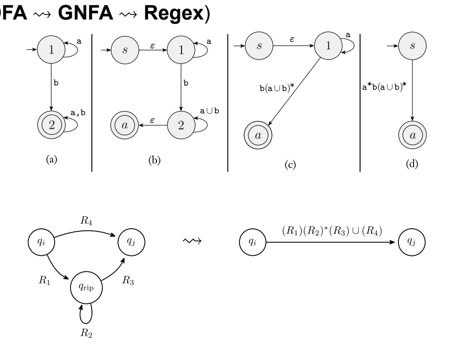
	$\overline{\text{REG}}$	REG	CFL	DEC.	REC.	P	NP	NPC
$L1 \cup L2$	no	✓	✓	✓	✓	✓	✓	no
$L1 \cap L2$	no	✓	no	✓	✓	✓	✓	no
$\overline{L}$	✓	✓	no	✓	no	✓	?	?
$L1 \cdot L2$	no	✓	✓	✓	✓	✓	✓	no
L^*	no	✓	✓	✓	✓	✓	✓	no
$L^{\mathcal{R}}$	✓	✓	✓	✓	✓	✓		
$L1 \setminus L2$	no	✓	no	✓	no	✓	?	
$L \cap R$	no	✓	✓	✓	✓	✓		

- **(DFA)** $M = (Q, \Sigma, \delta, q_0, F)$, $\delta : Q \times \Sigma \to Q$.
- **(NFA)** $M = (Q, \Sigma, \delta, q_0, F)$, $\delta : Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$.
- **(GNFA)** $(Q, \Sigma, \delta, q_0, q_a)$, $\delta : (Q \setminus \{q_a\}) \times (Q \setminus \{q_{\text{start}}\} \to \mathcal{R}$ (where $\mathcal{R} = \{\text{Regex over } \Sigma\}$)
- GNFA accepts $w \in \Sigma^*$ if $w = w_1 \cdots w_k$, where $w_i \in \Sigma^*$ and there exists a sequence of states $q_0, q_1, \ldots, q_k$ s.t. $q_0 = q_{\text{start}}$, $q_k = q_a$ and for each i, we have $w_i \in L(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$.
- n-state DFA A, m-state DFA $B \implies \exists\, nm$-state DFA C s.t. $L(C) = L(A) \Delta L(B)$.
- p-state DFA C, if $L(C) \neq \emptyset$ then $\exists\, s \in L(C)$ s.t. $|s| < p$.
- Every NFA has an equiv. NFA with a single accept state.
- **(DFA ⇝ GNFA ⇝ Regex)**



- If $A = L(N_{\text{NFA}})$, $B = (L(M_{\text{DFA}}))^{\complement}$ then $A \cdot B \in \text{REG}$.
- **(NFA ⇝ DFA)**
  - $N = (Q, \Sigma, \delta, q_0, F)$
  - $D = (Q' = \mathcal{P}(Q), \Sigma, \delta', q_0' = E(\{q_0\}), F')$
  - $F' = \{q \in Q' \mid \exists p \in F : p \in q\}$
  - $E(\{q\}) := \{q\} \cup \{\text{states reachable from } q \text{ via } \varepsilon\text{-arrows}\}$
  - $\forall R \subseteq Q, \forall a \in \Sigma, \delta'(R, a) = E\left(\bigcup_{r \in R} \delta(r, a)\right)$
- **Regular Expressions Examples**:
- $\{a^n w b^n : w \in \Sigma^*\} \equiv a(a \cup b)^* b$
- $\{w : \#_w(0) \geq 2 \vee \#_w(1) \leq 1\} \equiv (\Sigma^* 0 \Sigma^* 0 \Sigma^*) \cup (0^*(\varepsilon \cup 1)0^*)$
- $\{w : |w| \bmod n = m\} \equiv (a \cup b)^m ((a \cup b)^n)^*$
- $\{w : \#_b(w) \bmod n = m\} \equiv (a^* b a^*)^m \cdot ((a^* b a^*)^n)^*$
- $\{w : |w| \text{ is odd}\} \equiv (a \cup b)^*((a \cup b)(a \cup b)^*)^*$
- $\{w : \#_a(w) \text{ is odd}\} \equiv b^* a (a b^* a \cup b)^*$
- $\{w : \#_{ab}(w) = \#_{ba}(w)\} \equiv \varepsilon \cup a \cup b \cup a \Sigma^* a \cup b \Sigma^* b$
- $\{a^m b^n \mid m + n \text{ is odd}\} \equiv a(aa)^*(bb)^* \cup (aa)^* b (bb)^*$
- $\{aw : aba \nsubseteq w\} \equiv a(a \cup bb \cup bbb)^*(b \cup \varepsilon)$

Pumping lemma for regular languages: $A \in \text{REG} \implies \exists p : \forall s \in A, |s| \geq p, s = xyz$, **(i)** $\forall i \geq 0, xy^i z \in A$, **(ii)** $|y| > 0$ **and (iii)** $|xy| \leq p$.

