CH3 FUNCTIONAL METHODS

3.1 Generating functional for connected green's fct

→ The generating function for Green's function is

$$Z[J] = \int \mathcal{D}\phi \, \exp\left\{\frac{i}{\hbar}\left(S[\phi] + J_A \phi^A\right)\right\}$$

while the one for the normalized Green's function is

$$\frac{Z[J]}{Z[0]} = \sum_k \left(\frac{i}{\hbar}\right)^k \frac{1}{k!} J_{A_1} \cdots J_{A_k} \langle \hat{\phi}^{A_1} \cdots \hat{\phi}^{A_k} \rangle$$

with $$\langle \hat{\phi}^{A_1} \cdots \hat{\phi}^{A_k} \rangle = \frac{\langle +\infty, 0 | T \prod_k \hat{\phi}^{A_k} | 0, -\infty \rangle}{\langle +\infty, 0 | 0, \infty \rangle}$$

This corresponds to summing Feynman diagrams without vacuum parts.

We found that

$$\frac{Z[J]}{Z[0]} = \exp\left\{\frac{i}{\hbar} I_1\left[\frac{\hbar}{i} \delta/\delta J\right]\right\} \exp\left\{\frac{\hbar}{2i} J_A (\Delta^{-1})^{AB} J_B\right\}$$

DEF | We write Green's functions with external source J as

$$\langle \hat{\phi}^{A_1} \cdots \hat{\phi}^{A_k} \rangle^J = \frac{1}{Z[J]} \left(\frac{\hbar}{i}\right)^k \frac{\delta^k}{\delta J_{A_1} \cdots \delta J_{A_k}} Z[J]$$

DEF | The generating functional of connected Green's fct $W[J]$ is

$$W[J] = \frac{\hbar}{i} \ln\left\{\frac{Z[J]}{Z[0]}\right\} \Leftrightarrow \frac{Z[J]}{Z[0]} = e^{\frac{i}{\hbar} W[J]}$$

→ We can write $\frac{i}{\hbar} W[J] = \sum_k \frac{(\hbar/i)^k}{k!} J_{A_1} \cdots J_{A_k} \langle \hat{\phi}_{A_1} \cdots \hat{\phi}_{A_k} \rangle_c$

⊙ Classical field and invertibility

Not | We introduce the notation $\phi_J^A \equiv \langle \hat{\phi}^A \rangle^J = \frac{\delta W}{\delta J^A}$

→ One assume that this relation can be inverted to give J_A as a function of ϕ^A:

$$\forall \phi^A, \ \exists! \, J_A^\phi \ / \ \phi^A = \left.\frac{\delta W}{\delta J^A}\right|_{J = J^\phi}$$

→ The field ϕ^A is called the classical field. It's the conjugate variable of J^A.

3.2 Effective action

3.2.1 Legendre transform :

DEF The effective action $\Gamma[\phi]$ is defined as the Legendre transform of $W[J]$ with respect to J :

$$\phi^A = \frac{\delta W}{\delta J_A} \Leftrightarrow J_A = J_A^\phi, \quad \Gamma[\phi] = \left(W[J] - J_A \phi^A\right)\Big|_{J^\phi}$$

↳ Here, ϕ is not a "QFT field", it's an external source called classical field.

→ Performing a Taylor expansion, we get

$$\underset{\substack{\downarrow \\ \text{1-pt corr. fct}}}{\frac{\delta W}{\delta J_A} = 0} + \underbrace{\frac{\delta W}{\delta J_A \delta J_B}\Big|_{J=0}}_{\text{↳ propagator → invertible}} \cdot J_B + \mathcal{O}(J^2) = \phi^A_J$$

