CH2 PHYSICS WITHOUT TIME

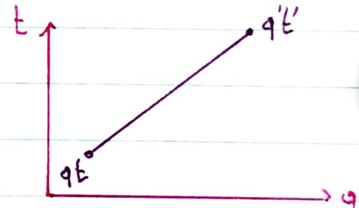
2.1 Hamilton function

→ Let a system be described by a configuration variable $q \in \mathcal{C}$ where $\mathcal{C}$ is the configuration space. The evolution of the system is given by $q(t)$, functions $\mathbb{R} \to \mathcal{C}$. The physical motions $q(t)$ are determined by a lagrangian $\mathcal{L}(q, \dot{q})$ where $\dot{q} = dq/dt$, as the ones that minimize the action $S[q] = \int dt\, \mathcal{L}(q(t), \dot{q}(t))$, a functional of $q(t)$.

→ Let us denote $q_{qt,q't'}(t)$ as the physical motion that starts at (q,t) and ends at (q',t'). That is, a function of time that solves the EOM and such that $q_{qt,q',t'}(t) = q$ and $q_{qt,q't'}(t') = q'$

DEF The Hamilton function $S(q,t,q',t)$ is defined by
$$S(q,t,q',t') \equiv \int_t^{t'} dt\, \mathcal{L}\left(q_{qt,q't'}(t), \dot{q}_{qt,q't'}(t)\right)$$

⊙ Example: free particle

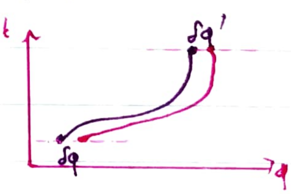


→ The action is $S[q] = \int dt\, m\dot{q}^2/2$

The EOM are $v \equiv \dot{q} = q'-q / t'-t$. The Hamilton function is then $S(qtq't') = \int_t^{t'} dt\, \frac{1}{2} m \left(\frac{q'-q}{t'-t}\right)^2 = \frac{m(q'-q)^2}{2(t'-t)}$

⊙ Properties of the Hamilton function

PROP $\frac{\partial S(qtq't')}{\partial q} = -p(qtq't')$ and $\frac{\partial S(qtq't')}{\partial q'} = p'(qtq't')$

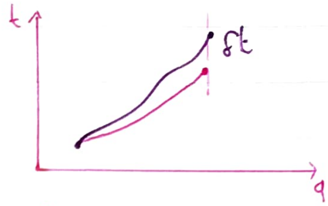


DEMO Let vary q, keeping t fixed: $q(t) \mapsto q(t) + \delta q(t)$:
$$\delta S = \int_t^{t'} dt \underbrace{\left(\frac{\partial \mathcal{L}}{\partial q}\delta q - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} \cdot \delta q\right)}_{=0 \text{ on the EOM}} + \left|\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q\right|_t^{t'}$$
$$= p'\delta q' - p\,\delta q \overset{!}{=} 0$$

■

→ Inverting $p(q,t,q',t')$, we get $q'(t'; q,p,t)$: the final position as a function the time t' and the initial data q, p, t.

PROP $\dfrac{\partial S(q,t,q',t')}{\partial t} = E(q,t,q',t')$ $\qquad$ $\dfrac{\partial S(q,t,q',t')}{\partial t'} = -E'(q,t,q',t')$



DEMO Let see how S vary wrt t':

? $\delta S = \delta \int_t^{t'} dt\, \mathcal{L} = \mathcal{L}\big|_{t'} \delta t' - \dfrac{\partial \mathcal{L}}{\partial \dot{q}}\Big|_{t'} \dot{q}' \delta t' = -(p'\dot{q}' - \mathcal{L}(t'))\, dt'$

$= -E'\delta t'$. Thus, $\vec{\nabla}_{(q',t')} S = (p', -E')$ and $-\vec{\nabla}_{(q,t)} S = (p, -E)$ ■

PROP The Hamilton function is a solution of the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0$$

⊙ Example: free particle

From $S(qtq't) = \frac{m}{2}(q'-q)^2/(t'-t)$, we see that:

$\partial_q S = -m \dfrac{q'-q}{t'-t} = -mv = -p$ and $\partial_t S = +\frac{m}{2} v^2 = E$

↳ Inverting the first relation:

$q' = q + p(t'-t)/m$ and the Hamilton-Jacobi equation is:

$$\frac{\partial S}{\partial t} = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2$$

⊙ Boundary terms:

→ We need to watch out for boundary terms when dealing with action with 2^{nd} derivative (ex: $S[q] = \int dt\, \frac{1}{2} m q \ddot{q}$) or in the 1^{st} order form (ex: $S[q,p] = \int dt\, (p\dot{q} - \frac{1}{2m}p^2)$). Indeed, the EOM might be the same, but the value of the action on them might differ. The "right" action is determined by the quantities fixed on the boundary

2.2 Transition amplitude

→ A quantum theory is defined by: a Hilbert space $\mathcal{H}$, operators $\hat{q}, \hat{p}$, a time variable t, and an hamiltonian $\hat{H}$ or the transition amplitudes it defines.

→ Let $\hat{q}$ be a set of operators that commute, complete, and whose corresponding classical variables coord. the config. space. Consider the basis where they're diagonals: $\hat{q}|q\rangle = q|q\rangle$.

DEF The transition amplitude $W(q,t,q',t')$ is defined as:
$W(q,t,q',t') = \langle q'| \exp\{\frac{-i}{\hbar} H(t'-t)\}|q\rangle$, the matrix element, of the evolution operator $U(t) \equiv e^{-\frac{i}{\hbar}Ht}$ in the $|q\rangle$ basis.

⊙ Free particle:

$$W(qtq't') = \langle q'| \exp\{\frac{-i}{\hbar}\frac{p^2}{2m}(t'-t)\}|q\rangle = \int dp \int dp' \langle q'|p'\rangle\langle p'|\exp\{\frac{-i}{\hbar}\frac{p^2}{2m}(t'-t)\}|p\rangle\langle p|q\rangle$$

$$= \frac{1}{2\pi\hbar}\int dp \exp\{\frac{i}{\hbar}p(q-q') - \frac{i}{\hbar}\frac{p^2}{2m}(t'-t)\}$$

$$= \sqrt{\frac{m}{2\pi\hbar i(t'-t)}} \exp\{\frac{i}{\hbar}\frac{m}{2}\frac{(q'-q)^2}{t'-t}\} \sim e^{\frac{i}{\hbar}S(qtq't')}$$

→ In general, $W \propto e^{\frac{i}{\hbar}S(qtq't')} + \mathcal{O}(\hbar^2)$

2.2.1 Transition amplitude as an integral over paths

→ We have $W(qtq't') = \langle q'|U(t'-t)|q\rangle = \langle q'|U(\epsilon)\cdots U(\epsilon)|q\rangle$ with $\epsilon \equiv \frac{t'-t}{N}$

We insert $\mathbb{1} = \int dq_n |q_n\rangle\langle q_n|$ between each U:

$W = \int dq_n \prod_{n=1}^{N} \langle q_n|U(\epsilon)|q_{n-1}\rangle$ and take the limit $N \to \infty$

→ We consider $H = \frac{p^2}{2m} + V(q)$. For $\epsilon \ll 1$, we have:

$$U(\epsilon) = \exp\{\frac{-i}{\hbar}(\frac{p^2}{2m}+V)\epsilon\} \simeq \exp\{\frac{-i}{\hbar}p^2/2m.\epsilon\}\exp\{\frac{-i}{\hbar}V\epsilon\}$$

