# CH5 QUANTUM GAUGE FIELDS

## 5.1 Gauge invariance

→ In QFT, the propagator encodes the dynamics of the fields and their interactions. However, for gauge field, gauge invariance imposes constraints on the EOM, leading to the non-invertibility of the quadratic kernel.

Recall: quadratic kernel $D$, ex: $\mathcal{L}_0 = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\phi D \phi$
so that $D = -\partial^2 - m^2$, or $D(p) = p^2 + m^2$

⊙ **Exercice: multiple scalar fields:**

→ Consider $S_0 \equiv -\frac{1}{2}\int d^n x \left(\partial_\mu \phi^i \partial^\mu \phi_i + m^2 \phi^i \phi_i\right) + i\epsilon$

$$= -\frac{1}{2}\int d^n x \int d^n x' \, \phi^i(x) \, \underbrace{\delta_{ij}\left(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x'^\mu}\delta(x-x') + m^2 - i\epsilon\right)}_{\equiv D_{ij}(x,x')} \phi^j(x')$$

→ The propagator is defined as $\Delta^{ij}(x,x') \equiv (D^{-1})^{ij}(x,x')$ so that

$$\int d^n x' \, D_{ij}(x,x')\Delta^{jk}(x',x'') = \delta_i{}^k \, \delta^n(x-x'')$$

↳ Since $\delta^n(x-x') = \frac{1}{(2\pi)^n}\int d^n p \, e^{ip(x-x')}$, one has

$$D_{ij}(x,x') = \delta_{ij}\frac{1}{(2\pi)^n}\int d^n p \, e^{ip(x-x')}(p^2 + m^2 - i\epsilon)$$

$$\Rightarrow \tilde{D}_{ij}(p) = \delta_{ij}(p^2 + m^2 - i\epsilon) \Leftrightarrow \tilde{\Delta}^{ij}(p) = \delta^{ij}\frac{1}{p^2 + m^2 - i\epsilon}$$

$$\Rightarrow \Delta^{ij}(x-x') = \delta^{ij}\frac{1}{(2\pi)^n}\int d^n p \, \frac{1}{p^2+m^2-i\epsilon} e^{ip(x-x')}$$

→ Scalar theories do not possess any gauge freedom, the field $\phi$ is unconstrained and all the field dof ⟺ physical dof.

→ The quadratic kernel $D$ is fully invertible because there are no "null directions" in the functional space of the scalar fields.

## • Exercise: massive vector field:

→ Consider $L_0 = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2} m^2 A^\mu A_\mu + i\epsilon$

$$= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu - \frac{1}{2} m^2 A^\mu A_\mu + i\epsilon$$

$$= -\frac{1}{2}\int d^n x\, d^n y\, A^\rho(x)\, D_{\rho\alpha}(x,y)\, A^\alpha(y)$$

with $D_{\rho\alpha}(x,y) = \left(\eta_{\rho\alpha}\frac{\partial^2}{\partial x^\mu \partial y_\mu} - \frac{\partial^2}{\partial x^\alpha \partial x^\rho} + m^2 \eta_{\rho\alpha}\right)\delta^n(x-y) - i\epsilon$

$$= \frac{1}{(2\pi)^n}\int d^n p\, e^{ip(x-y)}\left(\underbrace{\eta_{\rho\alpha}(p^2+m^2) - p_\rho p_\alpha - i\epsilon}_{=\tilde{D}_{\rho\alpha}}\right)$$

→ The propagator is then given by:

$$\Delta^{\rho\alpha}(x,y) = \frac{1}{(2\pi)^n}\int d^n p\, e^{ip(x-y)}\left(\frac{\eta^{\rho\alpha} + p^\rho p^\alpha/m^2}{p^2+m^2}\right)$$

Indeed, $\left(\eta_{\rho\alpha}(p^2+m^2) - p_\rho p_\alpha\right)\left(\frac{\eta^{\alpha\tau} + p^\alpha p^\tau/m^2}{p^2+m^2}\right)$

$$= (p^2+m^2)^{-1}\left(\delta_\rho^\tau(p^2+m^2) - p_\rho p^\tau + p_\rho p^\tau(p^2+m^2)/m^2 - p^2 p_\rho p^\tau/m^2\right) = \delta_\rho^\tau$$

