Effective Field Theory and Heavy Quark Physics (2004 TASI Lectures)

Matthias Neubert

^a Institute for High-Energy Phenomenology Newman Laboratory for Elementary-Particle Physics, Cornell University Ithaca, NY 14853, U.S.A.

> ^b Institut für Theoretische Physik, Universität Heidelberg Philosophenweg 16, D-69120 Heidelberg, Germany

Abstract

These notes are based on five lectures presented at the 2004 Theoretical Advanced Study Institute (TASI) on "Physics in $D \ge 4$ ". After a brief motivation of flavor physics, they provide a pedagogical introduction to effective field theory, the effective weak Lagrangian, and the technology of renormalization-group improved perturbation theory. These general methods are then applied in the context of heavy-quarks physics, introducing the concepts of heavy-quark and soft-collinear effective theory.

1 The Physics of Beauty

Many of the unsolved problems in particle physics have their origin in the fact that we do not yet fully understand the properties of matter. In flavor physics, we study aspects of matter connected with the observation that its fundamental constituents (quarks and leptons) come in replications known as generations. There exist some big, open questions in flavor physics, to which we would love to find some answers. Let me mention three of them:

What is the dynamics of flavor? The gauge forces in the Standard Model do not distinguish between fermions belonging to different generations. All charged leptons have the same electrical charge. All quarks carry the same color charge. In almost all respects the fermions belonging to different generations are equal – but not quite, since their masses are different. Today, we understand very little about the underlying dynamics responsible for the phenomenon of generations. Why do generations exist? Why are there three of them? Why are the hierarchies of the fermion masses and mixing angles the way they are? Why are these hierarchies different for quarks and leptons? We have good reasons to expect that the answers to these questions, if they can be found in the foreseeable future, will open the doors to some great discoveries, such as new symmetries, new forces, new dimensions, or something we have not yet thought about.

What is the origin of baryogenesis? The existential question about the origin of the matter-antimatter asymmetry provides a link between particle physics and the evolution of the Universe. The Standard Model satisfies the prerequisites for baryogenesis as spelled out in the Sakharov criteria: baryon-number violating processes are unsuppressed at high temperature; CP-violating interactions are present due to complex couplings in the quark (and presumably, the lepton) sector; non-equilibrium processes can occur during phase transitions driven by the expansion of the Universe. However, quantitatively the observed matter abundance cannot be explained in the Standard Model (by many orders of magnitude). Additional contributions, either due to new CP-violating phases or new mechanisms of CP violation, are required.

Are there connections between flavor physics and TeV-scale physics? What can flavor physics tell us about the origin of electroweak symmetry breaking? And, if the world is supersymmetric at some high energy scale, what can flavor physics teach us about the mechanism of Supersymmetry breaking? Virtually any extension of the Standard Model that can solve the gauge hierarchy problem (i.e., the fact that the electroweak scale is so much lower than the GUT scale) naturally contains a plethora of new flavor parameters. Some prominent examples are:

- Supersymmetry: hundreds of flavor- and/or CP-violating couplings, even in the MSSM and its next-to-minimal variants
- extra dimensions: flavor parameters of Kaluza-Klein states
- Technicolor: flavor couplings of Techni-fermions

- multi-Higgs models: CP-violating Higgs couplings
- Little Higgs models: flavor couplings of new gauge bosons (W', Z') and fermions (t')

If New Physics exists at or below the TeV scale, its effects should show up, at some level of precision, in flavor physics. Flavor- and/or CP-violating interactions can only be studied using precision measurements at highest luminosity. In the future, such studies will profit from the fact that the relevant mass scales will (hopefully) be known from the LHC.

To drive this last point home, let me recall some lessons from the past. Top quarks have been discovered through direct production at the Tevatron. In that way, their mass, spin, and color charge have been determined. Accurate predictions for the mass were available before, based on electroweak precision measurements at the Z pole, but also based on studies of B mesons. The rates for $B-\bar{B}$ mixing, as well as for rare flavor-changing neutral current (FCNC) processes such as $B \to X_s \gamma$, are very sensitive to the value of the top-quark mass. More importantly, everything else we know about the top quark, such as its generation-changing couplings $|V_{ts}| \approx 0.040$ and $|V_{td}| \approx 0.008$, as well as its CP-violating interactions (arg $(V_{td}) \approx -24^{\circ}$ with the standard choice of phase conventions), has come from studies of kaon and B physics. Next, recall the example of neutrino oscillations. The existence of neutrinos has been known for a long time, but it was the discovery of their flavor-changing interactions (neutrino oscillations) that has revolutionized our thinking about the lepton sector. We have learned that the hierarchy of the leptonic mixing matrix is very different from that of the quark mixing matrix, and we have discovered that leptogenesis and CP violation in the lepton sector may provide an alternative mechanism for baryogenesis.

Exploring flavor aspects of the New Physics, whatever it may be, is therefore not an exercise meant to fill the Particle Data Book. Rather, it is of crucial relevance to answer some profound, deep questions about Nature. Some questions for which we have a realistic chance of finding an answer with the help of a second-generation *B*-factory are:

- Do non-standard CP phases exist? If so, this may provide new clues about baryogenesis.
- Is the electroweak symmetry-breaking sector flavor blind (minimal flavor violation)?
- Is the Supersymmetry-breaking sector flavor blind?
- Do right-handed currents exist? This may provide clues about new gauge interactions and symmetries (left-right symmetry) at very high energy.

The interpretation of New Physics signals at present or future B-factories can be tricky. But since it is our hope to answer some important questions, we must try as hard as we can. Flavor physics will thus remain a valuable component in the comprehensive exploration of the TeV scale. It is a shame that this vital part of physics is currently being terminated in the U.S.

Precision measurements in the quark sector

At first sight, the presence of generations, i.e. the replication (triplication) of the fundamental fermions, makes the Standard Model more ugly than it needed to be. The fermion masses

and mixings constitute many of the parameters of the Standard Model Lagrangian. Unlike the parameters in the gauge sector, these masses and mixings exhibit strongly hierarchical patterns. Importantly, fermions of different generations can communicate via flavor-changing weak interactions. Indeed, these are the *only* flavor-changing (i.e., generation-nondiagonal) interactions in the Standard Model.

One of the main goals of the present B factories – one that has been achieved in a spectacular way! – is the precise determination of the parameters of the quark mixing matrix (the Cabibbo-Kobayashi-Maskawa matrix). Interestingly, the presence of at least three fermion generations allows for CP violation to occur in flavor-changing weak decays, which is one of the prerequisites for an explanation of the matter-antimatter asymmetry observed in the Universe. (Yet, as mentioned above, while baryogenesis is thus possible in the Standard Model, it has not been possible to explain the observed baryon asymmetry quantitatively.) Modern CKM physics is often described with the help of the unitarity-triangle relation

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0, (1)$$

which contains the two smallest entries in the matrix, V_{ub} and V_{td} .

Measuring the parameters of the unitarity triangle serves two purposes: first, to determine some fundamental parameters of the Standard Model, and secondly, to test whether the CKM mechanism of flavor and CP violation is indeed correct, or whether there are hints of deviations from the Standard Model. By now, there is a plethora of ways in which the sides and angles of the unitarity triangle have been constrained (see [1] for a comprehensive review). Until now no significant deviations from the CKM mechanism have been established. In particular, measurements of the sides of the triangle are consistent with measurements of the angles. Also, the CP-violating phase of $V_{td} \sim e^{-i\beta}$, which is probed in particle-antiparticle mixing, is consistent with the phase of $V_{ub} \sim e^{-i\gamma}$, which is probed in rare exclusive B-meson decays.

Developing theoretical methods for a systematic analysis of exclusive hadronic decays such as $B \to \pi\pi$ has been one of the greatest challenges to heavy-flavor theory. In the remaining lectures, I will discuss some of the theoretical concepts relevant in this context. Exclusive hadronic decays also feature prominently in searches for New Physics at the B factories, and indeed there exist some hints (not more) for possible deviations from the Standard Model in the measurements of CP asymmetries in some loop-dominated processes such as $B \to \phi K_S$ and $B \to \eta' K_S$. If confirmed, this could point toward the existence of some new FCNC transitions, which can effectively compete with Standard Model loop amplitudes. This would be the first signal we have for new TeV-scale physics, and it should provide enough motivation for you to follow the rest of these lectures.

2 Effective Field Theory

Effective field theory (EFT) is a very powerful tool in quantum field theory [2]. It provides a systematic formalism for the analysis of multi-scale problems. This is particularly important in QCD, where the value of the running coupling $\alpha_s(\mu)$ can change significantly between different energy scales. As such, EFT greatly simplifies practical calculations in field theory; indeed, it

often makes such calculations feasible. As we will discuss, EFT also provides a new, modern meaning to "renormalization".

The main idea of EFT is simply stated: Consider a quantum field theory with a large, fundamental scale M. This could be the mass of a heavy particle, or some large (Euclidean) momentum transfer. Suppose we are interested in physics at energies E (or momenta p) much smaller than M. How can we expand scattering or decay amplitudes in powers of E/M? The answer to this question proceeds in several steps:

1. Choose a cutoff $\Lambda < M$ and divide the fields of the theory into low-frequency and high-frequency modes,

$$\phi = \phi_L + \phi_H \,, \tag{2}$$

where ϕ_L contains the Fourier modes with frequency $\omega < \Lambda$, while ϕ_H contains the remaining modes with frequency $\omega > \Lambda$. We can think of the cutoff as a "threshold of ignorance" in the sense that we may pretend to know nothing about the theory for scales above Λ (which is indeed often the case). By construction, low-energy physics is described in terms of the ϕ_L fields. Everything we ever wish to know about the theory (Feynman diagrams, scattering amplitudes, cross sections, decay rates, etc.) can be derived from vacuum correlation functions of these fields. These correlators can be obtained using

$$\langle 0 | T\{\phi_L(x_1) \dots \phi_L(x_n)\} 0 \rangle = \frac{1}{Z[0]} \left(-i \frac{\delta}{\delta J_L(x_1)} \right) \dots \left(-i \frac{\delta}{\delta J_L(x_n)} \right) Z[J_L] \Big|_{J_L=0}, \quad (3)$$

where

$$Z[J_L] = \int \mathcal{D}\phi_L \, \mathcal{D}\phi_H \, e^{iS(\phi_L,\phi_H) + i \int d^D x \, J_L(x) \, \phi_L(x)} \tag{4}$$

is the generating functional of the theory. Here $S(\phi_L, \phi_H) = \int d^D x \mathcal{L}(x)$ is the action, D is the dimension of space-time, and we have only included sources J_L for the light fields, as this suffices to compute the correlation functions in (3).

2. In the next step, we perform the path integral over the high-frequency fields. This yields

$$Z[J_L] \equiv \int \mathcal{D}\phi_L \, e^{iS_\Lambda(\phi_L) + i \int d^D x \, J_L(x) \, \phi_L(x)} \,, \tag{5}$$

where

$$e^{iS_{\Lambda}(\phi_L)} = \int \mathcal{D}\phi_H \, e^{iS(\phi_L,\phi_H)} \tag{6}$$

is called the "Wilsonian effective action". Note that, by construction, this action depends on the choice of the cutoff Λ used to define the split between low- and high-frequency modes. S_{Λ} is non-local on scales $\Delta x^{\mu} \sim 1/\Lambda$, because high-frequency fluctuations have been removed from the theory. The process of removing these modes is often referred to as "integrating out" the high-frequency fields in the functional integral.

3. In the final step, we expand the non-local action functional in terms of local operators composed of light fields. This process is called the (Wilsonian) operator-product expansion (OPE). This expansion is possible because $E \ll \Lambda$ by assumption. The result can be cast in the form

$$S_{\Lambda}(\phi_L) = \int d^D x \, \mathcal{L}_{\Lambda}^{\text{eff}}(x) \,, \tag{7}$$

where

$$\mathcal{L}_{\Lambda}^{\text{eff}}(x) = \sum_{i} g_{i} Q_{i}(\phi_{L}(x)). \tag{8}$$

This object is called the "effective Lagrangian". It is an infinite sum over local operators Q_i multiplied by coupling constants g_i , which are referred to as Wilson coefficients. In general, all operators allowed by the symmetries of the theory are generated in the construction of the effective Lagrangian and appear in this sum.

Since there is always an infinite number of such operators, the question arises: How can the effective low-energy theory be predictive? This is where the simple, but powerful trick of "naive dimensional analysis" comes to play. As is common practice in high-energy physics, let us work in units where $\hbar = c = 1$. Then $[m] = [E] = [p] = [x^{-1}] = [t^{-1}]$ are all measured in the same units. We denote by $[g_i] = -\gamma_i$ the mass dimension of the effective couplings g_i . It follows that

$$g_i = C_i M^{-\gamma_i} \tag{9}$$

with dimensionless coefficients C_i . Since by assumption there is only a single fundamental scale M in the theory, we expect that $C_i = O(1)$. This assertion is known as the hypothesis of "naturalness". Unless there is a specific mechanism that could explain the smallness of the dimensionless numbers C_i , we should assume those numbers to be of O(1). The presence of unusually large (e.g. 10^6) or small (e.g. 10^{-6}) numbers in a theory would appear "unnatural" and call for further explanation.

At low energy $(E \ll \Lambda < M)$, the contribution of a given operator Q_i in the effective Lagrangian to an observable (which for simplicity we assume to be dimensionless) is expected to scale as

$$C_i \left(\frac{E}{M}\right)^{\gamma_i} = \begin{cases} O(1); & \text{if } \gamma_i = 0, \\ \ll 1; & \text{if } \gamma_i > 0, \\ \gg 1; & \text{if } \gamma_i < 0. \end{cases}$$
 (10)

It follows that only operators whose couplings have $\gamma_i \leq 0$ are important at low energy. This very fact is what makes the OPE a useful tool. Depending on the precision goal, one may truncate the series in (8) at a given order in E/M. Once this is done, only a finite (often small) number of operators Q_i and couplings g_i need to be retained.

Let us go through the above arguments once again, being slightly more careful. Assuming weak coupling, we can use the free action to assign a scaling behavior with E to all fields and

¹Interactions can change the naive scaling dimensions γ_i , as we will see later. For this reason, the γ_i are referred to as "anomalous dimensions".

