AFTER CHIRAL ANOMALIES WE AT LAST TURN TO GRAND UNIFIED THEORIES OR GUT.

THE FIRST EXAMPLE IS BASED ON $SU(5)$.

WHAT IS A GAUGE THEORY? A CHOICE OF GAUGE FIELDS + MATTER REPRESENTATIONS

STANDARD MODEL : $SU(3) \otimes SU(2) \otimes U(1)$ $\qquad Y/2$

TAKE QUARK DOUBLET $\quad Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \sim (3, 2, \tfrac{1}{6})_L$ LORENTZ

THEN QUARK U $\qquad U_R \sim (3, 1, 2/3)_R$

SO ONE FAMILY IS 15 FIELDS (IN COMPONENTS)

$(3\ 2\ \tfrac{1}{6})_L, (3, 1, 2/3)_R, (3, 1, -1/3)_R, (1, 2, -\tfrac{1}{2})_L, (1, 1, -1)_R$

THESE ARE THE BUILDING BLOCKS OF THE SM (3 TIMES AS THERE ARE 3 FAMILIES)

NOW $G \supset SU(3) \otimes SU(2) \otimes U(1)$ WITH (HOPEFULLY) FEWER BLOCKS?

FIRST MAKE THEM ALL L-HANDED (HENCE NO L SUBSCRIPT):

$(3\ 2\ \tfrac{1}{6}), (\bar{3}, 1, -2/3), (\bar{3}, 1, 1/3), (1, 2, -1/2), (1, 1, 1)$

$\bar{3}$ = COMPLEX CONJUGATE OF 3

SMALLEST SIMPLE GROUP THAT CONTAINS $SU(3) \otimes SU(2) \otimes U(1)$ IS $SU(5)$.

THE HAVE THE SAME RANK = NUMBER OF DIAGONAL GENERATORS.

2 FOR $SU(3)$ + 1 FOR $SU(2)$ + 1 FOR $U(1)$ (TRIVIALLY)

AND 4 FOR $SU(5)$

CLEARLY, THE RANK OF $SU(N)$ GROUPS IS $N-1$

NB: RANK? GENERATORS ARE HERMITIAN $\Rightarrow$ CAN BE DIAGONALIZED WITH REAL EIGENVALUES.
NOT POSSIBLE FOR ALL OF THEM (OR THE GROUP GENERATORS WOULD COMMUTE). THE LARGEST SET HAS DIMENSION = RANK

THE ARE ALSO TRACELESS $\Rightarrow$ $\sum_i \lambda_i^{(A)} = 0$ FOR A GIVEN (A) DIAGONALIZED GENERATORS.

$SU(N)$: T^A ARE $N\times N$ $\Rightarrow$ N EIGENVALUES -1
THUS THE SPACE OF COMMUTING GENERATORS HAS DIM $N-1$

NB: RANK OF $SO(2N)$ GROUPS IS N. SHOW THAT FOR $SU(5)$

$SU(5)$ HAS $N^2-1 = 24$ GENERATORS IN THE FUNDAMENTAL REP, 5×5 MATRICES; 8 ARE OF THE FORM

$$\begin{pmatrix} \boxed{3\times 3} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix}$$

AND 3 OF THE FORM

$$\begin{pmatrix} \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \boxed{2\times 2} \end{pmatrix}$$

THEY COMMUTE WITH EACH OTHERS $\Leftrightarrow$ SUBGROUPS $\sim SU(3)\otimes SU(2)$

FURTHERMORE, THEY COMMUTE WITH

$$\frac{Y}{2} = \begin{pmatrix} -\frac{1}{3} & & & & 0 \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ 0 & & & & \frac{1}{2} \end{pmatrix}$$

WE CALL THE HYPERCHARGE.

THAT MAKES 4 DIAGONAL GENERATORS USING

$$\lambda^3 = \frac{1}{2}\begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 0 \end{pmatrix} \quad \text{AND} \quad \lambda^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & -2 \end{pmatrix} \quad \text{FOR } SU(3)$$

$$\text{AND} \quad \frac{\sigma^3}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{FOR } SU(2)$$

REM: $\quad \mathrm{Tr}\left[\left(\frac{Y}{2}\right)^2\right] = \frac{1}{2}$ i.e. WE USE THE NORMALIZATION EXPLAINED IN THE PREVIOUS LECTURE.

