CH3 GRAVITY

3.1 Einstein's formulation

→ The gravitational field is a symmetric tensor field $g_{\mu\nu}(x)$, interpreted as a pseudo-Riemannian metric of spacetime. The dynamics is coded in the Einstein field equations (EFE), wich follow from the Einstein-Hilbert action

$$S[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R-2\Lambda)$$

In absence of matter, the EFE are $R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda g_{\mu\nu} = 0$
We call the greek indices $\mu,\nu = 0,1,2,3$, spacetime tangent indices.

→ The theory is defined by 2 constants:

$$G \simeq 6\cdot 10^{-6}\, cm^3 g^{-1} s^{-1} \quad \text{and} \quad \Lambda \simeq 10^{-52}\, m^{-2}$$

The presence of $\Lambda \Rightarrow \exists$ dimensionless constant in the quantum theory:

$$\frac{\Lambda\hbar G}{c^3} \sim 10^{-120}$$

The cosmological constant being close to the scale of the Hubble radius $R_{univ} \sim \Lambda^{-1/2}$, this number is +/- the ratio between the largest and the smallest thing in the universe we are aware of:

$$c^3/\Lambda\hbar G \sim R_{univers}^2 / L_{pl}^2 \sim 10^{120}$$

→ We'll drop Λ whenever it does not contribute conceptually, and we'll use units where $\hbar = 8\pi G = c = 1$

3.2 Tetrads and fermions

→ The existence of fermions show that the formulation of the grav. field as a pseudo-Riemannian metric cannot be fundamentally correct.
We need the tetrad formulation.

→ The dynamics of a fermion in flat space is governed by the Dirac equation: $i\gamma^I \partial_I \psi - m\psi = 0$
↳ We must use a tetrad $e^I_\mu(x)$ where $I, J, \ldots = 0,1,2,3$ are internal flat indices

PROP The relation between the tetrad and the metric is

$$\eta_{IJ} = g(e_I, e_J) = e_I^\mu e_J^\nu g_{\mu\nu} \Leftrightarrow g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ}$$

→ Geometrically, $e_\mu^I(x)$ is a map from the tangent space at x to Minkowski space. The metric $g_{\mu\nu}$ is the pull-back of the Minkowski metric to the tangent space.

The grav. action can be written replacing the metric with its expression in term of the tetrad $S[e] = S[g[e]]$

PROP The tetrad formalism satisfies an additional local Lorentz $SO(3,1)$ gauge invariance under the transformations

$$e_\mu^I(x) \mapsto \Lambda^I{}_J(x)\, e_\mu^J(x)$$

↳ The metric is not affected by this transfo because the Lorentz matrices transform the Minkowski metric η_{IJ} into itself.

→ Given a Riemannian manifold, the 4 tetrad fields are such that everywhere they provide an orthonormal frame. Conversely, if one has just the 4 vectors at each points, these define a Riemannian manifold by defining the frame that they determine to be orthonormal.

DEF Related to this local gauge invariance, we introduce the Lorentz connection $\omega_\mu{}^{IJ}$ such that $\omega_\mu{}^{IJ} = -\omega_\mu{}^{JI}$

Antisym because we have metricity: $0 = D_\mu \eta_{IJ} = \partial_\mu \eta_{IJ} - \omega_\mu{}^K{}_I \eta_{KJ} - \omega_\mu{}^K{}_J \eta_{IK}$
$= -2\,\omega_{\mu(IJ)} \overset{!}{=} 0$

↳ This connection defines a covariant derivative D_μ, and allow us to define a general covariant Dirac equation:

$$i\gamma^I e_I^\mu D_\mu \psi - m\psi = 0$$

DEF We introduce the form notation

$$e^I \equiv e_\mu^I dx^\mu \quad \text{and} \quad \omega^{IJ} = \omega_\mu^{IJ} dx^\mu$$

DEF The torsion 2-form T^I is defined as

$$T^I \equiv de^I + \omega^I{}_J \wedge e^J$$

Also called the 1st Cartan equation

The curvature 2-form $F^I{}_J$ is defined by

$$F^I{}_J = d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J$$

also called the 2nd Cartan equation

→ Given a tetrad, the condition $T^I = de^I + \omega^I{}_J \wedge e^J \doteq 0$ can be shown to have a unique solution $\omega[e]$ for the connection; the spin connection (or the Levi-Civita connection).

