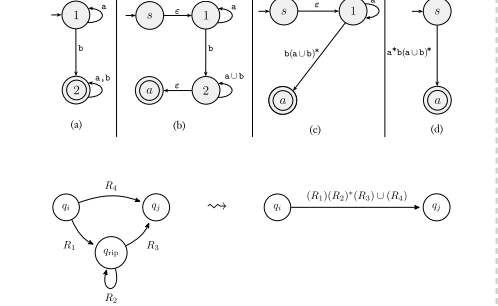


	REG	REG	CFL	DEC.	REC.	P	NP	NPC
$L_1 \cup L_2$	<b>no</b>	✓	✓	✓	✓	✓	✓	<b>no</b>
$L_1 \cap L_2$	<b>no</b>	✓	<b>no</b>	✓	✓	✓	✓	<b>no</b>
$\overline{L}$	✓	✓	<b>no</b>	✓	<b>no</b>	✓	?	?
$L_1 \cdot L_2$	<b>no</b>	✓	✓	✓	✓	✓	✓	<b>no</b>
$L^*$	<b>no</b>	✓	✓	✓	✓	✓	✓	<b>no</b>
$L\mathcal{R}$	✓	✓	✓	✓	✓	✓		
$L_1 \setminus L_2$	<b>no</b>	✓	<b>no</b>	✓	<b>no</b>	✓	?	
$L \cap R$	<b>no</b>	✓	✓	✓	✓	✓		

- **(DFA)**  $M = (Q, \Sigma, \delta, q_0, F)$ ,  $\delta : Q \times \Sigma \rightarrow Q$ .
- **(NFA)**  $M = (Q, \Sigma, \delta, q_0, F)$ ,  $\delta : Q \times \Sigma_\varepsilon \rightarrow \mathcal{P}(Q)$ .
- **(GNFA)**  $(Q, \Sigma, \delta, q_0, q_a)$ ,  
 $\delta : (Q \setminus \{q_a\}) \times (Q \setminus \{q_{\text{start}}\}) \rightarrow \mathcal{R}$  (where  $\mathcal{R} = \{\text{Regex over } \Sigma\}$ )
- **(DFA  $\rightsquigarrow$  GNFA  $\rightsquigarrow$  Regex)**



- GNFA accepts  $w \in \Sigma^*$  if  $w = w_1 \cdots w_k$ , where  $w_i \in \Sigma^*$  and there exists a sequence of states  $q_0, q_1, \dots, q_k$  s.t.  $q_0 = q_{\text{start}}$ ,  $q_k = q_a$  and for each  $i$ , we have  $w_i \in L(R_i)$ , where  $R_i = \delta(q_{i-1}, q_i)$ .
- $n$ -state DFA  $A$ ,  $m$ -state DFA  $B \implies \exists nm$ -state DFA  $C$  s.t.  $L(C) = L(A)\Delta L(B)$ .
- $p$ -state DFA  $C$ , if  $L(C) \neq \emptyset$  then  $\exists s \in L(C)$  s.t.  $|s| < p$ .
- Every NFA has an equiv. NFA with a single accept

**PL:**  $A \in \text{REG} \implies \exists p : \forall s \in A, |s| \geq p, s = xyz, \textbf{(i)} \forall i \geq 0, xy^iz \in A, \textbf{(ii)} |y| > 0 \text{ and } \textbf{(iii)} |xy| \leq p.$

- $\{w = a^{2^k}\}; \quad k = \lfloor \log_2 |w| \rfloor, s = a^{2^k} = xyz.$   
 $2^k = |xyz| < |xy^2z| \leq |xyz| + |xy| \leq 2^k + p < 2^{k+1}.$
- $\{w = w^{\mathcal{R}}\}; \quad s = 0^p10^p = xyz.$  then  
 $xy^2z = 0^{p+|y|}10^p \notin L.$
- $\{a^n b^n\}; \quad s = a^p b^p = xyz,$  where  $|y| > 0$  and  $|xy| \leq p.$

- Then  $xy^2z = a^{p+|y|}b^p \notin L.$
- $\{a^p : p \text{ is prime}\}; \quad s = a^t = xyz$  for prime  $t \geq p.$   
 $r := |y| > 0$
- $\{www : w \in \Sigma^*\}; s = a^p b a^p b a^p = xyz = a^{|x|+|y|+m} b a^p b a^p b$ ,  $m \geq 0$ , but  $xy^2z = a^{|x|+2|y|+m} b a^p b a^p b \notin L.$
- $\{a^{2n} b^{3n} a^n\}; s = a^{2p} b^{3p} a^p = xyz = a^{|x|+|y|+m+p} b^{3p} a^p$ ,  $m \geq 0$ , but  $xy^2z = a^{2p+|y|} b^{3p} a^p \notin L.$

- state.
- $A = L(N\text{NFA}), B = (L(M_{\text{DFA}}))^c$  then  $A \cdot B \in \text{REG}.$
- **(NFA  $\rightsquigarrow$  DFA)**
  - $N = (Q, \Sigma, \delta, q_0, F)$
  - $D = (Q' = \mathcal{P}(Q), \Sigma, \delta', q'_0 = E(\{q_0\}), F')$
  - $F' = \{q \in Q' \mid \exists p \in F : p \in q\}$
  - $E(\{q\}) := \{q\} \cup \{\text{states reachable from } q \text{ via } \varepsilon\text{-arrows}\}$
- $\forall R \subseteq Q, \forall a \in \Sigma, \delta'(R, a) = E\left(\bigcup_{r \in R} \delta(r, a)\right)$