So that $J_A = 0 \Leftrightarrow \phi^A = 0$, so $\Gamma[0] = W[0] = \frac{\hbar}{i} \ln 1 = 0$

$$\rightarrow \frac{\delta \Gamma}{\delta \phi^A} = \underbrace{\frac{\delta W}{\delta J_B}\Big|_{J=J^\phi}}_{= \phi^B} \cdot \frac{\delta J_B^\phi}{\delta \phi^A} - \frac{\delta J_B^\phi}{\delta \phi^A} \cdot \phi^B - J_A^\phi = - J_A^\phi$$

so that $\delta\Gamma / \delta\phi^A |_{J=0} = 0$

↳ The name effective action ⇐ the EOM of the classical action with sources $S[\phi] + J_A \phi^A$ are the same. Notice that the Taylor expansion of $\Gamma[\phi]$ start at $\mathcal{O}(2)$

$$\rightarrow \Gamma[\phi] = \Gamma[0] + \phi^A \cdot \frac{\delta\Gamma}{\delta\phi^A}\Big|_{\phi=0} + \frac{1}{2}\phi^A\phi^B \cdot \underbrace{\frac{\delta^2\Gamma}{\delta\phi^A\delta\phi^B}\Big|_{\phi=0}}_{\equiv \Gamma_{AB}} + \underbrace{\sum_{k\geq 3} \frac{1}{k!} \phi^{A_1} \cdots \phi^{A_k} \frac{\delta^k \Gamma}{\delta\phi^{A_1} \cdots \delta\phi^{A_k}}}_{\equiv \hat{\Gamma}[\phi]}$$

$$\Gamma[\phi] = \frac{1}{2}\phi^A\phi^B \Gamma_{AB} + \hat{\Gamma}[\phi]$$

PROP

$$\left(\frac{\delta^2\Gamma}{\delta\phi^A\delta\phi^B}\right)^{-1}\Bigg|_{\phi=0} = \frac{-i}{\hbar} \langle \hat{\phi}^A \hat{\phi}^B \rangle_c^{J^\phi}$$

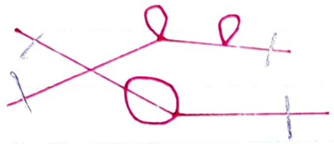
$\boxed{\text{DEMO}}$ → $\frac{\delta}{\delta J^A}\Big|_{J^\phi} = \frac{\delta \phi_J^B}{\delta J_A}\Big|_{J^\phi} \cdot \frac{\delta}{\delta \phi^B}$ and $\frac{\delta \phi_J^B}{\delta J_C}\Big|_{J^\phi} \cdot \frac{\delta J_A^\phi}{\delta \phi^B} = \delta_A{}^C$

Now, $\phi^B_J = \frac{\delta W}{\delta J}$ so that $\left(\frac{\delta^2 W}{\delta J_B \delta J_C}\right)\left(\frac{-\delta\Gamma}{\delta\phi^B\delta\phi^A}\right) = \delta_A{}^C$ ■

3.2.2 Connected diagrams and topological relations:

DEF
- An amputed diagram is a diagram from wich one has removed the external propagators
- A one-particle-irreducible (1PI) diagram is an amputed diagram that remains connected if one cuts an internal lines
- A tree diagram is a diagram without loops
- The order of a diagram is the number of external points.
- A proper vertex of order n $\Sigma_{A_1 \cdots A_n}$ is the sum over all 1PI diagrams of order n

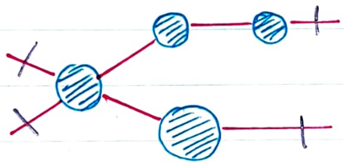
→ Ex :



→ 5 vertex
→ 4 external points
→ $g\,\phi^4/4!$ theory

PROP Every connected diagram may be decomposed into a tree diagram in which propagators connect 1PI diagrams.

→ Ex :



PROP Σ(connected diagrams) = Σ(connected tree diagram) where propagators relate (vertices of order $n \geq 2$) replaced by (proper vertices of order $n \geq 2$).

PROP In a connected diagram with L loops, V vertices and I internal lines, one has

$$L = I - V + 1$$

Or, if there are C connected components

$$\Sigma L = \Sigma I - \Sigma V + C$$

PROP For a connected Feynman diagram, the expansion in $\hbar$ is related to an expansion in the number of loops:

$$\mathcal{O}(\text{diagram}) = \hbar^{E+I-V} = \hbar^{E-1+L}$$

where E is the number of external lines.

