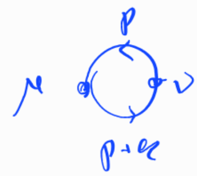
1) USE THE FEYNMAN RULES (FR) FROM SP TO CHECK THAT

$$i\Pi^{\mu\nu}(k) = -(-i)^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \frac{i}{(\not p - m)} \gamma^\nu \frac{i}{(\not p + \not k - m)}\right]$$

$=$ 

REMARK: in PS, THE COUPLING ENTER THE FR

↳ $-ie\gamma^\mu$

WE PUT INSTEAD e IN THE PROPAGATOR, SO THAT

↳ $-i\gamma^\mu$

- I SHOULD HAVE RECALLED THAT $i\Pi^{\mu\nu}(k) = \langle J^\mu(x) J^\nu(0)\rangle_{FT}$ $= \langle 0|\bar\psi(x)\gamma^\mu\psi(x)\bar\psi(0)\gamma^\nu\psi(0)|0\rangle$ is the FOURIER TRANSFORM OF EM CURRENT CORRELATOR.
- FROM THAT POINT ON, WE JUST NEED TO REMEMBER HOW TO USE F. RULES RECOGNIZING THAT THERE ARE 2 FERMION PROPAGATORS

$$\langle 0|\bar\psi(x)\gamma^\mu\psi(x)\bar\psi(0)\gamma^\nu\psi(0)|0\rangle$$

- THEN, THE FEYNMAN RULES GIVE A FACTOR OF $-i\gamma^{\mu(\nu)}$ AT EACH VERTEX, 2 PROP. AND AN OVERALL $(-)$ SIGN BECAUSE OF THE FERMION LOOP. THIS SIGN COMES FROM THE SPINOR $\psi(0)$ WHICH WE MUST BRING TO THE LEFT OF $\bar\psi(x)$ SO THAT $\underbrace{\psi(0)\bar\psi(x)}$ GIVES A FEYNMAN PROPAGATOR.
- THUS

$$i\Pi^{\mu\nu}(k) = -(-i)^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \frac{i}{\not p - m} \gamma^\nu \frac{i}{\not p + \not k - m}\right]$$

2) USE DIM. REG. TO CHECK THAT

$$i\Pi^{\mu\nu}(k) = i(k^2\eta^{\mu\nu} - k^\mu k^\nu)\,\Pi(k^2)$$

WITH $$\Pi(k^2) = -\frac{8}{16\pi^2}\int_0^1 dx\, x(1-x)\left[\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log\left(m^2 - k^2 x(1-x)\right)\right]$$

FROM (1), WE HAVE, USING THE PROPERTIES OF THE γ MATRICES,

$$i\Pi^{\mu\nu}(k) = -\int \frac{d^Dp}{(2\pi)^D}\, \frac{\mathrm{Tr}\left[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{p}+\not{k}+m)\right]}{(p^2-m^2)((p+k)^2-m^2)}$$

$$= -D\int\frac{d^Dp}{(2\pi)^D}\frac{\overbrace{\left[p^\mu(p+k)^\nu - p\cdot(p+k)\,\eta^{\mu\nu} + p^\nu(p+k)^\mu + m^2\eta^{\mu\nu}\right]}^{K^{\mu\nu}}}{(p^2-m^2)((p+k)^2-m^2)}$$

- THERE NO LINEAR TERMS IN $\not{k}, \not{p}$ BECAUSE ONLY QUADRATIC TERMS SURVIVE ($\mathrm{Tr}[\gamma^\mu\gamma^\alpha\gamma^\beta]=0$). ALL OF THIS REQUIRES SOME KNOWLEDGE OF THE PROPERTIES OF THE γ^μ...
- NEXT WE USE $$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{(Ax+B(1-x))^2}$$

  ANOTHER TRICK DUE TO FEYNMAN ...
- THEN

$$i\Pi^{\mu\nu}(k) = -D\int_0^1 dx\int\frac{d^Dp}{(2\pi)^D}\,\frac{K^{\mu\nu}}{\left((p^2-m^2)x + ((p+k)^2-m^2)(1-x)\right)^2}$$

$$= -D\int_0^1 dx\int\frac{d^Dp}{(2\pi)^D}\,\frac{K^{\mu\nu}}{\left(p^2 + 2p\cdot k\,(1-x) + k^2(1-x) - m^2\right)^2}$$

$$= -D\int_0^1 dx \int \frac{d^D p}{(2\pi)^D} \frac{K^{\mu\nu}}{\Big[\underbrace{(p+k(1-x))^2} - \underbrace{k^2(1-x)^2 + k^2(1-x)} - m^2\Big]^2}$$

$$+ \underbrace{k^2 x(1-x) - m^2}_{-M^2}$$

$$p' = p + k(1-x)$$

AND CHANGE $\int d^D p \to \int d^D p'$ (NOT OBVIOUS)

SO

$$i\Pi^{\mu\nu}(k) = -D\int_0^1 dx \int \frac{d^D p'}{(2\pi)^D} \frac{\left[(p'-k(1-x))^\mu (p'+kx)^\nu + (p'-k(1-x))^\nu (p'+kx)^\mu - (p'-k(1-x))\cdot(p'+kx)\eta^{\mu\nu} + m^2\eta^{\mu\nu}\right]}{(p'^2 - M^2)^2}$$

I DROP THE PRIMES

$$= -D\int_0^1 dx \int \frac{d^D p}{(2\pi)^D} \frac{\left[2p^\mu p^\nu - 2x(1-x)k^\mu k^\nu - (p^2 - m^2)\eta^{\mu\nu} + k^2 x(1-x)\eta^{\mu\nu}\right]}{(p^2 - M^2)^2}$$

THE TERMS LINEAR IN $p^\mu(\nu)$ DROP BECAUSE OF $\int_{-\infty}^{+\infty} d^D p \; p^\mu f(p^2) = 0$

LAST, $\int \frac{d^D p}{(2\pi)^D} p^\mu p^\nu f(p^2)$

$$= \frac{1}{D}\int \frac{d^D p}{(2\pi)^D} p^2 \eta^{\mu\nu} f(p^2)$$

THIS IS ALMOST TRANSVERSE

So

$$i\Pi^{\mu\nu}(k) = -D\int_0^1 dx \int \frac{d^D p}{(2\pi)^D} \frac{\left[\left(\frac{2}{D} - 1\right)p^2 + m^2\right]\eta^{\mu\nu} + \left[k^2\eta^{\mu\nu} - 2k^\mu k^\nu\right]x(1-x)}{p^2 - M^2}$$