If the connection is torsionless, it is uniquely determined by the tetrad and Cartan geometry → Riemannian geometry.

PROP The curvature of the spin-connection is related to the Riemann curvature tensor by

$$R^\mu{}_{\nu\rho\sigma} = e^\mu_I \, e_\nu^J \, F^I{}_{J\rho\sigma} \iff F^{IJ} = e^I_\mu \, e^J_\nu \, R^{\mu\nu}{}_{\rho\sigma} \, dx^\rho \wedge dx^\sigma$$

PROP We can rewrite the E-H action as the tetrad action

$$S[e] = \int d^4x \, |e| \, e^\mu_I \, e^\nu_J \, F_{\mu\nu}{}^{IJ}$$

DEF We introduce the Plebanski 2-form Σ as follow

$$\Sigma^{IJ} \equiv e^I \wedge e^J$$

→ Using the identity $2|e| \, e_I^{[\mu} e_J^{\nu]} = \epsilon^{\mu\nu\rho\sigma} \epsilon_{IJKL} \, e_\rho^K e_\sigma^L$, we can write

$$S = \int_M d^4x \sqrt{|g|} \, R = \int_M d^4x \, |e| \, e_I^\mu \, e_J^\nu \, F^{IJ}_{\mu\nu}$$

$$= \frac{1}{2} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} \epsilon_{IJKL} \, e_\rho^K \, e_\sigma^L \, F_{\mu\nu}{}^{IJ}$$

$$= \frac{1}{2} \int_M \epsilon_{IJKL} \, e^K \wedge e^L \wedge F^{IJ} = \int e \wedge e \wedge F^*$$

where $F^*_{IJ} \equiv *F_{IJ} = \frac{1}{2} \epsilon_{IJKL} F^{KL}$ is the Hodge dual in Minkowski space of F_{IJ}

→ The action of the fermion field interaction with gravity is

$$S_f[\psi, e] = \int \bar{\psi} \gamma^I D\psi \wedge e^J \wedge e^K \wedge e^L \epsilon_{IJKL} \equiv \int \bar{\psi} \gamma D\psi \wedge (e \wedge e \wedge e)^*$$

where $D \equiv D_\mu \, dx^\mu$

3.2.1 An important sign:

→ There is a difference between the Einstein-Hilbert action $S_{EH}[g] = \frac{1}{2}\int \sqrt{-\det g}\, R\, d^4x$ and the tetrad action $S_T[e] = \int e^I \wedge e^J \wedge F_{IJ}^*$. Indeed, let's perform a time-reversal operation:

$$^{(i)}Te^0 = -e^0 \quad \text{and} \quad ^{(i)}Te^i = e^i$$

Under this transformation, S_{EH} is invariant ($g = e^I e_I$), while S_T flips sign $S_T[^{(i)}Te] = -S_T[e]$

Writting both action in tensor notation and in term of tetrads:

$S_{EH}[e] = \frac{1}{2}\int |\det e|\, R[e]\, d^4x$ and $S_T[e] = \frac{1}{2}\int (\det e)\, R[e]\, d^4x$

They explicitly differ by $s \equiv \mathrm{sgn}(\det e)$

→ Since the fermions couple to the tetrad, their dynamic is sensitive to $s \equiv \mathrm{sgn}(\det e)$: the phase of the fermion evolves in the opposite direction in a region where s has the opposite sign.