- (the following are **non-reuglar but CFL**)
- $\{w = w^{\mathcal{R}}\}$; $s = 0^p 1 0^p = xyz$. but $xy^2z = 0^{p+|y|} 1 0^p \notin L$.
- $\{a^n b^n\}$; $s = a^p b^p = xyz$, $xy^2z = a^{p+|y|} b^p \notin L$.
- $\{w : \#_a(w) > \#_b(w)\}$; $s = a^p b^{p+1}$, $|s| = 2p + 1 \geq p$, $xy^2z = a^{p+|y|} b^{p+1} \notin L$.
- $\{w : \#_a(w) = \#_b(w)\}$; $s = a^p b^p = xyz$ but $xy^2z = a^{p+|y|} b^p \notin L$.
- $\{w : \#_w(a) \neq \#_w(b)\}$; (*pf.* by 'complement-closure', $\overline{L} = \{w : \#_w(a) = \#_w(b)\}$)
- $\{a^i b^j c^k : i < j \vee i > k\}$; $s = a^p b^{p+1} c^{2p} = xyz$, but $xy^2z = a^{p+|y|} b^{p+1} c^{2p}$, $p + |y| \geq p + 1$, $p + |y| \leq 2p$.
- (the following are both **non-CFL and non-reuglar**)
- $\{w = a^{2^k}\}$; $\quad k = \lfloor \log_2 |w| \rfloor, s = a^{2^k} = xyz$. $2^k = |xyz| < |xy^2z| \leq |xyz| + |xy| \leq 2^k + p < 2^{k+1}$.
- $\{a^p : p \text{ is prime}\}$; $\quad s = a^t = xyz$ for prime $t \geq p$. $r := |y| > 0$
- $\{www : w \in \Sigma^*\}$; $s = a^p b a^p b a^p = xyz = a^{|x|+|y|+m} b a^p b a^p b$, $m \geq 0$, but $xy^2z = a^{|x|+2|y|+m} b a^p b a^p b \notin L$.
- $\{a^{2n} b^{3n} a^n\}$; $s = a^{2p} b^{3p} a^p = xyz = a^{|x|+|y|+m+p} b^3 a^p$, $m \geq 0$, but $xy^2z = a^{2p+|y|} b^{3p} a^p \notin L$.

(PDA) $M = (Q, \underset{\text{input}}{\Sigma}, \underset{\text{stack}}{\Gamma}, \delta, q_0 \in Q, \underset{\text{accepts}}{F} \subseteq Q)$. $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$. $\quad L \in$ **CFL** $\Leftrightarrow \exists G_{\text{CFG}} : L = L(G) \Leftrightarrow \exists P_{\text{PDA}} : L = L(P)$

- **(CFG ⇝ CNF)** (**1.**) Add a new start variable S_0 and a rule $S_0 \to S$. (**2.**) Remove ε-rules of the form $A \to \varepsilon$ (except for $S_0 \to \varepsilon$). and remove A's occurrences on the RH of a rule (e.g.: $R \to uAvAw$ becomes $R \to uAvAw \mid uAvw \mid uvAw \mid uvw$. where $u, v, w \in (V \cup \Sigma)^*$). (**3.**) Remove unit rules $A \to B$ then whenever $B \to u$ appears, add $A \to u$, unless this was a unit rule previously removed. ($u \in (V \cup \Sigma)^*$). (**4.**) Replace each rule $A \to u_1 u_2 \cdots u_k$ where $k \geq 3$ and $u_i \in (V \cup \Sigma)$, with the rules $A \to u_1 A_1$, $A_1 \to u_2 A_2$, ..., $A_{k-2} \to u_{k-1} u_k$, where A_i are new variables. Replace terminals u_i with $U_i \to u_i$.
- If $G \in$ CNF, and $w \in L(G)$, then $|w| \leq 2^{|h|} - 1$, where h is the height of the parse tree for w.
- $\forall L \in$ CFL, $\exists G \in$ CNF : $L = L(G)$.
- **(derivation)** $S \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_n = w$, where each u_i is in $(V \cup \Sigma)^*$. (in this case, G **generates** w (or S **derives** w), $S \overset{*}{\Rightarrow} w$)
- M **accepts** $w \in \Sigma^*$ if there is a seq. $r_0, r_1, \ldots, r_m \in Q$ and $s_0, s_1, \ldots, s_m \in \Gamma^*$ s.t.: (1.) $r_0 = q_0$ and $s_0 = \varepsilon$; (2.) For $i = 0, 1, \ldots, m-1$, we have $(r_i, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$; (3.) $r_m \in F$.
- **(PDA transition)** "$a, b \to c$": **reads** a from the input (or read nothing if $a = \varepsilon$). **pops** b from the stack (or pops nothing if $b = \varepsilon$). **pushes** c onto the stack (or pushes nothing if $c = \varepsilon$)
- $R \in \text{REG} \wedge C \in \text{CFL} \implies R \cap C \in \text{CFL}$. (*pf.* construct PDA $P' = P_C \times D_R$.)

(CFG) $G = (V, \Sigma, R, S)$, $A \to w$, ($A \in V, w \in (V \cup \Sigma)^*$); **(CNF)** $A \to BC$, $A \to a$, $S \to \varepsilon$, ($A, B, C \in V$, $a \in \Sigma$, $B, C \neq S$).