→ Notice that $\lim_{m\to 0} D_{\rho\alpha}$ is not invertible: $\tilde{D}_{\rho\alpha}(p) = \eta_{\rho\alpha}p^2 - p_\rho p_\alpha$

Since it's a $n\times n$ symmetric matrix, $\exists$ at least one $p^\alpha$ an eigenvector of vanishing eigenvalue: $\tilde{D}_{\rho\alpha}p^\alpha = 0$

→ The quadratic part is non-invertible because of the gauge invariance. Indeed, it is equivalent to perform:

$$\int d^n y\, D_\rho{}^\alpha(x-y)\, A_\alpha^g(y) = 0$$

where $A_\alpha^g(y) \equiv \partial_\alpha \epsilon(y)$ (with $A_\nu \mapsto A_\nu + \partial_\nu \epsilon$ the gauge freedom)

A pure gauge field: $A_\alpha = A_\alpha^{phys} + A_\alpha^g = A_\alpha^{phys} + \partial_\alpha \epsilon = \partial_\alpha \epsilon$ annihilates the quadratic part; the pure gauge modes lie in the null space of the propagator due to gauge invariance.

# 5.2 Invariance BRST

→ Consider a gauge invariant action $S^{inv}[A, y^i] = \int d^n x\, L^{inv}[A, y^i]$ where $y^i \equiv (\phi, \xi, \chi)$ are the matter fields : scalar, Weyl fermions and Dirac fermions.

⊙ Yang-Mills theory:

DEF The Yang-Mills lagrangian $L^{YM}$ is defined as

$$L^{YM} \equiv \frac{-1}{4g^2} F_{\mu\nu}{}^a F^{\mu\nu b}\, g_{ab} \qquad \left(\text{with } f^d{}_{ac}\, g_{db} + f^d{}_{bc}\, g_{da} = 0\right)$$

where $g_{ab}$ is the Killing metric of the group, $g_{ab} \propto \mathrm{Tr}\{ad_a \circ ad_b\}$ and where $F_{\mu\nu}{}^a \equiv \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + f^a{}_{bc} A_\mu{}^b A_\nu{}^c$

$f^a{}_{bc}$ are the structure constants of the gauge group $[T_a, T_b] = i f^c_{ab} T_c$

→ Consider $L^{inv}[A, y^i] = L^{YM} + L_M[y^i, Dy^i]$, with the following infinitesimal gauge transformations :

$$\delta_\varepsilon S^{inv} = 0 \quad \Rightarrow \quad \begin{cases} \delta_\varepsilon A_\mu{}^a = D_\mu \varepsilon^a = \partial_\mu \varepsilon^a + g f^a{}_{bc} A_\mu{}^b \varepsilon^c \\ \delta_\varepsilon y^i = -\varepsilon^a (T_a)^i{}_j\, y^j \end{cases}$$

when $\varepsilon^a(x)$ is an arbitrary field.

→ If we rescale the gauge field $A_\mu^a \mapsto g A_\mu^a$, the structure constants $f^{abc} \mapsto g f^{abc}$ and the generator of the gauge group : $T_a^i \mapsto g T_a^i$, we get the canonical lagrangian normalization

⊙ Chern-Simons theory:

DEF The Chern-Simons lagrangian $L^{CS}$ in 3D is defined as :

$$L^{CS} = \frac{k}{8\pi} \epsilon^{\mu\nu\rho}\, g_{ab} A_\mu{}^a \left(\partial_\nu A_\rho{}^b + \frac{1}{3} f^b{}_{cd} A_\nu{}^c A_\rho{}^d\right)$$

where $k$ is the Chern-Simons level. It defines a topological field theory in 3-D of spacetime.