- **Regular Expressions Examples:**
  - $\{a^n w b^n : w \in \Sigma^*\} \equiv a(a \cup b)^* b$
  - $\{w : \#_w(0) \geq 2 \vee \#_w(1) \leq 1\} \equiv (\Sigma^* 0 \Sigma^* 0 \Sigma^*) \cup (0^* (\varepsilon \cup 1) 0^*)$
  - $\{w : |w| \bmod n = m\} \equiv (a \cup b)^m ((a \cup b)^n)^*$
  - $\{w : \#_b(w) \bmod n = m\} \equiv (a^* b a^*)^m \cdot ((a^* b a^*)^n)^*$
  - $\{w : |w| \text{ is odd}\} \equiv (a \cup b)^* ((a \cup b)(a \cup b)^*)^*$
  - $\{w : \#_a(w) \text{ is odd}\} \equiv b^* a (a b^* a \cup b)^*$
  - $\{w : \#_{ab}(w) = \#_{ba}(w)\} \equiv \varepsilon \cup a \cup b \cup a \Sigma^* a \cup b \Sigma^* b$

$L \in \text{CFL} \Leftrightarrow \exists G_{\text{CFG}} : L = L(G) \Leftrightarrow \exists M_{\text{PDA}} : L = L(M)$

- A derivation of  $w$  is a **leftmost derivation** if at every step the leftmost remaining variable is the one replaced;  $w$  is derived **ambiguously** in  $G$  if it has at least two different l.m. derivations.  $G$  is **ambiguous** if it generates at least one string ambiguously. A CFG is ambiguous iff it generates some string with two different parse trees. A CFL is **inherently ambiguous** if all CFGs that generate it are ambiguous.
- **(CFG  $\rightsquigarrow$  CNF) (1.)** Add a new start variable  $S_0$  and a rule  $S_0 \rightarrow S$ . **(2.)** Remove  $\varepsilon$ -rules of the form  $A \rightarrow \varepsilon$  (except for  $S_0 \rightarrow \varepsilon$ ). and remove  $A$ 's occurrences on the RH of a rule (e.g.:  $R \rightarrow uAvAw$  becomes  $R \rightarrow uAvAw \mid uAvw \mid uvAw \mid uvw$ . where

- $u, v, w \in (V \cup \Sigma)^*$ . **(3.)** Remove unit rules  $A \rightarrow B$  then whenever  $B \rightarrow u$  appears, add  $A \rightarrow u$ , unless this was a unit rule previously removed. ( $u \in (V \cup \Sigma)^*$ ). **(4.)** Replace each rule  $A \rightarrow u_1 u_2 \cdots u_k$  where  $k \geq 3$  and  $u_i \in (V \cup \Sigma)$ , with the rules  $A \rightarrow u_1 A_1, A_1 \rightarrow u_2 A_2, \dots, A_{k-2} \rightarrow u_{k-1} u_k$ , where  $A_i$  are new variables. Replace terminals  $u_i$  with  $U_i \rightarrow u_i$ .
- If  $G \in \text{CNF}$ , and  $w \in L(G)$ , then  $|w| \leq 2^{|h|} - 1$ , where  $h$  is the height of the parse tree for  $w$ .
- $\forall L \in \text{CFL}, \exists G \in \text{CNF} : L = L(G).$
- **(derivation)**  $S \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_n = w$ , where each  $u_i$  is in  $(V \cup \Sigma)^*$ . (in this case,  $G$  **generates**  $w$  (or  $S$  **derives**  $w$ ),  $S \xRightarrow{*} w$ )

- **(PDA)**  $M = (Q, \underset{\text{input}}{\Sigma}, \underset{\text{stack}}{\Gamma}, \delta, q_0 \in Q, \underset{\text{accepts}}{F} \subseteq Q)$ . (where  $Q, \Sigma, \Gamma, F$  finite).  $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$ .
- $M$  **accepts**  $w \in \Sigma^*$  if there is a seq.  $r_0, r_1, \dots, r_m \in Q$  and  $s_0, s_1, \dots, s_m \in \Gamma^*$  s.t.: (1.)  $r_0 = q_0$  and  $s_0 = \varepsilon$ ; (2.) For  $i = 0, 1, \dots, m - 1$ , we have  $(r_i, b) \in \delta(r_i, w_{i+1}, a)$ , where  $s_i = at$  and  $s_{i+1} = bt$  for some  $a, b \in \Gamma_\varepsilon$  and  $t \in \Gamma^*$ ; (3.)  $r_m \in F$ .
- **(PDA transition)** " $a, b \rightarrow c$ ": **reads**  $a$  from the input (or read nothing if  $a = \varepsilon$ ). **pops**  $b$  from the stack (or pops nothing if  $b = \varepsilon$ ). **pushes**  $c$  onto the stack (or pushes nothing if  $c = \varepsilon$ )
- $R \in \text{REG} \wedge C \in \text{CFL} \implies R \cap C \in \text{CFL}$ . (Prf. construct PDA  $P' = P_C \times D_R$ .)