- (the following are **CFL but non-reuglar**)
- $\{w : w = w^{\mathcal{R}}\}$; $S \to aSa \mid bSb \mid a \mid b \mid \varepsilon$
- $\{w : w \neq w^{\mathcal{R}}\}$; $S \to aSa \mid bSb \mid aXb \mid bXa; X \to aX \mid bX \mid \varepsilon$
- $\{ww^{\mathcal{R}}\} = \{w : w = w^{\mathcal{R}} \wedge |w| \text{ is even}\}$; $S \to aSa \mid bSb \mid \varepsilon$
- $\{wa^n w^{\mathcal{R}}\}$; $S \to aSa \mid bSb \mid M; M \to aM \mid \varepsilon$
- $\{w\#x : w^{\mathcal{R}} \subseteq x\}$; $S \to AX$; $A \to 0A0 \mid 1A1 \mid \#X$; $X \to 0X \mid 1X \mid \varepsilon$
- $\{w : \#_w(a) > \#_w(b)\}$; $S \to JaJ$; $J \to JJ \mid aJb \mid bJa \mid a \mid \varepsilon$
- $\{w : \#_w(a) \geq \#_w(b)\}$; $S \to SS \mid aSb \mid bSa \mid a \mid \varepsilon$
- $\{w : \#_w(a) = \#_w(b)\}$; $S \to SS \mid aSb \mid bSa \mid \varepsilon$
- $\{w : \#_w(a) \neq \#_w(b)\} = \{\#_w(a) > \#_w(b)\} \cup \{\#_w(a) < \#_w(b)\}$
- $\overline{\{a^n b^n\}}$; $S \to XbXaX \mid A \mid B$; $A \to aAb \mid Ab \mid b$; $B \to aBb \mid aB \mid a$; $X \to aX \mid bX \mid \varepsilon$.
- $\{a^n b^m \mid n \neq m\}$; $S \to aSb \mid A \mid B$; $A \to aA \mid a$; $B \to bB \mid b$
- $\{a^i b^j c^k \mid i \leq j \vee j \leq k\}$; $S \to S_1 C \mid AS_2$; $A \to Aa \mid \varepsilon$; $S_1 \to aS_1 b \mid S_1 b \mid \varepsilon$; $S_2 \to bS_2 c \mid S_2 c \mid \varepsilon$; $C \to Cc \mid \varepsilon$
- $\{x \mid x \neq ww\}$; $S \to A \mid B \mid AB \mid BA$; $A \to CAC \mid 0$; $B \to CBC \mid 1$; $C \to 0 \mid 1$
- $\{a^n b^m \mid m \leq n \leq 3m\}$; $S \to aSb \mid aaSb \mid aaaSb \mid \varepsilon$;
- $\{a^n b^n\}$; $S \to aSb \mid \varepsilon$
- $\{a^n b^m \mid n > m\}$; $S \to aSb \mid aS \mid a$
- $\{a^n b^m \mid n \geq m \geq 0\}$; $S \to aSb \mid aS \mid a \mid \varepsilon$
- $\{a^i b^j c^k \mid i + j = k\}$; $S \to aSc \mid X$; $X \to bXc \mid \varepsilon$
- $\{xy : |x| = |y|, x \neq y\}$; $S \to AB \mid BA$; $A \to a \mid aAa \mid aAb \mid bAa \mid bAb$; $B \to b \mid aBa \mid aBb \mid bBa \mid bBb$;
- (the following are both **CFL and regular**)
- $\{w : \#_w(a) \geq 3\}$; $S \to XaXaXaX$; $X \to aX \mid bX \mid \varepsilon$

Pumping lemma for context-free languages: $L \in \text{CFL} \implies \exists p : \forall s \in L, |s| \geq p, s = uvxyz$, **(i)** $\forall i \geq 0, uv^i xy^i z \in L$, **(ii)** $|vxy| \leq p$, **and (iii)** $|vy| > 0$.

- $\{w = a^n b^n c^n\}$; $s = a^p b^p b^p = uvxyz$. vxy can't contain all of a, b, c thus uv^2xy^2z must pump one of them less than the others.
- $\{ww : w \in \{a, b\}^*\}$;
- (**more example of not CFL**)
- $\{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$, $\{a^n b^n c^n \mid n \in \mathbb{N}\}$, $\{ww \mid w \in \{a, b\}^*\}$, $\{\mathtt{a}^{n^2} \mid n \geq 0\}$, $\{a^p \mid p \text{ is prime}\}$, $L = \{ww^{\mathcal{R}}w : w \in \{a, b\}^*\}$
- $\{w \mid \#_w(a) = \#_w(b) = \#_w(c)\}$: (*pf.* since Regular $\cap$ CFL $\in$ CFL, but $\{a^* b^* c^*\} \cap L = \{a^n b^n c^n\} \notin$ CFL)

$L \in \text{DECIDABLE} \iff (L \in \text{REC. and } L \in \text{co-REC.}) \iff \exists M_{\text{TM}}$ decides L.