OD 11 1	C1 'C '	c	1	1	1	•	. 1	or i	т .
Table I	Classification	\cap t c	merators ar	nd (combings i	ın	the	ettective	Lagrangian
rabic r.	Classification	OI C	perators ar	uu (coupings i	TTT	ULIC	CIICCUIVC	Dagrangian

Dimension	Importance for $E \to 0$	Terminology
$\delta_i < D, \gamma_i < 0$	grows	relevant operators
		(super-renormalizable)
$\delta_i = D, \gamma_i = 0$	constant	marginal operators
		(renormalizable)
$\delta_i > D, \gamma_i > 0$	falls	irrelevant operators
		$({\bf non\text{-}renormalizable})$

couplings in the low-energy effective theory. Consider scalar ϕ^4 theory as an example. The action is

$$S = \int d^D x \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{m^2}{2} \, \phi^2 - \frac{\lambda}{4!} \, \phi^4 \right). \tag{11}$$

Using that $x \sim E^{-1}$ and $\partial^{\mu} \sim E$, and requiring that the action scale like O(1) (in units of \hbar), we see that $\phi \sim E^{\frac{D}{2}-1}$. If we denote by δ_i the mass dimension of an operator Q_i , then $\gamma_i = \delta_i - D$. For the operators in the Lagrangian (11) we find:

$$\begin{array}{c|cccc} & \delta_i & \gamma_i & \text{Coupling} \\ \hline \partial_\mu \phi \, \partial^\mu \phi & D & 0 & 1 \\ \phi^4 & 2D - 4 & D - 4 & \lambda \sim \Lambda^{4-D} \\ \phi^2 & D - 2 & -2 & m^2 \sim \Lambda^2 \\ \end{array}$$

More generally, an operator with n_1 scalar fields and n_2 derivatives has

$$\delta_i = n_1 \left(\frac{D}{2} - 1 \right) + n_2, \qquad \gamma_i = (n_1 - 2) \left(\frac{D}{2} - 1 \right) + (n_2 - 2).$$
 (12)

It follows that for D > 2 only few operators have $\gamma_i \leq 0$.

A summary of these considerations is presented in Table 1. The common terminology of "relevant", "marginal", and "irrelevant" operators given there is without a doubt one of the worst misnomers is the history of physics. Really, "relevant" operators are usually unimportant, because they are forbidden by a symmetry (else they are disastrous, see below). "Marginal" operators are all there is in renormalizable quantum field theories. And "irrelevant" operators are those that are really interesting, because they teach us something about physics at the fundamental scale M.

A crucial insight, which one may term the "theorem of modesty", is that no quantum field theory is ever complete at arbitrarily high energy. At best it is an EFT valid up to some cutoff scale Λ . This "scale of ignorance" is often a physical scale, such as the mass of a new particle, which has not yet been discovered. When interpreted that way, many theories we know and love can be seen as EFTs:

High-energy theory	Fundamental scale	Low-energy theory	
Standard Model	$M_W \sim 80 {\rm GeV}$	Fermi theory	
GUT	$M_{\rm GUT} \sim 10^{16} {\rm GeV}$	Standard Model	
String theory	$M_S \sim 10^{18}\mathrm{GeV}$	QFT	
11-dim. M theory		String theory	

The arguments just presented provide a new perspective on renormalization. Instead of a paradigm of renormalizable theories based on the concept of systematic "cancellations of infinities", we should adopt the following, more physical point of view:

- Low-energy physics depends on the short-distance structure of the fundamental theory via relevant and marginal couplings, and possibly through some irrelevant couplings provided measurements are sufficiently precise.
- "Non-renormalizable" interactions are not forbidden; on the contrary, irrelevant operators always contribute at some level of precision. Their effects are simply numerically suppressed if the fundamental scale M is much larger than the typical energies achievable experimentally.
- These non-renormalizable, "irrelevant" interactions tell us something about the physics at the cutoff scale $\Lambda \sim M$.

A corrolary to the second item is that, at low energies, all EFTs are "automatically" renormalizable quantum field theories, provided that the cutoff scale Λ is large.

The comment about "irrelevant" interactions in the third item is very powerful, so let us illustrate it with two prominent examples: i) Early measurements of the magnitude and energy dependence of weak-interaction processes at low energy have indicated the relevance of a high mass scale $M \sim 100 \,\text{GeV}$. This was instrumental in finding the correct theory of the weak interactions. ii) The local gauge symmetries of the Standard Model allow us to write down a dimension-5 operator of the type $q \nu^T H H \nu$ with $q \sim 1/\Lambda$. After electroweak symmetry breaking, this operator gives rise to a neutrino Majorana mass term $m_{\nu} \sim v^2/\Lambda$, where $v \sim 246 \,\mathrm{GeV}$ is the vacuum expectation value of the Higgs field. Seen as an EFT, the Standard Model thus predicts the existence of neutrino masses, even though there are no righthanded neutrino fields in the theory. The seasaw mechanism provides an explicit example of how such a mass term might be realized in a more fundamental theory. But unless we forbid the dimension-5 operator by imposing a symmetry such a lepton-number conservation, the existence of neutrino masses is a generic prediction of the Standard Model. The fact that the observed neutrino masses imply $\Lambda \sim 10^{14}\,\mathrm{GeV}$ not far from the energy scale where the three gauge couplings approximately unify is a strong argument in favor of the idea of Grand Unification.

On the other hand, super-renormalizable terms in an effective Lagrangian are problematic. Consider as an example the operator ϕ^2 in scalar ϕ^4 theory (i.e., the mass term for the scalar field). In D=4 dimensions we have $\delta_i=2, \gamma_i=-2$, and so we expect that $m^2 \sim \Lambda^2$ by virtue of the hypothesis of naturalness. Since such large fluctuations are indeed generated in the

functional integral, we expect the mass of the scalar field to be enormous (assuming the cutoff scale is large). But this is a contradiction: the ϕ particle would be heavy, and so it would not be part of the low-energy effective theory. No experiment at low energy could produce such a heavy particle. This reasoning leads to a new paradigm of "naturalness": EFTs should be natural in the sense that all mass terms are forbidden by symmetries. These symmetries must be broken at low energy, since otherwise everything would be massless. In the Standard Model, the fact that the fundamental forces are derived from gauge interactions guarantees that the spin-1 bosons (photon, gluons, W and Z bosons) are massless, while the fact that the weak interactions act on chiral (left-handed) fermions forbids fermion mass terms. Some of these symmetries are broken at the electroweak scale, and so mass terms are generated at that scale.² Scalar particles, in particular the Higgs boson of the Standard Model, are not protected by any symmetry. In the context of EFTs they are not allowed in the low-energy effective Lagrangian. This leads to the notion that the Standard Model is not a consistent (better, natural) EFT. The Standard Model Higgs sector is thus not expected to be the correct theory of electroweak symmetry breaking.

There are several possible ways out of this dilemma. Two conventional routes are to invoke a new symmetry (Supersymmetry) to protect scalar particles from acquiring masses of order the cutoff scale, or to abandon the idea of a fundamental scalar and instead realize the Higgs boson as a bound state of some new, strong interaction (Technicolor). Alternatively, the Higgs boson can be made light if it is identified with the (pseudo-) Goldstone modes of a spontaneously broken global symmetry (little Higgs models). More recently, enlightened by the fact that there is no indication whatsoever for physics beyond the Standard Model, some theorists have given up on the idea of naturalness, and unnatural (e.g., fine-tuned) theories such as Split Supersymmetry [3, 4] have received a lot of attention. In these models the explanation and fine-tuning of parameters such as a light scalar mass (or the cosmological constant) is derived from anthropic reasoning [5, 6].

Let me finish this lecture with some comments on renormalization and running couplings. First, note that quantum corrections can alter the naive scaling relations for the dimensions of operators and couplings, giving rise to "anomalous dimensions" (a better term would be "quantum dimensions"). In a weakly coupled theory, these anomalous dimensions can be calculated using perturbation theory. Lecture 4 shows what they are good for. Next, while so far we thought of ϕ_H as describing some heavy-particle fields (such as the top quark, or the weak gauge bosons W, Z in the Standard Model), it is important to realize that these fields also describe the high-frequency quantum fluctuations of light or massless fields. Now consider an EFT with only light fields, in which we lower the cutoff scale Λ by some small amount $\delta\Lambda$. This corresponds to integrating out just a few high-frequency modes in a small slice between Λ and $\Lambda - \delta\Lambda$ in energy. Since the operators $Q_i(\phi_L)$ remain the same in such a situation (since no heavy particles are integrated out), the effects of lowering the cutoff must be absorbed entirely by a change of the effective couplings $C_i(\Lambda)$ in the effective Lagrangian $\mathcal{L}_{\Lambda}^{\text{eff}}$. This provides an intuitive understanding of why the effective couplings are "running couplings", whose values depend on the cutoff.

²This reasoning explains why $M_W \sim M_Z \sim 100\,\mathrm{GeV}$. However, the expectation that also the fermion masses are of order the weak scale only works for the top quark. This is known as the flavor puzzle.

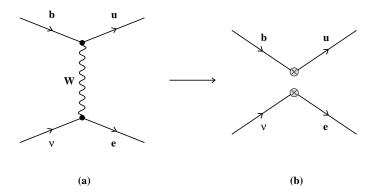


Figure 1: Example of an effective four-fermion interaction obtained by integrating out the W boson in the Standard Model. The two crossed circles in the second graph represent a local four-quark operator in the effective theory. (Courtesy of A. J. Buras [7])

3 Effective Weak Interactions

The couplings of the charged weak gauge bosons W^{\pm} to fermions (quarks and leptons) are the only flavor-changing interactions in the Standard Model. When studying flavor-changing processes at low energy (i.e., $E \ll M_W$), we can integrate out the heavy bosons from the Standard Model Lagrangian. As illustrated in Figure 1, this gives rise to local four-fermion interactions. The resulting Fermi theory of weak interactions is particularly simple if we ignore the effects of QCD. Indeed, at tree level the path integral is Gaussian, and integrating over the W^{\pm} fields gives the effective Lagrangian

$$\mathcal{L}_{\text{weak}}^{\text{eff}} = -\frac{g^2}{8M_W^2} \left[J_{\mu}^- J^{+\mu} + \frac{1}{M_W^2} J_{\mu}^- (\partial^{\mu} \partial^{\nu} - g^{\mu\nu} \,\Box) J_{\nu}^+ + \dots \right],\tag{13}$$

where $g^2/8M_W^2 \equiv G_F/\sqrt{2}$ defines the Fermi constant $(G_F = 1.16639 \cdot 10^{-5} \, \mathrm{GeV^{-2}})$, and

$$J_{\mu}^{+} = V_{ij} \,\bar{u}_{i} \gamma_{\mu} (1 - \gamma_{5}) d_{j} + \bar{\nu}_{i} \gamma_{\mu} (1 - \gamma_{5}) l_{i} \,, \qquad J_{\mu}^{-} = (J_{\mu}^{+})^{\dagger}$$
(14)

are the charged currents. V_{ij} are the elements of the CKM matrix, and a summation over flavor indices i, j is understood.

Already the leading term in the effective weak Lagrangian (13) contains irrelevant interactions ($\delta_i = 6$, $\gamma_i = 2$), and indeed the coupling constant $G_F \sim 1/M_W^2$ shows the expected suppression by two powers of the fundamental scale. Even before the discovery of the weak gauge bosons, experiments of low-energy weak interactions indicated that the fundamental scale of the weak force should be $(\sqrt{2}/G_F)^{1/2} \approx 110\,\text{GeV}$. It was a triumph of particle physics when the heavy gauge bosons were subsequently discovered at just that mass scale. The fact that there are no marginal operators in the effective weak Lagrangian explains the apparent "weakness" of the weak interactions at low energy. On the contrary, at high energy the weak force is unified with electromagnetism, and the two interactions are then governed by a single coupling constant. Subleading terms in the effective weak Lagrangian have dimension $\delta_i = 8$ and higher. Their effects are further suppressed by powers of $(E/M_W)^2$ and are tiny. They can be neglected for (almost) all practical purposes.

Beyond the tree approximation, the question arises of how to account for the effects of the strong interactions in the derivation of the effective Lagrangian [7]. They are due to the high-frequency modes of the quark and gluon fields. (The low-energy modes of these fields remain part of the low-energy theory, which still contains QCD.) Two problems arise: first, the path integral is no longer Gaussian once QCD effects are taken into account; secondly, the strong interactions are not perturbative at low energy due to the confinement of colored particles into hadrons. One deals with these difficulties using a general procedure called "matching", which consists of the following steps:

- 1. List all possible gauge-invariant operators of a given dimension allowed by the symmetries and quantum numbers associated with a given problem. The dimension of the operators is determined by the accuracy goal of the calculation, but generally $\delta_i = 6$ for calculations in flavor physics.
- 2. Write down the OPE for the effective Lagrangian with undetermined couplings C_i , in our case

$$\mathcal{L}_{\text{weak}}^{\text{eff}} = -\frac{G_F}{\sqrt{2}} \sum_{i} C_i Q_i.$$
 (15)

Note that the Wilson coefficients C_i are process independent, i.e., the same coefficients arise in the calculation of many different weak-interaction amplitudes.

3. Determine the values of the coefficients $C_i(\mu)$ such that³

$$\mathcal{A}_{n} = \langle f_{n} | \mathcal{L}_{SM} | i_{n} \rangle \stackrel{!}{=} \sum_{i} C_{i}(\mu) \langle f_{n} | Q_{i} | i_{n} \rangle + \text{higher power corrections}$$
 (16)

to a given order in perturbation theory. We need to study as many such matrix elements as necessary to determine the coefficients C_i once and forever. Note that the choice of matrix elements is not unique.

In a weakly coupled theory all calculations can be done using perturbation theory, but what if the theory becomes strongly coupled at low energy, such as QCD? The crucial point is that we can still determine the C_i perturbatively if the theory is weakly coupled at high energy (asymptotic freedom at short distances). This statement becomes obvious if we recall the construction of the Wilsonian effective action. The Wilson coefficient functions arise from integrating out high-frequency quantum fluctuations above the cutoff scale Λ . Below the scale Λ the effective theory is, by construction, equivalent to the fundamental theory order by order in power counting. Differences arise only at high energy, where the effective theory misses the high-frequency modes of the full theory. The corresponding contributions are absorbed into the Wilson coefficients. As along as the theory is perturbative (weakly coupled) at and above the scale Λ (or μ), the Wilson coefficients are calculable in perturbation theory. It follows that the Wilson coefficients in the effective Lagrangian are insensitive to any infrared (IR)

³I have switched notation here, such that μ takes the role of the cutoff scale Λ . In practice, one almost always uses dimensional regularization to perform calculations in quantum field theory, whereas a hard Wilsonian cutoff Λ is used in conceptual arguments.

physics – unlike the amplitudes themselves. As a result, matching calculations can be done using arbitrary IR regulators and working with free quark and gluon states. Obviously, this is a great advantage for actual calculations in QCD.

