

CHEAT SHEET: COMPUTATIONAL MODELS (20604)

<https://github.com/adieblm/20604>

N	δ	a	b	ε
1	3	{}	2	
A 2	1	{}	{}	
3	2	2, 3	{}	

$NFA \rightarrow DFA$

D	δ	a	b
1	3	{}	
A 2	1, 2	{}	
3	2	2, 3	
A 1, 2	1, 3, 2	{}	
A 1, 3	2, 3	2, 3	
A 2, 3	1, 2	2, 3	
A 1, 2, 3	1, 2, 3	2, 3	

Regular Expressions: Examples

- $\{a^n w b^n : w \in \Sigma^*\} \equiv a(a \cup b)^* b$
- $\{w : \#_w(0) \geq 2 \vee \#_w(1) \leq 1\} \equiv (\Sigma^* 0 \Sigma^* 0 \Sigma^*) \cup (0^* (\varepsilon \cup 1) 0^*)$
- $\{w : |w| \bmod n = m\} \equiv (a \cup b)^m ((a \cup b)^n)^*$
- $\{w : \#_b(w) \bmod n = m\} \equiv (a^* b a^*)^m \cdot ((a^* b a^*)^n)^*$
- $\{w : |w| \text{ is odd}\} \equiv (a \cup b)^* ((a \cup b)(a \cup b)^*)^*$
- $\{w : \#_a(w) \text{ is odd}\} \equiv b^* a (a b^* a \cup b)^*$
- $\{w : \#_{ab}(w) = \#_{ba}(w)\} \equiv \varepsilon \cup a \cup b \cup a \Sigma^* a \cup b \Sigma^* b$
- $\{a^m b^n \mid m + n \text{ is odd}\} \equiv a(aa)^*(bb)^* \cup (aa)^* b(bb)^*$
- $\{aw : aba \not\subseteq w\} \equiv a(a \cup bb \cup bbb)^*(b \cup \varepsilon)$

Pumping lemma for regular languages: $A \in \text{REG} \implies \exists p : \forall s \in A, |s| \geq p, s = xyz, \text{ (i) } \forall i \geq 0, xy^i z \in A, \text{ (ii) } |y| > 0 \text{ and (iii) } |xy| \leq p.$

(the following are **non-regular but CFL**)

- $\{w = w^R\}; s = 0^p 1 0^p = xyz.$ but $xy^2 z = 0^{p+|y|} 1 0^p \notin L.$
- $\{a^n b^n\}; s = a^p b^p = xyz, xy^2 z = a^{p+|y|} b^p \notin L.$
- $\{w : \#_a(w) > \#_b(w)\}; s = a^p b^{p+1}, |s| = 2p + 1 \geq p,$
 $xy^2 z = a^{p+|y|} b^{p+1} \notin L.$

- $\{w : \#_a(w) = \#_b(w)\}; s = a^p b^p = xyz$ but $xy^2 z = a^{p+|y|} b^p \notin L.$
- $\{w : \#_w(a) \neq \#_w(b)\};$ (pf. by 'complement-closure', $\bar{L} = \{w : \#_w(a) = \#_w(b)\})$
- $\{a^i b^j c^k : i < j \vee i > k\}; s = a^p b^{p+1} c^{2p} = xyz,$ but $xy^2 z = a^{p+|y|} b^{p+1} c^{2p}, p + |y| \geq p + 1, p + |y| \leq 2p.$
- (the following are both **non-CFL and non-regular**)

- $\{w = a^{2^k}\}; k = \lfloor \log_2 |w| \rfloor, s = a^{2^k} = xyz.$
 $2^k = |xyz| < |xy^2 z| \leq |xyz| + |xy| \leq 2^k + p < 2^{k+1}.$
- $\{a^p : p \text{ is prime}\}; s = a^t = xyz$ for prime $t \geq p.$
 $r := |y| > 0$
- $\{www : w \in \Sigma^*\}; s = a^p b a^p b a^p = xyz = a^{|x|+|y|+m} b a^p b a^p b$,
 $m \geq 0$, but $xy^2 z = a^{|x|+2|y|+m} b a^p b a^p b \notin L.$
- $\{a^{2^n} b^{3^n} a^n\}; s = a^{2^p} b^{3^p} a^p = xyz = a^{|x|+|y|+m+p} b^{3^p} a^p,$
 $m \geq 0$, but $xy^2 z = a^{2^p+|y|} b^{3^p} a^p \notin L.$

(PDA) $M = (Q, \Sigma, \Gamma, \delta, q_0 \in Q, F \subseteq Q). \delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \rightarrow \mathcal{P}(Q \times \Gamma_\varepsilon).$ $L \in \text{CFL} \iff \exists G_{\text{CFG}} : L = L(G) \iff \exists P_{\text{PDA}} : L = L(P)$