**(CFG)**  $G = (V, \Sigma, R, S), A \rightarrow w, (A \in V, w \in (V \cup \Sigma)^*)$ ; **(CNF)**  $A \rightarrow BC, A \rightarrow a, S \rightarrow \varepsilon, (A, B, C \in V, a \in \Sigma, B, C \neq S).$

- $\{w : w = w^{\mathcal{R}}\}; S \rightarrow aSa \mid bSb \mid a \mid b \mid \varepsilon$
- $\{w : w \neq w^{\mathcal{R}}\}; S \rightarrow aSa \mid bSb \mid aXb \mid bXa;$   
 $X \rightarrow aX \mid bX \mid \varepsilon$
- $\{ww^{\mathcal{R}} \mid w \in \{a, b\}^*\}$
- $\{w\#x : w^{\mathcal{R}} \subseteq x\}; S \rightarrow AX; A \rightarrow 0A0 \mid 1A1 \mid \#X; X \rightarrow 0X \mid 1X \mid \#$
- $\{w : \#_w(a) > \#_w(b)\}; S \rightarrow TaT; T \rightarrow TT \mid aTb \mid bTa \mid a \mid \varepsilon$
- $\{w : \#_w(a) \geq \#_w(b)\}; S \rightarrow SS \mid aSb \mid bSa \mid a \mid \varepsilon$

- $\{w : \#_w(a) = \#_w(b)\}; S \rightarrow SS \mid aSb \mid bSa \mid \varepsilon$
- $\{\overline{a^n b^n}\}; S \rightarrow XbXaX \mid A \mid B; A \rightarrow aAb \mid Ab \mid b;$   
 $B \rightarrow aBb \mid aB \mid a; X \rightarrow aX \mid bX \mid \varepsilon.$
- $\{a^n b^m \mid n \neq m\}; S \rightarrow aSb \mid A \mid B; A \rightarrow aA \mid a; B \rightarrow bB \mid b^*$
- $\{a^i b^j c^k \mid i \leq j \text{ or } j \leq k\}; S \rightarrow S_1 C \mid AS_2;$   
 $S_1 \rightarrow aS_1 b \mid S_1 b \mid \varepsilon; S_2 \rightarrow bS_2 c \mid S_2 c \mid \varepsilon;$   
 $A \rightarrow Aa \mid \varepsilon; C \rightarrow Cc \mid \varepsilon$

- $\{x \mid x \neq ww\}; S \rightarrow A \mid B \mid AB \mid BA; A \rightarrow CAC \mid 0;$   
 $B \rightarrow CBC \mid 1; C \rightarrow 0 \mid 1$
- $\{a^n b^m \mid m \leq n \leq 3m\}; S \rightarrow aSb \mid aaSb \mid aaaSb \mid \varepsilon;$
- $\{a^n b^n\}; S \rightarrow aSb \mid \varepsilon$
- $\{a^n b^m \mid n > m\}; S \rightarrow aSb \mid aS \mid a$
- $\{w : \#_w(a) \geq 3\}; S \rightarrow XaXaXaX; X \rightarrow aX \mid bX \mid \varepsilon$
- $\{w : w = w^{\mathcal{R}} \wedge |w| \text{ is even}\}; S \rightarrow aSa \mid bSb \mid \varepsilon$
- $\{a^i b^j c^k \mid i + j = k\}; S \rightarrow aSc \mid X; X \rightarrow bXc \mid \varepsilon$

**PL:**  $L \in \text{CFL} \implies \exists p : \forall s \in L, |s| \geq p, s = uvxyz, \textbf{(i)} \forall i \geq 0, uv^i xy^i z \in L, \textbf{(ii)} |vxy| \leq p, \text{ and } \textbf{(iii)} |vy| > 0.$

- $\{w = a^n b^n c^n\}; s = a^p b^p b^p = uvxyz.$   $vxy$  can't contain all of  $a, b, c$  thus  $uv^2 xy^2 z$  must pump one of them less than the others.

- $\{ww : w \in \{a, b\}^*\};$
- **(more example of not CFL)**
- $\{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}, \{a^n b^n c^n \mid n \in \mathbb{N}\},$   
 $\{ww \mid w \in \{a, b\}^*\}, \{a^{n^2} \mid n \geq 0\}, \{a^p \mid p \text{ is prime}\},$

- $L = \{ww^{\mathcal{R}}w : w \in \{a, b\}^*\}$
- $\{w \mid \#_w(a) = \#_w(b) = \#_w(c)\}$ : (Prf. since  $\text{Regular} \cap \text{CFL} \in \text{CFL}$ , but  $\{a^* b^* c^*\} \cap L = \{a^n b^n c^n\} \notin \text{CFL}$ )

$L \in \text{DECIDABLE} \iff (L \in \text{REC. and } L \in \text{co-REC.}) \iff \exists M_{\text{TM}} \text{ decides } L.$

- **(TM)**  $M = (Q, \underset{\text{input}}{\Sigma} \subseteq \Gamma, \underset{\text{tape}}{\Gamma}, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$ , where  $\sqcup \in \Gamma, \sqcup \notin \Sigma, q_{\text{rej}} \neq q_{\text{acc}}, \delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{\text{L}, \text{R}\}$
- **(recognizable)**  $\textcircled{\text{A}}$  if  $w \in L, \overline{\textcircled{\text{R}}}/\text{loops}$  if  $w \notin L$ ;  $A$  is **co-recognizable** if  $\overline{A}$  is recognizable.
- $L \in \text{RECOGNIZABLE} \iff L \leq_m A_{\text{TM}}.$
- Every inf. recognizable lang. has an inf. dec. subset.
- **(decidable)**  $\textcircled{\text{A}}$  if  $w \in L, \overline{\textcircled{\text{R}}}$  if  $w \notin L.$
- $L \in \text{DECIDABLE} \iff L \leq_m 0^* 1^*.$