→ In defining a path integral for the gravitational field, integration over configurations with $s<0$ contribute with a term of the form $\exp\{-i/\hbar\, S_{EH}[g]\}$ in addition to the term $\exp\{+\frac{i}{\hbar} S_{EH}[g]\}$

3.2.2 First-order formulation:

DEF | The Palatini action consider e and ω as 2 independent fields:

$$S[e,\omega] = \int e \wedge e \wedge F[\omega]^*$$

→ The variation of the tetrad gives the EFE:

$\delta\omega$: $\omega = \omega[e]$ and δe : EFE

→ This is called 1st order formulation. 1st and 2nd order formulat° are equiv. in pure gravity, but not so with minimally coupled fermions. Schematically,

$$S[\psi, e] = S[\psi, e, \omega] + \int \bar{\psi}\psi\, \bar{\psi}\psi$$

3.3 Holst action and Barbero-Immirzi coupling constant

→ Palatini action is polynomial (like action of Yang-Mills theory) and admits a purely geometric formulation in terms of differential forms.

→ Up to boundary terms, there is only one term we can add to the action: $\int e \wedge e \wedge F = \int e_I \wedge e_J \wedge F^{IJ}$. Let's add it to S with a coupling constant: $S[e,\omega] = \int e \wedge e \wedge F^* + \frac{1}{\gamma}\int e \wedge e \wedge F$

The EOM are the same as those of GR: $\delta\omega$ gives a torsionless connection, and the extra term becomes

$$\int e^I \wedge e^J \wedge F_{KL} = \int R_{\mu\nu\rho\sigma}\, \epsilon^{\mu\nu\rho\sigma}\, d^4x = 0$$

because of the symmetries of the Riemann. We can write

$$S[e,\omega] = \int e \wedge e \wedge \left(F^* + \tfrac{1}{\gamma}F\right) = \int e \wedge e \wedge (* + 1/\gamma)F$$

?

$$= \int \left(*e \wedge e + \tfrac{1}{\gamma} e \wedge e\right) \wedge F$$

DEF | We introduce the Holst action S_H and the Barbero-Immirzi constant γ:

$$S_H = \int *e \wedge e \wedge F + \underbrace{\frac{1}{\gamma}\int e \wedge e \wedge F}_{\text{Holst term.}}$$

→ This is the action with which we do 4-D QG.

→ We rewrite $B = \left(*e \wedge e + \frac{1}{\gamma} e \wedge e\right) = \left(* + \frac{1}{\gamma}\right)\Sigma$

? → On a $t=0$ boundary, $B = \partial S / (\partial\omega/\partial t)$ because the quadratic part of the action is $\sim B \wedge d\omega$), therefore B is the momentum conjugate to the connection. In a time gauge where the restriction of $*e \wedge e$ on the boundary vanishes, the momentum is the 2-form on the boundary taking values in the $SL(2,\mathbb{C})$ algebra $\Pi = \frac{1}{\gamma}B$

→ There is a similar term in QCD, which has no effect on the EOM, but is important in the quantum theory as well:

$$S_{QCD} = \int F \wedge F^* + \Theta_{QCD}\int F \wedge F$$

3.3.1 Linear simplicity constraint:

→ Consider a spacelike boundary surface Σ in the tetrad-connection formalism. $\forall \sigma \in \Sigma$, the tetrad maps the tangent space at σ into a 3-D linear spacelike subspace of the Minkowski space. $SO(3) \subsetneq SO(3,1)$ leaves this subspace invariant $\Rightarrow$ the boundary allows us to pick up a preffered Lorentz frame.

DEF The covector n_I normal to all vectors tangent to Σ is, in coord:

$$n_I \sim \epsilon_{IJKL}\, e_\mu^J e_\nu^K e_\rho^L \frac{\partial x^\mu}{\partial \sigma^1} \frac{\partial x^\nu}{\partial \sigma^2} \frac{\partial x^\rho}{\partial \sigma^3}$$

where $\{\sigma^i\}_{i=1}^3$ are the coord. of the point $\sigma \in \Sigma$ and $x^\mu(\sigma)$ is the embeding of the boundary Σ into spacetime.

→ We can use n_I to gauge-fix $SO(3,1)$ down to $SO(3)$: we orient the local Lorentz frame in such a way that the boundary is locally a fixed-time surface, so that $n_I = (1,0,0,0)$.

