Chapter 2

Neutrino masses & mixing

> There are several categories of scientists in the world; those of second or third rank do their best but never get very far. Then there is the first rank, those who make important discoveries, fundamental to scientific progress. But then there are the geniuses, like Galilei and Newton. Majorana was one of these.
>
> *Enrico Fermi*

2.1 Dirac and Majorana mass terms

Neutrinos are neutral particles, so there are 2 kind of mass term. First, as for any fermion, a neutrino can have a Dirac mass. This requires to have independent L and R Weyl spinors. In the SM, we can introduce the following gauge singlet spinor,

$$\nu_R \sim (0, 0, 0) \tag{2.1}$$

where as usual the entries correspond to $SU(3)$, $SU(2)$ and $U(1)_Y$ quantum numbers. The field has a subscript R to mean that ν_R transform as a R-handed spinor under Lorentz transformations but that's just a choice since ν_R^c is left-handed but has the same gauge numbers.[1] Since

$$\bar{L}\tilde{\Phi} \sim (0, 0, 0) \tag{2.2}$$

[1] We use the Dirac notation, so ψ_R is represented by a 4-components spinor; in the Weyl basis, these are two complex numbers.

we can write the following Yukawa coupling

$$\mathcal{L} \supset y_\nu \bar{L} \Phi \nu_R + \text{h.c.} \tag{2.3}$$

which, after SSB, give a Dirac mass to the neutrino component of the lepton doublet,

$$\mathcal{L} \supset -m_\nu^D (\bar{\nu}_L \nu_R + \bar{\nu}_R \nu_L) \qquad \text{(Dirac mass)} \tag{2.4}$$

with $m_\nu^D = y_\nu v/\sqrt{2}$. Like for the electron, a Dirac mass term involves 4 degrees of freedom, 2 (one for each spin) for a particle state and 2 for its antiparticle and they are distinct states. From an interaction/perturbative perspective, a Dirac mass term transforms a L-handed (anti)particle into a R-handed one, and vice versa. In Figure 2.1, the arrows tracks the flow of fermion number, here lepton number, which is thus conserved.

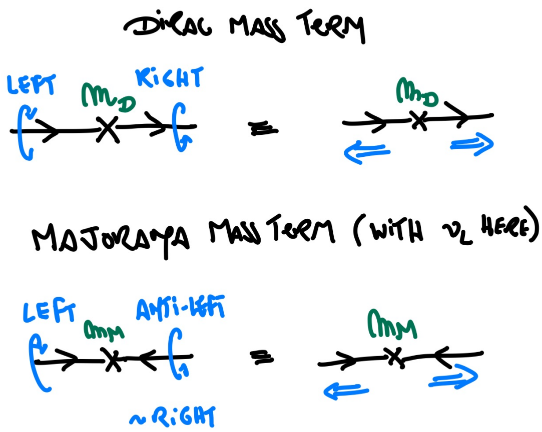

Figure 2.1: Dirac vs Majorana mass term (here shown for a ν_L. In both cases, the mass terms causes a spin/helicity flip. The arrows on the dark lines correspond to fermion number flow. In the Majorana case, fermion number is not conserved (neutrino Majorana mass cause lepton number violation). The blue arrows suggest the helicity directions. There are two more or less standard notations. I usually use the one on the right hand side (double arrows).

Alternatively, the neutrino could be its own antiparticle, like the photon. Since ν_L^c is R-handed (with our conventions), the following so-called Majorana[2] mass term is also allowed

$$\mathcal{L} \supset -\frac{1}{2} m_\nu^M (\bar{\nu}_L^c \nu_L + \bar{\nu}_L \nu_L^c) \qquad \text{(Majorana mass)} \tag{2.5}$$

[2]Dirac and Majorana have in common that they where un-ordinary geniuses, see Wikipedia,

The factor of $1/2$ is for normalisation, similar to the one for a real scalar. As Figure 2.1 suggests, a Majorana mass term transforms a L-handed particle into its antiparticle (and vice versa), so fermion number is not conserved. A neutrino Majorana mass term breaks lepton number.

2.2 Weinberg effective operator

As we have seen in the introduction, lepton number is a symmetry of the SM lagrangian. More precisely, there are 3 conserved lepton numbers, one for each generation (electron, muon and tau leptonic numbers). From the neutrino perspective, this implies that, through renormalizable Yukawa couplings akin to (2.3), we can only write down Dirac mass terms for the SM neutrinos. This strictly speaking only true at the classical level as chiral anomalies imply that only $B-L$ can be a true symmetry of the SM at the quantum level but the effects of the anomalies are non-perturbative and very suppressed at low energies for a weakly coupled theory like the electroweak sector.