- $L \in \text{DECIDABLE} \iff L^{\mathcal{R}} \in \text{DECIDABLE}.$
- **(decider)** TM that halts on all inputs.
- **(Rice)** Let  $P$  be a lang. of TM descriptions, s.t. **(i)**  $P$  is nontrivial (not empty and not all TM desc.) and **(ii)** for each two TM  $M_1$  and  $M_2$ , we have  $L(M_1) = L(M_2) \implies \langle M_1 \rangle \in P \iff \langle M_2 \rangle \in P$ . Then  $P$  is undecidable. (e.g.  $\text{INFINITE}_{\text{TM}}, \text{ALL}_{\text{TM}}$ )
- $\{\text{all TMs}\}$  is count.;  $\Sigma^*$  is count. (finite  $\Sigma$ );  $\{\text{all lang.}\}$  is uncount.;  $\{\text{all infinite bin. seq.}\}$  is uncount.
- $\text{DFA} \neq \text{NFA} \equiv \text{GNFA} \equiv \text{REG} \subset \text{NPDA} \subset \text{CFG} \subset \text{DTM} \equiv \text{NTM}$

- $f : \Sigma^* \rightarrow \Sigma^*$  is **computable** if  $\exists M_{\text{TM}} : \forall w \in \Sigma^*, M$  halts on  $w$  and outputs  $f(w)$  on its tape.
- If  $A \leq_m B$  and  $B$  is decidable, then  $A$  is dec.
- If  $A \leq_m B$  and  $A$  is undecidable, then  $B$  is undec.
- If  $A \leq_m B$  and  $B$  is recognizable, then  $A$  is rec.
- If  $A \leq_m B$  and  $A$  is unrecognizable, then  $B$  is unrec.
- (transitivity) If  $A \leq_m B$  and  $B \leq_m C$ , then  $A \leq_m C$ .
- $A \leq_m B \iff \overline{A} \leq_m \overline{B}$  (esp.  $A \leq_m \overline{A} \iff \overline{A} \leq_m A$ )
- If  $A \leq_m \overline{A}$  and  $A \in \text{RECOGNIZABLE}$ , then  $A \in \text{DEC}.$

FINITE  $\subset$  REGULAR  $\subset$  CFL  $\subset$  CSL  $\subset$  DECIDABLE  $\subset$  RECOGNIZABLE

<ul style="list-style-type: none"> <li><b>(unrecognizable)</b> <math>\overline{A_{TM}}, \overline{EQ_{TM}}, EQ_{CFG}, \overline{HALT_{TM}}, REG_{TM} = \{\langle M \rangle : L(M) \text{ is regular}\}, E_{TM}, EQ_{TM}, ALL_{CFG}, EQ_{CFG}</math></li> <li><b>(recognizable but undecidable)</b> <math>A_{TM}, HALT_{TM}, \overline{EQ_{CFG}}, E_{TM}, \{\langle M, k \rangle \mid \exists x (M(x) \text{ halts in } \geq k \text{ steps})\}</math></li> <li><b>(decidable)</b> <math>A_{DFA}, A_{NFA}, A_{REG}, E_{DFA}, EQ_{DFA}, A_{CFG}, E_{CFG}, A_{LBA}, ALL_{DFA} = \{\langle D \rangle \mid L(D) = \Sigma^*\}, A \varepsilon_{CFG} = \{\langle G \rangle \mid \varepsilon \in L(G)\}</math></li> <li><b>Examples of Deciders:</b></li> <li><math>INFINITE_{DFA}</math>: "On <math>n</math>-state DFA <math>\langle A \rangle</math>: const. DFA <math>B</math> s.t. <math>L(B) = \Sigma^{\geq n}</math>; const. DFA <math>C</math> s.t. <math>L(C) = L(A) \cap L(B)</math>; if</li> </ul>	$L(C) \neq \emptyset$ (via $E_{DFA}$ ) <b>A</b> ; O/W, $\overline{R}$ " <ul style="list-style-type: none"> <li><math>\{\langle D \rangle \mid \nexists w \in L(D) : \#_1(w) \text{ is odd}\}</math>: "On <math>\langle D \rangle</math>: const. DFA <math>A</math> s.t. <math>L(A) = \{w \mid \#_1(w) \text{ is odd}\}</math>; const. DFA <math>B</math> s.t. <math>L(B) = L(D) \cap L(A)</math>; if <math>L(B) = \emptyset</math> (via <math>E_{DFA}</math>) <b>A</b>; O/W, <math>\overline{R}</math>"</li> <li><math>\{\langle R, S \rangle \mid R, S \text{ are regex}, L(R) \subseteq L(S)\}</math>: "On <math>\langle R, S \rangle</math>: const. DFA <math>D</math> s.t. <math>L(D) = L(R) \cap \overline{L(S)}</math>; if <math>L(D) = \emptyset</math> (via <math>E_{DFA}</math>), <b>A</b>; O/W, <math>\overline{R}</math>"</li> <li><math>\{\langle D_{DFA}, R_{REG} \rangle \mid L(D) = L(R)\}</math>: "On <math>\langle D, R \rangle</math>: convert <math>R</math> to DFA <math>D_R</math>; if <math>L(D) = L(D_R)</math> (via <math>EQ_{DFA}</math>), <b>A</b>; O/W, <math>\overline{R}</math>"</li> <li><math>\{\langle D_{DFA} \rangle \mid L(D) = (L(D))^R\}</math>: "On <math>\langle D \rangle</math>: const. DFA <math>D^R</math> s.t. <math>L(D^R) = (L(D))^R</math>; if <math>L(D) = L(D^R)</math> (via <math>EQ_{DFA}</math>),</li> </ul>	<b>A</b> ; O/W, $\overline{R}$ " <ul style="list-style-type: none"> <li><math>\{\langle M, k \rangle \mid \exists x (M(x) \text{ runs for } \geq k \text{ steps})\}</math>: "On <math>\langle M, k \rangle</math>: (foreach <math>w \in \Sigma^{\leq k+1}</math>: if <math>M(w)</math> not halt within <math>k</math> steps, <b>A</b>); O/W, <math>\overline{R}</math>"</li> <li><math>\{\langle M, k \rangle \mid \exists x (M(x) \text{ halts in } \leq k \text{ steps})\}</math>: "On <math>\langle M, k \rangle</math>: (foreach <math>w \in \Sigma^{\leq k+1}</math>: run <math>M(w)</math> for <math>\leq k</math> steps, if halts, <b>A</b>); O/W, <math>\overline{R}</math>"</li> <li><math>\{\langle M_{DFA} \rangle \mid L(M) = \Sigma^*\}</math>: "On <math>\langle M \rangle</math>: const. DFA <math>M^c = (L(M))^c</math>; if <math>L(M^c) = \emptyset</math> (via <math>E_{DFA}</math>), <b>A</b>; O/W <math>\overline{R}</math>."</li> <li><math>\{\langle R_{REG} \rangle \mid \exists s, t \in \Sigma^* : w = s111t \in L(R)\}</math>: "On <math>\langle R \rangle</math>: const. DFA <math>D</math> s.t. <math>L(D) = \Sigma^*111\Sigma^*</math>; const. DFA <math>C</math> s.t. <math>L(C) = L(R) \cap L(D)</math>; if <math>L(C) \neq \emptyset</math> (via <math>E_{DFA}</math>), <b>A</b>; O/W <math>\overline{R}</math>."</li> </ul>
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Mapping Reduction:  $A \leq_m B$  if  $\exists f : \Sigma^* \rightarrow \Sigma^* : \forall w \in \Sigma^*, w \in A \iff f(w) \in B$  and  $f$  is computable.