→ The pull-back to Σ of the momentum of B^{IJ} can be decomposed into its electric $K^I \equiv n_J B^{IJ}$ and magnetic $L^I \equiv n_J (*B)^{IJ}$ parts. Since $B^{IJ} = B^{[IJ]}$, L^I and K^I do not have components normal to Σ: $n_I K^I = 0 = n_J L^J \Rightarrow$ they are 3-D vectors in the space normal to n. We can denote them $\vec{K}$ and $\vec{L}$.

→ In the gauge $n_I = (1,0,0,0)$, these are simply

$$K^i = B^{0i} \quad \text{and} \quad L^i = \frac{1}{2} \epsilon^i{}_{jk} B^{jk}$$

→ We have $n_I B^{IJ} = n_I \left(* e\wedge e + \frac{1}{\gamma} e \wedge e\right)^{IJ}$

$$= n_I \left(\epsilon^{IJ}{}_{KL}\, e^K \wedge e^L + \frac{1}{\gamma} e^I \wedge e^J\right)$$

but by def., $n_I e^I|_\Sigma = 0$, so: $n_I B^{IJ} = \frac{1}{\gamma} (* e \wedge e)^{IJ}$

Similarly $n_I (B^*)^{IJ} = n_I \left(\left(\frac{1}{\gamma} e^I \wedge e^J\right)*\right)^{IJ} = \frac{1}{\gamma} n_I (* e \wedge e)^{IJ}$

$$= \frac{1}{\gamma} n_I B^{IJ}$$

PROP We found the linear simplicity constraint:

$$\boxed{\vec{K} = \gamma \vec{L}}$$

3.3.2 Boundary term :

→ Consider gravity defined in a compact region R of spacetime, with the topology of a 4-D ball. We have $\Sigma \equiv \partial R \sim S^3$

→ In the metric formulation, the boundary term is

$$S_{EH,\partial} \equiv \int_\Sigma k^{ab} q_{ab} \sqrt{q}\, d^3\sigma \quad \text{where} \begin{cases} k^{ab} = \text{extrinsic curvature} \\ q_{ab} = \text{3-metric induced on } \Sigma \end{cases}$$

→ In pure gravity with $\Lambda = 0$, the Ricci scalar $= 0$ on the solution of the EOM, the bulk action vanishes and the Hamilton function is given by the boundary term :

$$S_{EH}[q] = \int_\Sigma k^{ab}[q]\, q_{ab} \sqrt{q}\, d^3\sigma$$

→ This a very non local functional to compute, because the extrinsic curvature $k^{ab}[q]$ is determined by the bulk solution singled out by the boundary intrinsic geometry $\Rightarrow k^{ab}[q]$ is going to be non local

↳ Knowing the general dependence of k^{ab} from q is $\Leftrightarrow$ to knowing the general solution of the EFE.

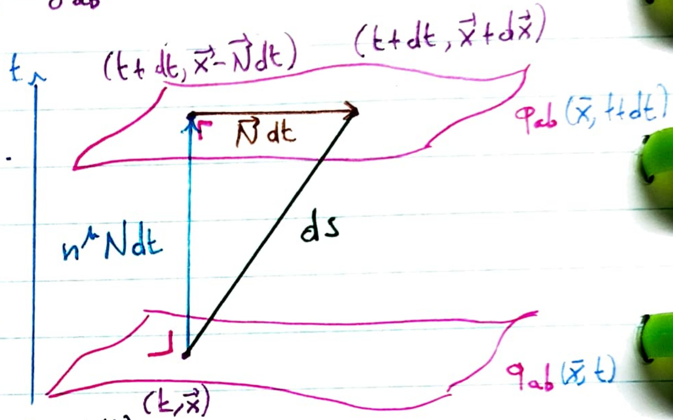
→ Remark :
- → $e \in \mathbb{R}^{3,1}$
- → $\omega \in sl(2, \mathbb{C})$
- → $B \in sl(2, \mathbb{C})$
- → time gauge: $g_{\mu\nu}(x,t) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g_{ab}(x,t) \end{array}\right)$

3.4 Hamiltonian general relativity

3.4.1 ADM variables:

DEF We define the ADM variables (for Arnowitt, Deser and Misner):
- → the Lapse function N as $N \equiv \sqrt{-g^{00}}$
- → the Shift function N^a as $N_a = g_{a0}$
- → the 3-metric q_{ab} as $q_{ab} = g_{ab}$

→ These variables have a nice intrinsic geometrical interpretation.



