5. CHIRAL ANOMALIES

CHIRAL ANOMALIES PROVIDE IMPORTANT CONSTRAINTS ON BSM PHYSICS.

IT IS A QUANTUM EFFECT WHOSE ORIGIN IS BASED ON TWO FEATURES OF QFT:

+ THE EXISTENCE OF CHIRAL FERMIONS AS FUNDAMENTAL BUILDING BLOCKS
+ THE INFINITE NUMBER OF DEGREES OF FREEDOM AND THE NEED TO REGULARIZE DIVERGENCES.

IT IS A VAST TOPIC. WE JUST COVER BASIC FEATURES.
THE BOTTOM LINE IS THAT A GOOD THEORY SHOULD BE FREE OF GAUGE ANOMALIES.
THIS PUTS CONSTRAINTS ON THE MATTER CONTENT OF A THEORY.

CHIRAL SYMMETRY RECAP

FROM $i\gamma^\mu\partial_\mu\psi - m\psi = 0$ AND $\{\gamma_5, \gamma^\mu\} = 0$

WE HAVE $i\partial_\mu\bar\psi\gamma^\mu + m\bar\psi = 0$ WITH $\bar\psi = \psi^\dagger\gamma^0$ (USING $\gamma^{\mu\dagger} = \gamma^\mu$)

$$\Rightarrow \quad i(\partial_\mu\bar\psi)\gamma^\mu\psi + i\bar\psi\gamma^\mu(\partial_\mu\psi) = 0 \iff \partial_\mu(\bar\psi\gamma^\mu\psi) = 0$$

BY NOETHER, THIS IS ASSOCIATED INVARIANCE UNDER $\psi \to e^{i\alpha}\psi$

$$* \quad \mathcal{L} = i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi \quad \Rightarrow \quad \delta\mathcal{L} = i\delta\bar\psi\,\partial\!\!\!/\,\psi + i\bar\psi\,\partial\!\!\!/\,\delta\psi - m\delta\bar\psi\psi - m\bar\psi\delta\psi$$

$$= \delta\bar\psi(i\partial\!\!\!/ - m)\psi + \underbrace{i\bar\psi\gamma^\mu\partial_\mu\delta\psi}_{i\partial_\mu(\bar\psi\gamma^\mu\delta\psi) - i\partial_\mu\bar\psi\gamma^\mu\delta\psi} - m\bar\psi\delta\psi$$

$$= \delta\bar\psi(i\partial\!\!\!/ - m)\psi$$

$$= \underbrace{\delta\bar\psi(i\partial\!\!\!/ - m)\psi}_{=0} - \underbrace{(i\partial_\mu\bar\psi\gamma^\mu + m\bar\psi)}_{=0 \text{ ON-SHELL}}\delta\psi + i\partial_\mu(\bar\psi\gamma^\mu\delta\psi)$$

$$\Rightarrow \delta\mathcal{L} = i\partial_\mu(\bar\psi\gamma^\mu\delta\psi)$$

REM: NON-ZERO AT BOUNDARIES!

if SYMMETRY $\delta\mathcal{L} = 0 \iff \delta\psi = i\alpha\psi \quad \Rightarrow -\alpha\partial_\mu(\bar\psi\gamma^\mu\psi) = 0$

MORE GENERALLY $\delta\mathcal{L} = \partial_\mu K^\mu$ TOTAL DERIVATIVE

ALSO $\quad + i\gamma^\mu \gamma_5 \partial_\mu \psi + m \gamma_5 \psi = 0$

$$i \partial_\mu \bar{\psi} \gamma^\mu \gamma_5 + m \bar{\psi} \gamma_5 = 0$$

$$i \partial_\mu (\bar{\psi} \gamma^\mu \gamma_5 \psi) + 2m \bar{\psi}\gamma_5 \psi = 0 \quad \Rightarrow \boxed{\partial_\mu (\bar{\psi}\gamma^\mu \gamma_5 \psi) = 2im \bar{\psi}\gamma_5 \psi}$$

THIS IS CONSERVED IF $m = 0$, ASSOCIATED TO $\psi \to e^{i\alpha\gamma_5}\psi$

CHIRAL TRANSFORMATION $\begin{cases} \chi_L \to e^{-i\alpha} \chi_L \\ \chi_R \to e^{i\alpha} \chi_R \end{cases}$

ex: CHECK THIS AT THE LEVEL OF THE LAGRANGIAN AS FOR THE VECTOR CURRENT $J^\mu = \bar{\psi}\gamma^\mu \psi$

$$\mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi}\psi$$

NOW, INTERESTING LOOP EFFECT; THE CHIRAL ANOMALY $\sim O(\hbar)$, CAN SPOIL THIS EVEN IF $m = 0$. IT IS $O(\hbar)$ IN THE SENSE THAT IT ARISES AT 1-LOOP ORDER.

CHIRAL ANOMALY IN 2D : SIMPLER, CLEARER

$2\,\mathrm{Tr}[1] = 2D$

TAKE $i\Pi_{\mu\nu} = (q^2\eta_{\mu\nu} - q_\mu q_\nu)\, i\Pi(q^2)$ with $\Pi(q^2) = -\frac{8}{(4\pi)^{D/2}}\Gamma(2-D/2)\int_0^1 dx \frac{x(1-x)}{(m^2 - q^2x(1-x))^{2-D/2}}$

$$= -i(q^2\eta_{\mu\nu} - q_\mu q_\nu)\frac{2\,\mathrm{Tr}[1]}{(4\pi)^{D/2}}\Gamma(2-D/2)\int_0^1 dx\, x(1-x)\frac{1}{(-x(1-x)q^2)^{2-D/2}}$$