- **(TM)** $M = (Q, \underset{\text{input}}{\Sigma} \subseteq \Gamma, \underset{\text{tape}}{\Gamma}, \delta, q_0, q_{Ⓐ}, q_{Ⓡ})$, where $\sqcup \in \Gamma$, $\sqcup \notin \Sigma$, $q_{Ⓡ} \neq q_{Ⓐ}$, $\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{\text{L}, \text{R}\}$
- **(recognizable)** Ⓐ if $w \in L$, Ⓡ/loops if $w \notin L$; A is **co-recognizable** if $\overline{A}$ is recognizable.
- $L \in \text{RECOGNIZABLE} \iff L \leq_{\text{m}} A_{\text{TM}}$.
- Every inf. recognizable lang. has an inf. dec. subset.
- **(decidable)** Ⓐ if $w \in L$, Ⓡ if $w \notin L$.
- $L \in \text{DECIDABLE} \iff L \leq_{\text{m}} 0^*1^*$.
- $L \in \text{DECIDABLE} \iff L^{\mathcal{R}} \in \text{DECIDABLE}$.
- **(decider)** TM that halts on all inputs.
- **(Rice)** Let P be a lang. of TM descriptions, s.t. (**i**) P is nontrivial (not empty and not all TM desc.) and (**ii**) for each two TM M_1 and M_2, we have $L(M_1) = L(M_2) \implies (\langle M_1 \rangle \in P \iff \langle M_2 \rangle \in P)$. Then P is undecidable. (*e.g.* $INFINITE_{\text{TM}}$, ALL_{TM}, E_{TM}, $\{\langle M_{\text{TM}} \rangle : 1 \in L(M)\}$)
- $\{\text{all TMs}\}$ is count.; Σ^* is count. (finite Σ); $\{\text{all lang.}\}$ is uncount.; $\{\text{all infinite bin. seq.}\}$ is uncount.
- DFA $\equiv$ NFA $\equiv$ GNFA $\equiv$ REG $\subset$ NPDA $\equiv$ CFG $\subset$ DTM $\equiv$ NTM
- $f : \Sigma^* \to \Sigma^*$ is **computable** if $\exists M_{\text{TM}} : \forall w \in \Sigma^*$, M halts on w and outputs $f(w)$ on its tape.
- If $A \leq_{\text{m}} B$ and B is decidable, then A is dec.
- If $A \leq_{\text{m}} B$ and A is undecidable, then B is undec.
- If $A \leq_{\text{m}} B$ and B is recognizable, then A is rec.
- If $A \leq_{\text{m}} B$ and A is unrecognizable, then B is unrec.
- (transitivity) If $A \leq_{\text{m}} B$ and $B \leq_{\text{m}} C$, then $A \leq_{\text{m}} C$.
- $A \leq_{\text{m}} B \iff \overline{A} \leq_{\text{m}} \overline{B}$ (esp. $A \leq_{\text{m}} \overline{A} \iff \overline{A} \leq_{\text{m}} A$)
- If $A \leq_{\text{m}} \overline{A}$ and $A \in \text{RECOGNIZABLE}$, then $A \in \text{DEC}$.

FINITE $\subset$ REGULAR $\subset$ CFL $\subset$ CSL $\subset$ DECIDABLE $\subset$ RECOGNIZABLE

- (**unrecognizable**) $\overline{A_{\mathsf{TM}}}$, $\overline{EQ_{\mathsf{TM}}}$, EQ_{CFG}, $\overline{HALT_{\mathsf{TM}}}$, REG_{TM}, E_{TM}, EQ_{TM}, ALL_{CFG}, EQ_{CFG}
- (**recognizable but undecidable**) A_{TM}, $HALT_{\mathsf{TM}}$, $\overline{EQ_{\mathsf{CFG}}}$, $\overline{E_{\mathsf{TM}}}$, $\{\langle M,k\rangle \mid \exists x\ (M(x) \text{ halts in } \geq k \text{ steps})\}$
- (**decidable**) A_{DFA}, A_{NFA}, A_{REX}, E_{DFA}, EQ_{DFA}, A_{CFG}, E_{CFG}, A_{LBA}, $ALL_{\mathsf{DFA}} = \{\langle D\rangle \mid L(D) = \Sigma^*\}$, $A\varepsilon_{\mathsf{CFG}} = \{\langle G\rangle \mid \varepsilon \in L(G)\}$

Examples of Deciders:

- $\text{INFINITE}_{\mathsf{DFA}}$: "On n-state DFA $\langle A\rangle$: const. DFA B s.t. $L(B) = \Sigma^{\geq n}$; const. DFA C s.t. $L(C) = L(A) \cap L(B)$; if $L(C) \neq \emptyset$ (by E_{DFA}) Ⓐ; O/W, $\boxed{R}$"
- $\{\langle D\rangle \mid \nexists w \in L(D) : \#_1(w) \text{ is odd}\}$: "On $\langle D\rangle$: const. DFA A s.t. $L(A) = \{w \mid \#_1(w) \text{ is odd}\}$; const. DFA B s.t. $L(B) = L(D) \cap L(A)$; if $L(B) = \emptyset$ (E_{DFA}) Ⓐ; O/W $\boxed{R}$"
- $\{\langle R,S\rangle \mid R,S \text{ are regex}, L(R) \subseteq L(S)\}$: "On $\langle R,S\rangle$: const. DFA D s.t. $L(D) = L(R) \cap \overline{L(S)}$; if $L(D) = \emptyset$ (by E_{DFA}), Ⓐ; O/W, $\boxed{R}$"
- $\{\langle D_{\mathsf{DFA}}, R_{\mathsf{REX}}\rangle \mid L(D) = L(R)\}$: "On $\langle D,R\rangle$: convert R to DFA D_R; if $L(D) = L(D_R)$ (by EQ_{DFA}), Ⓐ; O/W, $\boxed{R}$"
- $\{\langle D_{\mathsf{DFA}}\rangle \mid L(D) = (L(D))^{\mathcal{R}}\}$: "On $\langle D\rangle$: const. DFA $D^{\mathcal{R}}$ s.t. $L(D^{\mathcal{R}}) = (L(D))^{\mathcal{R}}$; if $L(D) = L(D^{\mathcal{R}})$ (by EQ_{DFA}), Ⓐ; O/W, $\boxed{R}$"
- $\{\langle M,k\rangle \mid \exists x\ (M(x) \text{ runs for} \geq k \text{ steps})\}$: "On $\langle M,k\rangle$: (foreach $w \in \Sigma^{\leq k+1}$: if $M(w)$ not halt within k steps, Ⓐ); O/W, $\boxed{R}$"
- $\{\langle M,k\rangle \mid \exists x\ (M(x) \text{ halts in} \leq k \text{ steps})\}$: "On $\langle M,k\rangle$: (foreach $w \in \Sigma^{\leq k+1}$: run $M(w)$ for $\leq k$ steps, if halts, Ⓐ); O/W, $\boxed{R}$"
- $\{\langle M_{\mathsf{DFA}}\rangle \mid L(M) = \Sigma^*\}$: "On $\langle M\rangle$: const. DFA $M^{\complement} = (L(M))^{\complement}$; if $L(M^{\complement}) = \emptyset$ (by E_{DFA}), Ⓐ; O/W $\boxed{R}$."
- $\{\langle R_{\mathsf{REX}}\rangle \mid \exists s,t \in \Sigma^* : w = s111t \in L(R)\}$: "On $\langle R\rangle$: const. DFA D s.t. $L(D) = \Sigma^*111\Sigma^*$; const. DFA C s.t. $L(C) = L(R) \cap L(D)$; if $L(C) \neq \emptyset$ (E_{DFA}) Ⓐ; O/W $\boxed{R}$"

Mapping Reduction: $A \leq_{\mathrm{m}} B$ if $\exists f : \Sigma^* \to \Sigma^* : \forall w \in \Sigma^*,\ w \in A \iff f(w) \in B$ and f is computable.