→ Assume spacetime is foliated by hypersurfaces for $t = cst$. Then q_{ab} is the 3-metric induced on the Σ_t by $g_{\mu\nu}$.

→ Let n^μ be a unit normal vector to $\Sigma(t)$.
Then $y^\mu \equiv x^\mu + N n^\mu dt \in \Sigma(t+dt)$: the Lapse function $\sim d\tau$, it determines the rate at which physical time elapses in the coord. chosen.

→ The shift $\vec{N} = \{N^a\} = \{q^{ab} N_b\}$ gives the separation between y^μ and $(t+dt, \vec{x})$; it measures the shift of the spatial coord. from $\Sigma(t)$ to $\Sigma(t+dt)$, with respect to the coord. that observer not moving on $\Sigma(t)$ would carry.

DEF(v2) The Lapse can equivalently be defined by
$$N^2 \det q = \det g \qquad \text{or} \qquad g_{00} = -N^2 + N_a N^a$$

→ The line element reads $ds^2 = -(N^2 - N_a N^a)dt^2 + 2N_a dx^a dt + q_{ab} dx^a dx^b$
The extrinsic curvature of Σ is given by
$$k_{ab} = \frac{1}{2N}\left(\dot{q}_{ab} - D_{(a} N_{b)}\right)$$

→ The action can be rewritten as
$$S = [N, \vec{N}, q] = \int dt \int d^3x \underbrace{\sqrt{q}\, N}_{\sqrt{-\det g}} \underbrace{\left(k_{ab} k^{ab} - k^2 + R[q]\right)}_{R[g]}$$

Replacing k_{ab} with its expression, we get

$$\mathcal{L}[N,\vec{N},q] = \frac{\sqrt{q}\,N}{4N^2}\left\{(\dot{q}_{ab}-D_{(a}N_{b)})(\dot{q}^{ab}-D^{(a}N^{b)}) - (\dot{q}_a{}^a - D_{(a}N^{a)})(\dot{q}_b{}^b - D_{(b}N^{b)})\right\} + \sqrt{q}NR$$

$$= \frac{\sqrt{q}}{4N}(g^{ac}g^{bd}-g^{ab}g^{cd})(\dot{q}_{ab}-D_{(a}N_{b)})(\dot{q}_{cd}-D_{(c}N_{d)}) + \sqrt{q}NR[q]$$

→ Since $\dot{N}, \dot{N}_a \notin \mathcal{L}$, we have $\pi_N = 0 = \pi_{N_a}$. The canonical momentum of q_{ab} is:

$$\pi^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{ab}} = \sqrt{q}\,G^{abcd}k_{cd} = \sqrt{q}(k^{ab} - kq^{ab})$$

where G^{abcd} is the Wheeler-Dewitt metric:

$$G^{abcd} = \frac{\sqrt{q}}{2}(g^{ac}g^{bd} + g^{ad}g^{bc} - 2g^{ab}g^{cd})$$

PROP The hamiltonian action reads:

$$S[N,\vec{N},q,\pi] = \int dt \int d^3x \left(\pi^{ab}\dot{q}_{ab} - N\,C(\pi,q) - 2N^a C_a(\pi,q)\right)$$

where $C \equiv G_{abcd}\pi^{ab}\pi^{cd} - \sqrt{q}\,R[q]$ is the scalar constraint (or hamiltonian constraint)

and $C_a = D_a \pi^{ab}$ is the vector constraint (or diffeo. constraint)

and where we inversed the WdW metric:

$$G_{abcd} = \frac{1}{2\sqrt{q}}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd})$$

such that $G^{abcd}G_{cdef} = \delta_e^{(a}\delta_f^{b)}$

→ The constraints must vanish with $\delta_N S = 0 = \delta_{N^a} S$, and the hamiltonian must vanish when the constraints are satisfied.

3.4.2 What does this mean? Dynamics

→ Let's compare the hamiltonian structure of 2 well-known theories.