PROP If there are N_i vertex of type i that involves n_i fields, and if there are E external lines, we have

$$2I + E = \sum_i N_i n_i$$

→ The type is for instance $\underset{n=3}{\phi^3}, \underset{n=4}{\phi^4}, \underset{n=3}{\bar{\psi}\psi\phi}$, the polynomiality of the field

3.2.3 Complete propagator and proper vertex of order 2 :

PROP For a connected diagram, we have, using $\sum_i N_i = V$, that

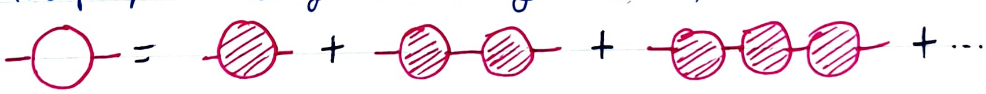
$$E - 2 = \sum_i N_i (n_i - 2) - 2L$$

→ In our theory, there is no 1PI with $n_i = 1$ since $\langle \hat{\Phi}^A \rangle^0 = 0$. Instead, we can consider the sum over connected tree diagrams ($L=0$) with 2 external legs ($E=2$). Then,

$$0 = E - 2 = \sum_i N_i (n_i - 2) \Rightarrow n_i = 2 \rightsquigarrow \text{only contains proper vertex of order 2}$$

Not Denoting the 2-function $\langle \hat{\Phi}^A \hat{\Phi}^B \rangle_c^0 = \frac{\hbar}{i} \left.\frac{\delta^2 W[J]}{\delta J_A \delta J_B}\right|_{J=0}$ by 

and the proper vertex of order 2 by , we have



→ Explicitly,
$$\langle \hat{\Phi}^A \hat{\Phi}^B \rangle_c^0 = \frac{\hbar}{i}(D^{-1})^{AB} + \frac{\hbar}{i}(D^{-1})^{AC} \Sigma_{CD} \frac{\hbar}{i}(D^{-1})^{DB}$$
$$+ \frac{\hbar}{i}(D^{-1})^{AC} \Sigma_{CD} \frac{\hbar}{i}(D^{-1})^{DE} \Sigma_{EF} \frac{\hbar}{i}(D^{-1})^{FB} + \dots$$

$1 + x + x^2 + \dots = \frac{1}{1-x}$

$$= \frac{\hbar}{i}(D^{-1})^{AC}\left(\delta_C^B - \Sigma_{CD} \frac{\hbar}{i}(D^{-1})^{DB}\right)^{-1}$$

PROP We have $\Gamma_{AB} = -D_{AB} + \frac{\hbar}{i}\Sigma_{AB}$

Corr $\sum$(connected diagrams) $= \sum$(connected tree diagrams) with (propagator replaced by complete propagators) and (vertices of order $n \geq 3$ replaced by proper vertex of order $n \geq 3$)

3.2.4 Semi-classical expansion of the effective action:

We now use the path integral representation for Green's function and we expand around classical solution in the presence of a source

$\rightarrow$ $\exp\{\frac{i}{\hbar} W[J]\} = \mathcal{N}^{-1} \int \mathcal{D}\phi \exp\{\frac{i}{\hbar} S[\phi] + J_A \phi^A\}$

with $\mathcal{N} = \int \mathcal{D}\phi \exp\{i/\hbar . S[\phi]\}$

DEF We denote ϕ_0^J the unique classical solution in the presence of a source: $\frac{\delta S_A}{\delta \phi^A} + J_A = 0$

$\rightarrow$ Consider the $g/3!\ \phi^3$ theory:

$$S[\phi] = -\frac{1}{2} \phi^A D_{AB} \phi^B - V[\phi] = -\frac{1}{2} \int d^n x\, d^n x'\ D(x,x') \phi(x) \phi(x') - \frac{g}{3!} \int d^n x\ V[\phi]$$

$$= -\frac{1}{2} \int d^n x \{ \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 + \frac{g}{3!} \phi^3 \}$$

$D(x,x') = (\partial_\mu \partial^\mu_{x'} + m^2) \delta^n(x,x')$ $+ i\epsilon$

Recall that $\delta^n(x,x') = \int \frac{d^n p}{(2\pi)^n} e^{ip\cdot(x-x')}$ and $\tilde{D}(p) = p^2 + m^2 + i\epsilon$