- (CFG \rightsquigarrow CNF) (1.)** Add a new start variable S_0 and a rule $S_0 \rightarrow S.$ **(2.)** Remove ε -rules of the form $A \rightarrow \varepsilon$ (except for $S_0 \rightarrow \varepsilon$). and remove A 's occurrences on the RH of a rule (e.g.: $R \rightarrow uAvAw$ becomes $R \rightarrow uAvAw \mid uAvw \mid uvAw \mid uvw.$ where $u, v, w \in (V \cup \Sigma)^*$). **(3.)** Remove unit rules $A \rightarrow B$ then whenever $B \rightarrow u$ appears, add $A \rightarrow u$, unless this was a unit rule previously removed. ($u \in (V \cup \Sigma)^*$). **(4.)** Replace each rule $A \rightarrow u_1 u_2 \dots u_k$ where $k \geq 3$ and $u_i \in (V \cup \Sigma)$, with the rules $A \rightarrow u_1 A_1, A_1 \rightarrow u_2 A_2, \dots,$

- $A_{k-2} \rightarrow u_{k-1} u_k$, where A_i are new variables. Replace terminals u_i with $U_i \rightarrow u_i.$
- If $G \in \text{CNF}$, and $w \in L(G)$, then $|w| \leq 2^{|h|} - 1$, where h is the height of the parse tree for $w.$
- $\forall L \in \text{CFL}, \exists G \in \text{CNF} : L = L(G).$
- (derivation)** $S \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \dots \Rightarrow u_n = w$, where each u_i is in $(V \cup \Sigma)^*$. (in this case, G **generates** w (or S **derives** w), $S \xrightarrow{*} w$)
- M **accepts** $w \in \Sigma^*$ if there is a seq. $r_0, r_1, \dots, r_m \in Q$ and $s_0, s_1, \dots, s_m \in \Gamma^*$ s.t.: (1.) $r_0 = q_0$ and $s_0 = \varepsilon$; (2.)

For $i = 0, 1, \dots, m-1$, we have $(r_i, b) \in \delta(r_i, w_{i+1}, a)$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$; (3.) $r_m \in F.$

- (PDA transition)** " $a, b \rightarrow c$ ": **reads** a from the input (or read nothing if $a = \varepsilon$). **pops** b from the stack (or pops nothing if $b = \varepsilon$). **pushes** c onto the stack (or pushes nothing if $c = \varepsilon$)
- $R \in \text{REG} \wedge C \in \text{CFL} \implies R \cap C \in \text{CFL}.$ (pf. construct PDA $P' = P_C \times D_R.$)

(CFG) $G = (V, \Sigma, R, S), A \rightarrow w, (A \in V, w \in (V \cup \Sigma)^*);$ **(CNF)** $A \rightarrow BC, A \rightarrow a, S \rightarrow \varepsilon, (A, B, C \in V, a \in \Sigma, B, C \neq S).$

the following are **CFL but non-regular**:

- $\{w : w = w^R\}; S \rightarrow aSa \mid bSb \mid a \mid b \mid \varepsilon$
- $\{w : w \neq w^R\}; S \rightarrow aSa \mid bSb \mid aXb \mid bXa; X \rightarrow aX \mid bX \mid \varepsilon$
- $\{ww^R\} = \{w : w = w^R \wedge |w| \text{ is even}\}; S \rightarrow aSa \mid bSb \mid \varepsilon$
- $\{wa^n w^R\}; S \rightarrow aSa \mid bSb \mid M; M \rightarrow aM \mid \varepsilon$
- $\{w\#x : w^R \subseteq x\}; S \rightarrow AX; A \rightarrow 0A0 \mid 1A1 \mid \#X;$
 $X \rightarrow 0X \mid 1X \mid \varepsilon$
- $\{w : \#_w(a) > \#_w(b)\}; S \rightarrow JaJ; J \rightarrow JJ \mid aJb \mid bJa \mid a \mid \varepsilon$
- $\{w : \#_w(a) \geq \#_w(b)\}; S \rightarrow SS \mid aSb \mid bSa \mid a \mid \varepsilon$
- $\{w : \#_w(a) = \#_w(b)\}; S \rightarrow SS \mid aSb \mid bSa \mid \varepsilon$