ⓐ Electromagnetism

→ In term of potential A_μ, the Maxwell field is $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ and the Maxwell equations are invariant under the gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu \lambda$. The action is:

$$S[A] = \frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu}$$

We see that $S[A]$ does not depend on $\dot{A}_0$. The gauge transfo. allows us to set $A_0 = 0$.
Analogously, Lapse and Shift can be chosen arbitrarily. They simply determine the position of the coord. of the next Σ, after the 1st is chosen.
The simplest choice is $N=1$, $\vec{N}=0$.

DEF | The time gauge is $N^\mu \equiv (N, \vec{N}) = (1, \vec{0})$
↳ We then have $ds^2 = -dt^2 + q_{ab}\, dx^a dx^b$

→ The momentum of A_0 vanishes, and the momentum of $\vec{A}$ is

$$\pi^a = \frac{\partial L}{\partial \dot{A}_a} = F^{0a} \sim \vec{E} \text{ the electric field}$$

PROP | The hamiltonian action reads:

$$S[A_0, \vec{A}, \vec{E}] = -\int dt \int d^3x \left(E^a \dot{A}_a - (E^2 + B^2) + A_0\, C(\vec{E}) \right)$$

where $C(E) \equiv \partial_a E^a$ is the Gauss constraint

→ We can still gauge transform $A_a \mapsto A_a + \partial_a \lambda$ with λ time independent: there is a residual gauge freedom in the initial data. (1st class constraints are related to gauge freedom). Indeed, defining $C[\lambda] \equiv \int d^3x \, \lambda(x)\, C(x)$, we have $\{A, C[\lambda]\} = d\lambda = \delta_\lambda A$: the Poisson bracket of the variable with the constraint gives the usual gauge transformation.
Analogously, the Scalar and Vector constraints of gravity generate gauge transfo. (deformations of Σ and the change of spatial coord. on this surface).

⊙ Special relativistic dynamics:

→ The action of a rel. relativistic particle can be written in the form

$$S = m\int d\tau \sqrt{\dot{x}^\mu \dot{x}_\mu}$$

The momenta are $p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} = m \frac{\dot{x}_\mu}{|\dot{x}|}$, and satisfy $C = p^2 - m^2 = 0$

This constraint generates the gauge transformations of the theory, wich are the reparametrizations in τ.

→ Let's now consider $S = \frac{1}{2}\int \left(\frac{\dot{x}^\mu \dot{x}_\mu}{N} + N C(p) \right) d\tau$

The momenta is now $p_\mu = \dot{x}_\mu / N$ and the hamiltonian action is

$$S = \frac{1}{2}\int d\tau \; p_\mu \dot{x}^\mu + N\, C(p)$$

↳ The hamiltonian vanishes when the constraint is satisfied.

→ In GR, the vanishing Hamiltonian is a csq of the fact that the evolution parameter is not a physical quantity.
The Vector constraint reduces the 6 dof to 3, and the Scalar one determines the dynamics between these 3 variables. therefore the theory has 2 dof per space point.

3.4.3 Ashtekar connection and triads

DEF | We introduce triads on each $\Sigma(t)$, defined by

$$q_{ab}(x) = e^i_a(x)\, e^j_b(x)\, \delta_{ij}$$

↳ The introduction of triads adds a local SO(3) invariance to the theory

DEF | We define the triad version of the extrinsic curvature by

$$k^a_i\, e^i_b \equiv k_{ab}$$

→ We can consider these variables as a 9+9 dimensional canonical conjugate pair, namely pose $\{k^a_i, e^j_b\} \sim \delta^j_i \delta^a_b$

This increases by 3 the number of variables. However, $k_{ab} = k_{(ab)}$ so $k_i^{[a} e_i^{b]} \stackrel{!}{=} 0$. If we want to recover the 6+6 dimensional (q_{ab}, k^{ab}) phase space, we must impose the constraint

$$G_i \equiv \epsilon_{cab} k_i^a e_i^b = 0$$

↳ This constraint generate the local SO(3) gauge rotations.

?