Four-fermion processes

To construct the leading terms in the effective weak Lagrangian we take note of the following facts:

- Four fermion fields already make dimension $\delta_i = 6$, so no derivatives or extra fields are allowed at this order.
- Weak interactions only involve left-handed fermion fields.
- Chirality is preserved in strong-interaction processes, since we can set $m_q = 0$ at leading power.
- For the quark bilinears $\bar{\psi}_L \Gamma \psi_L$ with Γ an element of the Dirac basis, only the possibility $\Gamma = \gamma_\mu$ remains.
- Operators must be gauge invariant (in particular, color singlets) and Lorentz invariant.

Example 1: Consider semileptonic decays such as $\bar{B}^0 \to \pi^+ e^- \bar{\nu}_e$, which are based on the quark transition $b \to u e^- \bar{\nu}_e$. From such processes one can extract the CKM element $|V_{ub}|$. The unique dimension-6 operator in the corresponding effective Lagrangian is

$$\bar{e}_L \gamma_\mu \nu_L \, \bar{u}_L^i \gamma^\mu b_L^i \, \hat{=} \, \bar{u}_L^i \gamma_\mu \nu_L \, \bar{e}_L \gamma^\mu b_L^i \,, \tag{17}$$

where the second form is related to the first one by a Fierz transformation. This operator is generated by integrating out the W boson in Figure 1. Below, we will often omit color indices i whenever they are contracted between neighboring quark fields. Tree-level matching yields

$$\mathcal{L}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{ub} C_1(\mu) \,\bar{e}_L \gamma_\mu \nu_L \,\bar{u}_L \gamma^\mu b_L \tag{18}$$

with $C_1 = 1 + O(\alpha_s)$.

Example 2: A more interesting case is offered by hadronic decays such as $\bar{B}^0 \to \pi^+ D_s^-$, which are based on the quark transition $b \to u \bar{c} s$. Similar arguments now allow two operators differing in their color structure. Specifically,

$$\mathcal{L}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{cs}^* V_{ub} \left[C_1(\mu) \, \bar{s}_L^j \gamma_\mu c_L^j \, \bar{u}_L^i \gamma^\mu b_L^i + C_2(\mu) \, \bar{s}_L^i \gamma_\mu c_L^j \, \bar{u}_L^j \gamma^\mu b_L^i \right], \tag{19}$$

where $C_1 = 1 + O(\alpha_s)$ and $C_2 = O(\alpha_s)$ follow again from tree-level matching. Using a Fierz rearrangement, the second operator above can also be written as $\bar{u}_L^j \gamma_\mu c_L^j \bar{s}_L^i \gamma^\mu b_L^i$. Note also that

$$\bar{s}_L \gamma_\mu t_a c_L \, \bar{u}_L \gamma^\mu t_a b_L = \frac{1}{2} \, \bar{s}_L^i \gamma_\mu c_L^j \, \bar{u}_L^j \gamma^\mu b_L^i - \frac{1}{2N_c} \, \bar{s}_L \gamma_\mu c_L \, \bar{u}_L \gamma^\mu b_L \tag{20}$$

gives nothing new. Here t_a are the generators of color SU(3).

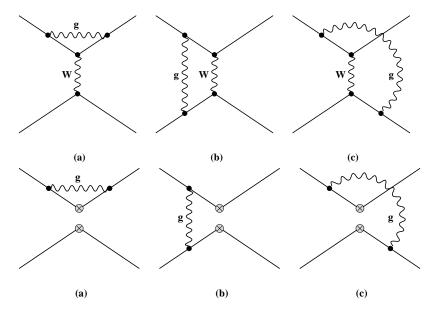


Figure 2: One-loop QCD corrections to four-fermion weak-interaction amplitudes, both in the Standard Model (top row) and in the low-energy effective theory (bottom row). For $b \to u e^- \bar{\nu}_e$ only the diagrams labeled (a) are possible, whereas for $b \to u \bar{c} s$ all six diagrams contribute. (Courtesy of A. J. Buras [7])

Matching at one-loop order

The one-loop QCD corrections to the $b \to u e^- \bar{\nu}_e$ transition in both the full theory and the effective theory are shown by the first diagram, labeled (a), in each row in Figure 2. (Diagrams (b) and (c) are absent if the lower fermion line represents a lepton pair, as in the present case.) One finds that the radiative corrections in the two theories are identical, and so $C_1(\mu) = 1$ to all orders in (QCD) perturbation theory. Note that in this example the Taylor expansion of the W-boson propagator trivially commutes with loop integrations.

The one-loop QCD corrections to the $b \to u \bar{c} s$ transition in both the full theory and the effective theory are represented by all six diagrams shown in Figure 2. The diagrams labeled (b) and (c), which now contribute, are more interesting, and their evaluation is considerably more difficult. One finds that in this example the two operations – expansion of the W propagator and integration over loop momenta – do not commute. The reason is that in the last two diagrams in the top row the loop momentum flows through the W-boson propagator. Rather than going through the full calculation in detail, we just note that

$$\int d^D p \, \frac{1}{M_W^2 - p^2} f(p) \neq \frac{1}{M_W^2} \int d^D p \, \left(1 + \frac{p^2}{M_W^2} + \dots \right) f(p) \,. \tag{21}$$

While the left-hand side is non-analytic in M_W , the right-hand side is obviously analytic. Differences between the two integrals arise from the region of large loop momenta where $p^2 \sim M_W^2$. But for such large momenta QCD is weakly coupled. Perturbation theory can thus be trusted to compute the differences between the matrix elements in the two theories, which are accounted for by the Wilson coefficient functions. Explicit calculation of the diagrams in

Figure 2 gives (in the $\overline{\text{MS}}$ subtraction scheme) [8]

$$C_{1}(\mu) = 1 + \frac{3}{N_{c}} \frac{\alpha_{s}(\mu)}{4\pi} \left(\ln \frac{M_{W}^{2}}{\mu^{2}} - \frac{11}{6} \right) + O(\alpha_{s}^{2}),$$

$$C_{2}(\mu) = -3 \frac{\alpha_{s}(\mu)}{4\pi} \left(\ln \frac{M_{W}^{2}}{\mu^{2}} - \frac{11}{6} \right) + O(\alpha_{s}^{2}).$$
(22)

Some important comments are in order:

- IR regulators (such as quark and gluon masses, external momenta, etc.) present in intermediate steps of the calculation in both theories cancel in the results for the Wilson coefficients C_i .
- Matrix elements in the effective theory are often more singular than those in the full theory (which are ultraviolet (UV) finite in the present case) and require additional UV subtractions (operator renormalization). This gives rise to the renormalization-scale and -scheme dependence of the Wilson coefficients.
- The physical reason for this is that the mass M_W acts as an UV regulator in the box diagrams of the full theory. Roughly speaking, the logarithms in the results for the Wilson coefficients arise as follows:

$$\underbrace{1 + \alpha_s \ln \frac{M_W^2}{-p^2}}_{\text{full theory}} = \underbrace{\left(1 + \alpha_s \ln \frac{M_W^2}{\mu^2}\right)}_{C(\mu)} \underbrace{\left(1 + \alpha_s \ln \frac{\mu^2}{-p^2}\right)}_{\langle Q(\mu) \rangle} + \dots, \tag{23}$$

where the expression on the left is the matrix element in the full theory (which is UV finite and regularized in the IR by an off-shell momentum p^2), while the expression on the right is the product of a Wilson coefficient and a matrix element in the effective theory. The EFT matrix element has the same dependence on the IR cutoff as the matrix element in the full theory, while all reference to the fundamental scale M_W resides in the Wilson coefficient. Another way of representing this result is in the form of logarithmic integrations:

$$\int_{-p^2}^{M_W^2} \frac{dk^2}{k^2} = \int_{\mu^2}^{M_W^2} \frac{dk^2}{k^2} + \int_{-p^2}^{\mu^2} \frac{dk^2}{k^2} \,. \tag{24}$$

In general, the Wilson coefficients absorb the high-frequency contributions of the loop integrals, while the low-frequency contributions reside in the EFT matrix elements.

FCNC transitions

As a more interesting example of an effective weak Lagrangian, let us now consider the important case of flavor-changing neutral current (FCNC) transitions [8]. Such interactions are absent at tree level in the Standard Model, while they can be mediated through loop processes. The corresponding suppression of FCNC amplitudes provides a window for New

Physics searches. The perhaps best known example is the rare radiative decay $B \to X_s \gamma$, which is the "standard candle" of quark flavor physics.

What are the effective operators that can mediate transitions of the type $b \to s_L + \text{anything}$, where "anything" must be flavor diagonal? At dimension $\delta_i = 4$, the two possible operators one can write down (we set $m_s = 0$ but consider the possibility of having $m_b \neq 0$ for the heavy b quark) are $m_b \, \bar{s}_L b_R$ and $\bar{s}_L \, i \not\!\!D \, b_L$. Note that the weak interactions only operate on left-handed fields, so a mass insertion $\mathcal{L}_m = m_b (\bar{b}_R b_L + \bar{b}_L b_R)$ is needed to turn the left-handed b quark into a right-handed one. The two operators are equivalent because of the equation of motion $i \not\!\!D b_L = m_b \, b_R$. However, the "flavor off-diagonal mass term" $m_b \, \bar{s}_L b_R$ can always be removed by a field redefinition of the quark fields. It follows that there are no dimension-4 terms in the effective Lagrangian. (Note also that the operator $\bar{s}_L H b_L$ is not gauge invariant.)

The leading operators in the effective weak Lagrangian once again have dimension $\delta_i = 6$. They can be arranged in three classes. The first type of operators is analogous to those in (19):

$$\bar{s}_L^i \gamma_\mu b_L^i \, \bar{q}_L^j \gamma^\mu q_L^j , \qquad (25)$$

$$\bar{s}_L^i \gamma_\mu b_L^j \, \bar{q}_L^j \gamma^\mu q_L^i \, \hat{=} \, \bar{s}_L^i \gamma_\mu q_L^i \, \bar{q}_L^j \gamma^\mu b_L^j .$$

However, since the flavor-singlet quark pair is not restricted to couple to weak gauge bosons, we can also have operators of the form

$$\bar{s}_L^i \gamma_\mu b_L^i \bar{q}_R^j \gamma^\mu q_R^j, \qquad (26)$$

$$\bar{s}_L^i \gamma_\mu b_L^j \bar{q}_R^j \gamma^\mu q_R^i = -2\bar{s}_L^i q_R^i \bar{q}_R^j b_L^j.$$

Finally, there are operators containing the gauge fields, namely

$$g_s m_b \,\bar{s}_L \sigma_{\mu\nu} \,G_a^{\mu\nu} t_a b_R \,, \qquad g_s \,\bar{s}_L \gamma_\nu \,i D_\mu \,G_a^{\mu\nu} t_a b_L \,. \tag{27}$$

In the first case a mass insertion is required. The second operator is redundant, since the equation of motion for the Yang-Mills field,

$$D_{\mu} G_a^{\mu\nu} = -\sum_a g_s \,\bar{q} \gamma^{\nu} t_a q \,, \tag{28}$$

implies

$$g_s \,\bar{s}_L \gamma_\nu \, iD_\mu \, G_a^{\mu\nu} t_a b_L = -g_s^2 \sum_q \,\bar{s}_L \gamma_\nu t_q b_L \,\bar{q} \gamma^\nu t_a q \,, \tag{29}$$

which can be written as a linear combination of the four-quark operators in (25) and (26). When electromagnetic interactions are included, we have in addition the operator

$$em_b \,\bar{s}_L \sigma_{\mu\nu} \, F^{\mu\nu} b_R \,, \tag{30}$$

where $F^{\mu\nu}$ is the electromagnetic field strength. Also, there are additional four-quark operators involving leptons, such as

$$\bar{s}_L \gamma_\mu b_L \, \bar{l}_L \gamma^\mu l_L \,, \qquad \bar{s}_L \gamma_\mu b_L \, \bar{l}_R \gamma^\mu l_R \,, \qquad \bar{s}_L \gamma_\mu b_L \, \bar{\nu}_{lL} \gamma^\mu \nu_{lL} \,.$$
 (31)

To summarize, the resulting Standard Model operator basis for FCNC processes (without leptons, for simplicity) contains the "current-current operators" (with p = u, c)

$$Q_1^{(p)} = (\bar{s}_i p_i)_{V-A} (\bar{p}_j b_j)_{V-A},$$

$$Q_2^{(p)} = (\bar{s}_i p_j)_{V-A} (\bar{p}_j b_i)_{V-A},$$
(32)

the "QCD penguin operators"

$$Q_{3} = (\bar{s}_{i}b_{i})_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_{j}q_{j})_{V-A} ,$$

$$Q_{4} = (\bar{s}_{i}b_{j})_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_{j}q_{i})_{V-A} ,$$

$$Q_{5} = (\bar{s}_{i}b_{i})_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_{j}q_{j})_{V+A} ,$$

$$Q_{6} = (\bar{s}_{i}b_{j})_{V-A} \sum_{q=u,d,s,c,b} (\bar{q}_{j}q_{i})_{V+A} ,$$

$$(33)$$

the "electroweak penguin operators" (with e_q the electric changes of the quarks in units of |e|)

$$Q_{7} = (\bar{s}_{i}b_{i})_{V-A} \sum_{q=u,d,s,c,b} \frac{3}{2} e_{q} (\bar{q}_{j}q_{j})_{V+A},$$

$$Q_{8} = (\bar{s}_{i}b_{j})_{V-A} \sum_{q=u,d,s,c,b} \frac{3}{2} e_{q} (\bar{q}_{j}q_{i})_{V+A},$$

$$Q_{9} = (\bar{s}_{i}b_{i})_{V-A} \sum_{q=u,d,s,c,b} \frac{3}{2} e_{q} (\bar{q}_{j}q_{j})_{V-A},$$

$$Q_{10} = (\bar{s}_{i}b_{j})_{V-A} \sum_{q=u,d,s,c,b} \frac{3}{2} e_{q} (\bar{q}_{j}q_{i})_{V-A},$$

$$(34)$$

and the electromagnetic and chromo-magnetic dipole operators

$$Q_{7\gamma} = -\frac{em_b}{8\pi^2} \bar{s}_L \sigma_{\mu\nu} F^{\mu\nu} b_R ,$$

$$Q_{8g} = -\frac{g_s m_b}{8\pi^2} \bar{s}_L \sigma_{\mu\nu} G_a^{\mu\nu} t_a b_R .$$
(35)

We use the short-hand notation $(\bar{q}_1q_2)_{V\pm A} \equiv \bar{q}_1\gamma^{\mu}(1\pm\gamma_5)q_2$. Note that (if only to confuse you) the standard convention for the electroweak penguin operators is such that $Q_{7,8}$ correspond to $Q_{5,6}$, while $Q_{9,10}$ correspond to $Q_{3,4}$. Some representative Feynman diagrams in the full theory from which these operators originate are shown in Figure 3. The names of the various penguin operators reflect the nature of the gauge bosons (gluons for QCD penguins, and γ or Z-bosons for electroweak penguins) emitted from the penguin loops.