THE KEY PART IS

$$\int \frac{d^D p}{(2\pi)^D} \frac{\left[\left(\frac{2}{D}-1\right)p^2 + m^2\right]}{(p^2-M^2)^2}$$

LOOKS LIKE IT IS QUADRATICALLY DIVERGENT

$$\sim \int d^4 p \frac{p^2}{(p^2-M^2)^2} \propto \Lambda^2$$

BUT DIMENSIONAL REGULARIZATION SHOWS THAT IT IS ONLY LOG-DIVERGENT $\sim \int d^4 p \frac{1}{(p^2-M^2)} \sim \log \Lambda$

FROM PESKIN-SCHROEDER APPENDIX (A.45) WE HAVE

$$\int \frac{d^D p}{(2\pi)^D} \left(\frac{2}{D}-1\right) \frac{p^2 \eta^{\mu\nu}}{(p^2-M^2)^2} = \frac{-i}{(4\pi)^{D/2}} \overbrace{\left(1-\frac{D}{2}\right)\Gamma\left(1-\frac{D}{2}\right)}^{\Gamma(2-D/2)} \left(\frac{1}{M^2}\right)^{1-\frac{D}{2}} \eta^{\mu\nu}$$

$$= \frac{-i}{(4\pi)^{D/2}} \Gamma\left(2-\frac{D}{2}\right) \left(\frac{1}{M^2}\right)^{2-D/2} \eta^{\mu\nu} M^2$$

AND

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2-M^2)^2} = \frac{i}{(4\pi)^{D/2}} \Gamma(2-D/2) \left(\frac{1}{M^2}\right)^{2-D/2}$$

SO

$$i\Pi^{\mu\nu} = -\frac{iD}{(4\pi)^{D/2}} \int_0^1 dx\, \Gamma(2-D/2) \frac{1}{(M^2)^{2-D/2}} \Big[\underbrace{(-M^2+m^2)}_{-m^2+k^2x(1-x)} \eta^{\mu\nu} + (k^2\eta^{\mu\nu} - 2k^\mu k^\nu)\, x(1-x) \Big]$$

$$= +i\,(k^2\eta^{\mu\nu} - k^\mu k^\nu)\, \Pi(k^2)$$

With $\boxed{\Pi(k^2) = -\frac{D}{(4\pi)^{D/2}} \Gamma(2-D/2) \int_0^1 dx \frac{x(1-x)}{(M^2)^{2-\frac{D}{2}}}}$

WE CAN KNOW TAKE $D = 4 - \varepsilon$ TO GET THE REGULARIZED POLARISATION TENSOR WITH $\Gamma(2 - D/2) = \Gamma(\varepsilon/2) = \frac{2}{\varepsilon} - \gamma + O(\varepsilon)$

With $\gamma \approx 0.5772$ is the EULER-MASCHERONI CONSTANT

$$\Pi(k^2) = -\frac{8}{16\pi^2} \left\{ \frac{2}{\varepsilon} \times \left(1 + \frac{\varepsilon}{2} \log 4\pi\right) \times \int_0^1 dx\, x(1-x)\left(1 - \frac{\varepsilon}{2} \log M^2\right) \right\}$$

$$+ O(\varepsilon^2)$$

OR $$\Pi(k^2) = -\frac{1}{2\pi^2}\left[\frac{2}{\varepsilon} + \gamma + \log 4\pi - \int_0^1 dx\, x(1-x) \log\left(m^2 - x(1-x)k^2\right)\right]$$

THE IMPLICATIONS ARE (PARTIALLY) COVERED IN THE NEXT EXERCISE

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3) THE COULOMB POTENTIAL BETWEEN TO OPPOSITE CHARGE PARTICLES (EG ELECTRON AND PROTON) IS GIVEN, AT LEADING ORDER, BY

$$V(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}} \left(-\frac{e^2}{\vec{k}^2}\right)$$

THE ELECTRIC CHARGE IS MODIFIED BY QUANTUM FLUCTUATIONS

$$e^2 \rightarrow \frac{e^2}{\left(1 - e^2 \Pi_R(-\vec{k}^2)\right)}$$

REM: $k^2 = w_k^2 - \vec{k}^2 \neq -\vec{k}^2$

WITH $\Pi_R(k^2) = \Pi(k^2) - \Pi(0)$

AND $\Pi(k^2)$ GIVEN IN 2)

SHOW THAT FOR $\vec{k}^2 \ll m^2$

$$V(\vec{x}) \simeq -\frac{\alpha}{2} - \frac{4\alpha^2}{15 m^2} \delta^3(\vec{x})$$

- MORE ATTRACTIVE AT SHORT DISTANCE!
- IMPORTANT FOR FINE STRUCTURE OF THE HYDROGEN SPECTRUM (DARWIN TERM)

WE HAVE FROM ex 2 THAT

$$\Pi_R(k^2) = -\frac{1}{2\pi^2}\left[\frac{2}{\varepsilon} + \gamma + \log(4\pi) - \int_0^1 dx\, x(1-x)\log\left(m^2 - k^2 x(1-x)\right)\right.$$

$$\left. - \left(\frac{2}{\varepsilon} + \gamma + \log(4\pi)\right) - \int_0^1 dx\, x(1-x)\log m^2\right]$$

$$= \frac{1}{2\pi^2}\int_0^1 dx\, x(1-x)\log\left(\frac{m^2}{m^2 - k^2 x(1-x)}\right)$$

THE LOGIC OF RENORMALIZATION IS THAT WE MEASURE THE ELECTRIC AT SOME SCALE (SAY $k^2 = 0$). THE LOG DEPENDENCE OF $e_R(k^2)$ EXPRESS THE FACT THAT THE MEASUREMENT WILL DIFFER AT A DIFFERENT SCALE THROUGH

$$e_R^2(k^2) = \frac{e_0^2}{1 - e_0^2\,\Pi_R(k^2)} \qquad *3$$

REM: $k^2 = 0$ IS NATURALLY THOUGHT AS

$$k^2 = \omega^2 - \vec{k}^2 = 0 - 0$$

MEANING STATIC $\Leftrightarrow \omega = 0$ AND LARGE DISTANCES

(MEANING distance $\gtrsim \frac{1}{m}$) $\Longleftrightarrow$ $|\vec{z}| \simeq 0$

The REST of the exercise is to explore this.