However, high energy/mass degrees of freedom may may leave an imprint on the SM low energy theory through their virtual effects. These are typically suppressed by (powers of) a large scale or so-called effective operators. In the SM, it is possible to write down the following, unique up to flavour indices, dim 5 effective operators

$$\mathcal{L} \supset \frac{c_{ij}}{\Lambda}(\bar{L}_i\tilde{\phi})(\tilde{\Phi}^T L_j^c) + h.c. \qquad \text{(Weinberg operator)} \tag{2.6}$$

with $i, j = e, \mu, \tau$. The first important feature is that this operator breaks lepton numbers (in general, all of them, baring extra symmetries). Accordingly, after SSB breaking, one get Majorana mass terms,

$$\mathcal{L} \supset \frac{1}{2} m_{ij}^M \bar{\nu}_i \nu_j^c + h.c. \tag{2.7}$$

with

$$m_{ij}^M = c_{ij}\frac{v^2}{\Lambda} \tag{2.8}$$

A priori, they form together a complex 3x3 matrix for the 3 generations of the SM. The most immediate feature of the entries of this mass matrix is that there are naturally "small", meaning that if the scale Λ is much larger than the EW scale v, $\Lambda \gg v$, then the scale of SM neutrino mass is parametrically much smaller than the EW scale, as is observed experimentally. For $c_{ij} = \mathcal{O}(1)$, $\Lambda \sim 10^{15}$ GeV leads to m_ν below the eV scale.

2.3 See-saw mechanism

The effective operator argument implies that neutrino should be naturally light if the SM is a good effective theory and furthermore that their mass should be of the Majorana kind, and by extension, that lepton number is an accidental symmetry, only valid a low energies. That neutrinos are massive is well-established, see following section. Whether they have a Majorana mass remains to be established experimentally. The key is to look for lepton number violating signatures, see forthcoming sections.

There are several possible UV completions of such effective operators, both at tree level and from loop corrections. The simplest possibility is the following. For simplicity of notation, consider just one SM generation and introduce a heavy singlet state, similar to (2.1), with a mass $M \gg v$,

$$\mathcal{L} \supset -\frac{1}{2} M \bar{\nu}_R^c \nu_R + \text{h.c.} \tag{2.9}$$

As the notation suggests, we take ν_R to transform as a R-handed Weyl spinor under Lorentz transformations. Next we couple this state to the SM through the Yukawa coupling of (2.3).[3] After SSB, we thus have both a Dirac mass term, with mass say $m = yv/\sqrt{2}$, that mixes the SM neutrino(s) to the heavy singlet, which has a Majorana mass term,

$$\mathcal{L} \supset -m \bar{\nu}_L \nu_R - \frac{1}{2} M \bar{\nu}_R^c \nu_R + \text{h.c.} \tag{2.10}$$

Since $\bar{\nu}_L \nu_R \equiv \bar{\nu}_R^c \nu_L^c$, we can rewrite these terms in the equivalent matrix form

$$\mathcal{L} \supset -\frac{1}{2} \left(\bar{\nu}_L^c \; \bar{\nu}_R \right) \begin{pmatrix} 0 & m \\ m & M \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \text{h.c.} \tag{2.11}$$

This (8x8 in our convention) mass matrix can be readily diagonalized. In the limit $M \gg m$, it has eigenvalues (see exercises)

$$m_{\text{light}} \approx -m^2/M \quad \text{and} \quad m_{\text{heavy}} \approx M \tag{2.12}$$

and eigenvectors that are essentially

$$\Psi_{\text{light}} \approx \nu_L + \nu_L^c \quad \text{and} \quad \Psi_{\text{heavy}} \approx \nu_R^c + \nu_R \tag{2.13}$$

They are thus Majorana state, $\Psi_{\text{light}}^c = \Psi_{\text{light}}$ and the same for the heavy one. The light neutrino mass come from EWSB (like all SM fermions) but is suppressed by the ratio m/M, a result which is referred to as the "see-saw mechanism".

[3] This is sometime called the neutrino portal, although in a slightly different context (hidden sector framework).