<ul style="list-style-type: none"> <li><math>A_{TM} \leq_m \{\langle M_{TM} \rangle \mid L(M) = (L(M))^R\}</math>; <math>f(\langle M, w \rangle) = \langle M' \rangle</math>, where <math>M' =</math>"On <math>x</math>, if <math>x \notin \{01, 10\}, \overline{R}</math>; if <math>x = 01</math>, return <math>M(x)</math>; if <math>x = 10</math>, <b>A</b>."</li> <li><math>A_{TM} \leq_m L = \{\langle M, D \rangle \mid L(M) = L(D)\}</math>; <math>f(\langle M, w \rangle) = \langle M', D \rangle</math>, where <math>M' =</math>"On <math>x</math>: if <math>x = w</math> return <math>M(x)</math>; O/W, <math>\overline{R}</math>;" <math>D</math> is DFA s.t. <math>L(D) = \{w\}</math>.</li> <li><math>A \leq_m HALT_{TM}</math>; <math>f(w) = \langle M, \varepsilon \rangle</math>, where <math>M =</math>"On <math>x</math>: if <math>w \in A</math>, halt; if <math>w \notin A</math>, loop;"</li> <li><math>A_{TM} \leq_m CFL_{TM} = \{\langle M \rangle \mid L(M) \text{ is CFL}\}</math>; <math>f(\langle M, w \rangle) = \langle N \rangle</math>, where <math>N =</math>"On <math>x</math>: if <math>x = a^n b^n c^n</math>, <b>A</b>; O/W, return <math>M(w)</math>;"</li> <li><math>A \leq_m B = \{0w : w \in A\} \cup \{1w : w \notin A\}</math>; <math>f(w) = 0w</math>.</li> <li><math>E_{TM} \leq_m USELESS_{TM}</math>; <math>f(\langle M \rangle) = \langle M, q_{\mathbf{A}} \rangle</math></li> </ul>	<ul style="list-style-type: none"> <li><math>A_{TM} \leq_m EQ_{TM}</math>; <math>f(\langle M, w \rangle) = \langle M_1, M_2 \rangle</math>, where <math>M_1 =</math> "<b>A</b> all"; <math>M_2 =</math>"On <math>x</math>: return <math>M(w)</math>;"</li> <li><math>A_{TM} \leq_m \overline{EQ_{TM}}</math>; <math>f(\langle M, w \rangle) = \langle M_1, M_2 \rangle</math>, where <math>M_1 =</math> "<math>\overline{R}</math> all"; <math>M_2 =</math>"On <math>x</math>: return <math>M(w)</math>;"</li> <li><math>ALL_{CFG} \leq_m EQ_{CFG}</math>; <math>f(\langle G \rangle) = \langle G, H \rangle</math>, s.t. <math>L(H) = \Sigma^*</math>.</li> <li><math>A_{TM} \leq_m \{\langle M_{TM} \rangle \mid  L(M)  = 1\}</math>; <math>f(\langle M, w \rangle) = \langle M' \rangle</math>, where <math>M' =</math>"On <math>x</math>: if <math>x = x_0</math>, return <math>M(w)</math>; O/W, <math>\overline{R}</math>;" (where <math>x_0 \in \Sigma^*</math> is fixed).</li> <li><math>\overline{A_{TM}} \leq_m E_{TM}</math>; <math>f(\langle M, w \rangle) = \langle M' \rangle</math>, where <math>M' =</math>"On <math>x</math>: if <math>x \neq w, \overline{R}</math>; O/W, return <math>M(w)</math>;"</li> <li><math>\overline{HALT_{TM}} \leq_m \{\langle M_{TM} \rangle \mid  L(M)  \leq 3\}</math>; <math>f(\langle M, w \rangle) = \langle M' \rangle</math>, where <math>M' =</math>"On <math>x</math>: <b>A</b> if <math>M(w)</math> halts"</li> <li><math>HALT_{TM} \leq_m \{\langle M_{TM} \rangle \mid  L(M)  \geq 3\}</math>; <math>f(\langle M, w \rangle) = \langle M' \rangle</math>, where <math>M' =</math>"On <math>x</math>: <b>A</b> if <math>M(w)</math> halts"</li> </ul>	<ul style="list-style-type: none"> <li><math>\overline{HALT_{TM}} \leq_m \{\langle M_{TM} \rangle : M \text{ <b>A</b> all even num.}\}</math>; <math>f(\langle M, w \rangle) = \langle M' \rangle</math>, where <math>M' =</math>"On <math>x</math>: <math>\overline{R}</math> if <math>M(w)</math> halts within <math> x </math>. O/W, <b>A</b>"</li> <li><math>\overline{HALT_{TM}} \leq_m \{\langle M_{TM} \rangle : L(M) \text{ is finite}\}</math>; <math>f(\langle M, w \rangle) = \langle M' \rangle</math>, where <math>M' =</math>"On <math>x</math>: <b>A</b> if <math>M(w)</math> halts"</li> <li><math>\overline{HALT_{TM}} \leq_m \{\langle M_{TM} \rangle : L(M) \text{ is infinite}\}</math>; <math>f(\langle M, w \rangle) = \langle M' \rangle</math>, where <math>M' =</math>"On <math>x</math>: <math>\overline{R}</math> if <math>M(w)</math> halts within <math> x </math> steps. O/W, <b>A</b>"</li> <li><math>HALT_{TM} \leq_m \{\langle M_1, M_2 \rangle : \varepsilon \in L(M_1) \cup L(M_2)\}</math>; <math>f(\langle M, w \rangle) = \langle M', M' \rangle</math>, where <math>M' =</math>"On <math>x</math>: <b>A</b> if <math>M(w)</math> halts"</li> <li><math>HALT_{TM} \leq_m \overline{E_{TM}}</math>; <math>f(\langle M, w \rangle) = \langle M' \rangle</math>, where <math>M' =</math> "On <math>x</math>: if <math>x \neq w \overline{R}</math>; else, <b>A</b> if <math>M(w)</math> halts"</li> <li><math>HALT_{TM} \leq_m \{\langle M_{TM} \rangle \mid \exists x : M(x) \text{ halts in } &gt;  M  \text{ steps}\}</math>; <math>f(\langle M, w \rangle) = \langle M' \rangle</math>, where <math>M' =</math>"On <math>x</math>: if <math>M(w)</math> halts, make <math> M  + 1</math> steps and then halt; O/W, loop"</li> </ul>
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$P = \bigcup_{k \in \mathbb{N}} \text{TIME}(n^k) \subseteq NP = \bigcup_{k \in \mathbb{N}} \text{NTIME}(n^k) = \{L \mid L \text{ is decidable by a PT verifier}\} \supseteq \text{NP-complete} = \{B \mid B \in NP, \forall A \in NP, A \leq_P B\}$ .