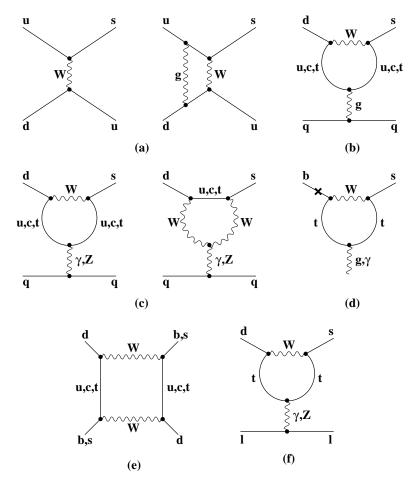


Figure 3: Typical diagrams in the Standard Model which generate the different operators in the effective weak Lagrangian. The current-current operators $Q_{1,2}$ result from graphs of type (a), the QCD penguin operators $Q_{3,\dots,6}$ from graphs of type (b), the electroweak penguin operators $Q_{7,\dots,10}$ from graphs of type (c), and the dipole operators from graphs of type (d). Digram (f) generates the operators with leptons shown in (31), while diagram (e) contributes to $B-\bar{B}$ and $K-\bar{K}$ mixing. (Figure taken from [8] with permission from the authors)

Some particular features of the Standard Model have been implicitly incorporated in the above considerations, namely that only left-handed fields are involved in flavor-changing weak interactions, that light (approximately massless) quarks have identical couplings with respect to the strong interactions, and that all up-type (u, c) and down-type (d, s, b) quark fields couple identically to the weak force.

The unitarity of the CKM matrix implies $\lambda_u + \lambda_c + \lambda_t = 0$, where $\lambda_p \equiv V_{pb}V_{ps}^*$. We will use this relation to eliminate CKM factors involving couplings of the top quark. Note also that in the limit $m_u = m_c = 0$ (which is justified at dimension-6 order) the penguin graphs always involve $\lambda_t = -(\lambda_u + \lambda_c)$. The final result for the effective weak Lagrangian reads

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} \left[\sum_{p=u,c} \lambda_p \left(C_1 Q_1^{(p)} + C_2 Q_2^{(p)} \right) + \sum_{i=3,\dots,10,7\gamma,8q} (\lambda_u + \lambda_c) C_i Q_i \right].$$
 (36)

Note that

$$\frac{\lambda_u}{\lambda_c} = \frac{V_{ub}V_{us}^*}{V_{cb}V_{cs}^*} \sim e^{-i\gamma} \tag{37}$$

has a non-zero, relative CP-violating phase. This allows for the phenomenon of CP violation from amplitude interference in FCNC processes – a phenomenon that is currently being studied extensively at the B-factories (see, e.g., [1]).

Let me finish this discussion by quoting the matching conditions for the various operators at the weak scale $\mu = M_W$. They are [8]:

$$C_{1}(M_{W}) = 1 - \frac{11}{6} \frac{\alpha_{s}(M_{W})}{4\pi},$$

$$C_{2}(M_{W}) = \frac{11}{2} \frac{\alpha_{s}(M_{W})}{4\pi},$$

$$C_{3}(M_{W}) = C_{5}(M_{W}) = -\frac{1}{6} \widetilde{E}_{0} \left(\frac{m_{t}^{2}}{M_{W}^{2}}\right) \frac{\alpha_{s}(M_{W})}{4\pi},$$

$$C_{4}(M_{W}) = C_{6}(M_{W}) = \frac{1}{2} \widetilde{E}_{0} \left(\frac{m_{t}^{2}}{M_{W}^{2}}\right) \frac{\alpha_{s}(M_{W})}{4\pi},$$

$$C_{7}(M_{W}) = f\left(\frac{m_{t}^{2}}{M_{W}^{2}}\right) \frac{\alpha(M_{W})}{6\pi},$$

$$C_{9}(M_{W}) = \left[f\left(\frac{m_{t}^{2}}{M_{W}^{2}}\right) + \frac{1}{\sin^{2}\theta_{W}} g\left(\frac{m_{t}^{2}}{M_{W}^{2}}\right)\right] \frac{\alpha(M_{W})}{4\pi},$$

$$C_{8}(M_{W}) = C_{10}(M_{W}) = 0,$$
(38)

with

$$\widetilde{E}_0(x) = -\frac{7}{12} + O(1/x),
f(x) = \frac{x}{2} + \frac{4}{3} \ln x - \frac{125}{36} + O(1/x),
g(x) = -\frac{x}{2} - \frac{3}{2} \ln x + O(1/x),$$
(39)

and

$$C_{7\gamma}(M_W) = -\frac{1}{3} + O(1/x),$$

 $C_{8g}(M_W) = -\frac{1}{8} + O(1/x).$ (40)

Note that despite of the fact that there is a heavy top-quark running in the penguin loops, the Wilson coefficients exhibit non-decoupling, i.e., they do not vanish in the limit where $m_t \to \infty$. The coefficients of the electroweak penguin operators are even proportional to m_t^2 in this limit. This makes electroweak penguin operators relevant for phenomenology, even though the are suppressed by small electroweak coupling constants.

4 RG-Improved Perturbation Theory

There are some important technical aspects which we have ignored in the discussion of the previous lecture. Recall the one-loop matching results for the Wilson coefficients C_1 and C_2 from (22):

$$C_{1}(\mu) = 1 + \frac{3}{N_{c}} \frac{\alpha_{s}(\mu)}{4\pi} \left(\ln \frac{M_{W}^{2}}{\mu^{2}} - \frac{11}{6} \right) + O(\alpha_{s}^{2}),$$

$$C_{2}(\mu) = -3 \frac{\alpha_{s}(\mu)}{4\pi} \left(\ln \frac{M_{W}^{2}}{\mu^{2}} - \frac{11}{6} \right) + O(\alpha_{s}^{2}).$$

$$(41)$$

Ideally, we would like to integrate out all high-frequency modes perturbatively and then evaluate the remaining EFT matrix elements $\langle Q_i(\mu) \rangle$ at some low scale $\mu \sim$ few GeV, below which perturbation theory becomes untrustworthy. The computation of these matrix elements must use a non-perturbative approach such as lattice QCD, heavy-quark expansions, or chiral perturbation theory. A glance at the above equations shows a potential problem: the expansion parameter is not $\frac{\alpha_s}{\pi} \sim 0.1$, but $\frac{\alpha_s}{\pi} \ln \frac{M_W^2}{\mu^2} \sim 0.8$. The problem is indeed generic: in the presence of widely separated scales $M \gg \mu$, perturbation theory often involves powers of $\alpha_s \ln \frac{M}{\mu}$ rather than powers of α_s . Such large logarithmic terms must be resummed to all orders.

While this problem is particularly acute for almost all practical calculations in QCD, it is also relevant to theories with smaller coupling constants. For instance, when the gauge couplings of the Standard Model are extrapolated from low energy up to the GUT scale $M_{\rm GUT} \sim 10^{16}\,{\rm GeV}$, the relevant logarithm is $\ln\frac{M_{\rm GUT}^2}{\mu^2}\approx 65$. Resummation is essential to control such large logarithms even if the coupling constants are as small as those for the electro-weak interactions of the Standard Model.

The general solution to the problem of large logarithms is called "renormalization-group (RG) improved perturbation theory". It provides a reorganization of perturbation theory in which $\alpha_s \ln \frac{M}{\mu}$ is treated as an O(1) parameter, while $\alpha_s \ll 1$. Large logarithms are resummed to all orders in perturbation theory by solving RG equations. The nomenclature of RG-improved perturbation theory is as follows: At leading order (LO) all terms of the form $(\alpha_s \ln \frac{M}{\mu})^n$ with $n=0,\ldots,\infty$ are resummed. The result is an O(1) contribution to the Wilson coefficient functions. At next-to-leading order (NLO), one also resums terms of the form $\alpha_s(\alpha_s \ln \frac{M}{\mu})^n$, all of which count as $O(\alpha_s)$, and so on. Note that in cases where the term with n=0 is absent (such as for C_2), there may be O(1) effects after resummation that not seen at tree level in perturbation theory. This happens also for the Wilson coefficients of the QCD penguin operators in the effective weak Lagrangian. As shown in (38) the matching conditions for the coefficients $C_{2,\ldots,6}$ start at $O(\alpha_s)$; yet, after RG resummation these coefficients become of O(1) and contribute at the same order as the Wilson coefficient C_1 of the leading current-current operator.

Before we can perform such resummations, we must study in some more detail the renormalization of the composite operators in the effective Lagrangian.

Anomalous dimensions

Consider a complete set (a basis)

$$\{Q_i(\mu)\}; \quad i = 1, \dots, n.$$
 (42)

of operators of dimension δ allowed by the symmetries (quantum numbers, etc.) of a problem. Recall that by changing the scale μ one reshuffles terms from the matrix elements $\langle Q_i \rangle$ into the coefficients C_i , leaving the result for any observable unchanged, i.e.

$$\mathcal{A} = \sum_{i=1}^{n} C_i(\mu) \langle Q_i(\mu) \rangle = \sum_{i=1}^{n} C_i(\mu - \delta \mu) \langle Q_i(\mu - \delta \mu) \rangle. \tag{43}$$

The fact that physical observables are scale independent implies that

$$\frac{d}{d\ln\mu} \sum_{i=1}^{n} C_i(\mu) \langle Q_i(\mu) \rangle = 0.$$
 (44)

Since the operator basis is complete, we can expand the logarithmic derivative of the operator matrix elements in terms of the same basis operators. We write

$$\frac{d}{d\ln\mu} \langle Q_i(\mu) \rangle \equiv -\sum_{j=1}^n \gamma_{ij}(\mu) \langle Q_j(\mu) \rangle. \tag{45}$$

If there is more than one operator present, we say that the operators mix under scale variation. The dimensionless coefficients γ_{ij} measure the incremental change under scale variation and are free of large logarithms. They are called anomalous dimensions. Using this definition, it follows from (44) that

$$\sum_{j=1}^{n} \left[\frac{d}{d \ln \mu} C_j(\mu) - \sum_{i=1}^{n} C_i(\mu) \gamma_{ij}(\mu) \right] \langle Q_j(\mu) \rangle = 0.$$
 (46)

Since by assumption the operators Q_i are linearly independent, we conclude that

$$\frac{d}{d \ln \mu} C_j(\mu) - \sum_{i=1}^n C_i(\mu) \gamma_{ij}(\mu) = 0.$$
 (47)

This is the RG equation obeyed by the Wilson coefficient functions. In matrix notation, we can rewrite this equation as

$$\frac{d}{d\ln\mu}\vec{C}(\mu) = \hat{\gamma}^T(\mu)\vec{C}(\mu). \tag{48}$$

The dimensionless anomalous-dimension matrix $\hat{\gamma}$ depends on the scale μ only through the running coupling $\alpha_s(\mu)$. Changing variables from $\ln \mu$ to $\alpha_s(\mu)$, we find

$$\frac{d}{d\alpha_s(\mu)}\vec{C}(\mu) = \frac{\hat{\gamma}^T(\alpha_s(\mu))}{\beta(\alpha_s(\mu))}\vec{C}(\mu), \qquad (49)$$

where $\beta = d\alpha_s(\mu)/d \ln \mu$ is the QCD β -function. The initial condition for the solution of the RG equation is set by the values $\vec{C}(M_W)$ of the Wilson coefficients at the weak scale.

Equation (49) has the same structure as the Heisenberg equation for the time dependence of the Hamiltonian in quantum field theory. The unique solution to this equation is

$$\vec{C}(\mu) = T_{\alpha} \exp \left[\int_{\alpha_s(M_W)}^{\alpha_s(\mu)} d\alpha \frac{\hat{\gamma}^T(\alpha)}{\beta(\alpha)} \right] \vec{C}(M_W).$$
 (50)

The matrix exponential is defined through its Taylor expansion, and the symbol T_{α} means an ordering such that $\hat{\gamma}^{T}(\alpha)$ with larger α stands to the left of those with smaller α . Such an ordering prescription is necessary because, in general, the matrices $\hat{\gamma}^{T}(\alpha)$ at different α values do not commute.

