CH 4 CLASSICAL DISCRETIZATION

→ A discretization is an approximation: a truncation in the number of dof where we disregard those likely to be irrelevant for a given problem. In QFT, for describing the weak field regime, we use Fock-space methods and perturbation theory. It's constructed by defining the N-particle state, then take $N \to \infty$. The same will be true for QG.

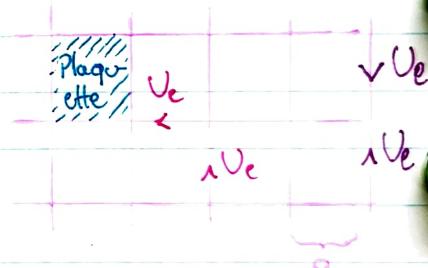
4.1 Lattice QCD

→ We consider a SU(2) Yang-Mills theory in 4-D. The field variable in the continuous theory is an SU(2) connection $A^i_\mu(x)$. We can write it as a one-form with value in $su(2)$:

$$A(x) = A^i_\mu(x)\, \tau_i\, dx^\mu \quad \text{where } \tau_i \text{ are the } su(2) \text{ gen.}$$

→ The key idea of Wilson, is that we must view the algebra as the tangent space to the group, and A as the log of the group variable.

→ Fix a cubic lattice with N vertices connected by E edges in spacetime. This breaks the rotation and Lorentz invariance of the theory, but it will be recover when taking the limit.



Let a be the lenght of the lattice edges, determined by $\eta_{\mu\nu}$.

→ We associate a group element $U_e \in SU(2)$ to each oriented edge e of the lattice, and U_e^{-1} to the opposite direction.

Then $\{U_e \,/\, e \in E\}$ provide a discritization of the continuous field A. Physical qties must be studied in the limit $\begin{cases} N \to \infty \\ a \to 0 \end{cases}$

DEF The formal def. of the group element is a holonomy:

$$U_e \equiv P \exp\{\textstyle\int_e A\}$$

defined by the solution of the differential equation

$$\frac{d}{ds} U(s) = \dot{\gamma}^a(s)\, A_a(\gamma(s)) \quad \text{for} \begin{cases} s = 1 \\ U(0) = \mathbb{1} \end{cases}$$

Expanding in the length a of the edge: $U_e = \mathbb{1} + a A_\mu(s_e)\dot{e}^\mu$
where $\dot{e}^\mu$ is the unit tangent to the edge and s_e is the initial point of e (source).

→ The holonomy is invariant under all local gauge transfo: $A \mapsto A + D\lambda$ except those at the boundary points of the edge.

Therefore, when truncating the theory to the group variables, the local $SU(2)$ gauge symmetry is reduced to a symmetry under $SU(2)$ rotations at the vertices of the lattice only.

↳ The gauge group of the lattice theory is $SU(2)^V$, where V = # vertices.

The group variables transform as:

$U_e \mapsto \lambda_{s_e} U_e \lambda_{t_e}^{-1}$, $\lambda_v \in SU(2)$ under such a gauge transfo.
where t_e is the final vertice of e (target).

→ The ordered product of 4 group elements around a plaquette f
$U_f = U_{e_1} U_{e_2} U_{e_3} U_{e_4}$ is the discrete version of the curvature.
Its trace is gauge invariant.

→ The action $S = \beta \sum_f Tr(U_f) + c.c.$ approximate the continuous action in the limit $a \to 0$.

→ Consider now the boundary of the lattice. Let l be the boundary edge or links. If we fix their value and integrate the exponential of the action over the bulk groups elements, we obtain the transition amplitude of the truncated theory:

$$W(U_l) = \int dU_e \exp\{ i/\hbar \, S[U]\}$$

→ This integral fully describe the theory. To get the continuous transition amplitudes, one must study the limit $\begin{cases} \beta \to 0 \\ N \to \infty \end{cases}$

4.1.1 Hamiltonian lattice theory

→ the hamiltonian formulation lives on a boundary of the lattice, which we assume to be spacelike. The hamiltonian coord. are the $\{U_e\}$.

DEF We call the boundary edges links ℓ and the boundary vertices nodes n.