<ul style="list-style-type: none"> <li><b>((Running time) decider <math>M</math> is a <math>f(n)</math>-time TM.)</b> <math>f : \mathbb{N} \rightarrow \mathbb{N}</math>, where <math>f(n)</math> is the max. num. of steps that DTM (or NTM) <math>M</math> takes on any <math>n</math>-length input (and any branch of any <math>n</math>-length input. resp.).</li> <li><math>\text{TIME}(t(n)) = \{L \mid L \text{ is dec. by } O(t(n)) \text{ DTM}\}</math>.</li> <li><math>\text{NTIME}(t(n)) = \{L \mid L \text{ is dec. by } O(t(n)) \text{ NTM}\}</math>.</li> <li><b>(verifier for <math>L</math>) TM <math>V</math> s.t. <math>L = \{w \mid \exists c : V(\langle w, c \rangle) = \mathbf{A}\}</math>;</b> <b>(certificate for <math>w \in L</math>) str. <math>c</math> s.t. <math>V(\langle w, c \rangle) = \mathbf{A}</math>.</b></li> </ul>	<ul style="list-style-type: none"> <li><math>f : \Sigma^* \rightarrow \Sigma^*</math> is <b>PT computable</b> if there exists a PT TM <math>M</math> s.t. for every <math>w \in \Sigma^*</math>, <math>M</math> halts with <math>f(w)</math> on its tape.</li> <li>If <math>A \leq_P B</math> and <math>B \in P</math>, then <math>A \in P</math>.</li> <li>If <math>A \leq_P B</math> and <math>B \leq_P A</math>, then <math>A</math> and <math>B</math> are <b>PT equivalent</b>, denoted <math>A \equiv_P B</math>. <math>\equiv_P</math> is an equiv. relation on NP. <math>P \setminus \{\emptyset, \Sigma^*\}</math> is an equiv. class of <math>\equiv_P</math>.</li> <li><math>ALL_{DFA}, CONNECTED, TRIANGLE, L(G_{CFG}), RELPRIME, \overset{\text{directed}}{PATH}_{s \rightarrow t} \in P</math></li> </ul>	<ul style="list-style-type: none"> <li><math>CLIQUE, SUBSET-SUM, SAT, 3SAT, \overset{VERTEX}{COVER}, HAMPATH, UHAMATH, 3COLOR \in \text{NP-complete}</math>. <math>\emptyset, \Sigma^* \notin \text{NP-complete}</math>.</li> <li>If <math>B \in \text{NP-complete}</math> and <math>B \in P</math>, then <math>P = NP</math>.</li> <li>If <math>B \in \text{NPC}</math> and <math>C \in \text{NP}</math> s.t. <math>B \leq_P C</math>, then <math>C \in \text{NPC}</math>.</li> <li>If <math>P = NP</math>, then <math>\forall A \in P \setminus \{\emptyset, \Sigma^*\}, A \in \text{NP-complete}</math>.</li> </ul>
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Polytime Reduction:  $A \leq_P B$  if  $\exists f : \Sigma^* \rightarrow \Sigma^* : \forall w \in \Sigma^*, w \in A \iff f(w) \in B$  and  $f$  is polytime computable.

