SO(10) UNIFICATION

- IN SU(5), ALL SM FERMIONS IN $\bar{5} \oplus 10$
- ALSO, ANOMALY CANCELLATION BETWEEN $\bar{5}$ AND 10
- UNIFICATION $SU(5) \supset SU(3) \times SU(2) \times U(1)$ BUT DOES NOT QUITE WORK (MISMATCH, UNLESS SUSY, AND CONSTRAINTS FROM PROTON DECAY)
- NEUTRINOS ARE MASSLESS !

RECAP SM NEUTRINOS ARE IN SU(2) DOUBLETS
$\Longrightarrow$ L-HANDED

- 1st WAY TO HAVE A MASS TERM : À LA DIRAC

$$\mathcal{L} \supset m \bar{\psi}\psi = m \left(\nu_L^+ \nu_R + \nu_R^+ \nu_L \right)$$

$\Longrightarrow$ NEED BOTH ν_L AND ν_R STATES
$\downarrow$
"ABSENT" IN SM

ANYWAY ν_L IN DOUBLET, SO WE NEED

$$\mathcal{L} \supset y_\nu \bar{L} \tilde{\phi} \nu_R + h.c.$$

WITH $\tilde{\phi} = i\sigma^2 \phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix}^* = \begin{pmatrix} \phi_0^* \\ -\phi^- \end{pmatrix}$

NOTE THAT $\bar{L}\tilde{\phi} \sim (1, 1, 0)$ SO THIS COMBINATION IS A SINGLET OF THE SM GAUGE GROUPS

SO, AS EXPECTED $\nu_R \sim (1,1,0)$ ALSO SINGLET

$\Rightarrow$ ν_R IS A TOTALLY INERT STATE, BUT FOR A POSSIBLE YUKAWA COUPLING.

. IN OTHER WORDS, NOTHING PREVENTS HAVING THE COUPLINGS IN THE SM

$$\mathcal{L} \supset y_{ij} \bar{L}_i \tilde{\phi} \nu_{R_j} + h.c.$$

$$\xrightarrow{SSB} \underbrace{y_{ij} \frac{v}{\sqrt{2}}}_{\text{NEUTRINO MASS MATRIX}} \nu^+_{L_i} \nu_{R_j} + h.c.$$

. TWO INTERESTING POINTS HERE:

1) TYPICAL ENTRY IS $m_\nu \sim y v$

FROM NEUTRINO OSCILLATIONS

$$\Delta m^2_{12} \simeq 7.6 \times 10^{-5}\, eV^2$$
$$\Delta m^2_{23} \simeq 2.4 \times 10^{-3}\, eV^2$$

DIRECT BOUNDS (LIGHTEST COULD STILL BE ZERO)

$\nu_e < 0.086\, eV$ (COSMO!)

$\nu_\mu < 170\, keV$

$\nu_\tau < 18\, MeV$

REGARDLESS, REQUIRES VERY SMALL YUKAWA COUPLINGS $y \lesssim 10^{-11}$ $\Leftrightarrow$ SEEN AS BEING UNNATURAL!

2) A REMINDER. IN "SM", NO CKM MATRIX IN LEPTONIC SECTOR

$\Rightarrow$ + DIAGONALIZE CHARGED LEPTON YUKAWA MATRIX $\mathcal{M}$

SAY $e_L' = U_e e_L$ AND $e_R' = V_e e_R$

$$m_e = V_e \mathcal{M} U_e^+$$

+ LEPTONIC CHARGED CURRENTS

$$\mathcal{L} \supset \frac{g}{\sqrt{2}} W_\mu^+ \bar{\nu}_L \gamma^\mu e_L + h.c.$$

$$\longrightarrow \frac{g}{\sqrt{2}} W_\mu^+ \bar{\nu}_L \gamma^\mu U_e^+ e_L' + h.c.$$

+ DEFINE $\nu_L' = U_e^+ \nu_L$

$\Longrightarrow$ UNOBSERVABLE IF NEUTRINOS ARE MASSLESS! (ACTUALLY MASS DEGENERATE)

$\Longrightarrow$ 3 CONSERVED LEPTONIC CURRENTS L_e, L_μ, L_τ

$\Longrightarrow$ ANY BREAKING MUST BE PROPORTIONAL TO ν MASS DIFFERENCES!

$\Longrightarrow$ TINY EFFECTS, DIFFICULT TO DETECT

* A DIRAC MASS, WHILE POSSIBLE, SEEMS UNNATURAL, GIVEN THE UNBEARABLE LIGHTNESS OF NEUTRINOS.
BUT THERE IS ANOTHER WAY.