$\rightarrow$ The classical solution ϕ_0^J is determined by the source. Writing

$J(x) = \int d^n y \{ D(x,y) \phi(y) + g/2 . \phi^2(x) \}$

$\hookrightarrow$ Without interaction (for $g=0$), one has

$$\phi(x) \overset{\circledast}{=} \int d^n y\ D^{-1}(x,y)\ J(y)$$

$\hookrightarrow$ Turning the interaction on, we have:

$$\phi(x) = \int d^n y\ D^{-1}(x,y)\ J(y) - \int d^n y\ D^{-1}(x,y) \frac{g}{2} \phi^2(y) \Big|_{\phi = (\circledast + \text{interaction})}$$

$$= \int d^n y\ D^{-1}(x,y)\ J(y) - \int d^n y\ D^{-1}(x,y) . \frac{g}{2} \Big(\int d^n z\ D^{-1}(y,z)\ J(z) - \int d^n z\ D^{-1}(y,z) \frac{g}{2} \phi^2(z) \Big)^2 = \ldots = \phi_0^J$$

$\rightarrow$ The solution $\phi_0^J(x)$ is a unique as a series in J.

$\rightarrow$ Each term of ϕ_0^J is invertible $\Rightarrow$ ϕ_0^J is perturbatively invertible.

→ Let's perform a perturbative expansion around the classical solution $\phi^A = \phi_0^{AJ} + \varphi^A$ (where $\delta S/\delta\phi_0^{AJ} + J_A = 0$) :

$$\exp\left\{\frac{i}{\hbar} W[J]\right\} = N^{-1} \exp\left\{\frac{i}{\hbar}\left(S[\phi_0^J] + J_A \phi_0^{JA}\right)\right\} \int \mathcal{D}\varphi \, \exp\left\{\frac{i}{\hbar}\left(\frac{1}{2} \frac{\delta^2 S}{\delta\phi^A \delta\phi^B}\Big|_{\phi_0^J} \varphi_A \varphi_B + \dots\right)\right\}$$

Indeed, recall that the term linear in φ^A vanishes on account of the definition of ϕ_0^J : $(\delta S/\delta\phi + J_A^A)|_{\phi_0^{AJ}} \cdot \varphi = 0$

→ Consider for instance

$$S = \int d^n x \left(-\frac{1}{2} \partial_\mu \phi^A \partial^\mu \phi_A - \frac{1}{2} m^2 \phi^A \phi_A - V[\phi]\right)$$

The new quadratic part in φ^A is :

$$S^{(2)} = \int d^n x \left(-\frac{1}{2} \partial_\mu \varphi^A \partial^\mu \varphi_A - \frac{1}{2} m^2 \varphi^A \varphi_A - \frac{\partial^2 V}{\partial\phi^A \partial\phi^B}\Big|_{\phi_0^J} \varphi^A \varphi^B\right)$$

$$\rightarrow \frac{\delta^2 S}{\delta\phi^A \delta\phi^B}\Big|_{\phi_0^J} = -D_{AB} - V''_{AB}[\phi_0^J]$$

→ Let $\varphi^A \mapsto \sqrt{\hbar}\,\varphi^A$ so that

$$\exp\left\{\frac{i}{\hbar} W[J]\right\} = N'^{-1} \exp\left\{\frac{i}{\hbar}\left(S[\phi_0^J] + J_A \phi_0^{AJ}\right)\right\} \int \mathcal{D}\varphi \, \exp\left\{\frac{i}{2} \frac{\delta^2 S}{\delta\phi^A \delta\phi^B}\Big|_{\phi_0^J} \varphi^A \varphi^B + \mathcal{O}(\hbar)\right\}$$

$$= N''^{-1} \exp\left\{\frac{i}{\hbar}\left(S[\phi_0^J] + J_A \phi_0^{AJ}\right)\right\} \frac{1}{\sqrt{\mathrm{Det}\left[-i \frac{\delta^2 S}{\delta\phi^A \delta\phi^B}\Big|_{\phi_0^J}\right]}} + \mathcal{O}(\hbar)$$

? where N'' is set such that $\exp\left\{\frac{i}{\hbar} W[0]\right\} = 1 \Leftrightarrow W[0] = 0 \Leftrightarrow \phi_0^0 = 0$

