completely. In particular, by direct substitution in equations (4), one verifies that to any solution one can add a gauge field

$$
\begin{aligned}
h_{\mu\nu}{}^{(\text{gauge})} &= \xi_{\mu,\nu} + \xi_{\nu,\mu}, \\
\bar{h}_{\mu\nu}{}^{(\text{gauge})} &= \xi_{\mu,\nu} + \xi_{\nu,\mu} - \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha},
\end{aligned}
\tag{5}
$$

without changing $T^{\mu\nu}$.

Let ξ_μ vanish outside some finite spacetime volume, but be otherwise arbitrary. Then $h_{\mu\nu}$ and $h_{\mu\nu} = h_{\mu\nu} + h_{\mu\nu}{}^{(\text{gauge})}$ both satisfy the source equation (4) for the same source $T^{\mu\nu}$ and the same boundary conditions at infinity. We therefore expect them to be physically equivalent.

By a specialization of the gauge analogous to the "Lorentz" specialization $A^{\alpha}{}_{,\alpha} = 0$ of electromagnetism (equation 3.58a; exercise 3.17), one imposes the condition

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0. \tag{6}$$

This reduces the field equations (4) to the simple d'Alembertian form

$$\Box\bar{h}^{\mu\nu} \equiv \bar{h}^{\mu\nu}{}_{,\alpha}{}^{\alpha} = -16\pi T^{\mu\nu} \tag{7}$$

(see exercise 18.2). Here and henceforth we set $G = 1$ ("geometrized units").

C. Field of a Point Mass (exercise 7.3C)

For a static source, the wave equation (7) reduces to a Laplace equation

$$\nabla^2\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}.$$

The stress-energy tensor for a static point mass (equation 7.10) is $T^{00} = M\delta^3(x)$ and $T^{\mu k} = 0$. Put this into the Laplace equation, solve for $\bar{h}_{\mu\nu}$, and use equation (7.8c) to obtain $h_{\mu\nu}$. The result is:

$$h_{00} = 2M/r; \qquad h_{0k} = 0; \qquad h_{ik} = \delta_{ik}(2M/r) \tag{8}$$

(see equation 18.15a).

D. Perihelion Precession (exercise 7.3D)

Direct substitution of the potential (8) into the equations of motion (2) is tedious and not very instructive. Variational principles are popular in mechanics because they simplify such calculations. Return to the basic variational principle $\delta I_{p+i} = 0$ (equation 1), and insert the potential (8) for the sun. Convert to spherical coordinates so oriented that the orbit lies in the equatorial ($\theta = \pi/2$) plane:

$$I_{p+i} = \int L\, d\tau; \tag{9}$$

$$L = \frac{1}{2} m[-(1 - 2Mr^{-1})\dot{t}^2 + (1 + 2Mr^{-1})(\dot{r}^2 + r^2\dot{\phi}^2)]. \tag{10}$$

Box 7.1 ***(continued)***

From the absence of explicit t-, ϕ-, and τ-dependence in L, infer three constants of motion: the canonical momenta

$$P_t \equiv -m\gamma = \partial L/\partial \dot{t}$$

(this defines γ) and

$$P_\phi \equiv m\alpha = \partial L/\partial \dot{\phi}$$

(this defines α); and the Hamiltonian

$$H = \dot{x}^\mu(\partial L/\partial \dot{x}^\mu) - L,$$

which can be set equal to $-\frac{1}{2}m$ by appropriate normalization of the path parameter τ. From these constants of the motion, derive an orbit equation as follows: (1) calculate $H = -\frac{1}{2}m$ in terms of r, $\dot{r}$, $\dot{\phi}$, and $\dot{t}$; (2) eliminate $\dot{t}$ and $\dot{\phi}$ in favor of the constants γ and α; (3) as in Newtonian orbit problems, define $u = M/r$, and write

$$\frac{du}{d\phi} = \frac{\dot{u}}{\dot{\phi}} = -\frac{M\dot{r}}{r^2\dot{\phi}} = -\frac{M}{\alpha}(1 + 2u)\dot{r};$$

(4) in H, eliminate $\dot{r}$ in favor of $du/d\phi$ via the above equation, and eliminate r in favor of u; (5) solve for $du/d\phi$. The result is

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = (\gamma^2 - 1 + 2u)\frac{M^2}{\alpha^2}\left[\frac{1 + 2u}{1 - 2u}\right]. \tag{11}$$

Neglecting cubic and higher powers of $u = GM/c^2r \sim (1 - \gamma^2)$ in this equation, derive the perihelion shift. (For details of method, see exercise 40.4, with the γ and α of this box renamed $\widetilde{E}$ and $\widetilde{L}$, and with the γ and β of that exercise set equal to 1 and 0.) The resulting shift per orbit is

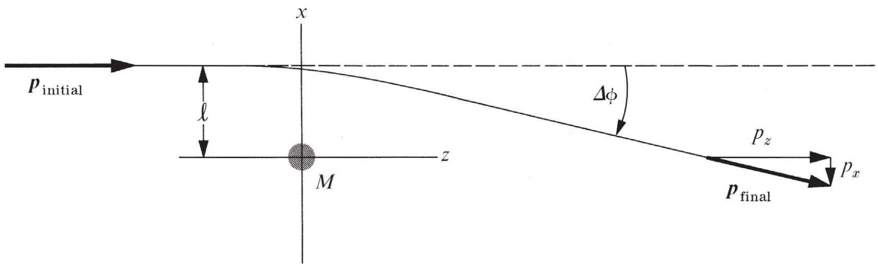
$$\Delta\phi = 8\pi M/r_0 + O([M/r_0]^2). \tag{12}$$

This is $\frac{4}{3}$ the prediction of general relativity; and it disagrees with the observations on Mercury (see Box 40.3).

E. Bending of Light (exercise 7.3E)

The deflection angle for light passing the sun is, on dimensional grounds, a small quantity, $\Delta\phi \sim M_\odot/R_\odot \sim 10^{-6}$; from the outset, one makes approximations based on this smallness. A diagram of the photon trajectory, set in the x, z-plane, shows that, for initial motion parallel to the z-axis, the deflection angle can be expressed in terms of the final momentum as $\Delta\phi = p_x/p_z$. Compute the final p_x by an integral along the trajectory,

$$p_x = \int_{-\infty}^{+\infty} (dp_x/dz)\, dz,$$



treating p_z as essentially constant. This computation requires generalization of the equation of motion (2) to the case of zero rest mass. To handle the limit $m \longrightarrow 0$, introduce a new parameter $\lambda = \tau/m$; then $p^\mu = m(dz^\mu/d\tau) = dz^\mu/d\lambda$. Also define $P_\mu = (\eta_{\mu\nu} + h_{\mu\nu})p^\nu$, since this quantity appears simply in equation (2) and agrees with p_μ in the limit $r \longrightarrow \infty$, where one will need to evaluate it. Then equation (2) reads, for any m, including $m = 0$,

$$\frac{dP_\mu}{d\lambda} = \frac{1}{2} h_{\alpha\beta,\mu} p^\alpha p^\beta.$$

On the righthand side here, since $h_{\alpha\beta,\mu}$ is small, a crude approximation to p^μ suffices; $p^1 = p^2 = 0$, $p^0 = p^3 = dz/d\lambda = \omega =$ constant. Thus,

$$\frac{dP_1}{d\lambda} = \frac{1}{2}(h_{00} + 2h_{03} + h_{33})_{,1}\omega^2$$

and

$$\frac{1}{p_3}\frac{dP_1}{dz} = \frac{1}{2}(h_{00} + 2h_{03} + h_{33})_{,1}.$$

For the sun, $h_{00} = h_{33} = 2M/r$, and $h_{03} = 0$ (equation 8), so

$$\Delta\phi = -\left(\frac{p_1}{p_3}\right)_{\text{final}} = -\left(\frac{P_1}{p_3}\right)_{\text{final}} = \int_{-\infty}^{\infty} \frac{2M\ell\, dz}{(\ell^2 + z^2)^{3/2}} = \frac{2M}{\ell}\int_{-\infty}^{\infty} \frac{d\zeta}{(1 + \zeta^2)^{3/2}} = \frac{4M}{\ell}. \quad (13)$$

For light grazing the sun, $\ell = R_\odot$, this gives $\Delta\phi = 4M_\odot/R_\odot$ radians $= 1''.75$, which is also the prediction of general relativity, and is consistent with the observations (see Box 40.1).

F. Gravitational Waves (exercise 7.3F)

The field equations (4) and gauge properties (5) of the present flat-spacetime theory are identical to those of Einstein's "linearized theory." Thus, the treatment of gravitational waves using linearized theory, which is presented in §§18.2, 35.3, and 35.4, applies here.

G. Positive Energy of the Waves (exercise 7.3G)

Computing a general formula for $\mathcal{H}$ from equation (7.17) is tedious, but it is sufficient to consider only the special case of a plane wave (equation 7.13)—or better still,

2

Box 7.1 (*continued*)

a plane wave with only $h_{12} = h_{21} = f(z - t)$. Any gravitational wave can be constructed as a superposition of such plane waves. First compute the Langrangian for this case. According to equation (7.8), it reads

$$\mathcal{L}_f = (32\pi)^{-1}[(h_{12,0})^2 - (h_{12,3})^2].$$

Now the full content of the formula (7.17) defining $\mathcal{H}$ is precisely the following: start from the Lagrangian; keep all terms that are quadratic in time derivatives; omit all terms that are linear in time derivatives; and reverse the sign of terms that contain no time derivatives. The result is

$$\mathcal{H} = (32\pi)^{-1}[(h_{12,0})^2 + (h_{12,3})^2], \tag{14}$$

which is positive.

H. Self-Inconsistency of the Theory (exercise 7.3H)

From equation (7.10), find

$$T^{\mu\nu}{}_{,\nu} = m \int \frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}\frac{\partial}{\partial x^\nu}\,\delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)]\,d\tau.$$

But $\delta^4(\boldsymbol{x} - \boldsymbol{z})$ depends only on the difference $x^\mu - z^\mu$, so $-\partial/\partial z^\nu$ can replace $\partial/\partial x^\nu$ when acting on the δ-function. Noting that

$$\frac{dz^\nu}{d\tau}\frac{\partial}{\partial z^\nu}\,\delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)] = \frac{d}{d\tau}\,\delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)],$$

rewrite $T^{\mu\nu}{}_{,\nu}$ as

$$T^{\mu\nu}{}_{,\nu} = -m \int \dot{z}^\mu (d/d\tau)\,\delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)]\,d\tau = +m \int \ddot{z}^\mu\,\delta^4[\boldsymbol{x} - \boldsymbol{z}(\tau)]\,d\tau.$$

(The last step is obtained by an integration by parts.) Thus $T^{\mu\nu}{}_{,\nu} = 0$ holds *if and only if* $\ddot{z}^\mu = 0$. But $\ddot{z}^\mu = 0$ means that the gravitational fields have no effect on the motion of the particle. But this contradicts the equation of motion (2), which follows from the theory's variational principle. Thus, this tensor theory of gravity is inconsistent. [Stated briefly, equation (4) requires $T^{\mu\nu}{}_{,\nu} = 0$, while equation (2) excludes it.]

The fact that, in this theory, gravitating bodies cannot be affected by gravity, also holds for bodies made of arbitrary stress-energy (e.g., rubber balls or the Earth). Since all bodies gravitate, since the field equations imply $T^{\mu\nu}{}_{,\nu} = 0$, and since this "equation of motion for stress-energy" implies conservation of a body's total 4-momentum $P^\mu = \int T^{\mu 0}\, d^3x$, no body can be accelerated by gravity. The Earth cannot be attracted by the sun; it must fly off into interstellar space!

Straightforward steps to repair this inconsistency in the theory lead inexorably to general relativity (see Box 17.2 part 5). Having adopted general relativity as *the* theory of gravity, one can then use the present flat-spacetime theory as an approximation to it ("Linearized general relativity"; treated in Chapters 18, 19, and 35; see especially discussion at end of §18.3).

§7.2. GRAVITATIONAL RED SHIFT DERIVED FROM ENERGY CONSERVATION

Einstein argued against the existence of any ideal, straight-line reference system such as is assumed in Newtonian theory. He emphasized that nothing in a natural state of motion, not even a photon, could ever give evidence for the existence or location of such ideal straight lines.

Gravitational redshift derived from energy considerations

That a photon must be affected by a gravitational field Einstein (1911) showed from the law of conservation of energy, applied in the context of Newtonian gravitation theory. Let a particle of rest mass m start from rest in a gravitational field g at point $\mathcal{A}$ and fall freely for a distance h to point $\mathcal{B}$. It gains kinetic energy mgh. Its total energy, including rest mass, becomes

$$m + mgh. \tag{7.18}$$

Now let the particle undergo an annihilation at $\mathcal{B}$, converting its total rest mass plus kinetic energy into a photon of the same total energy. Let this photon travel upward in the gravitational field to $\mathcal{A}$. If it does not interact with gravity, it will have its original energy on arrival at $\mathcal{A}$. At this point it could be converted by a suitable apparatus into another particle of rest mass m (which could then repeat the whole process) plus an excess energy mgh that costs nothing to produce. To avoid this contradiction of the principal of conservation of energy, which can also be stated in purely classical terms, Einstein saw that the photon must suffer a red shift. The energy of the photon must decrease just as that of a particle does when it climbs out of the gravitational field. The photon energy at the top and the bottom of its path through the gravitational field must therefore be related by

$$E_{\text{bottom}} = E_{\text{top}}(1 + gh) = E_{\text{top}}(1 + g_{\text{conv}}h/c^2). \tag{7.19}$$

The drop in energy because of work done against gravitation implies a drop in frequency and an increase in wavelength (red shift; traditionally stated in terms of a red shift parameter, $z = \Delta\lambda/\lambda$); thus,

$$1 + z = \frac{\lambda_{\text{top}}}{\lambda_{\text{bottom}}} = \frac{h\nu_{\text{bottom}}}{h\nu_{\text{top}}} = \frac{E_{\text{bottom}}}{E_{\text{top}}} = 1 + gh. \tag{7.20}$$

The redshift predicted by this formula has been verified to 1 percent by Pound and Snider (1964, 1965), refining an experiment by Pound and Rebka (1960).

§7.3. GRAVITATIONAL REDSHIFT IMPLIES SPACETIME IS CURVED

An argument by Schild (1960, 1962, 1967) yields an important conclusion: the existence of the gravitational redshift shows that a consistent theory of gravity cannot be constructed within the framework of special relativity.

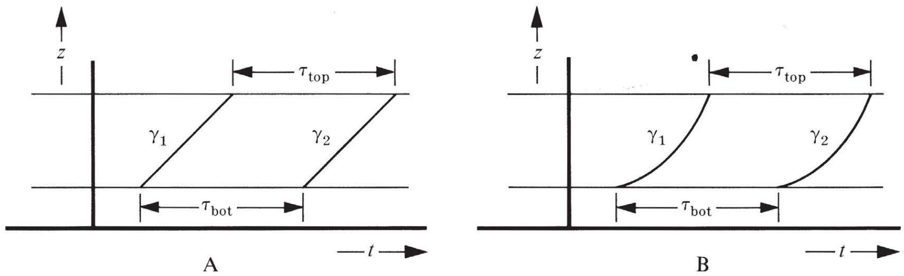


Figure 7.1.
Successive pulses of light rising from height z_1, to height $z_2 = z_1 + h$ against the gravitational field of the earth. The paths γ_1, and γ_2 must be exactly congruent, whether sloped at 45° (left) or having variable slope (right).

Assume gravity is described by an (unspecified) field in flat spacetime . . .

Whereas Einstein's argument (last section) was formulated in Newtonian theory, Schild's is formulated in special relativity. It analyzes gravitational redshift experiments in the field of the Earth, using a global Lorentz frame tied to the Earth's center. It makes no demand that free particles initially at rest remain at rest in this global Lorentz frame (except far from the Earth, where gravity is negligible). On the contrary, it demands that free particles be accelerated relative to the Lorentz frame by the Earth's gravitational field. It is indifferent to the mathematical nature of that field (scalar, vector, tensor, . . .), but it does insist that the gravitational accelerations agree with experiment. And, of course, it demands that proper lengths and times be governed by the metric of special relativity.

Schild's argument proceeds as follows. Consider one observer at rest on the Earth's surface at height z_1, and a second above the Earth's surface at height $z_2 = z_1 + h$ (Figure 7.1). The observers may verify that they are at rest relative to each other and relative to the Earth's Lorentz frame by, for instance, radar ranging to free particles that are at rest in the Earth's frame far outside its gravitational field. The bottom experimenter then emits an electromagnetic signal of a fixed standard frequency ω_b which is received by the observer on top. For definiteness, let the signal be a pulse exactly N cycles long. Then the interval of time* $\delta\tau_{bot}$ required for the emission of the pulse is given by $2\pi N = \omega_b\, \delta\tau_{bot}$. The observer at the top is then to receive these same N cycles of the electromagnetic wave pulse and measure the time interval* $\delta\tau_{top}$ required. By the definition of "frequency," it satisfies $2\pi N = \omega_t\, \delta\tau_{top}$. The redshift effect, established by experiment (for us) or by energy conservation (for Einstein), shows $\omega_t < \omega_b$; consequently the time intervals are different, $\delta\tau_{top} > \delta\tau_{bot}$. Transfer this information to the special-relativity spacetime diagram of the experiment (Figure 7.1). The waves are light rays; so one might show them as traveling along 45° null lines in the spacetime diagram (Figure 7.1,A). In this

*Proper time equals Lorentz coordinate time for both observers, since they are at rest in the Earth's Lorentz frame.

simplified but slightly inadequate form of the argument, one reaches a contradiction by noticing that here one has drawn a *parallelogram* in Minkowski spacetime in which two of the sides are unequal, $\tau_{\mathrm{top}} > \tau_{\mathrm{bot}}$, whereas a parallelogram in flat Minkowski spacetime cannot have opposite sides unequal. One concludes that *special relativity cannot be valid* over any sufficiently extended region. Globally, spacetime, as probed by the tracks of light rays and test particles, departs from flatness ("curvature"; Parts III and IV of this book), despite the fine fit that Lorentz-Minkowski flatness gives to physics locally.

This assumption is incompatible with gravitational redshift

Figure 7.1,B, repairs an oversimplification in this argument by recognizing that the propagation of light will be influenced by the gravitational field. Therefore photons might not follow straight lines in the diagram. Consequently, the world lines γ_1 and γ_2 of successive pulses are curves. However, the gravitational field is static and the experimenters do not move. Therefore nothing in the experimental setup changes with time. Whatever the path γ_1, the path γ_2 must be a *congruent* path of exactly the same shape, merely translated in time. On the basis of this congruence and the fact that the observers are moving on parallel world lines, one would again conclude, if flat Minkowski geometry were valid, that $\tau_{\mathrm{bot}} = \tau_{\mathrm{top}}$, thus contradicting the observed redshift experiment. The experimenters do not need to understand the propagation of light in a gravitational field. They need only use their radar apparatus to verify the fact that they are at rest relative to each other and relative to the source of the gravitational field. They know that, whatever influence the gravitational field has on their radar apparatus, it will not be a time-dependent influence. Moreover, they do not have to know how to compute their separation in order to verify that the separation remains constant. They only need to verify that the round-trip time required for radar pulses to go out to each other and back is the same every time they measure it.

Schild's redshift argument does not reveal what kind of curvature must exist, or whether the curvature exists in the neighborhood of the observational equipment or some distance away from it. It does say, however, quite unambigously, that the flat spacetime of special relativity is inadequate to describe the situation, and it should therefore motivate one to undertake the mathematical analysis of curvature in Part III.

Conclusion: spacetime is curved

§7.4. GRAVITATIONAL REDSHIFT AS EVIDENCE FOR THE PRINCIPLE OF EQUIVALENCE

Einstein (1908, 1911) elevated the idea of the universality of gravitational interactions to the status of a fundamental *principle of equivalence, that all effects of a uniform gravitational field are identical to the effects of a uniform acceleration of the coordinate system*. This principle generalized a result of Newtonian gravitation theory, in which a uniform acceleration of the coordinate system in equation (7.1) gives rises to a

Principle of equivalence: a uniform gravitational field is indistinguishable from a uniform acceleration of a reference frame supplementary uniform gravitational field. However, the Newtonian theory only gives this result for particle mechanics. Einstein's principle of equivalence asserts that a similar correspondence will hold for all the laws of physics, including Maxwell's equations for the electromagnetic field.

The rules of the game—the "scientific method"—require that experimental support be sought for any new theory or principle, and Einstein could treat the gravitational redshift as the equivalent of experimental confirmation of his principle of equivalence. There are two steps in such a confirmation: first, the theory or principle must predict an effect (the next paragraph describes how the equivalence principle implies the redshift); second, the predicted effect must be observed. With the Pound-Rebka-Snider experiments, one is in much better shape today than Einstein was for this second step. Einstein himself had to rely on the experiments supporting the general concept of energy conservation, plus the necessity of a redshift to preserve energy conservation, as a substitute for direct experimental confirmation.

Gravitational redshift derived from equivalence principle

The existence of the gravitational redshift can be deduced from the equivalence principle by considering two experimenters in a rocket ship that maintains a constant acceleration g. Let the distance between the two observers be h in the direction of the acceleration. Suppose for definiteness that the rocket ship was at rest in some inertial coordinate system when the bottom observer sent off a photon. It will require time $t = h$ for the photon to reach the upper observer. In that time the top observer acquires a velocity $v = gt = gh$. He will therefore detect the photon and observe a Doppler redshift $z = v = gh$. The results here are therefore identical to equation (7.20). The principle of equivalence of course requires that, if this redshift is observed in an experiment performed under conditions of uniform acceleration in the absence of gravitational fields, then the same redshift must be observed by an experiment performed under conditions where there is a uniform gravitational field, but no acceleration. Consequently, by the principle of equivalence, one can derive equation (7.20) as applied to the gravitational situation.

§7.5. LOCAL FLATNESS, GLOBAL CURVATURE

Equivalence principle implies nonmeshing of local Lorentz frames near Earth (spacetime curvature!)

The equivalence principle helps one to discern the nature of the spacetime curvature, whose existence was inferred from Schild's argument. Physics is the same in an accelerated frame as it is in a laboratory tied to the Earth's surface. Thus, an Earth-bound lab can be regarded as accelerating upward, with acceleration g, relative to the Lorentz frames in its neighborhood.* Equivalently, relative to the lab and the Earth's surface, all Lorentz frames must accelerate downward. But the downward (radial) direction is different at different latitudes and longitudes. Hence, local Lorentz frames, initially at rest with respect to each other but on opposite sides of the Earth, subsequently fall toward the center and go flying through each other. Clearly they cannot be meshed to form the single global Lorentz frame, tied to the

*This upward acceleration of the laboratory, plus equation (6.18) for the line element in an accelerated coordinate system, explains the nonequality of the bottom and top edges of the parallelograms in Figure 7.1.

Earth, that was assumed in Schild's argument. This nonmeshing of local Lorentz frames, like the nonmeshing of local Cartesian coordinates on a curved 2-surface, is a clear manifestation of spacetime curvature.

Geographers have similar problems when mapping the surface of the earth. Over small areas, a township or a county, it is easy to use a standard rectangular coordinate system. However, when two fairly large regions are mapped, each with one coordinate axis pointing north, then one finds that the edges of the maps overlap each other best if placed at a slight angle (spacetime analog: relative velocity of two local Lorentz frames meeting at center of Earth). It is much easier to start from a picture of a spherical globe, and then talk about how small flat maps might be used as good approximations to parts of it, than to start with a huge collection of small maps and try to piece them together to build up a picture of the globe. The exposition of the geometry of spacetime in this book will therefore take the first approach. Now that one recognizes that the problem is to fit together local, flat spacetime descriptions of physics into an over-all view of the universe, one should be happy to jump, in the next chapter, into a broadscale study of geometry. From this more advantageous viewpoint, one can then face the problem of discussing the relationship between the local inertial coordinate systems appropriate to two nearby regions that have slightly different gravitational fields.

Nonmeshing of local Lorentz frames motivates study of geometry

There are actually two distinguishable ways in which geometry enters the theory of general relativity. One is the geometry of lengths and angles in four-dimensional spacetime, which is inherited from the metric structure ds^2 of special relativity. Schild's argument already shows (without a direct appeal to the equivalence principle) that the special-relativistic ideas of length and angle must be modified. The modified ideas of metric structure lead to Riemannian geometry, which will be treated in Chapters 8 and 13. However, geometry also enters general relativity because of the equivalence principle. An equivalence principle can already be stated within Newtonian gravitational theory, in which no concepts of a *spacetime* metric enter, but only the Euclidean metric structure of three-dimensional *space*. The equivalence-principle view of Newtonian theory again insists that the local standard of reference be the freely falling particles. This requirement leads to the study of a *spacetime* geometry in which the curved world lines of freely falling particles are defined to be locally straight. They play the role in a curved spacetime geometry that straight lines play in flat spacetime. This "affine geometry" will be studied in Chapters 10–12. It leads to a quantitative formulation of the ideas of "covariant derivative" and "curvature" and even "curvature of Newtonian spacetime"!

Two types of geometry relevant to spacetime:

Riemannian geometry (lengths and angles)

Affine geometry ("straight lines" and curvature)

PART III

THE MATHEMATICS OF CURVED SPACETIME

Wherein the reader is exposed to the charms of a new temptress—
Modern Differential Geometry—and makes a decision:
to embrace her for eight full chapters; or,
having drunk his fill, to escape after one.

CHAPTER 8

DIFFERENTIAL GEOMETRY: AN OVERVIEW

I am coming more and more to the conviction that the necessity of our geometry cannot be demonstrated, at least neither by, nor for, the human intellect. . . . geometry should be ranked, not with arithmetic, which is purely aprioristic, but with mechanics. (*1817*)

We must confess in all humility that, while number is a product of our mind alone, space has a reality beyond the mind whose rules we cannot completely prescribe. (*1830*)

CARL FRIEDRICH GAUSS

§8.1. AN OVERVIEW OF PART III

Gravitation is a manifestation of spacetime curvature, and that curvature shows up in the deviation of one geodesic from a nearby geodesic ("relative acceleration of test particles"). The central issue of this part of the book is clear: *How can one quantify the "separation," and the "rate of change" of "separation," of two "geodesics" in "curved" spacetime?* A clear, precise answer requires new concepts.

Concepts to be developed in Part III:

"Separation" between geodesics will mean "vector." But the concept of vector as employed in flat Lorentz spacetime (a bilocal object: point for head and point for tail) must be sharpened up into the local concept of *tangent vector,* when one passes to curved spacetime. Chapter 9 does the sharpening. It also reveals how the passage to curved spacetime affects 1-forms and tensors.

Tangent vector

It takes one tool (vectors in curved geometry, Chapter 9) to define "separation" clearly as a vector; it takes another tool (parallel transport in curved spacetime, Chapter 10) to compare separation vectors at neighboring points and to define the "rate of change of separation." No transport, no comparison; no comparison, no meaning to the term "rate of change"! The notion of parallel transport founds itself

Geodesic

Covariant derivative

on the idea of *"geodesic,"* the world line of a freely falling particle. The special mathematical properties of a geodesic are explored in Chapter 10. That chapter uses geodesics to define parallel transport, uses parallel transport to define *covariant derivative,* and—completing the circle—uses covariant derivative to describe geodesics.

Geodesic deviation

Spacetime curvature

Chapter 11 faces up to the central issue: *geodesic deviation* ("rate of change of separation vector between two geodesics"), and its use in defining the *curvature* of spacetime.

But to define curvature is not enough. The man who would understand gravity deeply must also see curvature at work, producing relative accelerations of particles in Newtonian spacetime (Chapter 12); he must learn how, in Einstein spacetime, distances (metric) determine completely the curvature and the law of parallel transport (Chapter 13); he must be the master of powerful tools for computing curvature (Chapter 14); and he must grasp the geometric significance of the algebraic and differential symmetries of curvature (Chapter 15).

This chapter: a Track-1 overview of differential geometry

Unfortunately, such deep understanding requires time—far more time than one can afford in a ten-week or fifteen-week course, far more than a lone reader may wish to spend on first passage through the book. For the man who must rush on rapidly, this chapter contains a "Track-1" overview of the essential mathematical tools (§§8.4–8.7). From it one can gain an adequate, but not deep, understanding of spacetime curvature, of tidal gravitational forces, and of the mathematics of curved spacetime. This overview is also intended for the Track-2 reader; it will give him a taste of what is to come. The ambitious reader may also wish to consult other introductions to differential geometry (see Box 8.1).

Box 8.1 BOOKS ON DIFFERENTIAL GEOMETRY

There are several mathematics texts that may be consulted for a more detailed and extensive discussion of modern differential geometry along the line taken here. Bishop and Goldberg (1968) is the no. 1 reference. Hicks (1965) could be chosen as a current standard graduate-level text, with O'Neill (1966) at the undergraduate level introducing many of the same topics without presuming that the reader finds easy and obvious the current style in which pure mathematicians think and write. Auslander and MacKenzie (1963) at a somewhat more advanced level also allow for the reader to whom differential equations are more familiar than homomorphisms. Willmore (1959) is easy to read but presents no challenge, and leads to little progress in adapting to the style of current mathematics. Trautman (1965) and Misner (1964a, 1969a) are introductions somewhat similar to ours, except for deemphasis of pictures; like ours, they are aimed at the student of relativity. Flanders (1963) is easy and useful as an introduction to exterior differential forms; it also gives examples of their application to a wide variety of topics in physics and engineering.

§8.2. TRACK 1 VERSUS TRACK 2: DIFFERENCE IN OUTLOOK AND POWER

Nothing is more wonderful about the relation between Einstein's theory of gravity and Newton's theory than this, as discovered by Élie Cartan (1923, 1924): that both theories lend themselves to description in terms of curvature; that in both this curvature is governed by the density of mass-energy; and that this curvature allows itself to be defined and measured without any use of or reference to any concept of metric. The difference between the two theories shows itself up in this: Einstein's theory in the end (or in the beginning, depending upon how one presents it!) does define an interval between every event and every nearby event; Newton's theory not only does not, but even says that any attempt to talk of *spacetime* intervals violates Newton's laws. This being the case, Track 2 will forego for a time (Chapters 9–12) any use of a spacetime metric ("Einstein interval"). It will extract everything possible from a metric-free description of spacetime curvature (all of Newton's theory; important parts of Einstein's theory).

Preview of Track-2 differential geometry

Geodesic deviation is a measurer and definer of curvature, but the onlooker is forbidden to reduce a vector description of separation to a numerical measure of distance (no metric at this stage of the analysis): what an impossible situation! Nevertheless, that is exactly the situation with which Chapters 9–12 will concern themselves: how to do geometry without a metric. Speaking physically, one will overlook at this stage the fact that the geometry of the physical world is always and everywhere locally Lorentz, and endowed with a light cone, but one will exploit to the fullest the Galileo-Einstein principle of equivalence: in any given locality one can find a frame of reference in which every neutral test particle, whatever its velocity, is free of acceleration. The tracks of these neutral test particles define the geodesics of the geometry. These geodesics provide tools with which one can do much: define parallel transport (Chapter 10), define covariant derivative (Chapter 10), quantify geodesic deviation (Chapter 11), define spacetime curvature (Chapter 11), and explore Newtonian gravity (Chapter 12). Only after this full exploitation of metric-free geodesics will Track 2 admit the Einstein metric back into the scene (Chapters 13–15).

But to forego use of the metric is a luxury which Track 1 can ill afford; too little time would be left for relativistic stars, cosmology, black holes, gravitational waves, experimental tests, and the dynamics of geometry. Therefore, the Track-1 overview in this chapter keeps the Einstein metric throughout. But in doing so, it pays a heavy price: (1) no possibility of seeing curvature at work in Newtonian spacetime (Chapter 12); (2) no possibility of comparing and contrasting the geometric structures of Newtonian spacetime (Chapter 12) and Einstein spacetime (Chapter 13), and hence no possibility of grasping fully the Newtonian-based motivation for the Einstein field equations (Chapter 17); (3) no possibility of understanding *fully* the mathematical interrelationships of "geodesic," "parallel transport," "covariant derivative," "curvature," and "metric" (Chapters 9, 10, 11, 13); (4) no possibility of introducing the mathematical subjects *"differential topology"* (geometry without metric or covariant

What the Track-1 reader will miss

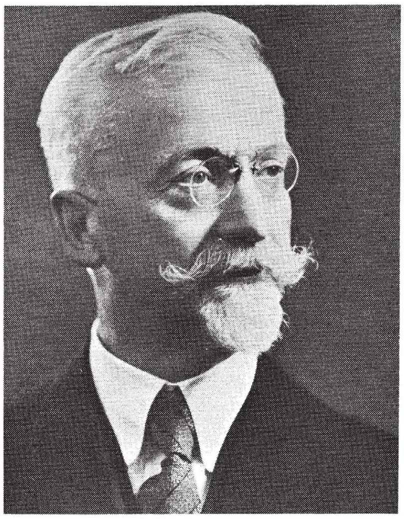
derivative, Chapter 9) and *"affine geometry"* (geometry with covariant derivative but no metric, Chapters 10 and 11), subjects which find major application in modern analytical mechanics [see, e.g., Arnold and Avez (1968); also exercise 4.11 of this book], in Lie group theory with its deep implications for elementary particle physics [see, e.g., Hermann (1966); also exercises 9.12, 9.13, 10.16, and 11.12 of this book], in the theory and solution of partial differential equations [see, e.g., Sternberg (1969)], and, of course, in gravitation theory.

§8.3. THREE ASPECTS OF GEOMETRY: PICTORIAL, ABSTRACT, COMPONENT

Geometry from three viewpoints: pictorial, abstract, component

Gain the power in §8.4 and Chapter 9 to discuss tangent vectors, 1-forms, tensors in curved spacetime; gain the power in §8.5 and Chapter 10 to parallel-transport vectors, to differentiate them, to discuss geodesics; use this power in §8.7 and Chapter 11 to discuss geodesic deviation, to define curvature; . . . . But full power this will be only if it can be exercised in three ways: in pictures, in abstract notation, and in component notation (Box 8.3). Élie Cartan (Box 8.2) gave new insight into both

Box 8.2 ÉLIE CARTAN, 1869–1951



Élie Cartan is a most remarkable figure in recent mathematical history. One learns from his obituary [Chern and Chevalley (1952)] that he was born a blacksmith's son in southern France, and proved the value of government scholarship aid by rising through the system to a professorship at the Sorbonne in 1912 when he was 43. At the age of 32 he invented the exterior derivative [Cartan (1901)], which he used then mostly in differential equations and the theory of Lie groups, where he had already made significant contributions. He was about fifty when he began applying it to geometry, and sixty before Riemannian geometry specifically was the object of this research, including his text [Cartan (1928)], which is still reprinted and worth studying. Although universally recognized, his work did not find responsive readers until he neared retirement around 1940, when the "Bourbaki" generation of French mathematicians began to provide a conceptual framework for (among other things) Cartan's insights and methods. This made Cartan communicable and teachable as his own writings never were, so that by the time of his death at 82 in 1951 his influence was obviously dominating the revolutions then in full swing in all the fields (Lie groups, differential equations, and differential geometry) in which he had primarily worked.

The modern, abstract, coordinate-free approach to geometry, which is used extensively in this book, is due largely to Élie Cartan. He also discovered the geometric approach to Newtonian gravity that is developed and exploited in Chapter 12.

Box 8.3 THREE LEVELS OF DIFFERENTIAL GEOMETRY

(1) Purely *pictorial* treatment of geometry:
tangent vector is conceived in terms of the separation of two points in the limit in which the points are indefinitely close;
vectors are added and subtracted locally as in flat space;
vectors at distinct points are compared by parallel transport from one point to another;
this parallel transport is accomplished by a "Schild's ladder construction" of geodesics (Box 10.2);
diagrams, yes; algebra, no;
it is tied conceptually as closely as possible to the world of test particles and measurements.

(2) *Abstract* differential geometry:
treats a tangent vector as existing in its own right, without necessity to give its breakdown into components,

$$\boldsymbol{A} = A^0\boldsymbol{e}_0 + A^1\boldsymbol{e}_1 + A^2\boldsymbol{e}_2 + A^3\boldsymbol{e}_3,$$

just as one is accustomed nowadays in electromagnetism to treat the electric vector $\boldsymbol{E}$, without having to write out its components;
uses a similar approach to differentiation (compare gradient operator $\boldsymbol{\nabla}$ of elementary vector analysis, as distinguished from coordinate-dependent pieces of such an operator, such as $\partial/\partial x$, $\partial/\partial y$, etc.);
is the quickest, simplest mathematical scheme one knows to derive general results in differential geometry.

(3) Differential geometry as expressed in the language of *components:*
is indispensible in programming large parts of general relativity for a computer;
is convenient or necessary or both when one is dealing even at the level of elementary algebra with the most simple applications of relativity, from the expansion of the Friedmann universe to the curvature around a static center of attraction.

Newtonian gravity (Chapter 12) and the central geometric simplicity of Einstein's field equations (Chapter 15), because he had full command of all three methods of doing differential geometry. Today, no one has full power to communicate with others about the subject who cannot express himself in all three languages. Hence the interplay between the three forms of expression in what follows.

Planetary orbit as example of three viewpoints

It is not new to go back and forth between the three languages, as witnesses the textbook treatment of the velocity and acceleration of a planet in Kepler motion around the sun. The velocity is written

$$\boldsymbol{v} = v^{\hat{r}}\boldsymbol{e}_{\hat{r}} + v^{\hat{\phi}}\boldsymbol{e}_{\hat{\phi}}. \tag{8.1}$$

(The hats "$\hat{}$" on $\boldsymbol{e}_{\hat{r}}$ and $\boldsymbol{e}_{\hat{\phi}}$ signify that these are unit vectors.) The acceleration is

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{dv^{\hat{r}}}{dt}\boldsymbol{e}_{\hat{r}} + \frac{dv^{\hat{\phi}}}{dt}\boldsymbol{e}_{\hat{\phi}} + v^{\hat{r}}\frac{d\boldsymbol{e}_{\hat{r}}}{dt} + v^{\hat{\phi}}\frac{d\boldsymbol{e}_{\hat{\phi}}}{dt}. \tag{8.2}$$

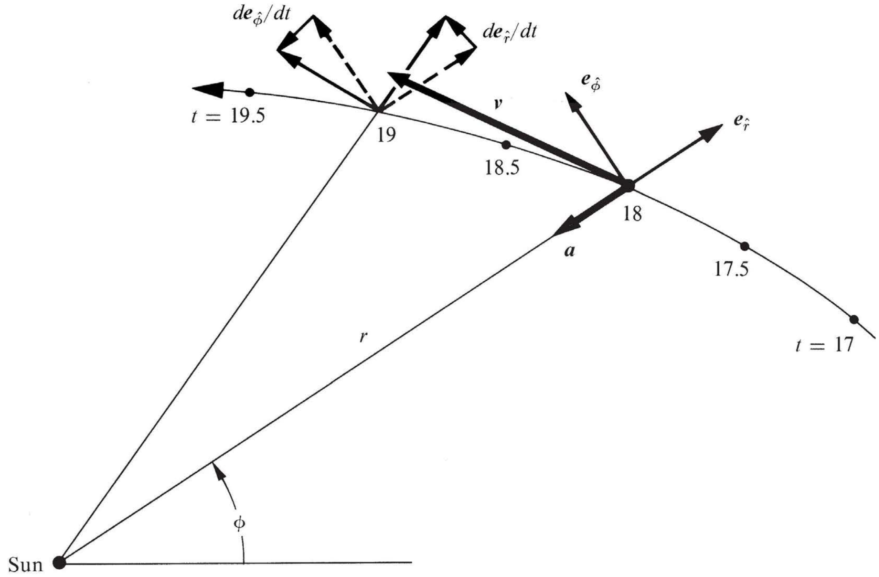


Figure 8.1.
A Keplerian orbit in the sun's gravitational field, as treated using the standard Euclidean-space version of Newtonian gravity. The basis vectors themselves change from point to point along the orbit [equations (8.3)]. This figure illustrates the pictorial aspect of differential geometry. Later (exercise 8.5) it will illustrate the concepts of "covariant derivative" and "connection coefficients."

The unit vectors are turning (Figure 8.1) with the angular velocity $\omega = d\phi/dt$; so

$$\begin{aligned} \frac{d\boldsymbol{e}_{\hat r}}{dt} &= \omega \boldsymbol{e}_{\hat\phi} = \frac{d\phi}{dt}\boldsymbol{e}_{\hat\phi}, \\ \frac{d\boldsymbol{e}_{\hat\phi}}{dt} &= -\omega \boldsymbol{e}_{\hat r} = -\frac{d\phi}{dt}\boldsymbol{e}_{\hat r}. \end{aligned} \tag{8.3}$$

Thus the components of the acceleration have the values

$$a^{\hat r} = \frac{dv^{\hat r}}{dt} - v^{\hat\phi}\frac{d\phi}{dt} = \frac{d^2 r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 \tag{8.4a}$$

and

$$a^{\hat\phi} = \frac{dv^{\hat\phi}}{dt} + v^{\hat r}\frac{d\phi}{dt} = \frac{d}{dt}\left(r\frac{d\phi}{dt}\right) + \frac{dr}{dt}\frac{d\phi}{dt}. \tag{8.4b}$$

Here is the acceleration in the language of components; $\boldsymbol{a}$ was the acceleration in abstract language; and Figure 8.1 shows the acceleration as an arrow. Each of these three languages will receive its natural generalization in the coming sections and chapters from two-dimensional flat space (with curvilinear coordinates) to four-dimensional curved spacetime, and from spacetime to more general manifolds (see §9.7 on manifolds).

Turn now to the Track-1 overview of differential geometry.

§8.4. TENSOR ALGEBRA IN CURVED SPACETIME

Tensor algebra:

To see spacetime curvature at work, examine tidal gravitational forces (geodesic deviation); and to probe these forces, make measurements in a finite-sized laboratory. Squeeze the laboratory to infinitesimal size; all effects of spacetime curvature become infinitesimal; the physicist cannot tell whether he is in flat spacetime or curved spacetime. Neither can the mathematician, in the limit as his domain of attention squeezes down to a single event, $\mathscr{P}_0$.

(1) occurs in infinitesimal neighborhood of an event

At the event $\mathscr{P}_0$ (in the infinitesimal laboratory) both physicist and mathematician can talk of vectors, of 1-forms, of tensors; and no amount of spacetime curvature can force the discussion to change from its flat-space form. A particle at $\mathscr{P}_0$ has a 4-momentum $\boldsymbol{p}$, with squared length

(2) is same in curved spacetime as in flat

$$\boldsymbol{p}^2 = \boldsymbol{p} \cdot \boldsymbol{p} = \boldsymbol{g}(\boldsymbol{p}, \boldsymbol{p}) = -m^2.$$

The squared length, as always, is calculated by inserting $\boldsymbol{p}$ into both slots of a linear machine, the metric $\boldsymbol{g}$ at $\mathscr{P}_0$. The particle also has a 4-acceleration $\boldsymbol{a}$ at $\mathscr{P}_0$; and, if the particle is charged and freely moving, then $\boldsymbol{a}$ is produced by the electromagnetic field tensor $\boldsymbol{F}$:

$$m\boldsymbol{a} = e\boldsymbol{F}(\ldots, \boldsymbol{u}).$$

In no way can curvature affect such local, coordinate-free, geometric relations. And in no way can it prevent one from introducing a local Lorentz frame at $\mathscr{P}_0$, and from performing standard, flat-space index manipulations in it:

$$\boldsymbol{p} = p^\alpha \boldsymbol{e}_\alpha, \qquad \boldsymbol{p}^2 = p^\alpha p^\beta \eta_{\alpha\beta} = p^\alpha p_\alpha, \qquad m a^\alpha = e F^{\alpha\beta} u_\beta.$$

But local Lorentz frames are not enough for the man who would calculate in curved spacetime. Non-Lorentz frames (nonorthonormal basis vectors $\{\boldsymbol{e}_\alpha\}$) often simplify calculations. Fortunately, no effort at all is required to master the rules of "index mechanics" in an arbitrary basis at a fixed event $\mathscr{P}_0$. The rules are identical to those in flat spacetime, except that (1) the covariant Lorentz components $\eta_{\alpha\beta}$ of the metric are replaced by

(3) rules for component manipulation change slightly when using nonorthonormal basis

Components of metric

$$g_{\alpha\beta} \equiv \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta \equiv \boldsymbol{g}(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta); \tag{8.5}$$

(2) the contravariant components $\eta^{\alpha\beta}$ are replaced by $g^{\alpha\beta}$, where

$$\|g^{\alpha\beta}\| \equiv \|g_{\alpha\beta}\|^{-1} \text{ (matrix inverse)}; \tag{8.6}$$

i.e.,

$$g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha{}^\gamma; \tag{8.6$'$}$$

(3) the Lorentz transformation matrix $\|\Lambda^{\alpha'}{}_\beta\|$ and its inverse $\|\Lambda^\beta{}_{\alpha'}\|$ are replaced by an arbitrary but nonsingular transformation matrix $\|L^{\alpha'}{}_\beta\|$ and its inverse $\|L^\beta{}_{\alpha'}\|$:

Transformation of basis

$$\boldsymbol{e}_\beta = \boldsymbol{e}_{\alpha'} L^{\alpha'}{}_\beta, \qquad p^\beta = L^\beta{}_{\alpha'} p^{\alpha'}, \tag{8.7}$$

$$\|L^\beta{}_{\alpha'}\| = \|L^{\alpha'}{}_\beta\|^{-1}; \tag{8.8}$$

(4) in the special case of "coordinate bases," $\boldsymbol{e}_\alpha = \partial\mathscr{P}/\partial x^\alpha$, $\boldsymbol{e}_{\beta'} = \partial\mathscr{P}/\partial x^{\beta'}$,

$$L^{\alpha'}{}_\beta = \partial x^{\alpha'}/\partial x^\beta, \qquad L^\beta{}_{\alpha'} = \partial x^\beta/\partial x^{\alpha'}; \tag{8.9}$$

Components of Levi-Civita tensor

and (5) the Levi-Civita tensor $\boldsymbol{\varepsilon}$, like the metric tensor, has components that depend on how nonorthonormal the basis vectors are (see exercise 8.3): if $\boldsymbol{e}_0$ points toward the future and $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are righthanded, then

$$\begin{aligned} \varepsilon_{\alpha\beta\gamma\delta} &= (-g)^{1/2}[\alpha\beta\gamma\delta], \\ \varepsilon^{\alpha\beta\gamma\delta} &= g^{-1}\varepsilon_{\alpha\beta\gamma\delta} = -(-g)^{-1/2}[\alpha\beta\gamma\delta], \end{aligned} \tag{8.10a}$$

where $[\alpha\beta\gamma\delta]$ is the completely antisymmetric symbol

$$[\alpha\beta\gamma\delta] \equiv \begin{cases} +1 \text{ if } \alpha\beta\gamma\delta \text{ is an even permutation of } 0123, \\ -1 \text{ if } \alpha\beta\gamma\delta \text{ is an odd permutation of } 0123, \\ 0 \text{ if } \alpha\beta\gamma\delta \text{ are not all different,} \end{cases} \tag{8.10b}$$

and where g is the determinant of the matrix $||g_{\alpha\beta}||$

$$g \equiv \det||g_{\alpha\beta}|| = \det||\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta||. \tag{8.11}$$

Read Box 8.4 for full discussion and proofs; work exercise 8.1 below for fuller understanding and mastery.

Several dangers are glossed over in this discussion. In flat spacetime one often does not bother to say where a vector, 1-form, or tensor is located. One freely moves geometric objects from event to event without even thinking. Of course, the unwritten rule of transport is: hold all lengths and directions fixed while moving; i.e., hold all Lorentz-frame components fixed; i.e., "parallel-transport" the object. But in

Box 8.4 TENSOR ALGEBRA AT A FIXED EVENT IN AN *ARBITRARY* BASIS

A. Bases

Tangent-vector basis: Pick $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ at $\mathscr{P}_0$ arbitrarily—but insist they be linearly independent.

"Dual basis" for 1-forms: The basis $\{\boldsymbol{e}_\alpha\}$ determines a 1-form basis $\{\boldsymbol{\omega}^\alpha\}$ (its "dual basis") by

$$\langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta \rangle = \delta^\alpha{}_\beta$$

[see equation (2.19)].

Geometric interpretation (Figure 9.2): $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, $\boldsymbol{e}_0$ lie parallel to surfaces of $\boldsymbol{\omega}^1$; $\boldsymbol{e}_1$ pierces precisely one surface of $\boldsymbol{\omega}^1$.

Function interpretation: $\langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta \rangle = \delta^\alpha{}_\beta$ determines the value of $\boldsymbol{\omega}^\alpha$ on any vector $\boldsymbol{u} = u^\beta \boldsymbol{e}_\beta$ (number of "bongs of bell" as $\boldsymbol{u}$ pierces $\boldsymbol{\omega}^\alpha$):

$$\langle \boldsymbol{\omega}^\alpha, \boldsymbol{u}\rangle = \langle \boldsymbol{\omega}^\alpha, u^\beta \boldsymbol{e}_\beta\rangle = u^\beta \langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta\rangle = u^\beta \delta^\alpha{}_\beta = u^\alpha.$$

Special case: *coordinate bases.* Choose an arbitrary coordinate system $\{x^\alpha(\mathscr{P})\}$. At $\mathscr{P}_0$ choose $\boldsymbol{e}_\alpha = \partial\mathscr{P}/\partial x^\alpha$ as basis vectors. Then the dual basis is $\boldsymbol{\omega}^\alpha = \boldsymbol{d}x^\alpha$. *Proof:* the general coordinate-free relation $\langle \boldsymbol{d}f, \boldsymbol{v}\rangle = \partial_{\boldsymbol{v}} f$ [equation (2.17)], with $f = x^\alpha$ and $\boldsymbol{v} = \partial\mathscr{P}/\partial x^\beta$, reads

$$\langle \boldsymbol{d}x^\alpha, \partial\mathscr{P}/\partial x^\beta\rangle = (\partial/\partial x^\beta)x^\alpha = \delta^\alpha{}_\beta.$$

B. Algebra of Tangent Vectors and 1-Forms

The Lorentz-frame discussion of equations (2.19) to (2.22) is completely unchanged when one switches to an arbitrary basis. Its conclusions:

expansion, $\boldsymbol{u} = \boldsymbol{e}_\alpha u^\alpha$, $\boldsymbol{\sigma} = \sigma_\alpha \boldsymbol{\omega}^\alpha$;

calculation of components, $u^\alpha = \langle \boldsymbol{\omega}^\alpha, \boldsymbol{u}\rangle$, $\sigma_\alpha = \langle \boldsymbol{\sigma}, \boldsymbol{e}_\alpha\rangle$;

value of form on vector, $\langle \boldsymbol{\sigma}, \boldsymbol{u}\rangle = \sigma_\alpha u^\alpha$.

Application to gradients of functions:

expansion, $\boldsymbol{d}f = f_{,\alpha}\boldsymbol{\omega}^\alpha$ [defines $f_{,\alpha}$];

calculation of components, $f_{,\alpha} = \langle \boldsymbol{d}f, \boldsymbol{e}_\alpha\rangle = \partial_{\boldsymbol{e}_\alpha} f$ [see equation (2.17)].

Raising and lowering of indices is accomplished with $g_{\alpha\beta}$ and $g^{\alpha\beta}$ [equations (8.5) and (8.6)]. Proof:

$\tilde{\boldsymbol{u}}$, the 1-form corresponding to $\boldsymbol{u}$, is defined by $\langle \tilde{\boldsymbol{u}}, \boldsymbol{v}\rangle = \boldsymbol{u}\cdot\boldsymbol{v}$ for all $\boldsymbol{v}$;

thus, $u_\alpha \equiv \langle \tilde{\boldsymbol{u}}, \boldsymbol{e}_\alpha\rangle = \boldsymbol{u}\cdot\boldsymbol{e}_\alpha = u^\beta \boldsymbol{e}_\beta\cdot\boldsymbol{e}_\alpha = u^\beta g_{\beta\alpha}$;

inverting this equation yields $u^\beta = g^{\beta\alpha}u_\alpha$.

C. Change of Basis

The discussion of Lorentz transformations in equations (2.39) to (2.43) is applicable to general changes of basis if one replaces $||\Lambda^{\alpha'}{}_\beta||$ by an arbitrary but nonsingular matrix $||L^{\alpha'}{}_\beta||$ [equations (8.7), (8.8)]. Conclusions:

$$\begin{aligned}
\boldsymbol{e}_{\alpha'} &= \boldsymbol{e}_\beta L^\beta{}_{\alpha'}, & \boldsymbol{e}_\beta &= \boldsymbol{e}_{\alpha'} L^{\alpha'}{}_\beta;\\
\boldsymbol{\omega}^{\alpha'} &= L^{\alpha'}{}_\beta \boldsymbol{\omega}^\beta, & \boldsymbol{\omega}^\beta &= L^\beta{}_{\alpha'}\boldsymbol{\omega}^{\alpha'};\\
v^{\alpha'} &= L^{\alpha'}{}_\beta v^\beta, & v^\beta &= L^\beta{}_{\alpha'} v^{\alpha'};\\
\sigma_{\alpha'} &= \sigma_\beta L^\beta{}_{\alpha'}, & \sigma_\beta &= \sigma_{\alpha'} L^{\alpha'}{}_\beta.
\end{aligned}$$

When both bases are coordinate bases, then $L^\beta{}_{\alpha'} = \partial x^\beta/\partial x^{\alpha'}$, $L^{\alpha'}{}_\beta = \partial x^{\alpha'}/\partial x^\beta$. *Proof:*

$$\boldsymbol{e}_{\alpha'} = \frac{\partial}{\partial x^{\alpha'}} = \frac{\partial x^\beta}{\partial x^{\alpha'}}\frac{\partial}{\partial x^\beta} = \frac{\partial x^\beta}{\partial x^{\alpha'}}\boldsymbol{e}_\beta; \qquad \text{similarly } \boldsymbol{e}_\beta = \frac{\partial x^{\alpha'}}{\partial x^\beta}\boldsymbol{e}_{\alpha'}.$$

Box 8.4 ***(continued)***

D. Algebra of Tensors

The discussions of tensor algebra given in §3.2 [equations (3.8) to (3.22)] and in §3.5 (excluding gradient and divergence) are unchanged, except that

$$\eta_{\alpha\beta} \longrightarrow g_{\alpha\beta} \qquad \eta^{\alpha\beta} \longrightarrow g^{\alpha\beta}, \qquad \Lambda^{\alpha'}{}_{\beta} \longrightarrow L^{\alpha'}{}_{\beta}, \qquad \Lambda^{\beta}{}_{\alpha'} \longrightarrow L^{\beta}{}_{\alpha'};$$

and the components of the Levi-Civita tensor are changed from (3.50) to (8.10) [see exercise 8.3[.