From the perspective of the Weinberg operator, $\Lambda \sim M$, the mass of the heavy Majorana neutrino. The introduction of heavy singlet Majorana fermions is referred as "type I see-saw mechanism". This suggests that there are other types (see exercices). Finally, this and similar setups can be readily generalized to 3 generations of fermions, in which case the light neutrino mass becomes a a complex 3x3 matrix. We should thus expect that light neutrinos (essentially SM neutrinos as we have seen) mass eigenstates are distinct from leptonic flavour eigenstates. This leads to a very important and quite fascinating phenomenology of "neutrino oscillations".

2.4 Neutrino mixing

Neutrinos masses generically imply that neutrino propagation or mass eigenstates (latin index $i = 1, 2, 3$) and interactions or flavour states (greek index $\alpha = e, \mu, \tau$) are distinct. If neutrinos have only a Dirac mass, the situation is similar to that of quarks in the SM. The 3x3 mass complex matrix $\mathcal{M}$ can diagonalized using a biunitary transformation,

$$\mathcal{L} \supset -(\bar{\nu}_{eR}\bar{\nu}_{\mu R}\bar{\nu}_{\tau R})\mathcal{M}_\nu \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} + \text{h.c.} = (\bar{\nu}_{1R}\bar{\nu}_{2R}\bar{\nu}_{3R})\mathcal{M}_{\nu,\text{diag}} \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix} + \text{h.c.} \tag{2.14}$$

with unitary transformations

$$\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} = U_\nu \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \nu_{eR} \\ \nu_{\mu R} \\ \nu_{\tau R} \end{pmatrix} = V_\nu \begin{pmatrix} \nu_{1R} \\ \nu_{2R} \\ \nu_{3R} \end{pmatrix} \tag{2.15}$$

or, in more compact form,

$$\nu_L = U_\nu \nu'_L \quad \text{and} \quad \nu_R = V_\nu \nu'_R \tag{2.16}$$

with prime mass eigenstates, so that

$$\mathcal{M}_{\nu,\text{diag}} = V_\nu \mathcal{M}_\nu U_\nu^\dagger \qquad \text{(Dirac mass matrix)} \tag{2.17}$$

As for the quark sector, this field redefinition affects the coupling of leptons to the W gauge boson,

$$\mathcal{L} \supset -\frac{g}{\sqrt{2}}\bar{e}_L\gamma^\mu\nu_L W^- + \text{h.c.} = -\frac{g}{\sqrt{2}}\bar{e}'_L\gamma^\mu U_{\text{PMNS}}\nu'_L W^- + \text{h.c.} \tag{2.18}$$

with

$$U_{\text{PMNS}} \hat{=} U_e^\dagger U_\nu \tag{2.19}$$

the equivalent for leptons of the CKM matrix, with PMNS standing for Pontecorvo-Maki-Nakagawa-Sakata (matrix). This unitary mixing matrix involves a priori 6 reals parameters, 3 mixing angles (for 3 generations) and 6 phases. As for the quarks, we can however absorb 5 out of these 3 phases (for 3 generations). So the the PMNS matrix can be expressed in terms of 3 mixing angles and 1 CP violating Dirac phase. As for quarks, this phase requires to involves the 3 generations of leptons.

As for quarks, if the neutrinos were massless or more generally, degenerate, we could redefine the neutrinos states at will and through this, absorb the unitary transformation of the charged leptons. Thus, lepton flavour processes should be suppressed by (power of) neutrino mass differences.

If the neutrinos are Majorana particles instead, then the neutrino mass matrix is symmetric

$$\mathcal{M}^T = \mathcal{M} \tag{2.20}$$

and thus can be diagonalized by a single unitary transformation,

$$\mathcal{L} \supset -(\bar{\nu}^c_{eL}\bar{\nu}^c_{\mu L}\bar{\nu}^c_{\tau L})\mathcal{M}_\nu \begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} + \text{h.c.} = (\bar{\nu}^c_{1L}\bar{\nu}^c_{2L}\bar{\nu}^c_{3L})\mathcal{M}_{\nu,\text{diag}} \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix} + \text{h.c.} \tag{2.21}$$

with

$$\mathcal{M}_{\nu,\text{diag}} = U_\nu \mathcal{M}_\nu U_\nu^\dagger \qquad \text{(Majorana mass matrix)} \tag{2.22}$$

The PMNS matrix has the same structure but, unlike the case of Dirac mass, because of lepton violation, it is no possible to further redefine the neutrino fields by a phase. Out of the 6 phases only 3 for the charged leptons can be removed, so 3 CP violating phases of the PMNS matrix are in principle observable, 1 being the Dirac phase as above. The 2 other are related to the Majorana nature of the neutrinos, so they can play only a role only if they are involved in lepton number violating processes. Notice also that, unlike the case of the quarks, the phases are in principle observable already for 1 neutrino generation.