LET US FIRST ACT ON A 5-DIMENSIONAL OBJECT ψ^a WITH $a=1\dots5$. WE SPLIT IT INTO $a = I, j$ WITH $I=1,2,3 \sim SU(3)$ AND $i=1,2 \sim SU(2)$

i.e.

$$\psi = \left(\psi^a\right) = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \\ \psi^5 \end{pmatrix} = \begin{pmatrix} \psi^I \\ \hdashline \psi^i \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} 3 \text{ NUMBERS (WELL, L-SPINORS)} \\ \left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right\} 2 \end{matrix}$$

IT IS NATURAL TO TAKE $\psi^I \sim 3$, A TRIPLET OF $SU(3)$, AND $\psi^i \sim 2$, A DOUBLET OF $SU(2)$.
BUT $\psi^I \sim 1$ UNDER $SU(2)$ TRANSFORMATIONS AND $\psi^i \sim 1$ UNDER $SU(3)$! HENCE

$$5 \longrightarrow (3,1,-1/3) \oplus (1,2,1/2)$$

OR

$$\bar{5} \longrightarrow \underbrace{(\bar{3},1,1/3)}_{\sim\, d^c} \oplus \underbrace{(1,2,-1/2)}_{L}$$

WE COULD FIT 2 OUT OF 5 REPRESENTATIONS OF THE SM!
ALSO, OUT OF 24 GENERATORS, $8+3+1$ DON'T MIX THE d^c AND L COMPONENTS, BUT 12 DO!
THEY TRANSFORM LEPTONS INTO (ANTI)QUARKS!
(THEY ARE SOMETIME CALLED LEPTO-QUARKS)

WE FITTED $\bar{5}$ FIELDS ; WE ARE LEFT WITH 10:

$$\underbrace{(3, 2, 1/6)}_{6} \oplus \underbrace{(\bar{3}, 1, -2/3)}_{3} \oplus \underbrace{(1, 1, 1)}_{1} = 10$$

THE ψ^a ARE DIM = 5. THE NEXT ONE IN DIMENSION IS A 10.
INDEED, AN ANTISYMMETRIC MATRIX $\psi^{ab} = -\psi^{ba}$ HAS

$\frac{N(N-1)}{2} = 10$ COMPONENTS!

THAT SOUNDS GOOD BUT THE QUANTUM NUMBER MUST COME RIGHT.

WE KNOW NOW THAT $\psi^a \sim 5 \to (3, 1, -\frac{1}{3}) \oplus (1, 2, \frac{1}{2})$

SO $\psi^{ab} \sim \psi^a \otimes_A \psi^b$ (TRANSFORMS AS)

↳ ANTISYMMETRIC TENSOR PRODUCT.

$$\Longrightarrow \quad 5 \otimes_A 5 \to \left((3, 1, -\tfrac{1}{3}) \oplus (1, 2, \tfrac{1}{2}) \right) \otimes_A \left((3, 1, -\tfrac{1}{3}) \oplus (1, 2, \tfrac{1}{2}) \right)$$

HERE $(3, 1, -\frac{1}{3}) \otimes (3, 1, -\frac{1}{3}) = (\bar{3}, 1, -2/3) \sim u^c$

(REMEMBER $u^i d^j - u^j d^i \sim \varepsilon^{ijk} d_k \sim \bar{3}$)

THIS STATE IS THE ANTI-QUARK UP (SINGLET OF SU(2) !
LEFT-HANDED,
CHARGE $-2/3$)

ALSO $(1, 2, 1/2) \otimes (1, 2, 1/2) = (1, 1, 1) \sim e^c$

HOW ABOUT

$$\underbrace{(3,1,-\tfrac{1}{3})}_{\psi^I} \otimes_A (1,2,\tfrac{1}{2}) \ ?$$

$$\psi^I \otimes_A \psi^i \sim 6 \text{ COMPONENTS}$$

$$\sim (3,2,\tfrac{1}{6}) \sim Q$$ (TRIPLET OF SU(3) DOUBLET OF SU(2) HYPERCHARGE = 1/6)