We find: $\langle q_{n+1}|U(\epsilon)|q_n\rangle \sim \exp\{\frac{i}{\hbar}\left(\frac{m}{2}\frac{(q_{n+1}-q_n)^2}{(t_{n+1}-t_n)^2} - V(q_n)\right)\epsilon\}$

and

$$W = \lim_{N\to\infty} \mathcal{N} \int dq_n \exp\{\frac{i}{\hbar}\sum_{n=1}^{N}\left(\frac{m}{2}\frac{(q_{n+1}-q_n)^2}{\epsilon^2} - V(q_n)\right)\epsilon\}$$

$$\equiv \lim_{N\to\infty} \mathcal{N}\int dq_n \exp\{\frac{i}{\hbar}S_N(q_n)\} \equiv \int \mathcal{D}[q(t)] \exp\{\frac{i}{\hbar}S[q]\}$$

→ In the regime where $\hbar \to 0$, we have an oscillating integral with a small parameter in the front: dominated by the saddle-point expansion.

$$W(q,t,q',t') = \int \mathcal{D}q \exp\{\tfrac{i}{\hbar} S[q]\} \sim e^{\frac{i}{\hbar} S[q_{q,t,q',t'}]} \sim e^{\frac{i}{\hbar} S(q,t,q',t')}$$

2.2.2 General properties of the transition amplitude

→ The transition amplitude $W(q,t,q',t')$ is also as a function of q' the wavefunction at time t' for a state at time t that was a delta function concentrated at q. Therefore it satisfies the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} W + H\left(i\hbar \frac{\partial}{\partial q}, q\right) W = 0$$

→ The Hamilton-Jacobi equation is the eikonal approximation to the $\ddot{S}$ equation.

→ While $S(q,t,q',t')$ depends on the configuration space variable $q \in \mathcal{C}$, $W(q,t,q',t)$ depends on the eigenstate of the $\hat{q}$ operator. If the spectrum is discrete, q is not a classical variable anymore.

→ The transition amplitude has a direct physical meaning: it determines the amplitude of a process, characterized by its boundary quantities: (q,t,q',t'). For a theory to make sense, they must be a way to assign numbers to these quantities: they must have some operational meaning.

2.3 General covariant form of mechanics

→ Let us write the dynamic of a simple system in parametric form. The action $S[q]=\int_t^{t'} dt\, \mathcal{L}(q(t),\dot{q}(t))$ can be rewritten as a functional of two functions q,t, by $t\mapsto t(\tau)$:

$$S[q,t]=\int_\tau^{\tau'} d\tilde{\tau}\, \frac{dt(\tilde{\tau})}{d\tilde{\tau}}\, \mathcal{L}\left(q(\tilde{\tau}), \frac{dq(\tilde{\tau})/d\tilde{\tau}}{dt(\tilde{\tau})/d\tilde{\tau}}\right)$$

↳ The motions $(q(\tau),t(\tau))$ that minimize $S[q,t]$ determine the motions $q(t)$ that minimize $S[q]$

⊙ Example: newtonian system

→ Take $\mathcal{L}(q,\dot{q})=\frac{1}{2}m\dot{q}^2-V(q)$. It gives $\frac{d}{dt}m\dot{q}=-\nabla_q V$

The parametric form is given by $q(\tau),t(\tau)$ evolving in τ with

$$\mathcal{L}(q,t,\dot{q},\dot{t})=\frac{1}{2}m\,\dot{q}^2/\dot{t}-\dot{t}V(q)$$

The EOMs are: $\underbrace{\frac{d}{d\tau}m\frac{\dot{q}}{\dot{t}}+\dot{t}\nabla_q V=0}_{\text{Newton eq.}}$ and $\underbrace{\frac{d}{d\tau}\left(\frac{1}{2}m\left(\frac{\dot{q}}{\dot{t}}\right)^2-V(q)\right)=0}_{\text{Energy conservation}}$

→ There is a gauge invariance: the arbitrariness in the choice of the parameter τ. The EOM are invariant under $\begin{cases} q(\tau)\mapsto q(\tau'(\tau)) \\ t(\tau)\mapsto t(\tau'(\tau)) \end{cases}$ for any differentiable invertible function τ'.

↳ τ is pure gauge, the physics is in the relation between q and t.