* 2nd WAY TO HAVE A MASS TERM: À LA MAJORANA

. IN PREVIOUS LECTURES, WE HAVE SEEN THAT

$\chi_L^c = \sigma_2 \chi_L^* \sim \chi_R$ MEANING, TRANSFORMS UNDER LORENTZ TRANSFORMATIONS AS A LEFT-HANDED WEYL SPINOR.

. BACK TO DIRAC

$$\mathcal{L}_D = m\, \chi_R^+ \chi_L + h.c.$$
$$= m\, \chi_L'^{T} \sigma_2 \chi_L + m\, \chi_L^+ \sigma_2 \chi_L'^*$$
$$= m\, \chi_L'^{c+} \chi_L + h.c.$$

$$\Longrightarrow \chi_L' \neq \chi_L \; !$$

BUT HOW ABOUT

$$\chi_L' = \chi_L \, ?$$

$$\mathcal{L}_M = \frac{1}{2} m \left[\chi_L^T \sigma_2 \chi_L + \chi_L^+ \sigma_2 \chi_L^* \right]$$
$$= \frac{1}{2} m\, \chi_L^{c+} \chi_L + h.c.$$

= MAJORANA MASS TERM

Ex: SHOW THAT IT IS CRUCIAL TO TAKE INTO ACCOUNT THE FACT THAT WEYL SPINORS ANTI-COMMUTE (ie ARE FERMIONS) TO HAVE A MAJORANA MASS TERM

FROM THERE, 2 ROADS

① EFFECTIVE OPERATOR PERSPECTIVE.

- $\bar{L}\tilde{\phi}$ is SINGLET $\sim \frac{v}{\sqrt{2}} \nu_L^+$ AFTER SSB

$$\mathcal{L}_{eff} \supset \frac{1}{\Lambda} \bar{L}^c \tilde{\phi}^* \underbrace{\tilde{\phi}^+ L}_{\text{DIMENSION } 1+3/2 = 5/2} + h.c.$$

$\Longrightarrow$ DIMENSION 5 OPERATOR = WEINBERG OPERATOR

$\Lambda \sim$ SOME (SUPPOSEDLY LARGE) SCALE $\gg v$

- AFTER SSB, THE WEINBERG OPERATOR GIVES

$$\mathcal{L}_{eff} \supset \frac{v^2}{\Lambda} \nu_L^{c+} \nu_L + h.c.$$

$\uparrow$ NEUTRINO MASS is $m_\nu \sim \frac{v^2}{\Lambda}$

Ex: TAKE $m_\nu \lesssim$ eV AND $v = 250$ GeV

$\Longrightarrow \Lambda \gtrsim$?

BEAUTIFUL, THE LOGIC OF EFT IS THAT THE WEINBERG OPERATOR SHOULD BE PRESENT AT SOME LEVEL. IF $\Lambda \gg v$ (SEE EX) THEN SM NEUTRINOS ARE <u>LIGHT</u> AND <u>MAJORANA</u>

$\Longrightarrow$ <u>LEPTON NUMBER IS NOT CONSERVED</u>!

$$\text{if } \boxed{\begin{array}{l} L \to e^{i\alpha} L \\ \phi \to \phi \end{array}} \qquad \begin{array}{l} \bar{L}^c \tilde{\phi}^* \longrightarrow e^{+i\alpha} \bar{L} \tilde{\phi}^* \\ \tilde{\phi}^\dagger L \longrightarrow e^{i\alpha} \tilde{\phi}^\dagger L \end{array}$$

$$\mathcal{L}_{\text{WEINBERG}} \longrightarrow e^{2i\alpha} \mathcal{L}_{\text{WEINBERG}}$$

- DIAGRAMMATICALLY

② <u>STERILE NEUTRINO = SINGLET STATE PERSPECTIVE</u>

$$\mathcal{L} \supset y \bar{L} \phi \nu_R + h.c. \quad \sim \text{DIRAC YUKAWA}$$

$$+ \frac{1}{2} M \bar{\nu}_R^c \nu_R + h.c. \quad \sim \text{MAJORANA MASS FOR } \nu_R$$

MASS MATRIX ? AFTER SSB

$$\mathcal{L} \supset \begin{pmatrix} \bar{\nu}_L^c & \bar{\nu}_R \end{pmatrix} \underbrace{\begin{pmatrix} 0 & gv \\ gv & M \end{pmatrix}}_{\mathcal{M}} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix}$$

$$\mathrm{Tr}\,\mathcal{M} = M = m_1 + m_2 \simeq m_1$$

$$\det \mathcal{M} = -g^2 v^2 = m_1 m_2 \implies m_2 \simeq -\frac{g^2 v^2}{M}$$

ex: CHECK BY DIAGONALIZING $\mathcal{M}$, CHECK THAT BOTH ARE MAJORANA

$\implies$ 2 MASS EIGENSTATES, ONE HEAVY $\sim M$

ONE LIGHT $\sim \frac{g^2 v^2}{M}$

COMPARING WITH WEINBERG OPERATOR

$$M \sim g\Lambda \quad \text{if } g \sim O(1) \quad M \sim \Lambda$$

BUT CAN BE PARAMETRICALLY LIGHTER

$\implies$ WEINBERG OPERATOR COMES FROM INTEGRATING OUT A HEAVY MAJORANA!