2.5 Neutrino oscillations

Neutrino mixing leads to the phenomenon of neutrino oscillations (Pontecorvo, 1957). Such processes are allow to determine neutrino mass difference (but not their absolute scale) and mixing angle. To are also sensible to the Dirac phase by comparing the behaviour of neutrinos and their antiparticles. As alluded to above, to measure the Majorana phases would requires to have neutrino-antineutrino transformations; these are however very suppressed at the energies available experimentally.

To simplify, we begin with the case of 2 neutrino flavours. The problem is akin to many oscillation phenomena in quantum mechanics (as well as in classical waves). Since this does not involve a change of number of degrees of freedom, it is useful to rely on the intuition (you may have) developed in non-relativistic quantum mechanics of 2 level system using the Schroedinger representation.

First, we define the neutrino flavour states to be those that couple through the W boson to the charged leptons mass eigenstates. The neutrino flavour and mass eigenstate are thus related by the PMNS matrix,

$$|\nu_\alpha\rangle = \sum_{i=1,2,3} U_{\mathrm{PMNS},\alpha i}|\nu_i\rangle \tag{2.23}$$

with

$$\langle\nu_\alpha|\nu_\beta\rangle = \delta_{\alpha\beta} \quad \text{and} \quad \langle\nu_i|\nu_j\rangle = \delta_{ij} \tag{2.24}$$

equivalent for 2 flavours (say the muon and tau neutrinos) to

$$\begin{aligned} |\nu_\mu\rangle &= \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle \\ |\nu_\tau\rangle &= -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle \end{aligned} \tag{2.25}$$

This involves only one mixing angle and if they are only 2 flavours, the Dirac CP phase has no impact. The mass eigenstates, treated as plane waves, evolves in time and space as

$$|\nu_i(t,x)\rangle = e^{-i(E_i t - p_i x)}|\nu_i\rangle \tag{2.26}$$

with

$$E_i = \sqrt{p_i^2 + m_i^2} \tag{2.27}$$

For the case that are experimentally accessible, the energy is in the MeV up to several GeV range, so $E_i \approx p_i \gg m_i$ and

$$E_i \approx p_i + \frac{m_i^2}{2p_i} \approx E + \frac{m_i^2}{2E} \tag{2.28}$$

Thus (2.25) at position $x = L \equiv t$ (neutrinos are very relativistic) from a source reads

$$\begin{aligned} |\nu_\mu(t,x)\rangle &\approx e^{-i(Et-px)}\left(e^{-im_1^2L/2E}\cos\theta|\nu_1\rangle + e^{-im_2^2L/2E}\sin\theta|\nu_2\rangle\right) \\ |\nu_\tau(t,x)\rangle &\approx e^{-i(Et-px)}\left(-\sin\theta e^{-im_1^2L/2E}|\nu_1\rangle + e^{-im_2^2L/2E}\cos\theta|\nu_2\rangle\right) \end{aligned} \tag{2.29}$$

Thus, the probability for, say a muon neutrino produced initially is observed at L as a tau neutrino is given by

$$\begin{aligned} P(\nu_\mu \to \nu_\tau)(L) &= |\langle\nu_\tau|\nu_\mu(L)\rangle|^2 = \sin^2\theta\cos^2\theta|-e^{-im_1^2/2EL} + e^{-im_2^2/2EL}|^2 \\ &= \sin^2(2\theta)\sin^2\left(\frac{\Delta m_\nu^2 L}{4E}\right) \end{aligned} \tag{2.30}$$

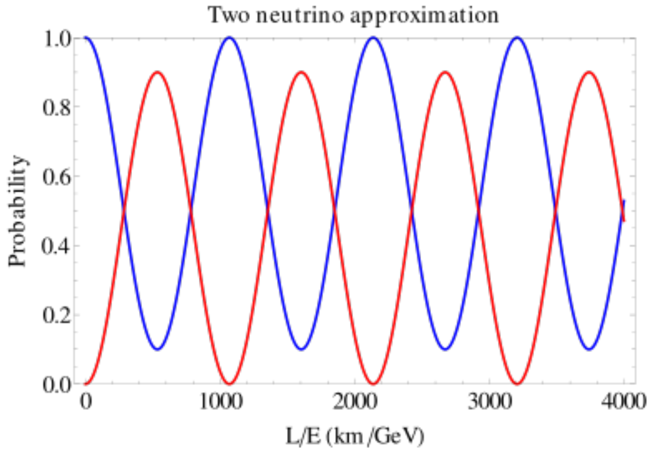

Figure 2.2: Oscillation probabilities for 2 generations, with $\theta = 0.9$ and $\Delta m^2 = 2 \times 10^{-3}$ eV^2.. This 2 flavour oscillatory pattern is a reasonably good approximation for the case of initial muon neutrino (blue) oscillating into a tau neutrino (red), as possibly from atmospheric neutrinos.