- FOR MORE, SEE PESKIN & SCHROEDER, CHAPTER 7.5
- WE START WITH COULOMB POTENTIAL ENERGY, IN FOURIER TRANSFORM FORM

$$V(\vec{x}) = \underbrace{-}_{\text{ATTRACTIVE}} \int \frac{d^3\vec{k}}{(2\pi)^3} \, e^{i\vec{k}\cdot\vec{x}} \, \frac{e_0^2}{\vec{k}^2}$$

- IF YOU ARE PUZZLED WITH THIS, CHECK THAT

$$\nabla^2 V(\vec{x}) = e_0^2 \, \delta^3(\vec{x})$$

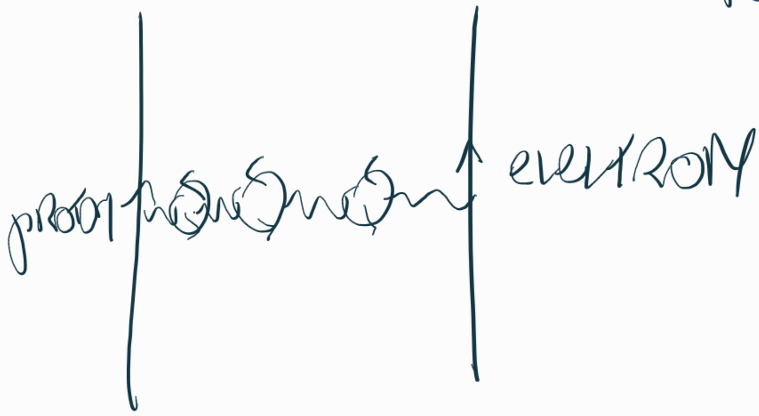
(OPPOSITE CHARGES) SAY PROTON AND ELECTRON

- THE INTEGRANT IS JUST THE ZERO FREQUENCY LIMIT OF

$$\frac{e_0^2}{\vec{k}^2} \simeq \frac{e_0^2}{\omega^2 - \vec{k}^2}$$

WHICH IS THE PROPAGATOR OF THE PHOTON. HERE $\omega \simeq 0$, SO THE PHOTON IS OFF-SHELL (ie VIRTUAL).

THIS IS THE TREE-LEVEL RESULT, BUT VACUUM POLARIZATION CHANGES THAT TO

$\sum_{0}$ 

$=$

OR

$$V_{BORN} = -e_0^2 \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2} = -\frac{e_0^2}{4\pi r}$$

$$\rightarrow V_{DRESSED} = -e_0^2 \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{k^2\left(1-e_0^2\,\Pi_R(-\vec{k}^2)\right)}$$

WITH

$$e_0^2\,\Pi_R(-\vec{k}^2) = -\frac{e_0^2}{2\pi^2}\int_0^1 dx\, x(1-x)\log\left(\frac{m^2}{m^2+\vec{k}^2 x(1-x)}\right)$$

FINE STRUCTURE CONSTANT

$$= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\log\left(\frac{1}{1+\frac{\vec{k}^2}{m^2}x(1-x)}\right)$$

LET'S US COMPUTE THIS ASSUMING THAT

$\vec{k}^2/m^2 \ll 1$

ROUGHLY, THIS MEANS

AN EXPANSION IN $\frac{m^2 N^2}{m^2} \approx N^2 \ll 1$ THAT IS NON-RELATIVISTIC

. THAT SHOULD BE RELEVANT TO THE ELECTRON IN, SAY, THE POTENTIAL OF A PROTON ~ HYDROGEN ATOM.

. THEN
$$\ln\left(\frac{1}{1+\frac{\vec{k}^2}{m^2}x(1-x)}\right) \simeq -\frac{\vec{k}^2}{m^2}x(1-x)$$

AND
$$e_0^2\, \Pi_R(-\vec{k}^2) \simeq \frac{2\alpha}{\pi}\int_0^1 dx\, x^2(1-x)^2\, \frac{\vec{k}^2}{m^2}$$
$$= \frac{2\alpha}{\pi}\cdot\frac{1}{30}\cdot\frac{\vec{k}^2}{m^2} \ll 1$$

WHICH WE CAN PUT BACK IN THE EXPRESSION OF THE POTENTIAL ENERGY
$$V(\vec{x}) = -e_0^2\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{e^{i\vec{k}\cdot\vec{x}}}{k^2\left(1-e_0^2\,\Pi_R(-\vec{k}^2)\right)}$$
$$\simeq -e_0^2\int\frac{d^3k}{(2\pi)^3}\,\frac{e^{i\vec{k}\cdot\vec{x}}}{k^2}\left(1+e_0^2\,\Pi_R(-\vec{k})\right)$$

OR

$$V(\vec{x}) = V_{\text{BORN}} - e_0^2 \frac{2\alpha}{30\pi} \int \frac{d^3k}{(2\pi)^3} \, e^{i\vec{k}\cdot\vec{x}} \frac{\cancel{\vec{k}^2}}{\cancel{\vec{k}^2}} \frac{1}{m^2}$$

$$= -\left[\frac{\alpha}{r} + \frac{4\alpha^2}{15 m^2} \delta^3(\vec{x})\right]$$

$$\downarrow$$

STRONG ATTRACTION AT THE ORIGIN

- THIS IS NOTHING BUT THE DARWIN TERM THAT LEADS TO A SHIFT OF THE S-MODES (HYPERFINE STRUCTURE) OF, SAY, THE HYDROGEN ATOM.
- ONE CAN GO FURTHER, BUT THIS IS MORE TECHNICAL, SEE PESKIN & SCHROEDER.

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4) FOR APPLICATION TO NON-ABELIAN THEORIES, CHECK FOR SU(2) THAT

- $\mathrm{Tr}\,[T^A T^B] = \frac{1}{2}\delta^{AB}$ WITH T^A GENERATORS IN THE FUNDAMENTAL REPRESENTATION

- DEFINE $(T_G^A)_{ij} = i f^{iAj}$

  AND CHECK THAT

  $$[T_G^A, T_G^B] = i f^{ABC} T_G^C$$

  FOR THIS, WRITE DOWN EXPLICITLY THE 3×3 MATRICES T_G^A

- CHECK THAT (AGAIN FOR SU(2) SO $N=2$)

  $$f^{ACD} f^{BCD} = N \delta^{AB}$$

SU(2) is THE REFERENCE WE MUST ALWAYS GO BACK in LIE GROUP / LIE ALGEBRA BUSINESS.