- $\{w : \#_w(a) = 2 \cdot \#_w(b)\};$
 $S \rightarrow SS \mid S_1 b S_1 \mid b S a a \mid a a S b \mid \varepsilon; S_1 \rightarrow a S \mid S S_1$
- $\{w : \#_w(a) \neq \#_w(b)\} = \{\#_w(a) > \#_w(b)\} \cup \{\#_w(a) < \#_w(b)\}$
- $\{a^n b^n\}; S \rightarrow XbXaX \mid A \mid B; A \rightarrow aAb \mid Ab \mid b;$
 $B \rightarrow aBb \mid aB \mid a; X \rightarrow aX \mid bX \mid \varepsilon.$
- $\{a^n b^m \mid n \neq m\}; S \rightarrow aSb \mid AB; A \rightarrow aA \mid a; B \rightarrow bB \mid b$
- $\{x \mid x \neq ww\}; S \rightarrow A \mid B \mid AB \mid BA; A \rightarrow CAC \mid 0;$
 $B \rightarrow CBC \mid 1; C \rightarrow 0 \mid 1$
- $\{a^n b^m \mid m \leq n \leq 3m\}; S \rightarrow aSb \mid aaSb \mid aaaSb \mid \varepsilon;$
- $\{a^n b^n\}; S \rightarrow aSb \mid \varepsilon$
- $\{a^n b^m \mid n > m\}; S \rightarrow aSb \mid aS \mid a$
- $\{a^n b^m \mid n \geq m \geq 0\}; S \rightarrow aSb \mid aS \mid a \mid \varepsilon$

- $\{a^i b^j c^k \mid i + j = k\}; S \rightarrow aSc \mid X; X \rightarrow bXc \mid \varepsilon$
- $\{a^i b^j c^k \mid i \leq j \vee j \leq k\}; S \rightarrow S_1 C \mid AS_2; A \rightarrow Aa \mid \varepsilon;$
 $S_1 \rightarrow aS_1 b \mid S_1 b \mid \varepsilon; S_2 \rightarrow bS_2 c \mid S_2 c \mid \varepsilon; C \rightarrow Cc \mid \varepsilon$
- $\{a^i b^j c^k \mid i = j \vee j = k\};$
 $S \rightarrow AX_1 \mid X_2 C; X_1 \rightarrow bX_1 c \mid \varepsilon; X_2 \rightarrow aX_2 b \mid \varepsilon; A \rightarrow aA \mid \varepsilon; C$
- $\{xy : |x| = |y|, x \neq y\}; S \rightarrow AB \mid BA;$
 $A \rightarrow a \mid aAa \mid aAb \mid bAa \mid bAb;$
 $B \rightarrow b \mid aBa \mid aBb \mid bBa \mid bBb;$

the following are both **CFL and regular**:

- $\{w : \#_w(a) \geq 3\}; S \rightarrow XaXaXaX; X \rightarrow aX \mid bX \mid \varepsilon$
- $\{w : |w| \text{ is odd}\}; S \rightarrow aaS \mid abS \mid bbS \mid baS \mid a \mid b$
- $\{w : |w| \text{ is even}\}; S \rightarrow aaS \mid abS \mid bbS \mid baS \mid \varepsilon$
- $\emptyset; S \rightarrow S$

Pumping lemma for context-free languages: $L \in \text{CFL} \implies \exists p : \forall s \in L, |s| \geq p, s = uvxyz, \text{ (i) } \forall i \geq 0, uv^i xy^i z \in L, \text{ (ii) } |vxy| \leq p, \text{ and (iii) } |vy| > 0.$

- $\{w = a^n b^n c^n\}; s = a^p b^p b^p = uvxyz. vxy$ can't contain all of a, b, c thus $uv^2 xy^2 z$ must pump one of them less than the others.

- $\{ww : w \in \{a, b\}^*\};$
- (more example of not CFL)**
- $\{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}, \{a^n b^n c^n \mid n \in \mathbb{N}\},$
 $\{ww \mid w \in \{a, b\}^*\}, \{a^{2^n} \mid n \geq 0\}, \{a^p \mid p \text{ is prime}\},$

- $L = \{ww^R w : w \in \{a, b\}^*\}$
- $\{w \mid \#_w(a) = \#_w(b) = \#_w(c)\}:$ (pf. since Regular \cap CFL \in CFL, but $\{a^* b^* c^*\} \cap L = \{a^n b^n c^n\} \notin \text{CFL}$)

$L \in \text{DECIDABLE} \iff (L \in \text{REC. and } L \in \text{co-REC.}) \iff \exists M_{\text{TM}} \text{ decides } L.$

- (TM)** $M = (Q, \Sigma_{\text{input}} \subseteq \Gamma, \Gamma_{\text{tape}}, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$, where $\sqcup \in \Gamma, \sqcup \notin \Sigma, q_{\text{rej}} \neq q_{\text{acc}}; \delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$
- (recognizable)** **A** if $w \in L, \bar{R}/\text{loops}$ if $w \notin L; A$ is **co-recognizable** if \bar{A} is recognizable.
- $L \in \text{RECOGNIZABLE} \iff L \leq_m A_{\text{TM}}.$
- Every inf. recognizable lang. has an inf. dec. subset.
- (decidable)** **A** if $w \in L, \bar{R}$ if $w \notin L.$
- $L \in \text{DECIDABLE} \iff L \leq_m 0^* 1^*.$

