLIQUE, SUBSET-SUM, SAT, 3SAT, VERTEX-COVER, HAMPATH, UHAMATH, $3COLOR \in$ NP-complete.
- $\emptyset, \Sigma^* \notin$ NP-complete.
- If $B \in$ NP-complete and $B \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$.
- If $B \in$ NP-complete and $C \in \mathrm{NP}$ s.t. $B \leq_{\mathrm{P}} C$, then $C \in$ NP-complete.
- If $\mathrm{P} = \mathrm{NP}$, then $\forall A \in \mathrm{P} \setminus \{\emptyset, \Sigma^*\}$, $A \in$ NP-complete.

## Examples: $A \leq_{\mathrm{P}} B$ and $f : A \to B$ s.t. $w \in A \iff f(w) \in B$ and $f$ is polytime computable

- SAT $\leq_{\mathrm{P}}$ DOUBLE-SAT
  - $f(\phi) = \phi \wedge (x \vee \neg x)$
- SUBSET-SUM $\leq_{\mathrm{P}}$ SET-PARTITION
  - $f(\langle x_1, \ldots, x_m, t \rangle) = \langle x_1, \ldots, x_m, S - 2t \rangle$, where $S$ sum of $x_1, \ldots, x_m$, and $t$ is the target subset-sum.
- $3COLOR \leq_{\mathrm{P}} 3COLOR_{almost}$
  - $f(\langle G \rangle) = \langle G' \rangle$, where $G' = G \cup K_4$
- VERTEX-COVER $\leq_{\mathrm{P}}$ WVC
  - $f(\langle G, k \rangle) = (G, w, k)$, $\forall v \in V, w(v) = 1$.
- $\underset{\text{length} \geq k}{\text{SimplePATH}} \leq_{\mathrm{P}}$ UHAMATH
- $\underset{\text{undir. } G \text{ has } k\text{-clique}}{\text{CLIQUE}} \leq_{\mathrm{P}} \underset{\text{undir. } G \text{ has } |V|/2\text{-clique}}{\text{HALF-CLIQUE}}$
  - $f(\langle G = (V, E), k \rangle) = \langle G' = (V', E') \rangle$, if $k = \frac{|V|}{2}$, $E = E'$, $V' = V$. if $k > \frac{|V|}{2}$, $V' = V \cup \{j = 2k - |V| \text{ new nodes}\}$. if $k < \frac{|V|}{2}$, $V' = V \cup \{j = |V| - 2k \text{ new nodes}\}$ and $E' = E \cup \{\text{edges for new nodes}\}$
- CLIQUE $\leq_{\mathrm{P}}$ INDEPENDENT-SET
- SET-COVER $\leq_{\mathrm{P}}$ VERTEX-COVER
- 3SAT $\leq_{\mathrm{P}}$ SET-SPLITTING
- INDEPENDENT-SET $\leq_{\mathrm{P}}$ VERTEX-COVER
- VERTEX-COVER $\leq_{\mathrm{p}}$ CLIQUE

## Counterexamples

- $A \leq_{\mathrm{m}} B$ and $B \in$ REG, but, $A \notin$ REG: $A = \{0^n 1^n \mid n \geq 0\}$, $B = \{1\}$, $f : A \to B$, $f(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A \end{cases}$.
- $L \in$ CFL but $\overline{L} \notin$ CFL: $L = \{x \mid \forall w \in \Sigma^*, x \neq ww\}$, $\overline{L} = \{ww \mid w \in \Sigma^*\}$.
- $L_1, L_2 \in$ CFL but $L_1 \cap L_2 \notin$ CFL: $L_1 = \{a^n b^n c^m\}$, $L_2 = \{a^m b^n c^n\}$, $L_1 \cap L_2 = \{a^n b^n c^n\}$.
- $L_1 \in$ CFL, $L_2$ is infinite, but $L_1 \setminus L_2 \notin$ REG : $L_1 = \Sigma^*$, $L_2 = \{a^n b^n \mid n \geq 0\}$, $L_1 \setminus L_2 = \{a^m b^n \mid m \neq n\}$.
- $L_1, L_2 \in$ REG, $L_1 \not\subset L_2$, $L_2 \not\subset L_1$, but, $(L_1 \cup L_2)^* = L_1^* \cup L_2^*$ : $L_1 = \{\mathtt{a}, \mathtt{b}, \mathtt{ab}\}$, $L_2 = \{\mathtt{a}, \mathtt{b}, \mathtt{ba}\}$.
- $L_1 \in$ REG, $L_2 \notin$ REG, but, $L_1 \cap L_2 \in$ REG, and $L_1 \cup L_2 \in$ REG : $L_1 = L(\mathtt{a}^*\mathtt{b}^*)$, $L_2 = \{\mathtt{a}^n \mathtt{b}^n \mid n \geq 0\}$.
- $L_1, L_2, L_3, \dots \in$ REG, but, $\bigcup_{i=1}^{\infty} L_i \notin$ REG : $L_i = \{\mathtt{a}^i \mathtt{b}^i\}$, $\bigcup_{i=1}^{\infty} L_i = \{\mathtt{a}^n \mathtt{b}^n \mid n \geq 0\}$.
- $L_1 \cdot L_2 \in$ REG, but $L_1 \notin$ REG : $L_1 = \{\mathtt{a}^n \mathtt{b}^n \mid n \geq 0\}$, $L_2 = \Sigma^*$.
- $L_2 \in$ CFL, and $L_1 \subseteq L_2$, but $L_1 \notin$ CFL : $\Sigma = \{a, b, c\}$, $L_1 = \{a^n b^n c^n \mid n \geq 0\}$, $L_2 = \Sigma^*$.
- $L_1, L_2 \in$ DECIDABLE, and $L_1 \subseteq L \subseteq L_2$, but $L \in$ UNDECIDABLE : $L_1 = \emptyset$, $L_2 = \Sigma^*$, $L$ is some undecidable language over $\Sigma$.
- $L_1 \in$ REG, $L_2 \notin$ CFL, but $L_1 \cap L_2 \in$ CFL : $L_1 = \{\varepsilon\}$, $L_2 = \{a^n b^n c^n \mid n \geq 0\}$.