Conversely, by direct calculation or from unitarity, the probability that it is observed as an muon neutrino is

$$\begin{aligned} P(\nu_\mu \to \nu_\mu)(L) &= |\langle \nu_\mu | \nu_\mu(L) \rangle|^2 = 1 - P(\nu_\tau \to \nu_\mu)(L) \\ &= 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m_\nu^2 L}{4E}\right) \end{aligned} \tag{2.31}$$

Precisely the same expressions hold for $P(\nu_\mu \to \nu_e)(L)$ and $P(\nu_\mu \to \nu_\mu)(L)$.

Oscillation requires $\theta \neq 0 \bmod (\pi/2)$ and non-degenerate neutrino masses or, in other words, only mass difference are relevant. Maximum oscillation occurs for an oscillation length such that

$$L_{max} = \frac{2\pi E}{\Delta m^2} \quad \text{with} \quad P(\nu_e \to \nu_\mu)(L_{max}) = \sin^2(2\theta) \tag{2.32}$$

with a periodicity $L_{osc} = 2L_{max}$, see figure 2.2 . For E ~GeV and $\Delta m \sim$ eV, the oscillation length is in the $L \sim$ km range,

$$\frac{\Delta m^2 L}{4E} \approx 1.27 \left(\frac{\Delta m^2}{eV^2}\right)\left(\frac{GeV}{E}\right)\left(\frac{L}{km}\right) \tag{2.33}$$

2.6 Neutrinos data

Oscillation patterns are much richer for the realistic case of 3 generations. There are 3 mixing angles, and potentially a Dirac CP phase that can be probed using various neutrino sources, and energy and distances. Here is just a summary.

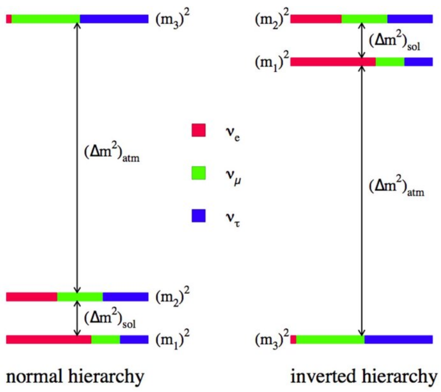

Figure 2.3: Normal and inverted neutrino mass hierarchy. The absolute scale is constrained from above, consistent with the lightest neutrino to be massless. The colour coding correspond to mixing fractions, e.g. for normal hierarchy, the lightest neutrino is mostly an electron neutrino; τ and μ neutrinos are essentially maximally mixed, $\theta_{23} \approx \pi/4$.

For more complete information, wikipedia is a good source for quick reference. A good general review of neutrino phenomenology remains "Neutrino Mass and Mixing: from Theory to Experiment" by S. King et al. See also the `https://pdg.lbl.gov/index.html`Particle Data Group.

Oscillations have been observed from solar neutrinos (MeV electron neutrinos disappearance from β processes at the core of the Sun, range : astronomical unit), atmospheric neutrinos (GeV range muon neutrinos disapearance, produced from muon decays in the atmosphere, km range), reactor (MeV electron antineutrino disappearance, range : km). For 3 generations, there are 2 mass differences measured to be (of the order of)

$$\Delta m^2_{12} = \Delta m^2_{sol} = 7.5 \times 10^{-5}\,\text{eV}^2 \qquad \text{(solar neutrinos)} \tag{2.34}$$

and

$$\Delta m^2_{23} = \Delta m^2_{atm} = 2.5 \times 10^{-3}\,\text{eV}^2 \qquad \text{(atmospheric neutrinos)} \tag{2.35}$$

Given these numbers, there are two possibilities for the mass spectrum, which are referred (for nothing but historical reasons) normal and inverted hierarchies, see figure 2.3.

Neutrino oscillations are not sensitive enough to determine whether neutrinos masses are of Dirac or Majorana type. This would require spin flips which are suppressed by $(m/E)^2$. Also, they are not sensitive to the overall neutrino mass scale.