LEAVES OPEN THE QUESTION OF THE ORIGIN OF THESE ν_R?
IN SM, ALL KNOWN FERMIONS HAVE GAUGE INTERACTIONS
$\implies$ GUT ORIGIN?

BACK TO SO(10)

- SU(5) is RANK 4 $\supset SU(3)\times SU(2)\times U(1)$
- NEXT IS $SU(6)\supset SU(5)\times U(1)$ = RANK 5

  ↳ FUNDAMENTAL = 6 AND $\bar{6}$

  DOES NOT GO TO INCLUDING $\bar{5}$ AND 10 of SU(5)

MORE INTERESTING IS $SO(10) \sim SO(2n)$ ← rank n

DEFINING REPRESENTATION IS VECTORS v^i $i=1\ldots10$

BUT THERE ARE ALSO SPINOR REPRESENTATIONS

↳ JUST LIKE FOR $SO(3)\sim SU(2)$

AND $SO(4)\sim SU(2)\times SU(2)$

CLIFFORD ALGEBRAS AND SPINOR REPS

→ $2n$ HERMITIAN MATRICES γ_i $i=1\ldots2n$

SO THAT

$$\{\gamma_i,\gamma_j\}=2\delta_{ij}$$

$\underline{n=1}$ $\gamma_1=\sigma_1$ AND $\gamma_2=\sigma_2$ (REMEMBER SO(1,1) FROM ANOMALIES IN D=2)

$\underline{n>1}$ $\gamma_i^{(n+1)}=\gamma_i^{(n)}\otimes\sigma_3=\begin{pmatrix}\gamma_i^{(n)} & 0\\ 0 & -\gamma_i^{(n)}\end{pmatrix}$ $i=1,\ldots,2n$

$2^n\times2^n$ MATRICES!

$$\gamma_{2n+1}^{(n+1)}=\mathbb{1}_n\otimes\sigma_1=\begin{pmatrix}0&\mathbb{1}_n\\ \mathbb{1}_n&0\end{pmatrix}$$

$$\gamma_{2n+2}^{(n+1)}=\mathbb{1}_n\otimes\sigma_2=\begin{pmatrix}0&-i\mathbb{1}_n\\ i\mathbb{1}_n&0\end{pmatrix}$$

Ex: CHECK THAT

THEN

$$\gamma_{2k-1} = \underbrace{1_2 \otimes \cdots \otimes 1_2}_{k-1} \otimes \sigma_1 \otimes \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-k}$$

$$\gamma_{2k} = \overbrace{1_2 \otimes \cdots \otimes 1_2} \otimes \sigma_2 \otimes \overbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}$$

Ex CHECK THIS FOR $SO(4)$, $SO(6)$, ...

LIKE FOR MINKOWSKI, DEFINE

$$\sigma_{ij} = \frac{i}{2}[\gamma_i, \gamma_j]$$

$J_{ij} = \frac{1}{2}\sigma_{ij}$ = GENERATORS (SPINORIAL REPS) OF $SO(2n)$

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ex. CHECK THAT

$\boxed{n=3}$ $$[\sigma_{12}, \sigma_{23}] = 2i\,\sigma_{13}$$

AND THAT

$$[\sigma_{12}, \sigma_{34}] = 0$$

THE $2^n \times 2^n$ MATRICES ACT ON 2^n COMPONENTS ψ = $SO(2n)$ SPINORS

$$\psi \to e^{i\frac{\omega_{ij}}{2}\sigma_{ij}}\psi \quad \text{WITH } \omega_{ij} = -\omega_{ji}$$

$= n(2n-1)$ GENERATORS

REM: $SO(2n)$ = GROUP THAT PRESERVES EUCLIDEAN SCALAR PRODUCT $\vec{x}' \cdot \vec{y}' = \vec{x}\cdot\vec{y}$

WITH $\quad \vec{x}' = R\vec{x}$

INF. FORM $\quad R = \mathbb{1} + \omega$

$$x'_i = (\delta_{ij} + \omega_{ij})\, x_j$$

$$\Longrightarrow \quad x'_i y'_j \delta_{ij} = (\delta_{ik} + \omega_{ik})(\delta_{je} + \omega_{je})\delta_{ke}\, x_k y_e$$

$$= (\delta_{ij} + \omega_{ij} + \omega_{ji})\, x_i y_j$$

$$= \delta_{ij}\, x_i y_j$$

$\Longrightarrow \quad \omega_{ij} + \omega_{ji} = 0 \quad \Longleftrightarrow \quad n(2n-1)$ PARAMETERS.