→ The momenta are: $p_t=\frac{\partial\mathcal{L}}{\partial\dot{t}}=\frac{-1}{2}m\left(\frac{\dot{q}}{\dot{t}}\right)^2-V(q)$; $p_q=\frac{\partial\mathcal{L}}{\partial\dot{q}}=m\frac{\dot{q}}{\dot{t}}$

↳ The map $(\dot{t},\dot{q})\mapsto(p_t,p_q)$ is not invertible, because the image of the map is not full (p_t,p_q), but a subspace, determined by $C(t,q,p_t,p_q)=0$. We have: $C=p_t+H_0(p_q,q)=0$ where

$$H_0(p_q,q)=\frac{p_q^2}{2m}+V(q)$$

→ The Legendre transform from the lagrangian gives the hamiltonian

$$H=\dot{q}\,\partial_{\dot{q}}\mathcal{L}+\dot{t}\,\partial_{\dot{t}}\mathcal{L}-\mathcal{L}$$

Using the constraint, we see that $H\sim C=0$. It's expected since H generate the evolution of τ, wich is pure gauge.

→ For any function on the phase space, we can compute the EOM in τ by taking the Poisson bracket with the constraint
$$\frac{dA}{d\tau} = \{A, C\}$$
and we must impose $C(q, t, p_q, p_t) = 0$. Thus the constraint allow us to derive all observable correlations between variables.

⊙ Example : relativistic particle
→ The action reads $S = m \int d\tau \sqrt{\dot{x}^\mu \dot{x}_\mu}$. The indices μ label 4 variables, but the system has only 3 dof, and S is invariant under $\tau \mapsto \tilde{\tau}(\tau)$.
↳ The hamiltonian is 0 and the constraint reads $C = p^2 - m^2 = 0$

⊙ Example : GR
→ The action reads : $S[g] = \int d^4x \sqrt{-\det g}\, R[g]$. It's invariant under any reparametrization of x. The canonical hamiltonian vanishes and the info about the dynamics is coded in the constraints.
↳ The dynamics is described by the relative evolution of the fields with respect to one another.
We call covariant this generalized formulation of mechanics.

2.3.1 Hamilton function of a general covariant system.

→ We consider $\mathcal{L} = \frac{1}{2} m \dot{q}^2/\dot{t} - \dot{t}\, V(q)$. Let's compute its hamilton function:
$$S(q t \tau q' t' \tau') = \int_{\tau_i}^{\tau_f} \mathcal{L}(q_{qt\tau,q't'\tau'}, t_{qt\tau,q't'\tau'}, \dot{q}_{qt\tau,q't'\tau'}, \dot{t}_{qt\tau,q't'\tau'})\, d\tau$$
$$= \int_t^{t'} d\tilde{t} \left(\frac{1}{2} m \left(\frac{dq}{dt}\right)^2 - V(q) \right) = S(q t q' t')$$

↳ $\partial S / \partial \tau = 0$
↳ The hamilton function of a covariant system does not depend on the evolution parameter, but only on the boundary values of q and t.

→ The Hamilton-Jacobi equation becomes
$$\frac{\partial S}{\partial \tau} + H\left(\frac{\partial S}{\partial t}, \frac{\partial S}{\partial q}, q, t\right) S(q, t, q', t') = 0 \mapsto C\left(\frac{\partial S}{\partial t}, \frac{\partial S}{\partial q}, q, t\right) S(q, t, q', t') = 0$$

2.3.2 Partial observables :

→ When treated equally, q and t are called partial observables.
The physics is about the relative evolution of partial observables.

→ # partial observables > # dof because the # of dof is given by the # of quantities whose evolution can be predicted by the theory (thus by relations among partial observables).