Note in passing that cosmology is an important source of information on neutrino properties and in particular their mass. There is an old bound on the sum

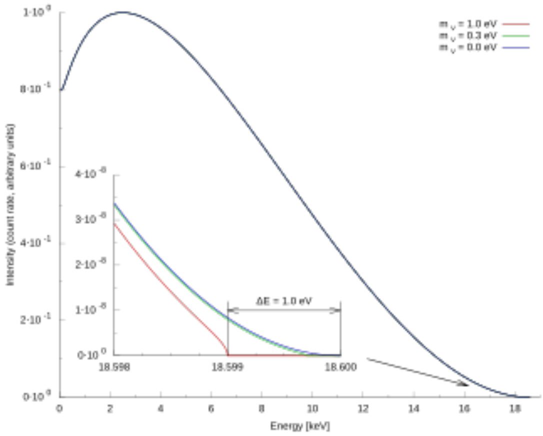

Figure 2.4: Energy spectrum of the electrons emitted in tritium beta decay, from wikipedia.

of all neutrino mass based on relic neutrinos and the mass fraction of dark matter, which is about 15 eV. Much more stringent bounds are obtained from cosmic anisotropies, $\sum_i m_{\nu,i} \lesssim 0.5$ eV (Planck), see e.g. "Massive neutrinos and cosmology" by Lesgourgues and Pastor.

A similar but laboratory constraint on the absolute neutrino mass scale is obtained from precise measurements of the endpoint of β decay, see figure 2.4. For exemple, the Katrin experiment studies the decay of tritium into helium-3, ${}^3\mathrm{H} \to^3 He + e^- + \bar{n}u_e$, with a bound $m_\nu \lesssim 1.1$ eV.

To determine whether neutrinos are Majorana or Dirac requires to look for lepton number violating processes. The key target is neutrinoless double beta decay, say $(A, Z) \to (A, Z+2) + 2e^-$, which is forbidden if lepton number is conserved, see figure 2.5. Negative results (Cuore, Gerda,...) set bounds are expressed as lower bounds on the lifetime of nuclei, for example on tellurium (${}^130Te$, Cuore), $T^{0\nu}_{\beta\beta} > 2.2 \cdot 10^{25}$ years.

The relation of such number to a Majorana mass is model dependent and also depend on whehter the hierarchy is normal or inverted. A summary of existing bounds and reach of future experiments is depicted in figure 2.6; similar, updated or specific, version of such figure can be found on the net).

2.7 Exercises

2.7.1 See-saw I

Find the eigenvalues and eigenvectors of the mass matrix (2.11) for 1 generation, both exactly and in the approximation $M \gg m$.

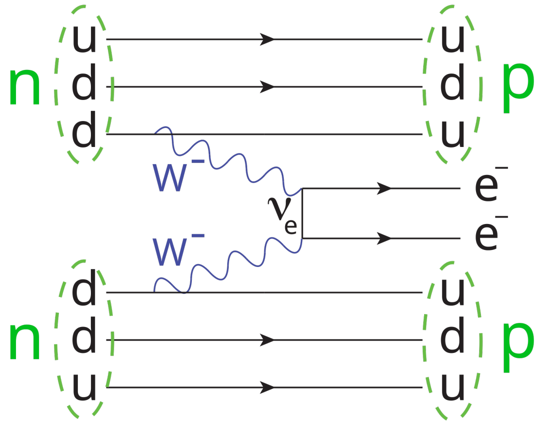

Figure 2.5: Feynman diagram for Majorana mass induced neutrinoless double beta decay, from wikipedia.

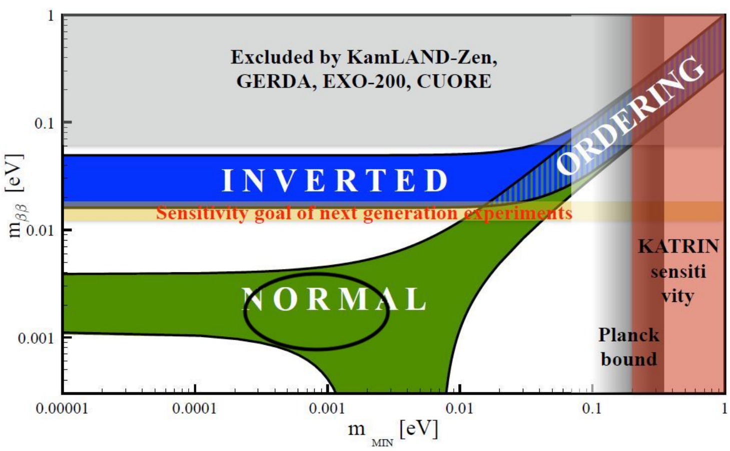

Figure 2.6: The grey area is excluded by direct searches for neutrinoless double beta decay (bounds on $m_{\beta\beta}$, vertical axis). Predictions of models depend strongly on the hierarchy of neutrino mass. Horizontal is the overall neutrino mass scale, with competing cosmological and laboratory limits (brown regions excluded).