THEN $T^A = \frac{\sigma^A}{2}$; PAULI MATRICES

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

BUT OF COURSE WE KNOW THAT

$$\sigma^i \sigma^j = \delta^{ij} + i\,\varepsilon^{ijk}\sigma^k$$

AND $$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\,\varepsilon^{ijk}\frac{\sigma^k}{2}$$

SO

(1) $\mathrm{Tr}\left[T^A T^B\right] = \mathrm{Tr}\left[\frac{\sigma^A}{2}\frac{\sigma^B}{2}\right] = \frac{1}{2}\delta^{AB}$

- ALSO $SU(2) \sim SO(3)$ IS A WELL-KNOWN ISOMORPHISM. WE EXPLORED A BIT OF THIS LOOKING AT SPINORS BECAUSE

$$SO(3,1) \sim SU(2) \times SU(2)$$

χ_L REPS $\qquad$ χ_R REPS

BUT NEVER MIND.

- NOW, ACCORDING TO OUR LIE ALGEBRA CRASH INTRO, THE $\frac{\sigma^A}{2}$ ARE THE GENERATORS OF THE FUNDAMENTAL REPRESENTATION, FROM WHICH WE READ THE LIE ALGEBRA STRUCTURE CONSTANTS,

$$\left[T^A, T^B\right] = i f^{ABC} T^C$$

HERE $f^{ABC} = \varepsilon^{ABC}$ SIMPLY.

NOW, TAKE THE 3x3 MATRIX.

$$\left(T_G^A\right)_{ij} = i\,\varepsilon^{iAj}$$

IN EXPLICIT FORM

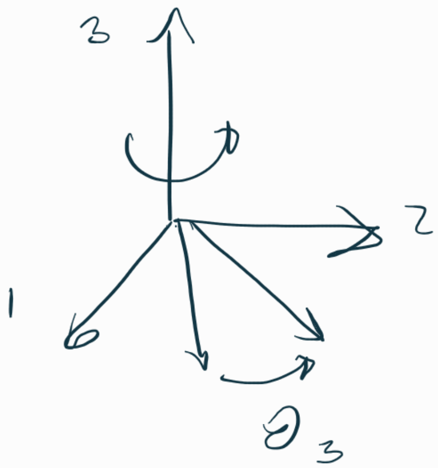
$$T_G^1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T_G^2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$T_G^3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- THE MATRIX ARE CLEARLY THE GENERATORS OF ROTATIONS AROUND AXIS LABELLED BY A.
- MORE CONCRETALLY, TAKE $A = 3$

$$R_3 = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



OR FOR $\theta \ll 1$

$$R_3 \simeq \begin{pmatrix} 1 & -\theta_3 & 0 \\ \theta_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_3 + \theta_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\equiv \mathbb{1}_3 - i\,\theta_3 T^3_G$$

- THE FUNNY FACTOR OF i (FUNNY FROM THE PERSPECTIVE OF ROTS IN $D=3$) IS BECAUSE PHYSICIST LIKE TO WORK WITH HERMITIAN GENERATORS (OBVIOUSLY, BECAUSE OF QM) (THE MATHEMATICIANS ARE PURER AND DON'T DO THAT)

- OF COURSE, IT IS EASY TO CHECK THAT

$$[T^A_G, T^B_G] = i\,\varepsilon^{ABC} T^C_G$$ AND $$\mathrm{Tr}[T^A_G T^B_G] = 2\delta^{AB}$$ N FOR SU(N)

THIS IS TRUE IN GENERAL FOR ANY LIE ALGEBRA BECAUSE, FOR ANY SQUARE MATRICES, WE HAVE THE JACOBI IDENTITY

$$[[A,B],C]+[[C,A],B]+[[B,C],A]=0$$

AND SO

$$[[T^A,T^B],T^C]+[[T^C,T^A],T^B]+[[T^B,T^C],T^A]=0$$

YOU CAN CHECK THAT THIS IMPLIES THAT

$$(T^A_G)_{ij}=i\,f^{iAj}$$

SATISFY $[T^A_G,T^B_G]=i\,f^{ABC}\,T^C_G$

AND SO DEFINES A REPRESENTATION OF A LIE GROUP CALLED THE ADJOINT. THE NAME MEANS WHAT IT MEANS. GIVEN THE STRUCTURE CONSTANT FROM THE FUNDAMENTAL REPRESENTATION, ALSO CALLED THE DEFINING REPRESENTATION

THEN WE CAN ALWAYS DEFINE THE ADJOINT (ASSOCIATED TO FUNDAMENTAL) REPRESENTATION.

THIS IS THE BEGINNINGS OF THE CLASSIFICATION OF LIE ALGEBRAS BASED ON THE REQUIRED PROPERTIES OF THE STRUCTURE CONSTANTS.

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5) RUNNING OF GAUGE COUPLINGS IN SM

FROM $\mu \frac{d}{d\mu} g = \beta(g) = \frac{b}{16\pi^2} g^3$ IN SU(N) THEORIES, WE GOT

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + \frac{2}{(4\pi)^2}\left[\frac{11}{3}N - \frac{2}{3}n_F\right]\log\frac{\mu}{\mu_0}$$

THE (1-LOOP) RUNNING OF THE GAUGE COUPLING FOR A SU(N) GAUGE THEORY COUPLED TO n_F DIRAC FERMIONS.