AND SET $D=2$

$i\Pi_{\mu\nu} = \langle J_\mu J_\nu\rangle \Rightarrow \boxed{q^\mu \Pi_{\mu\nu} = 0}$ WARD-TAKAHASHI IDENTITY IS VERIFIED

$$i\Pi_{\mu\nu} = -i(q^2\eta_{\mu\nu} - q_\mu q_\nu)\frac{4}{4\pi}\underbrace{\Gamma(2-1)}_{\text{FINITE}=1}\int_0^1 dx\, x(1-x)\frac{1}{(-x(1-x)q^2)}$$

$$= +i(q^2\eta_{\mu\nu} - q_\mu q_\nu)\frac{1}{\pi q^2}$$

$$= \frac{i}{\pi}\Big(\eta_{\mu\nu} - \underbrace{\frac{q_\mu q_\nu}{q^2}}_{\text{TRANSVERSE AT POLE } \frac{1}{q^2}\ldots!}\Big)$$

$$\Rightarrow iD_{\mu\nu} = \frac{-i\,\eta_{\mu\nu}e^2}{q^2\left(1-\frac{e^2}{\pi}\frac{1}{q^2}\right)} + \ldots$$

$$= \frac{-i\,\eta_{\mu\nu}e^2}{q^2 - e^2/\pi}$$

THE PHOTON IS MASSIVE IN $D=2$!

Rem: WHAT IS THE DIMENSION OF e IN $D=2$?

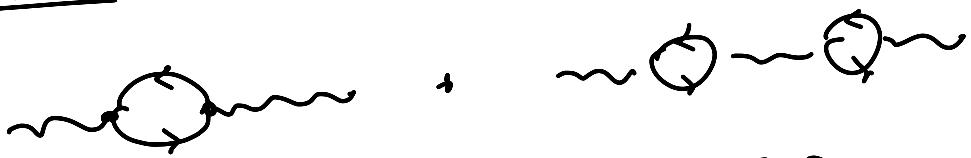
$$\mathcal{L} \supset e\bar{\psi}\gamma^\mu\psi A_\mu + i\bar{\psi}\not{\partial}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$\Rightarrow \boxed{[e] = E}$ $\qquad [\psi] = 1/2$ $\qquad [A_\mu] = 0$

DIMENSION OF MASS ...

$\longrightarrow$ • FROM PREVIOUS RESULT, $\boxed{m_\gamma^2 = \frac{e^2}{\pi}}$

• DYNAMICS GIVES MASS TO THE PHOTON (in $D=2$)

•  etc...

THIS IS INTERESTING STORY BUT WE FOCUS ON SOMETHING ELSE...

EX: CHECK THAT in $D=2$,

WE CAN TAKE

$$\gamma_5 = -\sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

$$\gamma^0 = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^1 = +i\sigma_2 = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} ?$$

$\gamma_5^2 = +1$

$\gamma^{0\,2} = +1$

$\Rightarrow \gamma^{1\,2} = -1$

THEN $\psi_2 = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$

$P_L = \frac{1-\gamma_5}{2}$ = LEFT-CHIRALITY

IMPLIES $\chi_L = \chi_L(t+x)$

INDEED $i\gamma^\mu\partial_\mu\psi_2 = 0$

$[i\gamma^0\partial_0 + i\gamma^1\partial_1]\psi_2 = 0$

$(\partial_0 - \partial_1)\chi_L = 0$

so $\chi_R(t-x)$, $(\partial_0 + \partial_1)\chi_R = 0$

CHECK ALSO THAT

in , $\quad \gamma^\mu \gamma_5 = -\varepsilon^{\mu\nu}\gamma_\nu$ with $\varepsilon^{01} = +1$

$\Longrightarrow$ THIS IMPLIES THAT $\quad J_5^\mu = -\varepsilon^{\mu\nu} J_\nu$

NOW, in VACUUM $\langle J^\mu \rangle = 0$

WHAT if WE TURN ON AN ELECTRIC FIELD ?

$$\langle J^\mu \rangle_{A^\nu} = \int \mathcal{D}\bar\psi \mathcal{D}\psi \, J^\mu \, e^{iS + i\int e J_\nu A^\nu d^2x}$$

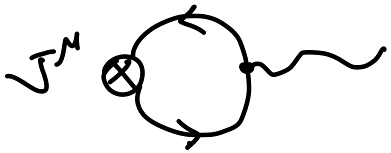
$$= \int \mathcal{D}\bar\psi \mathcal{D}\psi \, J^\mu \, e^{iS} \left(1 + ie\int J_\nu A^\nu d^2x + \dots\right)$$

$$= + ie \int \mathcal{D}\bar\psi \mathcal{D}\psi \int d^2x \, J^\mu J^\nu \, e^{iS} A^\nu$$

GRAPHICALLY

$$\overset{FT}{\Longrightarrow} \quad \langle J^\mu \rangle = + ie \langle J^\mu J^\nu \rangle A_\nu$$

$$= (i)^2 e \, \Pi^{\mu\nu} A_\nu$$



SO $\langle J^\mu \rangle = i^2 \Pi^{\mu\nu}(q) A_\nu(q) = -\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \frac{e}{\pi} A_\nu(q)$

NOW WE USE $\gamma^\mu \gamma_5 = -\varepsilon^{\mu\nu}\gamma_\nu$

$$\Longrightarrow \quad \langle J_5^\mu(q) \rangle = -\varepsilon^{\mu\nu} \langle J_\nu(q) \rangle$$

$$= +\varepsilon^{\mu\nu} \left(\delta_\nu^\alpha - \frac{q_\nu q^\alpha}{q^2} \right) \frac{e}{\pi} A_\alpha$$

$$= +\left(\varepsilon^{\mu\alpha} - \varepsilon^{\mu\nu} \frac{q_\nu q^\alpha}{q^2} \right) \frac{e}{\pi} A_\alpha$$

(THE SECOND TERM VANISHES)

SO $q_\mu \langle J_5^\mu(q) \rangle = +\left(q_\mu \varepsilon^{\mu\alpha} \frac{e}{\pi} A_\alpha \right)$

$$= \frac{e}{\pi} \varepsilon^{\mu\nu} q_\mu A_\nu$$

$$\Longleftrightarrow \quad \boxed{\partial_\mu J_5^\mu = \frac{e}{2\pi} \varepsilon^{\mu\nu} F_{\mu\nu}}$$

THE CHIRAL CURRENT IS NOT CONSERVED !