We now perform a (controlled) perturbative expansion of the quantities $\vec{C}(M_W)$, $\hat{\gamma}(\alpha)$, and $\beta(\alpha)$ entering the general solution, all of which are free of large logarithms. Consider, for simplicity, the case of a single Wilson coefficient (n = 1, no mixing). Writing

$$\gamma(\alpha_s) = \gamma_0 \frac{\alpha_s}{4\pi} + O(\alpha_s^2), \quad \beta(\alpha_s) = -2\alpha_s \left[\beta_0 \frac{\alpha_s}{4\pi} + O(\alpha_s^2) \right], \quad C(M_W) = 1 + O(\alpha_s), \quad (51)$$

we find the leading-order solution

$$C(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\gamma_0}{2\beta_0}} \left[1 + O(\alpha_s)\right]. \tag{52}$$

To see that this sums the leading logarithms, note that

$$\left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\gamma_0}{2\beta_0}} \approx \left(1 + \beta_0 \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{\mu^2}\right)^{-\frac{\gamma_0}{2\beta_0}} = 1 - \frac{\gamma_0}{2} \frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{\mu^2} + O(\alpha_s^2 \ln^2 \frac{M_W^2}{\mu^2}).$$
(53)

It is straightforward to go to higher orders in the expansion in α_s . For the case of a single operator, the NLO solution reads

$$C(\mu) = \left(\frac{\alpha_s(\mu)}{\alpha_s(M_W)}\right)^{-\frac{\gamma_0}{2\beta_0}} \left[1 + \frac{\alpha_s(\mu) - \alpha_s(M_W)}{4\pi} S + c_1 \frac{\alpha_s(M_W)}{4\pi} + O(\alpha_s^2) \right], \tag{54}$$

where

$$S = -\frac{\gamma_0}{2\beta_0} \left(\frac{\gamma_1}{\gamma_0} - \frac{\beta_1}{\beta_0} \right), \tag{55}$$

and we have expanded

$$\gamma(\alpha_s) = \sum_{n=0}^{\infty} \gamma_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1}, \qquad \beta(\alpha_s) = -2\alpha_s \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1},$$

$$C(M_W) = 1 + \sum_{n=1}^{\infty} c_n \left(\frac{\alpha_s(M_W)}{4\pi}\right)^n. \tag{56}$$

The generalization to the case of operator mixing is discussed in the literature [8].

The systematics of RG-improved perturbation theory is summarized in the following table:

Order	γ , β	$C(M_W)$
LO	1-loop	tree-level
NLO	2-loop	1-loop
NNLO	3-loop	2-loop

The LO approximation is really only good for illustration purposes. At NLO we achieve the same accuracy as in the case of a conventional one-loop calculation for a single-scale problem. Note, however, that two-loop anomalous dimensions are required at this order. The NNLO approximation is the state of the art for many applications, where high precision is of concern.

5 Effective Theories for Heavy Quarks

Heavy-quark systems provide prime examples for applications of the EFT technology, because the hierarchy $m_b \gg \Lambda_{\rm QCD}$ provides a natural separation of scales. Physics at the scale m_b is of a short-distance nature, while for heavy-quark systems there is always also some hadronic physics governed by the confinement scale $\Lambda_{\rm QCD}$. Being able to separate the short-distance and long-distance effects associated with these two scales is vital for any quantitative description in heavy-quark physics. For instance, if the long-distance hadronic matrix elements are obtained from lattice QCD, then it is necessary to analytically compute the short-distance effects, which come from short-wavelength modes that do not fit on present-day lattices. In many other instances, the long-distance hadronic physics can be encoded in a small number of universal parameters. To identify these parameters requires that one first extracts all short-distance effects.

Heavy-quark effective theory (HQET)

The simplest effective theory for heavy-quark systems is heavy-quark effective theory (HQET) [9]. It provides a simplified description of the soft interactions of a single heavy quark interacting with soft, light partons. This includes the interactions that bind the heavy quark with other light partons inside heavy mesons (B, B^*, \ldots) and baryons $(\Lambda_b, \Sigma_b, \ldots)$. I will only offer an HQET primer in this lecture, referring to reviews in the literature for a more detailed discussion [10, 11].

A softly interacting heavy quark is nearly on-shell. Its momentum may be decomposed as

$$p_Q^{\mu} = m_Q v^{\mu} + k^{\mu} \,, \tag{57}$$

where v is the 4-velocity of the hadron containing the heavy quark ($v^2 = 1$), and the "residual momentum" $k \sim \Lambda_{\rm QCD}$. This off-shell momentum results from the soft interactions of the heavy quark with its environment. (This assumes a reference frame in which the heavy meson has a small velocity, v = O(1).) A near on-shell Dirac spinor has two large and two small components. We define

$$Q(x) = e^{-im_Q v \cdot x} \left[h_v(x) + H_v(x) \right], \tag{58}$$

where

$$h_v(x) = e^{im_Q v \cdot x} \frac{1 + \psi}{2} Q(x) \tag{59}$$

are the large, "upper" components, while

$$H_v(x) = e^{im_Q v \cdot x} \frac{1 - \psi}{2} Q(x) \tag{60}$$

are the small, "lower" components. The extraction of the phase factor in (58) implies that the fields h_v and H_v carry the residual momentum k. These fields obey the projection relations

$$\psi h_v(x) = h_v(x), \qquad \psi H_v(x) = -H_v(x).$$
 (61)

Inserting these definitions into the Dirac Lagrangian yields

$$\mathcal{L}_{Q} = \bar{Q} (i \not\!\!D - m_{Q}) Q$$

$$= \bar{h}_{v} i \not\!\!D h_{v} + \bar{H}_{v} (i \not\!\!D - 2m_{Q}) H_{v} + \bar{h}_{v} i \not\!\!D H_{v} + \bar{H}_{v} i \not\!\!D h_{v}$$

$$= \bar{h}_{v} i v \cdot D h_{v} + \bar{H}_{v} (-i v \cdot D - 2m_{Q}) H_{v} + \bar{h}_{v} i \not\!\!D H_{v} + \bar{H}_{v} i \not\!\!D h_{v}, \qquad (62)$$

where $i\vec{D}^{\mu} = iD^{\mu} - v^{\mu} iv \cdot D$ is the "spatial" covariant derivative (note that $v^{\mu} = (1, \vec{0})$ in the heavy-hadron rest frame). We have used that between two h_v spinors a Dirac matrix γ^{μ} can be replaced with v^{μ} , while between two H_v spinors it can be replaced with $-v^{\mu}$. We have also used the projection properties (61), which in particular imply that $\bar{H}_v \psi h_v = 0$.

The interpretation of (62) is that the field h_v describes a massless fermion, while H_v describes a heavy fermion with mass $2m_Q$. Both modes are coupled to each other via the last two terms. Soft interactions cannot excite the heavy fermion, so we integrate it out from the generating functional of the theory. The light field which remains describes the fluctuations of the heavy quark about its mass shell. Solving the classical equation of motion for the field H_v yields

$$H_v = \frac{1}{2m_Q + iv \cdot D} i \vec{\cancel{D}} h_v = \frac{1}{2m_Q} \sum_{n=0}^{\infty} \left(-\frac{iv \cdot D}{2m_Q} \right)^n i \vec{\cancel{D}} h_v , \qquad (63)$$

which implies $H_v = O(\Lambda_{\rm QCD}/m_Q) h_v$ provided the residual momenta are small. The leading-order effective Lagrangian obtained from (62) then reads

$$\mathcal{L}_{\text{HQET}} = \bar{h}_v \, iv \cdot D_s \, h_v + O(1/m_Q) \,. \tag{64}$$

Note that the covariant derivative $iD_s^{\mu} = i\partial^{\mu} + g_s A_s^{\mu}$ contains only the soft gluon field. Hard gluons have been integrated out.

It is straightforward to include power corrections to the effective Lagrangian by keeping higher-order terms in (63). One finds that at subleading order in $1/m_Q$ two new operators arise, such that

$$\mathcal{L}_{\text{HQET}} = \bar{h}_v \, iv \cdot D_s \, h_v + \frac{1}{2m_Q} \left[\bar{h}_v \, (i\vec{D}_s)^2 h_v + C_{\text{mag}}(\mu) \, \frac{g_s}{2} \, \bar{h}_v \, \sigma_{\mu\nu} \, G_s^{\mu\nu} h_v \right] + \dots \, (65)$$

The new operators are referred to ask the "kinetic energy" and the "chromo-magnetic interaction". The kinetic-energy operator corresponds to the first correction term in the Taylor

expansion of the relativistic energy $E = \sqrt{\vec{p}^2 + m_Q^2} = m_Q + \vec{p}^2/2m_Q + \dots$, and Lorentz invariance ensures that its coefficient is not renormalized. The Wilson coefficient of the chromomagnetic interaction operators is, however, non-trivial. It has been calculated in [12] at NLO in RG-improved perturbation theory.

The leading term in the HQET Lagrangian exhibits a $SU(2n_Q)$ spin-flavor symmetry. Its physical meaning is that, in the infinite mass limit, the properties of hadronic systems containing a single heavy quark are insensitive to the spin and flavor of the heavy quark [13]. The flavor symmetry is broken by the operators arising at order $1/m_Q$ and higher. Note, however, that at this order only the chromo-magnetic operator breaks the spin symmetry. Many phenomenological implications of heavy-quark symmetry are explored in [10].

Scalings of fields

It is useful to set up a systematic power counting in $\lambda = \Lambda_{\rm QCD}/m_Q$ by assigning scaling properties for all objects in the effective theory. To do this, all dimensionful quantities are expressed in powers of the fundamental scale m_Q . For the residual momentum we find $k^{\mu} \sim \Lambda_{\rm QCD} = \lambda m_Q \sim \lambda$. All derivatives on the soft fields in HQET obey the same scaling as the residual momentum, i.e. $\partial^{\mu} \sim \lambda$.

The scaling property of the heavy-quark field h_v follows by considering the free quark propagator in position space. We have

$$\langle 0|T\{h_v(x)\bar{h}_v(0)\}|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{v\cdot k + i\epsilon} \sim \lambda^4 \cdot \frac{1}{\lambda} \sim \lambda^3, \tag{66}$$

from which it follows that $h_v \sim \lambda^{3/2}$. For the soft gluon field, a similar consideration shows that

$$\langle 0 | T \{ A_s^{\mu}(x) A_s^{\nu}(0) \} | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \frac{i}{k^2 + i\epsilon} \left[-g^{\mu\nu} + (1-a) \frac{k^{\mu}k^{\nu}}{k^2} \right] \sim \lambda^4 \cdot \frac{1}{\lambda^2} \sim \lambda^2 , \quad (67)$$

and hence $A_s^{\mu} \sim \lambda$. Note that this ensures a homogeneous scaling of the covariant derivative, $D_s^{\mu} \sim \lambda$. Using these relations, we see that the different terms in the HQET Lagrangian in (65) scale like

$$\mathcal{L}_{\text{HQET}} \sim \lambda^4 + \frac{\lambda^5}{m_Q} + \dots$$
 (68)

The scaling of the integration measure d^4x for operators containing soft fields follows from the requirement that $e^{-ik\cdot x} \sim O(1)$, since this defines what are "important contributions" to Fourier integrals. We conclude that $x^{\mu} \sim \lambda^{-1}$, $d^4x \sim \lambda^{-4}$, and hence

$$S_{\text{HQET}} = \int d^4x \, \mathcal{L}_{\text{HQET}}(x) \sim \lambda^0 + \frac{\lambda}{m_Q} + \dots,$$
 (69)

as it should be.

Residual gauge invariance

The effective Lagrangian \mathcal{L}_{HQET} defines an effective theory for soft interactions of heavy quarks. All hard interactions (including the couplings of heavy quarks to hard gluons) are integrated out in the construction of the effective Lagrangian. As a result, the effective theory no longer has the full gauge invariance of QCD, but only a residual gauge invariance with respect to gauge transformations that preserve the scaling properties of the fields. These are called "soft gauge transformations" and denoted by $U_s(x)$. The transformation rules are

$$h_v(x) \to U_s(x) h_v(x) ,$$

 $A_s^{\mu}(x) \to U_s(x) A_s^{\mu}(x) U_s^{\dagger}(x) + \frac{i}{g_s} U_s(x) [\partial^{\mu}, U_s^{\dagger}(x)] .$ (70)

Operationally, "soft" functions like $A_s^{\mu}(x)$ and $U_s(x)$ can be defined via a restriction to soft modes in their Fourier decomposition. In practice, however, the use of dimensional regularization makes it unnecessary to introduce the hard cutoffs associated with this construction.

Decoupling transformation

The couplings of soft gluons to heavy quarks can be "removed" by the (non-local, field-dependent) field redefinition

$$h_v(x) = S_v(x) h_v^{(0)}(x), (71)$$

with

$$S_v(x) = P \exp\left(ig_s \int_{-\infty}^0 dt \, v \cdot A_s(x + tv)\right) \tag{72}$$

a time-like Wilson line extending from minus infinity to the point x. The symbol P means an ordering with respect to t such that gauge fields are ordered from left to right in the order of decreasing t values. The Wilson line $S_v^{\dagger}(x)$ is given by a similar expression but with the opposite ordering prescription, and with ig_s replaced by $-ig_s$ in the exponent.

The soft Wilson line obeys the important property

$$S_v^{\dagger} iv \cdot D_s S_v = iv \cdot \partial. \tag{73}$$

Using this relation, it follows that in terms of the new fields the HQET Lagrangian becomes

$$\mathcal{L}_{\text{HOET}} = \bar{h}_v^{(0)} i v \cdot \partial h_v^{(0)} + O(1/m_Q). \tag{74}$$

At leading order in $1/m_Q$, this is a free theory as far as the strong interactions of heavy quarks are concerned! However, the theory is nevertheless non-trivial once we allow for the presence of external sources. Consider, e.g., what happens to a flavor-changing weak-interaction current, which turns a heavy b-quark into a c-quark (plus a W boson). At tree level, matching such a current onto HQET gives

$$\bar{c}\gamma^{\mu}(1-\gamma_5)b \to \bar{h}_{v'}\gamma^{\mu}(1-\gamma_5)h_v = \bar{h}_{v'}^{(0)}\gamma^{\mu}(1-\gamma_5)(S_{v'}^{\dagger}S_v)h_v^{(0)}. \tag{75}$$

Here v and v' are the (in general different) velocities of the heavy mesons containing the heavy quarks. Unless the two velocities are equal, the object $S_{v'}^{\dagger}S_v$ is non-trivial, and hence the soft gluons do not decouple from the heavy quarks inside the current operator. Indeed, if we close the integration contour at $t = -\infty$ we may interpret $S_{v'}^{\dagger}S_v$ as a Wilson loop with a cusp at the point x, where the two paths parallel to the different velocity vectors intersect. The presence of the cusp leads to non-trivial UV behavior (for $v \neq v'$), which is described by a cusp anomalous dimension $\Gamma_c(w)$ depending on the kinematical invariant $w = v \cdot v'$. This quantity was calculated at two-loop order as early as in 1987 [14]. The cusp anomalous dimension is nothing but the celebrated velocity-dependent anomalous dimension of heavy-quark currents, which was rediscovered three years later in the context of HQET [15].