- $L \in \text{DECIDABLE} \iff L^R \in \text{DECIDABLE}.$
- (decider)** TM that halts on all inputs.
- (Rice)** Let P be a lang. of TM descriptions, s.t. (i) P is nontrivial (not empty and not all TM desc.) and (ii) for each two TM M_1 and M_2 , we have $L(M_1) = L(M_2) \implies (\langle M_1 \rangle \in P \iff \langle M_2 \rangle \in P).$ Then P is undecidable. (e.g. $\text{INFINITE}_{\text{TM}}, \text{ALL}_{\text{TM}}, E_{\text{TM}}, \{\langle M_{\text{TM}} \rangle : 1 \in L(M)\}$)
- $\{\text{all TMs}\}$ is count.; Σ^* is count. (finite Σ); $\{\text{all lang.}\}$ is uncount.; $\{\text{all infinite bin. seq.}\}$ is uncount.

- $\text{DFA} \equiv \text{NFA} \equiv \text{GNFA} \equiv \text{REG} \subset \text{NPDA} \equiv \text{CFG} \subset \text{DTM} \equiv \text{NTM}$
- $f : \Sigma^* \rightarrow \Sigma^*$ is **computable** if $\exists M_{\text{TM}} : \forall w \in \Sigma^*, M$ halts on w and outputs $f(w)$ on its tape.
- If $A \leq_m B$ and B is decidable, then A is dec.
- If $A \leq_m B$ and A is undecidable, then B is undec.
- If $A \leq_m B$ and B is recognizable, then A is rec.
- If $A \leq_m B$ and A is unrecognizable, then B is unrec.
- (transitivity) If $A \leq_m B$ and $B \leq_m C$, then $A \leq_m C.$
- $A \leq_m B \iff \bar{A} \leq_m \bar{B}$ (esp. $A \leq_m \bar{A} \iff \bar{A} \leq_m A$)
- If $A \leq_m \bar{A}$ and $A \in \text{RECOGNIZABLE}$, then $A \in \text{DEC}.$

<ul style="list-style-type: none"> (unrecognizable) $\overline{A_{TM}}, \overline{EQ_{TM}}, EQ_{CFG}, \overline{HALT_{TM}}, REG_{TM}, E_{TM}, EQ_{TM}, ALL_{CFG}, EQ_{CFG}$ (recognizable but undecidable) $A_{TM}, HALT_{TM}, \overline{EQ_{CFG}}, \overline{E_{TM}}, \{ \langle M, k \rangle \mid \exists x (M(x) \text{ halts in } \geq k \text{ steps}) \}$ (decidable) $A_{DFA}, A_{NFA}, A_{REG}, E_{DFA}, EQ_{DFA}, A_{CFG}, E_{CFG}, A_{LBA}, ALL_{DFA} = \{ \langle D \rangle \mid L(D) = \Sigma^* \}, A_{\varepsilon_{CFG}} = \{ \langle G \rangle \mid \varepsilon \in L(G) \}$ Examples of Recognizers: $\overline{EQ_{CFG}}$: "On $\langle G_1, G_2 \rangle$: for each $w \in \Sigma^*$ (lexico.): Test (by A_{CFG}) whether $w \in L(G_1)$ and $w \notin L(G_2)$ (vice versa), if so A; O/W, continue" Examples of Deciders: 	<ul style="list-style-type: none"> INFINITE_{DFA}: "On n-state DFA $\langle A \rangle$: const. DFA B s.t. $L(B) = \Sigma^{\geq n}$; const. DFA C s.t. $L(C) = L(A) \cap L(B)$; if $L(C) \neq \emptyset$ (by E_{DFA}) A; O/W, R" $\{ \langle D \rangle \mid \nexists w \in L(D) : \#_1(w) \text{ is odd} \}$: "On $\langle D \rangle$: const. DFA A s.t. $L(A) = \{ w \mid \#_1(w) \text{ is odd} \}$; const. DFA B s.t. $L(B) = L(D) \cap L(A)$; if $L(B) = \emptyset$ (E_{DFA}) A; O/W R" $\{ \langle R, S \rangle \mid R, S \text{ are regex}, L(R) \subseteq L(S) \}$: "On $\langle R, S \rangle$: const. DFA D s.t. $L(D) = L(R) \cap \overline{L(S)}$; if $L(D) = \emptyset$ (by E_{DFA}) A; O/W, R" $\{ \langle D_{DFA}, R_{REG} \rangle \mid L(D) = L(R) \}$: "On $\langle D, R \rangle$: convert R to DFA D_R; if $L(D) = L(D_R)$ (by EQ_{DFA}) A; O/W, R" $\{ \langle D_{DFA} \rangle \mid L(D) = (L(D))^{\mathbb{R}} \}$: "On $\langle D \rangle$: const. DFA $D^{\mathbb{R}}$ s.t. $L(D^{\mathbb{R}}) = (L(D))^{\mathbb{R}}$; if $L(D) = L(D^{\mathbb{R}})$ (by EQ_{DFA}), 	<ul style="list-style-type: none"> A; O/W, R" $\{ \langle M, k \rangle \mid \exists x (M(x) \text{ runs for } \geq k \text{ steps}) \}$: "On $\langle M, k \rangle$: (foreach $w \in \Sigma^{\leq k+1}$: if $M(w)$ not halt within k steps, A); O/W, R" $\{ \langle M, k \rangle \mid \exists x (M(x) \text{ halts in } \leq k \text{ steps}) \}$: "On $\langle M, k \rangle$: (foreach $w \in \Sigma^{\leq k+1}$: run $M(w)$ for $\leq k$ steps, if halts, A); O/W, R" $\{ \langle M_{DFA} \rangle \mid L(M) = \Sigma^* \}$: "On $\langle M \rangle$: const. DFA $M^c = (L(M))^c$; if $L(M^c) = \emptyset$ (by E_{DFA}) A; O/W R." $\{ \langle R_{REG} \rangle \mid \exists s, t \in \Sigma^* : w = s111t \in L(R) \}$: "On $\langle R \rangle$: const. DFA D s.t. $L(D) = \Sigma^*111\Sigma^*$; const. DFA C s.t. $L(C) = L(R) \cap L(D)$; if $L(C) \neq \emptyset$ (E_{DFA}) A; O/W R"
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Mapping Reduction: $A \leq_m B$ if $\exists f : \Sigma^* \rightarrow \Sigma^* : \forall w \in \Sigma^*, w \in A \iff f(w) \in B$ and f is computable.