LEFT & RIGHT MOVERS

$$\Rightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & F_{01} \\ F_{10} & 0 \end{pmatrix} = \begin{pmatrix} 0 & E_x \\ -E_x & 0 \end{pmatrix}$$

COULOMB GAUGE

$$F_{01} = \partial_0 A_1 = -\partial_0 A^1 = E_x$$

ELECTRIC FIELD.

$$\varepsilon^{\mu\nu} F_{\mu\nu} = \varepsilon^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial_\mu (2\varepsilon^{\mu\nu} A_\nu)$$

$=$ BOUNDARY TERM ... NO EFFECT ?

$$\int d^2x \, \partial_\mu J^\mu_5 = \int dt \int dx \, (\partial_0 J^0_5 + \partial_1 J^1_5)$$

$\hookrightarrow$ SPATIAL

$$= \int dt \int dx \, \partial_0 J^0_5$$

$$= \int dt \frac{\partial}{\partial t} \int dx \, J^0_5 = \Delta N_R - \Delta N_L \quad (= 0 \text{ if conserved})$$

?

eg take periodic boundary condition, space of dimension L

BACKGROUND ? $\Rightarrow$ $A_1(x,t)$ SLOWLY VARYING

A CONSTANT FIELD

BUT PERIODIC

CANNOT BE GAUGED AWAY

$$e^{-ie\int_0^L dx \, A_1}$$

HAMILTONIAN ?

$$H = \int dx\ \psi^{+}(-i\,\alpha' D_1)\psi \quad \text{with } \alpha' = \gamma^0\gamma^1$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

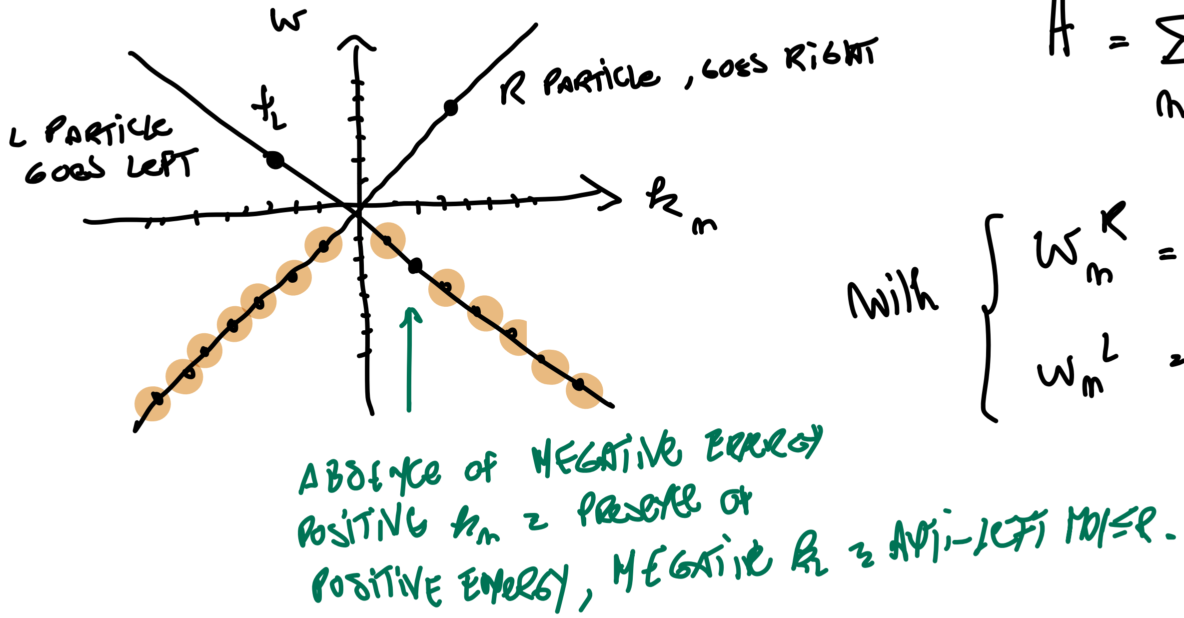
REMEMBER

$$\left(H = -i\,\vec{\alpha}\cdot\vec{\partial} + \beta m \right)$$

$$H = \int dx\ \psi^{+}\left(-i\,\gamma_5(\partial_1 + ieA_1)\right)\psi$$

$$= \int dx\left[i\psi_L^{+}(\partial_1 + ieA_1)\psi_L - i\psi_R^{+}(\partial_1 + ieA_1)\psi_R\right] \quad \text{with } A_1 = A_1(t)$$

ψ_L or left $\quad k_n = \frac{2\pi}{L} n \sim$ boundary conditions $\psi \sim e^{ik_n x}$



$$H = \sum_{n=-\infty}^{+\infty}\left[\omega_n^L\,\psi_L^{+}\psi_L + \omega_n^R\,\psi_R^{+}\psi_R\right]$$

with $\begin{cases} \omega_n^R = k_n + eA_1 \\ \omega_n^L = -(k_n + eA_1) \end{cases}$

$\Longrightarrow$ START WITH VACUUM AND INCREASE $A_1 > 0 \Longleftrightarrow$ INCREASING E FIELD