DEF The gauge field connection $A$ is defined as

$$A = A^a_\mu \, dx^\mu \, T_a$$

→ The gauge field is a Lie algebra-valued 1-form

The inner product on the Lie algebra is

$$\langle T_a, T_b \rangle = g_{ab}$$

The exterior derivative $d$ of a 1-form is given by

$$dA = \partial_\mu A_\nu \, dx^\mu \wedge dx^\nu \quad \text{where } A_\mu = A^a_\mu T_a$$

The graded commutator $[A,A]$ is defined as

$$[A,A] = f_{ab}{}^{c} A^a \wedge A^b T_c$$

→ Notice that $F = dA + \frac{1}{2}[A,A]$. Indeed,

$$dA + \frac{1}{2}[A,A] = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\, dx^\mu \wedge dx^\nu + \frac{1}{2} f_{ab}{}^{c} A^a \wedge A^b T_c$$

$$= \frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)\, dx^\mu \wedge dx^\nu T_a + \frac{1}{2} f_{ab}{}^{c} A^a_\mu A^b_\nu \, dx^\mu \wedge dx^\nu T_c$$

$$= \frac{1}{2}\left((\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) T_a + f_{ab}{}^{c} A_\mu{}^a A_\nu{}^b T_c\right) dx^\mu \wedge dx^\nu$$

so that $F_{\mu\nu}{}^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_a{}^{bc} A_\mu{}^b A_\nu{}^c$

→ Notice that $\langle A, dA \rangle = \langle A^a_\sigma \, dx^\sigma T_a, \partial_{(\mu} A^b_{\nu)} \, dx^\mu \wedge dx^\nu T_b \rangle$

$$= A^a_\sigma \partial_{(\mu} A^b_{\nu)} \, dx^\sigma \wedge dx^\mu \wedge dx^\nu \langle T_a, T_b \rangle = g_{ab} A^a \wedge dA^b$$

PROP The Chern-Simons action can be written as

$$S^{cs} = \frac{k}{4\pi}\int \langle A, dA \rangle + \frac{2}{3}\langle A^3 \rangle = \frac{k}{8\pi}\int \langle A, dA + \frac{2}{3}A^2 \rangle$$

[DEMO] $d^3x \, L^{cs} = \frac{k}{8\pi} \epsilon^{\mu\nu\rho} \Big( \underbrace{g_{ab} A^a_\mu \partial_\nu A^b_\rho}_{=2\langle A, dA\rangle} + \frac{1}{3} g_{ab} f^{bcd} A^a_\mu A^c_\nu A^d_\rho \Big) d^3x$

Now, $\langle A^3 \rangle = \langle A \wedge A \wedge A \rangle = \langle A^a_\mu A^b_\nu A^c_\rho \, dx^\mu \wedge dx^\nu \wedge dx^\rho \, T_a T_b T_c \rangle$

and $T_{[a} T_b T_{c]} = \frac{1}{3!} f^{abc} T_c$ so that $\langle A^3 \rangle = g_{ad} f^{dbc} A^a \wedge A^b \wedge A^c$ ∎

→ On can normalize $A^a{}_\mu \mapsto g A^a{}_\mu$, $f^a{}_{bc} \mapsto g f^a{}_{bc}$ so that the quadratic part of $L^{cs}$ becomes $L^{cs}_{quad} = \frac{1}{2}\epsilon^{\mu\nu\rho} A^a_\mu \partial_\nu A^b_\rho \, g_{ab}$

→ The gauge transformation $\delta_\epsilon A^a_\mu = \partial_\mu \epsilon^a$ leaves $L^{cs}_{quad}$ invariant up to a total derivative term → topological.

# ⊙ ghosts:

→ We replace the gauge parameter $\varepsilon^a(x)$ with a ghost field $C^a(x)$
$$\varepsilon^a(x) \to C^a(x)$$

→ These field are fermionic (anticomuting) scalars hence they violate the spin-statistics theorem. It's ok since they're not irrep of the Poincaré group.

→ The fields in the theory are now:

$\phi^A \equiv (A^a_\mu, y^i, C^a, \bar{C}^a, B^a)$ fields

$\phi^{*A} \equiv (A^{*\mu}_a, y^*_i, C^*_a, \bar{C}^*_a, B^*_a)$ conjugate fields (antifield)

with $B^a$ a auxiliary bosonic field.