SO $$10 \to (3,2,\tfrac{1}{6}) \oplus (\bar{3},1,-\tfrac{2}{3}) \oplus (1,1,1)$$

$$6 + 4 + 1 = 10$$

WE HAVE FITTED ALL THE SM REPS (1 FAMILY) INTO

$$\boxed{\bar{5} \oplus 10} \quad !!!$$

IN COMPONENTS

$$\psi_a \equiv \bar{5} = \begin{pmatrix} \psi_I \\ \psi_i \end{pmatrix} = \begin{pmatrix} \bar{d}_R \\ \bar{d}_G \\ \bar{d}_B \\ \nu \\ e \end{pmatrix}_L$$

$$\psi^{ab} \equiv 5 = \{ \psi^{IJ}, \psi^{Ij}, \psi^{ij} \}$$

WITH

$$\psi^{ab} = \begin{pmatrix} 0 & u^c & -u^c & u & d \\ -u^c & 0 & u^c & u & d \\ u^c & -u^c & 0 & u & d \\ -u & -u & -u & 0 & e^c \\ -d & -d & -d & -e^c & 0 \end{pmatrix}$$

FURTHER INSIGHTS OFFERED BY GRAND UNIFICATION

* CHARGE QUANTIZATION

- IN QED (EM FIELD)

$$\mathcal{L} = i\bar{\psi}_1(\partial\!\!\!/ - iA\!\!\!/ - m_1)\psi_1 + i\bar{\psi}_2(\partial\!\!\!/ - i\varepsilon A\!\!\!/ - m_2)\psi_2$$

IS NO PROBLEM WITH ε ARBITRARY NUMBER.

- IN $SO(5)$, A_μ COUPLES TO A GENERATOR!

$\Longleftrightarrow$ COUPLINGS ARE QUANTIZED!

i.e. EIGENVALUES OF $\frac{S^3}{2} = \pm\frac{1}{2}$

IDEM WITH λ^3 AND λ^8

* WHY IS THE ELECTRIC CHARGE OF THE PROTON OPPOSITE TO THAT OF THE ELECTRON?

GENERATORS ARE TRACELESS!

$$\Rightarrow \mathrm{Tr}[Q] = 3Q_{\bar{d}} + Q_e = 0$$

(ON $\bar{5}$)

$$\boxed{Q_{\bar{d}} = -\frac{1}{3}Q_e}$$

IF $Q_e = -1$

$Q_{\bar{d}} = +1/3$

THE UP QUARKS ARE IN THE 10 OF SU(5); THEIR CHARGE FOLLOW FROM GROUP THEORY, SO $Q_u = -2 Q_d$, etc...

THEN $Q_p = 2Q_u + Q_d = -3 Q_d = -Q_e$ CQFD

* WEINBERG ANGLE

in $SU(3) \times SU(2) \times U(1)$ $\tan \theta_W = g'/g$

FROM $g W^A_\mu T^A + g' \frac{Y}{2} B_\mu$

NORMALIZATION OF SU(5) GENERATORS TO

$Tr[T^a T^b] = \frac{1}{2}\delta^{ab}$

SO g AND g' MUST BE RELATED

$Tr[T_3^2] = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$

$Tr[(\frac{Y}{2})^2] = 3 \times (\frac{-1}{3})^2 + 2 (\frac{1}{2})^2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

$\Rightarrow$ THUS T_3 AND $\sqrt{\frac{3}{5}}\, \frac{Y}{2}$ ARE CORRECTLY NORMALIZED.

$\Rightarrow g_5 A^3_\mu T^3 + g_5 \sqrt{\frac{3}{5}}\, \frac{Y}{2} B_\mu$

$\Rightarrow g A^3_\mu T^3 + g' \frac{Y}{2} B_\mu \Rightarrow \boxed{\frac{g'}{g} = \sqrt{\frac{3}{5}}}$

$\sin^2 \theta_W \big|_{SU(5)} = \frac{g'^2}{g^2 + g'^2} = \frac{3/5}{1 + 3/5} = \frac{3}{8} = 0.375$ TO BE COMPARED TO 0.223 OR SO AT LOW ENERGIES

OF COURSE $\sin^2\theta_W = 3/8$ IS THE SUPPOSED VALUE AT SOME HIGH SCALE. HOW HIGH SHOULD IT BE?

TO BE DISCUSSED SOON.

- FURTHER INSIGHTS FROM ANOMALY CANCELLATION?

TO BE A GOOD THEORY, WE NEED TO CHECK THE ABSENCE OF ANOMALIES!

- WHAT IS $A^{abc} = \mathrm{Tr}[T^a\{T^b, T^c\}]$ FOR THE $\bar{5}$ AND 10?

(ALL L-HANDED SO IT IS A CHIRAL THEORY)

$\Longrightarrow$ CHECK P&S ?