→ These group elements → space components of the connection on the boundary (in time gauge): they code the magnetic field.
→ In the $\mathcal{L}$ language, $\vec{K}$ is coded by the edges normal to ∂R
→ In the $\mathcal{H}$ language, $\vec{L}$ is coded in the momentum conjugate to the boundary group elements.

→ The canonical configuration space is $SU(2)^L$, $L = \#$ links. The reduced gauge is given by the gauge transfo. at the nodes on the boundary. The corresponding phase space is the cotangent space $T^*SU(2)^L$. There is one conjugate momentum L_ℓ^i in $su(2)$ $\forall \ell$, and it is identified with the electric field.
A cotangent space T^*Q carries a natural symplectic structure. The corresponding Poisson brackets are:
$\{U_\ell, U_{\ell'}\} = 0 \qquad \{U_{\ell'}, L_\ell^i\} = \delta_{\ell\ell'} U_\ell \tau^i \qquad \{L_\ell^i, L_{\ell'}^j\} = \delta_{\ell\ell'} \epsilon^{ij}{}_k L_\ell^k$

→ The Hilbert space of the discrete theory $\ni \psi(U_\ell)$, function on the config. space. The natural scalar product associated is

$$(\phi, \psi) = \int_{SU(2)} dU_\ell \, \overline{\phi(U_\ell)} \, \psi(U_\ell)$$

where dU_ℓ is the Haar measure. $(,)$ is invariant under gauge transfo. at the boundary. The boundary gauge transfo. act at the links, and transform the state as $\psi(U_\ell) \mapsto \psi(\lambda_{s_\ell} U_\ell \lambda_{t_\ell})$ $\lambda_n \in SU(2)$.
↳ The states invariant under this transformation form the $\mathcal{H}$ of gauge invariant state, which has the structure $L_2[SU(2)^L / SU(2)^N]$ where $N = \#$ of node at the boundary. The operator ↔ U_ℓ is diag in this basis, and the one ↔ conjugate to U_ℓ is $\vec{L}_\ell$ the left inv. vector field. Classically, the conjugate momentum to the connection is $\vec{K}$. These operator realize the Poisson algebra above.

4.2 Discretization of covariant systems

→ We've seen that a quantum theory can be define as the limit of a discretization of the theory.

$$W(q,t,q',t') = \lim_{N\to\infty} \int dq_n \exp\left\{\frac{i}{\hbar} S_N(q_n)\right\}$$

where $S_N(q_n) = \sum_{n=1}^{N} \frac{m(q_{n+1}-q_n)^2}{2\epsilon} - \epsilon V(q_n)$ and $\epsilon = (t'-t)/N$

Taking $q_n \mapsto q_n\sqrt{\epsilon}$, we get

$$S_N(q_n) = \sum_{n=1}^{N} \frac{m}{2}(q_{n+1}-q_n)^2 - \frac{\Omega}{2} q_n^2 \quad \text{where} \begin{cases} \Omega = \epsilon\omega \\ V(q) = \frac{1}{2}\omega^2 q^2 \end{cases}$$

When we take $N\to\infty$, we do $\sum^{N} \to \sum^{\infty}$ and $\Omega \to \Omega_c = 0$

→ For discretizing covariant system, the discretization doesn't depend on the lattice spacing ϵ. Let's consider $S = \int d\tau \left(\frac{m}{2}\frac{\dot{q}^2}{\dot{t}} - \dot{t}\, V(q) \right)$.

Discretized, we get

$$S = \sum_{n=1}^{N} \epsilon \left(\frac{m}{2} \frac{(q_{n+1}-q_n)^2/\epsilon^2}{(t_{n+1}-t_n)/\epsilon} - \frac{t_{n+1}-t_n}{\epsilon} V(q_n) \right)$$

$$= \sum_{n=1}^{N} \frac{m}{2} \frac{(q_{n+1}-q_n)^2}{(t_{n+1}-t_n)} - (t_{n+1}-t_n) V(q_n)$$

This is logic since the original action doesn't depend the parametrization. The reparametrization invariant system are characterized by the fact that there is no parameter to rescale in the discrete action in the limit.

→ A "good" discretization of GR is reparametrization independant. It is at the root of the peculiarity of the covariant form of QG.

4.3 Regge calculus

→ Tullio Regge introduced a discretization of GR, called Regge calculus.

