Problem Set 2: GWs from Eccentric Binaries **PHYS-F484**

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Gravitational waves (GWs) from binary systems provide crucial insights into the physics of compact objects, such as neutron stars and black holes. In the previous session, we focused on quasi-circular inspirals, where the binary system evolves under the influence of gravitational radiation. However, many astrophysical binaries exhibit significant eccentricity, particularly in their early inspiral phases or if formed dynamically in dense stellar environments.

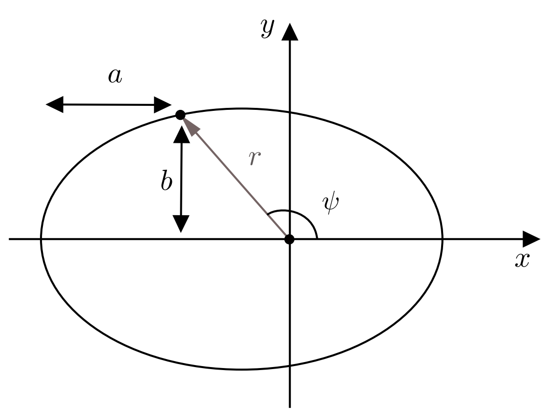
We will extend our analysis to eccentric binary systems. Specifically, we will derive the evolution of the orbital radius under the influence of gravitational radiation, compute the radiated power, and investigate the backreaction effects on elliptical orbits. We will compare our theoretical results with observational data from the Hulse-Taylor pulsar (PSR B1913+16), which provides one of the most well-studied examples of an eccentric compact binary system undergoing orbital decay due to gravitational wave emission. Moreover, we will find out that the quasi-circular approximation seen during the first exercise session is indeed relevant to consider.

1 Elliptic Orbits

In a Newtonian two-body motion, one can identify two conserved quantities: the energy and the angular momentum, respectively given by

$$\begin{cases} E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\psi}^2\right) - G\dfrac{\mu m}{r}, \\ L = \mu r^2\dot{\psi}, \end{cases} \tag{1}$$

with μ the reduced mass, ψ the angular position along the orbit and r measures the distance between the center of mass and the position on the orbit. The conservation of $\mathbf{L}$ implies that the orbit lies in a plane, described in polar coordinates through (r, ψ).



These yield differential equations that allow us to determine $r(\psi)$ in the polar plane in which the motion is confined. We have

$$r(\psi) = \frac{R}{1 + e\cos\psi}, \tag{2}$$

with the length scale of the system

$$R = \frac{L^2}{Gm\mu}, \tag{3}$$

and the eccentricity

$$e^2 = 1 + \frac{2EL^2}{G^2 m^2 \mu^3}, \tag{4}$$

such that $0 \leq e < 1$. The lower bond describing a circle and the upper one a parabola. Let us also remind ourselves that the semi-major and the semi-minor axis satisfy

$$a = \frac{R}{1 - e^2} \quad \text{and} \quad b = \frac{R}{(1 - e^2)^{1/2}}. \tag{5}$$

Problem 1.1. ***Elliptic Keplerian orbits***

1. *Show that one can express the semi-major axis in terms of the energy as*

$$a = \frac{Gm\mu}{2|E|}. \tag{6}$$

2. *Use Eq.* (6) *to rewrite the Eq.* (2) *in terms of* a *and* e.

3. *Compute the associated mass moments. Hint: remember that, for two point-like masses, we have*

$$M^{ij} = m x^i_{\mathrm{CM}} x^j_{\mathrm{CM}} + \mu x^i_0 x^j_0. \tag{7}$$

We are now ready to compute the total power radiated in GWs, integrating over all frequencies and over the solid angle.

Problem 1.2. ***Radiated Power***

1. *Compute the radiated power, considering the quadrupolar approximation.*

   Note that the GW energy is only well-defined by taking a temporal average over several periods of the wave. Hence, a well-defined quantity is the average of $P(\psi(t))$ over one period T.

2. *Use the periodicity of the considered two-body motion to time average the obtained radiated power. You should remind yourselves that for a Keplerian motion, we can relate the frequency of the GWs to its period* T *through*

$$\frac{1}{T} = \frac{\omega_0}{2\pi} = \left(\frac{Gm}{a^3}\right)^{1/2} \frac{1}{2\pi}. \tag{8}$$

3. *What can you conclude about the behaviour of the motion with respect to the eccentricity?*

   It is convenient to introduce

$$f(e) = \frac{1}{(1-e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right). \tag{9}$$

4. *Relate the time evolution of the orbital period T with the evolution of the orbital energy E. Express it in terms of $f(e)$.* You should end up with

$$\frac{\dot{T}}{T} = -\frac{96}{5}\frac{G^{5/3}\mu m^{2/3}}{c^5}\left(\frac{T}{2\pi}\right)^{-8/3} f(e). \tag{10}$$

This formula is closely related to the work of Russell Hulse and Joseph Taylor, who discovered the first binary pulsar, PSR B1913+16. They observed a gradual decrease in the pulsar's orbital period, which matched the predictions of general relativity for energy loss due to GW emission. Their precise measurements provided the first indirect evidence for GWs, earning them the 1993 Nobel Prize in Physics.

2 Backreaction on Elliptic Orbits

We will see that a binary system in a Keplerian orbit emits energy and angular momentum. In the point-mass approximation, this loss directly affects the orbital motion, leading to gradual changes in both the semi-major axis and eccentricity. Over time, the system evolves toward the merging phase, ultimately collapsing. In this section, we examine how the orbit's shape and size change for a general elliptical orbit.

Problem 2.1. ***Evolution Equations***
We already computed the energy radiated

$$\frac{dE}{dt} = -\frac{32}{5}\frac{G^4\mu^2 m^3}{c^5 a^5}\frac{1}{(1-e^2)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right). \tag{11}$$

Let us now focus on the evolution of the angular momentum.

1. *Compute the radiated angular momentum, still in the quadrupolar approximation and averaging over one period, knowing that*

$$\frac{dL_i}{dt} = -\frac{2G}{5c^5}\varepsilon_{ijk}\langle \ddot{M}^{ka}\dddot{M}^{la}\rangle. \tag{12}$$

   You should end up with

$$\frac{dL}{dt} = -\frac{32}{5}\frac{G^{7/2}\mu^2 m^{5/2}}{c^5 a^{7/2}}\frac{1}{(1-e^2)^2}\left(1 + \frac{7}{8}e^2\right). \tag{13}$$

2. *Rewrite Eqs.* (11) *and* (13) *in terms of the semi-major axis a and the eccentricity e, using Eqs.* (6) *and* (4)*. What can you observe, i.e. what is the effect of the GWs with respect to the orbit?*

Problem 2.2. ***Orbit Circularisation***

1. *Solve differential equations obtained in Point 2. of Problem 2.1.*

   Integrating those equations numerically is fairly difficult: the natural adimensional time scale of these differential equations is very large. For typical solar mass stars, this parameter reaches $\sim 10^{12}$.

2. *Find a more clever way to express the semi-major axis in terms of the eccentricity. Conclude what is the effect of the backreaction. Discuss limits for which $e \ll 1$ and for $e \simeq 1$.*

3. *How does gravitational wave emission affect the eccentricity of a compact binary system as it evolves toward coalescence? For example, consider the Hulse-Taylor binary, characterised by*

$$a_0 \simeq 2 \times 10^9 m \quad \textit{and} \quad e_0 \simeq 0.617. \tag{14}$$

   By the time that the two stars reach a short separation a, let's say $a = \mathcal{O}(10^2 R_{\rm NS}) \simeq 10^3$km, what happens to the eccentricity?