↳ We set $N'' = \mathrm{Det}\left[-i \frac{\delta^2 S}{\delta\phi^A \delta\phi^B}\Big|_{\phi_0^0 = 0}\right]^{-1/2} = \mathrm{Det}\left[-i\, D_{AB}[0]\right]^{-1/2}$

Since $(\mathrm{Det}\, A / \mathrm{Det}\, B)^{-1/2} = \mathrm{Det}(B^{-1} A)^{-1/2}$, we find

$$\exp\left\{\frac{i}{\hbar} W[J]\right\} = \exp\left\{\frac{i}{\hbar}\left(S[\phi_0^J] + J_A \phi_0^{JA}\right)\right\} \cdot \underbrace{\left(\mathrm{Det}\left[-(D^{-1})^{AB}(-D_{BC} - V''_{BC})\right]\right)^{-1/2}}_{\left(\mathrm{Det}\left[\delta^A{}_C + (D^{-1})^{AB} V''_{BC}[\phi_0^J]\right]\right)^{-1/2}}$$

Using $\mathrm{Det}\, A = e^{\mathrm{Tr}[\ln A]}$, we get

$$W[J] = S[\phi_0^J] + J_A \phi_0^{AJ} - \frac{\hbar}{2i} \mathrm{Tr}\left[\ln\left\{\delta^A{}_C + (D^{-1})^{AB} V''_{BC}[\phi_0^J]\right\}\right] + \mathcal{O}(\hbar^2)$$

→ For fermions, we have $-\hbar/2i \mapsto +\hbar/i$

→ For complex boson : $-\hbar/2i \mapsto -\hbar/i$

→ We now have our expansion around the classical action. To get the effective action, one needs to perform a Legendre transform.

$W = S + J_A\phi^A$

→ The classical field is defined as

$$\phi^{JA} \equiv \frac{\delta W}{\delta J^A} = \frac{\delta S}{\delta \phi^B}\Big|_{\phi_0^J} \frac{\delta \phi_0^{JB}}{\delta J_A} + \phi_0^{JA} + J_B \frac{\delta \phi_0^{JB}}{\delta J_A} + \mathcal{O}(\hbar) = \phi_0^{JA} + \mathcal{O}(\hbar)$$

$$\Leftrightarrow \phi^A = \frac{\delta W}{\delta J_A}\Big|_{J^\phi} = \phi_0^{J\phi A} + \mathcal{O}(\hbar)$$ the classical field $\overset{!}{=}$ independent variable

PROP | We find $\Gamma[\phi] = S[\phi] - \frac{\hbar}{2i} \mathrm{Tr}\left[\ln\left\{\delta^A{}_C + (D^{-1})^{AB} V''_{BC}[\phi]\right\}\right] + \mathcal{O}(\hbar^2)$

$\boxed{\text{DEMO}}$ Indeed, starting from the definition of $\Gamma[\phi] \equiv (W[J] - J_A\phi^A)|_{J=J^\phi}$;

$$\Gamma[\phi] = W[J^\phi] - J_A^\phi \phi^A = S[\phi_0^{J\phi}] + J_A^\phi \phi_0^{J\phi A} - \frac{\hbar}{i}\mathrm{Tr}[\cdots] - J_A^\phi \phi^A + \mathcal{O}(\hbar^2)$$

$$= S[\phi + (\phi_0^{J\phi} - \phi)] + J_A^\phi(\phi_0^{J\phi A} - \phi^A) - \frac{\hbar}{i}\mathrm{Tr}[\cdots] + \mathcal{O}(\hbar^2)$$

$$= S[\phi] + \left(\frac{\delta S}{\delta \phi^A_0}[\phi] + J_A^\phi\right)\underbrace{(\phi_0^{JA} - \phi^A)}_{\mathcal{O}(\hbar)} - \frac{\hbar}{2i}\mathrm{Tr}\ln(\delta + D^{-1}V''[\phi + \underbrace{(\phi^{J\phi} - \phi)}_{\mathcal{O}(\hbar)}])$$

$$= S[\phi] - \frac{\hbar}{2i}\mathrm{tr}\ln(\delta^A{}_C + D^{-1\,AB} V''_{BC}[\phi]) + \mathcal{O}(\hbar^2)$$