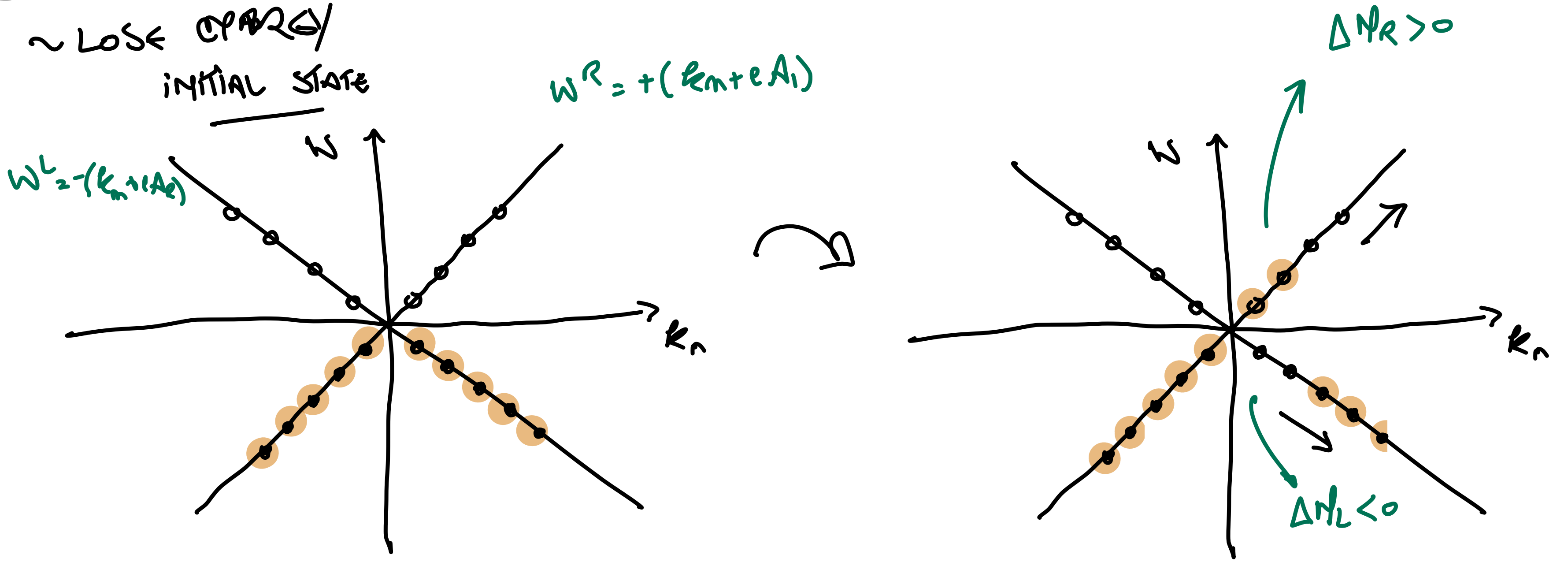
REM: DECREASING A_1

$\longrightarrow E \qquad e > 0$

LEFT MOVER $\longleftarrow$ DECELERATED $\sim$ LOSE ENERGY

RIGHT MOVER $\longrightarrow$ ACCELERATED $\sim$ INCREASING ENERGY

$\Longrightarrow$



NB : ELECTRIC CHARGE CONSERVED

$$\Delta N_R + \Delta N_L = 0 \ !$$

BUT CHANGE IN CHIRAL CHARGE...

DIGRESSION

$\longrightarrow$ WILSON LOOP $\quad W = e^{+ie\oint A_1\, dx^1}$

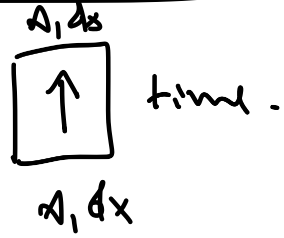
$$= e^{ie\int A_\mu\, dx^\mu}$$

WHEN ΔA_1 REACHES $\frac{2\pi}{eL}$

$\Longrightarrow$ SAME AS BEFORE

$\sim$ NEW VACUUM

$$W = e^{ie\oint A_\mu\, dx^\mu}$$



$$= e^{ie\int F_{\mu\nu}\, dS^{\mu\nu}}$$

BY STOKES' THEOREM

ANTISYMMETRIC $\sim \varepsilon^{\mu\nu}$

$$\int d^2x\, \frac{e}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu} = \frac{e}{\pi}\int dx\, dt\, \partial_0 A_1$$

$$= \frac{e}{\pi}\left[\int_0^L dx\, A_1\Big|_{t=+\infty} - \int_0^L dx\, A_1\Big|_{t=-\infty}\right]$$

$$= \Delta N_R - \Delta N_L = \int d^2x\, \partial_\mu J^\mu_5$$

$$= \int dx\, J^0_5\Big|_{t=+\infty} - \int dx\, J^0_5\Big|_{t=-\infty}$$

TRIANGULAR DIAGRAM

* AMPLITUDE FOR $\langle 0 | T J_5^\alpha(0) J^\mu(x_1) J^\nu(x_2) | 0 \rangle$?