See-saw II

Put the mass matrix (2.11) for N generations in block diagonal. Make the approximation that the entries in the Majorana mass terms are typically much more larger than those for the Dirac part, i.e. "$M \gg m$". Begin by showing that (2.11) becomes

$$\mathcal{L} \supset -\frac{1}{2}\left(\bar{\nu}_L^c\ \bar{\nu}_R\right)\begin{pmatrix} 0 & m^T \\ m & M \end{pmatrix}\begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} + \text{h.c.} \tag{2.36}$$

with m and M NxN matrices and find the light and heavy states. You can rely on properties of block matrices, see `https://en.wikipedia.org/wiki/Block_matrix` or, better, use Feynman diagrams.

Appendix A

Majorana spinors (D=4)

There are several, sometime conflicting, notations in the literature. Cheng and Li is not bad. It uses Dirac spinors (4 components), ψ in a basis independent way.

A.1 Dirac and Weyl spinors

Chiral states are defined as usual,

$$\psi_L = L\psi = \frac{1}{2}(1-\gamma_5)\psi \quad \text{and} \quad \psi_L = R\psi = \frac{1}{2}(1+\gamma_5)\psi \tag{A.1}$$

This is my notation (same as Zee, Cheng and Li, etc., but not Peskin and Schroeder) for the Dirac conjugate of these chiral projections,

$$\bar{\psi}_L \hat{=} (\psi_L)^\dagger \gamma^0 \quad \text{and} \quad \bar{\psi}_R \hat{=} (\psi_R)^\dagger \gamma^0 \tag{A.2}$$

Thus

$$\bar{\psi}_L \hat{=} \bar{\psi} R \quad \text{and} \quad \bar{\psi}_R = \bar{\psi} L \tag{A.3}$$

We can write a Dirac spinor as

$$\psi = \psi_L + \psi_R \quad \text{and} \quad \bar{\psi} = \bar{\psi}_R + \bar{\psi}_L \tag{A.4}$$

Consequently, a Dirac mass term is equivalent to

$$\bar{\psi}\psi = \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R \tag{A.5}$$

The notation emphasizes the fact a Dirac mass term couples spinors of different chiralities. In the SM, there are really different states (i.e. they have different gauge numbers). In the Weyl basis,

$$\psi_L = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} \tag{A.6}$$

where χ_L and χ_R are two components Weyl spinors. A Dirac mass term takes the form

$$\bar{\psi}\psi = \chi_R^\dagger \chi_L + \chi_L^\dagger \chi_R \tag{A.7}$$

in the Weyl basis.[1]

A.2 Majorana spinors

For Majorana spinors, it is useful to introduce the charge conjugate of ψ, noted ψ^c. It transforms the same way as ψ under Lorentz transformations. It is defined through the conjugation unitary transformation $\mathcal{C}^\dagger\mathcal{C} = \mathcal{C}\mathcal{C}^\dagger$,

$$\psi^c = \mathcal{C}\psi\mathcal{C}^\dagger \quad \text{and} \quad \bar{\psi}^c = \mathcal{C}\bar{\psi}\mathcal{C}^\dagger \tag{A.10}$$

which is such that the electric current change sign under conjugation,

$$\mathcal{C}J^\mu\mathcal{C}^\dagger = -J^\mu \tag{A.11}$$

In a way, the role of particles and antiparticles is exchanged. Charge conjugation invariance of QED requires that

$$\mathcal{C}A^\mu\mathcal{C}^\dagger = -A^\mu \tag{A.12}$$

Charge conjugation acting on Dirac spinors can be represented using the charge conjugation matrix C (not to be confused with the operator $\mathcal{C}$), built upon gamma matrices,

$$\psi^c \hat{=} C\bar{\psi}^T = i\gamma^2\psi^* \quad \text{with} \quad C = i\gamma^2\gamma^0 \tag{A.13}$$