<ul style="list-style-type: none"> <li><math>SAT \leq_P DOUBLE-SAT</math>; <math>f(\phi) = \phi \wedge (x \vee \neg x)</math></li> <li><math>3SAT \leq_P 4SAT</math>; <math>f(\phi) = \phi'</math>, where <math>\phi'</math> is obtained from the CNF <math>\phi</math> by adding a new var. <math>x</math> to each clause, and adding a new clause <math>(\neg x \vee \neg x \vee \neg x \vee \neg x)</math>.</li> <li><math>SUBSET-SUM \leq_P SET-PARTITION</math>; <math>f(\langle x_1, \dots, x_m, t \rangle) = \langle x_1, \dots, x_m, S - 2t \rangle</math>, where <math>S</math> sum of <math>x_1, \dots, x_m</math>, and <math>t</math> is the target subset-sum.</li> <li><math>3COLOR \leq_P \overset{\text{almost}}{3COLOR}</math>; <math>f(\langle G \rangle) = \langle G' \rangle</math>, <math>G' = G \cup K_4</math></li> <li><math>\overset{VERTEX}{COVER}_k \leq_P WVC</math>; <math>f(\langle G, k \rangle) = (G, w, k), \forall v \in V(G), w(v) = 1</math></li> <li><b>(dir.) <math>HAM-PATH \leq_P 2HAM-PATH</math>;</b> <math>f(\langle G, s, t \rangle) = \langle G', s', t' \rangle</math>, where <math>V' = V \cup \{s', t', a, b, c, d\}</math>, <math>E' = E \cup \{(s', a), (a, b), (b, s)\} \cup \{(s', b), (b, a), (a, s)\} \cup \{(t, c), (c, d), (d, t')\} \cup \{(t, d), (d, c), (c, t')\}</math>.</li> </ul>	<ul style="list-style-type: none"> <li>(undir.) <math>CLIQUE_k \leq_P \overset{ V /2\text{-clique}}{HALF-CLIQUE}</math>; <math>f(\langle G = (V, E), k \rangle) = \langle G' = (V', E') \rangle</math>, if <math>k = \frac{ V }{2}, E = E', V' = V</math>. if <math>k &gt; \frac{ V }{2}, V' = V \cup \{j = 2k -  V  \text{ new nodes}\}</math>. if <math>k &lt; \frac{ V }{2}, V' = V \cup \{j =  V  - 2k \text{ new nodes}\}</math> and <math>E' = E \cup \{\text{edges for new nodes}\}</math></li> <li>(dir.) <math>\overset{s \rightarrow t}{HAM-PATH} \leq_P HAM-CYCLE</math>; <math>f(\langle G, s, t \rangle) = \langle G', s, t \rangle</math> where <math>V' = V \cup \{x\}</math>, <math>E' = E \cup \{(t, x), (x, s)\}</math></li> <li><math>HAM-CYCLE \leq_P UHAMCYCLE</math>; <math>f(\langle G \rangle) = \langle G' \rangle</math>. For each <math>u, v \in V</math>: <math>u</math> is replaced by <math>u_{in}, u_{mid}, u_{out}</math>; <math>(v, u)</math> replaced by <math>\{v_{out}, u_{in}\}, \{u_{in}, u_{mid}\}</math>; and <math>(u, v)</math> by <math>\{u_{out}, v_{in}\}, \{u_{mid}, u_{out}\}</math>.</li> <li><math>UHAMPATH \leq_P PATH_{\geq k}</math>; <math>f(\langle G, a, b \rangle) = \langle G, a, b, k =  V(G)  - 1 \rangle</math></li> </ul>	<ul style="list-style-type: none"> <li><math>\overset{VERTEX}{COVER}_k \leq_P CLIQUE_k</math>; <math>f(\langle G, k \rangle) = \langle G^c = (V, E^c),  V  - k \rangle</math></li> <li><math>CLIQUE_k \leq_P \{\langle G, t \rangle : G \text{ has } 2t\text{-clique}\}</math>; <math>f(\langle G, k \rangle) = \langle G', t = \lceil k/2 \rceil \rangle</math>, <math>G' = G</math> if <math>k</math> is even; <math>G' = G \cup \{v\}</math> (<math>v</math> connected to all <math>G</math> nodes) if <math>k</math> is odd.</li> <li><math>CLIQUE_k \leq_P \overset{\text{almost}}{CLIQUE}_k</math>; <math>f(\langle G, k \rangle) = \langle G', k + 2 \rangle</math>, where <math>G' = G \cup \{v_{n+1}, v_{n+2}\}</math> and <math>v_{n+1}, v_{n+2}</math> are con. to all <math>G</math> nodes.</li> <li><math>\overset{VERTEX}{COVER}_k \leq_P DOMINATING-SET_k</math>; <math>f(\langle G, k \rangle) = \langle G', k \rangle</math>, where <math>V' = \{\text{non-isolated node in } V\} \cup \{v_e : e \in E\}</math>, <math>E' = E \cup \{(v_e, u), (v_e, w) : e = (u, w) \in E\}</math>.</li> <li><math>CLIQUE \leq_P INDEP-SET</math>; <math>SET-COVER \leq_P \overset{VERTEX}{COVER}</math>; <math>3SAT \leq_P SET-SPLITTING</math>; <math>INDEP-SET \leq_P \overset{VERTEX}{COVER}</math></li> </ul>
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## Counterexamples