DEFINE $\alpha_3 = \frac{g_3^2}{4\pi}$ WITH g_3 THE COUPLING OF $SU(3)_c$

$\alpha_2 = \frac{g^2}{4\pi}$ WITH g THE " $SU(2)_L$

$\alpha_1 = \frac{5}{3}\frac{g'^2}{4\pi}$ WITH g' THE " $U(1)_Y$

(THE FUNNY 5/3 WILL BE EXPLAINED LATER)

WRITE DOWN

$$\frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_i(\mu_0)} - \frac{1}{2\pi} b_i \log\left(\frac{\mu}{\mu_0}\right)$$

AND WORK OUT THE b_i

- THIS A BIT TRICKIER THAN IT LOOKS LIKE BECAUSE WE NEED TO KEEP IN MIND THE $SU(2)_L$ AND $U(1)_Y$ GROUP COUPLE DIFFERENTLY TO L AND RIGHT FERMIONS...
- WE START WITH $SU(3)_C$.
  - THERE ARE, PER FAMILY, u-QUARKS AND d-QUARKS.
  - $SU(3)_C$ COUPLES THE SAME TO u_L AND u_R AND d_L AND d_R

$SU(3)_C$ IS A VECTOR THEORY

↳ BY OPPOSITION TO A CHIRAL THEORY LIKE $SU(2)_L$

- SO THE FORMULA APPLIES DIRECTLY WITH $N=3$ AND $n_f = 3 = N_{\text{FAMILIES}} = N_F$

$$\boxed{\frac{1}{\alpha_s(\mu)} = \frac{1}{\alpha_s(\mu_0)} + \frac{2\cdot 4\pi}{(4\pi)^2}\left[\frac{11}{3}\times 3 - \frac{2}{3}\,2\cdot N_F\right]\log\frac{\mu}{\mu_0}}$$

WITH $\alpha_s = \frac{g_3^2}{4\pi}$

UPS AND DOWNS

SO $b_3 = -\left(11 - \frac{4}{3} N_f\right) < 0$

THE FUNNY SIGN IS BECAUSE $\beta_3 \propto b_3$

AND $\beta_3 < 0$ MEANS ASYMPTOTIC FREEDOM, IE A DECREASING COUPLING WITH ENERGY.

⊕ LET US CONSIDER NOW $SU(2)_L$

NOW $N = 2$ FROM THE GAUGE PART

BUT FOR THE FERMION PART WE MUST TAKE INTO ACCOUNT THE WE HAVE FERMIONS IN THE FUNDAMENTAL OF $SU(2)$ BUT ONLY OF THE LEFT TYPE. SO THEIR CONTRIBUTION TO THE RUNNING OF

$\alpha_2 = \frac{g_2^2}{4\pi}$ IS ONLY HALF OF

WHAT WE HAVE "COMPUTED"

+ ALSO WE HAVE LEPTON DOUBLETS (N_F OF THEM) BUT $(N_C = 3) \times N_F$

QUARK DOUBLET, BECAUSE THE COME

WITH A FACTOR OF COLOR.

SO

$$\boxed{\frac{1}{\alpha_2(\mu)} = \frac{1}{\alpha_2(\mu_0)} + \frac{1}{2\pi}\left[\frac{11}{3}\times 2 - \frac{2 N_F}{3\cdot 2}\left[1+3\right]\right] \times \log\frac{\mu}{\mu_0}}$$

AND $b_2 = -\left[\frac{22}{3} - \frac{4}{3} N_F\right] < 0$

SO α_2 ALSO IS ASYMPTOTICALLY FREE, THE COUPLING DECREASES WITH ENERGY.

④ LAST WE HAVE THE $U(1)_Y$. HERE NOTHING WORKS OUT-OF-THE BOX.

THIS IS BECAUSE

① THERE IS NO INTERACTION AMONG THE $U(1)_Y$ GAUGE BOSONS (LIKE FOR THE PHOTON)

② THE NORMALIZATION OF THE

* FERMION PART IS UNCLEAR.

HOW TO DO THINGS IS EXPLAINED IN THE INTRO TO THE EXERCISE. WE CAN ALSO RECONSTRUCT THE RESULT BECAUSE WE HAVE ACTUALLY COMPUTED THE RUNNING OF THE COUPLING IN THE U(1) CASE (ALBEIT FOR QED). SO LET'S TRY

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* CLEARLY NON-ABELIAN GAUGE THEORIES ARE RIGID IN THE SENSE THAT THE NORMALIZATION OF THE CONTRIBUTION AND OF THE FERMIONS ARE RELATED

FROM *3 IN EX.3 WE HAVE

$$e^2 = \frac{e_0^2}{1 - e_0^2 \Pi_R(k^2)}$$

SO

$$\frac{1}{e^2(k^2)} = \frac{1}{e_0^2} - \Pi_R(k^2)$$

WITH

$$\Pi_R(k^2) = -\frac{1}{2\pi^2}\int_0^1 dx\; x(1-x)\log\left(\frac{m^2}{m^2 - x(1-x)k^2}\right)$$

TAKE $k^2 \gg m^2$ AND SET $k^2 = -\mu^2$

$$\Pi_R(\mu^2) \simeq +\frac{1}{2\pi^2}\int_0^1 dx\; x(1-x)\log\mu^2 + \mu^2\text{-}$$

INDEPENDENT TERMS.

$$= \frac{1}{12\pi^2}\log\mu^2 + (\mu^2\text{-INDEPENDENT TERMS})$$

SO

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha_0} - \frac{2}{3\pi}\log\frac{\mu}{\mu_0}$$

$$\boxed{b_{QED} = \frac{4}{3} \text{ AND } \beta_{QED} = \frac{e^3}{12\pi^2}}$$

- OF COURSE $\alpha(\mu) \nearrow$ WITH ENERGY AS WE HAVE LEARNED BEFORE
- NOW, THIS IS FOR QED, WITH DIRAC FERMIONS. WE NEED TO TRANSLATE THAT TO THE EVOLUTION OF THE HYPERCHARGE GAUGE FIELD.

WHAT ARE THE HYPERCHARGE OF THE SM PARTICLES?

$\times \begin{pmatrix}\nu_L\\ e_L\end{pmatrix}_{-1=Y}$ SO THAT $Q_e = -\frac{1}{2} - \frac{1}{2} = -1$

$Q_\nu = -\frac{1}{2} + \frac{1}{2} = 0$

WITH COUPLING $\left(\frac{Y}{2}\right)^2$ IN

BOTH ν_L AND e_L

BUT ONLY LEFT-HANDED, SO AN OVERALL FACTOR

OF 1/2 $\Rightarrow 2\times\frac{1}{2}\left(\frac{-1}{2}\right)^2 = 2\times\frac{1}{8} = \boxed{\frac{1}{4}}$

$\times\ (e_R)_{-2} \Rightarrow \frac{1}{2}(-1)^2 = \boxed{\frac{1}{2}}$

$\times \begin{pmatrix}u_L\\ d_L\end{pmatrix}_{Y=1/3}$ SO THAT $Q_u = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

$Q_d = -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}$

BUT 3 COLOURS SO

$\Rightarrow 2\times\frac{1}{2}\times 3\times\left(\frac{1}{6}\right)^2 = \frac{3}{36} = \boxed{\frac{1}{12}}$