Then

$$\bar{\psi}^c = \psi^{c\dagger}\gamma^0 = \psi^T C \tag{A.14}$$

[1] Supersymmetry people (and Landau and Lifschitz) use the dotted-undotted Van der Waerden notation,

$$\psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad \text{and} \quad \bar{\psi} = (\chi^\alpha \, \bar{\psi}_{\dot{\alpha}}) \tag{A.8}$$

with a Dirac mass term

$$\bar{\psi}\psi = \chi^\alpha\psi_\alpha + \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \tag{A.9}$$

The bar in this notation has nothing to do with the bar in the Dirac adjoint of ψ. This may be confusing. This notation is closer to the fundamental fact that Weyl spinors are building blocks in the sense that they transform distinctly under Lorentz transformations. Yet, I have never seen the SM Lagrangian model written using this notation.

One can check (exercise) that the first identity leads to (A.11).[2] Useful properties (that you now know if you did the exercise) of C are

$$C^\dagger = C^{-1} = -C \quad \text{so that} \quad C^2 = -1 \quad \text{and} \quad C^{-1}\gamma_\mu C = -\gamma_\mu^T \tag{A.15}$$

It may be pertinent to write ψ^c in the Weyl basis,

$$\psi^c = \begin{pmatrix} i\sigma^2\chi_R^* \\ -i\sigma^2\chi_L^* \end{pmatrix} \tag{A.16}$$

which suggests (check) that $\sigma^2\chi_R^* \sim \chi_L$ (and vice-versa) under Lorentz boosts and so is a left-handed spinor.[3]

A Majorana spinor was introduced by Majorana based on the observation that the Dirac equation for a massive neutral fermion

$$i\gamma^\mu\partial_\mu\psi = m\psi \tag{A.19}$$

could be as well written as

$$i\gamma^\mu\partial_\mu\psi = m\psi^c \tag{A.20}$$

as $\psi^c \sim \psi$ under Lorentz transformations. The latter can be derived from the Lagrangian

$$\mathcal{L} = \bar{\psi}i\partial\!\!\!/\psi - \frac{1}{2}m(\bar{\psi}^c\psi + \bar{\psi}\psi^c) \tag{A.21}$$

This somewhat obscures the fact that only one chiral, or in other words, 2 degrees of freedom are sufficient to write down such a mass term. If we write a Majorana spinor as

$$\psi_M = \psi + \psi^c \tag{A.22}$$

then it is its own charge conjugate (like say the photon)

$$\psi_M^c = \psi_M \tag{A.23}$$

[2]This defines C up to a phase. Different choices exist in the literature.

[3]In the Van der Waerden notation,

$$\psi^c = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \tag{A.17}$$

This is obtained from (definition) $(\bar{\chi}^{\dot{\alpha}})^* = \chi^\alpha$ and $(\psi^\alpha)^* = \bar{\psi}^{\dot{\alpha}}$ and using raising and lowering anti-symmetric 2x2 matrices $\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} \hat{=} (i\sigma^2)_{\alpha\beta}$ and $\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} \hat{=} (-i\sigma^2)_{\alpha\beta}$. The summation convention is as follows

$$\bar{\chi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\chi}^{\dot{\beta}} \quad \text{and} \quad \bar{\psi}^\alpha = \epsilon^{\alpha\beta}\psi_\beta \tag{A.18}$$

There are different summation conventions in the literature.

Also, we could write it either in terms of left-handed or right-handed spinors. This is most transparent in the Weyl basis,

$$\psi_M = \begin{pmatrix} \chi_L \\ -i\sigma_2\chi_L^* \end{pmatrix} \quad \text{or} \quad \psi_M = \begin{pmatrix} i\sigma_2\chi_R^* \\ \chi_R \end{pmatrix} \tag{A.24}$$

Whether we see it as being built upon a left-handed or right-handed spinor is a matter of taste or context: "One Weyl equals one Majorana, and two Weyls equal one Dirac" (Zee, QFT in a nutshell, p.481)

One last comment. The Majorana mass term in the Lagrangian (A.21) can also be written as

$$\mathcal{L} \supset -\frac{1}{2}\left(\psi^T C\psi + h.c.\right) = -\frac{1}{2}\left(C_{\alpha\beta}\psi_\alpha\psi_\beta + h.c.\right) \tag{A.25}$$

Since the conjugation matrix is antisymmetric, $C^T = -C$, the mass term vanishes less we take into account the fact that fermionic fields are grassmannian or anticommutating numbers.