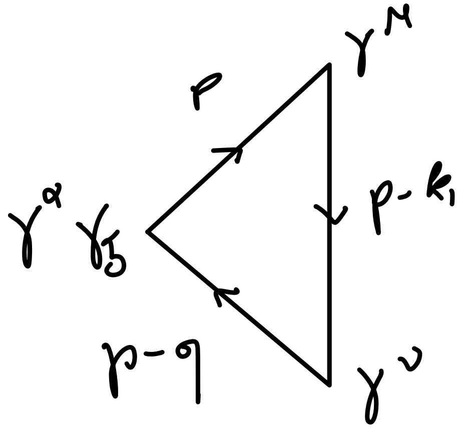
ITS FOURIER TRANSFORM IS

$$\Delta^{\alpha\mu\nu}(k_1, k_2) = (-) i^3 \int \frac{d^4 p}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\alpha \gamma_5 \frac{1}{\not p - \not q} \gamma^\nu \frac{1}{\not p - \not k_1} \gamma^\mu \frac{1}{\not p} + \gamma^\alpha \gamma_5 \frac{1}{\not p - \not q} \gamma^\mu \frac{1}{\not p - \not k_2} \gamma^\nu \frac{1}{\not p} \right]$$

FERMION LOOP

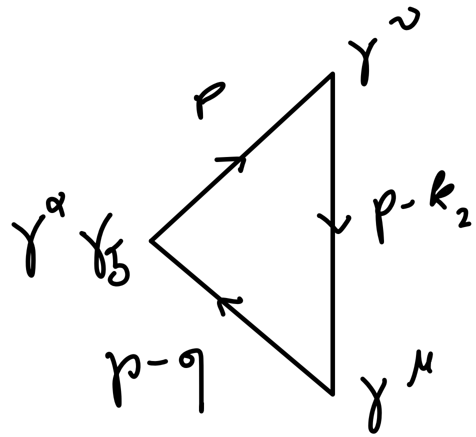
SYMMETRIC FOR $\mu \leftrightarrow \nu$, $k_1 \leftrightarrow k_2$ (BOSE STATISTICS)

OR



$q = k_1 + k_2$

+



$q = k_1 + k_2$

* CLASSICALLY $\partial_\mu J^\mu = 0$ AND $\partial_\mu J^\mu_5 = 0$ (massless fermion)

$\rightsquigarrow$

- conservation of fermion number $Q = \int d^3x\, J^0(\vec{x})$

less clear...

- For sure, we need that. Also J^μ couples to EM field A_μ

$\partial_\mu J^\mu = 0$ important for gauge invariance which is important for quantizing gauge field (ie QED)

* What do we get quantum mechanically?

* Check whether $k_{1,\mu}\Delta^{\alpha\mu\nu} = 0 = k_{2\nu}\Delta^{\alpha\mu\nu}$

$$\not{k}_1 \equiv -\not{p} + \not{k}_1 + \not{p}$$

$$k_{1\mu}\Delta^{\alpha\mu\nu}(k_1,k_2) = (-)\,i^3\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\Big[\gamma^\alpha\gamma_5\frac{1}{\not{p}-\not{q}}\gamma^\nu\frac{1}{\not{p}-\not{k}_1}\not{k}_1\frac{1}{\not{p}} + \gamma^\alpha\gamma_5\frac{1}{\not{p}-\not{q}}\not{k}_1\frac{1}{\not{p}-\not{k}_2}\gamma^\nu\frac{1}{\not{p}}\Big]$$

$$\not{k}_1 \equiv -\not{p} + \not{q} - \not{k}_2 + \not{p}$$

$$= (-)\,i^3\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\Big[\cancel{-\gamma^\alpha\gamma_5\frac{1}{\not{p}-\not{q}}\gamma^\nu\frac{1}{\not{p}}} + \gamma^\alpha\gamma_5\frac{1}{\not{p}-\not{q}}\gamma^\nu\frac{1}{\not{p}-\not{k}_1} - \gamma^\alpha\gamma_5\frac{1}{\not{p}-\not{k}_2}\gamma^\nu\frac{1}{\not{p}}$$
$$+ \cancel{\gamma^\alpha\gamma_5\frac{1}{\not{p}-\not{q}}\gamma^\nu\frac{1}{\not{p}}}\Big]$$

$$k_\mu \Delta^{\alpha\mu\nu} = (-) i^3 \int \frac{d^4 p}{(2\pi)^4} T_1 \left[\gamma^\alpha \gamma_5 \frac{1}{\not p - \not q} \gamma^\nu \frac{1}{\not p - \not k_1} - \gamma^\alpha \gamma_5 \frac{1}{\not p - \not k_2} \gamma^\nu \frac{1}{\not p} \right]$$

$$\int d^3x \, \vec{\nabla} \cdot \vec{E} = \int \vec{E} \cdot d\vec{S}$$

$p \to p - k_1$ SAME UP TO THIS SHIFT OF VARIABLE.

BUT NOT LEGITIMATE (NOT OBVIOUSLY) BECAUSE THE INTEGRAL IS ~~NOT CLEARLY CONVERGENT~~ A PRIORI DIVERGENT !

LOOK AT $\int_{-\infty}^{+\infty} dx \, f(x+a)$ and $\int_{-\infty}^{+\infty} dx \, f(x)$. WHEN ARE THEY THE SAME?

$$= \int_{-\infty}^{+\infty} dx \left(f(x) + a \frac{df}{dx}(x) + \dots \right) = \int_{-\infty}^{+\infty} dx \, f(x) + a \big(f(+\infty) - f(-\infty) \big) + \dots$$

$$\neq \int_{-\infty}^{+\infty} dx \, f(x)$$

CASE IF LINEARLY DIVERGENT...

in EUCLIDEAN D-SPACE

$$\int d^D p_E \big(f(p+a) - f(p) \big) = \int d^D p_E \left[a^\mu \partial_\mu f(p) + \dots \right] = \oint a^\mu f(p) dS_\mu$$

$$= a^\mu \hat{p}_\mu \bar{f}(p) S_{D-1}(p)$$

SPHERE IN D SPACE

$$\int d^D p \left(f(p+a) - f(p) \right) = i\, a^\mu \hat{P}_\mu \, f(P)\, 2\pi^2 P^3 \quad \text{in MINKOWSKI}$$