Chief conclusions:

expansion, $\boldsymbol{S} = S^{\alpha}{}_{\beta\gamma}\boldsymbol{e}_{\alpha} \otimes \boldsymbol{\omega}^{\beta} \otimes \boldsymbol{\omega}^{\gamma}$;
components, $S^{\alpha}{}_{\beta\gamma} = \boldsymbol{S}(\boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta}, \boldsymbol{e}_{\gamma})$;
raising and lowering indices, $S_{\mu\beta}{}^{\nu} = g_{\mu\alpha}g^{\nu\gamma}S^{\alpha}{}_{\beta\gamma}$;
change of basis, $S^{\lambda'}{}_{\mu'\nu'} = L^{\lambda'}{}_{\alpha}L^{\beta}{}_{\mu'}L^{\gamma}{}_{\nu'}S^{\alpha}{}_{\beta\gamma}$;
machine operation, $\boldsymbol{S}(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{v}) = S^{\alpha}{}_{\beta\gamma}\sigma_{\alpha}u^{\beta}v^{\gamma}$;
tensor product, $\boldsymbol{T} = \boldsymbol{u} \otimes \boldsymbol{v} \Longleftrightarrow T^{\alpha\beta} = u^{\alpha}v^{\beta}$;
contraction, "$\boldsymbol{M}$ = contraction of $\boldsymbol{R}$ on slots 1 and 3" $\Longleftrightarrow M_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$;
wedge product, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ has components $\alpha^{\mu}\beta^{\nu} - \beta^{\mu}\alpha^{\nu}$;
Dual, ${}^*J_{\alpha\beta\gamma} = J^{\mu}\varepsilon_{\mu\alpha\beta\gamma}$, ${}^*F_{\alpha\beta} = \frac{1}{2}F^{\mu\nu}\varepsilon_{\mu\nu\alpha\beta}$, ${}^*B_{\alpha} = \frac{1}{6}B^{\lambda\mu\nu}\varepsilon_{\lambda\mu\nu\alpha}$.

E. Commutators (exercise 8.2; §9.6; Box 9.2)

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are tangent vector fields, one takes the view that $\boldsymbol{u} = \partial_{\boldsymbol{u}}$ and $\boldsymbol{v} = \partial_{\boldsymbol{v}}$, and one defines

$$[\boldsymbol{u}, \boldsymbol{v}] \equiv [\partial_{\boldsymbol{u}}, \partial_{\boldsymbol{v}}] \equiv \partial_{\boldsymbol{u}}\,\partial_{\boldsymbol{v}} - \partial_{\boldsymbol{v}}\,\partial_{\boldsymbol{u}}.$$

This commutator is itself a tangent vector field.
Components in a coordinate basis:

$$[\boldsymbol{u}, \boldsymbol{v}] = (u^{\beta}v^{\alpha}{}_{,\beta} - v^{\beta}u^{\alpha}{}_{,\beta})\underbrace{(\partial/\partial x^{\alpha})}_{[=\boldsymbol{e}_{\alpha}]}.$$

Commutation coefficients of a basis:

$$[\boldsymbol{e}_{\alpha}, \boldsymbol{e}_{\beta}] \equiv c_{\alpha\beta}{}^{\gamma}\boldsymbol{e}_{\gamma}, \qquad c_{\alpha\beta\mu} \equiv c_{\alpha\beta}{}^{\gamma}g_{\gamma\mu}.$$

Coordinate basis ("holonomic") $c_{\alpha\beta}{}^{\gamma} = 0$;
Noncoordinate basis ("anholonomic") some $c_{\alpha\beta}{}^{\gamma} \neq 0$ (see exercise 9.9).

curved spacetime there is no global Lorentz coordinate system in which to hold components fixed; and objects initially parallel, after "parallel transport" along different curves cease to be parallel ("geodesic deviation"; Earth's meridians, parallel at equator, cross at north and south poles). Thus, in curved spacetime one must not blithely move a geometric object from point to point, without carefully specifying how it is to be moved and by what route. Each local geometric object has its own official place of residence (event $\mathscr{P}_0$); it can interact with other objects residing there (tensor algebra); but it cannot interact with any object at another event $\mathscr{Q}$, until it has been carefully transported from $\mathscr{P}_0$ to $\mathscr{Q}$.

Vectors and tensors must not be moved blithely from point to point

This line of reasoning, pursued further, leads one to speak of the "*tangent space*" at each event, in which that event's vectors (arrows) and 1-forms (families of surfaces) lie, and in which its tensors (linear machines) operate. One even draws heuristic pictures of the tangent space, as in Figure 9.1 (p. 231).

Tangent space defined

Another danger in curved spacetime is the temptation to regard vectors as arrows linking two events ("point for head and point for tail")—i.e., to regard the tangent space of Figure 9.1 as lying in spacetime itself. This practice can be useful for heuristic purposes, but it is incompatible with complete mathematical precision. (How is the tangent space to be molded into a warped surface?) Four definitions of a vector were given in Figure 2.1 (page 49): three definitions relying on "point for head and point for tail"; one, "$d\mathscr{P}/d\lambda$", purely local. Only the local definition is wholly viable in curved spacetime, and even it can be improved upon, in the eyes of mathematicians, as follows.

Definitions of vector in curved spacetime:

(1) as $d\mathscr{P}/d\lambda$

There is a one-to-one correspondence (complete "isomorphism") between vectors $\boldsymbol{u}$ and directional derivative operators $\partial_{\boldsymbol{u}}$. The concept of vector is a bit fuzzy, but "directional derivative" is perfectly well-defined. To get rid of all fuzziness, exploit the isomorphism to the full: *define* the tangent vector $\boldsymbol{u}$ to be equal to the corresponding directional derivative

(2) as directional derivative

$$\boldsymbol{u} \equiv \partial_{\boldsymbol{u}}. \tag{8.12}$$

(This practice, unfamiliar as it may be to a physicist at first, has mathematical power; so this book will use it frequently. For a fuller discussion, see §9.2.)

EXERCISES

Exercise 8.1. PRACTICE WITH TENSOR ALGEBRA

Let t, x, y, z be Lorentz coordinates in flat spacetime, and let

$$r = (x^2 + y^2 + z^2)^{1/2}, \qquad \theta = \cos^{-1}(z/r), \qquad \phi = \tan^{-1}(y/x)$$

be the corresponding spherical coordinates. Then

$$\boldsymbol{e}_0 = \partial\mathscr{P}/\partial t, \qquad \boldsymbol{e}_r = \partial\mathscr{P}/\partial r, \qquad \boldsymbol{e}_\theta = \partial\mathscr{P}/\partial\theta, \qquad \boldsymbol{e}_\phi = \partial\mathscr{P}/\partial\phi$$

is a coordinate basis, and

$$\boldsymbol{e}_{\hat{0}} = \frac{\partial\mathscr{P}}{\partial t}, \qquad \boldsymbol{e}_{\hat{r}} = \frac{\partial\mathscr{P}}{\partial r}, \qquad \boldsymbol{e}_{\hat{\theta}} = \frac{1}{r}\frac{\partial\mathscr{P}}{\partial\theta}, \qquad \boldsymbol{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial\mathscr{P}}{\partial\phi}$$

is a noncoordinate basis.

(a) Draw a picture of $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\phi$, $\boldsymbol{e}_{\hat\theta}$, and $\boldsymbol{e}_{\hat\phi}$ at several different points on a sphere of constant t, r. [Answer for $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\phi$ should resemble Figure 9.1.]

(b) What are the 1-form bases $\{\boldsymbol{\omega}^\alpha\}$ and $\{\boldsymbol{\omega}^{\hat\alpha}\}$ dual to these tangent-vector bases? [*Answer:* $\boldsymbol{\omega}^0 = \boldsymbol{d}t$, $\boldsymbol{\omega}^r = \boldsymbol{d}r$, $\boldsymbol{\omega}^\theta = \boldsymbol{d}\theta$, $\boldsymbol{\omega}^\phi = \boldsymbol{d}\phi$; $\boldsymbol{\omega}^{\hat 0} = \boldsymbol{d}t$, $\boldsymbol{\omega}^{\hat r} = \boldsymbol{d}r$, $\boldsymbol{\omega}^{\hat\theta} = r\,\boldsymbol{d}\theta$, $\boldsymbol{\omega}^{\hat\phi} = r \sin\theta\, \boldsymbol{d}\phi$.]

(c) What is the transformation matrix linking the original Lorentz frame to the spherical coordinate frame $\{\boldsymbol{e}_\alpha\}$? [*Answer:* nonzero components are

$$L^t{}_0 = 1, \qquad L^z{}_r = \frac{\partial z}{\partial r} = \cos\theta, \qquad L^z{}_\theta = \frac{\partial z}{\partial \theta} = -r\sin\theta,$$

$$L^x{}_r = \sin\theta\cos\phi, \qquad L^x{}_\theta = r\cos\theta\cos\phi, \qquad L^x{}_\phi = -r\sin\theta\sin\phi,$$
$$L^y{}_r = \sin\theta\sin\phi, \qquad L^y{}_\theta = r\cos\theta\sin\phi, \qquad L^y{}_\phi = r\sin\theta\cos\phi.]$$

(d) Use this transformation matrix to calculate the metric components $g_{\alpha\beta}$ in the spherical coordinate basis, and invert the resulting matrix to get $g^{\alpha\beta}$. [*Answer:*

$$g_{00} = -1, \quad g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2\sin^2\theta, \quad \text{all other } g_{\alpha\beta} = 0.$$
$$g^{00} = -1, \quad g^{rr} = 1, \quad g^{\theta\theta} = r^{-2}, \quad g^{\phi\phi} = r^{-2}\sin^{-2}\theta, \quad \text{all other } g^{\alpha\beta} = 0.]$$

(e) Show that the noncoordinate basis $\{\boldsymbol{e}_{\hat\alpha}\}$ is orthonormal everywhere; i.e., that $g_{\hat\alpha\hat\beta} = \eta_{\alpha\beta}$; i.e. that

$$\boldsymbol{g} = -\boldsymbol{\omega}^{\hat 0}\otimes\boldsymbol{\omega}^{\hat 0} + \boldsymbol{\omega}^{\hat r}\otimes\boldsymbol{\omega}^{\hat r} + \boldsymbol{\omega}^{\hat \theta}\otimes\boldsymbol{\omega}^{\hat \theta} + \boldsymbol{\omega}^{\hat \phi}\otimes\boldsymbol{\omega}^{\hat \phi}.$$

(f) Write the gradient of a function f in terms of the spherical coordinate and noncoordinate bases. [*Answer:*

$$\begin{aligned}\boldsymbol{d}f &= \frac{\partial f}{\partial t}\boldsymbol{d}t + \frac{\partial f}{\partial r}\boldsymbol{d}r + \frac{\partial f}{\partial \theta}\boldsymbol{d}\theta + \frac{\partial f}{\partial \phi}\boldsymbol{d}\phi \\ &= \frac{\partial f}{\partial t}\boldsymbol{\omega}^{\hat 0} + \frac{\partial f}{\partial r}\boldsymbol{\omega}^{\hat r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{\omega}^{\hat \theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{\omega}^{\hat \phi}.]\end{aligned}$$

(g) What are the components of the Levi-Civita tensor in the spherical coordinate and noncoordinate bases? [*Answer for coordinate basis:*

$$\varepsilon_{0r\theta\phi} = -\varepsilon_{r0\theta\phi} = +\varepsilon_{r\theta 0\phi} = \cdots = r^2\sin\theta,$$
$$\varepsilon^{0r\theta\phi} = -\varepsilon^{r0\theta\phi} = +\varepsilon^{r\theta\phi 0} = \cdots = -r^{-2}\sin^{-1}\theta.]$$

Exercise 8.2. COMMUTATORS

Take the mathematician's viewpoint that tangent vectors and directional derivatives are the same thing, $\boldsymbol{u} \equiv \partial_{\boldsymbol{u}}$. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vector fields, and define their commutator in the manner familiar from quantum mechanics

$$[\boldsymbol{u}, \boldsymbol{v}] \equiv [\partial_{\boldsymbol{u}}, \partial_{\boldsymbol{v}}] \equiv \partial_{\boldsymbol{u}}\partial_{\boldsymbol{v}} - \partial_{\boldsymbol{v}}\partial_{\boldsymbol{u}}. \tag{8.13a}$$

(a) Derive the following expression for $[\boldsymbol{u}, \boldsymbol{v}]$, valid in any *coordinate* basis,

$$[\boldsymbol{u}, \boldsymbol{v}] = (u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta})\boldsymbol{e}_\alpha. \tag{8.13b}$$

Thus, despite the fact that it looks like a second-order differential operator, $[\boldsymbol{u}, \boldsymbol{v}]$ is actually of first order—i.e., it is a tangent vector.

(b) For any basis $\{\boldsymbol{e}_\alpha\}$, one defines the *"commutation coefficients"* $c_{\beta\gamma}{}^\alpha$ and $c_{\beta\gamma\alpha}$ by

$$[\boldsymbol{e}_\beta, \boldsymbol{e}_\gamma] \equiv c_{\beta\gamma}{}^\alpha\boldsymbol{e}_\alpha; \qquad c_{\beta\gamma\alpha} \equiv g_{\alpha\mu}c_{\beta\gamma}{}^\mu. \tag{8.14}$$

Show that $c_{\beta\gamma}{}^\alpha = c_{\beta\gamma\alpha} = 0$ for any coordinate basis.

(c) Calculate $c_{\hat{\beta}\hat{\gamma}}{}^{\hat{\alpha}}$ for the spherical noncoordinate basis of exercise 8.1. [*Answer:* All vanish except

$$c_{\hat{r}\hat{\theta}}{}^{\hat{\theta}} = -c_{\hat{\theta}\hat{r}}{}^{\hat{\theta}} = -1/r,$$
$$c_{\hat{r}\hat{\phi}}{}^{\hat{\phi}} = -c_{\hat{\phi}\hat{r}}{}^{\hat{\phi}} = -1/r,$$
$$c_{\hat{\theta}\hat{\phi}}{}^{\hat{\phi}} = -c_{\hat{\phi}\hat{\theta}}{}^{\hat{\phi}} = -\cot\theta/r.]$$

Exercise 8.3. COMPONENTS OF LEVI-CIVITA TENSOR IN NONORTHONORMAL FRAME

(a) Show that expressions (8.10) are the components of $\boldsymbol{\varepsilon}$ in an arbitrary basis, with $\boldsymbol{e}_0$ pointing toward the future and $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ right-handed. [Hints: (1) Review the discussion of $\boldsymbol{\varepsilon}$ in Lorentz frames, given in exercise 3.13. (2) Calculate $\varepsilon_{\alpha\beta\gamma\delta}$ and $\varepsilon^{\alpha\beta\gamma\delta}$ by transforming from a local Lorentz frame $\{\boldsymbol{e}_{\hat{\mu}}\}$, e.g.,

$$\varepsilon_{\alpha\beta\gamma\delta} = L^{\hat{\mu}}{}_{\alpha} L^{\hat{\nu}}{}_{\beta} L^{\hat{\lambda}}{}_{\gamma} L^{\hat{\rho}}{}_{\delta} \varepsilon_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\rho}}.$$

(3) Show that these expressions reduce to

$$\varepsilon_{\alpha\beta\gamma\delta} = \det\|L^{\hat{\mu}}{}_{\nu}\| \varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}, \qquad \varepsilon^{\alpha\beta\gamma\delta} = \det\|L^{\nu}{}_{\hat{\mu}}\| \varepsilon^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}.$$

(4) Show, from the transformation law for the metric components, that

$$(\det\|L^{\nu}{}_{\hat{\mu}}\|)^2 \det\|g_{\alpha\beta}\| = -1.$$

(5) Combine these results to obtain expressions (8.10).]

(b) Show that the components of the permutation tensors [defined by equations (3.50h)–(3.50j)] have the same values [equations (3.50k)–(3.50m)] in arbitrary frames as in Lorentz frames.

Additional exercises on tensor algebra: exercises 9.3 and 9.4 (page 234).

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§8.5. PARALLEL TRANSPORT, COVARIANT DERIVATIVE, CONNECTION COEFFICIENTS, GEODESICS

The vehicle that carries one from classical mechanics to quantum mechanics is the correspondence principle. Similarly, the vehicle between flat spacetime and curved spacetime is the equivalence principle: "The laws of physics are the same in any local Lorentz frame of curved spacetime as in a global Lorentz frame of flat spacetime." But to apply the equivalence principle, one must first have a mathematical representation of a local Lorentz frame. The obvious choice is this: *A local Lorentz frame at a given event $\mathscr{P}_o$ is the closest thing there is to a global Lorentz frame at that event; i.e., it is a coordinate system in which*

Equivalence principle as vehicle between flat spacetime and curved

Local Lorentz frame: mathematical representation

$$g_{\mu\nu}(\mathscr{P}_o) = \eta_{\mu\nu}, \tag{8.15a}$$

and in which $g_{\mu\nu}$ holds as tightly as possible to $\eta_{\mu\nu}$ in the neighborhood of $\mathscr{P}_o$:

$$g_{\mu\nu,\alpha}(\mathscr{P}_o) = 0. \tag{8.15b}$$

More tightly than this it cannot hold in general [$g_{\mu\nu,\alpha\beta}(\mathscr{P}_o)$ cannot be set to zero]; spacetime curvature forces it to change. [Combine §11.5 with equations (8.24) and (8.44).]

Parallel transport defined

An observer in a local Lorentz frame in curved spacetime can compare vectors and tensors at neighboring events, just as he would in flat spacetime. But to make the comparison, he must parallel-transport them to a common event. For him the act of parallel transport is simple: he keeps all Lorentz-frame components fixed, just as if he were in flat spacetime. But for a man without a local Lorentz frame—perhaps with no coordinate system or basis vectors at all—parallel transport is less trivial. He must either ask his Lorentz-based friend the result, or he must use a more sophisticated technique. One technique he can use—a "Schild's ladder" construction that requires no coordinates or basis vectors whatsoever—is described in §10.2 and Box 10.2. But the Track-1 reader need not master Schild's ladder. He can always ask a local Lorentz observer what the result of any given parallel transport is, or he can use general formulas worked out below.

Comparison by parallel transport is the foundation on which rests the gradient of a tensor field, $\boldsymbol{\nabla T}$. No mention of parallel transport was made in §3.5, where the gradient was first defined, but parallel transport occurred implicitly: one defined $\boldsymbol{\nabla T}$ in such a way that its components were $T^{\alpha}{}_{\beta,\gamma} = \partial T^{\alpha}{}_{\beta}/\partial x^{\gamma}$ [for $\boldsymbol{T}$ a $\binom{1}{1}$ tensor]; i.e., one asked $\boldsymbol{\nabla T}$ to measure how much the Lorentz-frame components of $\boldsymbol{T}$ change from point to point. But "no change in Lorentz components" would have meant "parallel transport," so one was implicitly asking for the change in $\boldsymbol{T}$ relative to what $\boldsymbol{T}$ would have been after pure parallel transport.

Covariant derivative defined

To codify in abstract notation this concept of differentiation, proceed as follows. First define the "covariant derivative" $\boldsymbol{\nabla_u T}$ of $\boldsymbol{T}$ along a curve $\mathscr{P}(\lambda)$, whose tangent vector is $\boldsymbol{u} = d\mathscr{P}/d\lambda$:

$$(\boldsymbol{\nabla_u T})_{\text{at } \mathscr{P}(0)} = \lim_{\varepsilon\to 0} \left\{ \frac{\boldsymbol{T}[\mathscr{P}(\varepsilon)]_{\text{parallel-transported to } \mathscr{P}(0)} - \boldsymbol{T}[\mathscr{P}(0)]}{\varepsilon} \right\}. \tag{8.16}$$

Gradient defined

(See Figure 8.2 for the special case where $\boldsymbol{T}$ is a vector field $\boldsymbol{v}$.) Then define $\boldsymbol{\nabla T}$ to be the linear machine, that gives $\boldsymbol{\nabla_u T}$ when $\boldsymbol{u}$ is inserted into its last slot:

$$\boldsymbol{\nabla T}(\ldots,\ldots,\boldsymbol{u}) \equiv \boldsymbol{\nabla_u T}. \tag{8.17}$$

The result is the same animal ("gradient") as was defined in §3.5 (for proof see exercise 8.8). But this alternative definition makes clear the relationship to parallel transport, including the fact that

$$\boldsymbol{\nabla_u T} = 0 \Longleftrightarrow \boldsymbol{T} \text{ is parallel-transported along } \boldsymbol{u} = d\mathscr{P}/d\lambda. \tag{8.18}$$

In a local Lorentz frame, the components of $\boldsymbol{\nabla T}$ are directional derivatives of the components of $\boldsymbol{T}$: $T^{\beta}{}_{\alpha,\gamma}$. Not so in a general basis. If $\{\boldsymbol{e}_\beta(\mathscr{P})\}$ is a basis that varies arbitrarily but smoothly from point to point, and $\{\boldsymbol{\omega}^\alpha(\mathscr{P})\}$ is its dual basis, then $\boldsymbol{\nabla T} = \boldsymbol{\nabla}(T^{\beta}{}_{\alpha}\boldsymbol{e}_\beta \otimes \boldsymbol{\omega}^\alpha)$ will contain contributions from $\boldsymbol{\nabla e}_\beta$ and $\boldsymbol{\nabla \omega}^\alpha$, as well as from $\boldsymbol{\nabla} T^{\beta}{}_{\alpha} \equiv \boldsymbol{d}T^{\beta}{}_{\alpha} = T^{\beta}{}_{\alpha,\gamma}\boldsymbol{\omega}^\gamma$.

Connection coefficients defined

To quantify the contributions from $\boldsymbol{\nabla e}_\beta$ and $\boldsymbol{\nabla \omega}^\alpha$, i.e., to quantify the twisting, turning, expansion, and contraction of the basis vectors and 1-forms, one defines *"connection coefficients"*:

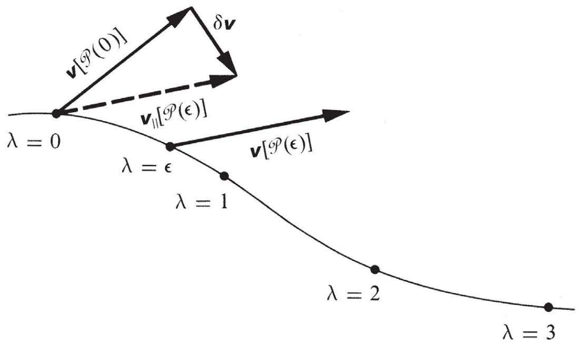


Figure 8.2.
Definition of the covariant derivative "$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$" of a vector field $\boldsymbol{v}$ along a curve $\mathscr{P}(\lambda)$, with tangent vector $\boldsymbol{u} \equiv d\mathscr{P}/d\lambda$: (1) choose a point $\mathscr{P}(0)$ on the curve, at which to evaluate $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$. (2) Choose a nearby point $\mathscr{P}(\varepsilon)$ on the curve. (3) Parallel-transport $\boldsymbol{v}[\mathscr{P}(\varepsilon)]$ along the curve back to $\mathscr{P}(0)$, getting the vector $\boldsymbol{v}_{\parallel}[\mathscr{P}(\varepsilon)]$. (4) Take the difference $\delta\boldsymbol{v} \equiv \boldsymbol{v}_{\parallel}[\mathscr{P}(\varepsilon)] - \boldsymbol{v}[\mathscr{P}(0)]$. (5) Then $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$ is defined by

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} \equiv \lim_{\varepsilon\to 0} \frac{\delta\boldsymbol{v}}{\varepsilon} = \lim_{\varepsilon\to 0} \left\{\frac{\boldsymbol{v}_{\parallel}[\mathscr{P}(\varepsilon)] - \boldsymbol{v}[\mathscr{P}(0)]}{\varepsilon}\right\}.$$

$$\Gamma^{\alpha}{}_{\beta\gamma} \equiv \langle \boldsymbol{\omega}^{\alpha}, \underbrace{\boldsymbol{\nabla}_{\gamma}}_{[\equiv \boldsymbol{\nabla}_{\boldsymbol{e}_\gamma}]} \boldsymbol{e}_{\beta} \rangle \qquad \begin{bmatrix}\text{Note reversal of } \beta \text{ and } \gamma \text{ to make the} \\ \text{differentiating index come last on } \Gamma\end{bmatrix} \tag{8.19a}$$

$$= \begin{pmatrix}\alpha \text{ component of change in } \boldsymbol{e}_{\beta}, \text{ relative} \\ \text{to parallel transport, along } \boldsymbol{e}_{\gamma}\end{pmatrix},$$

and one proves (exercise 8.12) that

$$\langle \boldsymbol{\nabla}_{\gamma}\boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta} \rangle = -\Gamma^{\alpha}{}_{\beta\gamma}. \tag{8.19b}$$

In terms of these coefficients and

$$T^{\beta}{}_{\alpha,\gamma} \equiv \boldsymbol{\nabla}_{\gamma} T^{\beta}{}_{\alpha} \equiv \partial_{\boldsymbol{e}_\gamma} T^{\beta}{}_{\alpha} \equiv \partial_{\gamma} T^{\beta}{}_{\alpha}, \tag{8.20}$$

the components of the gradient, denoted $T^{\beta}{}_{\alpha;\gamma}$, are

Components of gradient in arbitrary frame

$$T^{\beta}{}_{\alpha;\gamma} = T^{\beta}{}_{\alpha,\gamma} + \Gamma^{\beta}{}_{\mu\gamma} T^{\mu}{}_{\alpha} - \Gamma^{\mu}{}_{\alpha\gamma} T^{\beta}{}_{\mu} \tag{8.21}$$

(see exercise 8.13). If the basis at the event where $\boldsymbol{\nabla T}$ is calculated were a local Lorentz frame, the components of $\boldsymbol{\nabla T}$ would just be $T^{\beta}{}_{\alpha,\gamma}$. Because it is not, one must correct this "Lorentz-frame" value for the twisting, turning, expansion, and contraction of the basis vectors and 1-forms. The "ΓT" terms in equation (8.21) are the necessary corrections—one for each index of $\boldsymbol{T}$. The pattern of these correction terms is easy to remember: (1) "+" sign if index being corrected is up, "−" sign if it is down; (2) differentiation index (γ in above case) always at end of Γ; (3) index being corrected (β in first term, α in second) shifts from T onto Γ and gets replaced on T by a dummy summation index (μ).

Knowing the components (8.21) of the gradient, one can calculate the components of the covariant derivative $\nabla_{\boldsymbol{u}}\boldsymbol{T}$ by a simple contraction into u^γ [see equation (8.17)]:

$$\nabla_{\boldsymbol{u}}\boldsymbol{T} = (T^\beta{}_{\alpha;\gamma}u^\gamma)\boldsymbol{e}_\beta \otimes \boldsymbol{\omega}^\alpha. \tag{8.22}$$

Components of covariant derivative

When $\boldsymbol{u}$ is the tangent vector to a curve $\mathscr{P}(\lambda)$, $\boldsymbol{u} = d\mathscr{P}/d\lambda$, one uses the notation $DT^\beta{}_\alpha/d\lambda$ for the components of $\nabla_{\boldsymbol{u}}\boldsymbol{T}$:

$$\begin{aligned}\frac{DT^\beta{}_\alpha}{d\lambda} &\equiv T^\beta{}_{\alpha;\gamma}u^\gamma \underset{\uparrow}{=} T^\beta{}_{\alpha;\gamma}\frac{dx^\gamma}{d\lambda} \quad \text{[if basis is a coordinate basis so } u^\gamma = dx^\gamma/d\lambda\text{]}\\ &= (T^\beta{}_{\alpha,\gamma} + \text{“}\Gamma T\text{” corrections})\, dx^\gamma/d\lambda \\ &= \frac{dT^\beta{}_\alpha}{d\lambda} + (\Gamma^\beta{}_{\mu\gamma}T^\mu{}_\alpha - \Gamma^\mu{}_{\alpha\gamma}T^\beta{}_\mu)\frac{dx^\gamma}{d\lambda}.\end{aligned} \tag{8.23}$$

The “;” in $T^\beta{}_{\alpha;\gamma}$ reminds one to correct $T^\beta{}_{\alpha,\gamma}$ with “ΓT” terms; similarly, the “D” in $DT^\beta{}_\alpha/d\lambda$ reminds one to correct $dT^\beta{}_\alpha/d\lambda$ with “ΓT” terms.

Calculation of connection coefficients from metric and commutators

This is all well and good, but how does one find out the connection coefficients $\Gamma^\alpha{}_{\beta\gamma}$ for a given basis? The answer is derived in exercise 8.15. It says: (1) take the metric coefficients in the given basis; (2) calculate their directional derivatives along the basis directions

$$g_{\beta\gamma,\mu} \equiv \underset{\substack{\uparrow\\ [\equiv\partial_{\boldsymbol{e}_\mu}]}}{\partial_\mu} g_{\beta\gamma} \underset{\substack{\uparrow\\ \text{[if a coordinate basis, } \boldsymbol{e}_\mu = \partial\mathscr{P}/\partial x^\mu\text{, is being used]}}}{=} \partial g_{\beta\gamma}/\partial x^\mu; \tag{8.24a}$$

(3) calculate the commutation coefficients of the basis [equations (8.14) in general; $c_{\mu\beta\gamma} = 0$ in special case of coordinate basis]; (4) calculate the “covariant connection coefficients”

$$\Gamma_{\mu\beta\gamma} = \frac{1}{2}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu} + \underbrace{c_{\mu\beta\gamma} + c_{\mu\gamma\beta} - c_{\beta\gamma\mu}}_{\left[\substack{\text{these terms are 0 for}\\ \text{coordinate basis}}\right]}); \tag{8.24b}$$

(5) raise an index to get the connection coefficients:

$$\Gamma^\alpha{}_{\beta\gamma} = g^{\alpha\mu}\Gamma_{\mu\beta\gamma}. \tag{8.24c}$$

[*Note on terminology:* a coordinate basis always has $c_{\alpha\beta\gamma} = 0$, and is sometimes called *holonomic;* a noncoordinate basis always has some of its $c_{\alpha\beta\gamma}$ nonzero, and is sometimes called *anholonomic*. In the holonomic case, the connection coefficients are sometimes called *Christoffel symbols*.]

The component notation, with its semicolons, commas, D's, connection coefficients, etc., looks rather formidable at first. But it bears great computational power, one discovers as one proceeds deep into gravitation theory; and its rules of manipulation are simple enough to be learned easily. By contrast, the abstract notation ($\nabla \boldsymbol{T}$, $\nabla_{\boldsymbol{u}} \boldsymbol{T}$, etc.) is poorly suited to complex calculations; but it possesses great conceptual power.

This contrast shows clearly in the way the two notations handle the concept of *geodesic*. A geodesic of spacetime is a curve that is straight and uniformly parametrized, as measured in each local Lorentz frame along its way. If the geodesic is timelike, then it is a possible world line for a freely falling particle, and its uniformly ticking parameter λ (called "*affine parameter*") is a multiple of the particle's proper time, $\lambda = a\tau + b$. (Principle of equivalence: test particles move on straight lines in local Lorentz frames, and each particle's clock ticks at a uniform rate as measured by any Lorentz observer.) This definition of geodesic is readily translated into abstract, coordinate-free language: a geodesic is a curve $\mathscr{P}(\lambda)$ that parallel-transports its tangent vector $\boldsymbol{u} = d\mathscr{P}/d\lambda$ along itself—

Geodesic and affine parameter defined

$$\nabla_{\boldsymbol{u}} \boldsymbol{u} = 0. \tag{8.25}$$

(See Figure 10.1.) What could be simpler conceptually? But to compute the geodesic, given an initial event $\mathscr{P}_0$ and initial tangent vector $\boldsymbol{u}(\mathscr{P}_0)$ there, one must use the component formalism. Introduce a coordinate system $x^\alpha(\mathscr{P})$, in which $u^\alpha = dx^\alpha/d\lambda$, and write the component version of equation (8.25) as

$$0 = \frac{D(dx^\alpha/d\lambda)}{d\lambda} = \frac{d(dx^\alpha/d\lambda)}{d\lambda} + \left(\Gamma^\alpha{}_{\mu\gamma}\frac{dx^\mu}{d\lambda}\right)\frac{dx^\gamma}{d\lambda}$$

[see equation (8.23), with one less index on T]; i.e.,

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\mu\gamma}\frac{dx^\mu}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0. \tag{8.26}$$

Geodesic equation

This *geodesic equation* can be solved (in principle) for the coordinates of the geodesic, $x^\alpha(\lambda)$, when initial data [x^α and $dx^\alpha/d\lambda$ at $\lambda = \lambda_0$] have been specified.

The geodesics of the Earth's surface (great circles) are a foil against which one can visualize connection coefficients; see Figure 8.3.

The material of this section is presented more deeply and from a different viewpoint in Chapters 10 and 13. The Track-2 reader who plans to study those chapters is advised to ignore the following exercises. The Track-1 reader who intends to skip Chapters 9–15 will gain necessary experience with the component formalism by working exercises 8.4–8.7. Less important to him, but valuable nonetheless, are exercises 8.8–8.15, which develop the formalism of covariant derivatives and connection coefficients in a systematic manner. The most important results of these exercises will be summarized in Box 8.6 (pages 223 and 224).

EXERCISES

Exercise 8.4. PRACTICE IN WRITING COMPONENTS OF GRADIENT

Rewrite the following quantities in terms of ordinary derivatives ($f_{,\gamma} \equiv \partial_{\boldsymbol{e}_\gamma} f \equiv \nabla_\gamma f$) and "$\Gamma T$" correction terms: (a) $T_{;\gamma}$ where T is a function. (b) $T^\alpha{}_{;\gamma}$ where $\boldsymbol{T}$ is a vector. (c) $T_{\alpha;\gamma}$ where $\boldsymbol{T}$ is a 1-form. (d) $T^\alpha{}_{\beta\delta}{}^\varepsilon{}_{;\gamma}$. [*Answer:*

(a) $T_{;\gamma} = T_{,\gamma}$. (b) $T^\alpha{}_{;\gamma} = T^\alpha{}_{,\gamma} + \Gamma^\alpha{}_{\mu\gamma}T^\mu$. (c) $T_{\alpha;\gamma} = T_{\alpha,\gamma} - \Gamma^\mu{}_{\alpha\gamma}T_\mu$.

(d) $T^\alpha{}_{\beta\delta}{}^\varepsilon{}_{;\gamma} = T^\alpha{}_{\beta\delta}{}^\varepsilon{}_{,\gamma} + \Gamma^\alpha{}_{\mu\gamma}T^\mu{}_{\beta\delta}{}^\varepsilon - \Gamma^\mu{}_{\beta\gamma}T^\alpha{}_{\mu\delta}{}^\varepsilon - \Gamma^\mu{}_{\delta\gamma}T^\alpha{}_{\beta\mu}{}^\varepsilon + \Gamma^\varepsilon{}_{\mu\gamma}T^\alpha{}_{\beta\delta}{}^\mu$.]

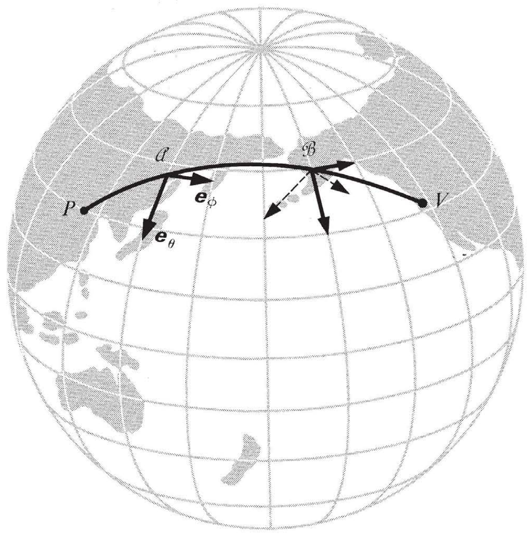


Figure 8.3.
The why of connection coefficients, schematically portrayed. The aviator pursuing his great circle route from Peking to Vancouver finds himself early going north, but later going south, although he is navigating the straightest route that is at all open to him (geodesic). The apparent change in direction indicates a turning, not in his route, but in the system of coordinates with respect to which his route is described. The vector $\boldsymbol{v}$ of his velocity (a vector defined not on spacetime but rather on the Earth's two-dimensional surface), carried forward by parallel transport from an earlier moment to a later moment, finds itself in agreement with the velocity that he is then pursuing; or, in the abstract language of coordinate-free differential geometry, the covariant derivative $\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{v}$ vanishes along the route ("equation of a geodesic"). Though $\boldsymbol{v}$ is in this sense constant, the individual pieces of which the navigator considers this vector to be built, $\boldsymbol{v} = v^\theta \boldsymbol{e}_\theta + v^\phi \boldsymbol{e}_\phi$, are not constant.

In the language of components, the quantities v^θ and v^ϕ are changing along the route at a rate that annuls the covariant derivative of $\boldsymbol{v}$; thus

$$\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{v} = \boldsymbol{a} = a^\phi \boldsymbol{e}_\phi + a^\theta \boldsymbol{e}_\theta = 0,$$

or

$$0 = a^\theta = \frac{dv^\theta}{dt} + \Gamma^\theta{}_{mn} v^m v^n,$$

$$0 = a^\phi = \frac{dv^\phi}{dt} + \Gamma^\phi{}_{mn} v^m v^n.$$

In this sense the connection coefficients $\Gamma^j{}_{mn}$ serve as "turning coefficients" to tell how fast to "turn" the components of a vector in order to keep that vector constant (against the turning influence of the base vectors).

Alternatively, the navigator can use an "automatic pilot system" which parallel-transports its own base vectors along the plane's route:

$$\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{e}_{\theta'} = \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{e}_{\phi'} = 0;$$

solid vectors at $\mathcal{A}$ become dotted vectors at $\mathcal{B}$. Then the components of $\boldsymbol{v}$ must be kept fixed to achieve a great-circle route,

$$\frac{dv^{\theta'}}{dt} = \frac{dv^{\phi'}}{dt} = 0;$$

and the turning coefficients are used to describe the turning of the lines of latitude and longitude relative to this parallel-transported basis:

$$\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{e}_\theta = \boldsymbol{e}_m \Gamma^m{}_{\theta n} v^n,$$
$$\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{e}_\phi = \boldsymbol{e}_m \Gamma^m{}_{\phi n} v^n.$$

The same turning coefficients enter into both viewpoints. The only difference is in how these coefficients are used.

Exercise 8.5. A SHEET OF PAPER IN POLAR COORDINATES

The two-dimensional metric for a flat sheet of paper in polar coordinates (r, θ) is $ds^2 = dr^2 + r^2\, d\phi^2$—or, in modern notation, $\boldsymbol{g} = \boldsymbol{d}r \otimes \boldsymbol{d}r + r^2\, \boldsymbol{d}\phi \otimes \boldsymbol{d}\phi$.

(a) Calculate the connection coefficients using equations (8.24). [*Answer:* $\Gamma^r{}_{\phi\phi} = -r$; $\Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = 1/r$; all others vanish.]

(b) Write down the geodesic equation in (r, ϕ) coordinates. [*Answer:* $d^2r/d\lambda^2 - r(d\phi/d\lambda)^2 = 0$; $d^2\phi/d\lambda^2 + (2/r)(dr/d\lambda)(d\phi/d\lambda) = 0$.]

(c) Solve this geodesic equation for $r(\lambda)$ and $\phi(\lambda)$, and show that the solution is a uniformly parametrized straight line ($x \equiv r\cos\phi = a\lambda + b$ for some a and b; $y \equiv r\sin\phi = j\lambda + k$ for some j and k).

(d) Verify that the noncoordinate basis $\boldsymbol{e}_{\hat r} \equiv \boldsymbol{e}_r = \partial\mathcal{P}/\partial r$, $\boldsymbol{e}_{\hat\phi} \equiv r^{-1}\boldsymbol{e}_\phi \quad r^{-1}\,\partial\mathcal{P}/\partial\phi$, $\boldsymbol{\omega}^{\hat r} = \boldsymbol{d}r$, $\boldsymbol{\omega}^{\hat\phi} = r\,\boldsymbol{d}\phi$ is orthonormal, and that $\langle \boldsymbol{\omega}^{\hat\alpha}, \boldsymbol{e}_{\hat\beta}\rangle = \delta^{\hat\alpha}{}_{\hat\beta}$. Then calculate the connection coefficients of this basis from a knowledge [part (a)] of the connection of the coordinate basis. [*Answer:*

$$
\begin{aligned}
\Gamma^{\hat\phi}{}_{\hat\phi\hat r} &= \langle \boldsymbol{\omega}^{\hat\phi}, \boldsymbol{\nabla}_{\hat r}\boldsymbol{e}_{\hat\phi}\rangle = \langle r\,\boldsymbol{d}\phi, \boldsymbol{\nabla}_r(r^{-1}\boldsymbol{e}_\phi)\rangle\\
&= r\langle \boldsymbol{d}\phi, (\boldsymbol{\nabla}_r r^{-1})\boldsymbol{e}_\phi + r^{-1}(\boldsymbol{\nabla}_r\boldsymbol{e}_\phi)\rangle = r\langle \boldsymbol{d}\phi, -r^{-2}\boldsymbol{e}_\phi\rangle + \langle \boldsymbol{d}\phi, \boldsymbol{\nabla}_r\boldsymbol{e}_\phi\rangle\\
&= -r^{-1} + \Gamma^\phi{}_{\phi r} = -r^{-1} + r^{-1} = 0;
\end{aligned}
$$

similarly, $\Gamma^{\hat\phi}{}_{\hat r\hat\phi} = +1/r$, $\Gamma^{\hat r}{}_{\hat\phi\hat\phi} = -1/r$; all others vanish.]

(e) Consider the Keplerian orbit of Figure 8.1 and §8.3 as a nongeodesic curve in the sun's two-dimensional, Euclidean, equatorial plane. In place of the old notation $d\boldsymbol{v}/dt$, $d\boldsymbol{e}_{\hat r}/dt$, etc., use the new notation $\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{v}$, $\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{e}_{\hat r}$, etc. Then $\boldsymbol{v} = d\mathcal{P}/dt$ is the tangent to the orbit, and $\boldsymbol{a} = \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{v}$ is the acceleration. Derive equations (8.4) for $a^{\hat r}$ and $a^{\hat\phi}$ using component manipulations and connection coefficients in the orthonormal basis.

Exercise 8.6. SPHERICAL COORDINATES IN FLAT SPACETIME

The spherical noncoordinate basis $\{\boldsymbol{e}_{\hat\alpha}\}$ of Exercise 8.1 was orthonormal, $g_{\hat\alpha\hat\beta} = \eta_{\hat\alpha\hat\beta}$, but had nonvanishing commutation coefficients [part (c) of Exercise 8.2].

(a) Calculate the connection coefficients for this basis, using equations (8.24). [*Answer:*

$$
\begin{aligned}
\Gamma^{\hat\theta}{}_{\hat r\hat\theta} &= \Gamma^{\hat\phi}{}_{\hat r\hat\phi} = -\Gamma^{\hat r}{}_{\hat\theta\hat\theta} = -\Gamma^{\hat r}{}_{\hat\phi\hat\phi} = 1/r;\\
\Gamma^{\hat\phi}{}_{\hat\theta\hat\phi} &= -\Gamma^{\hat\theta}{}_{\hat\phi\hat\phi} = \cot\theta/r;
\end{aligned}
$$

all others vanish.]

(b) Write down expressions for $\boldsymbol{\nabla}_{\hat\alpha}\boldsymbol{e}_{\hat\beta}$ in terms of $\boldsymbol{e}_{\hat\gamma}$, and verify the correctness of these expressions by drawing sketches of the basis vectors on a sphere of constant t and r. [*Answer:*

$$
\boldsymbol{\nabla}_{\hat\theta}\boldsymbol{e}_{\hat r} = r^{-1}\boldsymbol{e}_{\hat\theta}, \qquad \boldsymbol{\nabla}_{\hat\theta}\boldsymbol{e}_{\hat\theta} = -r^{-1}\boldsymbol{e}_{\hat r}, \qquad \boldsymbol{\nabla}_{\hat\phi}\boldsymbol{e}_{\hat r} = r^{-1}\boldsymbol{e}_{\hat\phi},
$$

$$
\boldsymbol{\nabla}_{\hat\phi}\boldsymbol{e}_{\hat\theta} = (\cot\theta/r)\boldsymbol{e}_{\hat\phi}, \qquad \boldsymbol{\nabla}_{\hat\phi}\boldsymbol{e}_{\hat\phi} = -r^{-1}\boldsymbol{e}_{\hat r} - (\cot\theta/r)\boldsymbol{e}_{\hat\theta}.
$$

All others vanish.]

(c) Calculate the divergence of a vector, $\boldsymbol{\nabla}\cdot\boldsymbol{A} = A^{\hat\alpha}{}_{;\hat\alpha}$, in this basis. [*Answer:*

$$
\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{A} &= A^{\hat t}{}_{,\hat t} + r^{-2}(r^2A^{\hat r})_{,\hat r} + (\sin\theta)^{-1}(\sin\theta A^{\hat\theta})_{,\hat\theta} + A^{\hat\phi}{}_{,\hat\phi}\\
&= \frac{\partial A^{\hat t}}{\partial t} + \frac{1}{r^2}\frac{\partial(r^2A^{\hat r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A^{\hat\theta})}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A^{\hat\phi}}{\partial\phi}.
\end{aligned}
$$

This answer should be familiar from flat-space vector analysis.]

Exercise 8.7. SYMMETRIES OF CONNECTION COEFFICIENTS

From equation (8.24b), the symmetry of the metric, and the antisymmetry ($c_{\beta\gamma\mu} = -c_{\gamma\beta\mu}$)

of the commutation coefficients, show that: $\Gamma_{\alpha[\beta\gamma]} = 0$ (last two indices are symmetric) in a coordinate basis; $\Gamma_{(\hat{\alpha}\hat{\beta})\hat{\gamma}} = 0$ (first two indices are antisymmetric) in a globally orthonormal basis, $g_{\hat{\alpha}\hat{\beta}} = \eta_{\alpha\beta}$.

SYSTEMATIC DERIVATION OF RESULTS IN §8.5

Exercise 8.8. NEW DEFINITION OF $\boldsymbol{\nabla T}$ COMPARED WITH OLD DEFINITION

The new definition of $\boldsymbol{\nabla T}$ is given by equations (8.16) and (8.17). Use the fact that parallel transport keeps local-Lorentz components fixed to derive, from (8.16), the Lorentz-frame equation $\boldsymbol{\nabla_u T} = T^{\beta}{}_{\alpha,\gamma} u^{\gamma} \boldsymbol{e}_{\beta} \otimes \boldsymbol{\omega}^{\alpha}$. From this and equation (8.17), infer that the Lorentz-frame components of $\boldsymbol{\nabla T}$ are $T^{\beta}{}_{\alpha,\gamma}$—which accords with the old definition of $\boldsymbol{\nabla T}$.

Exercise 8.9. CHAIN RULE FOR $\boldsymbol{\nabla_u T}$

(a) Use calculations in a local Lorentz frame to show that "$\boldsymbol{\nabla_u}$" obeys the standard chain rule for derivatives:

$$\boldsymbol{\nabla_u}(f\boldsymbol{A} \otimes \boldsymbol{B}) = (\boldsymbol{\nabla_u} f)\boldsymbol{A} \otimes \boldsymbol{B} + f(\boldsymbol{\nabla_u A}) \otimes \boldsymbol{B} + f\boldsymbol{A} \otimes (\boldsymbol{\nabla_u B}). \tag{8.27}$$

Here $\boldsymbol{A}$ and $\boldsymbol{B}$ are arbitrary vectors, 1-forms, or tensors; and f is an arbitrary function. [*Hint:* assume for concreteness that $\boldsymbol{A}$ is a $\binom{1}{1}$ tensor and $\boldsymbol{B}$ is a vector. Then this equation reads, in Lorentz-frame component notation,

$$(fA^{\alpha}{}_{\beta}B^{\gamma})_{,\delta}u^{\delta} = (f_{,\delta}u^{\delta})A^{\alpha}{}_{\beta}B^{\gamma} + f(A^{\alpha}{}_{\beta,\delta}u^{\delta})B^{\gamma} + fA^{\alpha}{}_{\beta}(B^{\gamma}{}_{,\delta}u^{\delta}).] \tag{8.27$'$}$$

(b) Rewrite equation (8.27) in component notation in an arbitrary basis. [*Answer:* same as (8.27′), except "," is replaced everywhere by ";". But *note* that $f_{;\delta}u^{\delta} = f_{,\delta}u^{\delta}$, because the function f "has no components to correct".]

Exercise 8.10. COVARIANT DERIVATIVE COMMUTES WITH CONTRACTION

(a) Let $\boldsymbol{S}$ be a $\binom{1}{2}$ tensor. Using components in a local Lorentz frame show that

$$\boldsymbol{\nabla_u}\,(\text{contraction on slots 1 and 2 of } \boldsymbol{S}) = (\text{contraction on slots 1 and 2 of } \boldsymbol{\nabla_u S}). \tag{8.28}$$

[Hint: in a local Lorentz frame this equation makes the trivial statement

$$\left(\sum_{\alpha} S^{\alpha}{}_{\alpha\beta}\right)_{,\gamma} u^{\gamma} = \sum_{\alpha} (S^{\alpha}{}_{\alpha\beta,\gamma}u^{\gamma}).]$$

Exercise 8.11. ALGEBRAIC PROPERTIES OF $\boldsymbol{\nabla}$

Use calculations in a local Lorentz frame to show that

$$\boldsymbol{\nabla}_{a\boldsymbol{u}+b\boldsymbol{v}}\boldsymbol{S} = a\,\boldsymbol{\nabla_u S} + b\,\boldsymbol{\nabla_v S} \tag{8.29}$$

for all tangent vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and numbers a, b; also that

$$\boldsymbol{\nabla_u}(\boldsymbol{S} + \boldsymbol{M}) = \boldsymbol{\nabla_u S} + \boldsymbol{\nabla_u M} \tag{8.30}$$

for any two tensor fields $\boldsymbol{S}$ and $\boldsymbol{M}$ of the same rank; also that

$$\nabla_{\boldsymbol{u}}\boldsymbol{w} - \nabla_{\boldsymbol{w}}\boldsymbol{u} = \underbrace{[\boldsymbol{u}, \boldsymbol{w}]}_{\left[\substack{\text{commutator of } \boldsymbol{u} \text{ and } \boldsymbol{w};\\ \text{discussed in exercise 8.2}}\right]}, \tag{8.31}$$

for any two vector fields $\boldsymbol{u}$ and $\boldsymbol{w}$.

Exercise 8.12. CONNECTION COEFFICIENTS FOR 1-FORM BASIS

Show that the same connection coefficients $\Gamma^{\alpha}{}_{\beta\gamma}$ that describe the changes in $\{\boldsymbol{e}_\beta\}$ from point to point [definition (8.19a)] also describe the changes in $\{\boldsymbol{\omega}^\alpha\}$, except for a change in sign [equation (8.19b)]. {*Answer:* (1) $\langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta \rangle = \delta^\alpha{}_\beta$ is a constant function (0 or 1, depending on whether $\alpha = \beta$). (2) Thus, $\nabla_\gamma \langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta \rangle = \partial_{\boldsymbol{e}_\gamma} \langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta \rangle = 0$. (3) But $\langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta \rangle$ is the contraction of $\boldsymbol{\omega}^\alpha \otimes \boldsymbol{e}_\beta$, so equation (8.28) implies $0 = \nabla_\gamma$(contraction of $\boldsymbol{\omega}^\alpha \otimes \boldsymbol{e}_\beta$) $=$ contraction of $[\nabla_\gamma(\boldsymbol{\omega}^\alpha \otimes \boldsymbol{e}_\beta)]$. (4) Apply the chain rule (8.27) to conclude $0 =$ contraction of $[(\nabla_\gamma \boldsymbol{\omega}^\alpha) \otimes \boldsymbol{e}_\beta + \boldsymbol{\omega}^\alpha \otimes (\nabla_\gamma \boldsymbol{e}_\beta)] = \langle \nabla_\gamma \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta \rangle + \langle \boldsymbol{\omega}^\alpha, \nabla_\gamma \boldsymbol{e}_\beta \rangle$. (5) Finally, use definition (8.19a) to arrive at the desired result, (8.19b).}

Exercise 8.13. "ΓT" CORRECTION TERMS FOR $T^{\beta}{}_{\alpha;\gamma}$

Derive equation (8.21) for $T^{\beta}{}_{\alpha;\gamma}$ in an arbitrary basis by first calculating the components of $\nabla_{\boldsymbol{u}}\boldsymbol{T}$ for arbitrary $\boldsymbol{u}$, and by then using equation (8.17) to infer the components of $\nabla\boldsymbol{T}$. [*Answer:* (1) Use the chain rule (8.27) to get

$$\begin{aligned}\nabla_{\boldsymbol{u}}\boldsymbol{T} &= \nabla_{\boldsymbol{u}}(T^{\beta}{}_{\alpha}\boldsymbol{e}_\beta \otimes \boldsymbol{\omega}^\alpha)\\ &= (\nabla_{\boldsymbol{u}}T^{\beta}{}_{\alpha})\boldsymbol{e}_\beta \otimes \boldsymbol{\omega}^\alpha + T^{\beta}{}_{\alpha}(\nabla_{\boldsymbol{u}}\boldsymbol{e}_\beta) \otimes \boldsymbol{\omega}^\alpha + T^{\beta}{}_{\alpha}\boldsymbol{e}_\beta \otimes (\nabla_{\boldsymbol{u}}\boldsymbol{\omega}^\alpha).\end{aligned}$$

(2) Write $\boldsymbol{u}$ in terms of its components, $\boldsymbol{u} = u^\gamma \boldsymbol{e}_\gamma$; use linearity of $\nabla_{\boldsymbol{u}}$ in $\boldsymbol{u}$ from equation (8.29), to get $\nabla_{\boldsymbol{u}} = u^\gamma \nabla_\gamma$; and use this in $\nabla_{\boldsymbol{u}}\boldsymbol{T}$:

$$\nabla_{\boldsymbol{u}}\boldsymbol{T} = u^\gamma \{T^{\beta}{}_{\alpha,\gamma}\boldsymbol{e}_\beta \otimes \boldsymbol{\omega}^\alpha + T^{\beta}{}_{\alpha}(\nabla_\gamma \boldsymbol{e}_\beta) \otimes \boldsymbol{\omega}^\alpha + T^{\beta}{}_{\alpha}\boldsymbol{e}_\beta \otimes (\nabla_\gamma \boldsymbol{\omega}^\alpha)\}.$$

(3) Use equations (8.19a,b), rewritten as

$$\nabla_\gamma \boldsymbol{e}_\beta = \Gamma^{\mu}{}_{\beta\gamma}\boldsymbol{e}_\mu, \qquad \nabla_\gamma \boldsymbol{\omega}^\alpha = -\Gamma^{\alpha}{}_{\mu\gamma}\boldsymbol{\omega}^\mu, \tag{8.32}$$

to put $\nabla_{\boldsymbol{u}}\boldsymbol{T}$ in the form

$$\nabla_{\boldsymbol{u}}\boldsymbol{T} = u^\gamma \{T^{\beta}{}_{\alpha,\gamma}\boldsymbol{e}_\beta \otimes \boldsymbol{\omega}^\alpha + \Gamma^{\mu}{}_{\beta\gamma}T^{\beta}{}_{\alpha}\boldsymbol{e}_\mu \otimes \boldsymbol{\omega}^\alpha - \Gamma^{\alpha}{}_{\mu\gamma}T^{\beta}{}_{\alpha}\boldsymbol{e}_\beta \otimes \boldsymbol{\omega}^\mu\}.$$

(4) Rename dummy indices so that the basis tensor $\boldsymbol{e}_\beta \otimes \boldsymbol{\omega}^\alpha$ can be factored out:

$$\nabla_{\boldsymbol{u}}\boldsymbol{T} = u^\gamma \{T^{\beta}{}_{\alpha,\gamma} + \Gamma^{\beta}{}_{\mu\gamma}T^{\mu}{}_{\alpha} - \Gamma^{\mu}{}_{\alpha\gamma}T^{\beta}{}_{\mu}\}\boldsymbol{e}_\beta \otimes \boldsymbol{\omega}^\alpha.$$

(5) By comparison with

$$\nabla_{\boldsymbol{u}}\boldsymbol{T} = \nabla\boldsymbol{T}(\ldots, \ldots, \boldsymbol{u}) = (T^{\beta}{}_{\alpha;\gamma}u^\gamma)\boldsymbol{e}_\beta \otimes \boldsymbol{\omega}^\alpha,$$

read off the value of $T^{\beta}{}_{\alpha;\gamma}$.]