AND ZEE'S TRICK ?

$$A^{abc} = A^{acb} = A^{cab} = A^{cba} \text{ etc.. full symmetric.}$$

$$\mathrm{Tr}[T^a,\{T^b,T^c\}] = \mathrm{Tr}[T^aT^bT^c] + \mathrm{Tr}[T^aT^cT^b]$$

$$= \mathrm{Tr}[\{T^a,T^b\}T^c] \text{ etc...}$$

A^{abc} is an invariant TENSOR OF $SU(N)$. COMES FROM

$$\{T^a, T^b\} = i\, d^{abc}\, T^c$$

THEN $\mathrm{Tr}[T^a_R, \{T^b_R, T^c_R\}] = A(R)\, d^{abc}$

$\downarrow$

JUST NEED THAT NUMBER!

GET IT FOR A SPECIFIC T.

A GOOD CHOICE IS

$$T = \frac{y}{2} = \begin{pmatrix} -1/3 & & & & 0 \\ & -1/3 & & & \\ & & -1/3 & & \\ & & & 1/2 & \\ 0 & & & & 1/2 \end{pmatrix}$$

$$T_1\left[\left(\frac{y}{2}\right)^3\right]\Big|_{\bar{5}} = 3 \times \left(+\frac{1}{3}\right)^3 + 2\left(\frac{-1}{2}\right)^3 = +\frac{1}{9} - \frac{1}{4} = -\frac{5}{36}$$

$$T_1\left[\left(\frac{y}{2}\right)^3\right]_{10} = 3 \times 2 \times \left(\frac{1}{6}\right)^3 + 3 \times \left(\frac{-2}{3}\right)^3 + (+1)^3$$

$$= \frac{1}{36} - \frac{8}{9} + 1 = \frac{1+36-32}{36}$$

$$= \frac{5}{36}$$

$$\boxed{\frac{5}{36} - \frac{5}{36} = 0}$$

VERY NICE

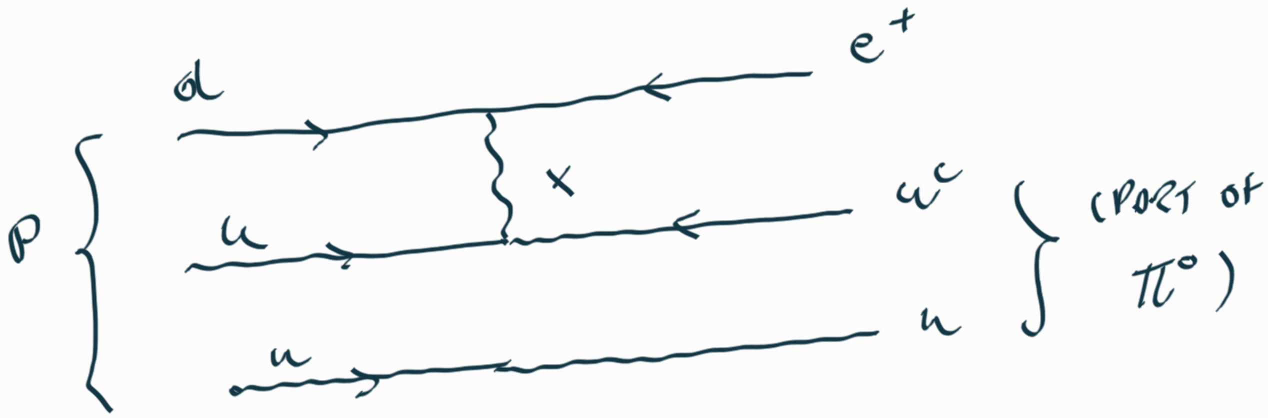
PROTON DECAY (IN $SU(5)$)

$$p \to \pi^0 e^+$$

IN $\psi_a = \begin{pmatrix} d^c \\ d^c \\ d^c \\ \nu \\ e \end{pmatrix}$ $\updownarrow$ THROUGH EXCHANGE OF $SU(5)$ GAUGE BOSONS

IN PARTICULAR $d \to e^+ + X_{(-4/3)}$

A SPECIFIC PROCESS IS



THIS PROCESS IS CHARACTERIZED BY

$$\Delta B = -1 = B_f - B_i$$

WITH $B_i = +1$ (PROTON IS A BARYON)

AND $\Delta L = -1 = L_f - L_i$

WITH $L_f = -1$ (POSITRON IS AN ANTI-LEPTON)

SO $\boxed{\begin{array}{c} \Delta(B+L) = -2 \\ \Delta(B-L) = 0 \end{array}}$