WITH $P \to \infty$ AND $\hat{P}_\mu = P_\mu / P$

WITH $f(p) = \mathrm{Tr}\left[\gamma^\alpha \gamma_5 \frac{1}{\cancel{p} - \cancel{k}_2} \gamma^\nu \frac{1}{\cancel{p}} \right] = \mathrm{Tr}\left[\frac{\gamma^\alpha \gamma_5 (\cancel{p} - \cancel{k}) \gamma^\nu \cancel{p}}{(p-k_2)^2 p^2} \right] = -4i\, \varepsilon^{\lambda\nu\sigma\alpha} \frac{(p-k)_\lambda p_\sigma}{(p-k_2)^2 p^2}$

$$\int d^D p \left[f(p-k_1) - f(p) \right] = \frac{-i^3}{(2\pi)^4} \lim_{P \to \infty} \left(-i k_1^\mu \frac{P_\mu}{P} 2\pi^2 P^3 \right) \frac{(+4i\, \varepsilon^{\lambda\nu\sigma\alpha} k_{2\lambda} P_\sigma)}{P^4}$$

WHAT IS $\frac{P_\mu P_\sigma}{P^2}$? THIS IS MEANT TO BE AVERAGED OVER THE VOLUME, SO $\frac{P_\mu P_\sigma}{P^2} = \frac{\eta_{\mu\sigma}}{D}$

$$\Rightarrow k_{1\mu} \Delta^{\alpha\mu\nu}(k_1, k_2) = \frac{i\, 4 \cdot 2\pi^2}{D\, (2\pi)^4} \varepsilon^{\lambda\nu\sigma\alpha} k_{1\sigma} k_{2\lambda} = \frac{i}{8\pi^2} \varepsilon^{\alpha\nu\sigma\lambda} k_{1\sigma} k_{2\lambda}$$

$\neq 0$ SO CURRENT IS NOT CONSERVED...

BUT WHAT WE GOT IS AMBIGUOUS. NOTHING PREVENTS US FROM STARTING WITH A TOTALLY DIFFERENT p IN THE INTEGRAND, e.g. DEFINE

$$\Delta^{\alpha\mu\nu}(a,k_1,k_2) = i\int \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\left[\gamma^\alpha\gamma_5 \frac{1}{\not p+\not a-\not q}\gamma^\nu \frac{1}{\not p+\not a-\not k_1}\gamma^\mu \frac{1}{\not p+\not a}\right] + (k_1 \leftrightarrow k_2, \mu\leftrightarrow\nu)$$

AND COMPARE TO

$$\Delta^{\alpha\mu\nu}(k_1,k_2) = i\int \frac{d^4p}{(2\pi)^4}\, \mathrm{Tr}\left[\gamma^\alpha\gamma_5 \frac{1}{\not p-\not q}\gamma^\nu \frac{1}{\not p-\not k_1}\gamma^\mu \frac{1}{\not p}\right] + (k_1 \leftrightarrow k_2, \mu\leftrightarrow\nu)$$

APPLYING THE MAGIC FORMULA GIVES

$$\Delta^{\alpha\mu\nu}(a,k_1,k_2) - \Delta^{\alpha\mu\nu}(k_1,k_2) = \lim_{P\to\infty} i\, a^\mu \frac{P_\mu}{P}$$

TO BE CONTINUED...

GROUP THEORY AND ANOMALIES

- FOR A LIE ALGEBRA, WE HAVE

$$[T^A, T^B] = i f^{ABC} T^C$$

WITH T^A THE GENERATORS FOR A GIVEN REPRESENTATION

- THEY ARE NORMALIZED, SO THAT

$$\mathrm{Tr}_n [T^A T^B] = C(n)\, \delta^{AB}$$

FOR INSTANCE $C(n) = 1/2$ FOR $SU(N)$ IN FUNDAMENTAL

$C(n) = N$ FOR n IN THE ADJOINT

- FOR A GIVEN REPRESENTATION R

$$\phi_R \to (1 + i \alpha^A T_R^A)\, \phi_R$$

- WE CAN CONJUGATE THIS

$$\phi_R^* \to (1 - i \alpha^A T_R^{A*})\, \phi_R^*$$

WE SAY THAT ϕ_R^* TRANSFORM AS THE CONJUGATE OF ϕ_R

- WE NOTE THIS AS $\bar{R}$. REMARK THAT

$$[(-T^{A*}), (-T^{B*})] = i f^{ABC} (-T^{C*})$$

SO $T^A_{\bar{R}} = -T_R^{A*}$

- SINCE THE T^A ARE HERMITIAN $T^{A+} = T^A$ (↓ HERMITIAN CONJUGATE)

$$T^A_{\bar{R}} = -T^{A*} = -T^{AT}$$

(↓ COMPLEX CONJUGATE) (↓ TRANSPOSED)

- IMPORTANTLY, IT MAY BE THAT

$$T^A_{\bar{R}} = U\, T^A_R\, U^+ \quad \text{WITH } U \text{ SOME UNITARY MATRIX.}$$

SUCH REPRESENTATION IS SAID TO BE REAL OR PSEUDOREAL. WE WILL SKIP THE DISTINCTION BETWEEN REAL AND PSEUDOREAL REPRESENTATION HERE.

- FOR INSTANCE, FOR $SU(2)$ IN THE FUNDAMENTAL, WE HAVE $T^A = \frac{\sigma^A}{2}$

BUT $(\sigma^a)^* = -\sigma^2 \sigma^a \sigma^2$

$$\sigma^{1*} = \sigma^1 = -\sigma^2\sigma^1\sigma^2 = -i\sigma^2\sigma^3 = \sigma^1$$

$$\sigma^{3*} = \sigma^3 = -\sigma^2\sigma^3\sigma^2 = -i\sigma^1\sigma^2 = \sigma^3$$

AND OF COURSE $\sigma^{2*} = -\sigma^2 = (-\sigma^2)(\sigma^2)^2$

$$i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = U$$

$$-i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = U^+$$

AND SO $\sigma^A_{\bar{R}} = U \sigma^A U^+$ SO $SU(2)$ IN THE FUNDAMENTAL IS (PSEUDO)REAL.