$\times\ d_R\ -2/3 \Rightarrow \frac{1}{2}\times 3\times\left(-\frac{1}{3}\right)^2 = \boxed{\frac{1}{6}}$

$\times\ u_R\ 4/3 \Rightarrow \frac{1}{2}\times 3\times\left(\frac{2}{3}\right)^2 = \boxed{\frac{2}{3}}$

ALL TOGETHER

$$\frac{1}{\alpha_1(\mu)} = \frac{1}{\alpha_1(\mu_0)} - \overset{\text{QED}}{\downarrow}\!\frac{2}{3\pi}\underbrace{\left[\frac{1}{4}+\frac{1}{2}+\frac{1}{12}+\frac{1}{6}+\frac{2}{3}\right]}_{\underbrace{\frac{3+6+1+2+8}{12}}_{12} = \frac{20}{12} = \frac{5}{3}} N_F \log\frac{\mu}{\mu_0}$$

$$\boxed{\frac{3}{5}\frac{1}{\alpha_1(\mu)} = \frac{3}{5\alpha_1(\mu_0)} - \frac{2}{3\pi} N_f \log\frac{\mu}{\mu_0}}$$

↳ WITH THIS FACTOR, THIS FACTOR IS $\frac{-1}{2\pi} \times b_1$ WITH

$b_1 = \frac{4}{3}$

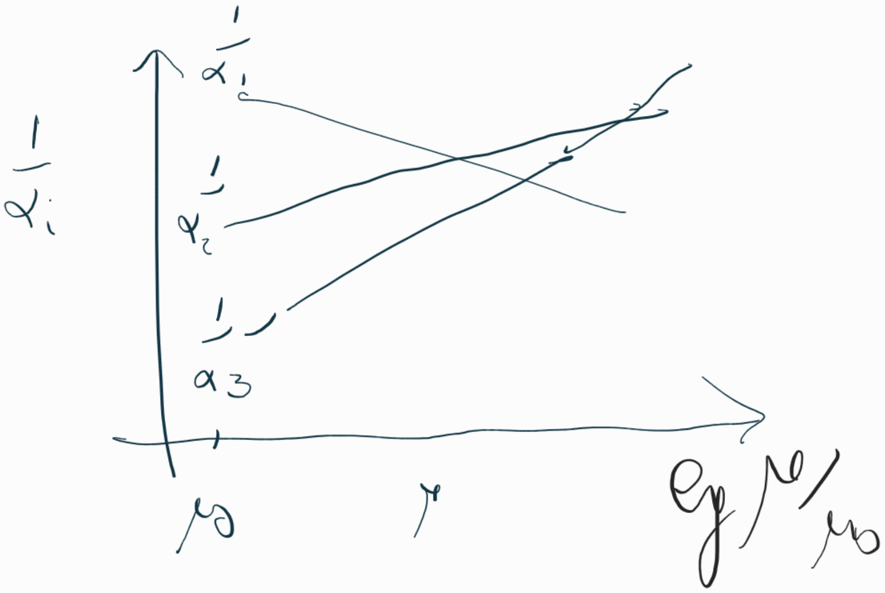
AS IN QED

SO, FINALLY,

$$\Rightarrow \begin{cases} b_3 = -11 + \frac{4}{3} N_F \\ b_2 = -\frac{22}{3} + \frac{4}{3} N_F \\ b_1 = \frac{4}{3} N_F \end{cases}$$

THERE IS A PATTERN HERE...

SCHEMATICALLY



PART 2 CHIRAL ANOMALIES

II CHIRAL ANOMALIES

1. CHECK THAT

$$\gamma^0 = \sigma_1$$
$$\gamma^1 = i\sigma_2$$

AND $\gamma_5 = -\sigma_3$

SATISFY $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

AND $\{\gamma^\mu, \gamma_5\} = 0$

IN $D=2$

2. CHECK THAT $\gamma^\mu \gamma_5 = -\varepsilon^{\mu\nu}\gamma_\nu$

$$\Rightarrow \quad \bar{\psi}\gamma^\mu\gamma_5\psi = -\varepsilon^{\mu\nu}\bar{\psi}\gamma_\nu\psi$$

IN $D=2$

① WE HAVE TO VERIFY THAT γ^0 AND γ^1 FORM A REPRESENTATION OF THE CLIFFORD ALGEBRA IN 2 DIMENSIONS

$$\gamma^0_{(2)} = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \text{LOOKS LIKE } \gamma^1_{(4)} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \text{ IN WEYL}$$

REPRESENTATION.

$$\gamma^1_{(2)} = i\sigma^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim \quad " \quad " \quad \gamma^i_{(4)} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

CHECK THAT $(\gamma^0)^2 = 1$; $(\gamma^1)^2 = -1$

AND $\{\gamma^0, \gamma^1\} = 0$

ONE MAY USE THE FOLLOWING PROPERTY OF THE PAULI MATRICES

$$\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k \quad \Rightarrow \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij}$$

SO, THE i IN FRONT OF σ^2 IS TO MAKE $(\gamma^1)^2 = \underset{\uparrow}{-}1$

- THE γS PROP. ARE ALSO OBVIOUS

REM WE CAN GO FROM $D=2$ TO $D=4$ BY THE FOLLOWING.

$$\gamma^0_{(4)} = \gamma^0_{(2)} \otimes \mathbb{1}_2 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} = \sigma^1 \otimes \mathbb{1}$$

$$\gamma^i_{(4)} = \gamma^1_{(2)} \otimes \sigma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = i\sigma^2 \otimes \sigma^i$$

INDEED
$$\{\gamma^0, \gamma^i\}_{(4)} = \{\sigma^1 \otimes \mathbb{1}, i\sigma^2 \otimes \sigma^i\}$$
$$= \sigma^1 (i\sigma^2) \otimes \sigma^i + (i\sigma^2)\sigma^1 \otimes \sigma^i$$
$$= \{\sigma^1, i\sigma^2\} \otimes \sigma^i = 0$$

AND $\{\gamma^i,\gamma^j\}_{(5\text{-}i)} = \{i\sigma^2\otimes\sigma^i, i\sigma^2\otimes\sigma^j\}$

$$= -\mathbb{1}_2\otimes\{\sigma^i,\sigma^j\}$$

$$= -\mathbb{1}_2\otimes\delta^{ij}\,\mathbb{1}_2 = -\delta^{ij}\,\mathbb{1}_2\otimes\mathbb{1}_2 = -\delta^{ij}\,\mathbb{1}_4$$