DEF | The space of partial observables is called the extended configuration space $\mathcal{C}_{ext} \equiv \mathcal{C} \times \mathbb{R}$

ex: $x = (q, t) \in \mathcal{C}_{ext}$

2.3.3 Classical physics without time.

→ Let $x \in \mathcal{C}_{ext}$, and $S[x(\tau)] = \int d\tau\, \mathcal{L}(x, \dot{x})$, invariant under $x(\tau) \mapsto x(\tau'(\tau))$.
This invariance ⇒ vanishing of the canonical hamiltonian, and the existence of a constraint $C(x, p) = 0$ where $p = \frac{\partial \mathcal{L}}{\partial \dot{x}}$

→ The hamiltonian function $S(x, x')$ is the value of the action on a solution of the EOM bounded by x and x'. It satisfies the constraint equation $C(\partial S/\partial x, x) = 0$, the covariant form of the Hamilton-Jacobi equation.
Knowledge of the hamilton function ⇒ general solution of the EOM.
The initial momenta are $p = -\frac{\partial S(x, x')}{\partial x} = p(x, x')$. This eq. gives the relation between partial observables x' if the x are fixed.
→ This relation is the predictive content of the theory.

2.4 Quantum physics without time

→ The quantum theory of a covariant system is defined by:
1) A kinematical Hilbert space $\mathcal{K}$, where self-adjoint operators x and $P_x \in \mathcal{K}$ correspond to classical variables in $x \in \mathcal{C}_{ext}$ and their momenta are defined.
2) A constraint operator C, whose classical limit is the constraint $C(x,p)$, or equivalently transition amplitudes $W(x,x')$

⊙ Discrete spectrum

→ If 0 is in the discrete spectrum of C, then $\{\psi / C\psi = 0\} \subset \mathcal{K}$ is a proper subspace of $\mathcal{K}$, therefore it's an Hilbert space.

DEF The Wheeler-de Witt equation is $C\psi = 0$
The subspace $\{\psi / C\psi = 0\} \subset \mathcal{K}$ is the physical state space $\mathcal{H} \subset \mathcal{K}$

→ The WdW eq. generalise the S eq. in the covariant case. Indeed:
→ it is the S eq. for the parametrized for of the Newtonian systems
→ it is the wave eq. whose classical limit is the Hamilton-Jacobi eq $C(\partial_x S, x) = 0$

→ Let's derive the transition amplitudes. There exist a map $P: \mathcal{K} \to \mathcal{H}$ given by the orthogonal projection

DEF The transition amplitude is $W(x,x') = \langle x' | P | x \rangle$

↳ Formally, $W(x,x') = \langle x' | \delta(C) | x \rangle \sim \int_{-\infty}^{\infty} d\tau \, \langle x' | e^{i\tau C} | x \rangle$.
Since C generates evolution in τ, we can write
$$W(x,x') = \int_{x}^{x'} \mathcal{D}[x(\tau)] \exp\left\{\frac{i}{\hbar} S[x]\right\}$$

→ Since S does not depend on the parametrisation, the integration includes a large gauge redundancy that needs to be factored out. For Newtonian system, gauge fixing $\tau = t$ takes $W(x,x') \mapsto W(q,t,q',t')$

⊙ Continuum spectrum:

→ Same construction holds if 0 is in the continuous spectrum of C. Let's pick a dense subset $S \subset \mathcal{K}$ whose dual S^* defines a space of generalized states (giving the Gelfand triple $S \subset \mathcal{K} \subset S^*$). Then $\mathcal{H} = \{\psi \in S^* / \psi(C\phi) = 0 \ \forall \phi \in S\}$, and the map $P: S \to \mathcal{H}$ is defined by $(P\phi)(\phi') = \int d\tau \langle \phi | e^{i\tau C} | \phi' \rangle$

→ The space $\mathcal{H}$ is still an Hilbert space, with scalar product
$\langle P\phi | P\phi' \rangle \equiv (P\phi)(\phi')$, where P is the Projector

→ Example: free Newtonian particle in 1-D.

→ $\mathcal{K} = L_2[\mathbb{R}^2, dq\,dt]$

→ The partial observable operators and their momenta are the diag. operators q and t, and the momenta operators $-i\hbar\partial_q$ and $-i\hbar\partial_t$.

→ The constraint operator is $C = -i\hbar\partial_t - \frac{\hbar^2}{2m}\partial_q^2$ so that the Wheeler DeWitt is precisely the $\ddot{S}$ eq.