DEF We introduce the parity $p$ that indicates if a field is comuting ($p=0$) or anticomuting ($p=1$), and the ghost number $gh$ that indicates if the field is a ghost ($gh=1$), an antighost ($gh=-1$) or neither ($gh=0$)
For the antifields, we have the following rules:
$$p(\phi^*_A) = p(\phi^A) + 1 \pmod 2 \qquad gh(\phi^*_A) = -gh(\phi^A) - 1$$

→ One has

| | $A^a_\mu$ | $\phi$ | $\xi$ | $\psi$ | $C^a$ | $\bar{C}^a$ | $B^a$ |
|---|---|---|---|---|---|---|---|
| $p$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| $gh$ | 0 | 0 | 0 | 0 | 1 | -1 | 0 |
| | $A^{*\mu}_a$ | $\phi^*$ | $\xi^*$ | $\psi^*$ | $C^*_a$ | $\bar{C}^*_a$ | $B^*_a$ |
| $p$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 |
| $gh$ | -1 | -1 | -1 | -1 | -2 | 0 | -1 |

→ Next, we introduce a graded analog of the Poisson bracket used in the Batalin-Vilkovsky formalism (BV).

DEF The antibracket $(\,,\,)$ is defined for two functionals F and G :

eq. 17.26 Henneaux

$$(F,G)=\int d^nx \left\{ \frac{\delta^R F}{\delta\phi^A(x)}\frac{\delta^L G}{\delta\phi^*_A(x)} - \frac{\delta^R F}{\delta\phi^*_A(x)}\frac{\delta^L G}{\delta\phi^A} \right\}$$

with the right derivative and left derivative such that

$$\int d^nx\, \delta\phi^B(x)\frac{\delta^L F}{\delta\phi^B(x)} = \int d^nx \frac{\delta^R F}{\delta\phi^B(x)}\delta\phi^B(x)$$

$$\Leftrightarrow \frac{\delta^R F}{\delta\phi^B(x)} = (-1)^{B(B+F)}\frac{\delta F}{\delta\phi^B(x)} \quad \text{with} \begin{cases} B \text{ the parity of } \phi^B(x) \\ F \text{ the parity of } F[\phi^A] \end{cases}$$

PROP ① Graded antisymmetry:

$$(F,G) = -(-1)^{(F+1)(G+1)}(G,F)$$

② Graded Jacobi identity

$$(F,(G,H)) = ((F,G),H) + (-1)^{(F+1)(G+1)}(G,(F,H))$$

$$\Leftrightarrow (F,(G,H)).(-1)^{(F+1)(G+1)} + \text{cyclic}(F,G,H) = 0$$

③ if $(-1)^F = +1$, then $\frac{1}{2}(F,F) = \int d^nx \frac{\delta^R F}{\delta\phi^A(x)}\frac{\delta^L F}{\delta\phi^*_A(x)} = -\int d^nx \frac{\delta^R F}{\delta\phi^*_A(x)}\frac{\delta^L F}{\delta\phi^A(x)}$

so $(F,F)\neq 0$ but $(F,(F,F))=0$ from ②

## ⊙ Master action and BRST transformations :

DEF The master action $S$ is given by

$$S[\phi^A,\phi^*_A] = S^{inv} + \int d^nx\left(-D_\mu C^a A^{*\mu}_a + C^a (T_a)^i{}_j y^j y^*_i + \frac{1}{2} f^a{}_{bc} C^b C^c C^*_a - B^a \bar{C}^*_a\right)$$

We denote the BRST differential as $s \equiv (S,\cdot)$

→ For instance, $s\phi^A(x) = (S,\phi^A) = -\frac{\delta^R S}{\delta\phi^*_A(x)}$ and $s\phi^*_A(x) = \frac{\delta^R S}{\delta\phi^A(x)} = (-1)^A \frac{\delta^L S}{\delta\phi^A(x)}$

→ One sees that

$$\begin{cases} sA^a_\mu = D_\mu C^a \qquad s y^i = -C^a (T_a)^i{}_j y^j \qquad sC^a = -\frac{1}{2} f^a{}_{bc} C^b C^c \\ s\bar{C}^a = B^a \qquad sB^a = 0 \end{cases}$$

PROP On the fields $A^a_\mu$, $y^i$, the BRST differential (or BRST transformation) act like gauge transformations with $\varepsilon^a(x) \mapsto C^a(x)$