Exercise 8.14. METRIC IS COVARIANTLY CONSTANT

Show on physical grounds (using properties of local Lorentz frames) that

$$\nabla\boldsymbol{g} = 0 \tag{8.33}$$

or, equivalently, that $\nabla_{\boldsymbol{u}}\boldsymbol{g} = 0$ for any vector $\boldsymbol{u}$. Then deduce as a mathematical consequence the obviously desirable product rule

$$\nabla_{\boldsymbol{u}}(\boldsymbol{A}\cdot\boldsymbol{B}) = (\nabla_{\boldsymbol{u}}\boldsymbol{A})\cdot\boldsymbol{B} + \boldsymbol{A}\cdot(\nabla_{\boldsymbol{u}}\boldsymbol{B}).$$

[Answer: (1) As discussed following equation (8.18), the components of $\nabla\boldsymbol{g}$ in a local Lorentz frame are $g_{\mu\nu,\alpha}$. Just use $\boldsymbol{g}$ for $\boldsymbol{T}$ in that discussion. But these components all vanish by equation (8.15b). Therefore equation (8.33) holds in this frame, and—as a tensor equation—in all frames. (2) The product rule is also a tensor equation, true immediately via components in a local Lorentz frame. (3) Prove the product rule also the hard way, to see where equation (8.33) enters. Use the chain rule of exercise 8.9 to write

$$\begin{aligned}\nabla_{\boldsymbol{u}}(\boldsymbol{g}\otimes\boldsymbol{A}\otimes\boldsymbol{B}) = (\nabla_{\boldsymbol{u}}\boldsymbol{g})\otimes\boldsymbol{A}\otimes\boldsymbol{B} &+ \boldsymbol{g}\otimes(\nabla_{\boldsymbol{u}}\boldsymbol{A})\otimes\boldsymbol{B}\\ &+ \boldsymbol{g}\otimes\boldsymbol{A}\otimes(\nabla_{\boldsymbol{u}}\boldsymbol{B}).\end{aligned}$$

Use equation (8.33) to drop one term, then contract, forming

$$\boldsymbol{A}\cdot\boldsymbol{B} = \text{contraction}\,(\boldsymbol{g}\otimes\boldsymbol{A}\otimes\boldsymbol{B})$$

and the other inner products. Exercise 8.10 is used to justify commuting the contraction with $\nabla_{\boldsymbol{u}}$ on the lefthand side.]

Exercise 8.15. CONNECTION COEFFICIENTS IN TERMS OF METRIC

Use the fact that the metric is covariantly constant [equation (8.33)] to derive equation (8.24b) for the connection coefficients. Treat equation (8.24c) as a definition of $\Gamma_{\mu\beta\gamma}$ in terms of $\Gamma^{\alpha}{}_{\beta\gamma}$. [*Answer:* (1) Calculate the components of $\nabla\boldsymbol{g}$ in an arbitrary frame:

$$\begin{aligned}g_{\alpha\beta;\gamma} = 0 &= g_{\alpha\beta,\gamma} - \Gamma^{\mu}{}_{\alpha\gamma}g_{\mu\beta} - \Gamma^{\mu}{}_{\beta\gamma}g_{\mu\alpha}\\ &\equiv g_{\alpha\beta,\gamma} - \Gamma_{\beta\alpha\gamma} - \Gamma_{\alpha\beta\gamma};\end{aligned}$$

thereby conclude that $g_{\alpha\beta,\gamma} = 2\Gamma_{(\alpha\beta)\gamma}$. (Round brackets denote symmetric part.) (2) Construct the metric terms in the claimed answer for $\Gamma_{\mu\beta\gamma}$:

$$\begin{aligned}\frac{1}{2}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) &= \Gamma_{(\mu\beta)\gamma} + \Gamma_{(\mu\gamma)\beta} - \Gamma_{(\beta\gamma)\mu}\\ &= \frac{1}{2}[\Gamma_{\mu\beta\gamma} + \Gamma_{\beta\mu\gamma} + \Gamma_{\mu\gamma\beta} + \Gamma_{\gamma\mu\beta} - \Gamma_{\beta\gamma\mu} - \Gamma_{\gamma\beta\mu}]\\ &= \Gamma_{\mu\beta\gamma} + (-\Gamma_{\mu[\beta\gamma]} + \Gamma_{\beta[\mu\gamma]} + \Gamma_{\gamma[\mu\beta]}).\end{aligned}$$

(3) Infer from equation (8.31), with $\boldsymbol{u}$ and $\boldsymbol{w}$ chosen as two basis vectors ($\boldsymbol{u} = \boldsymbol{e}_{\mu}$, $\boldsymbol{w} = \boldsymbol{e}_{\nu}$) that

$$c_{\mu\nu}{}^{\rho}\boldsymbol{e}_{\rho} \equiv [\boldsymbol{e}_{\mu}, \boldsymbol{e}_{\nu}] = \nabla_{\mu}\boldsymbol{e}_{\nu} - \nabla_{\nu}\boldsymbol{e}_{\mu} = (\Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu})\boldsymbol{e}_{\rho} = 2\Gamma^{\rho}{}_{[\nu\mu]}\boldsymbol{e}_{\rho};$$

i.e.,

$$\Gamma^{\rho}{}_{[\mu\nu]} = -\frac{1}{2}c_{\mu\nu}{}^{\rho}; \qquad \Gamma_{\rho[\mu\nu]} = -\frac{1}{2}c_{\mu\nu\rho}. \tag{8.34}$$

(4) This, combined with step (2) yields the desired formula for $\Gamma_{\mu\beta\gamma}$.]

§8.6. LOCAL LORENTZ FRAMES: MATHEMATICAL DISCUSSION

An observer falling freely in curved spacetime makes measurements in his local Lorentz frame. What he discovers has been discussed extensively in Parts I and II of this book. Try now to derive his basic discoveries from the formalism of the last section.

Local Lorentz frame:

Pick an event $\mathscr{P}_o$ on the observer's world line. His local Lorentz frame there is a coordinate system $x^\alpha(\mathscr{P})$ in which

$$g_{\alpha\beta} \equiv \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta \equiv \frac{\partial \mathscr{P}}{\partial x^\alpha} \cdot \frac{\partial \mathscr{P}}{\partial x^\beta} = \eta_{\alpha\beta} \text{ at } \mathscr{P}_o \tag{8.35a}$$

(Lorentz metric at $\mathscr{P}_o$), and in which

$$\partial g_{\alpha\beta}/\partial x^\mu = 0 \text{ at } \mathscr{P}_o \tag{8.35b}$$

(metric as Lorentz as possible near $\mathscr{P}_o$). [See equation (8.15).] In addition, by virtue of equations (8.24),

$$\Gamma^\alpha{}_{\beta\gamma} = 0 \text{ at } \mathscr{P}_o \tag{8.36}$$

(no "correction terms" in covariant derivatives). Of course, the observer must be at rest in his local Lorentz frame; i.e., his world line must be

$$x^j = x^j(\mathscr{P}_o) = \text{constant}; \qquad x^0 \text{ varying.} \tag{8.37}$$

Origin falls freely along a geodesic

Query: Equations (8.35) to (8.37) guarantee that the observer is at rest in a local Lorentz frame. Do they imply that he is freely falling? (They should!) *Answer:* Calculate the observer's 4-acceleration $\boldsymbol{a} = d\boldsymbol{u}/d\tau$ (notation of chapter 6) $= \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u}$ (notation of this chapter). His 4-velocity, calculated from equation (8.37) is

$$\boldsymbol{u} = (dx^\alpha/d\tau)\boldsymbol{e}_\alpha = (dx^0/d\tau)\boldsymbol{e}_0 \underset{\uparrow}{=} \boldsymbol{e}_0; \tag{8.38}$$
$$\begin{bmatrix}\text{because } \boldsymbol{u} \text{ and } \boldsymbol{e}_0 \text{ both} \\ \text{have unit length}\end{bmatrix}$$

so his 4-acceleration is

$$\begin{aligned}\boldsymbol{a} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = \boldsymbol{\nabla}_0\boldsymbol{e}_0 &= \Gamma^\alpha{}_{00}\boldsymbol{e}_\alpha \\ &= 0 \text{ at } \mathscr{P}_0.\end{aligned} \tag{8.39}$$

Thus, he is indeed freely falling ($\boldsymbol{a} = 0$); and he moves along a geodesic ($\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = 0$).

Freely falling particles move on straight lines

Query: Do freely falling particles move along straight lines ($d^2x^\alpha/d\tau^2 = 0$) in the observer's local Lorentz frame at $\mathscr{P}_o$? (They should!) *Answer:* A freely falling particle experiences zero 4-acceleration

$$\boldsymbol{a}_{\text{particle}} = \boldsymbol{\nabla}_{\boldsymbol{u}_{\text{particle}}}\boldsymbol{u}_{\text{particle}} = 0;$$

i.e., it parallel-transports its 4-velocity; i.e., it moves along a geodesic of spacetime with affine parameter equal to its proper time. The geodesic equation for its world line, in local Lorentz coordinates, says

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$
$$= 0 \text{ at } \mathscr{P}_o.$$

The particle's world line is, indeed, straight at $\mathscr{P}_o$.

Basis vectors at origin are Fermi-Walker transported

Query: Does the freely falling observer Fermi-Walker-transport his spatial basis vectors $\boldsymbol{e}_j$; i.e., can he attach them to gyroscopes that he carries? (He should be able to!) *Answer:* Fermi-Walker transport (Box 6.2) would say

$$\underbrace{\frac{d\boldsymbol{e}_j}{d\tau}}_{\text{old notation}} \equiv \underbrace{\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{e}_j}_{\text{new notation}} = \boldsymbol{u}(\boldsymbol{a}\cdot\boldsymbol{e}_j) - \boldsymbol{a}(\boldsymbol{u}\cdot\boldsymbol{e}_j).$$

But $\boldsymbol{u} = \boldsymbol{e}_0$, $\boldsymbol{e}_0\cdot\boldsymbol{e}_j = 0$, and $\boldsymbol{a} = 0$ for the observer; so Fermi-Walker transport in this case reduces to parallel transport along $\boldsymbol{e}_0$: thus $\boldsymbol{\nabla}_0\boldsymbol{e}_j = 0$. This is, indeed, how $\boldsymbol{e}_j$ is transported through $\mathscr{P}_o$, because

$$\boldsymbol{\nabla}_0\boldsymbol{e}_j = \Gamma^\alpha{}_{j0}\boldsymbol{e}_\alpha = 0 \text{ at } \mathscr{P}_o.$$

§8.7. GEODESIC DEVIATION AND THE RIEMANN CURVATURE TENSOR

"Gravitation is a manifestation of spacetime curvature, and that curvature shows up in the deviation of one geodesic from a nearby geodesic (relative acceleration of test particles)." To make this statement precise, first quantify the "deviation" or "relative acceleration" of neighboring geodesics.

Focus attention on a family of geodesics $\mathscr{P}(\lambda, n)$; see Figure 8.4. The smoothly varying parameter n ("selector parameter") distinguishes one geodesic from the next. For fixed n, $\mathscr{P}(\lambda, n)$ is a geodesic with affine parameter λ and with tangent vector

$$\boldsymbol{u} = \partial\mathscr{P}/\partial\lambda; \tag{8.40}$$

thus $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = 0$ (geodesic equation). The vector

$$\boldsymbol{n} \equiv \partial\mathscr{P}/\partial n \tag{8.41}$$

measures the separation between points with the same value of λ on neighboring geodesics.

An observer falling freely along the "fiducial geodesic" $n = 0$ watches a test particle fall along the "test geodesic" $n = 1$. The velocity of the test particle relative

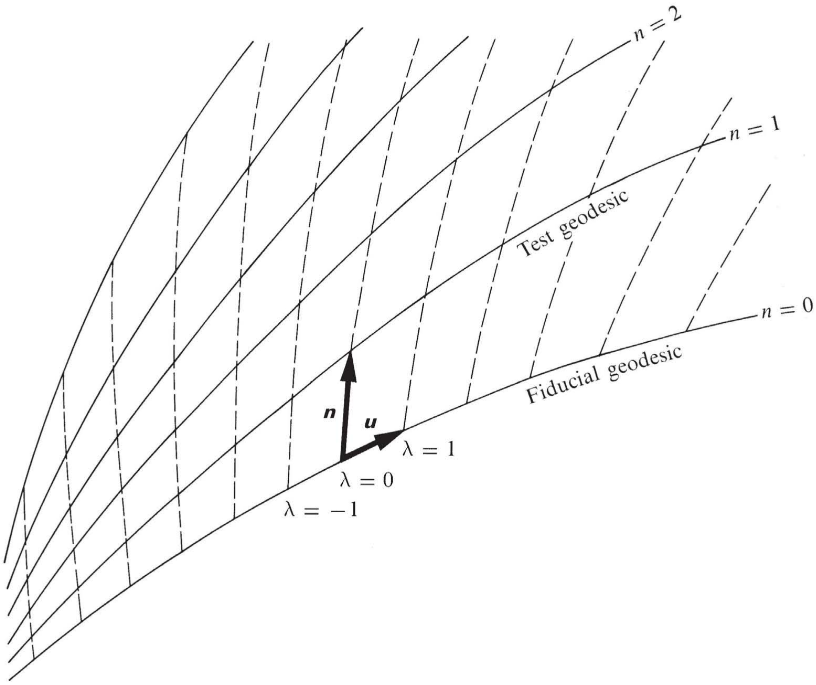


Figure 8.4.
A family of geodesics $\mathscr{P}(\lambda, n)$. The selector parameter n tells "which" geodesic; the affine parameter λ tells "where" on a given geodesic. The separation vector $\boldsymbol{n} \equiv \partial\mathscr{P}/\partial n$ at a point $\mathscr{P}(\lambda, 0)$ along the fiducial geodesic, $n = 0$, reaches (approximately) to the point $\mathscr{P}(\lambda, 1)$ with the same value of λ on the test geodesic, $n = 1$.

to him he quantifies by $\nabla_{\boldsymbol{u}}\boldsymbol{n}$. This relative velocity, like the separation vector $\boldsymbol{n}$, is an arbitrary "initial condition." Not arbitrary, however, is the "relative acceleration," $\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{u}}\boldsymbol{n}$ of the test particle relative to the observer (see Boxes 11.2 and 11.3). It would be zero in flat spacetime. In curved spacetime, it is given by

Riemann curvature tensor defined by relative acceleration of geodesics

$$\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{u}}\boldsymbol{n} + \boldsymbol{Riemann}\,(\ldots, \boldsymbol{u}, \boldsymbol{n}, \boldsymbol{u}) = 0, \tag{8.42}$$

or, in component notation,

$$\frac{D^2 n^\alpha}{d\lambda^2} + R^\alpha{}_{\beta\gamma\delta}u^\beta n^\gamma u^\delta = 0. \tag{8.43}$$

This equation serves as a definition of the "*Riemann curvature tensor;*" and it can also be used to derive the following expressions for the components of ***Riemann*** *in a coordinate basis:*

Components of ***Riemann***

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} &= \langle \boldsymbol{d}x^\alpha, [\nabla_\gamma, \nabla_\delta]\boldsymbol{e}_\beta \rangle \\ &= \frac{\partial \Gamma^\alpha{}_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha{}_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma}. \end{aligned} \tag{8.44}$$

(For proof, read Box 11.4, Box 11.5, and exercise 11.3, in that order.) For a glimpse of the man who first analyzed the curvature of spaces with three and more dimensions, see Box 8.5.

Effects of curvature

Spacetime curvature causes not only geodesic deviation, but also route dependence in parallel transport (parallel transport around a closed curve changes a vector or tensor—Box 11.7); it causes covariant derivatives to fail to commute [equation (8.44)]; and it prevents the existence of a global Lorentz coordinate system (§11.5).

At first sight one might think ***Riemann*** has $4 \times 4 \times 4 \times 4 = 256$ independent components. But closer examination (§13.5) reveals a variety of symmetries

Symmetries of ***Riemann***

$$R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]} = R_{[\gamma\delta][\alpha\beta]}, \qquad R_{[\alpha\beta\gamma\delta]} = 0, \qquad R_{\alpha[\beta\gamma\delta]} = 0 \tag{8.45}$$

Box 8.5 GEORG FRIEDRICH BERNHARD RIEMANN

September 17, 1826, Breselenz, Hanover—July 20, 1866, Selasca, Lake Maggiore

With his famous doctoral thesis of 1851, "Foundations for a general theory of functions of a single complex variable," Riemann founded one branch of modern mathematics (the theory of Riemann surfaces); and with his famous lecture of three years later founded another (Riemannian geometry). These and other writings will be found in his collected works, edited by H. Weber (1953).

"The properties which distinguish space from other conceivable triply-extended magnitudes are only to be deduced from experience.... At every point the three-directional measure of curvature can have an arbitrary value if only the effective curvature of every measurable region of space does not differ noticeably from zero." [G. F. B. Riemann, "On the hypotheses that lie at the foundations of geometry," *Habilitationsvorlesung* of June 10, 1854, on entry into the philosophical faculty of the University of Göttingen.]

Dying of tuberculosis twelve years later, occu-

(antisymmetry on first two indices; antisymmetry on last two; symmetry under exchange of first pair with last pair; vanishing of completely antisymmetric parts). These reduce ***Riemann*** (in four dimensions) from 256 to 20 independent components.

Besides these algebraic symmetries, ***Riemann*** possesses differential symmetries called *"Bianchi identities,"*

$$R^{\alpha}{}_{\beta[\lambda\mu;\nu]} = 0, \tag{8.46}$$

Bianchi identities

which have deep geometric significance (Chapter 15).

From ***Riemann*** one can form several other curvature tensors by contraction. The easiest to form are the *"Ricci curvature tensor,"*

pied with an attempt at a unified explanation of gravity and electromagnetism, Riemann communicated to Betti his system of characterization of multiply-connected topologies (which opened the door to the view of electric charge as "lines of force trapped in the topology of space"), making use of numbers that today are named after Betti but that are identified with a symbol, R_n, that honors Riemann.

"A more detailed scrutiny of a surface might disclose that what we had considered an elementary piece in reality has tiny handles attached to it which change the connectivity character of the piece, and that a microscope of ever greater magnification would reveal ever new topological complications of this type, *ad infinitum*. The Riemann point of view allows, also for real space, topological conditions entirely different from those realized by Euclidean space. I believe that only on the basis of the freer and more general conception of geometry which had been brought out by the development of mathematics during the last century, and with an open mind for the imaginative possibilities which it has revealed, can a philosophically fruitful attack upon the space problem be undertaken." H. Weyl (1949, p. 91).

"But . . . physicists were still far removed from such a way of thinking; space was still, for them, a rigid, homogeneous something, susceptible of no change or conditions. Only the genius of Riemann, solitary and uncomprehended, had already won its way by the middle of the last century to a new conception of space, in which space was deprived of its rigidity, and in which its power to take part in physical events was recognized as possible." A. Einstein (1934, p. 68).

Riemann formulated the first known model for superspace (for which see Chapter 43), a superspace built, however, not of the totality of all 3-geometries with positive definite Riemannian metric (the dynamic arena of Einstein's general relativity), but of all conformally equivalent closed Riemannian 2-geometries of the same topology, a type of superspace known today as Teichmüller space, for more on Riemann's contributions to which and the subsequent development of which, see the chapters by L. Bers and J. A. Wheeler in Gilbert and Newton (1970).

Ricci curvature tensor

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu} \underset{\text{[in a coordinate frame]}}{=} \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha}, \tag{8.47}$$

and the *"scalar curvature,"*

Scalar curvature

$$R \equiv R^{\mu}{}_{\mu}. \tag{8.48}$$

But of much greater geometric significance is the *"Einstein curvature tensor"*

Einstein curvature tensor

$$G^{\mu}{}_{\nu} \equiv \frac{1}{2}\,\varepsilon^{\mu\alpha\beta\gamma}R_{\beta\gamma}{}^{\rho\sigma}\,\frac{1}{2}\,\varepsilon_{\nu\alpha\rho\sigma} = R^{\mu}{}_{\nu} - \frac{1}{2}\,\delta^{\mu}{}_{\nu}R. \tag{8.49}$$

Of all second-rank curvature tensors one can form by contracting ***Riemann***, only ***Einstein*** = ***G*** retains part of the Bianchi identities (8.46): it satisfies

Contracted Bianchi identities

$$G^{\mu\nu}{}_{;\nu} = 0. \tag{8.50}$$

For the beautiful geometric meaning of these *"contracted Bianchi identities"* ("the boundary of a boundary is zero"), see Chapter 15.

Box 8.6 summarizes the above equations describing curvature, as well as the fundamental equations for covariant derivatives.

EXERCISE

[The following exercises from Track 2 are appropriate for the Track-1 reader who wishes to solidfy his understanding of curvature: 11.6, 11.9, 11.10, 13.7–11, and 14.3.]

Exercise 8.16. SOME USEFUL FORMULAS IN COORDINATE FRAMES

In any coordinate frame, define g to be the determinant of the matrix $g_{\alpha\beta}$ [equation 8.11]. Derive the following relations, valid in any coordinate frame.

(a) Contraction of connection coefficients:

$$\Gamma^{\alpha}{}_{\beta\alpha} = (\ln\sqrt{-g})_{,\beta}. \tag{8.51a}$$

[*Hint:* Use the results of exercise 5.5.]

(b) Components of Ricci tensor:

$$R_{\alpha\beta} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\,\Gamma^{\mu}{}_{\alpha\beta})_{,\mu} - (\ln\sqrt{-g})_{,\alpha\beta} - \Gamma^{\mu}{}_{\nu\alpha}\Gamma^{\nu}{}_{\beta\mu}. \tag{8.51b}$$

(c) Divergence of a vector A^{α} or *antisymmetric* tensor $F^{\alpha\beta}$:

$$A^{\alpha}{}_{;\alpha} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\,A^{\alpha})_{,\alpha}, \qquad F^{\alpha\beta}{}_{;\beta} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\,F^{\alpha\beta})_{,\beta}. \tag{8.51c}$$

(d) Integral of a scalar field Ψ over the proper volume of a 4-dimensional region $\mathscr{V}$:

$$\int_{\mathscr{V}} \Psi\, d(\text{proper volume}) = \int_{\mathscr{V}} \Psi\sqrt{-g}\,dt\,dx\,dy\,dz. \tag{8.51d}$$

[*Hint:* In a local Lorentz frame, d(proper volume) $= d\hat{t}\, d\hat{x}\, d\hat{y}\, d\hat{z}$. Use a Jacobian to transform this volume element to the given coordinate frame, and prove from the transformation law

$$g_{\alpha\beta} = \frac{\partial x^{\hat{\mu}}}{\partial x^{\alpha}} \frac{\partial x^{\hat{\nu}}}{\partial x^{\beta}} \eta_{\mu\nu}$$

that the Jacobian is equal to $\sqrt{-g}$.]

Box 8.6 COVARIANT DERIVATIVE AND CURVATURE: FUNDAMENTAL EQUATIONS

Entity	*Abstract notation*	*Component notation*
Covariant Derivative	$\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{T} = \boldsymbol{\nabla T}(\ldots, \ldots, \boldsymbol{u})$	$T^{\beta}{}_{\alpha;\gamma} u^{\gamma} = DT^{\beta}{}_{\alpha}/d\lambda \qquad (\boldsymbol{u} = d\mathscr{P}/d\lambda)$ $= \dfrac{dT^{\beta}{}_{\alpha}}{d\lambda} + (\Gamma^{\beta}{}_{\nu\mu} T^{\nu}{}_{\alpha} - \Gamma^{\nu}{}_{\alpha\mu} T^{\beta}{}_{\nu}) u^{\mu}$
	$\boldsymbol{\nabla}_{\boldsymbol{u}} f = \partial_{\boldsymbol{u}} f = \langle \boldsymbol{d}f, \boldsymbol{u} \rangle$	$f_{;\alpha} u^{\alpha} = f_{,\alpha} u^{\alpha}$
algebraic properties (Exercise 8.11)	$\boldsymbol{\nabla}_{a\boldsymbol{u}+b\boldsymbol{v}} \boldsymbol{T} = a \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{T} + b \boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{T}$ $\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{S} + \boldsymbol{M}) = \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{S} + \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{M}$ $\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{w} - \boldsymbol{\nabla}_{\boldsymbol{w}} \boldsymbol{u} = [\boldsymbol{u}, \boldsymbol{w}]$ for $\boldsymbol{u}, \boldsymbol{w}$ both vector fields	$T^{\beta}{}_{\alpha;\gamma}(au^{\gamma} + bv^{\gamma}) = aT^{\beta}{}_{\alpha;\gamma} u^{\gamma} + bT^{\beta}{}_{\alpha;\gamma} v^{\gamma}$ $(S^{\beta}{}_{\alpha} + M^{\beta}{}_{\alpha})_{;\gamma} u^{\gamma} = S^{\beta}{}_{\alpha;\gamma} u^{\gamma} + M^{\beta}{}_{\alpha;\gamma} u^{\gamma}$ $\Gamma^{\rho}{}_{[\mu\nu]} = -\frac{1}{2} c_{\mu\nu}{}^{\rho}$ [equation (8.34)]
chain rule	$\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{A} \otimes \boldsymbol{B}) = (\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{A}) \otimes \boldsymbol{B} + \boldsymbol{A} \otimes (\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{B})$ $\boldsymbol{\nabla}_{\boldsymbol{u}}(f\boldsymbol{A}) = (\boldsymbol{\nabla}_{\boldsymbol{u}} f)\boldsymbol{A} + f \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{A}$	$(A^{\alpha}{}_{\beta} B_{\gamma})_{;\mu} u^{\mu} = A^{\alpha}{}_{\beta;\mu} B_{\gamma} u^{\mu} + A^{\alpha}{}_{\beta} B_{\gamma;\mu} u^{\mu}$ $(fA^{\alpha}{}_{\beta})_{;\mu} u^{\mu} = f_{,\mu} A^{\alpha}{}_{\beta} u^{\mu} + f A^{\alpha}{}_{\beta;\mu} u^{\mu}$
$\boldsymbol{\nabla}_{\boldsymbol{u}}$ and contraction commute	$\boldsymbol{\nabla}_{\boldsymbol{u}}$(contraction of $\boldsymbol{S}$) = (contraction of $\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{S}$)	$\left(\sum_{\alpha} S^{\alpha}{}_{\alpha\gamma}\right)_{;\mu} u^{\mu} = \sum_{\alpha} (S^{\alpha}{}_{\alpha\gamma;\mu} u^{\mu})$
*metric covariantly constant	$\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{g} = 0$	$g_{\alpha\beta;\gamma} u^{\gamma} = 0$
Gradient	$\boldsymbol{\nabla T}$	$T^{\beta}{}_{\alpha;\gamma} = T^{\beta}{}_{\alpha,\gamma} + \Gamma^{\beta}{}_{\mu\gamma} T^{\mu}{}_{\alpha} - \Gamma^{\mu}{}_{\alpha\gamma} T^{\beta}{}_{\mu}$
Connection Coefficients	$\Gamma^{\alpha}{}_{\beta\gamma} = \langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{\nabla}_{\gamma} \boldsymbol{e}_{\beta} \rangle$ $= -\langle \boldsymbol{\nabla}_{\gamma} \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta} \rangle$	$\Gamma^{\alpha}{}_{\beta\gamma} = g^{\alpha\mu} \Gamma_{\mu\beta\gamma}$,* $\Gamma_{\mu\beta\gamma} = \frac{1}{2}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}$ $+ c_{\mu\beta\gamma} + c_{\mu\gamma\beta} - c_{\beta\gamma\mu})$* $c_{\beta\gamma\mu} = g_{\mu\alpha} c_{\beta\gamma}{}^{\alpha}$ $= g_{\mu\alpha} \langle \boldsymbol{\omega}^{\alpha}, [\boldsymbol{e}_{\beta}, \boldsymbol{e}_{\gamma}] \rangle$*
*Local Lorentz frame at $\mathscr{P}_0$		Coordinate system with $g_{\mu\nu}(\mathscr{P}_0) = \eta_{\mu\nu}, \qquad \Gamma^{\alpha}{}_{\beta\gamma}(\mathscr{P}_0) = 0$
Parallel transport	$\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{S} = 0$	$S^{\alpha}{}_{\beta;\gamma} u^{\gamma} = 0$

Box 8.6 (*continued*)

Entity	*Abstract notation*	*Component notation*
Geodesic Equation	$\nabla_{u} u = 0$	$d^2x^\alpha/d\lambda^2 + \Gamma^\alpha{}_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda) = 0$ in a coordinate basis
Riemann Curvature Tensor	$\textbf{Riemann}\,(\sigma, \boldsymbol{C}, \boldsymbol{A}, \boldsymbol{B}) \equiv \langle \sigma, \mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}\rangle$ $\mathcal{R}(\boldsymbol{A}, \boldsymbol{B}) \equiv [\nabla_{\boldsymbol{A}}, \nabla_{\boldsymbol{B}}] - \nabla_{[\boldsymbol{A},\boldsymbol{B}]}$ (not track-one formulas; see Chapter 11)	$R^\alpha{}_{\beta\gamma\delta} = \dfrac{\partial \Gamma^\alpha{}_{\beta\delta}}{\partial x^\gamma} - \dfrac{\partial \Gamma^\alpha{}_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma}$ in coordinate frame [see equation (11.13) for formula in non-coordinate frame]
Ricci Curvature Tensor	$\textbf{Ricci}$ = contraction on slots 1 and 3 of $\textbf{Riemann}$	$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\beta\alpha}\Gamma^\beta{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\nu}\Gamma^\beta{}_{\mu\alpha}$ in coordinate frame
*Curvature Scalar	R = (contraction of $\textbf{Ricci}$)	$R = R^\alpha{}_\alpha$
*Einstein Curvature Tensor	$\boldsymbol{G} = \textbf{Ricci} - \frac{1}{2}\boldsymbol{g}R$	$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ Useful formulas for computing $G^\alpha{}_\beta$ (derived in §14.2): $G^0{}_0 = -(R^{12}{}_{12} + R^{23}{}_{23} + R^{31}{}_{31})$, $G^0{}_1 = R^{02}{}_{12} + R^{03}{}_{13}$, $G^1{}_1 = -(R^{02}{}_{02} + R^{03}{}_{03} + R^{23}{}_{23})$, $G^1{}_2 = R^{10}{}_{20} + R^{13}{}_{23}$, etc.
*Symmetries of Curvature Tensors		$R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]} = R_{[\gamma\delta][\alpha\beta]}$, $R_{[\alpha\beta\gamma\delta]} = 0$, $R_{\alpha[\beta\gamma\delta]} = 0$ $R_{\alpha\beta} = R_{(\alpha\beta)}$, $G_{\alpha\beta} = G_{(\alpha\beta)}$
Bianchi Identities		$R^\alpha{}_{\beta[\mu\nu;\lambda]} = 0$
*Contracted Bianchi Identities		$G^{\alpha\beta}{}_{;\beta} = 0$
Geodesic Deviation	$\nabla_{u}\nabla_{u}\boldsymbol{n} + \textbf{Riemann}\,(\ldots, \boldsymbol{u}, \boldsymbol{n}, \boldsymbol{u}) = 0$	$\dfrac{D^2n^\alpha}{d\lambda^2} + R^\alpha{}_{\beta\gamma\delta}u^\beta n^\gamma u^\delta = 0$
Parallel Transport around closed curve (§11.4)	$\delta\boldsymbol{A} + \textbf{Riemann}\,(\ldots \boldsymbol{A}, \boldsymbol{u}, \boldsymbol{v}) = 0$ if $\boldsymbol{u}$, $\boldsymbol{v}$ are edges of curve	$\delta A^\alpha + R^\alpha{}_{\beta\gamma\delta}A^\beta u^\gamma v^\delta = 0$

*If metric is absent, these starred formulas cannot be formulated. All other formulas are valid in absence of metric.

CHAPTER 9

DIFFERENTIAL TOPOLOGY

In analytic geometry, many relations which are independent of any frame must be expressed with respect to some particular frame. It is therefore preferable to devise new methods—methods which lead directly to intrinsic properties without any mention of coordinates. The development of the topology of general spaces and of the objects which occur in them, as well as the development of the geometry of general metric spaces, are steps in this direction.

KARL MENGER, in Schilpp (1949), p. 467.

§9.1. GEOMETRIC OBJECTS IN METRIC-FREE, GEODESIC-FREE SPACETIME

This chapter is entirely Track 2.
It depends on no preceding Track-2 material.
It is needed as preparation for
(1) Chapters 10–13 (differential geometry; Newtonian gravity), and
(2) Box 30.1 (mixmaster cosmology).
It will be helpful in
(1) Chapter 14 (calculation of curvature) and in
(2) Chapter 15 (Bianchi identities).

Curved spacetime without metric or geodesics or parallel transport, i.e., "differential topology," is the subject of this easy chapter. It is easy because all the necessary geometric objects (event, curve, vector, 1-form, tensor) are already familiar from flat spacetime. Yet it is also necessary, because one's viewpoint must be refined when one abandons the Lorentz metric of flat spacetime.

Events

The primitive concept of an event $\mathcal{P}$ (Figure 1.2) needs no refinement. The essential property here is identifiability, which is not dependent on the Lorentz metric structure of spacetime.

Metric is abandoned

Geometric concepts must be refined

Curves

Again no refinement. A "curve" $\mathscr{P}(\lambda)$ is also too primitive to care whether spacetime has a metric—*except* that, with metric gone, there is no concept of "proper length" along the curve. This is in accord with Newton's theory of gravity, where one talks of the lengths of curves in "space," but never in "spacetime."

Vectors

Here refinement is needed. In special relativity one could dress primitive ("identifiable") events in enough algebraic plumage to talk of vectors as differences $\mathscr{P} - \mathscr{Q}$ between "algebraic" events. Now the plumage is gone, and the old bilocal ("point for head and point for tail") version of a vector must be replaced by a purely local version (§9.2). Also vectors cannot be moved around; each vector must be attached to a specific event (§§9.2 and 9.3).

1-Forms

Almost no refinement needed, except that, with metric gone, there is no way to tell which 1-form corresponds to a given vector (no way to raise and lower indices), and each 1-form must be attached to a specific event (§9.4).

Tensors

Again almost no refinement, except that each slot of a tensor is specific: if it accepts vectors, then it cannot accommodate 1-forms, and conversely (no raising and lowering of indices); also, each tensor must be attached to a specific event (§9.5).

§9.2. "VECTOR" AND "DIRECTIONAL DERIVATIVE" REFINED INTO TANGENT VECTOR

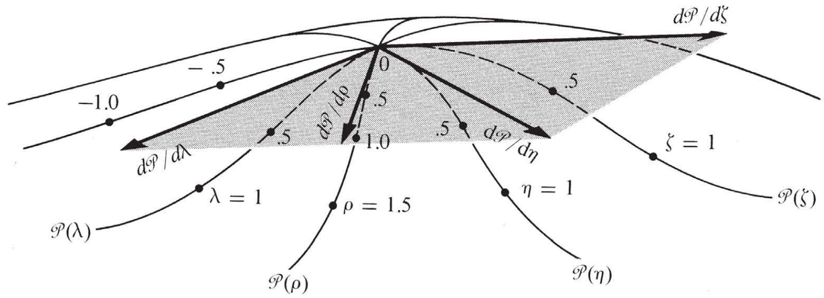
Flat spacetime can accommodate several equivalent definitions of a vector (§2.3): a vector is an arrow reaching from an event $\mathscr{P}_0$ to an event $\mathscr{Q}_0$; it is the parameterized straight line, $\mathscr{P}(\lambda) = \mathscr{P}_0 + \lambda(\mathscr{Q}_0 - \mathscr{P}_0)$ extending from $\mathscr{P}_0$ at $\lambda = 0$ to $\mathscr{Q}_0$ at $\lambda = 1$; it is the rate of change of the point $\mathscr{P}(\lambda)$ with increasing λ, $d\mathscr{P}/d\lambda$.

Old definitions of vector break down when metric is abandoned

With Lorentz metric gone, the "arrow" definition and the "parametrized-straight line" definition must break down. By what route is the arrow or line to be laid out between $\mathscr{P}_0$ and $\mathscr{Q}_0$? There is no concept of straightness; all routes are equally straight or bent.

Such fuzziness forces one to focus on the "rate-of-change-of-point-along-curve"

Box 9.1 TANGENT VECTORS AND TANGENT SPACE



A tangent vector $d\mathscr{P}/d\lambda$ is defined to be "the limit, when $N \longrightarrow \infty$, of N times the displacement of $\mathscr{P}$ as λ ranges from 0 to $1/N$." One cannot think of this final displacement $d\mathscr{P}/d\lambda$ as lying in spacetime; fuzziness forbids (no concept of straightness). Instead, one visualizes $d\mathscr{P}/d\lambda$ as lying in a "tangent plane" or "tangent space," which makes contact with spacetime only at $\mathscr{P}(0)$, the event where $d\mathscr{P}/d\lambda$ is evaluated. All other tangent vectors at $\mathscr{P}(0)$—e.g., $d\mathscr{P}/d\rho$, $d\mathscr{P}/d\eta$, $d\mathscr{P}/d\zeta$—lie in this same tangent space.

To make precise these concepts of tangent space and tangent vector, one may regard spacetime as embedded in a flat space of more than four dimensions. One can then perform the limiting process that leads to $d\mathscr{P}/d\lambda$, using straight arrows in the flat embedding space. The result is a higher-dimensional analog of the figure shown above.

But such a treatment is dangerous. It suggests, falsely, that the tangent vector $d\mathscr{P}/d\lambda$ and the tangent space at $\mathscr{P}_0$ depend on how the embedding is done, or depend for their existence on the embedding process. They do not. And to make clear that they do not is one motivation for defining the directional derivative operator "$d/d\lambda$" to be the tangent vector, rather than using Cartan's more pictorial concept "$d\mathscr{P}/d\lambda$".

definition, $d\mathscr{P}/d\lambda$. It, under the new name *"tangent vector,"* is explored briefly in Box 9.1, and in greater depth in the following paragraphs.

Even "$d\mathscr{P}/d\lambda$" is a fuzzy definition of tangent vector, most mathematicians would argue. More acceptable, they suggest, is this definition: *the tangent vector **u** to a curve $\mathscr{P}(\lambda)$ is the directional derivative operator along that curve*

Best new definition: "tangent vector equals directional derivative operator" $\boldsymbol{u} = d/d\lambda$

$$\boldsymbol{u} = \partial_{\boldsymbol{u}} = (d/d\lambda)_{\text{along curve}}. \tag{9.1}$$

Tangent vector equals directional derivative operator? Preposterous! A vector started out as a happy, irresponsible trip from $\mathscr{P}_0$ to $\mathscr{Q}_0$. It ended up loaded with the social responsibility to tell how something else changes at $\mathscr{P}_0$. At what point did the vector get saddled with this unexpected load? And did it really change its character all that much, as it seems to have done? For an answer, go back and try to redo the "rate-of-change-of-point" definition, $d\mathscr{P}/d\lambda$, in the form of a limiting process:

0. Choose a curve $\mathscr{P}(\lambda)$ whose tangent vector $d\mathscr{P}/d\lambda$ at $\lambda = 0$ is desired.
1. Take the displacement of $\mathscr{P}$ as λ ranges from 0 to 1; that is *not* $d\mathscr{P}/d\lambda$.
2. Take twice the displacement of $\mathscr{P}$ as λ ranges from 0 to $\frac{1}{2}$; that is *not* $d\mathscr{P}/d\lambda$.

N. Take N times the displacement of $\mathscr{P}$ as λ ranges from 0 to $1/N$; that is *not* $d\mathscr{P}/d\lambda$.

∞. Take the limit of such displacements as $N \longrightarrow \infty$; that *is* $d\mathscr{P}/d\lambda$.

This definition has the virtue that $d\mathscr{P}/d\lambda$ describes the properties of the curve $\mathscr{P}(\lambda)$, not over the huge range from $\lambda = 0$ to $\lambda = 1$, where the curve might be doing wild things, but only in an infinitesimal neighborhood of the point $\mathscr{P}_0 = \mathscr{P}(0)$.

Alternative definition, $\boldsymbol{u} = d\mathscr{P}/d\lambda$, requires embedding in flat space of higher dimensionality

The deficiency in this definition is that no meaning is assigned to steps $1, 2, \ldots, N, \ldots$, so there is nothing, yet, to take the limit of. To make each "displacement of $\mathscr{P}$" a definite mathematical object in a space where "limit" has a meaning, one can imagine the original manifold to be a low-dimensional surface in some much higher-dimensional *flat* space. Then $\mathscr{P}(1/N) - \mathscr{P}(0)$ is just a straight arrow connecting two points, i.e. a segment of a straight line, which, in general, will not lie in the surface itself—see Box 9.1. The resulting mental picture of a tangent vector makes its essential properties beautifully clear, but at the cost of some artifacts. The picture relies on a specific but arbitrary way of embedding the manifold of interest (metric-free spacetime) in an extraneous flat space. In using this picture, one must ignore everything that depends on the peculiarities of the embedding. One must think like the chemist, who uses tinkertoy molecular models to visualize many essential properties of a molecule clearly, but easily ignores artifacts of the model (colors of the atoms, diameters of the pegs, its tendency to collapse) that do not mimic quantum-mechanical reality.

Élie Cartan's approach to differential geometry, including the $d\mathscr{P}/d\lambda$ idea of a tangent vector, suggests that he always thought of manifolds as embedded in flat spaces this way, and relied on insights that he did not always formalize to separate the essential geometry of these pictures from their embedding-dependent details. Acceptance of his methods of calculation came late. Mathematicians, who mistrusted their own ability to distinguish fact from artifact, exacted this price for acceptance: stop talking about the movement of the point itself, and start dealing only with concrete measurable changes that take place within the manifold, changes in any or all scalar functions f as the point moves. The limiting process then reads:

Refinement of $d\mathscr{P}/d\lambda$ into $d/d\lambda$

0. Choose a curve $\mathscr{P}(\lambda)$ whose tangent vector at $\lambda = 0$ is desired.
1. Compute the number $f[\mathscr{P}(1)] - f[\mathscr{P}(0)]$, which measures the change in f as the point $\mathscr{P}(\lambda)$ moves from $\mathscr{P}_0 = \mathscr{P}(0)$ to $\mathscr{Q}_0 = \mathscr{P}(1)$.
2. Compute $2\{f[\mathscr{P}(\frac{1}{2})] - f[\mathscr{P}(0)]\}$, which is twice the change in f as the point goes from $\mathscr{P}(0)$ to $\mathscr{P}(\frac{1}{2})$.

N. Compute $N\{f[\mathscr{P}(1/N)] - f[\mathscr{P}(0)]\}$, which is N times the change in f as the point goes from $\mathscr{P}(0)$ to $\mathscr{P}(1/N)$.

∞. Same in the limit as $N \longrightarrow \infty$: (change in f) $= df/d\lambda$.

0. The vector is not itself the change in f. It is instead the operation $d/d\lambda$, which, when applied to f, gives the change $df/d\lambda$. Thus

$$\text{tangent vector} = d/d\lambda$$

[cf. definition (9.1)].

The operation $d/d\lambda$ clearly involves nothing but the last steps $N \longrightarrow \infty$ in this limiting process, and only those aspects of these steps that are independent of f. But this means it involves the infinitesimal displacements of the point $\mathcal{P}$ *and nothing more.*

One who wishes both to stay in touch with the present and to not abandon Cartan's deep geometric insight (Box 9.1) can seek to keep alive a distinction between:

(A) the tangent vector itself in the sense of Cartan, the displacement $d\mathcal{P}/d\lambda$ of a point; and
(B) the "tangent vector operator," or "directional derivative operator," telling what happens to a function in this displacement: (tangent vector operator) $= d/d\lambda$.

However, present practice drops (or, if one will, "slurs") the word "operator" in (B), and uses the phrase "tangent vector" itself for the operator, as will be the practice here from now on. The ideas (A) and (B) should also slur or coalesce in one's mind, so that when one visualizes an embedding diagram with arrows drawn tangent to the surface, one always realizes that the arrow characterizes an infinitesimal motion of a point $d\mathcal{P}/d\lambda$ that takes place purely within the surface, and when one thinks of a derivative operator $d/d\lambda$, one always visualizes this same infinitesimal motion of a point in the manifold, a motion that must occur in constructing any derivative $df(\mathcal{P})/d\lambda$. In this sense, one should regard a vector $d\mathcal{P}/d\lambda \equiv d/d\lambda$ as both "a displacement that carries attention from one point to another" and "a purely geometric object built on points and nothing but points."

The hard-nosed physicist may still be inclined to say "Tangent vector equals directional derivative operator? Preposterous!" Perhaps he will be put at ease by another argument. He is asked to pick an event $\mathcal{P}_0$. At that event he chooses any set of four noncoplanar vectors (vectors defined in whatever way seems reasonable to him); he names them $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$; and he uses them as a basis on which to expand all other vectors at $\mathcal{P}_0$:

$$\boldsymbol{u} = u^\alpha \boldsymbol{e}_\alpha, \qquad \boldsymbol{v} = v^\alpha \boldsymbol{e}_\alpha. \tag{9.2}$$

He is asked to construct the four directional derivative operators $\partial_\alpha \equiv \partial_{\boldsymbol{e}_\alpha}$ along his four basis vectors. As in flat spacetime, so also here; the same expansion coefficients that appear in $\boldsymbol{u} = u^\alpha \boldsymbol{e}_\alpha$ also appear in the expansion for the directional derivative:

Isomorphism between directional derivatives and vectors

$$\partial_{\boldsymbol{u}} = u^\alpha\, \partial_\alpha, \qquad \partial_{\boldsymbol{v}} = v^\alpha\, \partial_\alpha. \tag{9.3}$$

Hence, every relation between specific vectors at $\mathscr{P}_0$ induces an identical relation between their differential operators:

$$\begin{aligned} \boldsymbol{u} = a\boldsymbol{w} + b\boldsymbol{v} &\Longleftrightarrow u^\alpha = a w^\alpha + b v^\alpha \\ &\Longleftrightarrow \partial_{\boldsymbol{u}} = a\,\partial_{\boldsymbol{w}} + b\,\partial_{\boldsymbol{v}}. \end{aligned} \tag{9.4}$$

There is a complete "isomorphism" between the vectors and the corresponding directional derivatives. So how can the hard-nosed physicist deny the hard-nosed mathematician the right to identify completely each tangent vector with its directional derivative? No harm is done; no answer to any computation can be affected.

Tangent space defined

This isomorphism extends to the concept "*tangent space.*" Because linear relations (such as $\partial_{\boldsymbol{u}} = a\,\partial_{\boldsymbol{w}} + b\,\partial_{\boldsymbol{v}}$) among directional derivatives evaluated *at one and the same point* $\mathscr{P}_0$ are meaningful and obey the usual addition and multiplication rules, these derivative operators form an abstract (but finite-dimensional) vector space called the tangent space at $\mathscr{P}_0$. In an embedding picture (Box 9.1) one uses these derivatives (as operators in the flat embedding space) to construct tangent vectors $\boldsymbol{u} = \partial_{\boldsymbol{u}}\mathscr{P}$, $\boldsymbol{v} = \partial_{\boldsymbol{v}}\mathscr{P}$, in the form of straight arrows. Thereby one identifies the abstract tangent space with the geometrically visualized tangent space.

§9.3. BASES, COMPONENTS, AND TRANSFORMATION LAWS FOR VECTORS

Coordinate-induced basis defined

An especially useful basis in the tangent space at an event $\mathscr{P}_0$ is induced by any coordinate system [four functions, $x^0(\mathscr{P})$, $x^1(\mathscr{P})$, $x^2(\mathscr{P})$, $x^3(\mathscr{P})$]:

$$\boldsymbol{e}_0 \equiv \left(\frac{\partial}{\partial x^0}\right)_{x^1,x^2,x^3} = \begin{pmatrix}\text{directional derivative along the}\\ \text{curve with constant } (x^1, x^2, x^3)\\ \text{and with parameter } \lambda = x^0\end{pmatrix}_{\text{at }\mathscr{P}_0}, \tag{9.5}$$

$$\boldsymbol{e}_1 = \frac{\partial}{\partial x^1}, \qquad \boldsymbol{e}_2 = \frac{\partial}{\partial x^2}, \qquad \boldsymbol{e}_3 = \frac{\partial}{\partial x^3}.$$

(See Figure 9.1.)

Changes of basis: transformation matrices defined

A transformation from one basis to another in the tangent space at $\mathscr{P}_0$, like any change of basis in any vector space, is produced by a nonsingular matrix,

$$\boldsymbol{e}_{\alpha'} = \boldsymbol{e}_\beta L^\beta{}_{\alpha'}; \tag{9.6}$$

and, as always (including the Lorentz frames of flat spacetime), the components of a vector must transform by the inverse matrix

$$u^{\alpha'} = L^{\alpha'}{}_\beta u^\beta; \tag{9.7}$$

$$\|L^{\alpha'}{}_\beta\| = \|L^\beta{}_{\gamma'}\|^{-1}, \text{ i.e., } \begin{cases} L^{\alpha'}{}_\beta L^\beta{}_{\gamma'} = \delta^{\alpha'}{}_{\gamma'}, \\ L^\delta{}_{\alpha'} L^{\alpha'}{}_\beta = \delta^\delta{}_\beta. \end{cases} \tag{9.8}$$

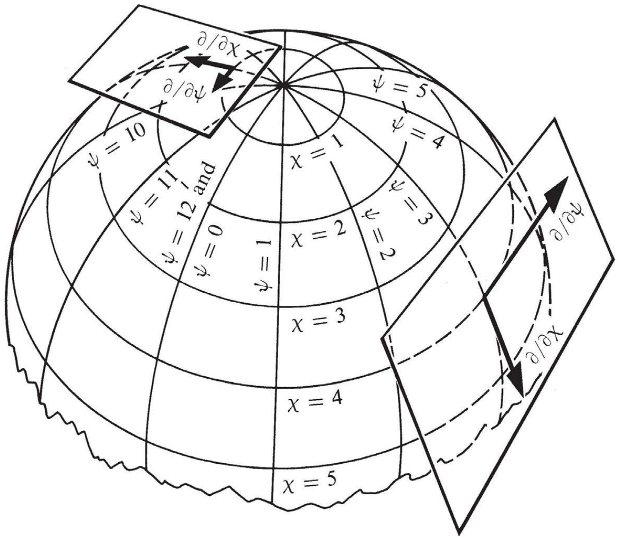


Figure 9.1.
The basis vectors induced, by a coordinate system, into the tangent space at each event. Here a truncated, two-dimensional spacetime is shown (two other dimensions suppressed), with coordinates $\chi(\mathscr{P})$ and $\psi(\mathscr{P})$, and with corresponding basis vectors $\partial/\partial\chi$ and $\partial/\partial\psi$.

This "inverse" transformation law guarantees compatibility between the expansions $\boldsymbol{u} = \boldsymbol{e}_{\alpha'}u^{\alpha'}$ and $\boldsymbol{u} = \boldsymbol{e}_{\beta}u^{\beta}$:

$$\begin{aligned}\boldsymbol{u} &= \boldsymbol{e}_{\alpha'}u^{\alpha'} = (\boldsymbol{e}_{\gamma}L^{\gamma}{}_{\alpha'})(L^{\alpha'}{}_{\beta}u^{\beta}) = \boldsymbol{e}_{\gamma}\delta^{\gamma}{}_{\beta}u^{\beta}\\ &= \boldsymbol{e}_{\beta}u^{\beta}.\end{aligned}$$

In the special case of transformations between coordinate-induced bases, the transformation matrix has a simple form:

$$\frac{\partial}{\partial x^{\alpha'}} = \frac{\partial x^{\beta}}{\partial x^{\alpha'}}\frac{\partial}{\partial x^{\beta}} \text{ (by usual rules of calculus),}$$

so

$$L^{\beta}{}_{\alpha'} = (\partial x^{\beta}/\partial x^{\alpha'})_{\text{at event } \mathscr{P}_0 \text{ where tangent space lies}}. \tag{9.9}$$

(*Note:* this generalizes the Lorentz-transformation law $x^{\beta} = \Lambda^{\beta}{}_{\alpha'}x^{\alpha'}$, which has the differential form $\Lambda^{\beta}{}_{\alpha'} = \partial x^{\beta}/\partial x^{\alpha'}$; also, it provides a good way to remember the signs in the Λ matrices.)

§9.4. 1-FORMS

When the Lorentz metric is removed from spacetime, one must sharpen up the concept of a 1-form $\boldsymbol{\sigma}$ by insisting that it, like any tangent vector $\boldsymbol{u}$, be attached to a specific event $\mathscr{P}_0$ in spacetime. The family of surfaces representing $\boldsymbol{\sigma}$ resides in the tangent space at $\mathscr{P}_0$, not in spacetime itself. The piercing of surfaces of $\boldsymbol{\sigma}$ by an arrow $\boldsymbol{u}$ to produce the number $\langle\boldsymbol{\sigma}, \boldsymbol{u}\rangle$ ("bongs of bell") occurs entirely in the tangent space.