- $A_{\mathsf{TM}} \leq_{\mathrm{m}} \{\langle M_{\mathsf{TM}}\rangle \mid L(M) = (L(M))^{\mathcal{R}}\}$; $f(\langle M,w\rangle) = \langle M'\rangle$, where M' ="On x, if $x \notin \{01,10\}$, $\boxed{R}$; if $x = 01$, return $M(x)$; if $x = 10$, Ⓐ;"
- $A_{\mathsf{TM}} \leq_{\mathrm{m}} L = \{\langle \underset{\mathsf{TM}}{M}, \underset{\mathsf{DFA}}{D}\rangle \mid L(M) = L(D)\}$; $f(\langle M,w\rangle) = \langle M', D\rangle$, where M' ="On x: if $x = w$ return $M(x)$; O/W, $\boxed{R}$;" D is DFA s.t. $L(D) = \{w\}$.
- $A \leq_{\mathrm{m}} HALT_{\mathsf{TM}}$; $f(w) = \langle M, \varepsilon\rangle$, where M ="On x: if $w \in A$, halt; if $w \notin A$, loop;"
- $A_{\mathsf{TM}} \leq_{\mathrm{m}} CFL_{\mathsf{TM}} = \{\langle M\rangle \mid L(M) \text{ is CFL}\}$; $f(\langle M,w\rangle) = \langle N\rangle$, where N ="On x: if $x = a^nb^nc^n$, Ⓐ; O/W, return $M(w)$;"
- $A \leq_{\mathrm{m}} B = \{0w : w \in A\} \cup \{1w : w \notin A\}$; $f(w) = 0w$.
- $E_{\mathsf{TM}} \leq_{\mathrm{m}} USELESS_{\mathsf{TM}}$; $f(\langle M\rangle) = \langle M, q_{Ⓐ}\rangle$
- $A_{\mathsf{TM}} \leq_{\mathrm{m}} REGULAR_{\mathsf{TM}}$; $f(\langle M,w\rangle) = \langle M'\rangle$, M' ="On $x \in \{0,1\}^*$: if $x = 0^n1^n$, Ⓐ; O/W, return $M(w)$;"
- $A_{\mathsf{TM}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$; $f(\langle M,w\rangle) = \langle M_1, M_2\rangle$, where M_1 = "Ⓐ all"; M_2 ="On x: return $M(w)$;"
- $A_{\mathsf{TM}} \leq_{\mathrm{m}} \overline{EQ_{\mathsf{TM}}}$; $f(\langle M,w\rangle) = \langle M_1, M_2\rangle$, where M_1 = "$\boxed{R}$ all"; M_2 ="On x: return $M(w)$;"
- $ALL_{\mathrm{CFG}} \leq_{\mathrm{m}} EQ_{\mathrm{CFG}}$; $f(\langle G\rangle) = \langle G, H\rangle$, s.t. $L(H) = \Sigma^*$.
- $A_{\mathsf{TM}} \leq_{\mathrm{m}} \{\langle M_{\mathsf{TM}}\rangle : |L(M)| = 1\}$; $f(\langle M,w\rangle) = \langle M'\rangle$, where M' ="On x: if $x = x_0$, return $M(w)$; O/W, $\boxed{R}$;" (where $x_0 \in \Sigma^*$ is fixed).
- $\overline{A_{\mathsf{TM}}} \leq_{\mathrm{m}} E_{\mathsf{TM}}$; $f(\langle M,w\rangle) = \langle M'\rangle$, where M' ="On x: if $x \neq w$, $\boxed{R}$; O/W, return $M(w)$;"
- $A_{\mathsf{TM}} \leq_{\mathrm{m}} \{\langle M_{\mathsf{TM}}\rangle : |L(M)| = 1\}$;
- $\overline{HALT_{\mathsf{TM}}} \leq_{\mathrm{m}} \{\langle M_{\mathsf{TM}}\rangle : |L(M)| \leq 3\}$; $f(\langle M,w\rangle) = \langle M'\rangle$, where M' ="On x: Ⓐ if $M(w)$ halts"
- $HALT_{\mathsf{TM}} \leq_{\mathrm{m}} \{\langle M_{\mathsf{TM}}\rangle : |L(M)| \geq 3\}$; $f(\langle M,w\rangle) = \langle M'\rangle$, where M' ="On x: Ⓐ if $M(w)$ halts"
- $\overline{HALT_{\mathsf{TM}}} \leq_{\mathrm{m}} \{\langle M_{\mathsf{TM}}\rangle : M \text{ Ⓐ all even num.}\}$; $f(\langle M,w\rangle) = \langle M'\rangle$, where M' ="On x: $\boxed{R}$ if $M(w)$ halts within $|x|$. O/W, Ⓐ"
- $\overline{HALT_{\mathsf{TM}}} \leq_{\mathrm{m}} \{\langle M_{\mathsf{TM}}\rangle : L(M) \text{ is finite}\}$; $f(\langle M,w\rangle) = \langle M'\rangle$, where M' ="On x: Ⓐ if $M(w)$ halts"
- $\overline{HALT_{\mathsf{TM}}} \leq_{\mathrm{m}} \{\langle M_{\mathsf{TM}}\rangle : L(M) \text{ is infinite}\}$; $f(\langle M,w\rangle) = \langle M'\rangle$, where M' ="On x: $\boxed{R}$ if $M(w)$ halts within $|x|$ steps. O/W, Ⓐ"
- $HALT_{\mathsf{TM}} \leq_{\mathrm{m}} \{\langle M_1, M_2\rangle : \varepsilon \in L(M_1) \cup L(M_2)\}$; $f(\langle M,w\rangle) = \langle M', M'\rangle$, M' ="On x: Ⓐ if $M(w)$ halts"
- $HALT_{\mathsf{TM}} \leq_{\mathrm{m}} \overline{E_{\mathsf{TM}}}$; $f(\langle M,w\rangle) = \langle M'\rangle$, where M' ="On x: if $x \neq w$ $\boxed{R}$; else, Ⓐ if $M(w)$ halts"
- $HALT_{\mathsf{TM}} \leq_{\mathrm{m}} \{\langle M_{\mathsf{TM}}\rangle \mid \exists x : M(x) \text{ halts in } > |\langle M\rangle| \text{ steps}\}$; $f(\langle M,w\rangle) = \langle M'\rangle$, where M' ="On x: if $M(w)$ halts, make $|\langle M\rangle| + 1$ steps and then halt; O/W, loop"

$\mathbf{P} = \bigcup_{k \in \mathbb{N}} \mathsf{TIME}(n^k) \subseteq \mathbf{NP} = \bigcup_{k \in \mathbb{N}} \mathsf{NTIME}(n^k) = \{L \mid L \text{ is decidable by a PT verifier}\} \supseteq \textbf{NP-complete} = \{B \mid B \in \mathrm{NP}, \forall A \in \mathrm{NP}, A \leq_{\mathrm{P}} B\}$.

- ((**Running time**) decider M is a **$f(n)$-time TM**.) $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the max. num. of steps that DTM (or NTM) M takes on any n-length input (and any branch of any n-length input. resp.).
- (**verifier** for L) TM V s.t. $L = \{w \mid \exists c : V(\langle w,c\rangle) = Ⓐ\}$; (**certificate** for $w \in L$) str. c s.t. $V(\langle w,c\rangle) = Ⓐ$.
- $f : \Sigma^* \to \Sigma^*$ is **PT computable** if there exists a PT TM M s.t. for every $w \in \Sigma^*$, M halts with $f(w)$ on its tape.
- If $A \leq_{\mathrm{P}} B$ and $B \in \mathrm{P}$, then $A \in \mathrm{P}$.
- If $A \leq_{\mathrm{P}} B$ and $B \leq_{\mathrm{P}} A$, then A and B are **PT equivalent**, denoted $A \equiv_P B$. $\equiv_P$ is an equiv. relation on NP. $\mathrm{P} \setminus \{\emptyset, \Sigma^*\}$ is an equiv. class of $\equiv_P$.
- ALL_{DFA}, $CONNECTED$, $\underset{\text{3-clique}}{TRIANGLE}$, $L(G_{\mathsf{CFG}})$, $RELPRIME$, $\overset{\text{directed}}{\underset{s \to t}{PATH}} \in \mathrm{P}$
- $CNF_2 \in \mathrm{P}$: (**alg.** $\forall x \in \phi$: (**1**) If x occurs 1-2 times in same clause $\to$ `del` cl.; (**2**) If x is twice in 2 cl. $\to$ `del` both cl.; (**3**) Similar to (2) for $\overline{x}$; (**4**) Replace any $(x \vee y)$, $(\neg x \vee z)$ with $(y \vee z)$; $(y, z$ may be $\varepsilon)$; (**5**) If $(x) \wedge (\neg x)$ found, $\boxed{R}$. (**6**) If $\phi = \varepsilon$, Ⓐ;)
- $CLIQUE$, $SUBSET\text{-}SUM$, SAT, $3SAT$, $\overset{VERTEX}{COVER}$, $HAMPATH$, $UHAMATH$, $3COLOR \in$ NP-complete. $\emptyset, \Sigma^* \notin$ NP-complete.
- If $B \in$ NP-complete and $B \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$.
- If $B \in$ NPC and $C \in$ NP s.t. $B \leq_{\mathrm{P}} C$, then $C \in$ NPC.
- If $\mathrm{P} = \mathrm{NP}$, then $\forall A \in \mathrm{P} \setminus \{\emptyset, \Sigma^*\}$, $A \in$ NP-complete.