<ul style="list-style-type: none"> $A_{TM} \leq_m \{ \langle M_{TM} \rangle \mid L(M) = (L(M))^{\mathbb{R}} \}$; $f(\langle M, w \rangle) = \langle M', w \rangle$, where $M' =$ "On x, if $x \notin \{01, 10\}$, R; if $x = 01$, return $M(x)$; if $x = 10$, A." $A_{TM} \leq_m L = \{ \langle M, D \rangle \mid L(M) = L(D) \}$; $f(\langle M, w \rangle) = \langle M', D \rangle$, where $M' =$ "On x: if $x = w$ return $M(x)$; O/W, R;" D is DFA s.t. $L(D) = \{ w \}$. $A \leq_m HALT_{TM}$; $f(w) = \langle M, \varepsilon \rangle$, where $M =$ "On x: if $w \in A$, halt; if $w \notin A$, loop;" $A_{TM} \leq_m CFL_{TM} = \{ \langle M \rangle \mid L(M) \text{ is CFL} \}$; $f(\langle M, w \rangle) = \langle N \rangle$, where $N =$ "On x: if $x = a^n b^n c^n$, A; O/W, return $M(w)$;" $A \leq_m B = \{ 0w : w \in A \} \cup \{ 1w : w \notin A \}$; $f(w) = 0w$. $A_{TM} \leq_m HALT_{TM}$; $f(\langle M, w \rangle) = \langle M', w \rangle$, where $M' =$ "On x: if $M(x)$ accepts, A. If rejects, loop" $E_{TM} \leq_m USELESS_{TM}$; $f(\langle M \rangle) = \langle M, q_{\mathbf{A}} \rangle$ 	<ul style="list-style-type: none"> $E_{TM} \leq_m EQ_{TM}$; $f(\langle M \rangle) = \langle M, M' \rangle$, $M' =$ "On x: R" $A_{TM} \leq_m REGULAR_{TM}$; $f(\langle M, w \rangle) = \langle M' \rangle$, $M' =$ "On $x \in \{0, 1\}^*$: if $x = 0^n 1^n$, A; O/W, return $M(w)$;" $A_{TM} \leq_m EQ_{TM}$; $f(\langle M, w \rangle) = \langle M_1, M_2 \rangle$, where $M_1 =$ "A all"; $M_2 =$ "On x: return $M(w)$;" $A_{TM} \leq_m \overline{EQ_{TM}}$; $f(\langle M, w \rangle) = \langle M_1, M_2 \rangle$, where $M_1 =$ "R all"; $M_2 =$ "On x: return $M(w)$;" $ALL_{CFG} \leq_m EQ_{CFG}$; $f(\langle G \rangle) = \langle G, H \rangle$, s.t. $L(H) = \Sigma^*$. $A_{TM} \leq_m \{ \langle M_{TM} \rangle : L(M) = 1 \}$; $f(\langle M, w \rangle) = \langle M' \rangle$, where $M' =$ "On x: if $x = x_0$, return $M(w)$; O/W, R;" (where $x_0 \in \Sigma^*$ is fixed). $\overline{A_{TM}} \leq_m E_{TM}$; $f(\langle M, w \rangle) = \langle M' \rangle$, where $M' =$ "On x: if $x \neq w$, R; O/W, return $M(w)$;" $HALT_{TM} \leq_m \{ \langle M_{TM} \rangle : L(M) \leq 3 \}$; $f(\langle M, w \rangle) = \langle M' \rangle$, where $M' =$ "On x: A if $M(w)$ halts" 	<ul style="list-style-type: none"> $HALT_{TM} \leq_m \{ \langle M_{TM} \rangle : L(M) \geq 3 \}$; $f(\langle M, w \rangle) = \langle M' \rangle$, where $M' =$ "On x: A if $M(w)$ halts" $\overline{HALT_{TM}} \leq_m \{ \langle M_{TM} \rangle : M \text{ all even num.} \}$; $f(\langle M, w \rangle) = \langle M' \rangle$, where $M' =$ "On x: R if $M(w)$ halts within x. O/W, A" $HALT_{TM} \leq_m \{ \langle M_{TM} \rangle : L(M) \text{ is finite} \}$; $f(\langle M, w \rangle) = \langle M' \rangle$, where $M' =$ "On x: A if $M(w)$ halts" $HALT_{TM} \leq_m \{ \langle M_{TM} \rangle : L(M) \text{ is infinite} \}$; $f(\langle M, w \rangle) = \langle M' \rangle$, where $M' =$ "On x: R if $M(w)$ halts within x steps. O/W, A" $HALT_{TM} \leq_m \{ \langle M_1, M_2 \rangle : \varepsilon \in L(M_1) \cup L(M_2) \}$; $f(\langle M, w \rangle) = \langle M', M' \rangle$, $M' =$ "On x: A if $M(w)$ halts" $HALT_{TM} \leq_m \overline{E_{TM}}$; $f(\langle M, w \rangle) = \langle M' \rangle$, where $M' =$ "On x: if $x \neq w$ R; else, A if $M(w)$ halts" $HALT_{TM} \leq_m \{ \langle M_{TM} \rangle \mid \exists x : M(x) \text{ halts in } > \langle M \rangle \text{ steps} \}$; $f(\langle M, w \rangle) = \langle M' \rangle$, where $M' =$ "On x: if $M(w)$ halts, make $\langle M \rangle + 1$ steps and then halt; O/W, loop"
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$$P = \bigcup_{k \in \mathbb{N}} \text{TIME}(n^k) \subseteq NP = \bigcup_{k \in \mathbb{N}} \text{NTIME}(n^k) = \{ L \mid L \text{ is decidable by a PT verifier} \} \supseteq \text{NP-complete} = \{ B \mid B \in NP, \forall A \in NP, A \leq_P B \}.$$