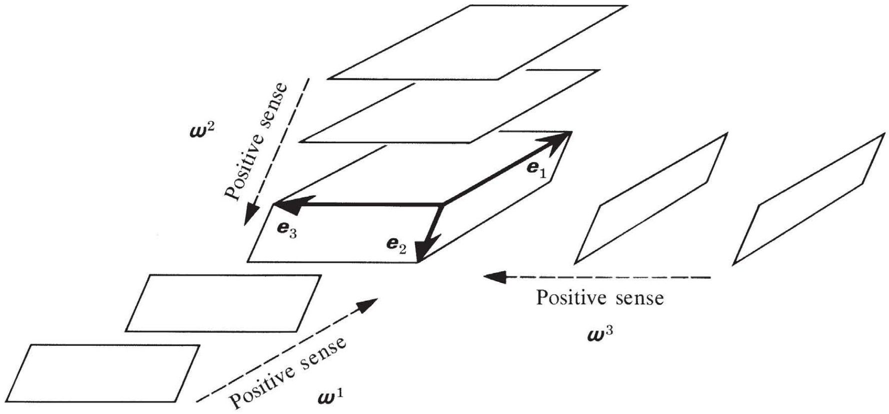


Figure 9.2.
The basis vectors $\boldsymbol{e}_\alpha$ and dual basis 1-forms $\boldsymbol{\omega}^\beta$ in the tangent space of an event $\mathscr{P}_0$. The condition

$$\langle \boldsymbol{\omega}^\beta, \boldsymbol{e}_\alpha \rangle = \delta^\beta{}_\alpha$$

dictates that the vectors $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ lie parallel to the surfaces of $\boldsymbol{\omega}^1$, and that $\boldsymbol{e}_1$ extend from one surface of $\boldsymbol{\omega}^1$ to the next (precisely 1.00 surfaces pierced).

Notice that this picture could fit perfectly well into a book on X-rays and crystallography. There the vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ would be the edges of a unit cell of the crystal; and the surfaces of $\boldsymbol{\omega}^1$, $\boldsymbol{\omega}^2$, $\boldsymbol{\omega}^3$ would be the surfaces of unit cells. Also, for an X-ray diffraction experiment, with wavelength of radiation and orientation of crystal appropriately adjusted, the successive surfaces of $\boldsymbol{\omega}^1$ would produce Bragg reflection. For other choices of wavelength and orientation, the surfaces of $\boldsymbol{\omega}^2$ or $\boldsymbol{\omega}^3$ would produce Bragg reflection.

Dual basis of 1-forms defined

Given any set of basis vectors $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ at an event $\mathscr{P}_0$, one constructs the *"dual basis"* of 1-forms $\{\boldsymbol{\omega}^0, \boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \boldsymbol{\omega}^3\}$ by choosing the surfaces of $\boldsymbol{\omega}^\beta$ such that that

$$\langle \boldsymbol{\omega}^\beta, \boldsymbol{e}_\alpha \rangle = \delta^\beta{}_\alpha. \tag{9.10}$$

See Figure 9.2. A marvelously simple formalism for calculating and manipulating components of tangent vectors and 1-forms then results:

Component-manipulation formulas

$\boldsymbol{u} = \boldsymbol{e}_\alpha u^\alpha$	(definition of components of $\boldsymbol{u}$),	(9.11a)
$\boldsymbol{\sigma} = \sigma_\beta \boldsymbol{\omega}^\beta$	(definition of components of $\boldsymbol{\sigma}$),	(9.11b)
$u^\alpha = \langle \boldsymbol{\omega}^\alpha, \boldsymbol{u} \rangle$	(way to calculate components of $\boldsymbol{u}$),	(9.11c)
$\sigma_\beta = \langle \boldsymbol{\sigma}, \boldsymbol{e}_\beta \rangle$	(way to calculate components of $\boldsymbol{\sigma}$),	(9.11d)
$\langle \boldsymbol{\sigma}, \boldsymbol{u} \rangle = \sigma_\alpha u^\alpha$	(way to calculate $\langle \boldsymbol{\sigma}, \boldsymbol{u} \rangle$ using components),	(9.11e)
$\boldsymbol{\omega}^{\alpha'} = L^{\alpha'}{}_\beta \boldsymbol{\omega}^\beta$	(transformation law for 1-form basis, corresponding to equation 9.6),	(9.11f)
$\sigma_{\alpha'} = \sigma_\beta L^\beta{}_{\alpha'}$	(transformation law for 1-form components).	(9.11g)

(Exercise 9.1 below justifies these equations.)

In the absence of a metric, there is no way to pick a specific 1-form $\tilde{\boldsymbol{u}}$ at an event $\mathscr{P}_0$ and say that it corresponds to a specific tangent vector $\boldsymbol{u}$ at $\mathscr{P}_0$. The correspondence set up in flat spacetime,

Correspondence between vectors and 1-forms rubbed out

$$\langle \tilde{\boldsymbol{u}}, \boldsymbol{v} \rangle = \boldsymbol{u} \cdot \boldsymbol{v} \qquad \text{for all } \boldsymbol{v},$$

was rubbed out when "·" was rubbed out. Restated in component language: the raising of an index, $u^\alpha = \eta^{\alpha\beta} u_\beta$, is impossible because the $\eta^{\alpha\beta}$ do not exist; similarly, lowering of an index, $u_\beta = \eta_{\beta\alpha} u^\alpha$, is impossible.

Gradient of a function

The 1-form gradient $\boldsymbol{d}f$ was introduced in §2.6 with absolutely no reference to metric. Consequently, it and its mathematical formalism are the same here, without metric, as there with metric, except that, like all other 1-forms, $\boldsymbol{d}f$ now resides in the tangent space rather than in spacetime itself. For example, there is no change in the fundamental equation relating the projection of the gradient to the directional derivative:

$$\langle \boldsymbol{d}f, \boldsymbol{u} \rangle = \underbrace{\partial_{\boldsymbol{u}} f}_{\left[\begin{array}{r}\text{old notation for}\\ \text{directional derivative}\end{array}\right]} = \underbrace{\boldsymbol{u}[f]}_{\left[\begin{array}{l}\text{new notation;}\\ \text{recall } \boldsymbol{u} = \partial_{\boldsymbol{u}}\end{array}\right]}. \tag{9.12}$$

Similarly, there are no changes in the component equations,

$$\boldsymbol{d}f = f_{,\alpha} \boldsymbol{\omega}^\alpha \qquad \text{(expansion of } \boldsymbol{d}f \text{ in arbitrary basis),} \tag{9.13a}$$

$$f_{,\alpha} = \partial_\alpha f = \boldsymbol{e}_\alpha[f] \qquad \text{(way to calculate components of } \boldsymbol{d}f\text{),} \tag{9.13b}$$

$$f_{,\alpha} = \partial f / \partial x^\alpha \qquad \text{if } \{\boldsymbol{e}_\alpha\} \text{ is a coordinate basis,}$$

except that they work in arbitrary bases, not just in Lorentz bases. And, as in Lorentz frames, so also in general: the one-form basis $\{\boldsymbol{d}x^\alpha\}$ and the tangent-vector basis $\{\partial/\partial x^\alpha\}$, which are induced into a tangent space by the same coordinate system, are the duals of each other,

$$\langle \boldsymbol{d}x^\alpha, \partial/\partial x^\beta \rangle = \delta^\alpha{}_\beta. \tag{9.14}$$

(See exercise 9.2 for proofs.) Also, most aspects of Cartan's "Exterior Calculus" (parts A, B, C of Box 4.1) are left unaffected by the removal of metric.

§9.5. TENSORS

A tensor $\boldsymbol{S}$, in the absence of Lorentz metric, differs from the tensors of flat, Lorentz spacetime in two ways. (1) $\boldsymbol{S}$ must reside at a specific event $\mathscr{P}_0$, just as any vector or 1-form must. (2) Each slot of $\boldsymbol{S}$ is specific; it will accept either vectors or 1-forms, but not both, because it has no way to convert a 1-form $\tilde{\boldsymbol{u}}$ into a "corresponding

Specificity of tensor slots

vector" $\boldsymbol{u}$ as it sends $\tilde{\boldsymbol{u}}$ through its linear machinery. Thus, if $\boldsymbol{S}$ is a $\binom{1}{2}$ tensor

$$\boldsymbol{S}(\ldots, \ldots, \ldots), \tag{9.15}$$

insert 1-form here — insert vector here — insert vector here

then it *cannot* be converted alternatively to a $\binom{2}{1}$ tensor, or a $\binom{3}{0}$ tensor, or a $\binom{0}{3}$ tensor by the procedure of §3.2. In component language, the indices of $\boldsymbol{S}$ cannot be raised and lowered.

Except for these two restrictions (attachment to a specific event; specificity of slots), a tensor $\boldsymbol{S}$ is the same linear machine as ever. And the algebra of component manipulations is the same:

$$S^{\alpha}{}_{\beta\gamma} = \boldsymbol{S}(\boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta}, \boldsymbol{e}_{\gamma}) \qquad (\boldsymbol{S}, \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta} \text{ must all reside at same event}) \tag{9.16}$$

$$\boldsymbol{S} = S^{\alpha}{}_{\beta\gamma} \boldsymbol{e}_{\alpha} \otimes \boldsymbol{\omega}^{\beta} \otimes \boldsymbol{\omega}^{\gamma}, \tag{9.17}$$

$$\boldsymbol{S}(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{v}) = S^{\alpha}{}_{\beta\gamma}\sigma_{\alpha}u^{\beta}v^{\gamma}. \tag{9.18}$$

EXERCISES

Exercise 9.1. COMPONENT MANIPULATIONS

Derive equations (9.11c) through (9.11g) from (9.10), (9.11a, b), (9.6), (9.7), and (9.8).

Exercise 9.2. COMPONENTS OF GRADIENT, AND DUALITY OF COORDINATE BASES

In an arbitrary basis, define $f_{,\alpha}$ by the expansion (9.13a). Then combine equations (9.11d) and (9.12) to obtain the method (9.13b) of computing $f_{,\alpha}$. Finally, combine equations (9.12) and (9.13b) to show that the bases $\{\boldsymbol{d}x^{\alpha}\}$ and $\{\partial/\partial x^{\beta}\}$ are the duals of each other.

Exercise 9.3. PRACTICE MANIPULATING TANGENT VECTORS

Let $\mathscr{P}_0$ be the point with coordinates $(x = 0, y = 1, z = 0)$ in a three-dimensional space; and define three curves through $\mathscr{P}_0$ by

$$\mathscr{P}(\lambda) = (\lambda, 1, \lambda),$$
$$\mathscr{P}(\zeta) = (\sin\zeta, \cos\zeta, \zeta),$$
$$\mathscr{P}(\rho) = (\sinh\rho, \cosh\rho, \rho + \rho^3).$$

(a) Compute $(d/d\lambda)f$, $(d/d\zeta)f$, and $(d/d\rho)f$ for the function $f = x^2 - y^2 + z^2$ at the point $\mathscr{P}_0$. (b) Calculate the components of the tangent vectors $d/d\lambda$, $d/d\zeta$, and $d/d\rho$ at $\mathscr{P}_0$, using the basis $\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$.

Exercise 9.4. MORE PRACTICE WITH TANGENT VECTORS

In a three-dimensional space with coordinates (x, y, z), introduce the vector field $\boldsymbol{v} = y^2\, \partial/\partial x - x\, \partial/\partial z$, and the functions $f = xy$, $g = z^3$. Compute

(a) $\boldsymbol{v}[f]$	(c) $\boldsymbol{v}[fg]$	(e) $\boldsymbol{v}[f^2 + g^2]$
(b) $\boldsymbol{v}[g]$	(d) $f\boldsymbol{v}[g] - g\boldsymbol{v}[f]$	(f) $\boldsymbol{v}\{\boldsymbol{v}[f]\}$

Exercise 9.5. PICTURE OF BASIS 1-FORMS INDUCED BY COORDINATES

In the tangent space of Figure 9.1, draw the basis 1-forms $\boldsymbol{d}\psi$ and $\boldsymbol{d}\chi$ induced by the ψ, χ-coordinate system.

Exercise 9.6. PRACTICE WITH DUAL BASES

In a three-dimensional space with spherical coordinates r, θ, ϕ, one often likes to use, instead of the basis $\partial/\partial r$, $\partial/\partial\theta$, $\partial/\partial\phi$, the basis

$$\boldsymbol{e}_{\hat{r}} = \frac{\partial}{\partial r}, \qquad \boldsymbol{e}_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial\theta}, \qquad \boldsymbol{e}_{\hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

(a) What is the 1-form basis $\{\boldsymbol{\omega}^{\hat{r}}, \boldsymbol{\omega}^{\hat{\theta}}, \boldsymbol{\omega}^{\hat{\phi}}\}$ dual to this tangent-vector basis? (b) On the sphere $r = 1$, draw pictures of the bases $\{\partial/\partial r, \partial/\partial\theta, \partial/\partial\phi\}$, $\{\boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\phi}}\}$, $\{\boldsymbol{d}r, \boldsymbol{d}\theta, \boldsymbol{d}\phi\}$, and $\{\boldsymbol{\omega}^{\hat{r}}, \boldsymbol{\omega}^{\hat{\theta}}, \boldsymbol{\omega}^{\hat{\phi}}\}$.

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§9.6. COMMUTATORS AND PICTORIAL TECHNIQUES

A vector $\boldsymbol{u}_0$ given only at one point $\mathscr{P}_0$ suffices to compute the derivative $\boldsymbol{u}_0[f] \equiv \partial_{\boldsymbol{u}_0} f$, which is simply a number associated with the point $\mathscr{P}_0$. In contrast, a vector field $\boldsymbol{u}$ provides a vector $\boldsymbol{u}(\mathscr{P})$—which is a differential operator $\partial_{\boldsymbol{u}(\mathscr{P})}$—at each point $\mathscr{P}$ in some region of spacetime. This vector field operates on a function f to produce not just a number, but another function $\boldsymbol{u}[f] \equiv \partial_{\boldsymbol{u}} f$. A second vector field $\boldsymbol{v}$ can perfectly well operate on this new function, to produce yet another function

$$\boldsymbol{v}\{\boldsymbol{u}[f]\} = \partial_{\boldsymbol{v}}(\partial_{\boldsymbol{u}} f).$$

Does this function agree with the result of applying $\boldsymbol{v}$ first and then $\boldsymbol{u}$? Equivalently, does the "commutator"

$$[\boldsymbol{u}, \boldsymbol{v}][f] \equiv \boldsymbol{u}\{\boldsymbol{v}[f]\} - \boldsymbol{v}\{\boldsymbol{u}[f]\} \tag{9.19}$$

Commutator defined

vanish?

The simplest special case is when $\boldsymbol{u}$ and $\boldsymbol{v}$ are basis vectors of a coordinate system, $\boldsymbol{u} = \partial/\partial x^\alpha$, $\boldsymbol{v} = \partial/\partial x^\beta$. Then the commutator does vanish, because partial derivatives always commute:

$$[\partial/\partial x^\alpha, \partial/\partial x^\beta][f] = \partial^2 f/\partial x^\beta\,\partial x^\alpha - \partial^2 f/\partial x^\alpha\,\partial x^\beta = 0.$$

But in general the commutator is nonzero, as one sees from a coordinate-based calculation:

$$\begin{aligned}[\boldsymbol{u}, \boldsymbol{v}]f &= u^\alpha \frac{\partial}{\partial x^\alpha}\left(v^\beta \frac{\partial f}{\partial x^\beta}\right) - v^\alpha \frac{\partial}{\partial x^\alpha}\left(u^\beta \frac{\partial f}{\partial x^\beta}\right) \\ &= \left[(u^\alpha v^\beta{}_{,\alpha} - v^\alpha u^\beta{}_{,\alpha})\frac{\partial}{\partial x^\beta}\right] f.\end{aligned}$$

Commutator of two vector fields is a vector field

Notice however, that the commutator $[\boldsymbol{u}, \boldsymbol{v}]$, like $\boldsymbol{u}$ and $\boldsymbol{v}$ themselves, is a vector field, i.e., a linear differential operator at each event:

$$[\boldsymbol{u}, \boldsymbol{v}] = (\boldsymbol{u}[v^\beta] - \boldsymbol{v}[u^\beta])\frac{\partial}{\partial x^\beta} = (u^\alpha v^\beta{}_{,\alpha} - v^\alpha u^\beta{}_{,\alpha})\frac{\partial}{\partial x^\beta}. \tag{9.20}$$

Such results should be familiar from quantum theory's formalism for angular momentum operators (exercise 9.8).

The three levels of geometry—pictorial, abstract, and component—yield three different insights into the commutator. (1) The abstract expression $[\boldsymbol{u}, \boldsymbol{v}]$ suggests the close connection to quantum theory, and brings to mind the many tools developed there for handling operators. But *recall* that the operators of quantum theory need not be first-order differential operators. The kinetic energy is second order and the potential is zeroth order in the familiar Schrödinger equation. Only first-order operators are vectors. (2) The component expression $u^\alpha v^\beta{}_{,\alpha} - v^\alpha u^\beta{}_{,\alpha}$, valid in any coordinate basis, brings the commutator into the reaches of the powerful tools of index mechanics. (3) The pictorial representation of $[\boldsymbol{u}, \boldsymbol{v}]$ (Box 9.2) reveals its fundamental role as a "closer of curves"—a role that will be important in Chapter 11's analysis of curvature.

Commutator as a "closer of curves"

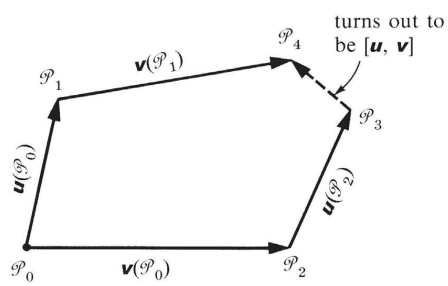
Commutators find application in the distinction between a coordinate-induced basis, $\{\boldsymbol{e}_\alpha\} = \{\partial/\partial x^\alpha\}$, and a noncoordinate basis. Because partial derivatives always commute,

$$[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = [\partial/\partial x^\alpha, \partial/\partial x^\beta] = 0 \text{ in any coordinate basis.} \tag{9.21}$$

Box 9.2 THE COMMUTATOR AS A CLOSER OF QUADRILATERALS

A. Pictorial Representation in Flat Spacetime

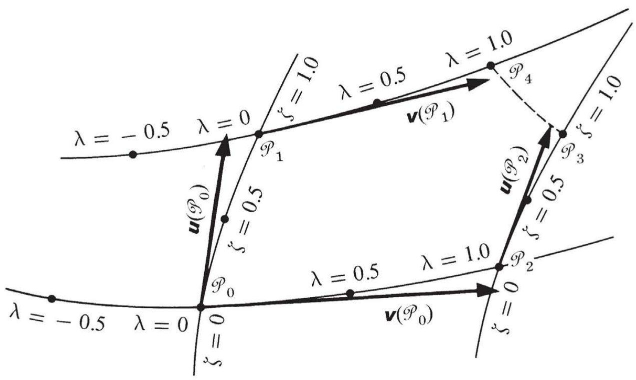
1. For ease of visualization, consider flat spacetime, so the two vector fields $\boldsymbol{u}(\mathscr{P})$ and $\boldsymbol{v}(\mathscr{P})$ can be laid out in spacetime itself.
2. Choose an event $\mathscr{P}_0$ where the commutator $[\boldsymbol{u}, \boldsymbol{v}]$ is to be calculated.
3. Give the names $\mathscr{P}_1$, $\mathscr{P}_2$, $\mathscr{P}_3$, $\mathscr{P}_4$ to the events pictured in the diagram.
4. Then the vector $\mathscr{P}_4 - \mathscr{P}_3$, which measures how much the four-legged curve fails to close, can be expressed in a coordinate basis as



$$\begin{aligned}\mathscr{P}_4 - \mathscr{P}_3 &= [\boldsymbol{u}(\mathscr{P}_0) + \boldsymbol{v}(\mathscr{P}_1)] - [\boldsymbol{u}(\mathscr{P}_2) + \boldsymbol{v}(\mathscr{P}_0)] \\ &= [\boldsymbol{v}(\mathscr{P}_1) - \boldsymbol{v}(\mathscr{P}_0)] - [\boldsymbol{u}(\mathscr{P}_2) - \boldsymbol{u}(\mathscr{P}_0)] \\ &= (v^{\beta}{}_{,\alpha} u^{\alpha} \boldsymbol{e}_{\beta})_{\mathscr{P}_0} - (u^{\beta}{}_{,\alpha} v^{\alpha} \boldsymbol{e}_{\beta})_{\mathscr{P}_0} + \text{errors} \\ &= [\boldsymbol{u}, \boldsymbol{v}]_{\mathscr{P}_0} + \underset{\uparrow\,[\text{terms such as } v^{\beta}{}_{,\mu\nu} u^{\mu} u^{\nu} \boldsymbol{e}_{\beta}]}{\text{errors}}.\end{aligned}$$

5. Notice that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are halved everywhere, then $[\boldsymbol{u}, \boldsymbol{v}]$ is cut down by a factor of 4, while the error terms in the above go down by a factor of 8. Thus, $[\boldsymbol{u}, \boldsymbol{v}]$ represents accurately the gap in the four-legged curve (“*quadrilateral*”) in the limit where $\boldsymbol{u}$ and $\boldsymbol{v}$ are sufficiently short; i.e., $[\boldsymbol{u}, \boldsymbol{v}]$ “*closes the quadrilateral*” whose edges are the vectors fields $\boldsymbol{u}$ and $\boldsymbol{v}$.

B. Pictorial Representation in Absence of Metric, or in Curved Spacetime with a Metric



1. The same picture must work, but now one dares not (at least initially) lay out the vector fields in spacetime itself. Instead one lays out two families of curves: the curves for which $\boldsymbol{u}(\mathscr{P})$ is the tangent vector; and the curves for which $\boldsymbol{v}(\mathscr{P})$ is the tangent vector.
2. The gap “$\mathscr{P}_4 - \mathscr{P}_3$” in the four-legged curve can be characterized by the difference $f(\mathscr{P}_4) - f(\mathscr{P}_3)$ in the values of an arbitrary function at $\mathscr{P}_4$ and $\mathscr{P}_3$. That difference is, in a coordinate basis,

Box 9.2 (*continued*)

$$
\begin{aligned}
f(\mathcal{P}_4) - f(\mathcal{P}_3) &= \underbrace{[f(\mathcal{P}_4) - f(\mathcal{P}_1)]}_{\left(f_{,\alpha}v^\alpha + \frac{1}{2}f_{,\alpha\beta}v^\alpha v^\beta\right)_{\mathcal{P}_1}} + \underbrace{[f(\mathcal{P}_1) - f(\mathcal{P}_0)]}_{\left(f_{,\alpha}u^\alpha + \frac{1}{2}f_{,\alpha\beta}u^\alpha u^\beta\right)_{\mathcal{P}_0}} \\
&\quad - \underbrace{[f(\mathcal{P}_2) - f(\mathcal{P}_0)]}_{\left(f_{,\alpha}v^\alpha + \frac{1}{2}f_{,\alpha\beta}v^\alpha v^\beta\right)_{\mathcal{P}_0}} - \underbrace{[f(\mathcal{P}_3) - f(\mathcal{P}_2)]}_{\left(f_{,\alpha}u^\alpha + \frac{1}{2}f_{,\alpha\beta}u^\alpha u^\beta\right)_{\mathcal{P}_2}} \\
&= [(f_{,\alpha}v^\alpha)_{,\beta}u^\beta - (f_{,\alpha}u^\alpha)_{,\beta}v^\beta]_{\mathcal{P}_0} + \text{“cubic errors”} \\
&= [(u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta})\partial f/\partial x^\alpha]_{\mathcal{P}_0} + \text{“cubic errors”} \\
&= \{[\boldsymbol{u}, \boldsymbol{v}][f]\}_{\mathcal{P}_0} + \text{“cubic errors.”}
\end{aligned}
$$

Here “cubic errors” are cut down by a factor of 8, while $[\boldsymbol{u}, \boldsymbol{v}]f$ is cut down by one of 4, whenever $\boldsymbol{u}$ and $\boldsymbol{v}$ are cut in half.

3. The result

$$f(\mathcal{P}_4) - f(\mathcal{P}_3) = \{[\boldsymbol{u}, \boldsymbol{v}][f]\}_{\mathcal{P}_0} + \text{“cubic errors”}$$

says that $[\boldsymbol{u}, \boldsymbol{v}]$ is a tangent vector at $\mathcal{P}_0$ that describes the separation between the points $\mathcal{P}_3$ and $\mathcal{P}_4$. Its description gets arbitrarily accurate when $\boldsymbol{u}$ and $\boldsymbol{v}$ get arbitrarily short. Thus, $[\boldsymbol{u}, \boldsymbol{v}]$ closes the quadrilateral whose edges are the projections of $\boldsymbol{u}$ and $\boldsymbol{v}$ into spacetime.

C. Philosophy of Pictures

1. Pictures are no substitute for computation. Rather, they are useful for (a) suggesting geometric relationships that were previously unsuspected and that one verifies subsequently by computation; (b) interpreting newly learned geometric results.
2. This usual noncomputational role of pictures permits one to be sloppy in drawing them. No essential new insight was gained in part B over part A, when one carefully moved the tangent vectors into their respective tangent spaces, and permitted only curves to lie in spacetime. Moreover, the original picture (part A) was clearer because of its greater simplicity.
3. This motivates one to draw “sloppy” pictures, with tangent vectors lying in spacetime itself—so long as one keeps those tangent vectors short and occasionally checks the scaling of errors when the lengths of the vectors are halved.

Conversely, if one is given a field of basis vectors ("frame field") $\{\boldsymbol{e}_\alpha(\mathscr{P})\}$, but one does not know whether a coordinate system $\{x^\alpha(\mathscr{P})\}$ exists in which $\{\boldsymbol{e}_\alpha\} = \{\partial/\partial x^\alpha\}$, one can find out by a simple test: calculate all $(4 \times 3)/2 = 6$ commutators $[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta]$; if they all vanish, then there exists such a coordinate system. If not, there doesn't. Stated more briefly, $\{\boldsymbol{e}_\alpha(\mathscr{P})\}$ *is a coordinate-induced basis if and only if* $[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = 0$ *for all* $\boldsymbol{e}_\alpha$ *and* $\boldsymbol{e}_\beta$. (See exercise 9.9 for proof; see §11.5 for an important application.) Coordinate-induced bases are sometimes called "*holonomic.*" In an "*anholonomic* basis" (noncoordinate basis), one defines the commutation coefficients $c_{\mu\nu}{}^\alpha$ by

Vanishing commutator: a test for coordinate bases

Commutation coefficients defined

$$[\boldsymbol{e}_\mu, \boldsymbol{e}_\nu] = c_{\mu\nu}{}^\alpha \boldsymbol{e}_\alpha. \tag{9.22}$$

They enter into the component formula for the commutator of arbitrary vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$[\boldsymbol{u}, \boldsymbol{v}] = (\boldsymbol{u}[v^\beta] - \boldsymbol{v}[u^\beta] + u^\mu v^\nu c_{\mu\nu}{}^\beta)\boldsymbol{e}_\beta \tag{9.23}$$

(see exercise 9.10).

[*Warning!* In notation for functions and fields, mathematicians and physicists often use the same symbols to mean contradictory things. The physicist may write ℓ when considering the length of some critical component in an instrument he is designing, then switch to $\ell(T)$ when he begins to analyze its response to temperature changes. Thus ℓ is a number, whereas $\ell(T)$ is a function. The mathematician, in contrast, will write f for a function that he may be considering as an element in some infinite-dimensional function space. Once the function is supplied with an argument, he then contemplates $f(x)$, which is merely a number: the value of f at the point x. Caught between these antithetical rituals of the physics and mathematics sects, the authors have adopted a clear policy: vacillation. Usually physics-sect statements, like "On a curve $\mathscr{P}(\lambda)$ . . . ," are used; and the reader can translate them himself into mathematically precise language: "Consider a curve $\mathscr{C}$ on which a typical point is $\mathscr{P} = \mathscr{C}(\lambda)$; on this curve . . . ." But on occasion the reader will encounter a pedantic-sounding paragraph written in mathematics-sect jargon (Example: Box 23.3). Such paragraphs deal with concepts and relationships so complex that standard physics usage would lead to extreme confusion. They also should prevent the reader from becoming so conditioned to physics usage that he is allergic to the mathematical literature, where great advantages of clarity and economy of thought are achieved by consistent reliance on wholly unambiguous notation.]

Physicists' notation vs. mathematicians' notation

EXERCISES

Exercise 9.7. PRACTICE WITH COMMUTATORS

Compute the commutator $[\boldsymbol{e}_{\hat\theta}, \boldsymbol{e}_{\hat\phi}]$ of the vector fields

$$\boldsymbol{e}_{\hat\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}, \qquad \boldsymbol{e}_{\hat\phi} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

Express your result as a linear combination of $\boldsymbol{e}_{\hat\theta}$ and $\boldsymbol{e}_{\hat\phi}$.

Exercise 9.8. ANGULAR MOMENTUM OPERATORS

In Cartesian coordinates of three-dimensional Euclidean space, one defines three "*angular-momentum operators*" (vector fields) $\boldsymbol{L}_j$ by

$$\boldsymbol{L}_j \equiv \epsilon_{jk\ell} x^k (\partial/\partial x^\ell).$$

Draw a picture of these three vector fields. Calculate their commutators both pictorially and analytically.

Exercise 9.9. COMMUTATORS AND COORDINATE-INDUCED BASES

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vector fields in spacetime. Show that in some neighborhood of any given point there exists a coordinate system for which

$$\boldsymbol{u} = \partial/\partial x^1, \qquad \boldsymbol{v} = \partial/\partial x^2,$$

if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent and commute:

$$[\boldsymbol{u}, \boldsymbol{v}] = 0.$$

First make this result plausible from the second figure in Box 9.2; then prove it mathematically. *Note:* this result can be generalized to four arbitrary vector fields $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$. There exists a coordinate system in which $\boldsymbol{e}_\alpha = \partial/\partial x^\alpha$ if and only if $\boldsymbol{e}_0$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are linearly independent and $[\boldsymbol{e}_\mu, \boldsymbol{e}_\nu] = 0$ for all pairs $\boldsymbol{e}_\mu$, $\boldsymbol{e}_\nu$.

Exercise 9.10. COMPONENTS OF COMMUTATOR IN NON-COORDINATE BASIS

Derive equation (9.23).

Exercise 9.11. LIE DERIVATIVE

The "Lie derivative" of a vector field $\boldsymbol{v}(\mathcal{P})$ along a vector field $\boldsymbol{u}(\mathcal{P})$ is defined by

$$\pounds_{\boldsymbol{u}} \boldsymbol{v} \equiv [\boldsymbol{u}, \boldsymbol{v}]. \tag{9.24}$$

Draw a space-filling family of curves (a "*congruence*") on a sheet of paper. Draw an arbitrary vector $\boldsymbol{v}$ at an arbitrary point $\mathcal{P}_0$ on the sheet. Transport that vector along the curve through $\mathcal{P}_0$ by means of the "*Lie transport law*" $\pounds_{\boldsymbol{u}} \boldsymbol{v} = 0$, where $\boldsymbol{u} = d/dt$ is the tangent to the curve. Draw the resulting vector $\boldsymbol{v}$ at various points $\mathcal{P}(t)$ along the curve.

Exercise 9.12. A CHIP OFF THE OLD BLOCK

(a) Prove the Jacobi identity

$$[\boldsymbol{u}, [\boldsymbol{v}, \boldsymbol{w}]] + [\boldsymbol{v}, [\boldsymbol{w}, \boldsymbol{u}]] + [\boldsymbol{w}, [\boldsymbol{u}, \boldsymbol{v}]] = 0 \tag{9.25}$$

by picking out all terms of the form $\partial_{\boldsymbol{u}} \partial_{\boldsymbol{v}} \partial_{\boldsymbol{w}}$, showing that they add to zero, and arguing from symmetry that all other terms, e.g., $\partial_{\boldsymbol{w}} \partial_{\boldsymbol{u}} \partial_{\boldsymbol{v}}$ terms, must similarly cancel.

(b) State this identity in index form.

(c) Draw a picture corresponding to this identity (see Box 9.2).

§9.7. MANIFOLDS AND DIFFERENTIAL TOPOLOGY

Spacetime is not the only arena in which the ideas of this chapter can be applied. Points, curves, vectors, 1-forms, and tensors exist in any "differentiable manifold."

Their use to study differentiable manifolds constitutes a branch of mathematics called *"differential topology"*—hence the title of this chapter.

The mathematician usually begins his development of differential topology by introducing some very primitive concepts, such as sets and topologies of sets, by building a fairly elaborate framework out of them, and by then using that framework to define the concept of a differentiable manifold. But most physicists are satisfied with a more fuzzy, intuitive definition of manifold: roughly speaking, an n-dimensional *differentiable manifold* is a set of "points" tied together continuously and differentiably, so that the points in any sufficiently small region can be put into a one-to-one correspondence with an open set of points of R^n. [R^n is the number space of n dimensions, i.e., the space of ordered n-tuples $(x^1, x^2, \ldots, x^n)$.] That correspondence furnishes a coordinate system for the neighborhood.

Differentiable manifold "defined"

A few examples will convey the concept better than this definition. Elementary examples (Euclidean 3-spaces, the surface of a sphere) bring to mind too many geometric ideas from richer levels of geometry; so one is forced to contemplate something more complicated. Let R^3 be a three-dimensional number space with the usual advanced-calculus ideas of continuity and differentiability. Points $\boldsymbol{\xi}$ of R^3 are triples, $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, of real numbers. Let a *ray* $\mathscr{P}$ in R^3 be any half-line from the origin consisting of all $\boldsymbol{\xi}$ of the form $\boldsymbol{\xi} = \lambda\boldsymbol{\eta}$ for some fixed $\boldsymbol{\eta} \neq 0$ and for all positive real numbers $\lambda > 0$. (See Figure 9.3.) A good example of a differentiable manifold then is the set S^2 of all distinct rays. If f is a real-valued function with a specific value $f(\mathscr{P})$ for any ray $\mathscr{P}$ [so one writes $f\colon S^2 \longrightarrow R\colon \mathscr{P} \longrightarrow f(\mathscr{P})$], it should be intuitively (or even demonstrably) clear that we can define what we mean by saying f is continuous or differentiable. In this sense S^2 itself is continuous and differentiable. Thus S^2 is a manifold, and the rays $\mathscr{P}$ are the points of S^2. There are many other manifolds that differential topology finds indistinguishable from S^2. The simplest is the two-dimensional spherical surface (2-sphere), which is the standard representation of S^2; it is the set of points $\boldsymbol{\xi}$ of R^3 satisfying $(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 = 1$. Clearly a different point $\mathscr{P}$ of S^2 (one ray in R^3) intersects each point of this standard 2-sphere surface, and the correspondence is continuous and differentiable in either direction (ray to point; point to ray). The same is true for any ellipsoidal surface in R^3 enclosing the origin, and for any other surface enclosing the origin that has

Examples of differentiable manifolds

The manifold S^2

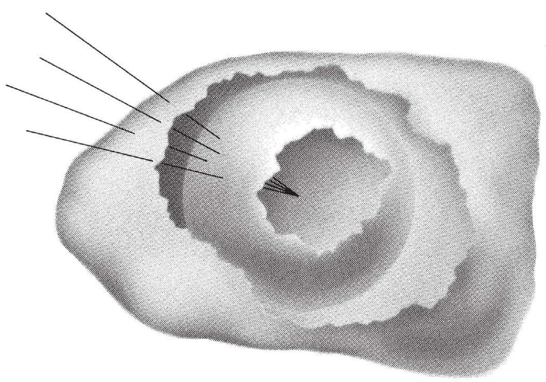


Figure 9.3.
Three different representations of the differentiable manifold S^2. The first is the set of all rays emanating from the origin; the second is the sphere they intersect; the third is an odd-shaped, closed surface that each ray intersects precisely once.

a different ray through each point of itself. They each embody the same global continuity and differentiability concepts, and represent the same abstract differentiable manifold S^2, the 2-sphere. They, and the bundle of rays we started with, all have the same geometric properties at this rudimentary level of geometry. A two-dimensional manifold that has a different geometric structure at this level (a different "differentiable structure") is the torus T^2, the surface of a donut. There is no way to imbed this surface smoothly in R^3 so that a distinct ray $\mathcal{P} \in S^2$ intersects each of its points; there is no invertible and differentiable correspondence between T^2 and S^2.

The manifold T^2

The manifold $SO(3)$ (rotation group)

Another example of a manifold is the rotation group $SO(3)$, whose points $\mathcal{P}$ are all the 3×3 orthogonal matrices of unit determinant, so $\mathcal{P} = ||P_{ij}||$ with $\mathcal{P}^T\mathcal{P} = 1$ and $\det \mathcal{P} = 1$. This is a three-dimensional space (one often uses the three Euler-angle parameters in computations), where differential ideas (e.g., angular velocity) are employed; hence, it is a manifold. So is the Lorentz group.

The differentiability of a manifold (i.e., the possibility of defining differentiable functions on it) permits one to introduce coordinate systems locally, if not globally, and also curves, tangent spaces, tangent vectors, 1-forms, and tensors, just as is done for spacetime. *But* the mere fact that a manifold is differentiable does not mean that such concepts as geodesics, parallel transport, curvature, metric, or length exist in it. These are additional layers of structure possessed by some manifolds, but not by all. Roughly speaking, every manifold has smoothness properties and topology, but without additional structure it is shapeless and sizeless.

Affine geometry and Riemannian geometry defined

That branch of mathematics which adds geodesics, parallel transport, and curvature (shape) to a manifold is called *affine geometry;* that branch which adds a metric is called *Riemannian geometry*. They will be studied in the next few chapters.

EXERCISES

EXERCISES ON THE ROTATION GROUP

As the exposition of differential geometry becomes more and more sophisticated in the following chapters, the exercises will return time and again to the rotation group as an example of a manifold. Then, in Box 30.1, the results developed in these exercises will be used to analyze the "Mixmaster universe," which is a particularly important cosmological solution to Einstein's field equation.

Before working these exercises, the reader may wish to review the Euler-angle parametrization for rotation matrices, as treated, e.g., on pp. 107–109 of Goldstein (1959).

Exercise 9.13. ROTATION GROUP: GENERATORS

Let $\mathcal{K}_\ell$ be three 3×3 matrices whose components are $(K_\ell)_{mn} = \epsilon_{\ell mn}$.

(a) Display the matrices $\mathcal{K}_1$, $(\mathcal{K}_1)^2$, $(\mathcal{K}_1)^3$, and $(\mathcal{K}_1)^4$.

(b) Sum the series

$$\mathcal{R}_x(\theta) \equiv \exp(\mathcal{K}_1\theta) = \sum_{n=0}^{\infty} \frac{\theta^n}{n!}(\mathcal{K}_1)^n. \tag{9.26}$$

Show that $\mathcal{R}_x(\theta)$ is a rotation matrix and that it produces a rotation through an angle θ about the x-axis.

(c) Show similarly that $\mathcal{R}_z(\phi) = \exp(\mathcal{K}_3\phi)$ and $\mathcal{R}_y(\chi) = \exp(\mathcal{K}_2\chi)$ are rotation matrices, and that they produce rotations through angles ϕ and χ about the z- and y-axes, respectively.

(d) Explain why $\mathcal{P} = \mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi)$ defines the Euler-angle coordinates, ψ, θ, ϕ for the generic element $\mathcal{P} \in SO(3)$ of the rotation group.

(e) Let $\mathcal{C}$ be the curve $\mathcal{P} = \mathcal{R}_z(t)$ through the identity matrix, $\mathcal{C}(0) = \mathcal{I} \in SO(3)$. Show that its tangent, $(d\mathcal{C}/dt)(0) \equiv \dot{\mathcal{C}}(0)$ does not vanish by computing $\dot{\mathcal{C}}(0)f_{12}$, where f_{12} is the function $f_{12}(\mathcal{P}) = P_{12}$, whose value is the 12 matrix element of $\mathcal{P}$.

(f) Define a vector field $\boldsymbol{e}_3$ on $SO(3)$ by letting $\boldsymbol{e}_3(\mathcal{P})$ be the tangent (at $t = 0$) to the curve $\mathcal{C}(t) = \mathcal{R}_z(t)\mathcal{P}$ through $\mathcal{P}$. Show that $\boldsymbol{e}_3(\mathcal{P})$ is nowhere zero. *Note:* $\boldsymbol{e}_3(\mathcal{P})$ is called the "generator of rotations about the z-axis," because it points from $\mathcal{P}$ toward neighboring rotations, $\mathcal{R}_z(t)\mathcal{P}$, which differ from $\mathcal{P}$ by a rotation about the z-axis.

(g) Show that $\boldsymbol{e}_3 = (\partial/\partial\psi)_{\theta\phi}$.

(h) Derive the following formulas, valid for $t \ll 1$:

$$\mathcal{R}_x(t)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi) = \mathcal{R}_z(\psi - t\sin\psi\cot\theta)\mathcal{R}_x(\theta + t\cos\psi)\mathcal{R}_z(\phi + t\sin\psi/\sin\theta);$$
$$\mathcal{R}_y(t)\mathcal{R}_z(\psi)\mathcal{R}_x(\theta)\mathcal{R}_z(\phi) = \mathcal{R}_z(\psi + t\cos\psi\cot\theta)\mathcal{R}_x(\theta + t\sin\psi)\mathcal{R}_z(\phi - t\cos\psi/\sin\theta).$$

(i) Define $\boldsymbol{e}_1(\mathcal{P})$ and $\boldsymbol{e}_2(\mathcal{P})$ to be the tangent vectors (at $t = 0$) to the curves $\mathcal{C}(t) = \mathcal{R}_x(t)\mathcal{P}$ and $\mathcal{C}(t) = \mathcal{R}_y(t)\mathcal{P}$, respectively. Show that

$$\boldsymbol{e}_1 = \cos\psi\frac{\partial}{\partial\theta} - \sin\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right),$$
$$\boldsymbol{e}_2 = \sin\psi\frac{\partial}{\partial\theta} + \cos\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right).$$

$\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the "generators of rotations about the x- and y-axes."

Exercise 9.14. ROTATION GROUP: STRUCTURE CONSTANTS

Use the three vector fields constructed in the last exercise,

$$\begin{aligned}\boldsymbol{e}_1 &= \cos\psi\frac{\partial}{\partial\theta} - \sin\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right),\\ \boldsymbol{e}_2 &= \sin\psi\frac{\partial}{\partial\theta} + \cos\psi\left(\cot\theta\frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right),\\ \boldsymbol{e}_3 &= \frac{\partial}{\partial\psi},\end{aligned} \tag{9.27}$$

as basis vectors for the manifold of the rotation group. The above equations express this "basis of generators" in terms of the Euler-angle basis. Show that the commutation coefficients for this basis are

$$c_{\alpha\beta}{}^{\gamma} = -\epsilon_{\alpha\beta\gamma}, \tag{9.28}$$

independently of location $\mathcal{P}$ in the rotation group. These coefficients are also called the *structure constants* of the rotation group.

CHAPTER 10

AFFINE GEOMETRY: GEODESICS, PARALLEL TRANSPORT, AND COVARIANT DERIVATIVE

Galilei's Principle of Inertia is sufficient in itself to prove conclusively that the world is affine in character.

HERMANN WEYL

§10.1. GEODESICS AND THE EQUIVALENCE PRINCIPLE

This chapter is entirely Track 2.
Chapter 9 is necessary preparation for it.
It will be needed as preparation for
(1) Chapters 11–13 (differential geometry; Newtonian gravity),
(2) the second half of Chapter 14 (calculation of curvature), and
(3) the details, but not the message, of Chapter 15 (Bianchi identities).

Free fall is the "natural state of motion," so natural, in fact, that *the path through spacetime of a freely falling, neutral test body is independent of its structure and composition* (the "weak equivalence principle" of Einstein, Eötvös, Dicke; see Box 1.2 and §38.3).

Picture spacetime as filled with free-fall trajectories. Pick an event. Pick a velocity there. They determine a unique trajectory.

Freely falling particles and their clocks

Be more precise. Ask for the maximum amount of information tied up in each trajectory. Is it merely the sequence of points along which the test body falls? No; there is more. Each test body can carry a clock with itself (same kind of clock—"good" clock in sense of Figure 1.9—regardless of structure or composition of test body). The clock ticks as the body moves, labeling each event on its trajectory with a number: the time λ the body was there. Result: the free-fall trajectory is not just a sequence of points; it is a parametrized sequence, a "*curve*" $\mathscr{P}(\lambda)$.

But is the parametrization unique? Not entirely. Quite arbitrary are (1) the choice of time origin, $\mathscr{P}(0)$; and (2) the units (centimeters, seconds, furlongs, . . .) in which clock time λ is measured. Hence, λ is unique only up to linear transformations

$$\lambda_{\text{new}} = a\lambda_{\text{old}} + b; \tag{10.1}$$

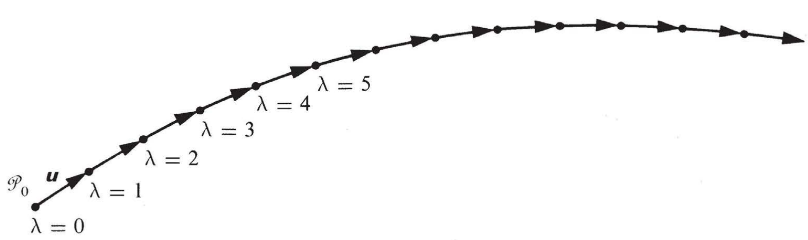


Figure 10.1.
A geodesic viewed as a rule for "straight-on parallel transport." Pick an event $\mathcal{P}_0$ and a tangent vector $\boldsymbol{u} = d/d\lambda$ there. Construct the unique geodesic $\mathcal{P}(\lambda)$ that (1) passes through $\mathcal{P}_0$: $\mathcal{P}(0) = \mathcal{P}_0$; and (2) has $\boldsymbol{u}$ as its tangent vector there: $(d\mathcal{P}/d\lambda)_{\lambda=0} = \boldsymbol{u}$. This geodesic can be viewed as a rule for picking up $\boldsymbol{u}$ from $\mathcal{P}(0)$ and laying it down again at its tip, $\mathcal{P}(1)$, in as straight a manner as possible,

$$\boldsymbol{u}_{\lambda=1} = (d\mathcal{P}/d\lambda)_{\lambda=1};$$

and for then picking it up and laying it down as straight as possible again at $\mathcal{P}(2)$,

$$\boldsymbol{u}_{\lambda=2} = (d\mathcal{P}/d\lambda)_{\lambda=2};$$

etc. This sequence of "straight as possible," "tail-on-tip" transports gives meaning to the idea that $(d\mathcal{P}/d\lambda)_{\lambda=17}$ and $\boldsymbol{u} = (d\mathcal{P}/d\lambda)_{\lambda=0}$ are "the same vector" at different points along the geodesic; or, equivalently, that one has been obtained from the other by "*straight-on parallel transport.*"

b ("new origin of clock time") is a number independent of location on this specific free-fall trajectory, and a ("ratio of new units to old") is also.

In the curved spacetime of Einstein (and in that of Cartan-Newton, Chapter 12), these parametrized free-fall trajectories are the straightest of all possible curves. Consequently, one gives these trajectories the same name, "*geodesics,*" that mathematicians use for the straight lines of a curved manifold; and like the mathematicians, one uses the name "*affine parameter*" for the parameter λ along a free-fall geodesic. Equation (10.1) then says "the affine parameter of a geodesic is unique up to linear transformations."

Geodesic defined as a free-fall trajectory

The affine parameter ("clock time") along a geodesic has nothing to do, à priori, with any metric. It exists even in the absence of metric (e.g., in Cartan-Newtonian spacetime). It gives one a method for comparing the separation between events on a geodesic ($\mathcal{B}$ and $\mathcal{A}$ are "twice as far apart" as $\mathcal{R}$ and $\mathcal{Q}$ if $[\lambda_{\mathcal{B}} - \lambda_{\mathcal{A}}] = 2[\lambda_{\mathcal{R}} - \lambda_{\mathcal{Q}}]$). But the affine parameter measures relative separations only along its own geodesic. It has no means of reaching off the geodesic.

Affine parameter defined as clock time along free-fall trajectory

The above features of geodesics, and others, are summarized in Figure 10.1 and Box 10.1.

§10.2. PARALLEL TRANSPORT AND COVARIANT DERIVATIVE: PICTORIAL APPROACH

Two test bodies, initially falling through spacetime on parallel, neighboring geodesics, get pushed toward each other or apart by tidal gravitational forces (spacetime curvature). To quantify this statement, one must quantify the concepts of "parallel" and "rate of acceleration away from each other." Begin with parallelism.

2

Box 10.1 GEODESICS

Geodesic in brief

Give point, give tangent vector; get unique, affine-parametrized curve ("geodesic").

Geodesic: in context of gravitation physics

World line of a neutral test particle ("Einstein's geometric theory of gravity"; also "Cartan's translation into geometric terms of Newton's theory of gravity"):

(1) "given point": some event on this world line;

(2) "given vector": vector ("displacement per unit increase of parameter") tangent to world line at instant defined by that event;

(3) "unique curve": every neutral test particle with a specified initial position and a specified initial velocity follows the same world line, regardless of its composition and regardless of its mass (small; test mass!; "weak equivalence principle of Einstein-Eötvös-Dicke");

(4) "affine parameter": in Cartan-Newton theory, Newton's "universal time" (which is measured by "good" clocks); in the real physical world, "proper time" (as measured by a "good" clock) along a timelike geodesic;

(5) "parametrized curve": (a) affine parameter unique up to a transformation of the form $\lambda \longrightarrow a\lambda + b$, where a and b are constants (no arbitrariness along a given geodesic other than zero of parameter and unit of parameter); or equivalently (b) given any three events $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ on the geodesic, one can find by well-determined physical construction ("clocking") a unique fourth event $\mathcal{D}$ on the geodesic such that $(\lambda_{\mathcal{D}} - \lambda_{\mathcal{C}})$ is equal to $(\lambda_{\mathcal{B}} - \lambda_{\mathcal{A}})$; or equivalently (c) [differential version] given a tangent vector with components $(dx^\alpha/d\lambda)_{\mathcal{A}}$ at point $\mathcal{A}$, one can find by physical construction (again "clocking") "the same tangent vector" at point $\mathcal{C}$ with uniquely determined components $(dx^\alpha/d\lambda)_{\mathcal{C}}$ (vector "equal"; components ordinarily not equal because of twisting and turning of arbitrary base vectors between $\mathcal{A}$ and $\mathcal{C}$).

Comparison of vectors at different events by parallel transport

Consider two neighboring events $\mathcal{A}$ and $\mathcal{B}$ connected by a curve $\mathcal{P}(\lambda)$. A vector $\boldsymbol{v}_{\mathcal{A}}$ lies in the tangent space at $\mathcal{A}$, and a vector $\boldsymbol{v}_{\mathcal{B}}$ lies in the tangent space at $\mathcal{B}$. How can one say whether $\boldsymbol{v}_{\mathcal{A}}$ and $\boldsymbol{v}_{\mathcal{B}}$ are parallel, and how can one compare their lengths? The equivalence principle gives an answer: an observer travels (using rocket power as necessary) through spacetime along the world line $\mathcal{P}(\lambda)$. He carries the vector $\boldsymbol{v}_{\mathcal{A}}$ with himself as he moves, and he uses flat-space Newtonian or Minkowskian standards to keep it always unchanging (flat-space physics is valid locally according to the equivalence principle!). On reaching event $\mathscr{B}$ the observer compares his "parallel-transported vector" $\boldsymbol{v}_{\mathscr{A}}$ with the vector $\boldsymbol{v}_{\mathscr{B}}$. If they are identical, then the original vector $\boldsymbol{v}_{\mathscr{A}}$ was (by definition) parallel to $\boldsymbol{v}_{\mathscr{B}}$, and they had the same length. (No metric means no way to quantify length; nevertheless, parallel transport gives a way to compare length!)

The equivalence principle entered this discussion in a perhaps unfamiliar way, applied to an observer who may be accelerated, rather than to one who is freely falling. But one cannot evade a basic principle by merely confronting it with an intricate application. (Ingenious perpetual-motion machines are as impossible as simpleminded ones!) The equivalence principle states that no local measurement that is insensitive to gravitational tidal forces can detect any difference whatsoever between flat and curved spacetime. The spaceship navigator has an inertial guidance system (accelerometers, gyroscopes, computers) capable of preserving an inertial reference frame in flat spacetime; and in flat spacetime it can compute the attitude and velocity of any object in the spaceship relative to a given inertial frame. The purchaser may specify whether he wants a guidance computer programmed with the laws of zero-gravity Newtonian mechanics, or with those of special-relativity physics. Use this same guidance system—including the same computer program—in curved spacetime. A vector is being parallel transported if the guidance system's computer says it is not changing.

Parallel transport defined using inertial guidance systems and equivalence principle

Will the result of transport in this way be independent of the curve used to link $\mathscr{A}$ and $\mathscr{B}$? Clearly yes, in gravity-free spacetime, since this is a principal performance criterion that the purchaser of an inertial guidance system can demand of the manufacturer. But in a curved spacetime, the answer is "NO!" If $\boldsymbol{v}_{\mathscr{A}}$ agrees with $\boldsymbol{v}_{\mathscr{B}}$ after parallel transport along one curve, it need not agree with $\boldsymbol{v}_{\mathscr{B}}$ after parallel transport along another. Spacetime curvature produces discrepancies. But one is not ready to study and quantify those discrepancies (Chapter 11), until one has developed the mathematical formalism of parallel transport, which, in turn, cannot be done until one has made precise the "flat-space standards for keeping the vector $\boldsymbol{v}_{\mathscr{A}}$ always unchanging" as it is transported along a curve.

Result of parallel transport depends on route

The flat-space standards are made precise in Box 10.2. They lead to (1) a "Schild's ladder" construction for performing parallel transport; (2) the concept "covariant derivative," $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$, of a vector field $\boldsymbol{v}$ along a curve with tangent $\boldsymbol{u}$; (3) the "equation of motion" $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = 0$ for a geodesic, which states that "a geodesic parallel transports its own tangent vector along itself;" and (4) a link between the tangent spaces at adjacent events (Figure 10.2).

Schild's ladder for performing parallel transport; its consequences

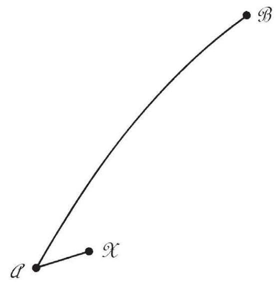
§10.3. PARALLEL TRANSPORT AND COVARIANT DERIVATIVE: ABSTRACT APPROACH

From the "Schild's ladder" construction of Box 10.2, one learns the following properties of spacetime's covariant derivative:

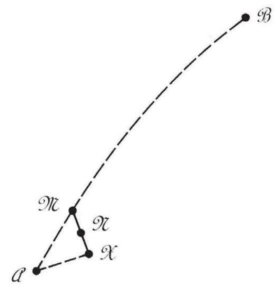
(*continued on page 252*)

Box 10.2 FROM GEODESICS TO PARALLEL TRANSPORT TO COVARIANT DIFFERENTIATION TO GEODESICS TO . . .