Polytime Reduction: $A \leq_{\mathrm{P}} B$ if $\exists f : \Sigma^* \to \Sigma^* : \forall w \in \Sigma^*,\ w \in A \iff f(w) \in B$ and f is polytime computable.

- $SAT \leq_{\mathrm{P}} DOUBLE\text{-}SAT$; $f(\phi) = \phi \wedge (x \vee \neg x)$
- $4SAT \leq_{\mathrm{P}} 4SAT$; $f(\phi) = \phi'$, where ϕ' is obtained from the CNF ϕ by adding a new var. x to each clause, and adding a new clause $(\neg x \vee \neg x \vee \neg x \vee \neg x)$.
- $3SAT \leq_{\mathrm{P}} CNF_3$; $f(\langle \phi\rangle) = \phi'$. If $\#_\phi(x) = k > 3$, replace x with $x_1, \ldots x_k$, and add $(\overline{x_1} \vee x_2) \wedge \cdots \wedge (\overline{x_k} \vee x_1)$.
- $SUBSET\text{-}SUM \leq_{\mathrm{P}} SET\text{-}PARTITION$; $f(\langle x_1, \ldots, x_m, t\rangle) = \langle x_1, \ldots, x_m, S - 2t\rangle$, where S sum of $x_1, \ldots, x_m$, and t is the target subset-sum.
- $3COLOR \leq_{\mathrm{P}} \overset{\text{almost}}{3COLOR}$; $f(\langle G\rangle) = \langle G'\rangle$, $G' = G \cup K_4$
- $\overset{VERTEX}{COVER}_k \leq_{\mathrm{P}} WVC$; $f(\langle G,k\rangle) = (G,w,k), \forall v \in V(G), w(v) = 1$
- (dir.) $HAM\text{-}PATH \leq_{\mathrm{P}} 2HAM\text{-}PATH$; $f(\langle G,s,t\rangle) = \langle G', s', t'\rangle$, where $V' = V \cup \{s', t', a, b, c, d\}$, $E' = E \cup \{(s',a), (a,b), (b,s)\} \cup \{(s',b), (b,a), (a,s)\} \cup \{(t,c), (c,d), (d,t')\} \cup \{(t,d), (d,c), (c,t')\}$.
- (undir.) $CLIQUE_k \leq_{\mathrm{P}} \underset{|V|/2\text{-clique}}{HALF\text{-}CLIQUE}$; $f(\langle G = (V,E), k\rangle) = \langle G' = (V', E')\rangle$, if $k = \frac{|V|}{2}$, $E = E'$, $V' = V$. if $k > \frac{|V|}{2}$, $V' = V \cup \{j = 2k - |V| \text{ new nodes}\}$. if $k < \frac{|V|}{2}$, $V' = V \cup \{j = |V| - 2k \text{ new nodes}\}$ and $E' = E \cup \{\text{edges for new nodes}\}$
- (dir.) $\underset{s \rightsquigarrow t}{HAM\text{-}PATH} \leq_{\mathrm{P}} HAM\text{-}CYCLE$; $f(\langle G,s,t\rangle) = \langle G', s, t\rangle$ where $V' = V \cup \{x\}$, $E' = E \cup \{(t,x), (x,s)\}$
- $HAM\text{-}CYCLE \leq_{\mathrm{P}} UHAMCYCLE$; $f(\langle G\rangle) = \langle G'\rangle$. For each $u, v \in V$: u is replaced by $u_{\mathsf{in}}, u_{\mathsf{mid}}, u_{\mathsf{out}}$; (v,u) replaced by $\{v_{\mathsf{out}}, u_{\mathsf{in}}\}, \{u_{\mathsf{in}}, u_{\mathsf{mid}}\}$; and (u,v) by $\{u_{\mathsf{out}}, v_{\mathsf{in}}\}, \{u_{\mathsf{mid}}, u_{\mathsf{out}}\}$.
- $UHAMPATH \leq_{\mathrm{P}} PATH_{\geq k}$; $f(\langle G,a,b\rangle) = \langle G, a, b, k = |V(G)| - 1\rangle$
- $\overset{VERTEX}{COVER}_k \leq_{\mathrm{P}} CLIQUE_k$; $f(\langle G,k\rangle) = \langle G^{\complement} = (V, E^{\complement}), |V| - k\rangle$
- $CLIQUE_k \leq_{\mathrm{P}} \{\langle G,t\rangle : G \text{ has } 2t\text{-clique}\}$; $f(\langle G,k\rangle) = \langle G', t = \lceil k/2 \rceil\rangle$, $G' = G$ if k is even; $G' = G \cup \{v\}$ (v connected to all G nodes) if k is odd.
- $CLIQUE_k \leq_{\mathrm{P}} \overset{\text{almost}}{CLIQUE}_k$; $f(\langle G,k\rangle) = \langle G', k+2\rangle$, where $G' = G \cup \{v_{n+1}, v_{n+2}\}$ and v_{n+1}, v_{n+2} are con. to all G nodes.
- $\overset{VERTEX}{COVER}_k \leq_{\mathrm{P}} DOMINATING\text{-}SET_k$; $f(\langle G,k\rangle) = \langle G', k\rangle$, where $V' = \{\text{non-isolated node in } V\} \cup \{v_e : e \in E\}$, $E' = E \cup \{(v_e, u), (v_e, w) : e = (u,w) \in E\}$.
- $CLIQUE \leq_{\mathrm{P}} INDEP\text{-}SET$; $SET\text{-}COVER \leq_{\mathrm{P}} \overset{VERTEX}{COVER}$; $3SAT \leq_{\mathrm{P}} SET\text{-}SPLITTING$; $INDEP\text{-}SET \leq_{\mathrm{P}} \overset{VERTEX}{COVER}$