<ul style="list-style-type: none"> (verifier for L) TM V s.t. $L = \{ w \mid \exists c : V(\langle w, c \rangle) = \mathbf{A} \}$; (certificate for $w \in L$) str. c s.t. $V(\langle w, c \rangle) = \mathbf{A}$. $f : \Sigma^* \rightarrow \Sigma^*$ is PT computable if there exists a PT TM M s.t. for every $w \in \Sigma^*$, M halts with $f(w)$ on its tape. If $A \leq_P B$ and $B \in P$, then $A \in P$. $A \equiv_P B$ if $A \leq_P B$ and $B \leq_P A$. \equiv_P is an equiv. relation on NP. $P \setminus \{ \emptyset, \Sigma^* \}$ is an equiv. class of \equiv_P. 	<ul style="list-style-type: none"> $ALL_{DFA}, CONNECTED, TRIANGLE, L(G_{CFG}), RELPRIME, \overset{directed}{PATH}_{s \rightarrow t} \in P$ $CNF_2 \in P$: (algo. $\forall x \in \phi$: (1) If x occurs 1-2 times in same clause \rightarrow remove cl.; (2) If x is twice in 2 cl. \rightarrow remove both cl.; (3) Similar to (2) for \bar{x}; (4) Replace any $(x \vee y), (\neg x \vee z)$ with $(y \vee z)$; (y, z may be ε); (5) If $(x) \wedge (\neg x)$ found, R. (6) If $\phi = \varepsilon$, A;)) 	<ul style="list-style-type: none"> $CLIQUE, SUBSET-SUM, SAT, 3SAT, \overset{VERTEX}{COVER}, HAMPATH, UHAMATH, 3COLOR \in \text{NP-complete}$. $\emptyset, \Sigma^* \notin \text{NP-complete}$. If $B \in \text{NP-complete}$ and $B \in P$, then $P = NP$. If $B \in \text{NPC}$ and $C \in \text{NP}$ s.t. $B \leq_P C$, then $C \in \text{NPC}$. If $P = NP$, then $\forall A \in P \setminus \{ \emptyset, \Sigma^* \}, A \in \text{NP-complete}$.
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Polytime Reduction: $A \leq_P B$ if $\exists f : \Sigma^* \rightarrow \Sigma^* : \forall w \in \Sigma^*, w \in A \iff f(w) \in B$ and f is polytime computable.