EX: SHOW THAT THE ADJOINT REPRESENTATION OF $SU(2)$ IS REAL

- THIS IS ACTUALLY TRUE FOR ALL ADJOINT REPRESENTATIONS.

INDEED

$$(T_G^A)_{ij} = i f^{iAj}$$

$$(T_G^A)_{ij}^* = -i f^{iAj}$$

$$\text{SO } T_{\bar{G}}^A = -(T_G^A)^* = +i f^{iAj} = T_G^A$$

THUS THE ADJOINT REPRESENTATION IS ALWAYS REAL

- CONSIDER R-HANDED FERMION IN A REPRESENTATION R.

IT IS USEFUL TO REWRITE THEM AS L-HANDED FIELDS.

OF COURSE, PARTICLES <-> ANTIPARTICLES.

FOR THIS, WE USE ($\neq$ AUTHORS USE $\neq$ PREFACTORS)

$$\psi_R \longrightarrow \psi_L = \sigma^2 \psi_R^*$$

- WE ALSO USE THE FACT JUST DISCUSSED THAT

$$\sigma^2 \sigma^A \sigma^2 = -(\sigma^A)^* = -(\sigma^A)^T$$

SO THAT $\boxed{\sigma^2 \sigma^{\mu T} \sigma^2 = \bar{\sigma}^\mu}$

AND $\boxed{\sigma^2 \bar{\sigma}^{\mu T} \sigma^2 = \sigma^\mu}$

THEN $\int \chi_R^\dagger\, i\sigma^\mu (\partial_\mu - ig A_\mu^a T_R^a)\, \chi_R$

$$= \int \big[-i\, \partial_\mu \chi_R^\dagger\, \sigma^\mu \chi_R + g\, \chi_R^\dagger \sigma^\mu T_R^a \chi_R A_\mu^a \big)$$

INTEGRATING BY PART

$$= \int \big[+i\, \chi_R^T\, \sigma^{\mu T} \partial_\mu \chi_R^* - g\, \chi_R^T \sigma^{\mu T} T_R^{aT} \chi_R^* A_\mu^a \big]$$

$\Rightarrow$ IMPORTANT TO TAKE INTO ACCOUNT THE FACT THAT FERMIONS ANTICOMMUTE

$$= \int \big[+i\, \chi_R^T \sigma^2 \bar\sigma^\mu \partial_\mu \sigma^2 \chi_R^* - g\, \chi_R^T \sigma_2 \bar\sigma^\mu \sigma_2 T_R^{aT} \chi_R^* A_\mu^a \big]$$

$$= \int \big(i\, \chi_L^\dagger \bar\sigma^\mu \partial_\mu \chi_L - g\, \chi_L^\dagger \bar\sigma^\mu T_R^{aT} \chi_L A_\mu^a \big]$$

$$= \int \big[i\, \chi_L^\dagger \bar\sigma^\mu \partial_\mu \chi_L + g\, \chi_L^\dagger \bar\sigma^\mu T_{\bar R}^a \chi_L \big]$$

SO THE ACTION OF χ_R TRANSFORMING AS R IS EQUIVALENT TO THE ACTION OF A χ_L TRANSFORMING AS $\bar R$.

SO WE CAN REWRITE A THEORY OF FERMIONS AS A THEORY OF L-HANDED FIELDS.

- FOR INSTANCE, IN QCD WE HAVE L AND R QUARKS IN THE 3 REPRESENTATION.
- WE CAN REWRITE THIS AS A THEORY OF

$$L \text{ in } 3 \quad \text{with} \quad L \text{ in } \bar{3}$$

(QUARKS L-HANDED) (ANTI-QUARKS LEFT-HANDED)

- HENCE A DIRAC IN THE R CAN BE EQUIVALENTLY AS L-HANDED FERMIONS IN $R \oplus \bar{R}$ REPRESENTATION. THIS IS REAL BECAUSE $(R \oplus \bar{R})^* = R \oplus \bar{R}$
- IN GENERAL A THEORY WITH L AND R IN $\neq$ REPS CANNOT BE WRITTEN AS A DIRAC THEORY AND IS THEN SAID TO BE CHIRAL.

- THIS IS RELATED TO THE MASS OF FERMIONS. A DIRAC MASS TERM IS OF THE FORM

$$m \bar{\psi} \psi = m (\chi_R^\dagger \chi_L + \chi_L^\dagger \chi_R)$$

$$= m (\chi_L'^T \sigma_2 \chi_L + \chi_L^\dagger \sigma_2 \chi_L'^*)$$

$$= m (\chi_L'^T \sigma_2 \chi_L + h.c.)$$

OF COURSE $\chi_L' \neq \chi_L$ FOR A DIRAC PARTICLE.

- IF $\chi'_L = \chi_L$ WE HAVE WHAT IS CALLED A MAJORANA MASS TERM. WE WILL COME BACK TO THAT.

- IN GENERAL, FOR A THEORY WITH i L-HANDED FERMIONS χ_i A MASS TERM CAN BE WRITTEN AS

$$M_{ij} \underbrace{\chi_i^T \sigma_2 \chi_j}_{} + h.c.$$

$$\begin{aligned} & \chi_{i\alpha}\, \sigma_{2\,\alpha\beta}\, \chi_{j\beta} \\ &= - \chi_{j\beta}\, \sigma_{2\,\alpha\beta}\, \chi_{i\alpha} \quad \text{(FERMIONS)} \\ &= + \chi_{j\beta}\, \sigma_{2\,\beta\alpha}\, \chi_{i\alpha} \quad (\sigma_2^T = -\sigma_2) \\ &= \chi_j^T \sigma_2 \chi_i \end{aligned}$$

SO $M_{ij} = M_{ji}$ MUST BE SYMMETRIC

- MORE IMPORTANTLY, THIS MASS TERM CAN ONLY BE WRITTEN FOR FERMIONS THAT ARE IN A REAL REPRESENTATION, LIKE IN QCD.