Counterexamples

- $A \leq_{\mathrm{m}} B$ and $B \in$ REG, but, $A \notin$ REG: $A = \{0^n1^n \mid n \geq 0\}$, $B = \{1\}$, $f : A \to B$, $f(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A \end{cases}$
- $L \in$ CFL but $\overline{L} \notin$ CFL: $L = \{x \mid \forall w \in \Sigma^*, x \neq ww\}$, $\overline{L} = \{ww \mid w \in \Sigma^*\}$.
- $L_1, L_2 \in$ CFL but $L_1 \cap L_2 \notin$ CFL: $L_1 = \{a^nb^nc^m\}$, $L_2 = \{a^mb^nc^n\}$, $L_1 \cap L_2 = \{a^nb^nc^n\}$.
- $L_1 \in$ CFL, L_2 is infinite, but $L_1 \setminus L_2 \notin$ REG : $L_1 = \Sigma^*$, $L_2 = \{a^nb^n \mid n \geq 0\}$, $L_1 \setminus L_2 = \{a^mb^n \mid m \neq n\}$.
- $L_1, L_2 \in$ REG, $L_1 \not\subset L_2$, $L_2 \not\subset L_1$, but, $(L_1 \cup L_2)^* = L_1^* \cup L_2^*$: $L_1 = \{\mathtt{a}, \mathtt{b}, \mathtt{ab}\}$, $L_2 = \{\mathtt{a}, \mathtt{b}, \mathtt{ba}\}$.
- $L_1 \in$ REG, $L_2 \notin$ REG, but, $L_1 \cap L_2 \in$ REG, and $L_1 \cup L_2 \in$ REG : $L_1 = L(\mathtt{a}^*\mathtt{b}^*)$, $L_2 = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$.
- $L_1, L_2, L_3, \dots \in$ REG, but, $\bigcup_{i=1}^{\infty} L_i \notin$ REG : $L_i = \{\mathtt{a}^i\mathtt{b}^i\}$, $\bigcup_{i=1}^{\infty} L_i = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$.
- $L_1 \cdot L_2 \in$ REG, but $L_1 \notin$ REG : $L_1 = \{\mathtt{a}^n\mathtt{b}^n \mid n \geq 0\}$, $L_2 = \Sigma^*$.
- $L_2 \in$ CFL, and $L_1 \subseteq L_2$, but $L_1 \notin$ CFL : $\Sigma = \{a,b,c\}$, $L_1 = \{a^nb^nc^n \mid n \geq 0\}$, $L_2 = \Sigma^*$.
- $L_1, L_2 \in$ DECIDABLE, and $L_1 \subseteq L \subseteq L_2$, but $L \in$ UNDECIDABLE : $L_1 = \emptyset$, $L_2 = \Sigma^*$, L is some undecidable language over Σ.
- $L_1 \in$ REG, $L_2 \notin$ CFL, but $L_1 \cap L_2 \in$ CFL : $L_1 = \{\varepsilon\}$, $L_2 = \{a^nb^nc^n \mid n \geq 0\}$.
- $L^* \in$ REG, but $L \notin$ REG : $L = \{a^p \mid p \text{ is prime}\}$, $L^* = \Sigma^* \setminus \{a\}$.
- $A \not\leq_{\mathrm{m}} \overline{A}$: $A = A_{\mathsf{TM}} \in$ RECOGNIZABLE, $\overline{A} = \overline{A_{\mathsf{TM}}} \notin$ RECOG.
- $A \notin$ DEC., $A \leq_{\mathrm{m}} \overline{A}$:
- $L \in$ CFL, $L \cap L^{\mathcal{R}} \notin$ CFL : $L = \{a^nb^na^m\}$.