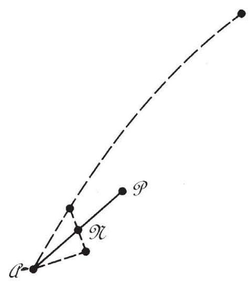
"Parallel transport" as defined by geodesics



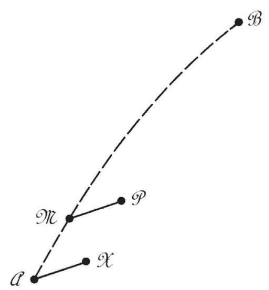
A. Transport any sufficiently short stretch of a curve $\mathcal{AX}$ (i.e., any tangent vector) parallel to itself along curve $\mathcal{AB}$ to point $\mathcal{B}$ as follows:



1. Take some point $\mathcal{M}$ along $\mathcal{AB}$ close to $\mathcal{A}$. Take geodesic $\mathcal{XM}$ through $\mathcal{X}$ and $\mathcal{M}$. Take any affine parametrization λ of $\mathcal{XM}$ and define a unique point $\mathcal{N}$ by the condition $\lambda_{\mathcal{N}} = \frac{1}{2}(\lambda_{\mathcal{X}} + \lambda_{\mathcal{M}})$ ("equal stretches of time in $\mathcal{XN}$ and $\mathcal{NM}$").



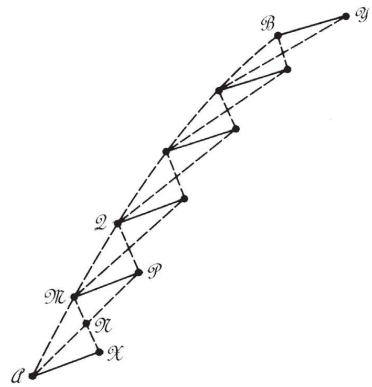
2. Take geodesic that starts at $\mathcal{A}$ and passes through $\mathcal{N}$, and extend it by an equal parameter increment to point $\mathcal{P}$.



3. Curve $\mathcal{MP}$ gives vector $\mathcal{AX}$ as propagated parallel to itself from $\mathcal{A}$ to $\mathcal{M}$ (for sufficiently short $\mathcal{AX}$ and $\mathcal{AM}$). This construction certainly yields parallel transport in flat spacetime (Newtonian or Einsteinian). Moreover, it is local (vectors $\mathcal{AX}$, $\mathcal{AM}$, etc., very short). Therefore, it must work even in curved spacetime. (It embodies the equivalence principle.)

2

4. Repeat process over and over, and eventually end up with $\mathcal{A}\mathcal{X}$ propagated parallel to itself from $\mathcal{A}$ to $\mathcal{B}$. Call this construction "Schild's Ladder," from Schild's (1970) similar construction. [See also Ehlers, Pirani, and Schild (1972).] Note that curve $\mathcal{A}\mathcal{B}$ need not be a geodesic. There is no requirement that $\mathcal{M}\mathcal{Q}$ be the straight-on continuation of $\mathcal{A}\mathcal{M}$ similar to the geodesic requirement in the "cross-brace" that $\mathcal{N}\mathcal{P}$ be the straight-on continuation of $\mathcal{A}\mathcal{N}$.



"Schild's Ladder"

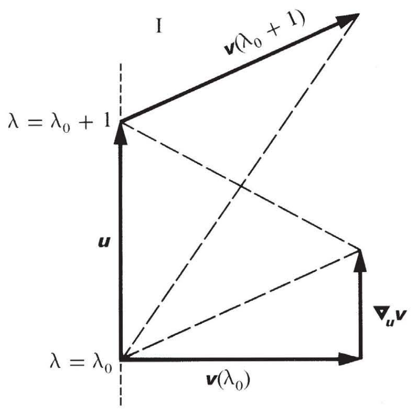
5. Result of propagating $\mathcal{A}\mathcal{X}$ parallel to itself from $\mathcal{A}$ to $\mathcal{B}$ depends on choice of world line $\mathcal{A}\mathcal{B}$ ("evidence of curvature of spacetime").

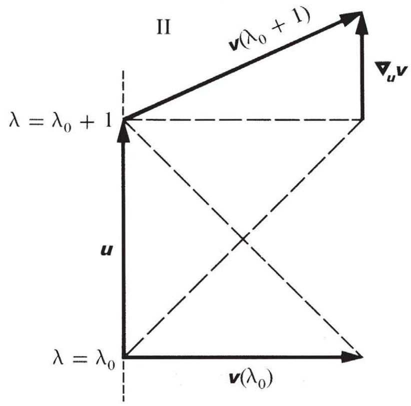
"Covariant differentiation" as defined by parallel transport

B. Ask how rapidly a vector field $\boldsymbol{v}$ is changing along a curve with tangent vector $\boldsymbol{u} = d/d\lambda$. The answer, $d\boldsymbol{v}/d\lambda \equiv \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} \equiv$ "rate of change of $\boldsymbol{v}$ with respect to λ" $\equiv$ "covariant derivative of $\boldsymbol{v}$ along $\boldsymbol{u}$," is constructed by the following obvious procedure: (1) Take $\boldsymbol{v}$ at $\lambda = \lambda_0 + \varepsilon$. (2) Parallel transport it back to $\lambda = \lambda_0$. (3) Calculate how much it differs from $\boldsymbol{v}$ there. (4) Divide by ε (and take limit as $\varepsilon \longrightarrow 0$):

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = \lim_{\varepsilon \to 0}\left\{\frac{[\boldsymbol{v}(\lambda_0 + \varepsilon)]_{\text{parallel transported to }\lambda_0} - \boldsymbol{v}(\lambda_0)}{\varepsilon}\right\}.$$

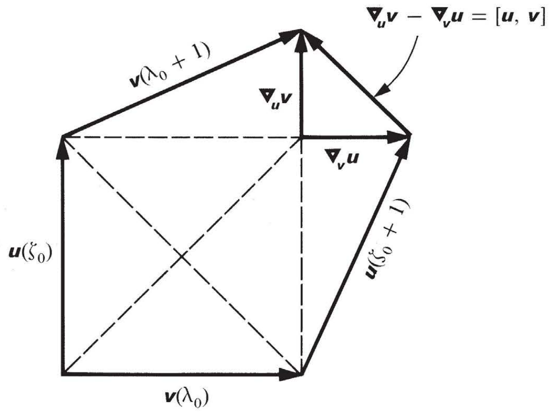
If $\boldsymbol{u} = d/d\lambda$ is short compared to scale of inhomogeneities in the vector field $\boldsymbol{v}$, then $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$ can be read directly off drawing I, or, equally well, off drawing II.





Box 10.2 (*continued*)

"Symmetry" of covariant differentiation



C. Take two vector fields. Combine into one the two diagrams for $\nabla_{\boldsymbol{u}}\boldsymbol{v}$ and $\nabla_{\boldsymbol{v}}\boldsymbol{u}$. Thereby discover that $\nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u}$ is the vector by which the ***v-u-v-u*** quadrilateral fails to close—i.e. (see Box 9.2), it is the commutator $[\boldsymbol{u}, \boldsymbol{v}]$: $\nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} = [\boldsymbol{u}, \boldsymbol{v}]$.

Terminology: ∇ is said to be a "*symmetric*" or "*torsion-free*" covariant derivative when $\nabla_{\boldsymbol{u}}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{u} = [\boldsymbol{u}, \boldsymbol{v}]$. Other types of covariant derivatives, as studied by mathematicians, have no relevance for any gravitation theory based on the equivalence principle.

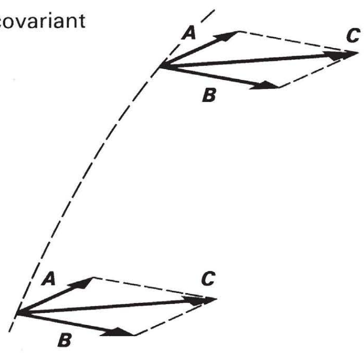
Chain rule for covariant differentiation

D. The "take-the-difference" and "take-the-limit" process used to define $\nabla_{\boldsymbol{u}}\boldsymbol{v}$ guarantees that it obeys the usual rule for differentiating products:

$$\nabla_{\boldsymbol{u}}(\underbrace{f}_{\text{scalar field}}\underbrace{\boldsymbol{v}}_{\text{vector field}}) = f\nabla_{\boldsymbol{u}}\boldsymbol{v} + (\underbrace{\boldsymbol{u}[f]}_{\substack{\text{“derivative of } f \text{ along } \boldsymbol{u}\text{,” denoted } \partial_{\boldsymbol{u}}f \text{ in first} \\ \text{part of book; actually equal to } df/d\lambda \text{ if } \boldsymbol{u} = d/d\lambda; \\ \text{also sometimes denoted } \nabla_{\boldsymbol{u}}f.}})\boldsymbol{v}$$

(for proof, see exercise 10.2.)

Additivity for covariant differentiation



E. In the real physical world, be it Newtonian or relativistic, parallel transport of a triangle cannot break its legs apart: (1) $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ initially such that $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{C}$; (2) $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ each parallel transported with himself by freely falling (inertial) observer; (3) then $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{C}$ always. Any other result would violate the equivalence principle!

1. Consequence of this (as seen by following through definition of covariant derivative, and by noting that any vector $\boldsymbol{u}$ can be regarded as the tangent vector to a freely falling world line):

$$\nabla_{\boldsymbol{u}}(\boldsymbol{v} + \boldsymbol{w}) = \nabla_{\boldsymbol{u}}\boldsymbol{v} + \nabla_{\boldsymbol{u}}\boldsymbol{w}$$

for any vector $\boldsymbol{u}$ and vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$.

2. Consequence of this, combined with symmetry of covariant derivative, and with additivity of the "closer of quadrilaterals" $[\boldsymbol{u}, \boldsymbol{v}]$:

$$\nabla_{\boldsymbol{u}+\boldsymbol{n}}\boldsymbol{v} = \nabla_{\boldsymbol{u}}\boldsymbol{v} + \nabla_{\boldsymbol{n}}\boldsymbol{v}.$$

(See exercise 10.1.) This can be inferred, alternatively, from the equivalence principle: in a local inertial frame, as in special relativity or Newtonian theory, the change in $\boldsymbol{v}$ along $\boldsymbol{u} + \boldsymbol{n}$ should equal the sum of the changes along $\boldsymbol{u}$ and along $\boldsymbol{n}$.

3. Consequence of above: choose $\boldsymbol{n}$ to be a multiple of $\boldsymbol{u}$; thereby conclude

$$\nabla_{a\boldsymbol{u}}\boldsymbol{v} = a\nabla_{\boldsymbol{u}}\boldsymbol{v}.$$

F. The "Schild's ladder" construction process for parallel transport (beginning of this box), applied to the tangent vector of a geodesic (exercise 10.6) guarantees: *a geodesic parallel transports its own tangent vector along itself.* Translated into covariant-derivative language:

$$\begin{pmatrix} \boldsymbol{u} = d/d\lambda \text{ is a tangent} \\ \text{vector to a curve, and} \\ \nabla_{\boldsymbol{u}}\boldsymbol{u} = 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \text{the curve is} \\ \text{a geodesic} \end{pmatrix}.$$

Thus closes the circle: geodesic to parallel transport to covariant derivative to geodesic.

Geodesics as defined by parallel transport or covariant differentiation

2

Covariant derivative: basic properties

Symmetry: $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} - \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{u} = [\boldsymbol{u}, \boldsymbol{v}]$ for any vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$; (10.2a)

Chain rule: $\boldsymbol{\nabla}_{\boldsymbol{u}}(f\boldsymbol{v}) = f\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{v}\,\partial_{\boldsymbol{u}}f$ for any function f, vector field $\boldsymbol{v}$, and vector $\boldsymbol{u}$; (10.2b)

Additivity: $\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}$ for any vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$, and vector $\boldsymbol{u}$; (10.2c)

$\boldsymbol{\nabla}_{a\boldsymbol{u}+b\boldsymbol{n}}\boldsymbol{v} = a\,\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + b\,\boldsymbol{\nabla}_{\boldsymbol{n}}\boldsymbol{v}$ for any vector field $\boldsymbol{v}$, vectors or vector fields $\boldsymbol{u}$ and $\boldsymbol{n}$, and numbers or functions a and b. (10.2d)

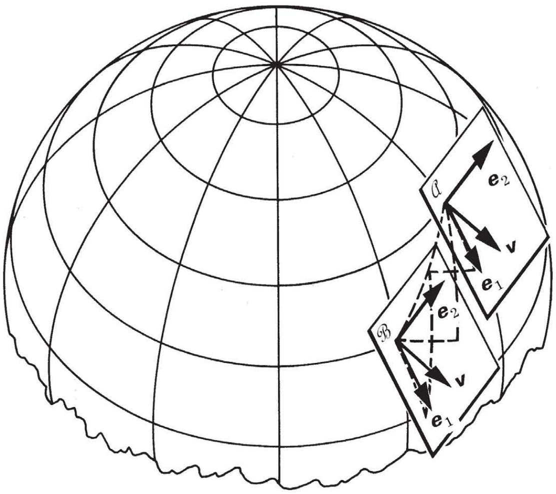


Figure 10.2.
The link between the tangent spaces at neighboring points, made possible by a parallel-transport law. Choose basis vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ at the event $\mathscr{A}$. Parallel transport them to a neighboring event $\mathscr{B}$. (Schild's ladder for transport of $\boldsymbol{e}_1$ is shown in the figure.) Then any other vector $\boldsymbol{v}$ that is parallel transported from $\mathscr{A}$ to $\mathscr{B}$ will have the same components at the two events (parallel transport cannot break the legs of a triangle; see Box 10.2):

[same numerically as at $\mathscr{A}$]

$$\boldsymbol{v} = v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 \text{ at } \mathscr{A} \Rightarrow \boldsymbol{v} = v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 \text{ at } \mathscr{B}.$$

[parallel transported from $\mathscr{A}$ to $\mathscr{B}$]

Thus, parallel transport provides a unique and complete link between the tangent space at $\mathscr{A}$ and the tangent space at $\mathscr{B}$. It identifies a unique vector at $\mathscr{B}$ with each vector at $\mathscr{A}$ in a way that preserves all algebraic relations. Similarly (see §10.3), it identifies a unique 1-form at $\mathscr{B}$ with each 1-form at $\mathscr{A}$, and a unique tensor at $\mathscr{B}$ with each tensor at $\mathscr{A}$, preserving all algebraic relations such as $\langle\boldsymbol{\sigma}, \boldsymbol{v}\rangle = 19.9$ and $\boldsymbol{S}(\boldsymbol{\sigma}, \boldsymbol{v}, \boldsymbol{w}) = 37\,1$.

Actually, all this is true only in the limit when $\mathscr{A}$ and $\mathscr{B}$ are arbitrarily close to each other. When $\mathscr{A}$ and $\mathscr{B}$ are close but not arbitrarily close, the result of parallel transport is slightly different for different paths; so the link between the tangent spaces is slightly nonunique. But the differences decrease by a factor of 4 each time the affine-parameter distance between $\mathscr{A}$ and $\mathscr{B}$ is cut in half; see Chapter 11.

2

Any "rule" $\boldsymbol{\nabla}$, for producing new vector fields from old, that satisfies these four conditions, is called by differential geometers a *"symmetric covariant derivative."* Such a rule is not inherent in the more primitive concepts (Chapter 9) of curves, vectors, tensors, etc. In the arena of a spacetime laboratory, there are as many ways of defining a covariant derivative rule $\boldsymbol{\nabla}$ as there are of rearranging sources of the gravitational field. Different free-fall trajectories (geodesics) result from different distributions of masses.

Given the geodesics of spacetime, or of any other manifold, one can construct a unique corresponding covariant derivative by the Schild's ladder procedure of Box 10.2. Given any covariant derivative, one can discuss parallel transport via the equation

Equation for parallel transport

$$d\boldsymbol{v}/d\lambda \equiv \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = 0 \Longleftrightarrow \text{the vector field } \boldsymbol{v} \text{ is parallel transported along the vector } \boldsymbol{u} = d/d\lambda; \tag{10.3}$$

and one can test whether any curve is a geodesic via

$$\begin{aligned} \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = 0 &\Longleftrightarrow \text{the curve } \mathscr{P}(\lambda) \text{ with tangent vector } \boldsymbol{u} = d/d\lambda \\ &\qquad \text{parallel transports its own tangent vector } \boldsymbol{u} \\ &\Longleftrightarrow \mathscr{P}(\lambda) \text{ is a geodesic.} \end{aligned} \tag{10.4}$$

Knowledge of all geodesics is equivalent to knowledge of covariant derivative

Thus a knowledge of all geodesics is completely equivalent to a knowledge of the covariant derivative.

Covariant derivative generalizes flat-space gradient

The covariant derivative $\boldsymbol{\nabla}$ generalizes to curved spacetime the flat-space gradient $\boldsymbol{\nabla}$. Like its flat-space cousin, it can be viewed as a machine for producing a number $\langle \boldsymbol{\sigma}, \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}\rangle$ out of a 1-form $\boldsymbol{\sigma}$, a vector $\boldsymbol{u}$, and a vector field $\boldsymbol{v}$. This machine viewpoint is explored in Box 10.3. Note there an important fact: despite its machine nature, $\boldsymbol{\nabla}$ is *not* a tensor; it is a nontensorial geometric object.

Action of covariant derivative on functions, 1-forms, and tensors

In curved as in flat spacetime, $\boldsymbol{\nabla}$ can be applied not only to vector fields, but also to functions, 1-form fields, and tensor fields. Its action on functions is defined in the obvious manner:

$$\boldsymbol{\nabla} f \equiv \boldsymbol{d}f; \qquad \boldsymbol{\nabla}_{\boldsymbol{u}} f \equiv \partial_{\boldsymbol{u}} f \equiv \boldsymbol{u}[f] \equiv \langle \boldsymbol{d}f, \boldsymbol{u}\rangle. \tag{10.5}$$

Its action on 1-form fields and tensor fields is defined by the curved-space generalization of equation (3.39): $\boldsymbol{\nabla S}$ is a linear machine for calculating the change in output of $\boldsymbol{S}$, from point to point, when "constant" (i.e., parallel transported) vectors are inserted into its slots. Example: the gradient of a $\binom{0}{1}$ tensor, i.e., of a 1-form field $\boldsymbol{\sigma}$. Pick an event $\mathscr{P}_0$; pick two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in the tangent space at $\mathscr{P}_0$; construct from $\boldsymbol{v}$ a "constant" vector field $\boldsymbol{v}(\mathscr{P})$ by parallel transport along the direction of $\boldsymbol{u}$, $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = 0$. Then $\boldsymbol{\nabla\sigma}$ is a $\binom{0}{2}$ tensor, and $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma}$ is a $\binom{0}{1}$ tensor defined at $\mathscr{P}_0$ by

$$\boldsymbol{\nabla\sigma}(\boldsymbol{v}, \boldsymbol{u}) \equiv \langle \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma}, \boldsymbol{v}\rangle \equiv \boldsymbol{\nabla}_{\boldsymbol{u}}(\langle \boldsymbol{\sigma}, \boldsymbol{v}\rangle) \equiv \frac{d}{d\lambda}\langle \boldsymbol{\sigma}, \boldsymbol{v}\rangle, \tag{10.6}$$

where $\boldsymbol{u} = d/d\lambda$. This defines $\boldsymbol{\nabla\sigma}$ and $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma}$, because it states their output for any

(continued on page 257)

Box 10.3 COVARIANT DERIVATIVE VIEWED AS A MACHINE; CONNECTION COEFFICIENTS AS ITS COMPONENTS

A. The Machine View

1. The covariant derivative operator $\boldsymbol{\nabla}$, like most other geometric objects, can be regarded as a machine with slots. There is one such machine at each event $\mathcal{P}_0$ in spacetime. In brief, the machine interpretation of $\boldsymbol{\nabla}$ at $\mathcal{P}_0$ says

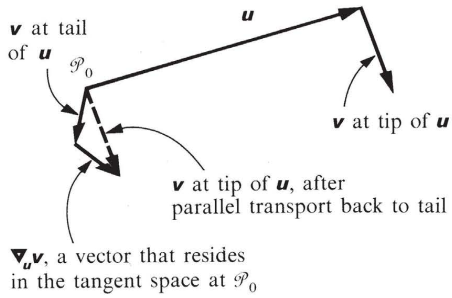
$$\boldsymbol{\nabla}(\boldsymbol{\sigma}, \boldsymbol{v}(\mathcal{P}), \boldsymbol{u}) \equiv \langle \boldsymbol{\sigma}, \underbrace{\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{v}} \rangle .$$

First slot; insert a 1-form $\boldsymbol{\sigma}$ that resides in the tangent space at $\mathcal{P}_0$	Second slot; insert a vector field $\boldsymbol{v}(\mathcal{P})$ defined on a neighborhood of $\mathcal{P}_0$	Third slot; insert a vector $\boldsymbol{u}$ that resides in the tangent space at $\mathcal{P}_0$	A new vector: "the covariant derivative of the vector field $\boldsymbol{v}$ with respect to $\boldsymbol{u}$."

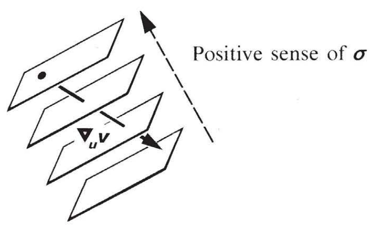
[Note: this slot notation for $\boldsymbol{\nabla}$ serves no useful purpose except to emphasize the "machine"-nature of $\boldsymbol{\nabla}$. This box is the only place it will be used.]

2. Geometrically, the output of the machine, $\langle \boldsymbol{\sigma}, \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{v} \rangle$, is obtained as follows:

(a) Calculate the rate of change of $\boldsymbol{v}$, $\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{v}$, along the vector $\boldsymbol{u}$; when $\boldsymbol{u}$ and $\boldsymbol{v}$ are infinitesimally small, the calculation can be represented pictorially:



(b) Count how many surfaces of the 1-form $\boldsymbol{\sigma}$ are pierced by the vector $\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{v}$ (piercing occurs in tangent space at $\mathcal{P}_0$)



$$\langle \boldsymbol{\sigma}, \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{v} \rangle = -2.8.$$

This number is the output of the machine $\boldsymbol{\nabla}$, when $\boldsymbol{\sigma}$, $\boldsymbol{v}(\mathscr{P})$ and $\boldsymbol{u}$ are inserted into its slots.

3. Another, equivalent, statement of covariant derivative as a machine. Leave first slot empty (no mention of any 1-form $\boldsymbol{\sigma}$); get a new vector field from original vector field $\boldsymbol{v}$:

$$\boldsymbol{\nabla}(\underbrace{\ \ \ }_{\text{empty}}, \boldsymbol{v}(\mathscr{P}), \boldsymbol{u}) \equiv \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$$

$$= \text{“covariant derivative of vector field } \boldsymbol{v} \text{ along vector } \boldsymbol{u}.\text{”}$$

4. A third machine operation. Leave first and third slots empty (no mention of any 1-form $\boldsymbol{\sigma}$; no mention of any vector $\boldsymbol{u}$ along which to differentiate); get a $\binom{1}{1}$ tensor field from original vector field $\boldsymbol{v}$:

$$\boldsymbol{\nabla}(\underbrace{\ \ \ }_{\text{empty}}, \boldsymbol{v}(\mathscr{P}), \underbrace{\ \ \ }_{\text{empty}}) \equiv \boldsymbol{\nabla v}$$

$$= \text{“covariant derivative” or “gradient” of vector field } \boldsymbol{v}.$$

This tensor field, $\boldsymbol{\nabla v}$, is the curved-space generalization of the flat-space $\boldsymbol{\nabla v}$ studied in §3.5. It has two slots (the two left empty in its definition). Its output for given input is

$$\boldsymbol{\nabla v}(\underbrace{\ \ \ }_{\text{empty}}, \boldsymbol{u}) = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$$

$$\boldsymbol{\nabla v}(\boldsymbol{\sigma}, \boldsymbol{u}) = \langle \boldsymbol{\sigma}, \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} \rangle.$$

5. Summary of the quantities defined above:
   (a) $\boldsymbol{\nabla}$ is a *covariant derivative operator;* to get a number from it, insert $\boldsymbol{\sigma}$, $\boldsymbol{v}(\mathscr{P})$, and $\boldsymbol{u}$; the result is $\langle \boldsymbol{\sigma}, \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} \rangle$.
   (b) $\boldsymbol{\nabla v}$ is the *gradient of* $\boldsymbol{v}$; to get a number from it, insert $\boldsymbol{\sigma}$ and $\boldsymbol{u}$; the result is $\langle \boldsymbol{\sigma}, \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} \rangle$ [same as in (a)].
   (c) $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$ is the *covariant derivative of* $\boldsymbol{v}$ *along* $\boldsymbol{u}$; to get a number from it, insert $\boldsymbol{\sigma}$; the result is $\langle \boldsymbol{\sigma}, \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} \rangle$ [same as in (a) and (b)].

B. How $\boldsymbol{\nabla}$ Differs from a Tensor

The machine $\boldsymbol{\nabla}$ differs from a tensor in two ways. (1) The middle slot of $\boldsymbol{\nabla}$ will not accept a vector; it demands a vector field—the vector field that is to be differentiated. (2) $\boldsymbol{\nabla}$ is not a linear machine (whereas a tensor must be linear!):

Box 10.3 (*continued*)

$$\begin{aligned}\boldsymbol{\nabla}(a\boldsymbol{\sigma}, f(\mathscr{P})\boldsymbol{v}(\mathscr{P}), b\boldsymbol{u}) &\equiv \langle a\boldsymbol{\sigma}, \boldsymbol{\nabla}_{b\boldsymbol{u}} f\boldsymbol{v}\rangle \\ &= abf\langle\boldsymbol{\sigma}, \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}\rangle + \underbrace{ab\langle\boldsymbol{\sigma}, \boldsymbol{v}\rangle\, \boldsymbol{\nabla}_{\boldsymbol{u}} f}_{\text{this would be absent if } \boldsymbol{\nabla} \text{ were a linear machine.}}\end{aligned}$$

C. The "Connection Coefficients" as Components of $\boldsymbol{\nabla}$

Given a tensor $\boldsymbol{S}$ of rank $\binom{1}{2}$, a basis of tangent vectors $\{\boldsymbol{e}_\alpha\}$ at the event $\mathscr{P}_0$ where $\boldsymbol{S}$ resides, and the dual basis of 1-forms $\{\boldsymbol{\omega}^\alpha\}$, one defines the components of $\boldsymbol{S}$ by

$$S^\alpha{}_{\beta\gamma} \equiv \boldsymbol{S}(\boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta, \boldsymbol{e}_\gamma).$$

One defines the components of $\boldsymbol{\nabla}$ similarly, except that for $\boldsymbol{\nabla}$ one needs not only a basis $\{\boldsymbol{e}_\alpha\}$ at the event $\mathscr{P}_0$, but also a basis $\{\boldsymbol{e}_\alpha(\mathscr{P})\}$ at each event $\mathscr{P}$ in its neighborhood:

$$\begin{aligned}\Gamma^\alpha{}_{\beta\gamma} &\equiv \text{components of } \boldsymbol{\nabla} = \boldsymbol{\nabla}(\boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta(\mathscr{P}), \boldsymbol{e}_\gamma) \\ &\equiv \langle\boldsymbol{\omega}^\alpha, \boldsymbol{\nabla}_{\boldsymbol{e}_\gamma}\boldsymbol{e}_\beta\rangle \\ &\simeq \left(\begin{array}{l}\text{“}\alpha\text{-component of change in basis vector } \boldsymbol{e}_\beta\text{, when} \\ \text{in evaluating } \boldsymbol{e}_\beta \text{ one moves from tail to tip of } \boldsymbol{e}_\gamma\text{”}\end{array}\right).\end{aligned}$$

These components of $\boldsymbol{\nabla}$ are called the *"connection coefficients"* of the basis $\{\boldsymbol{e}_\alpha\}$. They are the "coordinate representation" of the covariant derivative operator $\boldsymbol{\nabla}$.

The covariant derivative operator $\boldsymbol{\nabla}$ and the connection coefficients $\Gamma^\alpha{}_{\mu\nu}$ provide different mathematical representations of the same geometric animal? Preposterous! The one animal runs from place to place and barks, or at least bites (takes difference, for example, between vector fields at one place and at a nearby place). The other animal, endowed with forty faces (see exercise 10.9) sits quietly at one spot. It would be difficult for two animals to look more different. Yet they do the same jobs in any world compatible with the equivalence principle: (1) they summarize the properties of all geodesics that go through the point in question; and, so doing, (2) they provide a physical means (parallel transport) to compare the values of vector fields and tensor fields at two neighboring events.

given input vectors $\boldsymbol{v}$ and $\boldsymbol{u}$. If $\boldsymbol{v}(\mathscr{P})$ is not constrained to be "constant" along $\boldsymbol{u} = d/d\lambda$, then $(d/d\lambda)\,\langle\boldsymbol{\sigma}, \boldsymbol{v}\rangle$ has contributions from both the change in $\boldsymbol{v}$ and the change in $\boldsymbol{\sigma}$:

$$\frac{d}{d\lambda}\langle\boldsymbol{\sigma}, \boldsymbol{v}\rangle \equiv \boldsymbol{\nabla}_{\boldsymbol{u}}\langle\boldsymbol{\sigma}, \boldsymbol{v}\rangle = \langle\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma}, \boldsymbol{v}\rangle + \langle\boldsymbol{\sigma}, \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}\rangle \tag{10.7}$$

(see exercise 10.3).

Similarly, if $\boldsymbol{S}$ is a $\binom{1}{2}$ tensor field, then its gradient $\boldsymbol{\nabla S}$ is a $\binom{1}{3}$ tensor field defined as follows. Pick an event $\mathscr{P}_0$; pick three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, and a 1-form $\boldsymbol{\sigma}$ in the tangent space at $\mathscr{P}_0$; turn $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{\sigma}$ into "constant" vector fields and a "constant" 1-form field near $\mathscr{P}_0$ by means of parallel transport ($\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma} = 0$ at $\mathscr{P}_0$); then define

$$\begin{aligned}\boldsymbol{\nabla S}(\boldsymbol{\sigma}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}) \equiv (\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{S})(\boldsymbol{\sigma}, \boldsymbol{v}, \boldsymbol{w}) &\equiv \boldsymbol{\nabla}_{\boldsymbol{u}}[\boldsymbol{S}(\boldsymbol{\sigma}, \boldsymbol{v}, \boldsymbol{w})] \\ &= \partial_{\boldsymbol{u}}[\boldsymbol{S}(\boldsymbol{\sigma}, \boldsymbol{v}, \boldsymbol{w})].\end{aligned} \tag{10.8}$$

EXERCISES

Exercise 10.1. ADDITIVITY OF COVARIANT DIFFERENTIATION

Show that the commutator ("closer of quadrilaterals") is additive:

$$[\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w}] = [\boldsymbol{u}, \boldsymbol{v}] + [\boldsymbol{u}, \boldsymbol{w}]; \qquad [\boldsymbol{u} + \boldsymbol{n}, \boldsymbol{v}] = [\boldsymbol{u}, \boldsymbol{v}] + [\boldsymbol{n}, \boldsymbol{v}].$$

Use this result, the additivity condition $\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}$, and symmetry of the covariant derivative, $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} - \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{u} = [\boldsymbol{u}, \boldsymbol{v}]$, to prove that

$$\boldsymbol{\nabla}_{\boldsymbol{u}+\boldsymbol{n}}\boldsymbol{v} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\nabla}_{\boldsymbol{n}}\boldsymbol{v}.$$

Exercise 10.2. CHAIN RULE FOR COVARIANT DIFFERENTIATION

Use pictures, and the "take-the-difference-and-take-the-limit" definition of $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$ (Box 10.2) to show that

$$\boldsymbol{\nabla}_{\boldsymbol{u}}(f\boldsymbol{v}) = f\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{v}\,\partial_{\boldsymbol{u}}[f]. \tag{10.9}$$

Exercise 10.3. ANOTHER CHAIN RULE

Derive equation (10.7), using the "take-the-difference-and-take-the-limit" definitions of derivatives. *Hint:* Before taking the differences, parallel transport $\boldsymbol{\sigma}[\mathscr{P}(\lambda)]$ and $\boldsymbol{v}[\mathscr{P}(\lambda)]$ back from $\mathscr{P}(\lambda)$ to $\mathscr{P}(0)$.

Exercise 10.4. STILL ANOTHER CHAIN RULE

Show that, as in flat spacetime, so also in curved spacetime,

$$\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{v} \otimes \boldsymbol{w}) = (\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}) \otimes \boldsymbol{w} + \boldsymbol{v} \otimes (\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}). \tag{10.10}$$

Write down the more familiar component version of this equation in flat spacetime.

Solution to first part of exercise: Choose 1-forms $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ at the event $\mathscr{P}_0$ in question, and extend them along the vector $\boldsymbol{u} = d/d\lambda$ by parallel transport, $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\rho} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\sigma} = 0$. Then

$$
\begin{aligned}
[\nabla_{\boldsymbol{u}}(\boldsymbol{v} \otimes \boldsymbol{w})](\boldsymbol{\rho}, \boldsymbol{\sigma}) &= \frac{d}{d\lambda}[(\boldsymbol{v} \otimes \boldsymbol{w})(\boldsymbol{\rho}, \boldsymbol{\sigma})] && \text{(def of } \nabla_{\boldsymbol{u}} \text{ on a tensor)}\\
&= \frac{d}{d\lambda}[\langle \boldsymbol{\rho}, \boldsymbol{v}\rangle\langle \boldsymbol{\sigma}, \boldsymbol{w}\rangle] && \text{(def of tensor product “}\otimes\text{”)}\\
&= \frac{d\langle \boldsymbol{\rho}, \boldsymbol{v}\rangle}{d\lambda}\langle \boldsymbol{\sigma}, \boldsymbol{w}\rangle + \langle \boldsymbol{\rho}, \boldsymbol{v}\rangle \frac{d\langle \boldsymbol{\sigma}, \boldsymbol{w}\rangle}{d\lambda} && \text{(chain rule for derivatives)}\\
&= \langle \boldsymbol{\rho}, \nabla_{\boldsymbol{u}}\boldsymbol{v}\rangle\langle \boldsymbol{\sigma}, \boldsymbol{w}\rangle + \langle \boldsymbol{\rho}, \boldsymbol{v}\rangle\langle \boldsymbol{\sigma}, \nabla_{\boldsymbol{u}}\boldsymbol{w}\rangle\\
& && \text{(by equation 10.7 with } \boldsymbol{\rho}, \boldsymbol{\sigma} \text{ const)}\\
&= [(\nabla_{\boldsymbol{u}}\boldsymbol{v}) \otimes \boldsymbol{w}](\boldsymbol{\rho}, \boldsymbol{\sigma}) + [\boldsymbol{v} \otimes (\nabla_{\boldsymbol{u}}\boldsymbol{w})](\boldsymbol{\rho}, \boldsymbol{\sigma})\\
& && \text{(def of tensor product “}\otimes\text{”).}
\end{aligned}
$$

Exercise 10.5. ONE MORE CHAIN RULE

Show, using techniques similar to those in exercise 10.4, that

$$\nabla_{\boldsymbol{u}}(\boldsymbol{\sigma} \otimes \boldsymbol{\rho} \otimes \boldsymbol{v}) = (\nabla_{\boldsymbol{u}}\boldsymbol{\sigma}) \otimes \boldsymbol{\rho} \otimes \boldsymbol{v} + \boldsymbol{\sigma} \otimes (\nabla_{\boldsymbol{u}}\boldsymbol{\rho}) \otimes \boldsymbol{v} + \boldsymbol{\sigma} \otimes \boldsymbol{\rho} \otimes (\nabla_{\boldsymbol{u}}\boldsymbol{v}). \quad (10.11)$$

Exercise 10.6. GEODESIC EQUATION

Use the "Schild's ladder" construction process for parallel transport (beginning of Box 10.2) to show that a geodesic parallel transports its own tangent vector along itself (end of Box 10.2).

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§10.4. PARALLEL TRANSPORT AND COVARIANT DERIVATIVE: COMPONENT APPROACH

The pictorial approach motivates the mathematics; the abstract approach makes the pictorial ideas precise; but usually one must use the component approach in order to actually do complex calculations.

To work with components, one needs a set of basis vectors $\{\boldsymbol{e}_\alpha\}$ and the dual set of basis 1-forms $\{\boldsymbol{\omega}^\alpha\}$. In flat spacetime a single such basis suffices; all events can use the same Lorentz basis. Not so in curved spacetime! There each event has its own tangent space, and each tangent space requires a basis of its own. As one travels from event to event, comparing their bases via parallel transport, one sees the bases twist and turn. They must do so. In no other way can they accommodate themselves to the curvature of spacetime. Bases at points $\mathscr{P}_0$ and $\mathscr{P}_1$, which are the same when compared by parallel transport along one curve, must differ when compared along another curve (see "Curvature"; Chapter 11).

To quantify the twisting and turning of a "field" of basis vectors $\{\boldsymbol{e}_\alpha(\mathscr{P})\}$ and forms $\{\boldsymbol{\omega}^\alpha(\mathscr{P})\}$, use the covariant derivative. Examine the changes in vector fields along a basis vector $\boldsymbol{e}_\beta$, abbreviating

$$\nabla_{\boldsymbol{e}_\beta} \equiv \nabla_\beta \qquad \text{(def of } \nabla_\beta\text{);} \qquad (10.12)$$

and especially examine the rate of change of some basis vector: $\nabla_\beta \boldsymbol{e}_\alpha$. This rate of change is itself a vector, so it can be expanded in terms of the basis:

$$\underbrace{\nabla_{\beta} \boldsymbol{e}_{\alpha}} = \boldsymbol{e}_{\mu} \Gamma^{\mu}{}_{\underbrace{\alpha\beta}} \qquad (\text{def of } \Gamma^{\mu}{}_{\alpha\beta}); \tag{10.13}$$

note reversal of order of α and β!

Connection coefficients defined

and the resultant *"connection coefficients"* $\Gamma^{\mu}{}_{\alpha\beta}$ can be calculated by projection on the basis 1-forms:

$$\langle \boldsymbol{\omega}^{\mu}, \nabla_{\beta} \boldsymbol{e}_{\alpha} \rangle = \Gamma^{\mu}{}_{\alpha\beta}. \tag{10.14}$$

(See exercise 10.7; also Box 10.3.) Because the basis 1-forms are "locked into" the basis vectors ($\langle \boldsymbol{\omega}^{\nu}, \boldsymbol{e}_{\alpha} \rangle = \delta^{\nu}{}_{\alpha}$), these same connection coefficients $\Gamma^{\nu}{}_{\alpha\beta}$ tell how the 1-form basis changes from point to point:

$$\nabla_{\beta} \boldsymbol{\omega}^{\nu} = -\Gamma^{\nu}{}_{\alpha\beta} \boldsymbol{\omega}^{\alpha}, \tag{10.15}$$

$$\langle \nabla_{\beta} \boldsymbol{\omega}^{\nu}, \boldsymbol{e}_{\alpha} \rangle = -\Gamma^{\nu}{}_{\alpha\beta}. \tag{10.16}$$

(See exercise 10.8.)

The connection coefficients do even more. They allow one to calculate the components of the gradient of an arbitrary tensor $\boldsymbol{S}$. In a Lorentz frame of flat spacetime, the components of $\nabla \boldsymbol{S}$ are obtained by letting the basis vectors $\boldsymbol{e}_{\alpha} = \partial \mathscr{P}/\partial x^{\alpha} = \partial/\partial x^{\alpha}$ act on the components of $\boldsymbol{S}$. Thus for a $\binom{1}{2}$ tensor field $\boldsymbol{S}$ one finds that

$$\nabla \boldsymbol{S} \text{ has components } S^{\alpha}{}_{\beta\gamma,\delta} = \frac{\partial}{\partial x^{\delta}} [S^{\alpha}{}_{\beta\gamma}].$$

Not so in curved spacetime, or even in a non-Lorentz basis in flat spacetime. There the basis vectors turn, twist, expand, and contract, so even if $\boldsymbol{S}$ were constant ($\nabla \boldsymbol{S} = 0$), its components on the twisting basis vectors would vary. The connection coefficients, properly applied, will compensate for this twisting and turning. As one learns in exercise 10.10, the components of $\nabla \boldsymbol{S}$, called $S^{\alpha}{}_{\beta\gamma;\delta}$ so that

Components of the gradient of a tensor field

$$\nabla \boldsymbol{S} = S^{\alpha}{}_{\beta\gamma;\delta} \boldsymbol{e}_{\alpha} \otimes \boldsymbol{\omega}^{\beta} \otimes \boldsymbol{\omega}^{\gamma} \otimes \boldsymbol{\omega}^{\delta}, \tag{10.17}$$

can be calculated from those of $\boldsymbol{S}$ by the usual flat-space method, plus a correction applied to each index (i.e., to each basis vector):

["+" when correcting "up" indices]

[interchange and sum on index being corrected]

[differentiating index]

$$S^{\alpha}{}_{\beta\gamma;\delta} = S^{\alpha}{}_{\beta\gamma,\delta} + S^{\mu}{}_{\beta\gamma} \Gamma^{\alpha}{}_{\mu\delta} - S^{\alpha}{}_{\mu\gamma} \Gamma^{\mu}{}_{\beta\delta} - S^{\alpha}{}_{\beta\mu} \Gamma^{\mu}{}_{\gamma\delta}. \tag{10.18}$$

["−" when correcting "down" indices]

[interchange and sum on index being corrected]

[differentiating index]

Here

$$S^{\alpha}{}_{\beta\gamma,\delta} \equiv \boldsymbol{e}_{\delta}[S^{\alpha}{}_{\beta\gamma}] \equiv \partial_{\boldsymbol{e}_{\delta}} S^{\alpha}{}_{\beta\gamma}. \tag{10.19}$$

Equation (10.18) looks complicated; but it is really very simple, once the pattern has been grasped.

Components of the covariant derivative of a tensor field

Just as one uses special notation, $S^{\alpha}{}_{\beta\gamma;\delta}$, for the components of $\boldsymbol{\nabla S}$, so one introduces special notation, $DS^{\alpha}{}_{\beta\gamma}/d\lambda$, for components of the covariant derivative $\boldsymbol{\nabla_u S}$ along $\boldsymbol{u} = d/d\lambda$:

$$\boldsymbol{\nabla_u S} = (DS^{\alpha}{}_{\beta\gamma}/d\lambda)\boldsymbol{e}_{\alpha} \otimes \boldsymbol{\omega}^{\beta} \otimes \boldsymbol{\omega}^{\gamma}; \tag{10.20}$$

$$\frac{DS^{\alpha}{}_{\beta\gamma}}{d\lambda} = S^{\alpha}{}_{\beta\gamma;\delta}u^{\delta} = (S^{\alpha}{}_{\beta\gamma,\delta} + \text{correction terms})u^{\delta}.$$

Since for any f

$$f_{,\delta}u^{\delta} = \partial_{\boldsymbol{u}} f = df/d\lambda$$

this reduces to

$$\frac{DS^{\alpha}{}_{\beta\gamma}}{d\lambda} = \frac{dS^{\alpha}{}_{\beta\gamma}}{d\lambda} + S^{\mu}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\mu\delta}u^{\delta} - S^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta}u^{\delta} - S^{\alpha}{}_{\beta\mu}\Gamma^{\mu}{}_{\gamma\delta}u^{\delta}. \tag{10.21}$$

Chain rule for gradient

The power of the component approach shows up clearly when one discusses chain rules for covariant derivatives. The multitude of abstract-approach chain rules (equations 10.2b, 10.7, 10.10, 10.11) all boil down into a single rule for components: *The gradient operation ";" obeys the standard partial-differentiation chain rule of ordinary calculus.* Example:

$$(fv^{\alpha})_{;\mu} = \underbrace{f_{;\mu}}_{[= f_{,\mu} \text{ because } f \text{ has no indices to correct}]}v^{\alpha} + fv^{\alpha}{}_{;\mu} \tag{10.22a}$$

(contract this with u^{μ} to get chain rule 10.2b). Another example:

$$\underbrace{(\sigma_{\alpha}v^{\alpha})_{;\mu}}_{[= (\sigma_{\alpha}v^{\alpha})_{,\mu} \text{ because } \sigma_{\alpha}v^{\alpha} \text{ has no free indices to correct}]} = \sigma_{\alpha;\mu}v^{\alpha} + \sigma_{\alpha}v^{\alpha}{}_{;\mu} \tag{10.22b}$$

(contract this with u^{μ} to get chain rule 10.7). Another example:

$$(\sigma_{\alpha}\rho_{\beta}v^{\gamma})_{;\mu} = \sigma_{\alpha;\mu}\rho_{\beta}v^{\gamma} + \sigma_{\alpha}\rho_{\beta;\mu}v^{\gamma} + \sigma_{\alpha}\rho_{\beta}v^{\gamma}{}_{;\mu} \tag{10.22c}$$

(contract this with u^{μ} to get chain rule 10.11). Another example: see Exercise (10.12) below.

EXERCISES

Exercise 10.7. COMPUTATION OF CONNECTION COEFFICIENTS

Derive equation (10.14) for $\Gamma^{\mu}{}_{\alpha\beta}$ from equation (10.13).

Exercise 10.8. CONNECTION FOR 1-FORM BASIS

Derive equations (10.15) and (10.16), which relate $\boldsymbol{\nabla}_{\beta}\boldsymbol{\omega}^{\nu}$ *to* $\Gamma^{\nu}{}_{\alpha\beta}$, from equation (10.14). *Hint:* use equation (10.7).

Exercise 10.9. SYMMETRY OF CONNECTION COEFFICIENTS

Show that the symmetry of spacetime's covariant derivative (equation 10.2a) is equivalent to the following symmetry condition on the connection coefficients:

$$\begin{aligned}(\text{antisymmetric part of } \Gamma^{\mu}{}_{\alpha\beta}) &\equiv \frac{1}{2}(\Gamma^{\mu}{}_{\alpha\beta} - \Gamma^{\mu}{}_{\beta\alpha}) \\ &\equiv \Gamma^{\mu}{}_{[\alpha\beta]} = -\frac{1}{2}\langle \boldsymbol{\omega}^{\mu}, \underbrace{[\boldsymbol{e}_{\alpha}, \boldsymbol{e}_{\beta}]}_{\text{[commutator of basis vectors]}}\rangle \equiv -\frac{1}{2}c_{\alpha\beta}{}^{\mu}. \end{aligned} \tag{10.23}$$

As a special case, $\Gamma^{\mu}{}_{\alpha\beta}$ *is symmetric in* α *and* β *when a coordinate basis* ($\boldsymbol{e}_{\alpha} = \partial/\partial x^{\alpha}$) *is used.* Show that in a coordinate basis this symmetry reduces the number of independent connection coefficients at each event from $4 \times 4 \times 4 = 64$ to $4 \times 10 = 40$.

Exercise 10.10. COMPONENTS OF GRADIENT

Derive equation (10.18) for the components of the gradient, $S^{\alpha}{}_{\beta\gamma;\delta}$. *Hint:* Expand $\boldsymbol{S}$ in terms of the given basis, and then evaluate the righthand side of

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{S} = \boldsymbol{\nabla}_{\boldsymbol{u}}(S^{\alpha}{}_{\beta\gamma}\boldsymbol{e}_{\alpha} \otimes \boldsymbol{\omega}^{\beta} \otimes \boldsymbol{\omega}^{\gamma}),$$

for an arbitrary vector $\boldsymbol{u}$. Use the chain rules (10.2b) and (10.11). By comparing the result with

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{S} = S^{\alpha}{}_{\beta\gamma;\delta}u^{\delta}\boldsymbol{e}_{\alpha} \otimes \boldsymbol{\omega}^{\beta} \otimes \boldsymbol{\omega}^{\gamma},$$

read off the components $S^{\alpha}{}_{\beta\gamma;\delta}$.

Exercise 10.11. DIVERGENCE

Let $\boldsymbol{T}$ be a $\binom{2}{0}$ tensor field, and define the divergence on its second slot by the same process as in flat spacetime: $\boldsymbol{\nabla} \cdot \boldsymbol{T}$ = contraction of $\boldsymbol{\nabla T}$; i.e.,

$$(\boldsymbol{\nabla} \cdot \boldsymbol{T})^{\alpha} = T^{\alpha\beta}{}_{;\beta}. \tag{10.24}$$

Write the components $T^{\alpha\beta}{}_{;\beta}$ in terms of $T^{\alpha\beta}{}_{,\beta}$ plus correction terms for each of the two indices of $\boldsymbol{T}$.

[*Answer:*

$$T^{\alpha\beta}{}_{;\beta} = T^{\alpha\beta}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\beta}T^{\mu\beta} + \Gamma^{\beta}{}_{\mu\beta}T^{\alpha\mu}.]$$

Exercise 10.12. VERIFICATION OF CHAIN RULE

Let $S^{\alpha\beta}{}_{\gamma}$ be components of a $\binom{2}{1}$ tensor field, and $M_{\beta}{}^{\gamma}$ be components of a $\binom{1}{1}$ tensor field. By contracting these tensor fields, one obtains a vector field $S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma}$. The chain rule for the divergence of this vector field reads

$$(S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma})_{;\alpha} = S^{\alpha\beta}{}_{\gamma;\alpha}M_{\beta}{}^{\gamma} + S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma}{}_{;\alpha}.$$

Verify the validity of this chain rule by expressing both sides of the equation in terms of directional derivatives ($\partial_{\boldsymbol{e}_{\alpha}}$) plus connection-coefficient corrections. *Hint:* the left side becomes

$$(S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma})_{;\alpha} = \underbrace{(S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma})_{,\alpha}}_{\substack{[S^{\alpha\beta}{}_{\gamma,\alpha}M_{\beta}{}^{\gamma} + S^{\alpha\beta}{}_{\gamma}M_{\beta}{}^{\gamma}{}_{,\alpha} \\ \text{by chain rule for directional derivative}]}} + \Gamma^{\alpha}{}_{\mu\alpha}(S^{\mu\beta}{}_{\gamma}M_{\beta}{}^{\gamma}).$$

The right side has many more correction terms (three on $S^{\alpha\beta}{}_{\gamma;\alpha}$; two on $M_\beta{}^\gamma{}_{;\alpha}$), but they must cancel against each other, leaving only one.

Exercise 10.13. TRANSFORMATION LAW FOR CONNECTION COEFFICIENTS

Let $\{\boldsymbol{e}_\alpha\}$ and $\{\boldsymbol{e}_{\mu'}\}$ be two different fields of basis vectors related by the transformation law

$$\boldsymbol{e}_{\mu'}(\mathscr{P}) = L^\alpha{}_{\mu'}(\mathscr{P})\boldsymbol{e}_\alpha(\mathscr{P}). \tag{10.25}$$

Show that the corresponding connection coefficients are related by

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = \underbrace{L^{\alpha'}{}_\rho L^\mu{}_{\beta'} L^\nu{}_{\gamma'} \Gamma^\rho{}_{\mu\nu}}_{\substack{\text{standard transformation law}\\ \text{for components of a tensor}}} + L^{\alpha'}{}_\mu L^\mu{}_{\beta',\gamma'} \tag{10.26}$$

Exercise 10.14. POLAR COORDINATES IN FLAT 2-DIMENSIONAL SPACE

On a sheet of paper draw an (r, ϕ) polar coordinate system. At neighboring points, draw the basis vectors $\boldsymbol{e}_{\hat{r}} = \partial/\partial r$ and $\boldsymbol{e}_{\hat{\phi}} \equiv r^{-1}\,\partial/\partial\phi$. (a) Use this picture, and Euclid's version of parallel transport, to justify the relations

$$\nabla_{\hat{r}}\boldsymbol{e}_{\hat{r}} = 0, \qquad \nabla_{\hat{r}}\boldsymbol{e}_{\hat{\phi}} = 0, \qquad \nabla_{\hat{\phi}}\boldsymbol{e}_{\hat{r}} = r^{-1}\boldsymbol{e}_{\hat{\phi}}, \qquad \nabla_{\hat{\phi}}\boldsymbol{e}_{\hat{\phi}} = -r^{-1}\boldsymbol{e}_{\hat{r}}.$$

(b) From these relations write down the connection coefficients. (c) Let $\boldsymbol{A} = A^{\hat{r}}\boldsymbol{e}_{\hat{r}} + A^{\hat{\phi}}\boldsymbol{e}_{\hat{\phi}}$ be a vector field. Show that its divergence, $\nabla \cdot \boldsymbol{A} = A^{\hat{\alpha}}{}_{;\hat{\alpha}} = A^{\hat{\alpha}}{}_{,\hat{\alpha}} + \Gamma^{\hat{\alpha}}{}_{\hat{\mu}\hat{\alpha}}A^{\hat{\mu}}$, can be calculated using the formula

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r}\frac{\partial A^{\hat{\phi}}}{\partial \phi} + \frac{1}{r}\frac{\partial (rA^{\hat{r}})}{\partial r}$$

(which should be familiar to most readers).

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§10.5. GEODESIC EQUATION

Geodesics—the parametrized paths of freely falling particles—were the starting point of this chapter. From them parallel transport was constructed (Schild's ladder; Box 10.2); and parallel transport in turn produced the covariant derivative and its connection coefficients. Given the covariant derivative, one recovered the geodesics: they were the curves whose tangent vectors, $\boldsymbol{u} = d\mathscr{P}/d\lambda$, satisfy $\nabla_{\boldsymbol{u}}\boldsymbol{u} = 0$ ($\boldsymbol{u}$ is parallel transported along itself).