→ Consider the connection $A_a^i \equiv \Gamma_a^i[e] + \beta k_a^i$, where $\Gamma[e]^i_a$ is the torsionless spin connection of the triad (the unique solution to the 3-D 1st Cartan equation $de^i + \epsilon^i{}_{jk} \Gamma^j \wedge e^k = 0$), β is an arbitrary parameter.

DEF The Ashtekar electric field $E_i^a(x)$ is defined as

$$E_i^a(x) \equiv \frac{1}{2} \epsilon_{ijk} \epsilon^{abc} e_b^j e_c^k = \sqrt{\det q} \, . \, e_i^a$$

PROP The Ashtekar electric field is the conjugate momentum variable of the connection A_a^i (a 3-D SU(2) gauge field), in that it satisfies the Poisson bracket relation

$$\{A^i_a(x), A^j_b(y)\} = 0 \quad \text{and} \quad \{A^i_a(x), E^b_j(y)\} = \beta \, \delta^b_a \, \delta^i_j \, \delta(x,y)$$

DEF The (complex) Ashtekar connection is the case where $\beta = i$:

(sign?) $=i$

$$A^i_a \equiv \Gamma^i_a - i k^i_a$$

→ The Ashtekar connection can be shown to be the pull back to Σ of the self-dual component of the spacetime spin connection.

With this choice, the scalar and vector constraints turn out to have an impressively simple form:

$$C = \epsilon_{ijk} F^i_{ab} E^{aj} E^{bk} \quad \text{and} \quad C_a = F^i_{ab} E^{bi}$$

where F is the curvature of the Ashtekar connection

$$F^i_{ab} \equiv \partial_a A_b^i - \partial_b A_a^i + \epsilon^i{}_{jk} A^j_a A^k_b$$

and the constraint that generates the additional SO(3) local rot. is the same as in Yang-Mills theory:

$$G^i = D_a E^{ai}$$

→ If we take $\gamma = \beta \in \mathbb{R}$ instead, the hamiltonian reads

$$H = E_i^{\,a} E_j^{\,b} \left\{ \epsilon^{ijk} F_{ab}^{k} - 2(1+\beta^2) k^i_{[a} k^j_{b]} + \frac{\Lambda}{3} \epsilon_{abc} \epsilon^{ijk} E^c_k \right\}$$

PROP The area of a two surface S in a $\Sigma(t = cst)$ hypersurface is:

$$A_S = \int_S d^2\sigma \sqrt{E^a_i n_a E^b_i n_b}$$

DEMO $A_S = \int_S d^2\sigma \sqrt{\det^{(2)} q} = \int_S d^2\sigma \sqrt{q_{11} q_{22} - q_{12}^2} = \int_S d^2\sigma \sqrt{\det q \cdot q^{33}}$

$= \int_S d^2\sigma \sqrt{E^3_i E^3_i} = \int_S d^2\sigma \sqrt{E_i^a n_a E_i^b n_b}$ where we picked $\Sigma = \{x^3 = 0\}$ ■

→ By introducing the 2-form $E^i = \frac{1}{2} \epsilon_{abc} E^{ai} dx^b dx^c$, we can write

$$A_S = \int_S |E|$$

→ In the limit $S \to 0$, the vector $\vec{E}_S \equiv \int_S \vec{E}$ is normal to the surface, whose lenght is the area of the surface. In terms of the triad,

$$E^i_S = \frac{1}{2} \epsilon^i{}_{jk} \int_S e^j \wedge e^k$$

→ The momentum $\Pi = (8\pi\gamma G)^{-1} B$ is conjugate to the connection, and therefore it is the canonical generator of Lorentz transformation.

↳ The generator of a boost in the z direction is

$$\mathbb{K}_z = (8\pi\gamma G)^{-1} K_z = (8\pi G)^{-1} L_z \qquad (\vec{K} = \gamma \vec{L})$$

But now we know that L_z is the area element A of a surface normal to $\vec{z}$. Therefore A is related to the generator $\mathbb{K}$ of a boost in the direction normal to the surface by

$$\mathbb{K} = \frac{A}{8\pi G}$$

This equation encloses, in a sense, the full dynamics of GR.