→ We take S to be the Schwartz space and S^* the space of tempered distribution

→ $\mathcal{H}$ is the space of solution in S^* and not in $\mathcal{K}$, because they are not L^2 in $dq\,dt$.

→ The transition amplitude is given by

$$W(x,x') = W(q,t,q',t') = \int d\tau \langle q',t' | e^{\frac{i}{\hbar}\tau C} | q,t \rangle$$

$$= \frac{1}{2\pi}\int d\tau \int dp \int dp_t \exp\{ip(x-x')\}\exp\{ip_t(t'-t)\}\underbrace{\exp\{\tfrac{i}{\hbar}\tau(p_t + p^2/2m)\}}_{}$$

$\delta(p_t + p^2/2m)$

$$\sim \exp\left\{\frac{i}{\hbar}\frac{m(q'-q)^2}{2(t'-t)}\right\}$$

→ The operator P is simply given by $\delta(p_t + p^2/2m)$ in Fourier transform.

⊙ Interpretation:

→ The equation $W(q t q' t') = \langle q', t' | P | q, t \rangle$ can be interpreted as follow.

→ The unphysical state $|q,t\rangle \in \mathcal{K}$ is a δ on (q,t). The projector P send it to a solution of the $\ddot{S}$ eq., concentrated in $x = (q,t)$

→ Contraction with $\langle q', t'|$ gives the value of this wave at $x' = (q', t')$

→ This is physical overlap of the kinematical state representing the event "particle at x" and the kinematical state rep. the event "particle at x'".

→ The difference between q and t is how they enter in C, and therefore the solution of the WdW eq have different properties for them.

2.4.1 Observability in quantum gravity

→ GR: describe the world it term of relative evolution of partial observables. A fixed configuration of the gravitational field provides a convenient time, but this doesn't work when considering the full quantum dynamics of spacetime.

→ We call **process** what happen to a system between an initial and a final interaction. The dynamics has been presented in term of finite portions of the trajectory of a system, expressed in terms of relation between the value of physical variables at the boundaries of a process.

→ In classical regim, dynamics ⇒ relation between x and x'

→ In quantum regim, W determines probabilities of alternative set of boundaries values.

$\Sigma = \partial M$

2.4.2 Boundary formalism

→ Consider a compact region of spacetime M. The transition amplitude W is a function of the entire boundary $\Sigma = \partial M$.
The quantum state of the field on the entire boundary is an element of the boundary Hilbert space $\mathcal{H}$. The transition amplitude W is a linear functional $\langle W|$ on this space.

→ In the non relativistic case, $\mathcal{H}$ can be identified with the tensor product of the initial and final Hilbert space: $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_t^*$ and

$$W_t(\psi \otimes \phi^*) = \langle \phi | e^{-iHt} | \psi \rangle$$

→ In gravity, a transition amplitude $\langle W | \Psi \rangle$ depends on the state Ψ of the gravitational field on the boundary Σ: it will be given by the Feynman path integral in the internal region, at fixed boundary values of the gravitational fields on Σ.
→ the grav. field on Σ is the quantity that specifies the shape of Σ. We expect $\langle W | \Psi \rangle$ to be a function of Ψ and nothing else // $\partial S / \partial \varepsilon = 0$

→ In QG, dynamics is captured by a transition amplitude W that is the "sum over geometries" on a finite bulk region bounded by Σ.

2.4.3 Relational quanta, relational space.

→ A process is what happens between interactions. Q theory describes the universe in terms of the way systems affect each other.
States are descriptions of ways a system can affect another one.
↳ QM is based on relations between systems

→ In GR, the relevant relation that builds the spacetime structure is contiguity

→ An important discovery of XX century physics is locality: interactions are local, they require spacetime contiguity. And the only way to ascertain 2 objects are contiguous is to have them interacting.

→ In QG, a process is a spacetime region.

QM		GR
Process	locality	Spacetime region
State		Boundary

→ The physical theory is a description of how arbitrary partitions of nature affect one another.
A spacetime region is a process, a state is what happens at its boundary.