<ul style="list-style-type: none"> $SAT \leq_P DOUBLE-SAT$; $f(\phi) = \phi \wedge (x \vee \neg x)$ $3SAT \leq_P 4SAT$; $f(\phi) = \phi'$, where ϕ' is obtained from the CNF ϕ by adding a new var. x to each clause, and adding a new clause $(\neg x \vee \neg x \vee \neg x \vee \neg x)$. $3SAT \leq_P CNF_3$; $f(\langle \phi \rangle) = \phi'$. If $\# \phi(x) = k > 3$, replace x with x_1, \dots, x_k, and add $(\overline{x_1} \vee \overline{x_2}) \wedge \dots \wedge (\overline{x_k} \vee x_1)$. $SUBSET-SUM \leq_P SET-PARTITION$; $f(\langle x_1, \dots, x_m, t \rangle) = \langle x_1, \dots, x_m, S - 2t \rangle$, where S sum of x_1, \dots, x_m, and t is the target subset-sum. $3COLOR \leq_P \overset{\text{almost}}{3COLOR}$; $f(\langle G \rangle) = \langle G' \rangle$, $G' = G \cup K_4$ $\overset{VERTEX}{COVER}_k \leq_P WVC$; $f(\langle G, k \rangle) = (G, w, k)$, $\forall v \in V(G), w(v) = 1$ (dir.) $HAM-PATH \leq_P 2HAM-PATH$; $f(\langle G, s, t \rangle) = \langle G', s', t' \rangle$, where $V' = V \cup \{ s', t', a, b, c, d \}$, 	<ul style="list-style-type: none"> $E' = E \cup \{ (s', a), (a, b), (b, s) \} \cup \{ (s', b), (b, a), (a, s) \} \cup \{ (t, c), (c, d), (d, t') \} \cup \{ (t, d), (d, c), (c, t') \}$. (undir.) $CLIQUE_k \leq_P \overset{ V /2\text{-clique}}{HALF-CLIQUE}$; $f(\langle G = (V, E), k \rangle) = \langle G' = (V', E') \rangle$, if $k = \frac{ V }{2}$, $E = E'$, $V' = V$. If $k > \frac{ V }{2}$, $V' = V \cup \{ j = 2k - V \text{ new nodes} \}$. if $k < \frac{ V }{2}$, $V' = V \cup \{ j = V - 2k \text{ new nodes} \}$ and $E' = E \cup \{ \text{edges for new nodes} \}$ (dir.) $\overset{s \rightarrow t}{HAM-PATH} \leq_P HAM-CYCLE$; $f(\langle G, s, t \rangle) = \langle G', s, t \rangle$ where $V' = V \cup \{ x \}$, $E' = E \cup \{ (t, x), (x, s) \}$ $HAM-CYCLE \leq_P UHAMCYCLE$; $f(\langle G \rangle) = \langle G' \rangle$. For each $u, v \in V$: u is replaced by u_{in}, u_{mid}, u_{out}; (v, u) replaced by $\{ v_{out}, u_{in} \}, \{ u_{in}, u_{mid} \}$; and (u, v) by $\{ u_{out}, v_{in} \}, \{ u_{mid}, u_{out} \}$. 	<ul style="list-style-type: none"> $UHAMPATH \leq_P PATH_{\geq k}$; $f(\langle G, a, b \rangle) = \langle G, a, b, k = V - 1 \rangle$ $\overset{VERTEX}{COVER}_k \leq_P CLIQUE_k$; $f(\langle G, k \rangle) = \langle G^c = (V, E^c), V - k \rangle$ $CLIQUE_k \leq_P \{ \langle G, t \rangle : G \text{ has } 2t\text{-clique} \}$; $f(\langle G, k \rangle) = \langle G', t = \lceil k/2 \rceil \rangle$, $G' = G$ if k is even; $G' = G \cup \{ v \}$ (v connected to all G nodes) if k is odd. $CLIQUE_k \leq_P \overset{\text{almost}}{CLIQUE}_k$; $f(\langle G, k \rangle) = \langle G', k + 2 \rangle$, $G' = G \cup \{ v_{n+1}, v_{n+2} \}$; v_{n+1}, v_{n+2} are con. to all V $\overset{VERTEX}{COVER}_k \leq_P DOMINATING-SET_k$; $f(\langle G, k \rangle) = \langle G', k \rangle$, where $V' = \{ \text{non-isolated node in } V \} \cup \{ v_e : e \in E \}$, $E' = E \cup \{ (v_e, u), (v_e, w) : e = (u, w) \in E \}$. $CLIQUE \leq_P INDEP-SET$; $SET-COVER \leq_P \overset{VERTEX}{COVER}$; $3SAT \leq_P SET-SPLITTING$; $INDEP-SET \leq_P \overset{VERTEX}{COVER}$
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Counterexamples