3.5 Euclidean general relativity in three spacetime dimensions

→ In the following chapter, we use the euclidean version of GR in 3-D as a toy model.

→ In 3-D, the Riemann tensor is fully determined by the Ricci tensor, therefore the vacuum EFE Ricci $=0 \Rightarrow$ Riemann $=0$: spacetime is flat. The theory has no local degrees of freedom. But it still an interesting theory:

→ If space has a non-trivial topology, it can be locally flat but have a global dynamics

→ Even on trivial topology, where there is a single physical state, it determines nontrivial relations between boundary partial observables.

→ In metric formalism, we have:

$$S[g] = \underbrace{\frac{1}{16\pi G}\int_{\mathcal{M}} d^3x \sqrt{-g}\, R}_{\text{bulk}} + \underbrace{\frac{1}{8\pi G}\int_{\Sigma=\partial\mathcal{M}} d^2\sigma\, k^{ab} q_{ab}\sqrt{q}}_{\text{boundary}}$$

The bulk term doesn't contribute to the Hamilton function (Ricci scalar vanishes on the solution of the EOM), we then have

$$S[q] = \frac{1}{8\pi G}\int_\Sigma d^2\sigma\, k^{ab}[q]\, q_{ab}\sqrt{q}$$

The dependence of k^{ab} on q_{ab} is what codes the dynamics, and is highly non local. The solution of the dynamics is expressed by giving the boundary extrinsic curvature $k^{ab}(\sigma)$ as a function of the boundary partial observable $q_{ab}(\sigma)$. This is analogous to giving $p(q,t,q',t') \leftrightarrow q'(t';q,p,t)$.

→ In the triad-connection formalism, the grav. is described by a triad field $e^i \equiv e^i_a dx^a$ and a $SO(3)$ connection $\omega^i{}_j = \omega^i_{a\,j} dx^a$

In 3D, the bulk action is $S[e,\omega] = (16\pi G)^{-1}\int \epsilon_{ijk}\, e^i \wedge F^{jk}[\omega]$

where $F^i{}_j \equiv d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$

→ The EOM are
$$\begin{cases} T^i = de^i + \omega^i{}_j \wedge e^j = 0 \\ F^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = 0 \end{cases}$$

→ The 4-D local Lorentz invariance of the tetrad formalism is replaced by a local $SO(3)$ gauge invariance under

$$e^i_a(x) \mapsto R^i{}_j(x)\, e^j_a(x) \,, \quad R \in SO(3)$$

This is the 3-D euclidean GR.

→ Notation:

→ For all 2 index antisymmetric tensor, we use the single index notation

$$\omega^i \equiv \frac{1}{2} \epsilon^i{}_{jk}\, \omega^{kj}$$

The index i labels a basis in $so(3) = su(2)$ Lie algebra.

→ We write the connection as the $su(2)$ generator in the fundamental rep. of $su(2)$, by using Pauli matrices $\sigma_i = (\sigma_i{}^A{}_B)$ basis:

$$\omega = \omega^i \tau_i = \omega^i \left(\frac{-i}{2} \sigma_i\right)$$

→ We consider this theory on a compact region R of spacetime, with trivial topology, bounded by a 2D boundary Σ with topology of a sphere. The pull back of the connection on the boundary is an $SO(3)$ connection ω on Σ.

The pull back of the triad is a 1-form e on Σ with values in the $su(2)$ algebra.

Those two variables are canonically conjugate to one another.

→ The derivative term ∂_0 normal to the boundary Σ in S has the structure $\sim (8\pi G)^{-1} n_c\, \epsilon^{abc}\, e^i_a\, \partial_0 \omega_b{}^i$, where $n_a \perp \Sigma$.

Using $\epsilon_{ab} \equiv n^c \epsilon_{abc}$, the Poisson bracket between ω and e is

$$\{e^i_a(\sigma), \omega^j_b(\sigma')\} = 8\pi G\, \delta^{ij}\, \epsilon_{ab}\, \delta^2(\sigma, \sigma')$$

where $\sigma, \sigma' \in \Sigma$.