DEF A d-simplex is the convex hull of its $d+1$ vertices. These vertices are connected by $d(d+1)/2$ line segments whose lengths L_s fully specify the shape of the simplex.

ex: for $d=3$ (a tetrahedron), a 3-simplex has $d+1=4$ vertices connected by $d(d+1)/2 = 6$ line segments whose length $L_1, L_2, .., L_6$ fully determine its shape.

DEF A Regge space (M, L_s) in d-dimensions is a d-dimensional metric space obtained by gluing d-simplices along matching boundary $(d-1)$-simplices

ex: in $d=2$, we can obtain a surface by gluing triangles, bounded by segments, which meet at points.

→ In $d=4$, we chop spacetime into 4-simplices, bounded by tetrahedra, in turn bounded by segments, which meet at points. These structure are called triangulations and denoted Δ.

→ We assume that the triangulation is oriented.

→ Gluing flat d-simplices can generate curvature on the $(d-2)$ simplices (called hinges)

ex: in $d=2$, we can glue 4 equilateral Δ as in the boundary of a tetrahedron, and there is a clear curvature on the vertices of the tetrahedron.

In $d=3$, we can glue several tetrahedra all around a common segment, and obtain a manifold flat everywhere except at this segment

→ In 4-D, the curvature is on the triangles. The metric of the resulting space is uniquely determined by the length L_s of all its segments.

$d=2$			triangle	segment	point
$d=3$		tetrahedron	triangle	segment	point
$d=4$	4-simplex	tetrahedron	triangle	segment	point

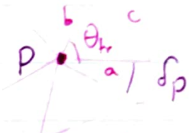
→ A Riemannian manifold (M,g) can be approximated arbitrarily well by a Regge manifold: $\forall (M,g), \forall \epsilon, \exists (M, L_s)$ such that $\forall x,y \in M$, $d_{Ri}(x,y) - d_{Re}(x,y) < \varepsilon$.

→ We approximate GR by a theory of Regge manifolds.

⊙ Regge's notion of curvature:

→ Consider the case $d=2$. Let $P \in$ triangulation. Around it, there are a certain number of triangle tr, of angle θ_{tr}.

This angle is given by
$$\cos(\theta_{tr}) = \frac{c^2 - a^2 - b^2}{2ab}$$



→ The Regge curvature at P can be define as the angle $\delta_P(L_s) = 2\pi - \sum_t \theta_t(L_s)$, called the deficit angle at P.

→ In d dimensions, P is replaced by a $(d-2)$ simplex (a segment if $d=3$ and a triangle if $d=4$). The sum is over the $(d-1)$-simplices around it (triangle in 3D, thetrahedra in 4D) and the angles become the dihedral angles of the flat d-simplices.

For instance, the dihedral angle on the ab side of a tetrahedron with vertices (a,b,c,d) is $\cos\theta_{ab} = (\cos\theta_{acd} - \cos\theta_{abc}\cos\theta_{abd}) / \sin\theta_{abc}\sin\theta_{abd}$ where θ_{abc} is the angle at vertex a of the triangle of vertices a, b, c.

→ If we parallel transport a vector in a loop around a $d-2$ simplex, the vector gets back rotated by the deficit angle.

DEF Let (M, L_s) be a Regge manifold, h the hinges ($d-2$ simplices of the triangulation), A_h the $(d-2)$-volume of h. The Regge action $S_M(L_s)$ is
$$S_M(L_s) \equiv \sum_h A_h(L_s)\,\delta_h(L_s)$$

PROP $S_M(L_s) \to S[g]$ when $(M, L_s) \to (M, g)$.

The Regge action converges to the Einstein-Hilbert action $S[g]$ when the Regge manifold (M, L_s) converges to the Riemann manifold (M, g).

→ the EOM are obtained by varying the action with respect to the length.

? $\delta_{L_s} S_M(L_s) = \sum_h \left(\delta_L A_h \cdot \delta_h + A_h \cancel{\delta_L \delta_h} \right) \overset{!}{=} 0$

$\Leftrightarrow \sum_h \frac{\partial A_h}{\partial L_s} \cdot \delta_h(L_s) = 0$ the Regge equations

↳ sum over the hinges adjacent to the segment s

↳ $\delta_h(L_s) \sim$ measure of the discrete Riemann curvature

↳ $\sum \partial_L A_h \cdot \delta_h \sim$ measure of the discrete Ricci tensor.