$\Rightarrow$ IN OTHER WORDS, WE NEED BOTH R AND $\bar{R}$ REPRESENTATIONS TO WRITE A MASS TERM SO THAT

$$R \otimes \bar{R} = 1 \oplus$$

$\downarrow$ SINGLET, OR GAUGE INVARIANT MASS TERM.

- THIS IS A KEY FEATURE OF GAUGE THEORIES WITH FERMIONS

  $\Longrightarrow$ GAUGE BOSONS CAN ONLY GET A MASS THROUGH SSB : GAUGE SYMMETRY PROTECTS THE MASS

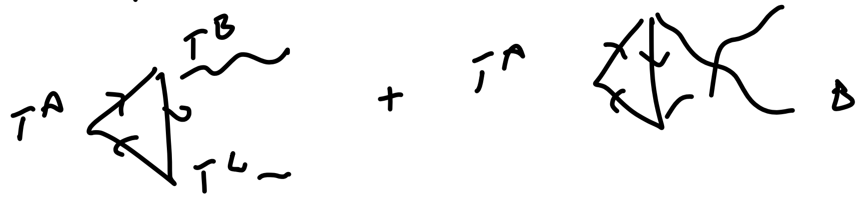
  $\Longrightarrow$ CHIRAL THEORIES (COUPLED TO GAUGE FIELDS) PROTECT THE MASS.

- BUT FOR THIS TO WORK, WE NEED TO MAKE SURE THAT CHIRAL SYMMETRIES THAT ARE GAUGED ARE NOT SPOILED BY QUANTUM CORRECTIONS.
- HENCE THE NECESSITY OF ANOMALY CANCELLATION.

- IN D=4 WE MUST ASK THAT

$$J^{\mu A}_L = \bar{\psi}_L \gamma^\mu T^A \psi_L$$

IS SUCH THAT $\partial_\mu J^{\mu A}_L = 0$

$\Longleftrightarrow$ 

$\Longrightarrow$ SUMMED OVER ALL REPRESENTATIONS !

$$\Longrightarrow \quad A^{ABC} = \mathrm{Tr}\left[T^A \{T^B, T^C\}\right]$$

= FULL SYMMETRIC IN ABC EXCHANGE.

- Key question: When is $A^{ABC} = 0$?

① AUTOMATIC

* ~~ex~~ SU(2)

$$\mathrm{Tr}\left[\sigma^a \{\sigma^b, \sigma^c\}\right] = \mathrm{Tr}\left[\sigma^a\right] \delta^{bc}$$

$$= 0$$

MORE GENERALLY,

* REAL REPRESENTATION

OR $\quad T^A_{\bar{R}} = U\, T^A_R\, U^\dagger = - T^{A\,T}_R$

THEN $\quad A^{ABC} = \bar{A}^{ABC} = \mathrm{Tr}\left[U\, T^A_R \{T^B_{\bar{R}}, T^C_{\bar{R}}\} U^\dagger\right]$

$$= \mathrm{Tr}\left[-T^A_R \{T^B_R, T^C_R\}\right]$$

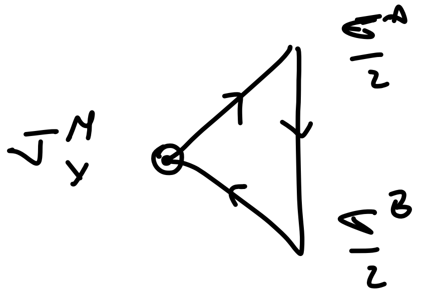
$$= - A^{ABC}$$

SO $\boxed{A^{ABC} = 0}$ IF REAL REPRESENTATION.

* THAT APPLIES TO $R \oplus \bar{R}$

SINCE $\quad A^{ABC} + \bar{A}^{ABC} = A^{ABC} - A^{ABC} = 0$

* MIXED ANOMALIES, FOR INSTANCE



$$\Rightarrow \quad \mathrm{Tr}\left[Y \left\{ \frac{\sigma^A}{2}, \frac{\sigma^B}{2} \right\} \right]$$

$$= \frac{1}{2}\, \delta^{AB}\, \mathrm{Tr}[Y] \equiv \delta^{AB}\, \mathrm{Tr}[Q]$$

SINCE $Q = \frac{\sigma^3}{2} + \frac{Y}{2}$

SM:

$\longrightarrow \quad \mathrm{Tr}[Q] = \left[-1 + \frac{2}{3}\times 3 + -\frac{1}{3}\times 3 \right] = 0$

NOTE THAT THIS RELATES THE CHARGE OF THE ELECTRON TO THAT OF THE QUARKS AND SO TO THAT OF THE PROTON.

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* CANCELLATION MAY ARISE BETWEEN $\neq$ REPRESENTATIONS

ex $SU(N)$

$$\{ T^A_n, T^B_n \} = \frac{1}{n}\, \delta^{AB} + \underbrace{d^{ABC}}_{\text{UNIQUE } (=0 \text{ FOR } SU(2))} T^C_n$$

THEN $\mathrm{Tr}\left[T^A_R \{ T^B_R, T^C_R \} \right] = \frac{1}{2} A(R)\, d^{ABC}$ POTENTIAL ANOMALY!

$\Rightarrow A(\bar{R}) = -A(R)$ SO ABSENT FOR REAL $R \oplus \bar{R}$ REPS.

WE WILL MORE SOPHISTAGATED CASE DISCUSSING SU(3)