NOTICE THAT $\gamma_5\big|_{D=2} = -\sigma^3 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \gamma^0\gamma^1$

$$= \sigma^1 i\sigma^2 = i(i\sigma^3) = -\sigma^3$$

SIMILARLY $\gamma_5\big|_{D=4} = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

FROM $\gamma^i = i\sigma^2\otimes\sigma^i$ WE HAVE

$$i\gamma^0\gamma^1\gamma^2\gamma^3 = i(\sigma^1\otimes\mathbb{1})(i\sigma^2\otimes\sigma^1)(i\sigma^2\otimes\sigma^2)(i\sigma^2\otimes\sigma^3)$$

$$= ((i)^4\sigma^1(\sigma^2)^3)\otimes\sigma^1\sigma^2\sigma^3$$

$$= i\sigma^3\otimes i(\sigma^1)^2 = -\sigma^3\otimes\mathbb{1}_2 = \begin{pmatrix}-\mathbb{1}_2 & 0\\ 0 & \mathbb{1}_2\end{pmatrix}$$

AS FOR EVERY D, $[\gamma^\mu] = 2^{D/2}$, WE CAN ITERATE THIS CONSTRUCTION FROM $D=4$ TO $D=6$ TO $D=8$, ... etc., SEE eg WIKIPEDIA "HIGHER DIMENSIONAL GAMMA MATRICES"

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(2) CHECK THAT $\gamma^\mu\gamma_5 = -\varepsilon^{\mu\nu}\gamma_\nu$

$$\Rightarrow \bar\psi\gamma^\mu\gamma_5\psi = -\varepsilon^{\mu\nu}\bar\psi\gamma_\nu\psi$$

IN $D=2$

EASY BUT IMPORTANT, SINCE THEN

$$\bar{\psi}\gamma^\mu\gamma_5\psi = -\varepsilon^{\mu\nu}\bar{\psi}\gamma_\nu\psi \quad \text{in } D=2$$

REM: THIS RELATION IS SPECIFIC TO $D=2$ SINCE $\varepsilon^{\mu\nu}$ RELATES 2 CURRENTS ONLY IN $D=2$.

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③ LET $\phi \to (1+i\alpha^A T_R^A)\phi$

WITH $[T_R^A, T_R^B] = i f^{ABC} T_R^C$ (f^{ABC} REAL AND COMPLETELY ANTISYMMETRIC)

SHOW THAT THE CONJUGATE FIELD ϕ^* TRANSFORM WITH GENERATORS $T_{\bar{R}}^A = -(T_R^A)^* = -(T_R^A)^T$

SHOW THAT

$$[T_{\bar{R}}^A, T_{\bar{R}}^B] = i f^{ABC} T_{\bar{R}}^C$$

- IF $\phi \Rightarrow \phi' = (1+i\alpha^A T_R^A)\phi$

THEN $\phi^* \Rightarrow \phi'^* = (1+i\alpha^A(-T_R^{A*}))\phi^*$

NOTICE THAT $-(T_R^A)^* = -(T_R^A)^T$ SINCE $(T_R^A)^\dagger = T_R^A$

WE WRITE $T_{\bar{R}}^A = -(T_R^A)^* = -(T_R^A)^T$

CONJUGATE REPRESENTATION

- THEN $\left([T_R^A, T_R^B] = i f^{ABC} T_R^B\right)^*$

GIVES

$$[-T_R^{A*}, -T_R^{B*}] = i f^{ABC} (-T_R^{B*})$$

So $$[T_{\bar{R}}^A, T_{\bar{R}}^B] = i f^{ABC} T_{\bar{R}}^B$$

The f^{ABC} are real

So when we have T_R^A, we have a new representation by taking the complex conjugate.

Rem: Sometime, there is no much difference between T_R^A and $T_{\bar{R}}^A$

- if $T_{\bar{R}}^A = U T_R^A U^{-1}$ for some unitary matrix

then T_R^A is said to be real (or pseudoreal)

otherwise it is complex.

- An important example is $SU(2)$. As an extra exercise you can show that there is a 2×2 unitary matrix such that

$$-\sigma^{a*} = U \sigma^a U^+$$

Answer: $U = i\sigma^2$ $(U^+ = -i\sigma^2)$

$$\phi^+ T_R^A \phi$$
$$= \phi'^+ T_{\bar{R}}^A \phi'$$
with $\phi' = U\phi$

(4) CONSIDER A CHIRAL (WEYL) FERMION

χ_L OR χ_R

REMEMBER THAT UNDER A BOOST ALONG, SAY x, THEY

THEY TRANSFORM AS

$$\chi_L \to e^{-\eta/2\,\sigma^1}\chi_L$$

$$\chi_R \to e^{\eta/2\,\sigma^1}\chi_R$$

SO AS INEQUIVALENT REPRESENTATIONS OF LORENTZ

SHOW THAT THE CONJUGATES TRANSFORM AS

$$\sigma^2\chi_R^* \to e^{-\eta/2\,\sigma^1}\sigma^2\chi_R^*$$

AND

$$-\sigma^2\chi_L^* \to e^{\eta/2\,\sigma^1}(-\sigma^2\chi_L^*)$$

(THE MINUS SIGN IS FOR LATER CONVENIENCE)

SO $\sigma^2\chi_R^* \sim \chi_L$

AND $-\sigma^2\chi_L^* \sim \chi_R$

SHOW THAT THIS CAN BE WRITTEN AS

$$\psi = \begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix} \to \psi_c = \gamma^2\psi^* = \gamma^2\gamma^0\bar{\psi}^T$$

FOR A DIRAC SPINORS. ψ_c IS THE CHARGE CONJUGATE OF ψ

THIS USES THE RELATION $\sigma^{a*} = -\sigma^2\sigma^a\sigma^2$

OR $\sigma^a = -\sigma^2\sigma^{a*}\sigma^2$

(cf PREVIOUS EXERCISE)

THEN IF $\chi_R \rightarrow e^{\eta/2 \sigma_1} \chi_R$ UNDER A BOOST ALONG DIRECTION $x' = x$

THEN $\sigma^2 \chi_R^* \rightarrow \sigma^2 \left(e^{\eta/2 \sigma_1^*}\right) \chi_R^*$

$$= \sigma^2 \left(e^{\eta/2 \sigma_1^*}\right) \sigma_2^2 \chi_R^*$$

$$= \left[\sigma^2 \left(e^{\eta/2 \sigma_1^*}\right) \sigma_2\right] \sigma_2 \chi_R^*$$

$$= e^{-\eta/2 \sigma_1} \sigma_2 \chi_R^* \sim \text{TRANSFORMS LIKE A } \chi_L$$

SO, FROM A RIGHT-HANDED SPINOR, WE CAN INSTEAD TREAT IT AS A LEFT-HANDED ONE BY COMPLEX CONJUGATING. THIS OF COURSE MEANS THAT WE EXCHANGE THE RÔLE OF PARTICLES AND ANTIPARTICLES.