→ In 3-D, hinges = segments, therefore the Regge equations reduces to $\delta_h(L_s) = 0$: flatness, as in the continuum case.

→ The Regge curvature is concentrated at the hinges. If we //-transport a vector around a hinge in a Regge manifold, it comes back in the same plane it was before.
ex: if the plane normal to the hinge is the (x^1, x^2) plane, the only non-vanishing component of the Riemann tensor would be $R^{12}{}_{12}$.
Since the physical curvature is the average of the Regge curvature over the region, and since the latter is not a linear property, the average curvature won't inherit this special property.

PROP The Regge action can be rewritten as a sum over the d-simplices σ of the triangulation:
$S_M(L_s) = 2\pi \sum_h A_h(L_s) - \sum_\sigma S_\sigma(L_s)$ where the action of a d-simplex is $S_\sigma(L_s) \equiv \sum_h A_h(L_s)\, \Theta_h(L_s)$

→ If we fix the triangulation Δ, we obtain a finite approx of GR: namely a truncation of GR.
In Regge calculus, the continuum limit is obtained by refining the Δ, with no parameter to be taken to 0 ⇒ covariant character of the theory.
→ The physical size of the segments of the Regge triangulation are the dynamical variables themselves.

4.4 Discretization of GR on a 2-complex

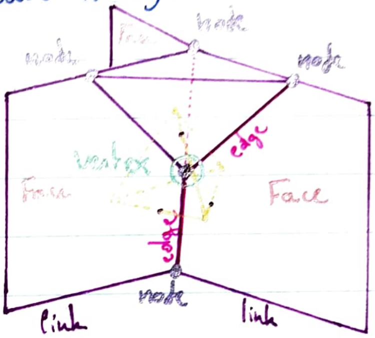
→ We have 2 main problems with the Regge discretization:

1) it's based on the metric variables, and we want tetrads

2) segments of a Regge Δ are constrained by triangular inequalities:
$d(P,Q) \leq d(P,R) + d(R,Q)$. It makes the space of configuration a complicated space.

→ Here we limit ourself to 3-D euclidean theory.

→ The key notion is the dual of the triangulation.



DEF | Let Δ be a 3-D triangulation.
The dual Δ^* of the triangulation Δ is defined as follow:
we place a **vertex** within each tetrahedron, joining the vertices of 2 adjacent tetrahedra by an **edge** "dual" to the triange that separates the 2 tetrahedra, and associating a **face** to each segment of the Δ, bounded by the edges that surround the segment. These object inherit an orientation from their dual.
The set of vertices, edges and faces, with their boundary relations, is called a **2-complex**.

⊙ Boundary:

→ The discretization of the bulk of a spacetime region $\mathcal{R}$ induces one on $\Sigma = \partial\mathcal{R}$. Σ is discretized by the boundary triangles of Δ, separated by the boundary segments of Δ. The end points of the edges dual to these triangles are called **nodes**, and edges between nodes are **links**.

DEF | A **graph** Γ is a set of nodes and links: $\Gamma \equiv \{\text{nodes}, \text{links}\} = \{n, \ell\}$

PROP | $\Gamma = \partial(\Delta^*) = (\partial\Delta)^*$: the boundary graph is the boundary of the 2-complex and the dual of the boundary of the Δ.

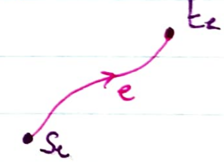
Bulk Δ		2-complex Δ^*		Boundary graph Γ	
tetrahedron	τ	vertex	v	node	n
triangle	t	edge	e	link	ℓ
segment	s	face	f		
point	p				

→ Recall: the grav. field is described by a triad field $e^i = e^i_a \, dx^a$ and an $SO(3)$ connection $\omega^i{}_j = \omega^i{}_{aj} \, dx^a$ where $a, b, \ldots = 1,2,3$ are spacetime indices and $i,j = 1,2,3$ are internal indices, raised by δ_{ij}.
To define a discretization of euclidean GR, we introduce discrete variables on the 2-complex.