Geodesic equation: abstract version

Let a coordinate system $\{x^\alpha(\mathscr{P})\}$ be given. Let it induce basis vectors $\boldsymbol{e}_\alpha = \partial/\partial x^\alpha$ into the tangent space at each event. Let the connection coefficients $\Gamma^\alpha{}_{\beta\gamma}$ for this "coordinate basis" be given. Then the component version of the *"geodesic equation"* $\nabla_{\boldsymbol{u}}\boldsymbol{u} = 0$ becomes a differential equation for the geodesic $x^\alpha(\lambda)$:

(1) $$\boldsymbol{u} = \frac{d}{d\lambda} = \frac{dx^\alpha}{d\lambda}\frac{\partial}{\partial x^\alpha} \quad \Longrightarrow \quad \text{components of } \boldsymbol{u} \text{ are } u^\alpha = \frac{dx^\alpha}{d\lambda};$$

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(2) then components of $\nabla_{\boldsymbol{u}}\boldsymbol{u} = 0$ are

$$0 = u^\alpha{}_{;\beta}u^\beta = (u^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\gamma\beta}u^\gamma)u^\beta$$
$$= \frac{\partial}{\partial x^\beta}\left(\frac{dx^\alpha}{d\lambda}\right)\frac{dx^\beta}{d\lambda} + \Gamma^\alpha{}_{\gamma\beta}\frac{dx^\gamma}{d\lambda}\frac{dx^\beta}{d\lambda},$$

which reduces to the differential equation

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\gamma\beta}\frac{dx^\gamma}{d\lambda}\frac{dx^\beta}{d\lambda} = 0. \tag{10.27}$$

Component version

This component version of the geodesic equation gives an analytic method ("translation" of Schild's ladder) for constructing the parallel transport law from a knowledge of the geodesics. Pick an event $\mathscr{P}_0$ and set up a coordinate system in its neighborhood. Watch many clock-carrying particles pass through (or arbitrarily close to) $\mathscr{P}_0$. For each particle read off the values of $d^2x^\alpha/d\lambda^2$ and $dx^\alpha/d\lambda$ at $\mathscr{P}_0$. Insert all the data for many particles into equation (10.27), and solve for the connection coefficients. Do not be disturbed that only the symmetric part of $\Gamma^\alpha{}_{\gamma\beta}$ is obtained thereby; the antisymmetric part, $\Gamma^\alpha{}_{[\gamma\beta]}$, vanishes identically in any coordinate frame! (See exercise 10.9.) Knowing $\Gamma^\alpha{}_{\gamma\beta}$, use them to parallel transport any desired vector along any desired curve through $\mathscr{P}_0$:

How to construct parallel transport law from knowledge of geodesics

$$\nabla_{\boldsymbol{u}}\boldsymbol{v} = 0 \quad\Longleftrightarrow\quad \frac{dv^\alpha}{d\lambda} + \Gamma^\alpha{}_{\gamma\beta}v^\gamma\frac{dx^\beta}{d\lambda} = 0. \tag{10.28}$$

EXERCISES

Exercise 10.15. COMPONENTS OF PARALLEL-TRANSPORT LAW

Show that equation (10.28) is the component version of the law for parallel transporting a vector $\boldsymbol{v}$ along the curve $\mathscr{P}(\lambda)$ with tangent vector $\boldsymbol{u} = d\mathscr{P}/d\lambda$.

Exercise 10.16. GEODESICS IN POLAR COORDINATES

In rectangular coordinates on a flat sheet of paper, Euclid's straight lines (geodesics) satisfy $d^2x/d\lambda^2 = d^2y/d\lambda^2 = 0$. Transform this geodesic equation into polar coordinates ($x = r\cos\phi$, $y = r\sin\phi$); and read off the resulting connection coefficients by comparison with equation (10.27). These are the connection coefficients for the coordinate basis ($\partial/\partial r$, $\partial/\partial\phi$). From them calculate the connection coefficients for the basis

$$\boldsymbol{e}_{\hat{r}} = \frac{\partial}{\partial r}, \qquad \boldsymbol{e}_{\hat{\phi}} = \frac{1}{r}\frac{\partial}{\partial\phi}.$$

The answer should agree with the answer to part (b) of Exercise 10.14. *Hint:* Use such relations as

$$\nabla_{\boldsymbol{e}_{\hat{\phi}}}\boldsymbol{e}_{\hat{r}} = \nabla_{(1/r)(\partial/\partial\phi)}(\partial/\partial r) = \frac{1}{r}\nabla_{(\partial/\partial\phi)}(\partial/\partial r).$$

2

Exercise 10.17. ROTATION GROUP: GEODESICS AND CONNECTION COEFFICIENTS

[Continuation of exercises 9.13 and 9.14.] In discussing the rotation group, one must make a clear distinction between the *Euclidean space* (coordinates x, y, z; basis vectors $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$) in which the rotation matrices act, and the *group manifold* $SO(3)$ (coordinates ψ, θ, ϕ; coordinate basis $\partial/\partial\psi$, $\partial/\partial\theta$, $\partial/\partial\phi$; basis of "generators" $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$), whose points $\mathcal{P}$ are rotation matrices.

(a) Pick a vector

$$\boldsymbol{n} = n^x\,\partial/\partial x + n^y\,\partial/\partial y + n^z\,\partial/\partial z$$

in Euclidean space. Show that

$$\mathcal{R}_{\boldsymbol{n}}(t) \equiv \exp[(n^x\mathcal{K}_1 + n^y\mathcal{K}_2 + n^z\mathcal{K}_3)t] \tag{10.29}$$

is a rotation matrix that rotates the axes of Euclidean space by an angle

$$t|\boldsymbol{n}| = t[(n^x)^2 + (n^y)^2 + (n^z)^2]^{1/2}$$

about the direction $\boldsymbol{n}$. ($\mathcal{K}_j$ are matrices defined in exercise 9.13.)

(b) In the group manifold $SO(3)$, pick a point (rotation matrix) $\mathcal{P}$, and pick a tangent vector $\boldsymbol{u} = u^\alpha\boldsymbol{e}_\alpha$ at $\mathcal{P}$. Let $\boldsymbol{u}$ be a vector in Euclidean space with the same components as $\boldsymbol{u}$ has in $SO(3)$:

$$\boldsymbol{u} = u^1\boldsymbol{e}_1 + u^2\boldsymbol{e}_2 + u^3\boldsymbol{e}_3; \qquad \boldsymbol{u} = u^1\,\partial/\partial x + u^2\,\partial/\partial y + u^3\,\partial/\partial z. \tag{10.30}$$

Show that $\boldsymbol{u}$ is the tangent vector (at $t = 0$) to the curve

$$\mathcal{C}(t) = \mathcal{R}_{\boldsymbol{u}}(t)\mathcal{P}. \tag{10.31}$$

The curve $\mathcal{C}(t)$ through the arbitrary point $\mathcal{P}$ with arbitrary tangent vector $\boldsymbol{u} = (d\mathcal{C}/dt)_{t=0}$ is a very special curve: every point on it differs from $\mathcal{P}$ by a rotation $\mathcal{R}_{\boldsymbol{u}}(t)$ about one and the same direction $\boldsymbol{u}$. No other curve in $SO(3)$ with "starting conditions" $\{\mathcal{P}, \boldsymbol{u}\}$ has such beautiful simplicity. Hence it is natural to decree that each such $\mathcal{C}(t)$ is a geodesic of the group manifold $SO(3)$. This decree adds new geometric structure to $SO(3)$; it converts $SO(3)$ from a differentiable manifold into something more special: an *affine manifold.*

One has no guarantee that an arbitrarily chosen family of curves in an arbitrary manifold *can* be decreed to be geodesics. Most families of curves simply do not possess the right geometric properties to function as geodesics. Most will lead to covariant derivatives that violate one or more of the fundamental conditions (10.2). To learn whether a given choice of geodesics is possible, one can try to derive connection coefficients $\Gamma^\alpha{}_{\beta\gamma}$ (for some given basis) corresponding to the chosen geodesics. If the derivation is successful, the choice of geodesics was a possible one. If the derivation produces inconsistencies, the chosen family of curves have the wrong geometric properties to function as geodesics.

(c) For the basis of generators $\{\boldsymbol{e}_\alpha\}$ derive connection coefficients corresponding to the chosen geodesics, $\mathcal{C}(t) = \mathcal{R}_{\boldsymbol{u}}(t)\mathcal{P}$, of $SO(3)$. *Hint:* show that the components $u^\alpha = \langle\boldsymbol{\omega}^\alpha, \boldsymbol{u}\rangle$ of the tangent $\boldsymbol{u} = d\mathcal{C}/dt$ to a given geodesic are independent of position $\mathcal{C}(t)$ along the geodesic. Then use the geodesic equation $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = 0$, expanded in the basis $\{\boldsymbol{e}_\alpha\}$, to calculate the symmetric part of the connection $\Gamma^\alpha{}_{(\beta\gamma)}$. Finally use equation (10.23) to calculate $\Gamma^\alpha{}_{[\beta\gamma]}$. [*Answer:*

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\epsilon_{\alpha\beta\gamma}, \tag{10.32}$$

where $\epsilon_{\alpha\beta\gamma}$ is the completely antisymmetric symbol with $\epsilon_{123} = +1$. This answer is independent of location $\mathcal{P}$ in $SO(3)$!]

CHAPTER 11

GEODESIC DEVIATION AND SPACETIME CURVATURE

§11.1. CURVATURE, AT LAST!

Spacetime curvature manifests itself as gravitation, by means of the deviation of one geodesic from a nearby geodesic (relative acceleration of test particles).

This chapter is entirely Track 2. Chapters 9 and 10 are necessary preparation for it.
It will be needed as preparation for
(1) Chapters 12 and 13 (Newtonian gravity; Riemannian geometry),
(2) the second half of Chapter 14 (calculation of curvature), and
(3) the details, but not the message, of Chapter 15 (Bianchi identities).

Let the geodesics of spacetime be known. Then the covariant derivative ∇ and its connection coefficients $\Gamma^{\alpha}{}_{\beta\gamma}$ are also known. How, from this information, does one define, calculate, and understand geodesic deviation and spacetime curvature? The answer unfolds in this chapter, and is summarized in Box 11.1. To disclose the answer one must (1) define the "relative acceleration vector" $\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{n}$, which measures the deviation of one geodesic from another (§11.2); (2) derive an expression in terms of ∇ or $\Gamma^{\alpha}{}_{\beta\gamma}$ for the "Riemann curvature tensor," which produces the geodesic deviation (§11.3); (3) see Riemann curvature at work, producing changes in vectors that are parallel transported around closed circuits (§11.4); (4) see Riemann curvature test whether spacetime is flat (§11.5); and (5) construct a special coordinate system, "Riemann normal coordinates," which is tied in a special way to the Riemann curvature tensor (§11.6).

Overview of chapter

§11.2. THE RELATIVE ACCELERATION OF NEIGHBORING GEODESICS

Geometry of a family of geodesics:

Selector parameter

Focus attention on a family of geodesics (Figure 11.1). Let one geodesic be distinguished from another by the value of a "selector parameter" n. The family includes not only geodesics $n = 0, 1, 2, \ldots$ but also geodesics for all intervening values of

2

Box 11.1 GEODESIC DEVIATION AND RIEMANN CURVATURE IN BRIEF

"Geodesic separation" $\boldsymbol{n}$ is displacement (tangent vector) from point on fiducial geodesic to point on nearby geodesic characterized by same value of affine parameter λ.

Geodesic separation changes with respect to λ (i.e., changes along the tangent vector $\boldsymbol{u} = d/d\lambda$) at a rate given by the *equation of geodesic deviation*

$$\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{u}} \boldsymbol{n} + \textbf{Riemann}\,(\ldots, \boldsymbol{u}, \boldsymbol{n}, \boldsymbol{u}) = 0 \quad (1)$$

(second-order equation; see §§1.6 and 1.7; Figures 1.10, 1.11, 1.12).

In terms of components of the Riemann tensor the driving force ("tidal graviational force") is

$$\textbf{Riemann}\,(\ldots, \boldsymbol{u}, \boldsymbol{n}, \boldsymbol{u}) = \boldsymbol{e}_\alpha R^\alpha{}_{\beta\gamma\delta} u^\beta n^\gamma u^\delta. \quad (2)$$

The components of the Riemann curvature tensor in a coordinate frame are given in terms of the connection coefficients by the formula

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} = {} & \frac{\partial \Gamma^\alpha{}_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha{}_{\beta\gamma}}{\partial x^\delta} \\ & + \Gamma^\alpha{}_{\mu\gamma} \Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta} \Gamma^\mu{}_{\beta\gamma}. \end{aligned} \quad (3)$$

This curvature tensor not only quantifies the concept of "tidal gravitational force," but also enters into Einstein's law, by which "matter tells spacetime how to curve." That law, to be studied in later chapters, takes the following operational-computational form in a given coordinate system:

(a) Write down trial formula for dynamic evolution of metric coefficients $g_{\mu\nu}$ with time.

(b) Calculate the connection coefficients from

$$\Gamma^\alpha{}_{\mu\nu} = g^{\alpha\beta} \Gamma_{\beta\mu\nu}; \quad (4)$$

$$\Gamma_{\beta\mu\nu} = \frac{1}{2}\left(\frac{\partial g_{\beta\nu}}{\partial x^\mu} + \frac{\partial g_{\beta\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right) \quad (5)$$

(derived in Chapter 13).

(c) Calculate Riemann curvature tensor from equation (3).

(d) Calculate Einstein curvature tensor from

$$G_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} - \frac{1}{2} g_{\mu\nu} g^{\sigma\tau} R^\alpha{}_{\sigma\alpha\tau} \quad (6)$$

(geometric significance in Chapter 15).

(e) Insert into Einstein's equations (Chapter 17):

$G_{\mu\nu} = 0$ (empty space),

$G_{\mu\nu} = 8\pi T_{\mu\nu}$ (when mass-energy is present).

(f) Test whether the trial formula for the dynamic evolution of the geometry was correct, and, if not, change it so it is.

Affine parameter

n. The typical point $\mathscr{P}$ on the typical geodesic will be a continuous, doubly differentiable function of the selector parameter n and the affine parameter λ; thus

$$\mathscr{P} = \mathscr{P}(\lambda, n). \quad (11.1)$$

Tangent vector

The tangent vector

$$\boldsymbol{u} = \frac{\partial \mathscr{P}}{\partial \lambda} \quad \text{(Cartan notation)}$$

or

$$\boldsymbol{u} = \frac{\partial}{\partial \lambda} \quad \text{(notation of this book)} \quad (11.2)$$

is constant along any given geodesic in this sense: the vector $\boldsymbol{u}$ at any point, trans-

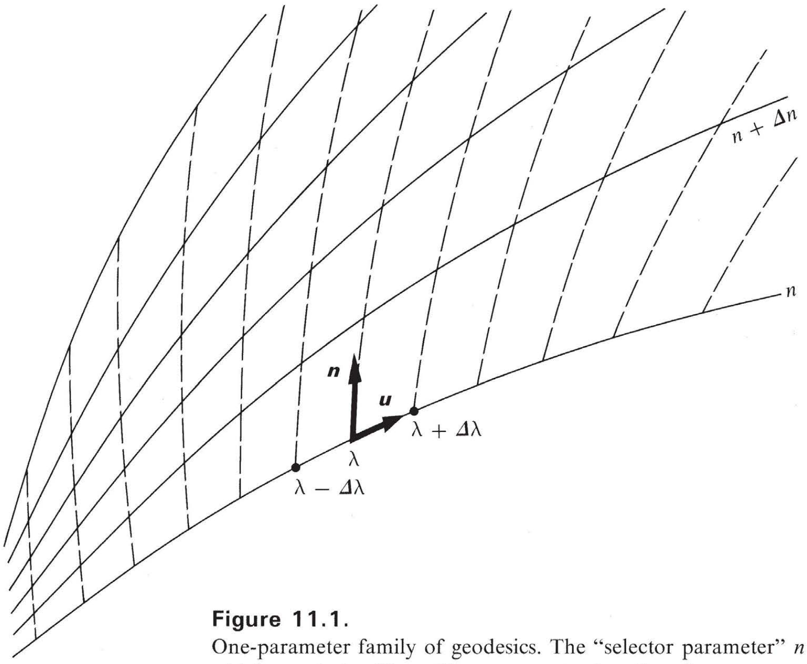


Figure 11.1.
One-parameter family of geodesics. The "selector parameter" n tells which geodesic. The affine parameter λ tells where on a given geodesic. The two tangent vectors indicated in the diagram are $\boldsymbol{u} = \partial/\partial\lambda$ (Cartan: $\partial\mathscr{P}/\partial\lambda$) and $\boldsymbol{n} = \partial/\partial n$ (Cartan: $\partial\mathscr{P}/\partial n$).

ported parallel to itself along the geodesic, arrives at a second point coincident in direction and length with the $\boldsymbol{u}$ already existing at that point.

Separation vector

The *"separation vector"*

$$\boldsymbol{n} = \frac{\partial\mathscr{P}}{\partial n} \qquad \text{(Cartan notation)}$$

or

$$\boldsymbol{n} = \frac{\partial}{\partial n} \qquad \text{(notation of this book)} \tag{11.3}$$

measures the separation between the geodesic n, regarded as the fiducial geodesic, and the typical nearby geodesic, $n + \Delta n$ (for small Δn), in the sense that

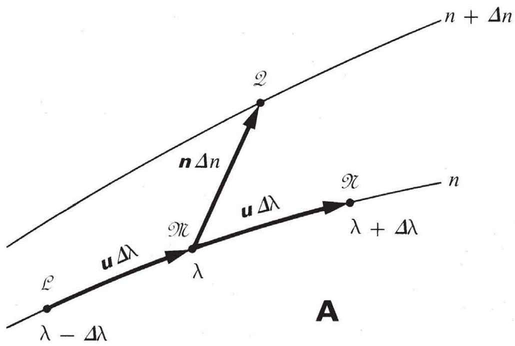
$$(\Delta n)\boldsymbol{n} = \begin{Bmatrix} \Delta n \dfrac{\partial\mathscr{P}}{\partial n} \\ \\ \Delta n \dfrac{\partial}{\partial n} \end{Bmatrix} \begin{matrix} \text{measures the} \\ \text{change in} \end{matrix} \begin{Bmatrix} \text{position} \\ \\ \text{any} \\ \text{function} \end{Bmatrix} \tag{11.4}$$

brought about by transfer of attention from the one geodesic to the other at a fixed value of the affine parameter λ. This vector is represented by the arrow $\mathscr{M}\mathscr{Q}$ in the first diagram in Box 11.2.

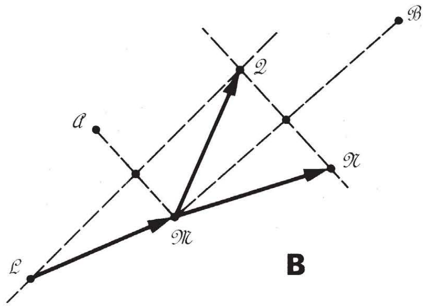
(continued on page 270)

Box 11.2 GEODESIC DEVIATION REPRESENTED AS AN ARROW

"Fiducial geodesic" n. Separation vector $\boldsymbol{n}\,\Delta n = \mathcal{M}\mathcal{Q}$ leads from point $\mathcal{M}$ on it, to point $\mathcal{Q}$ with same value of affine parameter λ (timelike quantity) on neighboring "test geodesic" $n + \Delta n$.



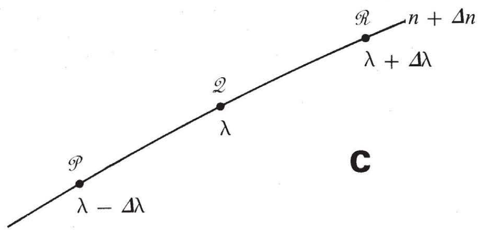
A



B

Parallel transport of $\mathcal{M}\mathcal{Q}$ by "Schild's ladder construction" (Box 10.2) to $\mathcal{N}\mathcal{B}$ and $\mathcal{L}\mathcal{A}$. If the test geodesic $n + \Delta n$ had kept a constant separation from the fiducial geodesic n, its tracer point would have arrived at $\mathcal{A}$ at the value $(\lambda - \Delta\lambda)$ of the affine parameter, and at $\mathcal{B}$ at $(\lambda + \Delta\lambda)$.

Actual location of tracer point of test geodesic at values of the timelike affine parameter $(\lambda - \Delta\lambda)$, λ, and $(\lambda + \Delta\lambda)$.



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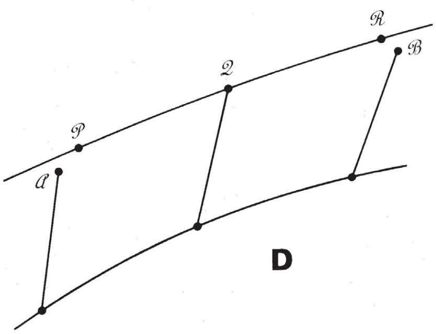
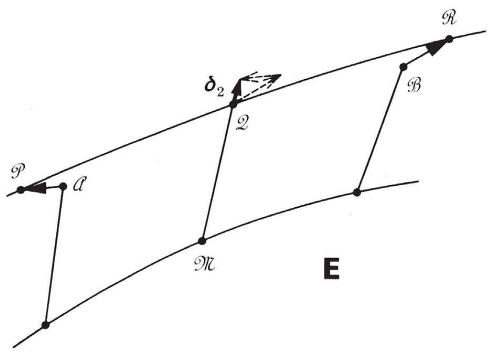
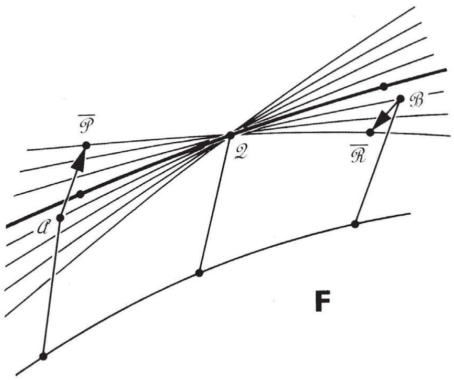


D

Confrontation between actual course of tracer point on test geodesic and "canonical course": course it would have had to take to keep constant separation from the tracer point moving along the fiducial geodesic.

Test geodesic same as before, except for uniform stretchout in scale of affine parameter. Any measure of departure of the actual course of geodesic from the canonical course ($\mathcal{A}\mathcal{Q}\mathcal{B}$), to be useful, should be independent of this stretchout. Hence, take as measure of geodesic deviation, not the vector $\mathcal{B}\mathcal{R}$ alone, nor the vector $\mathcal{A}\mathcal{P}$, but the stretch-independent combination $\boldsymbol{\delta}_2 = (\mathcal{B}\mathcal{R}) + (\mathcal{A}\mathcal{P})$. Here the sign of addition implies that the two vectors have been transported parallel to themselves, before addition, to a common point ($\mathcal{Q}$ in the diagram; $\mathcal{M}$ in the differential calculus limit $\Delta n \longrightarrow 0$, $\Delta\lambda \longrightarrow 0$).





Alternative courses that the test geodesic of **D** could have taken through $\mathcal{Q}$ (families of geodesics characterized by different degrees of divergence from the left or convergence towards the right). Tilt changes values of $\mathcal{A}\mathcal{P}$ (to $\mathcal{A}\bar{\mathcal{P}}$) and $\mathcal{B}\mathcal{R}$ (to $\mathcal{B}\bar{\mathcal{R}}$) individually, but not value of the sum $\boldsymbol{\delta}_2 = (\mathcal{B}\mathcal{R}) + (\mathcal{A}\mathcal{P})$ ("lever principle").

Note that arrow $\mathcal{B}\mathcal{R}$ is of first order in $\Delta\lambda$ and of first order in Δn; similarly for $\mathcal{A}\mathcal{P}$; hence the combination $\boldsymbol{\delta}_2$ is of second order in $\Delta\lambda$ and first order in Δn. Conclude that *the arrow* $\boldsymbol{\delta}_2/(\Delta\lambda)^2(\Delta n)$ *is the desired measure of geodesic deviation* in the sense that:

size of mesh (ultimately to go to zero) cancels out;
parameterization of test geodesic cancels out;
slope of test geodesic cancels out.

Give this arrow the name *"relative-acceleration vector";* and by examining it more closely (Box 11.3), discover the formula

$$\boldsymbol{\delta}_2/(\Delta\lambda)^2(\Delta n) = \boldsymbol{\nabla}_{\boldsymbol{u}}\, \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{n}$$

for it.

Box 11.2 illustrates what it means to speak of geodesic deviation. One transports the separation $\boldsymbol{n}\,\Delta n = \mathcal{MQ}$ parallel to itself along the fiducial geodesic. The tip of this vector traces out the canonical course that the nearby tracer point would have to pursue if it were to maintain constant separation from the fiducial tracer point. The actual course of the test geodesic deviates from this "canonical" course. The deviation, a vector ($\mathcal{AP}$ of Box 11.2), changes with the affine parameter ($\mathcal{AP}$ at $\mathcal{A}$, 0 at $\mathcal{Q}$, $\mathcal{BR}$ at $\mathcal{B}$). The first derivative of this vector with respect to the affine parameter is sensitive to the scale of parameterization along the test geodesic, and to its slope (Box 11.2, **F**). Not so the second derivative. It depends only on the tangent vector $\boldsymbol{u}$ of the fiducial geodesic, and on the separation vector $\boldsymbol{n}\,\Delta n$. Divide this second derivative of the deviation by Δn and give it a name: the "*relative-acceleration vector*". Discover (Box 11.3) a simple formula for it

Relative-acceleration vector

$$\text{(relative-acceleration vector)} = \boldsymbol{\nabla}_{\boldsymbol{u}}\,\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n}. \tag{11.5}$$

§11.3. TIDAL GRAVITATIONAL FORCES AND RIEMANN CURVATURE TENSOR

With "relative acceleration" now defined, turn to the "tidal gravitational force" (i.e., "spacetime curvature") that produces it. Use a Newtonian analysis of tidal forces

Box 11.3 GEODESIC DEVIATION: ARROW CORRELATED WITH SECOND COVARIANT DERIVATIVE

The arrow $\boldsymbol{\delta}_2$ in Box 11.2 measures, not the rate of change of the separation of the test geodesic $n + \Delta n$ from the "canonical course" $\mathcal{AQB}$ as baseline, but the second derivative:

$$\left(\text{first derivative at } \lambda + \frac{1}{2}\Delta\lambda\right) = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n} = \frac{\mathcal{NR} - \mathcal{NB}}{\Delta\lambda\,\Delta n} = \frac{\mathcal{BR}}{\Delta\lambda\,\Delta n};$$

$$\left(\text{first derivative at } \lambda - \frac{1}{2}\Delta\lambda\right) = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n} = \frac{\mathcal{LA} - \mathcal{LP}}{\Delta\lambda\,\Delta n} = \frac{-\mathcal{AP}}{\Delta\lambda\,\Delta n}.$$

Transpose to common location λ, take difference, and divide it by $\Delta\lambda$ to obtain the second covariant derivative with respect to the vector $\boldsymbol{u}$; thus

$$\begin{aligned}\boldsymbol{\nabla}_{\boldsymbol{u}}\,\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n} &= \frac{(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n})_{\lambda+\frac{1}{2}\Delta\lambda} - (\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n})_{\lambda-\frac{1}{2}\Delta\lambda}}{\Delta\lambda} \\ &= \frac{(\mathcal{BR} + \mathcal{AP})_{\substack{\text{vectors transported to}\\ \text{common location}}}}{(\Delta\lambda)^2\,\Delta n} = \frac{\boldsymbol{\delta}_2}{(\Delta\lambda)^2\,\Delta n} \\ &= \text{"relative acceleration vector" for neighboring geodesics.}\end{aligned}$$

(left half of Box 11.4) to motivate the geometric analysis (right half of same box). Thereby arrive at the remarkable equation

Tide-producing gravitational forces expressed in terms of a commutator

$$\underbrace{\nabla_u \nabla_u \boldsymbol{n}}_{\text{"relative acceleration"}} + \underbrace{[\nabla_n, \nabla_u]\boldsymbol{u}}_{\text{"tide-producing gravitational forces"}} = 0. \tag{11.6}$$

This equation is remarkable, because at first sight it seems crazy. The term $[\nabla_n, \nabla_u]\boldsymbol{u}$ involves second derivatives of $\boldsymbol{u}$, and a first derivative of ∇_n:

$$[\nabla_n, \nabla_u]\boldsymbol{u} \equiv \nabla_n \nabla_u \boldsymbol{u} - \nabla_u \nabla_n \boldsymbol{u}. \tag{11.7}$$

It thus must depend on how $\boldsymbol{u}$ and $\boldsymbol{n}$ vary from point to point. But the relative acceleration it produces, $\nabla_u \nabla_u \boldsymbol{n}$, is known to depend only on the values of $\boldsymbol{u}$ and $\boldsymbol{n}$ at the fiducial point, not on how $\boldsymbol{u}$ and $\boldsymbol{n}$ vary (see Box 11.2, **F**). How is this possible?

Somehow all derivatives must drop out of the tidal-force quantity $[\nabla_n, \nabla_u]\boldsymbol{u}$. One must be able to regard $[\nabla_{\ldots}, \nabla_{\ldots}]\ldots$ as a purely local, algebraic machine with three slots, whose output is a vector. If it is purely local and not differential, then it is even linear (as one sees from the additivity properties of ∇), so it must be a tensor. Give this tensor the name ***Riemann***, and give it a fourth slot for inputting a 1-form:

$$\textbf{\textit{Riemann}}\,(\ldots, \boldsymbol{C}, \boldsymbol{A}, \boldsymbol{B}) \equiv [\nabla_A, \nabla_B]\boldsymbol{C};$$
$$\textbf{\textit{Riemann}}\,(\boldsymbol{\sigma}, \boldsymbol{C}, \boldsymbol{A}, \boldsymbol{B}) \equiv \langle \boldsymbol{\sigma}, [\nabla_A, \nabla_B]\boldsymbol{C}\rangle.$$

This is only a tentative definition of ***Riemann***. Before accepting it, one should verify that it is, indeed, a tensor. Does it *really* depend on only the values of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ at the point of evaluation, and not on how they are changing there? The answer (derived in Box 11.5) is "almost." It fails the test, but with a slight modification it will pass. The modification is to replace the commutator $[\nabla_A, \nabla_B]$ by the *"curvature operator"*

Curvature operator defined

$$\mathcal{R}(\boldsymbol{A}, \boldsymbol{B}) \equiv [\nabla_A, \nabla_B] - \nabla_{[A,B]}, \tag{11.8}$$

where $\nabla_{[A,B]}$ is the derivative along the vector $[\boldsymbol{A}, \boldsymbol{B}]$ (commutator of $\boldsymbol{A}$ and $\boldsymbol{B}$). ($\mathcal{R}(\boldsymbol{A}, \boldsymbol{B}) \equiv [\nabla_A, \nabla_B]$ for the fields $\boldsymbol{A} = \boldsymbol{n}$ and $\boldsymbol{B} = \boldsymbol{u}$ of the geodesic-deviation problem, because $[\boldsymbol{n}, \boldsymbol{u}] = 0$.) Then the modified and acceptable *definition of the Riemann curvature tensor is*

Riemann curvature tensor defined

$$\begin{aligned} \textbf{\textit{Riemann}}\,(\ldots, \boldsymbol{C}, \boldsymbol{A}, \boldsymbol{B}) &\equiv \mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}; \\ \textbf{\textit{Riemann}}\,(\boldsymbol{\sigma}, \boldsymbol{C}, \boldsymbol{A}, \boldsymbol{B}) &\equiv \langle \boldsymbol{\sigma}, \mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}\rangle. \end{aligned} \tag{11.9}$$

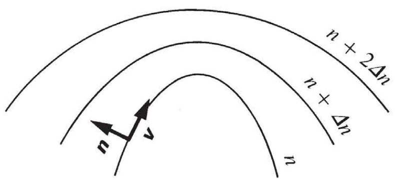
To define ***Riemann*** thus, and to verify its tensorial character (exercise 11.2), does not by any means teach one what curvature is all about. To understand curvature, one must scrutinize ***Riemann*** from all viewpoints. That is the task of the rest of this chapter.

(continued on page 275)

Box 11.4 RELATIVE ACCELERATION OF TEST PARTICLES—GEOMETRIC ANALYSIS PATTERNED ON NEWTONIAN ANALYSIS

Newtonian Analysis

1. Consider a family of test-particle trajectories $x^j(t, n)$ in ordinary, three-dimensional space: "t" is time measured by particle's clock, or any clock; "n" is "selector parameter."



2. Equation of motion for each trajectory:

$$\left(\frac{\partial^2 x^j}{\partial t^2}\right)_n + \frac{\partial \Phi}{\partial x^j} = 0,$$

where Φ is Newtonian potential.

3. Take difference between equations of motion for neighboring trajectories, n and $n + \Delta n$, and take limit as $\Delta n \longrightarrow 0$—i.e., take derivative

$$\left(\frac{\partial}{\partial n}\right)_t \left[\left(\frac{\partial^2 x^j}{\partial t^2}\right)_n + \frac{\partial \Phi}{\partial x^j}\right] = 0.$$

4. When $\partial/\partial n$ acts on second term, rewrite it as

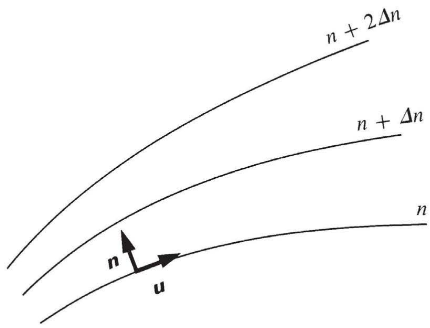
$$\left(\frac{\partial}{\partial n}\right)_t = \left(\frac{\partial x^k}{\partial n}\right)_t \frac{\partial}{\partial x^k} = n^k \frac{\partial}{\partial x^k};$$

Thereby obtain

$$\left(\frac{\partial}{\partial n}\right)_t \left(\frac{\partial}{\partial t}\right)_n \left(\frac{\partial x^j}{\partial t}\right)_n + \frac{\partial^2 \Phi}{\partial x^j \partial x^k} n^k = 0.$$

Geometric Analysis

1. Consider a family of test-particle trajectories (geodesics), $\mathscr{P}(\lambda, n)$, in spacetime: "λ" is affine-parameter, i.e., time measured by particle's clock; "n" is "selector parameter."



2. Geodesic equation for each trajectory:

$$\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{u} = 0.$$

[Looks like first-order equation; is actually second-order because the "$\boldsymbol{u}$" being differentiated is itself a derivative, $\boldsymbol{u} = (\partial \mathscr{P}/\partial \lambda)_n$.]

3. Take difference between geodesic equations for neighboring geodesics n and $n + \Delta n$, and take limit as $\Delta n \longrightarrow 0$—i.e., take covariant derivative

$$\boldsymbol{\nabla}_{\boldsymbol{n}} [\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{u}] = 0.$$

4. There is no second term, so leave equation in form

$$\boldsymbol{\nabla}_{\boldsymbol{n}} [\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{u}] = 0.$$

5. To obtain equation for relative acceleration, move $(\partial/\partial n)_t$ through both of the $(\partial/\partial t)_n$ terms (permissible because partial derivatives commute!):

$$\left(\frac{\partial}{\partial t}\right)_n \left(\frac{\partial}{\partial t}\right)_n \left(\frac{\partial x^j}{\partial n}\right)_t + \frac{\partial^2 \Phi}{\partial x^j\, \partial x^k} n^k = 0.$$

This is equivalent to

$$\underbrace{\left(\frac{\partial^2 n^j}{\partial t^2}\right)}_{\text{“relative acceleration”}} + \underbrace{\frac{\partial^2 \Phi}{\partial x^j\, \partial x^k} n^k}_{\text{“tide-producing gravitational forces”}} = 0.$$

5. To obtain equation for relative acceleration, $\nabla_u \nabla_u \boldsymbol{n}$, move ∇_n through ∇_u and through the $\partial/\partial\lambda$ of $\boldsymbol{u} = \partial\mathscr{P}/\partial\lambda$:

a. *First step:* In $\nabla_n \nabla_u \boldsymbol{u} = 0$, move ∇_n through ∇_u. The result:

$$(\nabla_u \nabla_n + \underbrace{[\nabla_n, \nabla_u]}_{\substack{\text{commutator; must be included}\\ \text{as protection against possibility}\\ \text{that } \nabla_u \nabla_n \neq \nabla_n \nabla_u.}})\boldsymbol{u} = 0.$$

b. *Second step:* Move ∇_n through $\partial/\partial\lambda$ of $\boldsymbol{u} = \partial\mathscr{P}/\partial\lambda$; i.e., write

$$\nabla_n \frac{\partial\mathscr{P}}{\partial\lambda} \underset{\text{def. of } \boldsymbol{u}}{=} \nabla_n \boldsymbol{u} = \nabla_u \boldsymbol{n} \underset{\text{def. of } \boldsymbol{n}}{=} \nabla_u \frac{\partial\mathscr{P}}{\partial n}$$

Why? Because symmetry of covariant derivative says $\nabla_n \boldsymbol{u} - \nabla_u \boldsymbol{n} = [\boldsymbol{n}, \boldsymbol{u}]$

$$= \left[\frac{\partial}{\partial n}, \frac{\partial}{\partial\lambda}\right] = \frac{\partial^2}{\partial n\, \partial\lambda} - \frac{\partial^2}{\partial\lambda\, \partial n} = 0.$$

c. Result:

$$\underbrace{\nabla_u \nabla_u \boldsymbol{n}}_{\text{“relative acceleration”}} + \underbrace{[\nabla_n, \nabla_u]\boldsymbol{u}}_{\substack{\text{“tide-producing gravitational forces”;}\\ \text{i.e., “spacetime curvature”}}} = 0$$

Box 11.5 RIEMANN CURVATURE TENSOR

A. Definition of *Riemann* Motivated by Tidal Gravitational Forces:

1. Tidal forces (spacetime curvature) produce relative acceleration of test particles (geodesics) given by

$$\nabla_u \nabla_u \boldsymbol{n} + [\nabla_n, \nabla_u]\boldsymbol{u} = 0. \qquad (1)$$

Box 11.5 (*continued*)

2. This motivates the definition

$$\boldsymbol{Riemann}\,(\ldots, \boldsymbol{C}, \boldsymbol{A}, \boldsymbol{B}) = [\boldsymbol{\nabla}_{\boldsymbol{A}}, \boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}. \tag{2}$$

[empty slot for inserting a one-form]

B. Failure of this Definition

1. Definition acceptable only if ***Riemann*** $(\ldots, \boldsymbol{C}, \boldsymbol{A}, \boldsymbol{B})$ is a linear machine, independent of how $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ vary from point to point.
2. Check, in part: change variations of $\boldsymbol{C}$, but not $\boldsymbol{C}$ itself, at event $\mathscr{P}_0$:

$$\boldsymbol{C}_{\mathrm{NEW}}(\mathscr{P}) = f(\mathscr{P})\boldsymbol{C}_{\mathrm{OLD}}(\mathscr{P}).$$

[arbitrary function except $f(\mathscr{P}_0) = 1$]

3. Does this change $[\boldsymbol{\nabla}_{\boldsymbol{A}}, \boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}$? *Yes!* Exercise 11.1 shows

$$\{[\boldsymbol{\nabla}_{\boldsymbol{A}}, \boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}_{\mathrm{NEW}}\}_{\mathrm{at}\,\mathscr{P}_0} - \{[\boldsymbol{\nabla}_{\boldsymbol{A}}, \boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}_{\mathrm{OLD}}\}_{\mathscr{P}_0} = \boldsymbol{C}_{\mathrm{OLD}}\,\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}f.$$

C. Modified Definition of *Riemann*:

1. The term causing trouble, $\boldsymbol{C}_{\mathrm{OLD}}\,\boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]}f$, can be disposed of by subtracting a "correction term" resembling it from ***Riemann***—i.e., by redefining

$$\boldsymbol{Riemann}\,(\ldots, \boldsymbol{C}, \boldsymbol{A}, \boldsymbol{B}) \equiv \mathscr{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}, \tag{3}$$

$$\mathscr{R}(\boldsymbol{A}, \boldsymbol{B}) \equiv [\boldsymbol{\nabla}_{\boldsymbol{A}}, \boldsymbol{\nabla}_{\boldsymbol{B}}] - \boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]} \tag{4}$$

2. The above calculation then gives a result independent of the "modifying function" f:

$$\{\mathscr{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}_{\mathrm{NEW}}\}_{\mathrm{at}\,\mathscr{P}_0} = \{\mathscr{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}_{\mathrm{OLD}}\}_{\mathrm{at}\,\mathscr{P}_0}.$$

D. Is Modified Definition Compatible with Equation for Tidal Gravitational Forces?

1. One would like to write $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n} + \boldsymbol{Riemann}\,(\ldots, \boldsymbol{u}, \boldsymbol{n}, \boldsymbol{u}) = 0$.
2. This works just as well for modified definition of ***Riemann*** as for original definition, because

$$\mathcal{R}(\boldsymbol{n}, \boldsymbol{u}) = [\nabla_{\boldsymbol{n}}, \nabla_{\boldsymbol{u}}] - \underbrace{\nabla_{[\boldsymbol{n},\boldsymbol{u}]}}_{\substack{= 0 \text{ because } \boldsymbol{n} = (\partial/\partial n)_\lambda \text{ and} \\ \boldsymbol{u} = (\partial/\partial \lambda)_n \text{ commute}}} = [\nabla_{\boldsymbol{n}}, \nabla_{\boldsymbol{u}}].$$

Geodesic deviation and tidal forces cannot tell the difference between $\mathcal{R}(\boldsymbol{n}, \boldsymbol{u})$ and $[\nabla_{\boldsymbol{n}}, \nabla_{\boldsymbol{u}}]$, nor consequently between old and new definitions of ***Riemann***.

E. Is Modified Definition Acceptable?

I.e., is ***Riemann*** $(\ldots, \boldsymbol{C}, \boldsymbol{A}, \boldsymbol{B}) \equiv \mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}$ a linear machine with output independent of how $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ vary near point of evaluation? YES! (See exercise 11.2.)

Take stock, first, of what one knows already about the Riemann curvature tensor. (1) ***Riemann*** *is* a tensor; despite the appearance of ∇ in its definition (11.9), no derivatives actually act on the input vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$. (2) ***Riemann*** is a $\binom{1}{3}$ tensor; its first slot accepts a 1-form; the others, vectors. (3) ***Riemann*** is determined entirely by ∇, or equivalently by the geodesics of spacetime, or equivalently by spacetime's parallel transport law; nothing but ∇ and the input vectors and 1-form are required to fix ***Riemann***'s output. (4) ***Riemann*** produces the tidal gravitational forces that pry geodesics (test-particle trajectories) apart or push them together; i.e., it characterizes the "curvature of spacetime":

Tide-producing gravitational forces expressed in terms of ***Riemann***

$$\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{u}} \boldsymbol{n} + \textit{\textbf{Riemann}}\,(\ldots, \boldsymbol{u}, \boldsymbol{n}, \boldsymbol{u}) = 0. \tag{11.10}$$

(This *"equation of geodesic deviation"* follows from equations 11.6, 11.8, and 11.9, and the relation $[\boldsymbol{n}, \boldsymbol{u}] = 0$.)

All these facets of ***Riemann*** are *pictorial* (e.g., geodesic deviation; see Boxes 11.2 and 11.3) or *abstract* (e.g., equations 11.8 and 11.9 for ***Riemann*** in terms of ∇). ***Riemann***'s *component* facet,

Components of ***Riemann*** expressed in terms of connection coefficients

$$R^\alpha{}_{\beta\gamma\delta} \equiv \textit{\textbf{Riemann}}\,(\boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta, \boldsymbol{e}_\gamma, \boldsymbol{e}_\delta) \equiv \langle \boldsymbol{\omega}^\alpha, \mathcal{R}(\boldsymbol{e}_\gamma, \boldsymbol{e}_\delta)\, \boldsymbol{e}_\beta \rangle, \tag{11.11}$$

is related to the component facet of ∇ by the following equation, valid in any coordinate basis $\{\boldsymbol{e}_\alpha\} = \{\partial/\partial x^\alpha\}$:

$$R^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha{}_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha{}_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha{}_{\mu\gamma} \Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta} \Gamma^\mu{}_{\beta\gamma}. \tag{11.12}$$

(See exercise 11.3 for derivation, and exercise 11.4 for the extension to noncoordinate bases.) These components of ***Riemann***, with no sign of any derivative operator anywhere, may leave one with a better feeling in one's stomach than the definition (11.8) with its nondifferentiating derivatives!

EXERCISES

Exercise 11.1. $[\boldsymbol{\nabla}_{\boldsymbol{A}}, \boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}$ DEPENDS ON DERIVATIVES OF $\boldsymbol{C}$

(Based on Box 11.5.) Let $\boldsymbol{C}_{\mathrm{NEW}}$ and $\boldsymbol{C}_{\mathrm{OLD}}$ be vector fields related by

$$\boldsymbol{C}_{\mathrm{NEW}}(\mathcal{P}) = \underbrace{f(\mathcal{P})}_{\text{arbitrary function, except } f(\mathcal{P}_0) = 1}\boldsymbol{C}_{\mathrm{OLD}}(\mathcal{P}).$$

Show that

$$\{[\boldsymbol{\nabla}_{\boldsymbol{A}}, \boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}_{\mathrm{NEW}}\}_{\mathrm{at}\,\mathcal{P}_0} - \{[\boldsymbol{\nabla}_{\boldsymbol{A}}, \boldsymbol{\nabla}_{\boldsymbol{B}}]\boldsymbol{C}_{\mathrm{OLD}}\}_{\mathrm{at}\,\mathcal{P}_0} = \boldsymbol{C}_{\mathrm{OLD}}\, \boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]} f.$$

Exercise 11.2. PROOF THAT *Riemann* IS A TENSOR

Show from its definition (11.8, 11.9) that ***Riemann*** is a tensor. Hint: Use the following procedure.

(a) If $f(\mathcal{P})$ is an arbitrary function, show that

$$\mathcal{R}(\boldsymbol{A}, \boldsymbol{B}) f\boldsymbol{C} = f\mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}.$$

(b) Similarly show that

$$\mathcal{R}(f\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C} = f\mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C} \qquad \text{and} \qquad \mathcal{R}(\boldsymbol{A}, f\boldsymbol{B})\boldsymbol{C} = f\mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}.$$

(c) Show that $\mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}$ is linear; i.e.,

$$\mathcal{R}(\boldsymbol{A} + \boldsymbol{a}, \boldsymbol{B})\boldsymbol{C} = \mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C} + \mathcal{R}(\boldsymbol{a}, \boldsymbol{B})\boldsymbol{C};$$
$$\mathcal{R}(\boldsymbol{A}, \boldsymbol{B} + \boldsymbol{b})\boldsymbol{C} = \mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C} + \mathcal{R}(\boldsymbol{A}, \boldsymbol{b})\boldsymbol{C};$$
$$\mathcal{R}(\boldsymbol{A}, \boldsymbol{B})(\boldsymbol{C} + \boldsymbol{c}) = \mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C} + \mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{c}.$$

(d) Now use the above properties to prove the most crucial feature of $\mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}$: Modify the variations (gradients) of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ in an arbitrary manner, but leave $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ unchanged at $\mathcal{P}_0$:

$$\left.\begin{aligned} \boldsymbol{A} &\longrightarrow \boldsymbol{A} + a^\alpha \boldsymbol{e}_\alpha \\ \boldsymbol{B} &\longrightarrow \boldsymbol{B} + b^\alpha \boldsymbol{e}_\alpha \\ \boldsymbol{C} &\longrightarrow \boldsymbol{C} + c^\alpha \boldsymbol{e}_\alpha \end{aligned}\right\} \quad \begin{aligned} &a^\alpha(\mathcal{P}),\ b^\alpha(\mathcal{P}),\ c^\alpha(\mathcal{P}) \text{ arbitrary except} \\ &\text{they all vanish at } \mathcal{P} = \mathcal{P}_0. \end{aligned}$$

Show that this modification leaves $\mathcal{R}(\boldsymbol{A}, \boldsymbol{B})\boldsymbol{C}$ unchanged at $\mathcal{P}_0$.

(e) From these facts, conclude that ***Riemann*** is a tensor.

Exercise 11.3. COMPONENTS OF *Riemann* IN COORDINATE BASIS

Derive equation (11.12) for the components of the Riemann tensor in a coordinate basis. [*Solution:*

$$\begin{aligned} R^\alpha{}_{\beta\gamma\delta} &= \textit{\textbf{Riemann}}\,(\boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta, \boldsymbol{e}_\gamma, \boldsymbol{e}_\delta) && \begin{bmatrix}\text{standard way to} \\ \text{calculate components}\end{bmatrix} \\ &= \langle \boldsymbol{\omega}^\alpha, \mathcal{R}(\boldsymbol{e}_\gamma, \boldsymbol{e}_\delta)\boldsymbol{e}_\beta \rangle && [\text{by definition } (11.9)] \\ &= \langle \boldsymbol{\omega}^\alpha, (\boldsymbol{\nabla}_\gamma \boldsymbol{\nabla}_\delta - \boldsymbol{\nabla}_\delta \boldsymbol{\nabla}_\gamma)\boldsymbol{e}_\beta \rangle && \begin{bmatrix}\text{by definition (11.8) plus} \\ [\boldsymbol{e}_\gamma, \boldsymbol{e}_\delta] = 0 \text{ in coord. basis}\end{bmatrix} \\ &= \langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\mu \Gamma^\mu{}_{\beta\delta,\gamma} + (\boldsymbol{e}_\nu \Gamma^\nu{}_{\mu\gamma})\Gamma^\mu{}_{\beta\delta} - \boldsymbol{e}_\mu \Gamma^\mu{}_{\beta\gamma,\delta} - (\boldsymbol{e}_\nu \Gamma^\nu{}_{\mu\delta})\Gamma^\mu{}_{\beta\gamma} \rangle \\ &= (\Gamma^\mu{}_{\beta\delta,\gamma} - \Gamma^\mu{}_{\beta\gamma,\delta})\langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\mu \rangle + (\Gamma^\nu{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\nu{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma})\langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\nu \rangle, \end{aligned}$$

which reduces (upon using $\langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\mu \rangle = \delta^\alpha{}_\mu$) to (11.12).]

Exercise 11.4. COMPONENTS OF *RIEMANN* IN NONCOORDINATE BASIS

In a noncoordinate basis with commutation coefficients $c_{\alpha\beta}{}^{\gamma}$ defined by equation (9.22), derive the following equation for the components of ***Riemann***:

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\mu\gamma}\Gamma^{\mu}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\mu\delta}\Gamma^{\mu}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\beta\mu}c_{\gamma\delta}{}^{\mu}. \tag{11.13}$$

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§11.4. PARALLEL TRANSPORT AROUND A CLOSED CURVE

What are the effects of spacetime curvature, and how can one quantify them? One effect is geodesic deviation (relative acceleration of test bodies), quantified by equation (11.10). Another effect, almost as important, is the change in a vector caused by parallel transport around a closed curve. This effect shows up most clearly in the same problem, geodesic deviation, that motivated curvature in the first place. The relative acceleration vector $\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{u}}\boldsymbol{n}$ is also the change $\delta\boldsymbol{u}$ in the vector $\boldsymbol{u}$ caused by parallel transport around the curve whose legs are the vectors $\boldsymbol{n}$ and $\boldsymbol{u}$:

Change in a vector due to parallel transport around a closed curve:

Related to geodesic deviation

$$\nabla_{\boldsymbol{u}}\nabla_{\boldsymbol{u}}\boldsymbol{n} = \delta\boldsymbol{u}.$$

(See Box 11.6 for proof.) Hence, in this special case one can write

$$\delta\boldsymbol{u} + \boldsymbol{Riemann}\,(\ldots, \boldsymbol{u}, \boldsymbol{n}, \boldsymbol{u}) = 0.$$

The expected generalization is obvious: pick a closed quadrilateral with legs $\boldsymbol{u}\,\Delta a$ and $\boldsymbol{v}\,\Delta b$ (Figure 11.2; Δa and Δb are small parameters, to go to zero at end of discussion). Parallel transport the vector $\boldsymbol{A}$ around this quadrilateral. The resultant change in $\boldsymbol{A}$ should satisfy the equation

$$\delta\boldsymbol{A} + \boldsymbol{Riemann}\,(\ldots, \boldsymbol{A}, \boldsymbol{u}\,\Delta a, \boldsymbol{v}\,\Delta b) = 0; \tag{11.14}$$

Equation for change

or, equivalently,

$$\delta\boldsymbol{A} + \Delta a\,\Delta b\,\mathcal{R}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{A} = 0; \tag{11.14$'$}$$

or, more precisely,

$$\lim_{\substack{\Delta a\to 0\\ \Delta b\to 0}}\left(\frac{\delta\boldsymbol{A}}{\Delta a\,\Delta b}\right) + \boldsymbol{Riemann}\,(\ldots, \boldsymbol{A}, \boldsymbol{u}, \boldsymbol{v}) = 0. \tag{11.14$''$}$$

The proof is enlightening, for it reveals the geometric origin of the correction term $\nabla_{[\boldsymbol{u},\boldsymbol{v}]}$ in the curvature operator.

The circuit of transport (Figure 11.2) is to be made from two arbitrary vector fields $\boldsymbol{u}\,\Delta a$ and $\boldsymbol{v}\,\Delta b$. However, a circuit made only of these fields has a gap in it, for a simple reason. The magnitude of $\boldsymbol{u}$ varies the wrong way from place to place. The displacement $\boldsymbol{u}\,\Delta a$ that reaches across at the bottom of the quadrilateral from

Derivation of equation for change

2

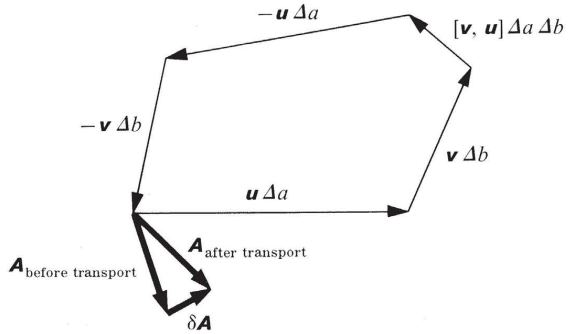


Figure 11.2.
The change $\delta\boldsymbol{A}$ in a vector $\boldsymbol{A}$ as a result of parallel transport around a closed curve. The edges of the curve are the vector fields $\boldsymbol{u}\,\Delta a$ and $\boldsymbol{v}\,\Delta b$, plus the "closer of the quadrilateral" $[\boldsymbol{v}\,\Delta b, \boldsymbol{u}\,\Delta a] = [\boldsymbol{v}, \boldsymbol{u}]\,\Delta a\,\Delta b$ (see Box 9.2).

one line of $\boldsymbol{v}$'s to another cannot make the connection at the top of the quadrilateral. Similarly the $\boldsymbol{v}$'s vary the wrong way from place to place to connect the $\boldsymbol{u}$'s. To close the gap and complete the circuit, insert the "closer of quadrilaterals" $[\boldsymbol{v}\,\Delta b, \boldsymbol{u}\,\Delta a] = [\boldsymbol{v}, \boldsymbol{u}]\,\Delta a\,\Delta b$. (See Box 9.2 for why this vector closes the gap.)