<ul style="list-style-type: none"> <li><math>A \leq_m B</math> and <math>B \in \text{REG}</math>, but, <math>A \notin \text{REG}</math>: <math>A = \{0^n 1^n \mid n \geq 0\}, B = \{1\}, f : A \rightarrow B</math>, <math>f(w) = \begin{matrix} 1 &amp; \text{if } w \in A \\ 0 &amp; \text{if } w \notin A \end{matrix}</math></li> <li><math>L \in \text{CFL}</math> but <math>\overline{L} \notin \text{CFL}</math>: <math>L = \{x \mid \forall w \in \Sigma^*, x \neq ww\}, \overline{L} = \{ww \mid w \in \Sigma^*\}</math>.</li> <li><math>L_1, L_2 \in \text{CFL}</math> but <math>L_1 \cap L_2 \notin \text{CFL}</math>: <math>L_1 = \{a^n b^n c^m\}, L_2 = \{a^m b^n c^n\}</math>.</li> <li><math>L_1 \in \text{CFL}, L_2</math> is infinite, but <math>L_1 \setminus L_2 \notin \text{REG}</math>: <math>L_1 = \Sigma^*, L_2 = \{a^n b^n \mid n \geq 0\}, L_1 \setminus L_2 = \{a^m b^n \mid m \neq n\}</math>.</li> </ul>	<ul style="list-style-type: none"> <li><math>L_1, L_2 \in \text{REG}, L_1 \not\subseteq L_2, L_2 \not\subseteq L_1</math>, but, <math>(L_1 \cup L_2)^* = L_1^* \cup L_2^*</math>: <math>L_1 = \{a, b, ab\}, L_2 = \{a, b, ba\}</math>.</li> <li><math>L_1 \in \text{REG}, L_2 \notin \text{REG}</math>, but, <math>L_1 \cap L_2 \in \text{REG}</math>, and <math>L_1 \cup L_2 \in \text{REG}</math>: <math>L_1 = L(a^* b^*)</math>, <math>L_2 = \{a^n b^n \mid n \geq 0\}</math>.</li> <li><math>L_1, L_2, L_3, \dots \in \text{REG}</math>, but, <math>\bigcup_{i=1}^{\infty} L_i \notin \text{REG}</math>: <math>L_i = \{a^i b^i\}, \bigcup_{i=1}^{\infty} L_i = \{a^n b^n \mid n \geq 0\}</math>.</li> <li><math>L_1 \cdot L_2 \in \text{REG}</math>, but <math>L_1 \notin \text{REG}</math>: <math>L_1 = \{a^n b^n \mid n \geq 0\}, L_2 = \Sigma^*</math>.</li> <li><math>L_2 \in \text{CFL}</math>, and <math>L_1 \subseteq L_2</math>, but <math>L_1 \notin \text{CFL}</math>: <math>\Sigma = \{a, b, c\}, L_1 = \{a^n b^n c^n \mid n \geq 0\}, L_2 = \Sigma^*</math>.</li> </ul>	<ul style="list-style-type: none"> <li><math>L_1, L_2 \in \text{DECIDABLE}</math>, and <math>L_1 \subseteq L_2 \subseteq L_2</math>, but <math>L \in \text{UNDECIDABLE}</math>: <math>L_1 = \emptyset, L_2 = \Sigma^*, L</math> is some undecidable language over <math>\Sigma</math>.</li> <li><math>L_1 \in \text{REG}, L_2 \notin \text{CFL}</math>, but <math>L_1 \cap L_2 \in \text{CFL}</math>: <math>L_1 = \{\varepsilon\}, L_2 = \{a^n b^n c^n \mid n \geq 0\}</math>.</li> <li><math>L^* \in \text{REG}</math>, but <math>L \notin \text{REG}</math>: <math>L = \{a^p \mid p \text{ is prime}\}, L^* = \Sigma^* \setminus \{a\}</math>.</li> <li><math>A \not\leq_m \overline{A}</math>: <math>A = A_{TM} \in \text{RECOGNIZABLE}, \overline{A} = \overline{A_{TM}} \notin \text{RECOG}</math>.</li> <li><math>A \notin \text{DEC.}, A \leq_m \overline{A}</math>:</li> </ul>
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