DEF We assign an $SU(2)$ group element U_e to each edge e of the 2-complex, and a vector $L^i_s \in \mathbb{R}^3$ to each segment s of the original Δ

$$\omega \longrightarrow U_e \equiv P\exp\{\textstyle\int_e \omega\} \in SU(2)$$
$$e \longrightarrow L^i_s = \textstyle\int_s e^i \qquad \in \mathbb{R}^3$$

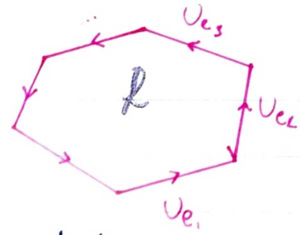
→ U_e is the holonomy of the connection along the edge, the matrix of //-transport generated by the connection along the edge, in the fundamental representation of $SU(2)$.
L^i_s is the line integral of the 1-form e^i along the segment.



→ Under a gauge transformation, $U_e \mapsto R_{s_e} U_e R^{-1}_{t_e}$, $R_v \in SO(3)$
↳ The continuous local $SO(3)$ invariance is reduced to rotations at the vertices.
→ The def. $L^i_s = \int_s e^i$ is taken in a gauge where the connection is constant along the segment s, as well as along the 1^{st} half of each edge e.
This way, L^i_s are invariant under all gauge transfo, exept those at the vertices. They transform covariantly in the adjoint rep. under R_v at v.

→ Each face $f \in \Delta^*$ correspond to $s = s_f \in \Delta$, so we can view $L^i_f = L^i_{s_f}$.
We define $L_f \equiv L^i_f \tau_i$
Therefore, the variable of the discretized theory are:
→ An $SU(2)$ group element $U_e \ \forall e \in \Delta^*$
→ An $su(2)$ algebra element $L_f \ \forall f \in \Delta^*$

Action:



→ Given a face f bounded by the edges $e_1, \dots, e_n$, we define the group element associated with the face itself:

$$U_f \equiv U_{e_1} \cdots U_{e_n}$$

↳ This is the holonomy of the connection going around the segment s_f dual to the face f

→ If $U_f \neq \mathbb{1}$, there is curvature, that we associate to the segment s.

PROP The discretize action then reads:

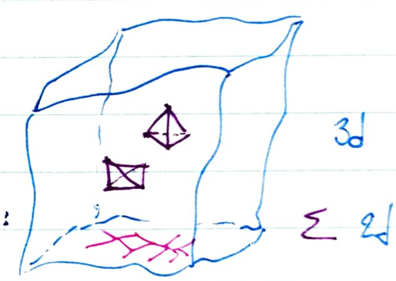
$$S = \frac{1}{8\pi G} \sum_f \mathrm{Tr}\{ L_f U_f \}, \quad \begin{cases} L_f \in su(2) \\ U_f \in SU(2) \end{cases}$$

→ On the boundary, we must close the perimeter of the faces in order to write the quantity U_f for the faces that end on the boundary.

↳ We associate the group qty U_e to the links on $\Sigma = \partial R$.

→ $\delta_{L_f} S = 0 \Rightarrow U_f = \mathbb{1} \Leftrightarrow$ flatness.

Boundary variables:



→ On the boundary, there are 2 types of variables:

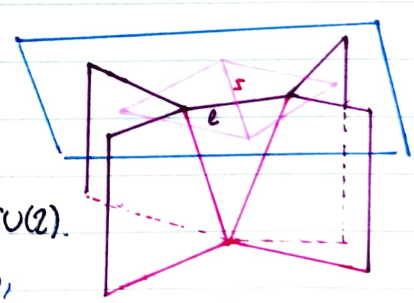
→ the group elements U_e (boundary edge)

→ the algebra element L_s (boundary segment)

↳ There is one segment s per link l:

We rename $L_e \equiv L_s$ when l is crossing s

→ The boundary variables are $(L_e, U_e) \in su(2) \times SU(2)$.



→ Since the algebra is the cotangent space of the group, $su(2) \times SU(2) \sim T^* SU(2)$, and the classical phase space of the discretized theory is $T^* SU(2)^L$. The Poisson brackets are:

$$\{U_e, U_{e'}\} = 0$$
$$\{U_e, L^i_{e'}\} = 8\pi G \, \delta_{ee'} U_e \tau^i$$
$$\{L^i_e, L^j_{e'}\} = 8\pi G \, \delta_{ee'} \epsilon^{ij}{}_k L^k_e$$