The technology introduced in the last three subsections is usually not taught in courses on HQET, and indeed it is not required necessarily. However, the interpretation of heavy quarks as Wilson lines is very useful, and it was put forward in some of the very first papers on the subject [16]. I have emphasized this technology here, because it will be useful in the study of the interactions of heavy quarks with collinear degrees of freedom, to which we turn now.

Soft-collinear effective theory (SCET)

An long-standing problem in QCD is how to systematically parameterize long-distance effects (power corrections) for processes that do not admit a local OPE. In problems with a large mass M or a large Euclidean momentum transfer Q^2 , the OPE provides a rigorous framework for an expansion of matrix elements in powers and logarithms of the large scale. However, processes involving energetic light particles pose new challenges. Here some components of p^{μ} are large, but $p^2 \approx 0$ is small. The kinematics in these "jet-like" processes is intrinsically Minkowskian, and the separation of short-distance from long-distance physics becomes a tricky issue. In particular, lattice QCD cannot be used (not even in principle) to study non-perturbative physics near the light cone.

There are many important examples of physical systems where energetic light partons play an important role. They include jet physics (e.g., the determination of α_s), rare exclusive B-meson decays such as $B \to \pi\pi$ and $B \to \phi K_S$ (unitarity-triangle physics, physics beyond the Standard Model), inclusive B decays such as $B \to X_s \gamma$, and many more. For concreteness, consider the two examples $B \to X_s \gamma$ and $B \to \pi\pi$. In the first case, working in the rest frame of the initial B meson and choosing the z-direction as the direction of the jet X_s , the momenta of the outgoing particles can be written (now in ordinary 4-vector notation) as

$$p_X^{\mu} = (M_B - E_{\gamma}, 0, 0, E_{\gamma}), \qquad p_{\gamma}^{\mu} = (E_{\gamma}, 0, 0, -E_{\gamma}).$$
 (76)

Experimentally one is forced to cut on large photon energy, $E_{\gamma} \approx M_B/2$, such that $M_B - 2E_{\gamma} = O(\Lambda_{\rm QCD})$. This implies large energy $E_X \approx M_B/2$ and small invariant mass $M_X^2 = M_B(M_B - 2E_{\gamma}) = O(M_B\Lambda_{\rm QCD})$ for the hadronic final state. The example $B \to \pi\pi$ is even simpler. In this case, we have

$$p_1^{\mu} = (E_{\pi}, 0, 0, \sqrt{E_{\pi}^2 - m_{\pi}^2}), \qquad p_2^{\mu} = (E_{\pi}, 0, 0, -\sqrt{E_{\pi}^2 - m_{\pi}^2}),$$
 (77)

where $E_{\pi} = M_B/2$. The final-state pions are on-shell, $p_1^2 = p_2^2 = m_{\pi}^2$.

The question is: What is there to integrate out in these processes? The large scales are not just given by the masses of heavy particles, but also by the large energies of some fast light particles. Nevertheless, the presence of several different scales means that we can classify quantum fluctuations as hard, hard-collinear, collinear, or soft. For the two examples mentioned above, the corresponding scales are

hard:
$$M_B, E_{\gamma}$$

hard-collinear: $M_X^2 \sim M_B \Lambda_{\rm QCD}$
soft: $\Lambda_{\rm QCD}$ (78)

in the case of $B \to X_s \gamma$, and

hard:
$$M_B$$
, E_{π} collinear: m_{π} soft: $\Lambda_{\rm OCD}$ (79)

in the case of $B \to \pi\pi$.

Our first goal is to integrate out all hard quantum fluctuations. For processes with light-like kinematics this will lead to a *non-local* EFT [17, 18, 19, 20, 21, 22], which will provide us with a systematic generalization of the concept of a local OPE. Moreover, because there are several short-distance scales in the problem (hard and hard-collinear), the construction of the EFT often proceeds in several steps. The last subsection in this lecture provides an explicit example.

Power counting

The expansion parameter of SCET is $\lambda = \Lambda_{\rm QCD}/E$, where E is the typical energy of the (hard-) collinear particles. In B decays, $E \sim m_b$, but it is useful to keep these two parameters separate for the time being. Because of the presence of fast, collinear particles it is convenient to decompose 4-vectors in a light-cone basis spanned by two light-like vectors n and \bar{n} and two transverse coordinates. We have $n^2 = \bar{n}^2 = 0$ and choose $n \cdot \bar{n} = 2$. If the collinear particles are aligned along the z direction, we choose $n^{\mu} = (1, 0, 0, 1)$ and $\bar{n}^{\mu} = (1, 0, 0, -1)$. In other words, n points in the direction of the collinear momenta, and \bar{n} points in the opposite direction. For an arbitrary 4-vector p^{μ} , we write

$$p^{\mu} = \frac{\bar{n}^{\mu}}{2} n \cdot p + \frac{n^{\mu}}{2} \bar{n} \cdot p + p_{\perp}^{\mu}$$
$$\equiv p_{+}^{\mu} + p_{-}^{\mu} + p_{\perp}^{\mu} \equiv (p_{+}, p_{-}, p_{\perp}), \qquad (80)$$

where I have introduced several useful notations in the second line. Let us perform the light-cone decomposition for the relevant momenta in the two examples discussed above. For $B \to X_s \gamma$ decay, the momenta of the final-state particles are decomposed as

$$p_X^{\mu} = M_B \frac{n^{\mu}}{2} + (M_B - 2E_{\gamma}) \frac{\bar{n}^{\mu}}{2}, \qquad p_{\gamma}^{\mu} = 2E_{\gamma} \frac{\bar{n}^{\mu}}{2}.$$
 (81)

For the hadronic final-state jet we have $\bar{n} \cdot P_X = M_B$ and $n \cdot P_X = M_B - 2E_{\gamma} \sim \Lambda_{\rm QCD}$. For the example of $B \to \pi\pi$ decay, we have

$$p_1^{\mu} = \frac{M_B}{2} \left(1 + \sqrt{1 - \frac{4m_{\pi}^2}{M_B^2}} \right) \frac{\bar{n}^{\mu}}{2} + \frac{M_B}{2} \left(1 - \sqrt{1 - \frac{4m_{\pi}^2}{M_B^2}} \right) \frac{\bar{n}^{\mu}}{2} ,$$

$$p_2^{\mu} = \frac{M_B}{2} \left(1 - \sqrt{1 - \frac{4m_{\pi}^2}{M_B^2}} \right) \frac{\bar{n}^{\mu}}{2} + \frac{M_B}{2} \left(1 + \sqrt{1 - \frac{4m_{\pi}^2}{M_B^2}} \right) \frac{\bar{n}^{\mu}}{2} ,$$
(82)

and hence $\bar{n} \cdot p_1 = n \cdot p_2 \approx M_B$ and $n \cdot p_2 = \bar{n} \cdot p_1 \approx m_\pi^2/M_B$.

We now distinguish between different types of momenta, classified according to their scaling properties with the large energy $E \gg \Lambda_{\rm QCD}$. For the example of $B \to X_s \gamma$ decay, the relevant momenta are (with $\Lambda \sim \Lambda_{\rm QCD}$):⁴

hard:
$$p_h^{\mu} \sim (E, E, E) \sim (1, 1, 1)$$

hard-collinear: $p_{hc}^{\mu} \sim (\Lambda, E, \sqrt{E\Lambda}) \sim (\lambda, 1, \lambda^{1/2})$
soft: $p_s^{\mu} \sim (\Lambda, \Lambda, \Lambda) \sim (\lambda, \lambda, \lambda)$ (83)

The corresponding virtualities are $p_h^2 \sim E^2$, $p_{hc}^2 \sim E\Lambda$, and $p_s^2 \sim \Lambda^2$. The partons of the final-state hadronic jet typically carry hard-collinear momenta, however, the jet may also contain some partons with soft momenta. The effective theory results from integrating out all fields carrying hard momenta, keeping fields with hard-collinear or soft momenta as dynamical degrees of freedom.

In exclusive processes such as $B \to \pi\pi$ one also encounters particles with collinear momenta, whose virtualities are $p_c^2 \sim \Lambda^2$. The corresponding scaling relations are:

collinear-1:
$$p_{c1}^{\mu} \sim (\Lambda^2/E, E, \Lambda) \sim (\lambda^2, 1, \lambda)$$

collinear-2: $p_{c2}^{\mu} \sim (E, \Lambda^2/E, \Lambda) \sim (1, \lambda^2, \lambda)$ (84)

While this looks similar to the scaling relation for a hard-collinear momentum, just with λ replaced by λ^2 , the difference is that the same replacement is *not* done for the soft fields. Hence, in a situation where both soft and collinear fields are present, integrating out hard modes results in a different EFT than the one mentioned above. It is conventional to call the EFT for soft and hard-collinear fields SCET-1, and that for soft and collinear fields SCET-2. In the latter case, the theory also contains so-called messenger modes with momentum scaling $p_m^{\mu} \sim (\lambda^2, \lambda, \lambda^{3/2})$. We will not discuss the many subtleties of SCET-2 here but refer the interested reader to the literature [23, 24]. From now on we study the theory SCET-1 in further detail, calling it SCET for simplicity.

As mentioned above, after integrating out hard modes (i.e., modes carrying hard momenta) we obtain an EFT for soft and hard-collinear fields. The momentum scalings in (83) allow

⁴Unfortunately, the terminology for the different momentum modes in SCET is not unique. Some authors refer to hard-collinear modes as collinear ones, and to soft modes as ultra-soft ones.

for interactions among these modes subject to the constraint that interactions among hard-collinear and soft fields are only allowed if the soft fields couple to at least two hard-collinear fields. The reason is that, when we couple one or more soft fields to a hard-collinear field, the resulting total momentum scales like a hard-collinear momentum: $(\lambda, 1, \lambda^{1/2}) + (\lambda, \lambda, \lambda) \sim (\lambda, 1, \lambda^{1/2})$. Momentum conservation then implies that there is another hard-collinear particle that can absorb this momentum.

Let us now introduce the various fields of SCET obtained by decomposing the quark and gluon fields into various momentum modes. The soft fields and their scalings are the same as in HQET:

soft heavy quark:
$$h_v \sim \lambda^{3/2}$$

soft light quark: $q_s \sim \lambda^{3/2}$
soft gluon: $A_s^{\mu} \sim \lambda$ (85)

Next, we introduce the following hard-collinear fields:

hard-collinear light quark:
$$\xi \sim \lambda^{1/2}$$

hard-collinear gluon: $A_{bc}^{\mu} \sim (\lambda, 1, \lambda^{1/2})$ (86)

There are no hard-collinear heavy-quark fields, since the corresponding virtualities would be hard, $p_b^2 - m_b^2 = (m_b v + k_{hc})^2 - m_b^2 \approx 2m_b v \cdot k_{hc} \approx 2m_b E_{hc}$, and so these fields have already been removed from the effective theory. Note that the scaling of the hard-collinear gluons is such that the covariant derivative $iD_{hc}^{\mu} = i\partial^{\mu} + g_s A_{hc}^{\mu}$ has a homogeneous scaling when acting on hard-collinear fields.

As in the case of HQET, the scaling properties of the hard-collinear fields can be derived most easily by analyzing the corresponding two-point functions [21]. For the gluon propagator, we have

$$\langle 0 | T\{A_{hc}^{\mu}(x) A_{hc}^{\nu}(y)\} | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} \frac{i}{p^2 + i\epsilon} \left[-g^{\mu\nu} + (1-a) \frac{p^{\mu}p^{\nu}}{p^2} \right]. \tag{87}$$

Generically (i.e., for $a \neq 1$) this scales like $p^{\mu}p^{\nu}$, and hence in an arbitrary gauge A^{μ}_{hc} scales like a hard-collinear momentum p^{μ} . Note that it would be a mistake to assign the scaling $A^{\mu}_{hc} \sim \lambda^{1/2}$ to the gluon field that one finds in Feynman gauge (a=1), even if Feynman gauge is adopted in practical calculations. (This mistake has been made in the literature.)

Let us now discuss the hard-collinear fermion field. For simplicity we will assume zero mass from now on, even though it would be straightforward to include a mass term of $O(\lambda^{1/2})$ or smaller in the effective Lagrangian. We decompose the 4-component Dirac spinor field into two fields

$$\psi_{hc}(x) = \xi(x) + \eta(x), \quad \text{with} \quad \hbar \xi = 0, \quad \hbar \eta = 0.$$
(88)

Using that

$$\frac{\hbar \hbar}{4} + \frac{\hbar \hbar}{4} = \frac{2n \cdot \bar{n}}{4} = 1, \tag{89}$$

it is easy to see that $\frac{1}{4} \not h \not h$ and $\frac{1}{4} \not h \not h$ are projection operators onto these two 2-component spinors, such that

$$\xi = \frac{\hbar \hbar}{4} \psi_{hc}, \qquad \eta = \frac{\hbar \hbar}{4} \psi_{hc}. \tag{90}$$

Consider now the two-point function

$$\langle 0 | T\{\psi_{hc}(x)\,\bar{\psi}_{hc}(y)\} | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \,e^{-ip\cdot x} \,\frac{i}{p^2 + i\epsilon} \,(\not p_+ + \not p_- + \not p_\perp) \,. \tag{91}$$

Projecting onto the various components, we find

$$\langle 0 | T\{\xi(x)\,\bar{\xi}(y)\} | 0 \rangle \sim \lambda ,$$

$$\langle 0 | T\{\eta(x)\,\bar{\eta}(y)\} | 0 \rangle \sim \lambda^2 ,$$

$$\langle 0 | T\{\xi(x)\,\bar{\eta}(y)\} | 0 \rangle , \langle 0 | T\{\eta(x)\,\bar{\xi}(y)\} | 0 \rangle \sim \lambda^{3/2} .$$

$$(92)$$

Obviously, this implies $\xi \sim \lambda^{1/2}$ and $\eta \sim \lambda$. It follows that a hard-collinear fermion spinor has two "large" and two "small" components. The small-component field η can be integrated out from the theory, leaving ξ as the field describing hard-collinear fermions in SCET. Like the soft heavy quarks in HQET, hard-collinear quarks in SCET are thus described by effective 2-component fields.