Ex CHECK THAT $\psi \to e^{i\frac{\omega_{ij}}{2}J_{ij}}\psi$ WITH $J_{ij} = \frac{1}{2}\sigma_{ij}$ IMPLIES THAT

$\psi^\dagger \gamma_k \psi$ TRANSFORMS AS A VECTOR OF $SO(2n)$

DEFINE

$$\gamma_5 = \underbrace{\sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3}_{n}$$

$\Longrightarrow \quad P_L = \frac{1-\gamma_5}{2} \qquad$ AND $\qquad \psi_L = P_L\psi \ ; \ \psi_R = P_R\psi$

$\qquad\quad P_R = \frac{1+\gamma_5}{2}$

$\uparrow$

⚠ IN $SO(2n)$

$\neq$ CHIRALITY IN SPACE-TIME

eg $SO(10)$ $n=5$

$\psi \sim 32 = 2^5 \Rightarrow \psi_L$ (or ψ_R) ~ 16

AND $16 = 5 + 10 + \boxed{1}$

FOR THIS TO BE OF ANY USE, WE NEED TO CHECK IF $SO(10) \supset SU(3) \times SU(2) \times U(1)$

ACTUALLY, IT IS NATURAL TO EMBED $SU(\underline{5})$ IN $SO(\underline{10})$ (BECAUSE $5 = 10/2$)

REM $SO(2) \longleftrightarrow \begin{pmatrix} x \\ y \end{pmatrix}$

$U(1) \longleftrightarrow z = x + iy$

$SO(2n)$: $\vec{x} = (x_1 \ldots x_n, y_1 \ldots y_n)$

$\vec{x}' = (x'_1 \ldots x'_n, y'_1 \ldots y'_n)$

$\vec{x} \cdot \vec{x}' = \sum_{i=1}^{n} (x_i x'_i + y_i y'_i)$ INVARIANT UNDER $SO(2n)$

$\Rightarrow$ $z = (x_1 + i y_1, \ldots, x_n + i y_n)$

$z' = (x'_1 + i y'_1, \ldots, x'_n + i y'_n)$ ACT ON BY $SU(n)$

$$\bar{z}' \cdot z = \sum_{i=1}^{n} (x_i' - i y_i')(x_i + i y_i)$$

CONJUGATE

$$= \sum_{i=1}^{n} (x_i' x_i + y_i' y_i)$$

$$+ i \sum_{i=1}^{n} (x_i' y_i - x_i y_i')$$

$\Rightarrow$ $SO(2n)$ LEAVES

$$\sum_{i=1}^{n} (x_i' x_i + y_i' y_i) \text{ INVARIANT}$$

WHILE $SU(n)$ LEAVES BOTH

$$\sum_{i=1}^{n} (x_i' x_i + y_i' y_i) \text{ AND } \sum_{i=1}^{n} (x_i' y_i - x_i y_i')$$

INVARIANT: MORE RESTRICTIVE $\Rightarrow$ $SU(n)$ = SUBGROUP OF $SO(2n)$

FROM THIS

$$2n \text{ OF } SO(2n) \longrightarrow n + \bar{n} \text{ OF } SU(n)$$

EG

VECTOR REPRESENTATION

10 OF $SO(10)$ $\longrightarrow$ $5 \oplus \bar{5}$ OF $SU(5)$

$(x_1 \ldots x_5, y_1 \ldots y_5) \to (x_1 + iy_1 \ldots x_5 + iy_5)$

AND

$(x_1 - iy_1 \ldots x_5 - iy_5)$

? SPINOR REPRESENTATION ?