With the route now specified, the vector $\boldsymbol{A}$ is to be transported around it. One way to do this, "geometrical construction" by the method of Schild's ladder applied over and over, is the foundation for planning a possible experiment. For planning an abstract and coordinate-free calculation (the present line of action), introduce a "fiducial field," only to take it away at the end of the calculation. *Plan:* Conceive of $\boldsymbol{A}$, not as a localized vector defined solely at the start of the trip, but as a vector field (defined throughout the trip). *Purpose:* To provide a standard of reference (comparison of $\boldsymbol{A}$ transported from the origin with $\boldsymbol{A}$ at the place in question). *Principle:* The standard of reference will cancel out in the end. *Procedure:*

$-\delta\boldsymbol{A} = -\Bigg($ Net change made in taking the vector $\boldsymbol{A}$, originally localized at the start of the circuit, and transporting it parallel to itself ("mobile $\boldsymbol{A}$") around the closed circuit. This quantity cannot be evaluated until completion of circuit because there is no preexisting standard of reference along the way. $\Bigg)$

$= +\Bigg($ A quantity subject to analysis for each leg of circuit individually. This new quantity is defined by introducing throughout the whole region a vector field $\boldsymbol{A}^{(\text{field})}$, smoothly varying, and in agreement at starting point with the original localized $\boldsymbol{A}$, but otherwise arbitrary. This new quantity is then given by $\boldsymbol{A}^{(\text{field})}$ at starting point (same as $\boldsymbol{A}^{(\text{localized})}$ at starting point) minus $\boldsymbol{A}^{(\text{mobile})}$ at finish point (after transit). $\Bigg)$

$= \sum_{\substack{\text{legs of}\\ \text{circuit}}} \Bigg($ Change in $\boldsymbol{A}^{(\text{field})}$ relative to $\boldsymbol{A}^{(\text{mobile})}$ in the course of transport along specified leg. Value for any one leg depends on the arbitrary choice of $\boldsymbol{A}^{(\text{field})}$, but this arbitrariness cancels out in end because of closure of circuit. $\Bigg)$

$$
= \left(\begin{array}{l}
\text{Change in } \boldsymbol{A}^{(\text{field})} \text{ relative to the parallel-transported } \boldsymbol{A}^{(\text{mobile})} \text{ as standard} \\
\text{of reference, made up of contributions along following legs of Figure} \\
\text{11.2:} \\
\quad \boldsymbol{v}\,\Delta b, \text{ giving } \quad \boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{A}^{(\text{field})}\,\Delta b \text{ (on line displaced } \boldsymbol{u}\,\Delta a \text{ from start)} \\
-\boldsymbol{v}\,\Delta b, \text{ giving } - \boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{A}^{(\text{field})}\,\Delta b \text{ (on line through starting point)} \\
-\boldsymbol{u}\,\Delta a, \text{ giving } - \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{A}^{(\text{field})}\,\Delta a \text{ (on line displaced } \boldsymbol{v}\,\Delta b \text{ from start)} \\
+\boldsymbol{u}\,\Delta a, \text{ giving } \quad \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{A}^{(\text{field})}\,\Delta a \text{ (on line through starting point)} \\
+[\boldsymbol{v}, \boldsymbol{u}]\,\Delta a\,\Delta b, \text{ giving } \boldsymbol{\nabla}_{[\boldsymbol{v},\boldsymbol{u}]} \boldsymbol{A}^{(\text{field})}\,\Delta a\,\Delta b
\end{array}\right)
$$

$$
= \{\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{\nabla}_{\boldsymbol{v}} - \boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{\nabla}_{\boldsymbol{u}} + \boldsymbol{\nabla}_{[\boldsymbol{v},\boldsymbol{u}]}\} \boldsymbol{A}^{(\text{field})}\,\Delta a\,\Delta b
$$

$$
= \textbf{Riemann}\,(\ldots, \boldsymbol{A}^{(\text{field})}, \boldsymbol{u}, \boldsymbol{v})\,\Delta a\,\Delta b = \mathcal{R}(\boldsymbol{u}, \boldsymbol{v}) \boldsymbol{A}^{(\text{field})}\,\Delta a\,\Delta b. \tag{11.15}
$$

Profit: The curvature operator

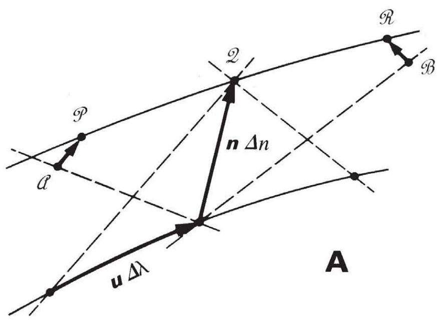
$$
\textbf{Riemann}\,(\ldots, \ldots, \boldsymbol{u}, \boldsymbol{v}) = \mathcal{R}(\boldsymbol{u}, \boldsymbol{v}) = [\boldsymbol{\nabla}_{\boldsymbol{u}}, \boldsymbol{\nabla}_{\boldsymbol{v}}] - \boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]},
$$

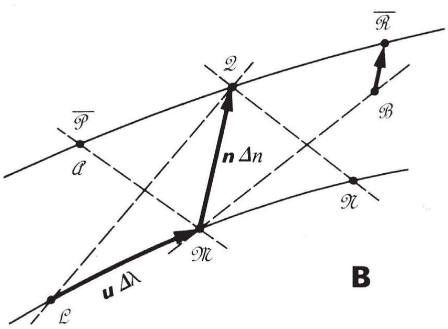
Box 11.6 GEODESIC DEVIATION AND PARALLEL TRANSPORT AROUND CLOSED CURVE: TWO ASPECTS OF SAME CONSTRUCTION

Geodesic Deviation

$$
\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{n} = \lim_{\substack{\Delta\lambda \to 0 \\ \Delta n \to 0}} \left\{ \frac{\mathcal{AP} + \mathcal{BR}}{(\Delta\lambda)^2\,\Delta n} \right\}.
$$

(See Boxes 11.2 and 11.3)





Geodesic Deviation

Same result; different construction. To simplify the connection with closed-curve transport, change the tilt and dilate the parametrization of geodesic $\mathcal{PQR}$ in **A**. The result: **B**, where $\overline{\mathcal{P}}$ and $\mathcal{A}$ coincide. From **F** of Box 11.2 one knows $\mathcal{AP} + \mathcal{BR} = \mathcal{A}\overline{\mathcal{P}} + \mathcal{B}\overline{\mathcal{R}}$—i.e. $\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{n}$ is the

Box 11.6 (*continued*)

same for this family of geodesics as for the original family

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\, \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{n} = \lim_{\substack{\Delta\lambda \to 0 \\ \Delta n \to 0}} \left\{ \frac{\mathcal{B}\overline{\mathcal{R}}}{(\Delta\lambda)^2\, \Delta n} \right\}.$$

Also, to simplify discussion set $\Delta n = \Delta\lambda = 1$, and assume $\boldsymbol{n}$ and $\boldsymbol{u}$ are small enough that one can evaluate $\boldsymbol{\nabla}_{\boldsymbol{u}}\, \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{n}$ without taking the limit:

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\, \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{n} = \mathcal{B}\overline{\mathcal{R}}.$$

Parallel Transport Around Closed Curve, Performed by Same Construction

Plan: Parallel transport the vector $\boldsymbol{u}\, \Delta\lambda = \mathcal{Q}\overline{\mathcal{R}}$ counterclockwise around the curve $\mathcal{Q} \longrightarrow \overline{\mathcal{P}} \longrightarrow \mathcal{L} \longrightarrow \mathcal{M} \longrightarrow \mathcal{Q}$. *Execution:* (1) Call transported vector $\boldsymbol{u}^{(\mathrm{m})}$ ("m" for "mobile"). (2) At $\mathcal{Q}$, $\boldsymbol{u}^{(\mathrm{m})} = \mathcal{Q}\overline{\mathcal{R}}$. (3) At $\overline{\mathcal{P}}$, $\boldsymbol{u}^{(\mathrm{m})} = \overline{\mathcal{P}}\mathcal{Q}$ because $\overline{\mathcal{P}}\mathcal{Q}\overline{\mathcal{R}}$ is a geodesic and $\boldsymbol{u}^{(\mathrm{m})}$ is its tangent vector. (4) At $\mathcal{L}$, $\boldsymbol{u}^{(\mathrm{m})} = \mathcal{L}\mathcal{M}$ according to Schild's ladder of the picture. (5) At $\mathcal{M}$, $\boldsymbol{u}^{(\mathrm{m})} = \mathcal{M}\mathcal{N}$ because $\mathcal{L}\mathcal{M}\mathcal{N}$ is a geodesic and $\boldsymbol{u}^{(\mathrm{m})}$ is now its tangent vector. (6) At $\mathcal{Q}$, $\boldsymbol{u}^{(\mathrm{m})} = \mathcal{Q}\mathcal{B}$ according to Schild's ladder. Result: The change in $\boldsymbol{u}^{(\mathrm{m})}$ is $-\mathcal{B}\overline{\mathcal{R}}$. Had the curve been circuited in opposite direction ($\mathcal{L} \longrightarrow \overline{\mathcal{P}} \longrightarrow \mathcal{Q} \longrightarrow \mathcal{M} \longrightarrow \mathcal{L}$), the change would have been $+\mathcal{B}\overline{\mathcal{R}}$:

$$(\delta \boldsymbol{u})_{\substack{\text{due to parallel transport up } \boldsymbol{n}\text{, out } \boldsymbol{u}\text{, down } -\boldsymbol{n}\text{, and} \\ \text{back along } -\boldsymbol{u} \text{ to starting point}}} = \mathcal{B}\overline{\mathcal{R}} = \boldsymbol{\nabla}_{\boldsymbol{u}}\, \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{n}.$$

applied to the vector field $\boldsymbol{A}^{(\text{field})}$, gives the negative of the change in the localized vector $\boldsymbol{A}^{(\text{localized})}$ (called $\boldsymbol{A}^{(\text{mobile})}$ during the phase of travel) on parallel transport around the closed circuit. It does not give the change in $\boldsymbol{A}^{(\text{field})}$ on traversal of that circuit, for $\boldsymbol{A}^{(\text{field})}$ has the same value at the end of the journey as at the beginning. Equation (11.14′) expresses that change in terms of the conveniently calculated differential operator, $\mathcal{R}(\boldsymbol{u}, \boldsymbol{v}) = [\boldsymbol{\nabla}_{\boldsymbol{u}}, \boldsymbol{\nabla}_{\boldsymbol{v}}] - \boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]}$. *Paradox:* Neither wanted nor evaluated is the change in the quantity $\boldsymbol{A}^{(\text{field})}$ acted on by this operator. *Payoff:* Ostensibly differential in the character of its action on $\boldsymbol{A}$, the operator ***Riemann*** $(\ldots, \ldots, \boldsymbol{u}, \boldsymbol{v}) = \mathcal{R}(\boldsymbol{u}, \boldsymbol{v})$ is actually local. Thus, replace the proposed smoothly varying vector field $\boldsymbol{A}^{(\text{field})}$ by a quite different but also smoothly varying vector field $\boldsymbol{A}^{(\text{field, new})}$. Then the two fields need agree only at the one point in question for them to give the same output ***Riemann*** $(\ldots, \boldsymbol{A}, \boldsymbol{u}, \boldsymbol{v}) = \mathcal{R}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{A}$ at that point. This one knows from the fact that $\delta \boldsymbol{A}$, the quantity calculated, has an existence and value independent of the choice of $\boldsymbol{A}^{(\text{field})}$. This one can also verify by detailed calculation (exercise 11.2). *Power:* Although they cancel out in their response to any change of $\boldsymbol{A}$ with location, the several differentiations in the curvature operator respond directly to the "rate of change of geometry with location" ("geodesic deviation"). *Prolongation:* The closed curve need not be a quadrilateral. The curvature operator tells how a vector changes on parallel transport about small curves of arbitrary shape (Box 11.7).

EXERCISE

Exercise 11.5. COPLANARITY OF CLOSED CURVES

Let $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ be the bivectors (see Box 11.7) for two small closed curves at the same event. Show that the curves are coplanar if and only if $\boldsymbol{f}_1 = a\boldsymbol{f}_2$ for some number a.

Box 11.7 THE LAW FOR PARALLEL TRANSPORT ABOUT A CLOSED CURVE

A. Special Case

Curve is closed quadrilateral formed by vector fields $\boldsymbol{u}$ and $\boldsymbol{v}$.

1. Law says (in component form)

$$\delta A^\alpha + R^\alpha{}_{\beta\gamma\delta} A^\beta u^\gamma v^\delta = 0. \tag{1}$$

2. On what characteristics of the closed curve does this depend?
   a. Notice that $R^\alpha{}_{\beta\gamma\delta} = -R^\alpha{}_{\beta\delta\gamma}$ (antisymmetry in last two indices; obvious in equation 11.12 for components; also obvious because reversing the direction the curve is traversed—i.e., interchanging $\boldsymbol{u}$ and $\boldsymbol{v}$—should reverse sign of $\delta\boldsymbol{A}$).
   b. Equation (1) contracts $\boldsymbol{u} \otimes \boldsymbol{v}$ into these antisymmetric, last two indices. The symmetric part of $\boldsymbol{u} \otimes \boldsymbol{v}$ must give zero. Only the antisymmetric part, $\boldsymbol{u} \wedge \boldsymbol{v} = \boldsymbol{u} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{u}$ can contribute:

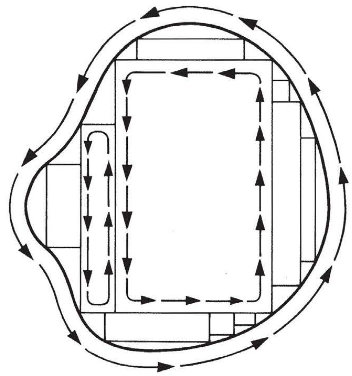
$$\delta A^\alpha + \frac{1}{2} R^\alpha{}_{\beta\gamma\delta} A^\beta (\boldsymbol{u} \wedge \boldsymbol{v})^{\gamma\delta} = 0. \tag{2}$$

3. This antisymmetric part is a *"bivector."* It is independent of the curve's shape; it depends only on (a) the plane the curve lies in, and (b) the area enclosed by the curve. [Although without metric "area" is meaningless, "relative areas at an event in a given plane" have just as much meaning as "relative lengths at an

event along a given direction." Two vectors at the same event lie on the same line if they are multiples of each other; their relative length in that case is their ratio. Similarly, two small closed curves at the same event lie in the same plane if their bivectors are multiples of each other (exercise 11.5); their relative area in that case is the ratio of their bivectors.]

B. General Case

Arbitrary but small closed curve.



1. Break the curve down into a number of quadrilaterals, all lying in the same plane as the curve.
2. Traverse each quadrilateral once in the same sense as the curve is to be traversed. Result: all interior edges get traversed twice in opposite directions (no net traversal); the outer edge (the curve itself) gets traversed once.
3. Thus, $\delta\boldsymbol{A}$ due to traversing curve is the sum of the $\delta\boldsymbol{A}$'s from traversal of each quadrilateral:

$$\delta A^{\alpha} = -\frac{1}{2} \sum_{\text{quadrilaterals}} R^{\alpha}{}_{\beta\gamma\delta} A^{\beta} (\boldsymbol{u} \wedge \boldsymbol{v}_{\text{for given quadrilateral}})^{\gamma\delta}.$$

Define the bivector $\boldsymbol{f}$ for the curve as the sum of the bivectors for its component quadrilaterals:

$$\boldsymbol{f} \equiv \sum_{\text{quadrilaterals}} (\mathbf{u} \wedge \boldsymbol{v})_{\text{quadrilateral}}$$

(add "areas"; keep plane the same).

4. Then

$$\delta A^{\alpha} + \frac{1}{2} R^{\alpha}{}_{\beta\gamma\delta} A^{\beta} f^{\gamma\delta} \equiv \delta A^{\alpha} + R^{\alpha}{}_{\beta|\gamma\delta|} A^{\beta} f^{\gamma\delta} = 0.$$

C. Warning

This is valid only for closed curves of small compass: $\delta\boldsymbol{A}$ doubles when the area doubles; but the error increases by a factor $\sim 2^{3/2}$ [$\delta\boldsymbol{A} \propto \Delta a\, \Delta b$ in calculation of §11.4; but error $\propto (\Delta a)^2\, \Delta b$ or $\Delta a\, (\Delta b)^2$].

§11.5. FLATNESS IS EQUIVALENT TO ZERO RIEMANN CURVATURE

Flatness of a manifold defined

To say that space or spacetime or any other manifold is flat is to say that there exists a coordinate system $\{x^\alpha(\mathscr{P})\}$ in which all geodesics appear straight:

$$x^\alpha(\lambda) = a^\alpha + b^\alpha\lambda. \tag{11.16}$$

(Example: Lorentz spacetime of special relativity, where test bodies move on such straight lines.) They can appear so if and only if the connection coefficients in the geodesic equation

$$\frac{d^2x^\beta}{d\lambda^2} + \Gamma^\beta{}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{11.17}$$

expressed in the same coordinate system, all vanish:

$$\Gamma^\beta{}_{\mu\nu} = 0. \tag{11.18}$$

Flatness implies ***Riemann*** = 0

From the vanishing of these connection coefficients, it follows immediately (equation 11.12) that all the components of the curvature tensor are zero:

$$R^\beta{}_{\gamma\mu\nu} = 0. \tag{11.19}$$

[Geometric restatement of (11.16) ⟶ (11.18) ⟶ (11.19): For all geodesics to be straight in a given coordinate system means that initially parallel geodesics preserve their separation; the geodesic deviation is zero; and therefore the curvature vanishes.]

Is the converse true? Does zero Riemann curvature imply the existence of a coordinate system in which all geodesics appear straight? Yes, as one sees by the following construction.

Proof that ***Riemann*** = 0 implies flatness

Transport a vector parallel to itself from $\mathscr{P}_0$ to $\mathscr{Q}$, and then back from $\mathscr{Q}$ to $\mathscr{P}_0$ along a slightly different route. It returns to its starting point with no alteration in magnitude or direction, *because* ***Riemann*** everywhere vanishes. Therefore parallel transport of a base vector $\boldsymbol{e}_\mu$ from $\mathscr{P}_0$ to $\mathscr{Q}$ yields at $\mathscr{Q}$ a base vector $\boldsymbol{e}_\mu$ that is independent, both in magnitude and in direction, of the route of transportation (for routes obtainable one from the other by any continuous sequence of deformations). As for $\mathscr{Q}$, so for all points of the manifold; and as for the one base vector $\boldsymbol{e}_\mu$, so for a complete set of base vectors ($\mu = 0, 1, 2, 3$): Parallel transport of a basis $\{\boldsymbol{e}_\alpha(\mathscr{P}_0)\}$ yields everywhere a field of frames ("frame field"), each base vector of which suffers zero change (relative to the frame field) on parallel transport from any point to any nearby point: thus,

$$\boldsymbol{\nabla}\boldsymbol{e}_\mu = 0; \tag{11.20}$$

or

$$\boldsymbol{\nabla}_\nu\boldsymbol{e}_\mu\,(\equiv \boldsymbol{\nabla}_{\boldsymbol{e}_\nu}\boldsymbol{e}_\mu) = 0. \tag{11.21}$$

With the vanishing of these individual derivatives, there also vanishes the commutator of any two basis-vector fields:

$$[\boldsymbol{e}_\mu, \boldsymbol{e}_\nu] = \boldsymbol{\nabla}_\mu\boldsymbol{e}_\nu - \boldsymbol{\nabla}_\nu\boldsymbol{e}_\mu = 0 - 0 = 0. \tag{11.22}$$

The gap in the quadrilateral of Figure 11.2 (there read "$\boldsymbol{e}_\mu$" for "$\boldsymbol{u}$," "$\boldsymbol{e}_\nu$" for "$\boldsymbol{v}$") closes up completely. Thereupon one can introduce coordinates x^μ, each of which increases with a motion in the direction of the corresponding vector field; and with appropriate scaling of these coordinates, one can write

$$\boldsymbol{e}_\mu = \frac{\partial}{\partial x^\mu} \tag{11.23}$$

(see exercise 9.9). With this coordinate basis in hand, one can employ the formula

$$\boldsymbol{\nabla}_\alpha \boldsymbol{e}_\beta = \boldsymbol{e}_\mu \Gamma^\mu{}_{\beta\alpha} \tag{11.24}$$

to calculate the connection coefficients. From the vanishing of the quantities on the left, one concludes that all the connection coefficients on the right ("bending of geodesics") must be zero; so spacetime is indeed flat.

Summary: Spacetime is flat—i.e., there exist "flat coordinates" in which $\Gamma^\mu{}_{\alpha\beta} = 0$ everywhere and geodesics are straight lines, $x^\alpha(\lambda) = a^\alpha + b^\alpha\lambda$—*if and only if* ***Riemann*** $= 0$.

Lorentz coordinates exist if and only if ***Riemann*** $= 0$

Note: In the spacetime of Einstein, which has a metric, one can choose $\{\boldsymbol{e}_\mu(\mathscr{P}_0)\}$ in the above argument to be orthonormal, $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = \eta_{\mu\nu}$ at $\mathscr{P}_0$. The resulting field of frames will then be orthonormal everywhere, and the resulting coordinate system will be Lorentz. Thus, in Einsteinian gravity the above summary can be rewritten: *spacetime is flat (there exists a Lorentz coordinate system) if and only if* ***Riemann*** $= 0$.

Flatness does not imply Euclidean topology

Warning: Flatness does not necessarily imply Euclidean topology. Take a sheet of paper. It is flat. Roll it up into a cylinder. It is still flat, intrinsically. The tracks of geodesics over it have not changed. Distances between neighboring points have not changed. Only the topology has changed, so far as an observer confined forever to the sheet is concerned. (The "extrinsic geometry"—the way the sheet is embedded in the surrounding three-dimensional space—has also changed; but an observer on the sheet knows nothing of this, and it is not the subject of the present chapter. See, instead §21.5.)

Take this cylinder. Bend it around and glue its two ends together, without changing its flat intrinsic geometry. Doing so is impossible if the cylinder remains embedded in flat, three-dimensional Euclidean space; perfectly possible if it is embedded in a Euclidean space of 4 dimensions. However, embedding is unimportant to observers confined to the cylinder, since all they ever measure is intrinsic geometry; so all that matters to them is the *topological identification* of the two ends of the cylinder with each other. The result is topologically a torus; but the tracks of geodesics are still unchanged; the intrinsic geometry is flat; ***Riemann*** vanishes.

By analogy, take flat Minkowskii spacetime. Pick some Lorentz frame, and in it pick a cube 10^{10} light years on each side ($0 < x < 10^{10}$ light years; similarly for y and z). Identify opposite faces of the cube so that a geodesic exiting across one face enters across the other. The result is topologically a three-torus: a "closed universe" with finite volume, with flat, Minkowskii geometry, and with a form that changes not at all as Lorentz time t passes (no expansion, no contraction).

§11.6. RIEMANN NORMAL COORDINATES

In curved spacetime one can never find a coordinate system with $\Gamma^{\alpha}{}_{\beta\gamma} = 0$ everywhere. But one can always construct local inertial frames at a given event $\mathscr{P}_0$; and as viewed in such frames, free particles must move along straight lines, at least locally—which means $\Gamma^{\alpha}{}_{\beta\gamma}$ must vanish, at least locally.

Riemann normal coordinates: a realization of local inertial frames

A very special and useful realization of such a local inertial frame is a *Riemann-normal coordinate system.* Pick an event $\mathscr{P}_0$ and a set of basis vectors $\{\boldsymbol{e}_\alpha(\mathscr{P}_0)\}$ to be used there by an inertial observer. Fill spacetime, near $\mathscr{P}_0$, with geodesics radiating out from $\mathscr{P}_0$ like the quills of a hedgehog or porcupine. Each geodesic is determined by its tangent vector $\boldsymbol{v}$ at $\mathscr{P}_0$; and the general point on it can be denoted

Geometric construction of Riemann normal coordinates

$$\mathscr{P} = \mathscr{G}(\underset{\uparrow}{\lambda}; \underset{\uparrow}{\boldsymbol{v}}). \tag{11.25}$$

$$\underbrace{\left[\begin{array}{l}\text{affine parameter;}\\ \text{tells ``where'' on geodesic}\end{array}\right]}_{\lambda} \qquad \underbrace{\left\{\begin{array}{l}\text{tangent vector at } \mathscr{P}_0;\\ \text{tells ``which geodesic''}\end{array}\right\}}_{\boldsymbol{v}}$$

Actually, this gives more geodesics than are needed. One reaches the same point after parameter length $\frac{1}{2}\lambda$ if the initial tangent vector is $2\boldsymbol{v}$, as one reaches after λ if the tangent vector is $\boldsymbol{v}$:

$$\mathscr{G}(\lambda; \boldsymbol{v}) = \mathscr{G}\left(\frac{1}{2}\lambda; 2\boldsymbol{v}\right) = \mathscr{G}(1; \lambda\boldsymbol{v}).$$

Thus, by fixing $\lambda = 1$ and varying $\boldsymbol{v}$ in all possible ways, one can reach every point in some neighborhood of $\mathscr{P}_0$. This is the foundation for constructing Riemann normal coordinates. Choose an event $\mathscr{P}$. Find that tangent vector $\boldsymbol{v}$ at $\mathscr{P}_0$ for which $\mathscr{P} = \mathscr{G}(1; \boldsymbol{v})$. Expand that $\boldsymbol{v}$ in terms of the chosen basis and give its components the names x^α:

$$\mathscr{P} = \mathscr{G}(1; x^\alpha \boldsymbol{e}_\alpha). \tag{11.26}$$

The point $\mathscr{P}$ determines x^α uniquely (if $\mathscr{P}$ is near enough to $\mathscr{P}_0$ that spacetime curvature has not caused geodesics to cross each other). Similarly, x^α determines $\mathscr{P}$ uniquely. Hence, x^α can be chosen as the coordinates of $\mathscr{P}$—its "Riemann-normal coordinates, based on the event $\mathscr{P}_0$ and basis $\{\boldsymbol{e}_\alpha(\mathscr{P}_0)\}$."

Mathematical properties of Riemann normal coordinates

Equation (11.26) summarizes Riemann-normal coordinates concisely. Other equations, derived in exercise 11.9, summarize their powerful properties:

$$\boldsymbol{e}_\alpha(\mathscr{P}_0) = (\partial/\partial x^\alpha)_{\mathscr{P}_0}; \tag{11.27}$$

$$\Gamma^{\alpha}{}_{\beta\gamma}(\mathscr{P}_0) = 0; \tag{11.28}$$

$$\Gamma^{\alpha}{}_{\beta\gamma,\mu}(\mathscr{P}_0) = -\frac{1}{3}(R^{\alpha}{}_{\beta\gamma\mu} + R^{\alpha}{}_{\gamma\beta\mu}). \tag{11.29}$$

If spacetime has a metric (as it does in actuality), and if the observer's frame at $\mathscr{P}_0$ has been chosen orthonormal ($\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta = \eta_{\alpha\beta}$), then

$$g_{\alpha\beta}(\mathscr{P}_0) = \eta_{\alpha\beta}, \tag{11.30}$$

$$g_{\alpha\beta,\mu}(\mathscr{P}_0) = 0, \tag{11.31}$$

$$\begin{aligned} g_{\alpha\beta,\mu\nu}(\mathscr{P}_0) &= -\frac{1}{3}(R_{\alpha\mu\beta\nu} + R_{\alpha\nu\beta\mu}) \\ &= -\frac{2}{3}J_{\alpha\beta\mu\nu}, \end{aligned} \tag{11.32}$$

$$R_{\alpha\beta\gamma\delta}(\mathscr{P}_0) = g_{\alpha\delta,\beta\gamma}(\mathscr{P}_0) - g_{\alpha\gamma,\beta\delta}(\mathscr{P}_0). \tag{11.32$'$}$$

Here $J_{\alpha\beta\mu\nu}$ are components of the Jacobi curvature tensor (see exercise 11.7).

Other mathematical realizations of a local inertial frame

Is this the only coordinate system that is locally inertial at $\mathscr{P}_0$ (i.e., has $\Gamma^{\alpha}{}_{\beta\gamma} = 0$ there) and is tied to the basis vectors $\boldsymbol{e}_\alpha$ there (i.e., has $\partial/\partial x^\alpha = \boldsymbol{e}_\alpha$ there)? No. But all such coordinate systems (called "*normal coordinates*") will be the same to second order:

$$x^{\alpha}_{\mathrm{NEW}}(\mathscr{P}) = x^{\alpha}_{\mathrm{OLD}}(\mathscr{P}) + \text{corrections of order } (x^{\alpha}_{\mathrm{OLD}})^3.$$

Moreover, only those the same to third order,

$$x^{\alpha}_{\mathrm{NEW}}(\mathscr{P}) = x^{\alpha}_{\mathrm{OLD}}(\mathscr{P}) + \text{corrections of order } (x^{\alpha}_{\mathrm{OLD}})^4,$$

will preserve the beautiful ties (11.29) and (11.32) to the Riemann curvature tensor.

EXERCISES

Exercise 11.6. SYMMETRIES OF *Riemann*

(To be discussed in Chapter 13). Show that ***Riemann*** has the following symmetries:

$$R^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta[\gamma\delta]} \quad \text{(antisymmetric on last 2 indices)} \tag{11.33a}$$

$$R^{\alpha}{}_{[\beta\gamma\delta]} = 0 \quad \text{(vanishing of completely antisymmetric part)} \tag{11.33b}$$

Exercise 11.7. GEODESIC DEVIATION MEASURES ALL CURVATURE COMPONENTS

The equation of geodesic deviation, written up to now in the form

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n} + \textbf{\textit{Riemann}}(\ldots, \boldsymbol{u}, \boldsymbol{n}, \boldsymbol{u}) = 0$$

or

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n} + \mathscr{R}(\boldsymbol{n}, \boldsymbol{u})\boldsymbol{u} = 0,$$

also lets itself be written in the Jacobi form $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{n} + \mathscr{J}(\boldsymbol{u}, \boldsymbol{u})\boldsymbol{n} = 0$. Here $\mathscr{J}(\boldsymbol{u}, \boldsymbol{v})$, the "*Jacobi curvature operator,*" is defined by

$$\mathscr{J}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{n} \equiv \frac{1}{2}[\mathscr{R}(\boldsymbol{n}, \boldsymbol{u})\boldsymbol{v} + \mathscr{R}(\boldsymbol{n}, \boldsymbol{v})\boldsymbol{u}], \tag{11.34}$$

and is related to the "Jacobi curvature tensor" by

$$\textbf{\textit{Jacobi}}(\ldots, \boldsymbol{n}, \boldsymbol{u}, \boldsymbol{v}) \equiv \mathscr{J}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{n}, \tag{11.35}$$

which implies

$$J^{\mu}{}_{\nu\alpha\beta} = J^{\mu}{}_{\nu\beta\alpha} = \frac{1}{2}(R^{\mu}{}_{\alpha\nu\beta} + R^{\mu}{}_{\beta\nu\alpha}). \tag{11.36}$$

(a) Show that $J^{\mu}{}_{(\alpha\beta\gamma)} = 0$ follows from $R^{\mu}{}_{\alpha\beta\gamma} = R^{\mu}{}_{\alpha[\beta\gamma]}$.

(b) Show that by studying geodesic deviation (allowing arbitrary $\boldsymbol{u}$ and $\boldsymbol{n}$ in $\boldsymbol{\nabla_u \nabla_u n} + \mathscr{J}(\boldsymbol{u}, \boldsymbol{u})\boldsymbol{n} = 0$) one can measure *all* components of ***Jacobi***.

(c) Show that ***Jacobi*** contains precisely the same information as ***Riemann***. [Hint: show that

$$R^{\mu}{}_{\alpha\nu\beta} = \frac{2}{3}(J^{\mu}{}_{\nu\alpha\beta} - J^{\mu}{}_{\beta\alpha\nu}); \tag{11.37}$$

this plus equation (11.36) for $J^{\mu}{}_{\nu\alpha\beta}$ proves "same information content".] Hence, by studying geodesic deviation one can also measure all the components of ***Riemann***.

(d) Show that the symmetry of $R^{\mu}{}_{[\nu\alpha\beta]} = 0$ is essential in the equivalence between ***Jacobi*** and ***Riemann*** by exhibiting proposed values for $R^{\mu}{}_{\nu\alpha\beta} = -R^{\mu}{}_{\nu\beta\alpha}$ for which $R^{\mu}{}_{[\nu\alpha\beta]} \neq 0$, and from which one would find $J^{\mu}{}_{\nu\alpha\beta} = 0$.

Exercise 11.8. GEODESIC DEVIATION IN GORY DETAIL

Write out the equation of geodesic deviation in component form in a coordinate system. Expand all covariant derivatives (semicolon notation) in terms of ordinary (comma) derivatives and in terms of Γ's to show all Γ and ∂ terms explicitly.

Exercise 11.9. RIEMANN NORMAL COORDINATES IN GENERAL

Derive properties (11.27), (11.28), (11.29), (11.31), (11.32), and (11.32′) of Riemann normal coordinates. *Hint:* Proceed as follows.

(a) From definition (11.26), derive $(\partial\mathscr{P}/\partial x^{\alpha})_{\mathscr{P}_0} = \boldsymbol{e}_{\alpha}$.

(b) Similarly, from definition (11.26), show that each of the curves $x^{\alpha} = v^{\alpha}\lambda$ (where the v^{α} are constants) is a geodesic through $\mathscr{P}_0$, with affine parameter λ.

(c) Show that $\Gamma^{\alpha}{}_{\beta\gamma}(\mathscr{P}_0) = 0$ by substituting $x^{\alpha} = v^{\alpha}\lambda$ into the geodesic equation.

(d) Since the curves $x^{\alpha} = v^{\alpha}\lambda$ are geodesics for every choice of the parameters v^{α}, they provide not only a geodesic tangent $\boldsymbol{u} \equiv (\partial/\partial\lambda)_{v^{\alpha}}$, but also several deviation vectors $\boldsymbol{N}_{(\alpha)} \equiv (\partial/\partial v^{\alpha})_{\lambda}$. Compute the components of these vectors in the Riemann normal coordinate system, and substitute into the geodesic deviation equation as written in exercise 11.8.

(e) Equate to zero the coefficients of the zeroth and first powers of λ in the geodesic deviation equation of part (d), using

$$\Gamma^{\alpha}{}_{\beta\gamma}\Big|_{x^{\mu}=v^{\mu}\lambda} = \lambda v^{\mu}\Gamma^{\alpha}{}_{\beta\gamma,\mu}(\mathscr{P}_0) + O(\lambda^2),$$

which is a Taylor series for Γ. In this way arrive at equation (11.29) for $\Gamma^{\alpha}{}_{\beta\gamma,\mu}$ in terms of the Riemann tensor.

(f) From equations (11.28), (11.29), and (8.24) for the connection coefficients in terms of the metric, derive equations (11.31), (11.32), and (11.32′).

Exercise 11.10. BIANCHI IDENTITIES

Show that the Riemann curvature tensor satisfies the following "Bianchi identities"

$$R^{\alpha}{}_{\beta[\gamma\delta;\epsilon]} = 0. \tag{11.38}$$

The geometric meaning of these identities will be discussed in Chapter 15. [*Hint:* Perform the calculation at the origin of a Riemann normal coordinate system.]

Exercise 11.11. CURVATURE OPERATOR ACTS ON 1-FORMS

Let $\mathcal{R}(\boldsymbol{u}, \boldsymbol{v})$ be the operator $\mathcal{R}(\boldsymbol{u}, \boldsymbol{v}) = [\boldsymbol{\nabla}_{\boldsymbol{u}}, \boldsymbol{\nabla}_{\boldsymbol{v}}] - \boldsymbol{\nabla}_{[\boldsymbol{u}, \boldsymbol{v}]}$ when acting on 1-forms $\boldsymbol{\sigma}$ (or other tensors) as well as on tangent vectors. Show that

$$\langle \mathcal{R}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{\sigma}, \boldsymbol{w} \rangle = -\langle \boldsymbol{\sigma}, \mathcal{R}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w} \rangle.$$

Exercise 11.12. ROTATION GROUP: RIEMANN CURVATURE

[Continuation of exercises 9.13, 9.14, and 10.17.] Calculate the components of the Riemann curvature tensor for the rotation group's manifold $SO(3)$; use the basis of generators $\{\boldsymbol{e}_\alpha\}$. [*Answer:*

$$R^{\alpha}{}_{\beta\gamma\delta} = \frac{1}{2}\delta^{\alpha\beta}_{\gamma\delta}, \tag{11.39}$$

where $\delta^{\alpha\beta}_{\gamma\delta}$ is the permutation symbol defined in equation (3.50l):

$$\delta^{\alpha\beta}_{\gamma\delta} \equiv (\delta^{\alpha}{}_{\gamma}\delta^{\beta}{}_{\delta} - \delta^{\alpha}{}_{\delta}\delta^{\beta}{}_{\gamma}).$$

Note that this answer is independent of location $\mathscr{P}$ in the group manifold.]

CHAPTER 12

NEWTONIAN GRAVITY IN THE LANGUAGE OF CURVED SPACETIME

The longest period of time for which a modern painting has hung upside down in a public gallery unnoticed is 47 days. This occurred to Le Bateau *by Matisse in the Museum of Modern Art, New York City. In this time 116,000 people had passed through the gallery.*

McWHIRTER AND McWHIRTER (1971)

§12.1. NEWTONIAN GRAVITY IN BRIEF

This chapter is entirely Track 2. Chapters 9–11 are necessary preparation for it.
It is not needed for any later chapter, but it will be helpful in
(1) Chapter 17 (Einstein field equations) and
(2) Chapters 38 and 39 (experimental tests and other theories of gravity).

The equivalence principle is not unique to Einstein's description of the facts of gravity. What is unique to Einstein is the combination of the equivalence principle and local Lorentz geometry. To return to the world of Newton, forget everything discovered in the last century about special relativity, light cones, the limiting speed of light, and proper time. Return to the "universal time" t of earlier centuries. In terms of that universal time, and of rectangular, "Galilean" space coordinates, Newtonian theory gives for the trajectories of neutral test particles

Newtonian gravity: original formulation

$$\frac{d^2x^j}{dt^2} + \frac{\partial \Phi}{\partial x^j} = 0; \tag{12.1}$$

$$\Phi \text{ (sometimes denoted } -U) = \text{Newtonian potential.} \tag{12.2}$$

Customarily one interprets these equations as describing the "curved paths" $x^j(t)$ along which test particles fall in Euclidean space (*not* spacetime). These curved paths include circular orbits about the Earth and the parabolic trajectory of a baseball. Cartan (1923, 1924) asks one to abandon this viewpoint. Instead, he says, regard these trajectories as geodesics $[t(\lambda), x^j(\lambda)]$ in curved spacetime. (This change of viewpoint was embodied in Figures B and C of Box 1.6.) Since the "affinely ticking"

Newtonian gravity: translation into language of curved spacetime

Newtonian clocks carried by test particles read universal time (or some multiple, $\lambda = at + b$, thereof), the equation of motion (12.1) can be rewritten

$$\frac{d^2t}{d\lambda^2} = 0, \qquad \frac{d^2x^j}{d\lambda^2} + \frac{\partial\Phi}{\partial x^j}\left(\frac{dt}{d\lambda}\right)^2 = 0. \tag{12.3}$$

By comparing with the geodesic equation

$$d^2x^\alpha/d\lambda^2 + \Gamma^\alpha{}_{\beta\gamma}(dx^\beta/d\lambda)(dx^\gamma/d\lambda) = 0,$$

one can read off the values of the connection coefficients:

$$\Gamma^j{}_{00} = \partial\Phi/\partial x^j; \qquad \text{all other } \Gamma^\alpha{}_{\beta\gamma} \text{ vanish.} \tag{12.4}$$

And by inserting these into the standard equation (11.12) for the components of the Riemann tensor, one learns (exercise 12.1)

$$R^j{}_{0k0} = -R^j{}_{00k} = \frac{\partial^2\Phi}{\partial x^j\,\partial x^k}; \qquad \text{all other } R^\alpha{}_{\beta\gamma\delta} \text{ vanish.} \tag{12.5}$$

Finally, the source equation for the Newtonian potential

$$\nabla^2\Phi \equiv \sum_j \Phi_{,jj} = 4\pi\rho \tag{12.6}$$

one can rewrite with the help of the *"Ricci curvature tensor"*

$$R_{\alpha\beta} \equiv R^\mu{}_{\alpha\mu\beta} \text{ (contraction of } \textbf{\textit{Riemann}}) \tag{12.7}$$

in the geometric form (exercise 12.2)

$$R_{00} = 4\pi\rho; \qquad \text{all other } R_{\alpha\beta} \text{ vanish.} \tag{12.8}$$

Equation (12.4) for $\Gamma^\alpha{}_{\beta\gamma}$, equation (12.5) for $R^\alpha{}_{\beta\gamma\delta}$, equation (12.8) for $R_{\alpha\beta}$, plus the law of geodesic motion are the full content of Newtonian gravity, rewritten in geometric language.

It is one thing to pass quickly through these component manipulations. It is quite another to understand fully, in abstract and pictorial terms, the meanings of these equations and the structure of Newtonian spacetime. To produce such understanding, and to compare Newtonian spacetime with Einsteinian spacetime, are the goals of this chapter, which is based on the work of Cartan (1923, 1924), Trautman (1965), and Misner (1969a).

EXERCISES

Exercise 12.1. RIEMANN CURVATURE OF NEWTONIAN SPACETIME

Derive equation (12.5) for $R^\alpha{}_{\beta\gamma\delta}$ from equation (12.4) for $\Gamma^\alpha{}_{\beta\gamma}$.

Exercise 12.2. NEWTONIAN FIELD EQUATION

Derive the geometric form (12.8) of the Newtonian field equation from (12.5) through (12.7).

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§12.2. STRATIFICATION OF NEWTONIAN SPACETIME

Galileo and Newton spoke of a flat, Euclidean "absolute space" and of an "absolute time," two concepts distinct and unlinked. In absolute space Newtonian physics took place; and as it took place, absolute time marched on. No hint was there that space and time might be two aspects of a single entity, a curved *"spacetime"*—until Einstein made the unification in relativity physics, and Cartan (1923) followed suit in Newtonian physics in order to provide clearer insight into Einstein's ideas.

The geometry of Newtonian spacetime:

How do the absolute space of Galileo and Newton, and their absolute time, fit into Cartan's *"Newtonian spacetime"?* The key to the fit is *stratification;* stratification produced by the universal time coordinate t.

"Universal time" as a scalar field

Regard t as a function (scalar field) defined once and for all in Newtonian spacetime

$$t = t(\mathscr{P}). \tag{12.9}$$

Without it, spacetime could not be Newtonian, for "t" is every bit as intrinsic to Newtonian spacetime as the metric "$\boldsymbol{g}$" is to Lorentz spacetime. The layers of spacetime are the slices of constant t—the *"space slices"*—each of which has an identical geometric structure: the old "absolute space."

Adopting Cartan's viewpoint, ask what kind of geometry is induced onto each space slice by the surrounding geometry of spacetime. A given space slice is endowed, by the Galilean coordinates of §12.1, with basis vectors $\boldsymbol{e}_j = \partial/\partial x^j$; and this basis has vanishing connection coefficients, $\Gamma^j{}_{k\ell} = 0$ [cf. equation (12.4)]. Consequently, *the geometry of each space slice is completely flat.*

Space slices with Euclidean geometry

"Absolute space" is Euclidean in its geometry, according to the old viewpoint, and the Galilean coordinates are Cartesian. Translated into Cartan's language, this says: not only is each space slice (t = constant) flat, and not only do its Galilean coordinates have vanishing connection coefficients, but also *each space slice is endowed with a three-dimensional metric, and its Galilean coordinate basis is orthonormal,*

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = (\partial/\partial x^i)\cdot(\partial/\partial x^j) = \delta_{ij}. \tag{12.10}$$

Curvature acts in spacetime, not in space slices

If the space slices are really so flat, where do curvature and geodesic deviation enter in? They are properties of *spacetime.* Parallel transport a vector around a closed curve lying entirely in a space slice; it will return to its starting point unchanged. But transport it forward in time by Δt, northerly in space by Δx^k, back in time by $-\Delta t$, and southerly by $-\Delta x^k$ to its starting point; it will return changed by

$$\delta\boldsymbol{A} = -\mathscr{R}\left(\Delta t\frac{\partial}{\partial t}, \Delta x^k\frac{\partial}{\partial x^k}\right)\boldsymbol{A};$$

i.e.,

$$\delta A^0 = 0, \qquad \delta A^j = -R^j{}_{00k}A^0(\Delta t)(\Delta x^k) = \frac{\partial^2\Phi}{\partial x^j\,\partial x^k}A^0(\Delta t)(\Delta x^k). \tag{12.11}$$

Geodesics of a space slice (Euclid's straight lines) that are initially parallel remain

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always parallel. But geodesics of spacetime (trajectories of freely falling particles) initially parallel get pried apart or pushed together by spacetime curvature,

$$\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{n} + \mathcal{R}(\boldsymbol{n}, \boldsymbol{u})\boldsymbol{u} = 0,$$

or equivalently in Galilean coordinates:

$$n^0 = dn^0/dt = 0 \text{ initially} \Longrightarrow n^0 = 0 \text{ always}; \tag{12.12a}$$

$$\frac{d^2 n^j}{dt^2} + \frac{\partial^2 \Phi}{\partial x^j\, \partial x^k} n^k = 0 \tag{12.12b}$$

(see Box 12.1 and exercise 12.3).

EXERCISE

Exercise 12.3. GEODESIC DEVIATION DERIVED

Produce a third column for Box 11.4, one that carries out the "geometric analysis" in component notation using the Galilean connection coefficients (12.4) of Newtonian spacetime. Thereby achieve a deeper understanding of how the geometric analysis parallels the old Newtonian analysis.

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§12.3. GALILEAN COORDINATE SYSTEMS

The Lorentz spacetime of special relativity has an existence and structure completely independent of any coordinate system. But a special property of its geometry (zero curvature) allows the introduction of a special class of coordinates (Lorentz coordinates), which cling to spacetime in a special way

$$(\partial/\partial x^\alpha) \cdot (\partial/\partial x^\beta) = \eta_{\alpha\beta} \text{ everywhere.}$$

By studying these special coordinate systems and the relationships between them (Lorentz transformations), one learns much about the structure of spacetime itself (breakdown in simultaneity; Lorentz contraction; time dilatation; . . .).

Galilean coordinates defined

Similarly for Newtonian spacetime. Special properties of its geometry (explored in abstract later; Box 12.4) permit the introduction of special coordinates (Galilean coordinates), which cling to spacetime in a special way

$$x^0(\mathscr{P}) = t(\mathscr{P});$$

$$(\partial/\partial x^j) \cdot (\partial/\partial x^k) = \delta_{jk};$$

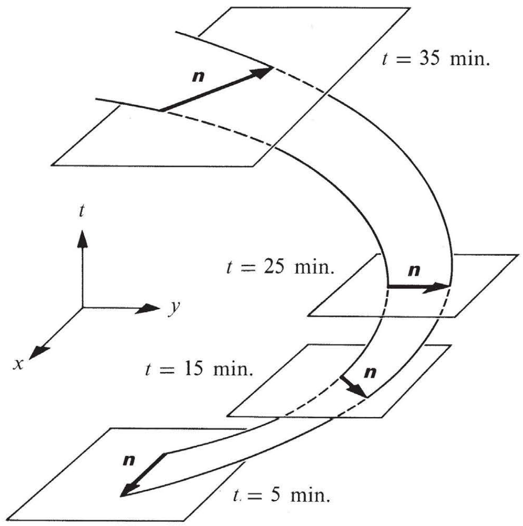
$$\Gamma^j{}_{00} = \Phi_{,j} \text{ for some scalar field } \Phi, \text{ and all other } \Gamma^\alpha{}_{\beta\gamma} \text{ vanish.}$$

To understand Newtonian spacetime more deeply, study the relations between these Galilean coordinate systems.

Box 12.1 GEODESIC DEVIATION IN NEWTONIAN SPACETIME

Coordinate system for calculation: Galilean space coordinates x^j and universal time coordinate t. General component form of equation:

$$\frac{D^2 n^\alpha}{d\lambda^2} + R^\alpha{}_{\beta\gamma\delta}\frac{dx^\beta}{d\lambda} n^\gamma \frac{dx^\delta}{d\lambda} = 0.$$



Special conditions for this calculation: let the particles' clocks (affine parameters) all be normalized to read universal time, $\lambda = t$. This means that the separation vector

$$n^\alpha = (\partial x^\alpha/\partial n)_\lambda$$

between geodesics has zero time component, $n^0 = 0$; i.e., in abstract language,

$$\langle \boldsymbol{d}t, \boldsymbol{n}\rangle = t_{,\alpha} n^\alpha = n^0 = 0;$$

i.e., in geometric language, $\boldsymbol{n}$ lies in a space slice (surface of constant t).

Evaluation of covariant derivative:

$$\frac{Dn^\alpha}{D\lambda} = \frac{dn^\alpha}{d\lambda} + \underbrace{\Gamma^\alpha{}_{\beta\mu}}_{[0 \text{ unless } \beta = 0]} \underbrace{n^\beta}_{[0 \text{ unless } \beta \text{ is a space index } (n^0=0)]} \frac{dx^\mu}{d\lambda} = \frac{dn^\alpha}{d\lambda},$$

$$\frac{D^2 n^\alpha}{d\lambda^2} = \frac{d(Dn^\alpha/d\lambda)}{d\lambda} + \underbrace{\Gamma^\alpha{}_{\beta\mu}}_{[0 \text{ unless } \beta = 0]} \underbrace{\frac{dn^\beta}{d\lambda}}_{[0 \text{ unless } \beta \text{ is space index}]} \frac{dx^\mu}{d\lambda} = \frac{d(Dn^\alpha/d\lambda)}{d\lambda} = \frac{d^2 n^\alpha}{d\lambda^2} \underset{[\text{since } \lambda = t]}{=} \frac{d^2 n^\alpha}{dt^2}.$$

Evaluation of tidal accelerations:

$$R^0{}_{\beta\gamma\delta}\frac{dx^\beta}{d\lambda} n^\gamma \frac{dx^\delta}{d\lambda} = 0 \qquad \text{since } R^j{}_{0k0} \text{ and } R^j{}_{00k} \text{ are only nonzero components.}$$

$$\underbrace{R^j{}_{\beta\gamma\delta}}_{[\text{for } \gamma \text{ a space index: 0 unless } \beta = \delta = 0]} \frac{dx^\beta}{d\lambda} \underbrace{n^\gamma}_{[0 \text{ unless } \gamma \text{ is space index}]} \frac{dx^\delta}{d\lambda} = R^j{}_{0k0}\frac{dt}{d\lambda} n^k \frac{dt}{d\lambda} = R^j{}_{0k0} n^k = \frac{\partial^2 \Phi}{\partial x^j\,\partial x^k} n^k.$$

Resultant equation of geodesic deviation:

$$\frac{d^2 n^0}{dt^2} = 0 \qquad \left(\begin{array}{l}\text{agrees with result } n^0 = 0 \text{ always, which} \\ \text{followed from choice } \lambda = t \text{ for all particles}\end{array}\right)$$

$$\frac{d^2 n^j}{dt^2} + \frac{\partial^2 \Phi}{\partial x^j\,\partial x^k} n^k = 0 \qquad \left(\begin{array}{l}\text{agrees with Newton-type calculation} \\ \text{in Box 11.4; see also exercise 12.3}\end{array}\right).$$

Point of principle: how can one write down the laws of gravity and properties of spacetime in Galilean coordinates first (§12.1), and only afterward (here) come to grip with the nature of the coordinate system and its nonuniqueness? Answer: (a quotation from §3.1, slightly modified): "Here and elsewhere in science, as emphasized not least by Henri Poincaré, that view is out of date which used to say 'Define your terms before you proceed.' All the laws and theories of physics, including Newton's laws of gravity, have this deep and subtle character, that they both define the concepts they use (here Galilean coordinates) and make statements about these concepts."

The Newtonian laws of gravity, written in a Galilean coordinate system

$$x^0 = t, \qquad (\partial/\partial x^j)\cdot(\partial/\partial x^k) = \delta_{jk}$$

make the statement "$\Gamma^j{}_{00} = \Phi_{,j}$ and all other $\Gamma^\alpha{}_{\beta\gamma} = 0$" about the geometry of spacetime. This statement in turn gives information about the relationships between different Galilean systems. Let one Galilean system $\{x^\alpha(\mathscr{P})\}$ be given, and seek the most general coordinate transformation leading to another, $\{x^{\alpha'}(\mathscr{P})\}$. The following constraints exist: (1) $x^{0'} = x^0 = t$ (both time coordinates must be universal time); (2) at fixed t (i.e., in a fixed space slice) both sets of space coordinates must be Euclidean, so they must be related by a rotation and a translation:

$$x^{j'} = \underbrace{A_{j'k}}_{\text{[rotation matrix, i.e., } A_{j'\ell}A_{k'\ell} = \delta_{j'k'}]} x^k + \underbrace{a^{j'}}_{\text{[translation]}} \tag{12.13a}$$

$$x^k = A_{j'k}x^{j'} - a^k, \text{ with } a^k \equiv A_{j'k}a^{j'}. \tag{12.13b}$$

The rotation and translation might, *a priori*, be different on different slices, $A_{j'k} = A_{j'k}(t)$ and $a^j = a^j(t)$; but (3) they must be constrained by the required special form of the connection coefficients. Calculate the connection coefficients in the new coordinate system, given their form in the old. The result (exercise 12.4) is:

$$\begin{aligned}
&\Gamma^{j'}{}_{0'k'} = \Gamma^{j'}{}_{k'0'} = A_{j'\ell}\dot{A}_{k'\ell} \qquad \text{(produces "Coriolis forces");}\\
&\Gamma^{j'}{}_{0'0'} = \frac{\partial\Phi}{\partial x^{j'}} + A_{j'k}(\underbrace{\ddot{A}_{\ell'k}x^{\ell'}}_{\text{["centrifugal forces"]}} - \underbrace{\ddot{a}^k}_{\text{["inertial forces"]}});\\
&\text{all other } \Gamma^{\alpha'}{}_{\beta'\gamma'} \text{ vanish}
\end{aligned} \tag{12.14}$$

("Euclidean" index conventions; repeated space indices to be summed even if both are down; dot denotes time derivative). These have the standard Galilean form (12.4) if and only if

$$\dot{A}_{j'k} = 0, \qquad \underbrace{\Phi'}_{\text{[Newtonian potential in new coordinate system]}} = \underbrace{\Phi}_{\text{[Newtonian potential in old coordinate system]}} - \ddot{a}^k x^k + \text{constant}. \tag{12.15}$$

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These results can be restated in words: any two Galilean coordinate systems are related by (1) a time-independent rotation of the space grid (*same* rotation on each space slice), and (2) a time-dependent translation of the space grid (translation possibly *different* on different slices)

Transformations linking Galilean coordinate systems

$$x^{j'} = \underbrace{A_{j'k}}_{\text{[constant]}} x^k + \underbrace{a^{j'}(t)}_{\text{[time-dependent]}}. \tag{12.16}$$

The Newtonian potential is not a function defined in spacetime with existence independent of all coordinate systems. (There is no coordinate-free way to measure it.) Rather, it depends for its existence on a particular choice of Galilean coordinates; and if the choice is changed via equation (12.16), then Φ is changed:

Newtonian potential depends on choice of Galilean coordinate system

$$\Phi' = \Phi - \ddot{a}^k x^k. \tag{12.17}$$

(By contrast, an existence independent of all coordinates is granted to the universal time $t(\mathscr{P})$ and the covariant derivative $\boldsymbol{\nabla}$.)