THE FUNNY SIGN BETWEEN $\chi_R^c = \sigma_2 \chi_R^* \sim \chi_L'$

AND $\chi_L^c = -\sigma_2 \chi_L^* \sim \chi_R'$

IS CONVENTIONAL BUT IS HANDY BECAUSE

$$(\chi_R^c)^c = -\sigma_2 (\sigma_2 \chi_R^*)^* = -\sigma_2 (\sigma^{2*} \chi_R)$$

$$= \sigma_2^2 \chi_R = \chi_R$$

AND THE SAME FOR $(\chi_L^c)^c = \chi_L$

. ALSO, WE CAN THEN WRITE

$$\psi^c = \gamma^2 \psi^* = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \chi_L^* \\ \chi_R^* \end{pmatrix} = \begin{pmatrix} \sigma^2 \chi_R^* \\ -\sigma^2 \chi_L^* \end{pmatrix}$$

④ CONSIDER $\chi_R^\dagger\, i\sigma^\mu(\partial_\mu - ig A_\mu^A T_R^A)\chi_R$

WITH χ_R TRANSFORMING ACCORDING TO REPRESENTATION R UNDER SOME GAUGE GROUP.

SHOW THAT THIS CAN REWRITTEN AS

$$\chi_R^\dagger\, i\sigma^\mu(\partial_\mu - ig A_\mu^A T_R^A)\chi_R \equiv i\,\psi_L^\dagger \bar\sigma^\mu(\partial_\mu - ig A_\mu^A T_{\bar R}^A)\psi_L$$

WITH $\psi_L = \sigma^2 \chi_R^*$

HINTS: . TAKE THE COMPLEX CONJUGATE OF THE LHS (OR THE TRANSPOSE)

. USE THAT $(\sigma^2)^2 = \mathbb{1}_2$

. TREAT THE χ_R AS <u>ANTICOMMUTING</u> FIELDS !!!

$$\Rightarrow \chi_{R_i}\chi_{R_j}^* = -\chi_{R_j}^*\chi_{R_i}$$

WITH $i=1,2$. THE WEYL INDICES

THIS IS IMPORTANT.

$$\mathcal{L}_R = i\,\chi_R^\dagger \sigma^\mu(\partial_\mu - ig T_r^A A_\mu^A)\chi_R$$

I WRITE r (A REPRESENTATION) INSTEAD OF R (RIGHT) TO AVOID CONFUSING

WE HAVE A R-HANDED SPINOR COUPLED GAUGE FIELD A_μ^A WITH χ_R IN THE REPRESENTATION T_r^A (SOME $A=1\ldots n$ HERMITIAN MATRICES)

(NOTE THAT $\mathcal{L} = \mathcal{L}^\dagger$ BY CONSTRUCTION)

↳ UP TO TOTAL DERIVATIVE)

LET TRY TO REWRITE THIS IN TERMS OF $\sigma^2\chi_R^*$. FOR THIS WE EXCHANGE

$\chi_R^\dagger$ AND χ_R

$$\mathcal{L} = i\, \chi_R^\dagger \sigma^\mu (\partial_\mu - i g T_r^A A_\mu^A)\, \chi_R$$

$$= -i \partial_\mu \chi_R^T \sigma^{\mu T} \chi_R^* - g\, \chi_R^T \sigma^{\mu T} (T_r^A)^T \chi_R^* A_\mu^A$$

$\Rightarrow$ WE TRANSPOSED THE SPINORS, BUT TAKING INTO ACCOUNT THAT THEY ANTICOMMUTE.

$$= + i\, \chi_R^T \sigma^{\mu T} \partial_\mu \chi_R^* + g\, \chi_R^T \sigma^{\mu T} \underbrace{T_{\bar r}^A}_{\text{CONJUGATE REP.}} \chi_R^* A_\mu^a$$

$$= i\, \chi_R^T \sigma_2 \underbrace{(\sigma_2 \sigma^{\mu T} \sigma_2)} \partial_\mu \sigma_2 \chi_R^* + g\, \chi_R^T \sigma_2 (\sigma_2 \sigma^{\mu T} \sigma_2) T_{\bar r}^A \sigma^2 \chi_R^* A_\mu^a$$

NOW THIS GIVES $\sigma_2 \sigma^{0T} \sigma_2 = \sigma^0 = \mathbb{1}_2$

$$\sigma_2 \sigma^{iT} \sigma_2 = \begin{cases} -\sigma_2 & \text{if } i = 2 \\ -\sigma_1 & \text{if } i = 1 \\ -\sigma_3 & \text{if } i = 3 \end{cases}$$

SO $\sigma_2 \sigma^{\mu T} \sigma_2 = \bar\sigma^\mu = (1, -\sigma^i)$

So $\mathcal{L}_R = i\, \chi_L'^\dagger \bar\sigma^\mu \partial_\mu \chi_L' + g\, \chi_L'^\dagger \bar\sigma^\mu T_{\bar r}^A A_\mu^A \chi_L'$

With $\chi_L' = \sigma_2 \chi_R^*$ (THE PRIME IS TO INSIST THAT IT IS NOT χ_L BUT $\sigma_2 \chi_R^*$)

$$\chi_R^T \sigma_2 = \chi_L'^\dagger$$

SO WE HAVE MANAGED TO REWRITE $\mathcal{L}_R$ FOR WEYL R-HANDED SPINORS IN A REPRESENTATION r AS $\mathcal{L}$ FOR A WEYL L-HANDED IN REP. $\bar r$ (CONJUGATE)

Rem: IF WE HAD DISCUSSED AN ABELIAN GAUGE FIELD WE WOULD HAVE CHANGED THE SIGN OF THE COUPLING $\Rightarrow$ PARTICLE $\Leftrightarrow$ ANTIPARTICLE