<ul style="list-style-type: none"> $A \leq_m B$ and $B \in \text{REG}$, but, $A \notin \text{REG}$: $A = \{ 0^n 1^n \mid n \geq 0 \}, B = \{ 1 \}, f : A \rightarrow B$, $f(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A \end{cases}$ $L \in \text{CFL}$ but $\overline{L} \notin \text{CFL}$: $L = \{ x \mid \forall w \in \Sigma^*, x \neq ww \}$, $\overline{L} = \{ ww \mid w \in \Sigma^* \}$. $L_1, L_2 \in \text{CFL}$ but $L_1 \cap L_2 \notin \text{CFL}$: $L_1 = \{ a^n b^n c^m \}$, $L_2 = \{ a^m b^n c^n \}$, $L_1 \cap L_2 = \{ a^n b^n c^n \}$. $L_1 \in \text{CFL}$, L_2 is infinite, but $L_1 \setminus L_2 \notin \text{REG}$: $L_1 = \Sigma^*$, $L_2 = \{ a^n b^n \mid n \geq 0 \}$, $L_1 \setminus L_2 = \{ a^m b^n \mid m \neq n \}$. 	<ul style="list-style-type: none"> $L_1, L_2 \in \text{REG}$, $L_1 \not\subseteq L_2$, $L_2 \not\subseteq L_1$, but, $(L_1 \cup L_2)^* = L_1^* \cup L_2^* : L_1 = \{ a, b, ab \}$, $L_2 = \{ a, b, ba \}$. $L_1 \in \text{REG}$, $L_2 \notin \text{REG}$, $L_1 \cap L_2 \in \text{REG}$, and $L_1 \cup L_2 \in \text{REG}$: $L_1 = L(a^* b^*)$, $L_2 = \{ a^n b^n \mid n \geq 0 \}$. $L_1, L_2, L_3, \dots \in \text{REG}$, $\bigcup_{i=1}^{\infty} L_i \notin \text{REG}$: $L_i = \{ a^i b^i \}$, $\bigcup_{i=1}^{\infty} L_i = \{ a^n b^n \mid n \geq 0 \}$. $L_1 \cdot L_2 \in \text{REG}$, $L_1 \notin \text{REG} : L_1 = \{ a^n b^n \}$, $L_2 = \Sigma^*$. $L_2 \in \text{CFL}$, and $L_1 \subseteq L_2$, but $L_1 \notin \text{CFL}$: $\Sigma = \{ a, b, c \}$, $L_1 = \{ a^n b^n c^n \mid n \geq 0 \}$, $L_2 = \Sigma^*$. $L_1, L_2 \in \text{DECIDABLE}$, and $L_1 \subseteq L \subseteq L_2$, but $L \in \text{UNDECIDABLE}$: $L_1 = \emptyset$, $L_2 = \Sigma^*$, L is some 	<ul style="list-style-type: none"> undecidable language over Σ. $L_1 \in \text{REG}$, $L_2 \notin \text{CFL}$, but $L_1 \cap L_2 \in \text{CFL}$: $L_1 = \{ \varepsilon \}$, $L_2 = \{ a^n b^n c^n \mid n \geq 0 \}$. $L^* \in \text{REG}$, but $L \notin \text{REG}$: $L = \{ a^p \mid p \text{ is prime} \}$, $L^* = \Sigma^* \setminus \{ a \}$. $A \not\leq_m \overline{A} : A = A_{TM} \in \text{RECOGNIZABLE}$, $\overline{A} = \overline{A_{TM}} \notin \text{RECOG}$. $A \notin \text{DEC.}$, $A \leq_m \overline{A} : f(0x) = 1x, f(1y) = 0y$, $A = \{ w \mid \exists x \in A_{TM} : w = 0x \vee \exists y \in \overline{A_{TM}} : w = 1y \}$ $L \in \text{CFL}$, $L \cap L^{\mathbb{R}} \notin \text{CFL} : L = \{ a^n b^n a^m \}$.
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