3.2.5 Effective action as generating functional for proper vertex:

PROP | ⓐ Connected Green's functions may be computed using $S[\phi] \mapsto \Gamma[\phi]$ in order to derive the Feynman rules and by summing only over connected tree diagrams

ⓑ The generating functional for proper vertex of order bigger than 3, is $\frac{i}{\hbar}\hat{\Gamma}[\phi]$. We have $\frac{i}{\hbar}\Gamma_{A_1\cdots A_k} = \Sigma_{A_1\cdots A_k}$, $k \geq 3$

→ Let's compute the generative functional for connected Green's function $W_\Gamma[J;g]$ computed with $\Gamma[\phi]$ (not $S[\phi]$) and $\hbar \mapsto g\hbar$:

$$\exp\left\{\frac{i}{g\hbar} W_\Gamma[J;g]\right\} = N^{-1}\int D\phi \exp\left\{\frac{i}{\hbar g}(\Gamma[\phi] + J_A\phi^A)\right\}$$

→ As before, writing ϕ_Γ^J such that $\frac{\delta\Gamma}{\delta\phi^A}[\phi_\Gamma^J] + J_A = 0$, we have

$$W_\Gamma[J;g] = \Gamma[\phi_\Gamma^J] + J_A\phi_\Gamma^{JA} + \mathcal{O}(g)$$

We can invert the Legendre transform. Writing $\Gamma[\phi] = (W[J] - J\phi)|_{J^\phi}$, we get $W[J] = (\Gamma[\phi] + J_A\phi^A)|_{\phi=\phi^J}$ with $-J_A = \frac{\delta\Gamma[\phi]}{\delta\phi^A}\Big|_{\phi^J}$

? These same relations are satisfied by $W_\Gamma[J;0]$ which implies that

$$W[J] = W_\Gamma[J;0]$$

→ The green's functions are:

$$\langle \hat{\phi}^{A_1} \dots \hat{\phi}^{A_k} \rangle_{c,\Gamma} = \left(\frac{g\hbar}{i}\right)^k \frac{\delta^k \frac{i}{g\hbar} W_\Gamma[J;g]}{\delta J_{A_1} \cdots \delta J_{A_k}}\bigg|_{J=0}$$

$$= g^{k-1} \left(\frac{\hbar}{i}\right)^k \frac{\delta^k \frac{i}{\hbar} W_\Gamma[J;g]}{\delta J_{A_1} \cdots \delta J_{A_k}}\bigg|_{J=0}$$

↳ Before: order of a diagram $= \hbar^{E-1+L}$

↳ Now: order in $g = g^{k-1+L}$ where k external propagator

→ this mean that

$$g^{1-k} \langle \hat{\phi}^{A_1} \dots \hat{\phi}^{A_k} \rangle_{c,\Gamma} = \sum_{L=0} g^L \left(\text{connected diagrams with } L \text{ loops computed with } \Gamma[\phi]\right)$$

$$= \left(\frac{\hbar}{i}\right)^k \frac{\delta^k (i/\hbar) W_\Gamma[J;g]}{\delta J_{A_1} \cdots \delta A_k}\bigg|_{J=0}$$

↳ Setting $g=0$, we proved ⓐ.

→ We had that $\sum(\langle \dots \rangle_c) = \sum \begin{pmatrix} \text{connected tree, with prop} \mapsto \text{complete} \\ \text{prop, and vertices } n \geq 3 \mapsto \text{proper vertices} \end{pmatrix}$

On the other hand, $\sum(\langle \dots \rangle_c) = \sum(\text{connected tree} \Leftarrow \Gamma[\phi])$

? Since prop. are determined by $(\Gamma_{AB})^{-1} = \left(\frac{\delta^2 \Gamma}{\delta \phi^A \delta \phi^B}\right)^{-1}$, but we have shown that $\frac{\delta^2 W}{\delta J_A \delta J_B} = \frac{\hbar}{i} \langle \hat{\phi}^A \hat{\phi}^B \rangle_c$, the complete propagator.

↳ Since vertices are determined by $\frac{i}{\hbar} \hat{\Gamma}[\phi]$, we thus have shown ⓑ