Effective Lagrangian

Let us first derive how hard-collinear quarks interact with hard-collinear or soft gluons, following closely the derivation presented in [21] (see also [17, 19]). When expressed in terms of ξ and η , the Dirac Lagrangian becomes

$$\mathcal{L} = \bar{\psi} i \mathcal{D} \psi = (\bar{\xi} + \bar{\eta}) i \mathcal{D} (\xi + \eta)$$

$$= \bar{\xi} \frac{\bar{\eta}}{2} i n \cdot D \xi + \bar{\eta} \frac{\dot{\eta}}{2} i \bar{n} \cdot D \eta + \bar{\xi} i \mathcal{D}_{\perp} \eta + \bar{\eta} i \mathcal{D}_{\perp} \xi.$$
(93)

Here the covariant derivative D still contains both the soft and the hard-collinear gluon fields. We now "integrate out" the small-component field η by solving its equation of motion:

$$\frac{\hbar}{2}i\bar{n}\cdot D\,\eta + i\not\!\!D_{\perp}\,\xi = 0 \quad \Rightarrow \quad \eta = -\frac{\hbar}{2}\frac{1}{i\bar{n}\cdot D + i\epsilon}i\not\!\!D_{\perp}\,\xi\,,\tag{94}$$

where the " $i\epsilon$ " prescription is arbitrary but must be fixed once and forever. Note that the result is highly non-local, involving an inverse differential operator acting on the hard-collinear quark field.

It is instructive to compare this solution with the corresponding expression (63) for the "small" components of the heavy-quark field in HQET. In this case, the result was "almost local" in that the derivative in the denominator produces a power of the residual momentum $v \cdot k$, which is much smaller than $2m_Q$. It was therefore possible to expand the expression (63) in an infinite series of local operators, and this produced the terms in the (local) HQET

Lagrangian. In the case of SCET such a local expansion is not possible. In order to proceed, it is useful to introduce the Wilson line

$$W(x) = P \exp\left(ig_s \int_{-\infty}^{0} dt \,\bar{n} \cdot A(x + t\bar{n})\right),\tag{95}$$

where $A^{\mu} = A^{\mu}_{hc} + A^{\mu}_{s}$ still contains both soft and hard-collinear gluons. The ordering prescription is the same as in (72). In analogy with (73), this object obeys the relations

$$W^{\dagger} i \bar{n} \cdot D W = i \bar{n} \cdot \partial , \qquad \frac{1}{i \bar{n} \cdot D + i \epsilon} = W \frac{1}{i \bar{n} \cdot \partial + i \epsilon} W^{\dagger} .$$
 (96)

This allows us to turn the inverse covariant derivative in (94) into an ordinary inverse derivative, i.e., an integral. The result is

$$\eta(x) = W(x) \frac{\rlap/n}{2} \frac{(-1)}{i\bar{n} \cdot \partial + i\epsilon} \left(W^{\dagger} i \not \!\!D_{\perp} \xi \right) (x)
= W(x) \frac{\rlap/n}{2} i \int_{-\infty}^{0} dt \left(W^{\dagger} i \not \!\!D_{\perp} \xi \right) (x + t\bar{n}).$$
(97)

Inserting this into (93) yields the non-local effective Lagrangian

$$\mathcal{L} = \bar{\xi}(x) in \cdot D(x) \, \xi(x) + \left(\bar{\xi} \, i \not\!\!D_{\perp} W\right) \left(x\right) \, \frac{\not\!\!n}{2} \, i \int_{-\infty}^{0} dt \, \left(W^{\dagger} \, i \not\!\!D_{\perp} \, \xi\right) \left(x + t\bar{n}\right). \tag{98}$$

At this stage our job is almost done. The remaining problem we need to deal with is that the above expression does not yet respect a proper power counting, since terms of different orders in λ are mixed up. In order to avoid double counting, it is imperative to expand all objects in the effective Lagrangian such that the power counting is consistent. For instance, the different components of the covariant derivative acting on hard-collinear fields scale like

$$iD^{\mu} = i\partial^{\mu} + g_s A^{\mu}_{hc} + g_s A^{\mu}_s \sim (\lambda, 1, \lambda^{1/2}) + (\lambda, 1, \lambda^{1/2}) + (\lambda, \lambda, \lambda).$$
 (99)

While for $n \cdot D$ all three contributions are of order λ , for $\bar{n} \cdot D$ and D_{\perp} the contributions involving the soft gluon field are power suppressed and should be neglected at leading order. Likewise, in the definition of the Wilson line W in (95) the contribution of the soft gluon field in the exponent is power suppressed and should be neglected. It follows that $W = W_{hc} + O(\lambda)$, where

$$W_{hc}(x) = P \exp\left(ig_s \int_{-\infty}^{0} dt \,\bar{n} \cdot A_{hc}(x + t\bar{n})\right). \tag{100}$$

Finally, in interactions with hard-collinear fields, soft fields must be multi-pole expanded in order to ensure a proper power counting [21]. To see why, consider as an example the phase factor associated with the coupling of a hard-collinear field (incoming momentum p_{hc}) to a soft field (incoming momentum p_s), producing a hard-collinear field (outgoing momentum p_{hc}):

$$S_{\text{int}} \ni \int d^4x \, \phi'_{hc}(x) \, \phi_{hc}(x) \, \phi_s(x) \sim \int d^4x \, e^{i(p'_{hc} - p_{hc} - p_s) \cdot x} \, \phi'_{hc}(0) \, \phi_{hc}(0) \, \phi_s(0) \,. \tag{101}$$

The combined momentum in the exponent scales like a hard-collinear momentum, since $(p'_{hc} - p_{hc} - p_s)^{\mu} \sim (\lambda, 1, \lambda^{1/2})$. Consequently, O(1) contributions to the action arise if $x^{\mu} \sim (1, \lambda^{-1}, \lambda^{-1/2})$. However, in this case the phase factor involving the soft momentum can be Taylor expanded:

$$e^{-ip_s \cdot x} = e^{-ip_{s+} \cdot x_{-}} \left(1 - ip_{s\perp} \cdot x_{\perp} - ip_{s-} \cdot x_{+} + \dots \right), \tag{102}$$

where the terms inside the bracket scale like 1, $\sqrt{\lambda}$, and λ , respectively. It follows that in interactions with hard-collinear fields we should expand soft fields as

$$\phi_s(x) = (1 + x_{\perp} \cdot \partial_{\perp} + \dots) \,\phi_s(x_{-}). \tag{103}$$

It is understood that the derivatives are evaluated before setting $x = x_{-}$ in the argument of the field ϕ_s . At leading order, only the first term on the right-hand side contributes.

With all this insight, we are now prepared to write down the leading-power terms in the SCET Lagrangian. The result is

$$\mathcal{L}_{\text{SCET}} = \bar{\xi}(x) \, in \cdot D_{hc}(x) \, \xi(x) + \bar{\xi}(x) \, g_s n \cdot A_s(x_-) \, \xi(x)$$

$$+ \left(\bar{\xi} \, i \not\!\!D_{hc\perp} W_{hc} \right) (x) \, \frac{\not\!\!n}{2} \, i \int_{-\infty}^0 dt \, \left(W_{hc}^{\dagger} \, i \not\!\!D_{hc\perp} \, \xi \right) (x + t\bar{n}) \, .$$

$$(104)$$

It is easy to check that each term in this Lagrangian scales like λ^2 , which when integrated with the measure $d^4x \sim \lambda^{-2}$ gives a leading, O(1) contribution to the action. It is in principle straightforward to work out the higher-order corrections to the effective Lagrangian, but I will spare you the rather lengthy details and instead refer to the literature [20, 21].

At leading order in power counting, there are no terms in the SCET Lagrangian containing interactions of soft quark fields with hard-collinear fields. There are, however, interactions among soft and hard-collinear gluons. Their explicit form at leading and subleading power can be found in [21]. From the SCET Lagrangian, one can derive Feynman rules for soft and hard-collinear fields in the usual way [19]. While these rules are more complicated that in HQET or even in full QCD, they nevertheless allow for a straightforward calculation of Feynman diagrams in SCET.

Residual gauge invariance

The effective Lagrangian of SCET (including power corrections) is invariant, order by order in λ , under a set of residual hard-collinear and soft gauge transformations, U_{hc} and U_s , that preserve the scaling properties of the various fields [18, 19, 21]. In analogy with (70), one finds that under a soft gauge transformation

$$h_{v}(x) \to U_{s}(x) h_{v}(x) ,$$

$$q_{s}(x) \to U_{s}(x) q_{s}(x) ,$$

$$A_{s}^{\mu}(x) \to U_{s}(x) A_{s}^{\mu}(x) U_{s}^{\dagger}(x) + \frac{i}{g_{s}} U_{s}(x) \left[\partial^{\mu}, U_{s}^{\dagger}(x) \right] ,$$

$$\xi(x) \to U_{s}(x_{-}) \xi(x) ,$$

$$A_{hc}^{\mu}(x) \to U_{s}(x_{-}) A_{hc}^{\mu}(x) U_{s}^{\dagger}(x_{-}) .$$
(105)

Note the multi-pole expansion whenever soft fields are coupled to hard-collinear ones. Similarly, under a hard-collinear gauge transformation

$$\xi(x) \to U_{hc}(x) \, \xi(x) \,,$$

$$n \cdot A_{hc}(x) \to U_{hc}(x) \, n \cdot A_{hc}(x) \, U_{hc}^{\dagger}(x) + \frac{i}{g_s} \, U_{hc}(x) \, [n \cdot D_s(x_-), U_{hc}^{\dagger}(x)] \,,$$

$$A_{hc}^{\mu}(x) \to U_{hc}(x) \, A_{hc}^{\mu}(x) \, U_{hc}^{\dagger}(x) + \frac{i}{g_s} \, U_{hc}(x) \, [\partial^{\mu}, U_{hc}^{\dagger}(x)] \,; \quad \mu \neq + \,, \tag{106}$$

while all soft fields remain invariant. As before, "soft" and "hard-collinear" functions like $U_s(x)$ and $U_{hc}(x)$ can be defined via a restriction to soft and hard-collinear modes in their Fourier decompositions, respectively.

Decoupling transformation

At leading order in λ , soft gluons couple to hard-collinear fields only via interactions involving $n \cdot A_s$. The reason is simply that this is the only component of the soft gluon field which scales in the same way as the corresponding component of the hard-collinear gluon field. In analogy with our discussion for HQET, these interactions can be removed by a field redefinition [19]. Let us define new fields

$$\xi(x) = S_n(x_-) \, \xi^{(0)}(x) \,,$$

$$A_{hc}^{\mu}(x) = S_n(x_-) \, A_{hc}^{\mu(0)}(x) \, S_n^{\dagger}(x_-), \tag{107}$$

where

$$S_n(x_-) = P \exp\left(ig_s \int_{-\infty}^0 dt \, n \cdot A_s(x_- + tn)\right) \tag{108}$$

is a light-like soft Wilson line along the n direction. In terms of the new fields, the SCET Lagrangian

$$\mathcal{L}_{\text{SCET}} = \bar{\xi}^{(0)}(x) \, in \cdot D_{hc}^{(0)}(x) \, \xi^{(0)}(x)$$

$$+ \left(\bar{\xi}^{(0)} \, i \not\!\!D_{hc\perp} W_{hc}^{(0)} \right) (x) \, \frac{\not\!\bar{n}}{2} \, i \int_{-\infty}^{0} dt \, \left(W_{hc}^{\dagger(0)} \, i \not\!\!D_{hc\perp}^{(0)} \, \xi^{(0)} \right) (x + t\bar{n})$$
(109)

has decoupled from soft gluons. As in the case of HQET, to see whether this is useful we will have to study what happens to external operators containing hard-collinear fields.

The decoupling transformation (107), and a similar transformation for the effective theory SCET-2 [24], play an important role in proofs of soft-collinear factorization theorems, see e.g. [19, 25, 26, 27, 28].

Heavy-light currents

Weak-interaction processes are mediated by flavor-changing quark currents, which in QCD have the generic form $J = \bar{\psi} \Gamma Q$, where Q is a heavy quark and Γ denotes some Dirac

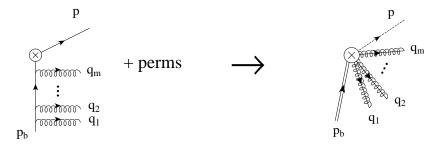


Figure 4: Tree-level matching for heavy-light currents from QCD onto SCET. In the diagram on the right, the double line represents the heavy quark, the dashed line the hard-collinear quark, and the crossed circle the effective current operator. (Figure taken from [17] with permission from the authors)

structure. An important questions is what happens to such operators after matching onto SCET. The naive guess, $J \to \bar{\xi} \Gamma h_v$ is not gauge invariant with respect to the soft and hard-collinear gauge transformations discussed above. Tree-level matching shows that the correct answer (at tree level only!) is [17]

$$J(x) \to (\bar{\xi} W_{hc})(x) \Gamma h_v(x_-),$$
 (110)

which is gauge invariant. This matching is illustrated in Figure 4. Remarkably, the effective current operator in SCET contains an infinite number of vertices involving an arbitrary number of $\bar{n} \cdot A_{hc} \sim 1$ gluons, whose couplings are unsuppressed. All of these vertices are present at leading power in λ .