- DO IT FOR $SO(10)$ SO 16 SPINORIAL REPS
- MOST INVOLVE 1, 5, 10 OR 15 OF $SU(5)$

① ADJOINT OF $SO(10)$ IS $45 = 5 \times 9$

$45 \sim x^i y^j - x^j x^i$ ~ ANTISYMMETRIC TENSOR

SO $45 \sim 10 \otimes_A 10$ (~ TRANSFORMS LIKE → DECOMPOSE INTO)

BUT $10 \longrightarrow 5 \oplus \bar{5}$

SO $10 \otimes_A 10 \longrightarrow (5 \oplus \bar{5}) \otimes_A (5 \oplus \bar{5})$

$5 \otimes_A 5 \sim 10$

$\bar{5} \otimes_A \bar{5} \sim \bar{10}$

THAT LEAVES $45 - 2 \times 10 = 25 = 24 + 1$

$\downarrow$ ADJOINT $\searrow$ SINGLET

SO $45 \longrightarrow 24 \oplus 10 \oplus \bar{10} \oplus 1$

ADJOINT OF SO(10)

ADJOINT OF SU(5)

$\Downarrow$ TRANSFORM SU(5) REPS INTO THEMSELVES (BECAUSE THIS IS WHAT THE ADJOINT DOES)

SINGLET = DOES NOTHING = PHASE

HOW ABOUT THE 10 ACTING ON 16?

- ASSUME $16 \supset 1$ OF SU(5)

THEN $10 \otimes 1 = 10$ OF SU(5)

SO $16 \supset 10$

- HOW ABOUT 10 ON 10?

= 2 ANTISYMMETRIC INDICES

= ANTISYMMETRIC INDICES

$10 \otimes 10 \sim$ 4 UPPER INDICES

MUST CONTAIN u^{ijke}

TOTALLY ANTISYMMETRIC

BUT BY

$$\varepsilon_{ijkem}\, u^{ijke} \sim u_m \sim \bar{5}$$

SO $\boxed{16 = 1 \oplus \bar{5} \oplus 10}$ (SAY χ_L)

OF COURSE

$$\overline{16} = 1 \oplus 5 \oplus \overline{10}$$

SO 16 IS A COMPLEX REPRESENTATION

(CAN CONTAIN CHIRAL MODES)

THAT DECOMPOSES INTO $\bar{5}$ AND 10

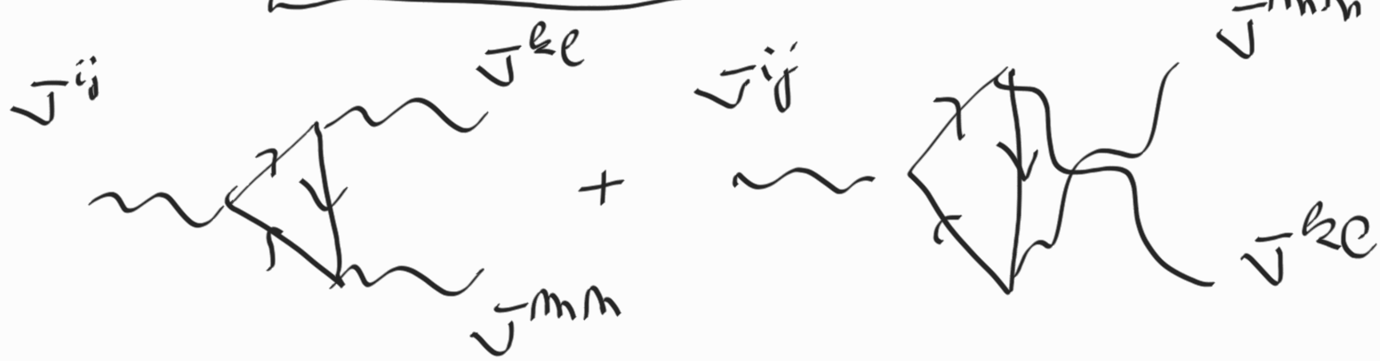
OF $SU(5)$. JUST ONE WE NEED, PLUS

AN EXTRA, 1, SINGLET OF $SU(5)$

AND THUS $SU(3) \times SU(2) \times U(1)$ = STERILE

NEUTRINO !!...

ANOMALIES?



MUST CANCEL FOR $SO(10)$ AND FERMIONS IN 16

$$\Rightarrow \quad \mathrm{Tr}\left[J^{ij} \{ J^{kl}, J^{mn} \} \right]$$

$$= A^{ijklmn}$$ = INVARIANT TENSOR of $SO(10)$

SUCH THAT $A^{ijklmn} = -A^{jiklmn}$ etc.

ONE CAN SHOW THAT THERE IS NO SUCH TENSOR FOR $SO(2n)$ EXCEPT FOR $SO(6) \sim SU(4)$

$SU(n)$ THEORIES HAVE

$$\mathrm{Tr}\left[T_r^A \{ T_r^B, T_r^C \} \right] = A(r)\, d^{ABC}$$

SO 16 IS ANOMALY FREE, AUTOMATICALLY!