Beyond tree level, the correct matching relation is yet more complicated. One can show that the most general gauge-invariant form is [21]

$$J(x) \to \sum_{i} \int dt \, \widetilde{C}_{i}(t,\mu) \left(\bar{\xi} W_{hc}\right) (x+t\bar{n}) \, \Gamma_{i} \, h_{v}(x_{-})$$

$$= \sum_{i} C_{i}(\bar{n} \cdot \boldsymbol{P}_{hc},\mu) \left(\bar{\xi} W_{hc}\right) (x) \, \Gamma_{i} \, h_{v}(x_{-}) \,. \tag{111}$$

Here Γ_i are Dirac matrices constructed with the help of the 4-vectors v, n, and \bar{n} , which must have the same quantum numbers as the original matrix Γ . The symbol \boldsymbol{P}_{hc} represents the operator of total hard-collinear momentum, which by definition is the total momentum of all hard-collinear fields. In the last step in (111) we have used translational invariance. Finally, we have defined the Fourier-transformed coefficients

$$C_i(\bar{n} \cdot p, \mu) = \int dt \, e^{i\bar{n} \cdot pt} \, \widetilde{C}_i(t, \mu) \,. \tag{112}$$

Let us see what happens to the result (111) when we apply the decoupling transformations (71) and (107). In terms of the new fields, we find

$$J(x) = \sum_{i} C_{i}(\bar{n} \cdot \boldsymbol{P}_{hc}, \mu) (\bar{\xi}^{(0)} W_{hc}^{(0)})(x) \Gamma_{i} \left[S_{n}^{\dagger}(x_{-}) S_{v}(x_{-}) \right] h_{v}^{(0)}(x_{-}).$$
 (113)

Recall that the redefined heavy-quark field $h_v^{(0)}$ is sterile (it does not couple to anything), while the redefined hard-collinear fields do not couple to soft gluons. Nevertheless, similar to (75), the soft gluon fields have not disappeared from the final expression. Rather, they are once again contained in a soft Wilson loop, $[S_n^{\dagger}(x_-) S_v(x_-)]$. The presence of a cusp at the point x_- implies that the anomalous dimension of the effective current operators contains a term proportional to the cusp anomalous dimension [24],

$$\gamma_{J_{\text{eff}}} = \Gamma_{\text{cusp}}(\alpha_s) \ln \frac{\bar{n} \cdot P_{hc}^{\text{tot}}}{\mu} + \gamma'(\alpha_s).$$
 (114)

The remaining term γ' results from the residual hard-collinear self interactions of the composite field $\bar{\xi}^{(0)}W^{(0)}$ located at point x. The anomalous dimension γ' (114) is known at two-loop order [29], while Γ_{cusp} is known to three loops [30]. With the help of these results, it is possible to resum Sudakov double and single logarithms at NLO in RG-improved perturbation theory (see [29] for an explicit example).

Sample application: Factorization in $B \to X_s \gamma$ decay

As a final, important example I discuss how soft-collinear factorization works for the rare, inclusive decay $B \to X_s \gamma$, following the discussions in [19, 27, 29]. This will provide a prototype application of a two-step matching procedure. In the first step, hard quantum fluctuations at scales $\mu \sim m_b$ or above are integrated out by matching the effective weak Lagrangian onto a bilocal operator in SCET. In the second step, fluctuations at the hard-collinear scale are integrated out by matching this bilocal SCET operator onto a bilocal operator in heavy-quark effective theory. While we will discuss this two-step matching at leading power in $\Lambda_{\rm QCD}/m_b$, the procedure can be extended systematically to higher orders in the heavy-quark expansion [31, 32, 33].

The derivation of the factorization formula proceeds in several steps, which I can only sketch here. In the first step, we use the optical theorem to relate the total decay rate to the imaginary part of the forward scattering amplitude:

$$\Gamma(B \to X_s \gamma) \propto \operatorname{Im} \int d^4 x \langle B | T \{ \mathcal{L}_{\text{eff}}^{b \to s \gamma}(x) \mathcal{L}_{\text{eff}}^{b \to s \gamma}(0) \} | B \rangle.$$
 (115)

In the second step we integrate out hard fluctuations by matching the effective weak Lagrangian $\mathcal{L}_{\text{eff}}^{b\to s\gamma}$ given in (36) onto SCET operators containing a heavy-quark field h_v and a hard-collinear field ξ describing the strange quark. At leading power all other fields are hard and integrated out. The result is (setting $V_{ub} \to 0$ for simplicity) [29]

$$\mathcal{L}_{\text{eff}}^{b \to s \gamma} = \frac{G_F}{\sqrt{2}} V_{cb} V_{cs}^* \frac{e}{2\pi^2} E_{\gamma} \overline{m}_b(\mu) H_{\gamma}(\mu) \left(\bar{\xi} W_{hc} \right) (x) \not\in_{\perp}^* (q) (1 - \gamma_5) h_v(x_-) + \dots , \qquad (116)$$

with

$$H_{\gamma}(\mu) = C_{7\gamma}(\mu) - \frac{1}{3}C_5(\mu) - C_6(\mu) + O(\alpha_s). \tag{117}$$

I won't bother writing down the $O(\alpha_s)$ corrections to the hard function, which have been calculated in the same paper. In the third step, we integrate out hard-collinear fields by

matching the two-point correlator in SCET onto bilocal operators in HQET. Before doing this, we decouple soft interactions from the hard-collinear fields by performing the field redefinition (107). Then all hard-collinear fields are contained in the "jet function" [19, 27]

$$J(x) \equiv \langle 0 | T \left\{ \left(W_{hc}^{\dagger(0)} \xi^{(0)} \right) (x) \left(\bar{\xi}^{(0)} W_{hc}^{(0)} \right) (0) \right\} | 0 \rangle.$$
 (118)

Since the hard-collinear fields live at a perturbative scale of order $\mu_{hc}^2 \sim E_{\gamma} \Lambda_{\rm QCD}$, the jet function can be calculated in perturbation theory. In fact, in the particular gauge $\bar{n} \cdot A_{hc} = 0$ it is nothing but the quark propagator.

After computing the jet function we are left with a *B*-meson forward matrix element of soft heavy-quark fields in HQET, which is a non-perturbative object. The relevant matrix element is

$$\langle B | \bar{h}_v(x_-) [x_-, 0] h_v(0) | B \rangle$$
, (119)

where the product $[x_-, 0] \equiv S_n(x_-) S_n^{\dagger}(0)$ is a straight soft Wilson line along the light-cone. This product arises from the field redefinition of the hard-collinear fields under the decoupling transformation (107). The Fourier transform of the above matrix element is called the shape function of the B meson and denoted by $S(\hat{\omega}, \mu)$ [34]. It is a conventional parton distribution function, defined however in the context of HQET. The shape function is a simple, non-perturbative function describing the internal dynamics of the B meson.

We are now ready to collect the result for the inclusive decay rate or, more precisely, for the photon-energy spectrum in the region of large photon energy, where the collinear expansion is justified. After Fourier transforming to momentum space, one obtains [29, 35]

$$\frac{d\Gamma(B \to X_s \gamma)}{dE_{\gamma}} = \frac{G_F^2 \alpha}{2\pi^4} |V_{cb}V_{cs}^*|^2 \overline{m}_b^2(\mu) |H_{\gamma}(\mu)|^2 E_{\gamma}^3$$

$$\times \int_0^{M_B - 2E_{\gamma}} d\hat{\omega} J(m_b(M_B - 2E_{\gamma} - \hat{\omega}), \mu) S(\hat{\omega}, \mu) + \dots, \qquad (120)$$

where the ellipses represent power-suppressed terms. This formula achieves a complete factorization (separation) of short- and long-distance physics. The short-distance physics resides in the perturbative hard function H_{γ} and in the perturbative jet function J. The characteristic scales associated with these objects are $\mu_h \sim E_{\gamma} \sim m_b$ and $\mu_{hc} \sim \sqrt{E_{\gamma}\Lambda_{\rm QCD}}$, respectively. The hadronic physics is parameterized by the non-perturbative shape function. Large logarithms associated with the various scales can be resummed to all orders in perturbation theory by solving the RG evolution equations for the hard, jet, and shape functions. This is discussed in detail in the literature [29, 36].

Relation (120) is a beautiful example of a QCD factorization formula. The technology of SCET and HQET, and of EFT in general, have made the derivation of such factorization statements a bit more straightforward then it was previously. Similar relations can be established for many other processes in B physics [25, 26, 28, 37] and elsewhere (e.g., in DIS and jet physics [38]). In many cases factorization is a property of the amplitude in the large-energy limit, but non-factorizable corrections arise at subleading order in the power expansion.

Prominent examples are exclusive B decays such as $B \to \pi\pi$ or $B \to K^*\gamma$, which play an important role in the physics program of the B-factories. In the case of inclusive B decays considered in this lecture, a factorization formula can be established at every order in the 1/E expansion [31, 32, 33]. Inclusive B-decay distributions are therefore examples of a small class of observables for which a systematic, field-theoretic description of power corrections can be given in terms of non-local string operators (operators whose component fields are separated in space-time), in generalization of the local OPE valid in Euclidean space.

Acknowledgments

I am grateful to the organizers of TASI 2004 for the invitation to present these lectures and for a flawless organization of the school. I am grateful to the students for their interest in the material presented here and for many lively discussions. I am particularly grateful to John Terning for his patience in waiting for these notes to appear. I like to thank the Institute for Theoretical Physics at the University of Heidelberg, Germany, for its hospitality during the completion of these notes. Work on this project was supported in part by a Research Award of the Alexander von Humboldt Foundation, and by the National Science Foundation under Grant PHY-0355005.

References

- [1] J. Charles et al. [CKMfitter Group], Eur. Phys. J. C 41, 1 (2005) [hep-ph/0406184].
- [2] An excellent introduction to this subject, which in part overlaps with the material presented in this lecture, can be found in the TASI 1992 lectures by J. Polchinski, "Effective field theory and the Fermi surface", hep-th/9210046.
- [3] N. Arkani-Hamed and S. Dimopoulos, JHEP **0506**, 073 (2005) [hep-th/0405159].
- [4] N. Arkani-Hamed, S. Dimopoulos, G. F. Giudice and A. Romanino, Nucl. Phys. B **709**, 3 (2005) [hep-ph/0409232].
- [5] S. Weinberg, Phys. Rev. Lett. **59**, 2607 (1987).
- [6] V. Agrawal, S. M. Barr, J. F. Donoghue and D. Seckel, Phys. Rev. D 57, 5480 (1998) [hep-ph/9707380].
- [7] All one would ever want to know about the formalism of the effective weak Lagrangian (and more) can be found in the Les Houches lectures by A. J. Buras, "Weak Hamiltonian, CP violation and rare decays", hep-ph/9806471.
- [8] For a review, see: G. Buchalla, A. J. Buras and M. E. Lautenbacher, Rev. Mod. Phys. 68, 1125 (1996) [hep-ph/9512380].
- [9] H. Georgi, Phys. Lett. B **240**, 447 (1990).
- [10] M. Neubert, Phys. Rept. 245, 259 (1994) [hep-ph/9306320].

- [11] A. V. Manohar and M. B. Wise, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. 10, 1 (2000).
- [12] G. Amoros, M. Beneke and M. Neubert, Phys. Lett. B **401**, 81 (1997) [hep-ph/9701375].
- [13] N. Isgur and M. B. Wise, Phys. Lett. B 232, 113 (1989); N. Isgur and M. B. Wise, Phys. Lett. B 237, 527 (1990).
- [14] G. P. Korchemsky and A. V. Radyushkin, Nucl. Phys. B 283, 342 (1987).
- [15] A. F. Falk, H. Georgi, B. Grinstein and M. B. Wise, Nucl. Phys. B 343, 1 (1990).
- [16] E. Eichten and B. Hill, Phys. Lett. B 234, 511 (1990). E. Eichten and B. Hill, Phys. Lett. B 243, 427 (1990).
- [17] C. W. Bauer, S. Fleming, D. Pirjol and I. W. Stewart, Phys. Rev. D 63, 114020 (2001) [hep-ph/0011336].
- [18] C. W. Bauer and I. W. Stewart, Phys. Lett. B **516**, 134 (2001) [hep-ph/0107001].
- [19] C. W. Bauer, D. Pirjol and I. W. Stewart, Phys. Rev. D **65**, 054022 (2002) [hep-ph/0109045].
- [20] J. Chay and C. Kim, Phys. Rev. D 65, 114016 (2002) [hep-ph/0201197].
- [21] M. Beneke, A. P. Chapovsky, M. Diehl and T. Feldmann, Nucl. Phys. B 643, 431 (2002) [hep-ph/0206152].
- [22] R. J. Hill and M. Neubert, Nucl. Phys. B 657, 229 (2003) [hep-ph/0211018].
- [23] T. Becher, R. J. Hill and M. Neubert, Phys. Rev. D 69, 054017 (2004) [hep-ph/0308122].
- [24] T. Becher, R. J. Hill, B. O. Lange and M. Neubert, Phys. Rev. D 69, 034013 (2004) [hep-ph/0309227].
- [25] C. W. Bauer, D. Pirjol and I. W. Stewart, Phys. Rev. Lett. 87, 201806 (2001) [hep-ph/0107002].
- [26] S. W. Bosch, R. J. Hill, B. O. Lange and M. Neubert, Phys. Rev. D 67, 094014 (2003) [hep-ph/0301123].
- [27] S. W. Bosch, B. O. Lange, M. Neubert and G. Paz, Nucl. Phys. B 699, 335 (2004) [hep-ph/0402094].
- [28] T. Becher, R. J. Hill and M. Neubert, Phys. Rev. D **72**, 094017 (2005) [hep-ph/0503263].
- [29] M. Neubert, Eur. Phys. J. C $\bf 40$, 165 (2005) [hep-ph/0408179].
- [30] S. Moch, J. A. M. Vermaseren and A. Vogt, Nucl. Phys. B **688**, 101 (2004) [hep-ph/0403192].

- [31] K. S. M. Lee and I. W. Stewart, Nucl. Phys. B **721**, 325 (2005) [hep-ph/0409045].
- [32] S. W. Bosch, M. Neubert and G. Paz, JHEP **0411**, 073 (2004) [hep-ph/0409115].
- [33] M. Beneke, F. Campanario, T. Mannel and B. D. Pecjak, JHEP 0506, 071 (2005) [hep-ph/0411395].
- [34] M. Neubert, Phys. Rev. D 49, 3392 (1994) [hep-ph/9311325]; M. Neubert, Phys. Rev. D 49, 4623 (1994) [hep-ph/9312311].
- [35] G. P. Korchemsky and G. Sterman, Phys. Lett. B **340**, 96 (1994) [hep-ph/9407344].
- [36] M. Neubert, Phys. Rev. D **72**, 074025 (2005) [hep-ph/0506245].
- [37] M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, Phys. Rev. Lett. 83, 1914 (1999) [hep-ph/9905312]; M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, Nucl. Phys. B 591, 313 (2000) [hep-ph/0006124]; M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, Nucl. Phys. B 606, 245 (2001) [hep-ph/0104110].
- [38] C. W. Bauer, S. Fleming, D. Pirjol, I. Z. Rothstein and I. W. Stewart, Phys. Rev. D 66, 014017 (2002) [hep-ph/0202088].