Were all the matter in the universe concentrated in a finite region of space and surrounded by emptiness ("island universe"), then one could impose the global boundary condition

Absolute Galilean coordinates defined

$$\Phi \longrightarrow 0 \text{ as } r \equiv (x^k x^k)^{1/2} \longrightarrow \infty. \tag{12.18}$$

This would single out a subclass of Galilean coordinates ("*absolute*" Galilean coordinates), with a unique, common Newtonian potential. The transformation from one absolute Galilean coordinate system to any other would be

Transformations linking absolute Galilean coordinate systems

$$x^{j'} = \underbrace{A_{j'k}}_{\text{constant rotation}} x^k + \underbrace{a^{j'}}_{\text{constant displacement}} + \underbrace{v^{j'}}_{\text{constant velocity}} t \tag{12.19}$$

("Galilean transformation"). *But,* (1) by no local measurements could one ever distinguish these absolute Galilean coordinate systems from the broader class of Galilean systems (to distinguish, one must integrate the locally measurable quantity $\Phi_{,j} = \Gamma^j{}_{00}$ out to infinity); and (2) astronomical data deny that the real universe is an island of matter surrounded by emptiness.

It is instructive to compare Galilean coordinates and Newtonian spacetime as described above with Lorentz coordinates and the Minkowskii spacetime of special relativity, and with the general coordinates and Einstein spacetime of general relativity; see Boxes 12.2 and 12.3.

(*continued on page 298*)

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Box 12.2 NEWTONIAN SPACETIME, MINKOWSKIIAN SPACETIME, AND EINSTEINIAN SPACETIME: COMPARISON AND CONTRAST

Query	*Newtonian spacetime*	*Minkowskiian spacetime (special relativity)*	*Einsteinian spacetime (general relativity)*
What *a priori* geometric structures does spacetime possess?	(1) Universal time function t (2) Covariant derivative ∇ (3) Spatial metric "$\cdot$"; but spacetime metric can *not* be defined (exercise 12.10)	A spacetime metric that is flat (vanishing Riemann curvature)	A spacetime metric
What preferred coordinate systems are present?	(1) Galilean coordinates in general (2) Absolute Galilean coordinates in an island universe (this case not considered here)	Lorentz coordinates	In general, every coordinate system is equally preferred (though in special cases with symmetry there are special preferred coordinates)
What is required to select out a particular preferred coordinate system?	(1) A single spatial orientation, the same throughout all spacetime (three Euler angles) (2) The arbitrary world line of the origin of space coordinates (three functions of time)	(1) A single spatial orientation, the same throughout all spacetime (three Euler angles) (2) The location of the origin of coordinates (four numbers) (3) The velocity of the origin of space coordinates (three numbers)	All four functions of position $x^{\alpha}(\mathscr{P})$
Under what conditions is "$\mathscr{P}$ and $\mathscr{Q}$ are simultaneous" well-defined?	In general; it is a coordinate-free geometric concept	Only after a choice of Lorentz frame has been made; "simultaneity" depends on the frame's velocity	Only after arbitrary choice of time coordinate has been made
Under what conditions is "$\mathscr{P}$ and $\mathscr{Q}$ occur at same point in space" well-defined?	Only after choice of Galilean coordinates has been made	Only after choice of Lorentz coordinates has been made	Only after arbitrary choice of space coordinates has been made
Under what conditions is "$\boldsymbol{u}$ and $\boldsymbol{v}$, at different events, point in same direction" well-defined?	Only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are both spatial vectors ($\langle \boldsymbol{d}t, \boldsymbol{u}\rangle = \langle \boldsymbol{d}t, \boldsymbol{v}\rangle = 0$); or if they lie in the same space slice and are arbitrary vectors; or if there exists a preferred route connecting their locations, along which to compare them by parallel transport	Always	Only if $\boldsymbol{u}$ and $\boldsymbol{v}$ lie at events infinitesimally close together; or if there exists a preferred route (e.g., a unique geodesic) connecting their locations, along which to compare them by parallel transport
Under what conditions is "the invariant distance between $\mathscr{P}$ and $\mathscr{Q}$" well-defined?	Only if $\mathscr{P}$ and $\mathscr{Q}$ lie in the same space slice	Always	Only if $\mathscr{P}$ and $\mathscr{Q}$ are sufficiently close together; or if there exists a unique preferred world line (e.g., a geodesic) linking them, along which to measure the distance

Box 12.3 NEWTONIAN GRAVITY Á LA CARTAN, AND EINSTEINIAN GRAVITY: COMPARISON AND CONTRAST

Property	*Newton-Cartan*	*Einstein*
Idea in brief (formulations of the equivalence principle of very different scope)	Laws of motion of free particles in a local, freely falling, nonrotating frame are identical to Newton's laws of motion as expressed in a gravity-free Galilean frame	Laws of physics in a local, freely falling, nonrotating frame are identical with the laws of physics as formulated in special relativity in a Lorentz frame
Idea even more briefly stated	Point mechanics simple in a local inertial frame	Everything simple in a local inertial frame
Consequence (tested to one part in 10^{11} by Roll-Krotkov-Dicke experiment)	Test particles of diverse composition started with same initial position and same initial velocity follow the same world line ("definition of geodesic")	Test particles of diverse composition started with same initial position and same initial velocity follow the same world line ("definition of geodesic")
Another consequence	In every local region, there exists a local frame ("freely falling frame") in which all geodesics appear straight (all $\Gamma^\alpha{}_{\mu\nu} = 0$)	In every local region there exists a local frame ("freely falling frame") in which all geodesics appear straight (all $\Gamma^\alpha{}_{\mu\nu} = 0$)
Consequence of way light rays travel in real physical world?	Disregarded or evaded. All light rays have same velocity? Speed depend on motion of source? Speed depend on motion of observer? Possible to move fast enough to catch up with a light ray? No satisfactory position on any of these issues	Spacetime always and everywhere has local Lorentz character
Summary of spacetime structure	Stratified into spacelike slices; geometry in each slice Euclidean; each slice characterized by value of universal time (geodesic parameter); displacement of one slice with respect to another not specified; no such thing as a spacetime interval	No stratification. Well-defined interval between every event and every nearby event; spacetime has everywhere local Lorentz character, with one local frame (specific space and time axes) as good as another (other space and time axes); "homogeneous" rather than stratified
This structure expressed in mathematical language	$\Gamma^\alpha{}_{\mu\nu}$'s, yes; spacetime metric $g_{\mu\nu}$, no; $\Gamma^i{}_{00} = \dfrac{\partial\Phi}{\partial x^i}\ (i = 1, 2, 3)$; all other $\Gamma^\alpha{}_{\mu\nu}$ vanish	$\Gamma^\alpha{}_{\mu\nu}$'s have no independent existence; all derived from $\Gamma^\alpha{}_{\mu\nu} = g^{\alpha\beta}\dfrac{1}{2}\left(\dfrac{\partial g_{\beta\nu}}{\partial x^\mu} + \dfrac{\partial g_{\beta\mu}}{\partial x^\nu} - \dfrac{\partial g_{\mu\nu}}{\partial x^\beta}\right)$ ("metric theory of gravity")

EXERCISES

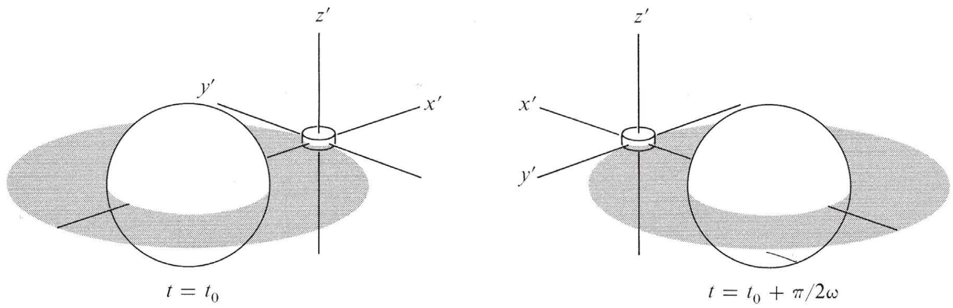

Figure 12.1.
The coordinate system carried by an orbital laboratory as it moves in a circular orbit about the Earth.

Exercise 12.4. CONNECTION COEFFICIENTS FOR ROTATING, ACCELERATING COORDINATES

Beginning with equation (12.4) for the connection coefficients of a Galilean coordinate system $\{x^\alpha(\mathcal{P})\}$, derive the connection coefficients (12.14) of the coordinate system $\{x^{\alpha'}(\mathcal{P})\}$ of equations (12.13). From this, verify that (12.15) are necessary and sufficient for $\{x^{\alpha'}(\mathcal{P})\}$ to be Galilean.

Exercise 12.5. EINSTEIN'S ELEVATOR

Use the formalism of this chapter to discuss "Einstein's elevator"—i.e., the equivalence of "gravity" to an acceleration of one's reference frame. Which aspects of "gravity" are equivalent to an acceleration, and which are not?

Exercise 12.6. GEODESIC DEVIATION ABOVE THE EARTH

A manned orbital laboratory is put into a circular orbit about the Earth [radius of orbit $= r_0$, angular velocity $= \omega = (M/r_0{}^3)^{1/2}$—why?]. An astronaut jetisons a bag of garbage and watches it move along its geodesic path. He observes its motion relative to (non-Galilean) space coordinates $\{x^{j'}(\mathcal{P})\}$ which—see Figure 12.1—(1) are Euclidean at each moment of universal time $[(\partial/\partial x^{j'}) \cdot (\partial/\partial x^{k'}) = \delta_{jk}]$, (2) have origin at the laboratory's center, (3) have $\partial/\partial x'$ pointing away from the Earth, (4) have $\partial/\partial x'$ and $\partial/\partial y'$ in the plane of orbit. Use the equation of geodesic deviation to calculate the motion of the garbage bag in this coordinate system. Verify the answer by examining the Keplerian orbits of laboratory and garbage. *Hints:* (1) Calculate $R^{\alpha'}{}_{\beta'\gamma'\delta'}$ in this coordinate system by a trivial transformation of tensorial components. (2) Use equation (12.14) to calculate $\Gamma^{\alpha'}{}_{\beta'\gamma'}$ at the center of the laboratory (i.e., on the fiducial geodesic).

§12.4. GEOMETRIC, COORDINATE-FREE FORMULATION OF NEWTONIAN GRAVITY

To restate Newton's theory of gravity in coordinate-independent, geometric language is the principal goal of this chapter. It has been achieved, thus far, with extensive assistance from a special class of coordinate systems, the Galilean coordinates. To climb out of Galilean coordinates and into completely coordinate-free language is straightforward in principle. One merely passes from index notation to abstract notation.

Example: Restate in coordinate-free language the condition $\Gamma^0{}_{\alpha\beta} = 0$ of Galilean coordinates.

Solution: Write $\Gamma^0{}_{\alpha\beta} = -\langle \boldsymbol{\nabla}_\beta\, \boldsymbol{d}t, \boldsymbol{e}_\alpha \rangle$; the vanishing of this for all α means $\boldsymbol{\nabla}_\beta\, \boldsymbol{d}t = 0$ for all β, which in turn means $\boldsymbol{\nabla}_{\boldsymbol{u}}\, \boldsymbol{d}t = 0$ for all $\boldsymbol{u}$. In words: *the gradient of universal time is covariantly constant.*

Coordinate-free, geometric axioms for Newton's theory of gravity

By this process one can construct a set of coordinate-free statements about Newtonian spacetime (Box 12.4) that are completely equivalent to the standard, nongeometric version of Newton's gravitation theory. From standard Newtonian theory, one can deduce these geometric statements (exercise 12.7); from these geometric statements, regarded as axioms, one can deduce standard Newtonian theory (exercise 12.8).

EXERCISES

Exercise 12.7. FROM NEWTON TO CARTAN

From the standard axioms of Newtonian theory (last part of Box 12.4) derive the geometric axioms (first part of Box 12.4). *Suggested procedure:* Verify each of the geometric axioms by a calculation in the Galilean coordinate system. Make free use of the calculations and results in §12.1.

Exercise 12.8. FROM CARTAN TO NEWTON

From the geometric axioms of Newtonian theory (first part of Box 12.4) derive the standard axioms (last part of Box 12.4). *Suggested procedure:* (1) Pick three orthonormal, spatial basis vectors ($\boldsymbol{e}_j$ with $\boldsymbol{e}_j \cdot \boldsymbol{e}_k = \delta_{jk}$) at some event $\mathscr{P}_0$. Parallel transport each of them by arbitrary routes to all other events in spacetime.

(2) Use the condition $\mathscr{R}(\boldsymbol{u}, \boldsymbol{n})\boldsymbol{e}_j = 0$ for all $\boldsymbol{u}$ and $\boldsymbol{n}$ [axiom (3)] and an argument like that in §11.5 to conclude: (a) the resultant vector fields $\boldsymbol{e}_j$ are independent of the arbitrary transport routes, (b) $\boldsymbol{\nabla}\boldsymbol{e}_j = 0$ for the resultant fields, and (c) $[\boldsymbol{e}_j, \boldsymbol{e}_k] = 0$.

(3) Pick an arbitrary "time line", which passes through each space slice (slice of constant t) once and only once. Parametrize it by t and select its tangent vector as the basis vector $\boldsymbol{e}_0$ at each event along it. Parallel transport each of these $\boldsymbol{e}_0$'s throughout its respective space slice by arbitrary routes.

(4) From axiom (4) conclude that the resultant field is independent of the transport routes; also show that the above construction process guarantees $\boldsymbol{\nabla}_j \boldsymbol{e}_0 = \boldsymbol{\nabla}_0 \boldsymbol{e}_j = 0$.

(5) Show that $[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = 0$ for all pairs of the four basis-vector fields, and conclude from this that there exists a coordinate system ("Galilean coordinates") in which $\boldsymbol{e}_\alpha = \partial/\partial x^\alpha$ (see §11.5 and exercise 9.9).

(6) Show that in this coordinate system $\boldsymbol{e}_j \cdot \boldsymbol{e}_k = \delta_{jk}$ everywhere (space coordinates are Euclidean), and the only nonzero components of the connection coefficient are $\Gamma^j{}_{00}$; here axioms (6) and (2) will be helpful.

(7) From the self-adjoint property of the Jacobi curvature operator (axiom 7) show that $R^j{}_{0k0} = R^k{}_{0j0}$; show that in terms of the connection coefficients this reads $\Gamma^j{}_{00,k} = \Gamma^k{}_{00,j}$; and from this conclude that there exists a potential Φ such that $\Gamma^j{}_{00} = \Phi_{,j}$.

(8) Show that the geometric field equation (axiom 5) reduces to Poisson's equation $\nabla^2\Phi = 4\pi\rho$.

(9) Show that the geodesic equation for free fall (axiom 8) reduces to the Newtonian equation of motion $d^2x^j/dt^2 + \Phi_{,j} = 0$.

(*continued on page 302*)

Box 12.4 NEWTONIAN GRAVITY: GEOMETRIC FORMULATION CONTRASTED WITH STANDARD FORMULATION

Geometric Formulation

Newton's theory of gravity and the properties of Newtonian spacetime can be derived from the following axioms. (For derivation see exercise 12.8.)

(1) There exists a function t called "universal time", and a symmetric covariant derivative $\boldsymbol{\nabla}$ (with associated geodesics, parallel transport law, curvature operator, etc.).

(2) The 1-form $\boldsymbol{d}t$ is covariantly constant; i.e.,

$$\boldsymbol{\nabla}_{\boldsymbol{u}}\, \boldsymbol{d}t = 0 \text{ for all } \boldsymbol{u}.$$

[Consequence: if $\boldsymbol{w}$ is a spatial vector field (i.e., $\boldsymbol{w}$ lies everywhere in a surface of constant t; i.e. $\langle \boldsymbol{d}t, \boldsymbol{w}\rangle = 0$ everywhere), then $\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}$ is also spatial for every $\boldsymbol{u}$,

$$\langle \boldsymbol{d}t, \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w}\rangle = \boldsymbol{\nabla}_{\boldsymbol{u}} \underbrace{\langle \boldsymbol{d}t, \boldsymbol{w}\rangle}_{\text{[0 always]}} - \langle \underbrace{\boldsymbol{\nabla}_{\boldsymbol{u}}\, \boldsymbol{d}t}_{\text{[0 always]}}, \boldsymbol{w}\rangle = 0.]$$

(3) Spatial vectors are unchanged by parallel transport around infinitesimal closed curves; i.e.,

$$\mathcal{R}(\boldsymbol{u}, \boldsymbol{n})\boldsymbol{w} = 0 \text{ if } \boldsymbol{w} \text{ is spatial, for every } \boldsymbol{u} \text{ and } \boldsymbol{n}.$$

(4) All vectors are unchanged by parallel transport around infinitesimal, spatial, closed curves; i.e.,

$$\mathcal{R}(\boldsymbol{v}, \boldsymbol{w}) = 0 \text{ for every spatial } \boldsymbol{v} \text{ and } \boldsymbol{w}.$$

(5) The Ricci curvature tensor, $R_{\alpha\beta} \equiv R^{\mu}{}_{\alpha\mu\beta}$, has the form

$$\boldsymbol{Ricci} = 4\pi\rho\, \boldsymbol{d}t \otimes \boldsymbol{d}t,$$

where ρ is the density of mass.

(6) There exists a metric "$\cdot$" defined on spatial vectors only, which is compatible with the covariant derivative in this sense: for any spatial $\boldsymbol{w}$ and $\boldsymbol{v}$, and for any $\boldsymbol{u}$ whatsoever,

$$\boldsymbol{\nabla}_{\boldsymbol{u}}(\boldsymbol{w}\cdot\boldsymbol{v}) = (\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{w})\cdot\boldsymbol{v} + \boldsymbol{w}\cdot(\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}).$$

[Note: axioms (1), (2), and (3) guarantee that such a spatial metric can exist; see exercise 12.9.]

(7) The Jacobi curvature operator $\mathcal{J}(\boldsymbol{u}, \boldsymbol{n})$, defined for any vectors $\boldsymbol{u}$, $\boldsymbol{n}$, $\boldsymbol{p}$ by

$$\mathcal{J}(\boldsymbol{u}, \boldsymbol{n})\boldsymbol{p} = \frac{1}{2}[\mathcal{R}(\boldsymbol{p}, \boldsymbol{n})\boldsymbol{u} + \mathcal{R}(\boldsymbol{p}, \boldsymbol{u})\boldsymbol{n}],$$

is *"self-adjoint"* when operating on spatial vectors; i.e.,

$$\boldsymbol{v} \cdot [\mathcal{J}(\boldsymbol{u}, \boldsymbol{n})\boldsymbol{w}] = \boldsymbol{w} \cdot [\mathcal{J}(\boldsymbol{u}, \boldsymbol{n})\boldsymbol{v}] \text{ for all spatial } \boldsymbol{v}, \boldsymbol{w}; \text{ and for any } \boldsymbol{u}, \boldsymbol{n}.$$

(8) "Ideal rods" measure the lengths that are calculated with the spatial metric; "ideal clocks" measure universal time t (or some multiple thereof); and "freely falling particles" move along geodesics of $\boldsymbol{\nabla}$. [Note: this can be regarded as a definition of "ideal rods," "ideal clocks," and "freely falling particles." A more complete theory (e.g., general relativity; see §16.4) would predict in advance whether a given physical rod or clock is ideal, and whether a given real particle is freely falling.]

Note: For an alternative but equivalent set of axioms, see pp. 106–107 of Trautman (1965).

Standard Formulation

The following standard axioms are equivalent to the above.

(1) There exist a universal time t, a set of Cartesian space coordinates x^j (called "Galilean coordinates"), and a Newtonian gravitational potential Φ.

(2) The density of mass ρ generates the Newtonian potential by Poisson's equation,

$$\boldsymbol{\nabla}^2\Phi \equiv \frac{\partial^2\Phi}{\partial x^j\,\partial x^j} = 4\pi\rho.$$

(3) The equation of motion for a freely falling particle is

$$\frac{d^2x^j}{dt^2} + \frac{\partial\Phi}{\partial x^j} = 0.$$

(4) "Ideal rods" measure the Galilean coordinate lengths; "ideal clocks" measure universal time.

Exercise 12.9. SPATIAL METRIC ALLOWED BY OTHER AXIOMS

Show that the geometric axioms (1), (2), and (3) of Box 12.4 permit one to introduce a spatial metric satisfying axiom (6). *Hint:* Pick an arbitrary spatial basis $\{\boldsymbol{e}_i\}$ at some event. *Define* it to be orthonormal, $\boldsymbol{e}_j \cdot \boldsymbol{e}_k \equiv \delta_{jk}$. Extend this basis through all spacetime by the method used in (1) of exercise 12.8. Define $\boldsymbol{e}_j \cdot \boldsymbol{e}_k \equiv \delta_{jk}$ everywhere in spacetime for this basis. Then prove that the resulting metric satisfies the compatibility condition of axiom (6).

Exercise 12.10. SPACETIME METRIC FORBIDDEN BY OTHER AXIOMS

Show that in Newtonian spacetime it is impossible to construct a nondegenerate spacetime metric $\boldsymbol{g}$, defined on *all* vectors, that is compatible with the covariant derivative in the sense that

$$\nabla_{\boldsymbol{u}}\boldsymbol{g}(\boldsymbol{n}, \boldsymbol{p}) = \boldsymbol{g}(\nabla_{\boldsymbol{u}}\boldsymbol{n}, \boldsymbol{p}) + \boldsymbol{g}(\boldsymbol{n}, \nabla_{\boldsymbol{u}}\boldsymbol{p}). \tag{12.20}$$

Note: to prove this requires mastery of the material in Chapter 8 or 13; so study either 8 or 13 before tackling it. *Hint:* Assume that such a $\boldsymbol{g}$ exists. Show, by the methods of exercise 12.8, that in a Galilean coordinate system the spatial components g_{jk} are independent of position in spacetime. Then use this and the form of $R^{\alpha}{}_{\beta\gamma\delta}$ in Galilean coordinates to prove R_{j0k0} and $-R_{0jk0}$ are not identical, a result that conflicts with the symmetries of the Riemann tensor [eq. (8.45)] in a manifold with compatible metric and covariant derivative.

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§12.5. THE GEOMETRIC VIEW OF PHYSICS: A CRITIQUE

An important digression is in order.

"Every physical quantity must be describable by a (coordinate-free) geometric object, and the laws of physics must all be expressible as geometric relationships between these geometric objects." This view of physics, sometimes known as the *"principle of general covariance,"* pervades twentieth-century thinking. But does it have any forcible content? No, not at all, according to one viewpoint that dates back to Kretschmann (1917). Any physical theory originally written in a special coordinate system can be recast in geometric, coordinate-free language. Newtonian theory is a good example, with its equivalent geometric and standard formulations (Box 12.4). Hence, as a sieve for separating viable theories from nonviable theories, the principle of general covariance is useless.

The principle of general covariance has no forcible content

But another viewpoint is cogent. It constructs a powerful sieve in the form of a slightly altered and slightly more nebulous principle: "Nature likes theories that are simple when stated in coordinate-free, geometric language."* According to this principle, Nature must love general relativity, and it must hate Newtonian theory. Of all theories ever conceived by physicists, general relativity has the simplest, most elegant geometric foundation [three axioms: (1) there is a metric; (2) the metric is governed by the Einstein field equation $\boldsymbol{G} = 8\pi\boldsymbol{T}$; (3) all special relativistic laws of physics are valid in local Lorentz frames of metric]. By contrast, what diabolically

Twentieth-century viewpoint judges a theory by simplicity of its geometric formulation

Einstein's theory of gravity is simple; Newton's is complex

*Admittedly, this principle is anthropomorphic: twentieth-century physicists like such theories and even find them effective in correlating observational data. Therefore, Nature must like them too!

clever physicist would ever foist on man a theory with such a complicated geometric foundation as Newtonian theory?

Of course, from the nineteenth-century viewpoint, the roles are reversed. It judged simplicity of theories by examining their coordinate formulations. In Galilean coordinates, Newtonian theory is beautifully simple. Expressed as differential equations for the metric coefficients in a specific coordinate system, Einstein's field equations (ten of them now!) are horrendously complex.

The geometric, twentieth-century view prevails because it accords best with experimental data (see Chapters 38–40). In Chapter 17 it will be applied ruthlessly to make Einstein's field equation seem compelling.

CHAPTER 13

RIEMANNIAN GEOMETRY: METRIC AS FOUNDATION OF ALL

Philosophy is written in this great book (by which I mean the universe) which stands always open to our view, but it cannot be understood unless one first learns how to comprehend the language and interpret the symbols in which it is written, and its symbols are triangles, circles, and other geometric figures, without which it is not humanly possible to comprehend even one word of it; without these one wanders in a dark labyrinth.

GALILEO GALILEI (1623)

§13.1. NEW FEATURES IMPOSED ON GEOMETRY BY LOCAL VALIDITY OF SPECIAL RELATIVITY

This chapter is entirely Track 2. Chapters 9–11 are necessary preparation for it. It will be needed as preparation for
(1) the second half of Chapter 14 (calculation of curvature), and
(2) the details, but not the message, of Chapter 15 (Bianchi identities).
§13.6 (proper reference frame) will be useful throughout the applications of gravitation theory (Chapters 18–40).

Freely falling particles (geodesics) define and probe the structure of spacetime. This spacetime is curved. Gravitation is a manifestation of its curvature. So far, so good, in terms of Newton's theory of gravity as translated into geometric language by Cartan. What is absolutely unacceptable, however, is the further consequence of the Cartan-Newton viewpoint (Chapter 12): stratification of spacetime into slidable slices, with no meaning for the spacetime separation between an event in one slice and an event in another.

Constraints imposed on spacetime by special relativity

Of all the foundations of physics, none is more firmly established than special relativity; and of all the lessons of special relativity none stand out with greater force than these. (1) Spacetime, far from being stratified, is homogeneous and isotropic throughout any region small enough ("local region") that gravitational tide-producing effects ("spacetime curvatures") are negligible. (2) No local experiment whatsoever can distinguish one local inertial frame from another. (3) The speed of light is the same in every local inertial frame. (4) It is not possible to give frame-independent meaning to the separation in time ("no Newtonian stratification"). (5) Between every event and every nearby event there exists a frame-independent, coordinate-independent spacetime interval ("Riemannian geometry"). (6) Spacetime is always and everywhere locally Lorentz in character ("local Lorentz character of this Riemannian geometry").

Metric: the instrument which imposes the constraints

What mathematics gives all these physical properties? A metric; a metric that is locally Lorentz (§§13.2 and 13.6). All else then follows. In particular, the metric destroys the stratified structure of Newtonian spacetime, as well as its gravitational potential and universal time coordinate. But not destroyed are the deepest features of Newtonian gravity: (1) the equivalence principle (as embodied in geodesic description of free-fall motion, §§13.3 and 13.4); and (2) spacetime curvature (as measured by tidal effects, §13.5).

The skyscraper of vectors, forms, tensors (Chapter 9), geodesics, parallel transport, covariant derivative (Chapter 10), and curvature (Chapter 11) has rested on crumbling foundations—Newtonian physics and a geodesic law based on Newtonian physics. But with metric now on the scene, the whole skyscraper can be transferred to new foundations without a crack appearing. Only one change is necessary: the geodesic law must be selected in a new, relativistic way; a way based on metric (§§13.3 and 13.4). Resting on metric foundations, spacetime curvature acquires additional and stronger properties (the skyscraper is redecorated and extended), which are studied in §13.5 and in Chapters 14 and 15, and which lead almost inexorably to Einstein's field equation.

§13.2. METRIC

A spacetime metric; a curved spacetime metric; a locally Lorentz, curved spacetime metric. This is the foundation of spacetime geometry in the real, physical world. Therefore take a moment to recall what "metric" is in three contrasting languages.

Metric described in three languages

In the language of elementary geometry, "metric" is a table giving the interval between every event and every other event (Box 13.1 and Figure 13.1). In the language of coordinates, "metric" is a set of ten functions of position, $g_{\mu\nu}(x^\alpha)$, such that the expression

$$\Delta s^2 = -\Delta\tau^2 = g_{\mu\nu}(x^\alpha)\,\Delta x^\mu\,\Delta x^\nu \tag{13.1}$$

gives the interval between any event x^α and any nearby event $x^\alpha + \Delta x^\alpha$. In the language of abstract differential geometry, metric is a bilinear machine, $\boldsymbol{g} \equiv (\ldots\cdot\ldots)$, to produce a number ["scalar product $\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) \equiv (\boldsymbol{u}\cdot\boldsymbol{v})$"] out of two tangent vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$.

The link between the abstract, machine viewpoint and the concrete coordinate viewpoint is readily exhibited. Let the tangent vector

$$\boldsymbol{\xi} \equiv \Delta x^\alpha \boldsymbol{e}_\alpha = \Delta x^\alpha(\partial/\partial x^\alpha)$$

represent the displacement between two neighboring events. The abstract viewpoint gives

$$\Delta s^2 \equiv \boldsymbol{\xi}\cdot\boldsymbol{\xi} \equiv \boldsymbol{g}(\Delta x^\mu \boldsymbol{e}_\mu, \Delta x^\nu \boldsymbol{e}_\nu) = \Delta x^\mu\,\Delta x^\nu \boldsymbol{g}(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu)$$

(continued on page 310)

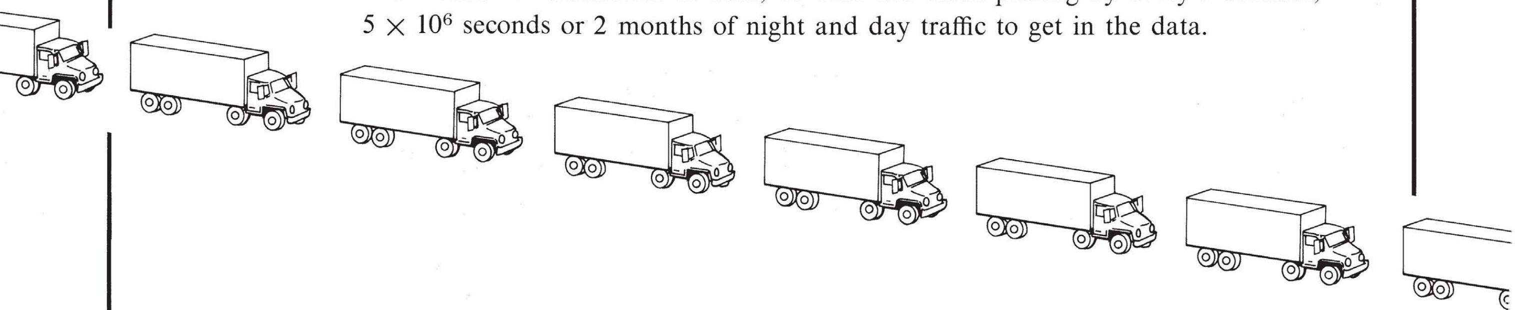
Box 13.1 METRIC DISTILLED FROM DISTANCES

Raw Data on Distances

Let the shape of the earth be described as in Figure 13.1, by giving distances between some of the principal identifiable points: buoys, ships, icebergs, lighthouses, peaks, and flags: points to a total of $n = 2 \times 10^7$. The total number of distances to be given is $n(n-1)/2 = 2 \times 10^{14}$. With 200 distances per page of printout, this means

First point	Second point	Distance (Nautical miles)	First point	Seco…
9,316,434	14,117,103	1410.316	9,316,434	
9,316,434	14,117,104	1812.717	9,316,434	
9,316,434	14,117,105	1629.291	9.3…	

10^{12} pages weighing 6 g each, or 6×10^6 metric tons of data. With 6 tons per truck this means 10^6 truckloads of data; or with one truck passing by every 5 seconds, 5×10^6 seconds or 2 months of night and day traffic to get in the data.

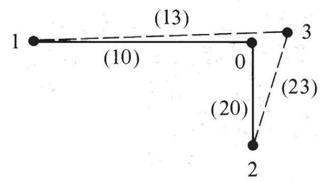


First Distillation: Distances to Nearby Points Only

Get distances between faraway points by adding distances covered on the elementary short legs of the trip. Boil down the table of distances to give only the distance between each point and the hundred nearest points. Now have $100\,n = 2 \times 10^9$ distances, or $2 \times 10^9/200 = 10^7$ pages of data, or 60 tons of records, or 10 truckloads.

Second Distillation: Distances Between Nearby Points in Terms of Metric

Idealize the surface of the earth as smooth. Then in any sufficiently limited region the geometry is Euclidean. This circumstance has a happy consequence. It is enough to know a few distances between the nearby points to be able to determine all the



distances between the nearby points. Locate point 2 so that (102) is a right triangle; thus, $(12)^2 = (10)^2 + (20)^2$. Consider a point 3 close to 0. Define

$$x(3) = (13) - (10)$$

and

$$y(3) = (23) - (20).$$

Then the distance (03) does not have to be supplied independently; it can be calculated from the formula*

$$(03)^2 = [x(3)]^2 + [y(3)]^2.$$

Similarly for a point 4 and its distance (04) from the local origin 0. Similarly for the distance (mn) between any two points m and n that are close to 0:

$$(mn)^2 = [x(m) - x(n)]^2 + [y(m) - y(n)]^2.$$

Thus it is only needful to have the distance $(1m)$ (from point 1) and $(2m)$ (from point 2) for each point m close to 0 ($m = 3, 4, \ldots, N + 2$) to be able to work out

---

*If the distance (03) is given arbitrarily, the resulting four-vertex figure will burst out of the plane. Regarded as a tetrahedron in a three-dimensional Euclidean space, it has a volume given by the formula of Niccolo Fontana Tartaglia (1500–1557), generalized today (Blumenthal 1953) to

$$\begin{pmatrix} \text{volume of} \\ n\text{-dimensional} \\ \text{simplex} \\ \text{spanned by} \\ (n+1) \text{ points} \end{pmatrix} = \left(\frac{(-1)^{n+1}}{2^n}\right)^{1/2} \frac{1}{n!} \begin{vmatrix} 0 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 0 & (01)^2 & (02)^2 & \ldots & (0n)^2 \\ 1 & (10)^2 & 0 & (12)^2 & \ldots & (1n)^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & (n0)^2 & (n1)^2 & (n2)^2 & \ldots & 0 \end{vmatrix}^{1/2},$$

which reduces for three points to the standard textbook formula of Hero of Alexandria (A.D. 62 to A.D. 150).

$$\text{area} = \{s[s - (01)][s - (02)][s - (12)]\}^{1/2},$$
$$2s = (01) + (02) + (12),$$

for the area of a triangle. Conversely, if the four points are to remain in two-dimensional Euclidean space, the tetrahedron must collapse to zero volume. This requirement supplies one condition on the one distance (03). It simplifies the discussion of this condition to take (03) small and (102) to be a right triangle, as above. However, the general principle is independent of such approximations, and follows directly from the extended Hero-Tartaglia formula. It is enough in a locally Euclidean or Lorentz space of n dimensions to have laid down $(n + 1)$ fiducial points $0, 1, 2, \ldots, n$, and to know the distance of every other point $j, k, \ldots$ from these fiducial points, in order to be able to calculate the distance of these points $j, k, \ldots$ from one another ("distances between nearby points in terms of coordinates"; metric as distillation of distance data).

Box 13.1 (*continued*)

its distance from every point n close to 0. The prescription to determine the $N(N-1)/2$ distances between these N nearby points can be reexpressed to advantage in these words: (1) each point has two coordinates, x and y; and (2) the distance is given in terms of these coordinates by the standard Euclidean metric; thus

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2.$$

Having gone this far on the basis of "distance geometry" (for more on which, see Robb 1914 and 1936), one can generalize from a small region (Euclidean) to a large region (not Euclidean). Introduce any arbitrary smooth pair of everywhere-independent curvilinear coordinates x^k, and express distance, not only in the immediate neighborhood of the point 0, but also in the immediate neighborhood of every point of the surface (except places where one has to go to another coordinate patch; at least two patches needed for 2-sphere) in terms of the formula

$$ds^2 = g_{jk}\, dx^j\, dx^k.$$

Thus out of the table of distances between nearby points one has distilled now five numbers per point (two coordinates, x^1, x^2, and three metric coefficients, g_{11}, $g_{12} = g_{21}$, and g_{22}), down by a factor of $100/5 = 20$ from what one had before (now 3 tons of data, or half a truckload).

Third Distillation: Metric Coefficients Expressed as Analytical Functions of Coordinates

Instead of giving the three metric coefficients at each of the 2×10^7 points of the surface, give them as functions of the two coordinates x^1, x^2, in terms of a power series or an expansion in spherical harmonics or otherwise with some modest number, say 100, of adjustable coefficients. Then the information about the geometry itself (as distinct from the coordinates of the 2×10^7 points located on that geometry) is caught up in these three hundred coefficients, a single page of printout. Goodbye to any truck! In brief, metric provides a shorthand way of giving the distance between every point and every other point—but its role, its justification and its meaning lies in these distances and only in these many distances.

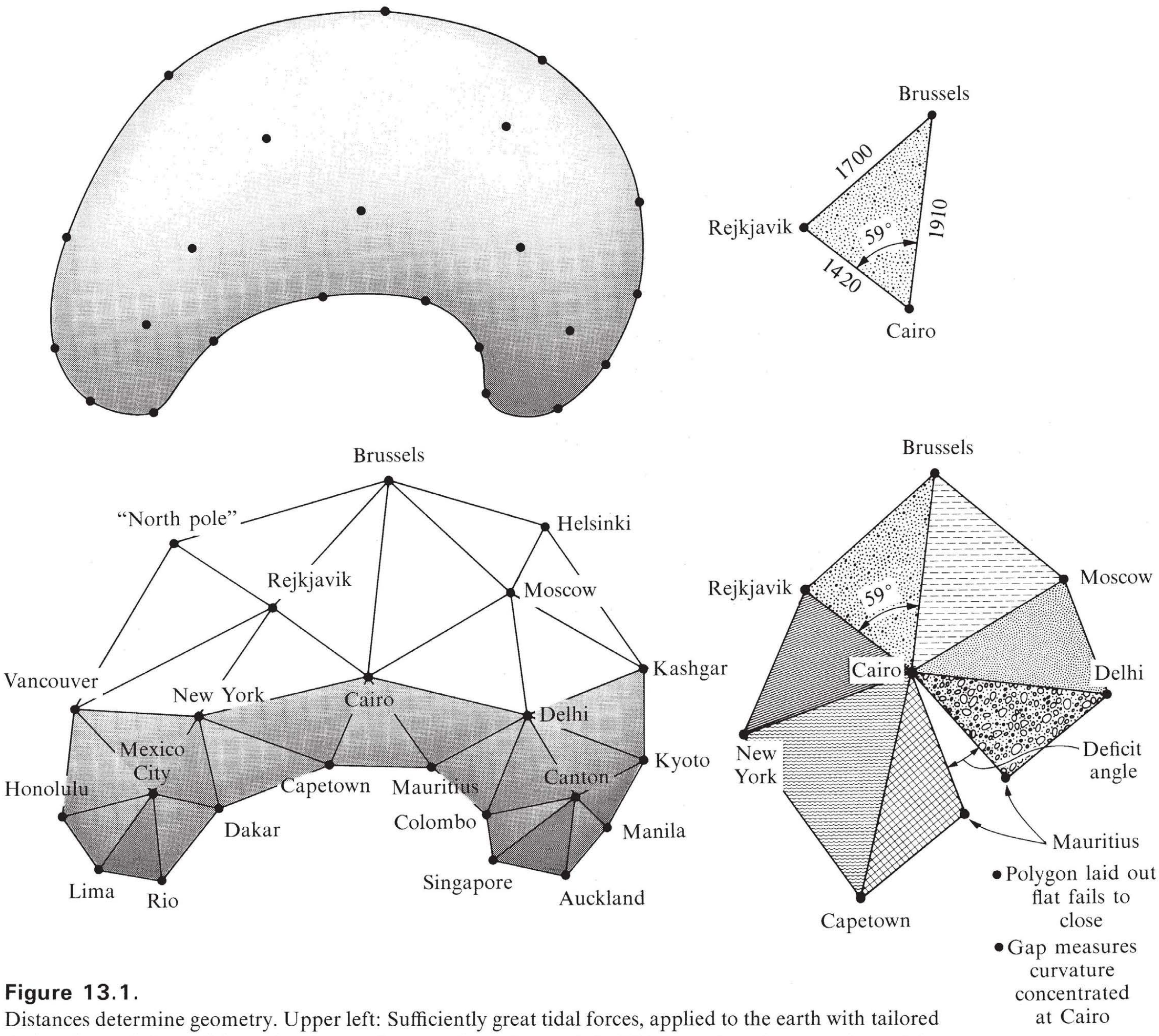


Figure 13.1.
Distances determine geometry. Upper left: Sufficiently great tidal forces, applied to the earth with tailored timing, have deformed it to the shape of a tear drop. Lower left: This tear drop is approximated by a polyhedron built out of triangles ("skeleton geometry"). The approximation can be made arbitrarily good by making the number of triangles sufficiently great and the size of each sufficiently small. Upper right: The geometry in each triangle is Euclidean: giving the three edge lengths fixes all the features of the figure, including the indicated angle. Lower right: The triangles that belong to a given vertex, laid out on a flat surface, fail to meet. The deficit angle measures the amount of curvature concentrated at that vertex on the tear-drop earth. The sum of these deficit angles for all vertices of the tear drop equals 4π. This "Gauss-Bonnet theorem" is valid for any figure with the topology of the 2-sphere; for the simplest figure of all, a tetrahedron, four vertices with a deficit angle at each of 180° are needed—3 triangles $\times$ 60° per triangle available = 180° deficit. In brief, the shape of the tear drop, in the given skeleton-geometry approximation, is determined by its 50 visible edge lengths plus, say, 32 more edge lengths hidden behind the figure, or a total of 82 edge lengths, and by nothing more ("distances determine geometry"). "Metric" tells the distance between every point and every nearby point. If volcanic action raises Rejkjavik, the distances between that Icelandic capital and nearby points increase accordingly; distances again reveal shape. Conversely, that there is not a great bump on the earth in the vicinity of Iceland, and that the earth does not now have a tear-drop shape, can be unambiguously established by analyzing the pattern of distances from point to point in a sufficiently well-distributed network of points, with no call for any observations other than measurements of distance.

for the interval between those events; comparison with the coordinate viewpoint [equation (13.1)] reveals

Covariant components of metric

$$g_{\mu\nu} = \boldsymbol{g}(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu) = \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu \tag{13.2}$$

(standard equation for calculating components of a tensor).

"Line element" compared with "metric as bilinear machine"

Just as modern differential geometry replaces the old style "differential" df by the "differential form" $\boldsymbol{d}f$ (Box 2.3, page 63), so it also replaces the old-style "line element"

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = (\text{“interval between } x^\alpha \text{ and } x^\alpha + dx^\alpha\text{”}) \tag{13.3}$$

by the bilinear machine ("metric tensor")

$$\boldsymbol{g} \equiv \boldsymbol{ds}^2 \equiv g_{\mu\nu}\, \boldsymbol{d}x^\mu \otimes \boldsymbol{d}x^\nu. \tag{13.4}$$

The output $\boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{\xi})$ of this machine, for given displacement-vector input, is identical to the old-style interval. Hence, $\boldsymbol{ds}^2 = g_{\mu\nu}\, \boldsymbol{d}x^\mu \otimes \boldsymbol{d}x^\nu$ represents the interval of an unspecified displacement; and the act of inserting $\boldsymbol{\xi}$ into the slots of $\boldsymbol{ds}^2$ is the act of making explicit the interval $\boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) = g_{\mu\nu}\, \Delta x^\mu\, \Delta x^\nu$ of an explicit displacement.

Metric produces a correspondence between 1-forms and tangent vectors

In curved spacetime with metric, just as in flat spacetime with metric (§2.5), a particular 1-form $\tilde{\boldsymbol{u}}$ corresponds to any given tangent vector $\boldsymbol{u}$:

$$\tilde{\boldsymbol{u}} \text{ is defined by “} \langle \tilde{\boldsymbol{u}}, \boldsymbol{v} \rangle \equiv \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) \text{ for all } \boldsymbol{v}\text{”} \tag{13.5}$$

("representation of the same physical quantity in the two alternative versions of vector and 1-form"; "corresponding representations" as $\binom{1}{0}$-tensor and as $\binom{0}{1}$-tensor). Example: the 1-form $\tilde{\boldsymbol{u}}$ corresponding to a basis vector $\boldsymbol{u} = \boldsymbol{e}_\alpha$ has components

$$u_\beta \underset{\uparrow}{=} \langle \tilde{\boldsymbol{u}}, \boldsymbol{e}_\beta \rangle \underset{\uparrow}{\equiv} \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{e}_\beta) \underset{\uparrow}{=} \boldsymbol{g}(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) \underset{\uparrow}{=} g_{\alpha\beta};$$

[standard way to compute u_β] [definition (13.5)] [by $\boldsymbol{u} = \boldsymbol{e}_\alpha$] [equation (13.2)]

thus

$$g_{\alpha\beta}\boldsymbol{\omega}^\beta \text{ is the 1-form } \tilde{\boldsymbol{e}}_\alpha \text{ corresponding to } \boldsymbol{e}_\alpha. \tag{13.6}$$

Also as in flat spacetime (§3.2), a tensor can accept either a vector or a 1-form into any given slot

$$\boldsymbol{S}(\tilde{\boldsymbol{u}}, \boldsymbol{\sigma}, \boldsymbol{v}) \equiv \boldsymbol{S}(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{v}). \tag{13.7}$$

Lowering indices

Equivalently, in component language, the indices of a tensor can be lowered with the covariant components of the metric

$$S_\alpha{}^\beta{}_\gamma \underset{\uparrow}{=} \boldsymbol{S}(\boldsymbol{e}_\alpha, \boldsymbol{\omega}^\beta, \boldsymbol{e}_\gamma) = \boldsymbol{S}(\tilde{\boldsymbol{e}}_\alpha, \boldsymbol{\omega}^\beta, \boldsymbol{e}_\gamma) \underset{\uparrow}{=} \boldsymbol{S}(g_{\alpha\mu}\boldsymbol{\omega}^\mu, \boldsymbol{\omega}^\beta, \boldsymbol{e}_\gamma) = g_{\alpha\mu} S^{\mu\beta}{}_\gamma. \tag{13.8}$$

[definition of $S_\alpha{}^\beta{}_\gamma$] [by equation (13.6)]

The basis vectors $\{\boldsymbol{e}_\alpha\}$ can be chosen arbitrarily at each event. Therefore the corresponding components $g_{\alpha\beta}$ of the metric are quite arbitrary (though symmetric: $g_{\alpha\beta} = g_{\beta\alpha}$). But the mixed components $g^\alpha{}_\beta$ are not arbitrary. In particular, equations (13.5) and (13.7) imply

$$\boldsymbol{g}(\tilde{\boldsymbol{u}}, \boldsymbol{v}) \equiv \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) \equiv \langle \tilde{\boldsymbol{u}}, \boldsymbol{v} \rangle. \tag{13.9}$$

Mixed and contravariant components of metric

Therefore one concludes that the metric tensor in mixed representation is identical with the unit matrix:

$$g^\alpha{}_\beta \equiv \boldsymbol{g}(\boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta) \equiv \langle \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\beta \rangle = \delta^\alpha{}_\beta. \tag{13.10}$$

This feature of the metric in turn fixes the contravariant components of the metric:

$$g^{\alpha\mu} g_{\mu\beta} \underset{\uparrow}{=} g^\alpha{}_\beta = \delta^\alpha{}_\beta; \tag{13.11}$$

[“lowering an index” of $g^{\alpha\mu}$]

i.e.,

$$\|g^{\alpha\beta}\| \text{ is the matrix inverse of } \|g_{\alpha\beta}\|. \tag{13.12}$$

Raising indices

This reciprocity enables one to undo the lowering of tensor indices (i.e., raise indices) with $g^{\alpha\beta}$:

$$S^{\mu\beta}{}_\gamma = \delta^\mu{}_\alpha S^{\alpha\beta}{}_\gamma = g^{\mu\nu} g_{\nu\alpha} S^{\alpha\beta}{}_\gamma = g^{\mu\nu} S_\nu{}^\beta{}_\gamma. \tag{13.13}$$

The last two paragraphs may be summarized in brief:

(1) $g^\alpha{}_\beta = \delta^\alpha{}_\beta$;
(2) $\|g^{\alpha\beta}\| = \|g_{\alpha\beta}\|^{-1}$;
(3) tensor indices are lowered with $g_{\alpha\beta}$;
(4) tensor indices are raised with $g^{\alpha\beta}$.

Metric must be locally Lorentz

In this formalism of metric and index shuffling, a big question demands attention: how can one tell whether the metric is locally Lorentz rather than locally Euclidean or locally something else? Of course, one criterion (necessary; not sufficient!) is dimensionality—a locally Lorentz spacetime must have four dimensions. (Recall the method of §1.2 to determine dimensionality.) Confine attention, then, to four-dimensional manifolds. What else must one demand? One must demand that at every event $\mathscr{P}$ there exist an orthonormal frame (orthonormal set of basis vectors $\{\boldsymbol{e}_{\hat\alpha}\}$) in which the components of the metric have their flat-spacetime form

$$g_{\hat\alpha\hat\beta} \equiv \boldsymbol{e}_{\hat\alpha} \cdot \boldsymbol{e}_{\hat\beta} = \eta_{\alpha\beta} \equiv \text{diagonal } (-1, 1, 1, 1). \tag{13.14}$$

To test for this is straightforward (exercise 13.1). (1) Search for a timelike vector $\boldsymbol{u}$ ($\boldsymbol{u} \cdot \boldsymbol{u} < 0$). If none exist, spacetime is not locally Lorentz. If one is found, then (2) examine all non-zero vectors $\boldsymbol{v}$ perpendicular to $\boldsymbol{u}$. If they are all spacelike ($\boldsymbol{v} \cdot \boldsymbol{v} > 0$), then spacetime is locally Lorentz. Otherwise it is not.

EXERCISES

Exercise 13.1. TEST WHETHER SPACETIME IS LOCAL LORENTZ

Prove that the above two-step procedure for testing whether spacetime is locally Lorentz is valid: i.e., prove that if the procedure says "yes," then there exists an orthonormal basis with $g_{\hat{\alpha}\hat{\beta}} = \eta_{\alpha\beta}$ at the event in question; if it says "no," then no such basis exists.

Exercise 13.2. PRACTICE WITH METRIC

A four-dimensional manifold with coordinates v, r, θ, ϕ has line element (old-style notation)

$$ds^2 = -(1 - 2M/r)\, dv^2 + 2\, dv\, dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

corresponding to metric (new-style notation)

$$\boldsymbol{ds^2} = -(1 - 2M/r)\, \boldsymbol{dv} \otimes \boldsymbol{dv} + \boldsymbol{dv} \otimes \boldsymbol{dr} + \boldsymbol{dr} \otimes \boldsymbol{dv} + r^2(\boldsymbol{d\theta} \otimes \boldsymbol{d\theta} + \sin^2\theta\, \boldsymbol{d\phi} \otimes \boldsymbol{d\phi}),$$

where M is a constant.

(a) Find the "*covariant*" components $g_{\alpha\beta}$ and "*contravariant*" components $g^{\alpha\beta}$ of the metric in this coordinate system. [Answer: $g_{vv} = -(1 - 2M/r)$, $g_{vr} = g_{rv} = 1$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2 \sin^2\theta$; all other $g_{\alpha\beta}$ vanish; $g^{vr} = g^{rv} = 1$, $g^{rr} = (1 - 2M/r)$, $g^{\theta\theta} = r^{-2}$, $g^{\phi\phi} = r^{-2}\sin^{-2}\theta$, all other $g^{\alpha\beta}$ vanish.]

(b) Define a scalar field t by

$$t \equiv v - r - 2M \ln[(r/2M) - 1].$$

What are the covariant and contravariant components (u_α and u^α) of the 1-form $\tilde{\boldsymbol{u}} \equiv \boldsymbol{d}t$? What is the squared length $\boldsymbol{u}^2 \equiv \boldsymbol{u} \cdot \boldsymbol{u}$, of the corresponding vector? Show that $\boldsymbol{u}$ is timelike in the region $r > 2M$. [Answer: $u_v = 1$, $u_r = -1/(1 - 2M/r)$, $u_\theta = u_\phi = 0$; $u^v = -1/(1 - 2M/r)$, $u^r = 0$, $u^\theta = u^\phi = 0$; $\boldsymbol{u}^2 = -1/(1 - 2M/r)$.]

(c) Find the most general non-zero vector $\boldsymbol{w}$ orthogonal to $\boldsymbol{u}$ in the region $r > 2M$, and show that it is spacelike. Thereby conclude that spacetime is locally Lorentz in the region $r > 2M$. [Answer: Since $\boldsymbol{w} \cdot \boldsymbol{u} = w_\alpha u^\alpha = -w_v/(1 - 2M/r)$, w_v must vanish, but w_r, w_θ, w_ϕ are arbitrary, and $\boldsymbol{w}^2 = (1 - 2M/r) {w_r}^2 + r^{-2} {w_\theta}^2 + r^{-2} \sin^{-2}\theta {w_\phi}^2 > 0$.]

(d) Let t, r, θ, ϕ be new coordinates for spacetime. Find the line element in this coordinate system. [Answer: This is the "Schwarzschild" line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2.]$$

(e) Find an orthonormal basis, for which $g_{\hat{\alpha}\hat{\beta}} = \eta_{\alpha\beta}$ in the region $r > 2M$. [Answer: $\boldsymbol{e}_{\hat{0}} \equiv (1 - 2M/r)^{-1/2}\, \partial/\partial t$, $\boldsymbol{e}_{\hat{r}} \equiv (1 - 2M/r)^{1/2}\, \partial/\partial r$, $\boldsymbol{e}_{\hat{\theta}} \equiv r^{-1}\, \partial/\partial\theta$, $\boldsymbol{e}_{\hat{\phi}} = (r \sin\theta)^{-1}\, \partial/\partial\phi$.]

§13.3. CONCORD BETWEEN GEODESICS OF CURVED SPACETIME GEOMETRY AND STRAIGHT LINES OF LOCAL LORENTZ GEOMETRY

More could be said about the mathematical machinery and physical implications of "metric," but an issue of greater urgency presses for attention. What has metric (or spacetime interval) to do with geodesic (or world line of test particle)? Answer:

Two mathematical objects ("straight line in a local Lorentz frame" and "geodesic of the over-all global curved spacetime geometry") equal to the same physical object ("world line of test particle") must be equal to each other ("condition of consistency"). As a first method to spell out this consistency requirement, examine the two mathematical representations of the world line of a test particle in the neighborhood of a given event $\mathscr{P}_0$. The local-Lorentz representation says:

Local-Lorentz description of straight lines

"Pick a local Lorentz frame at $\mathscr{P}_0$. [As spelled out in exercise 13.3, such a local Lorentz frame is the closest thing there is to a global Lorentz frame at $\mathscr{P}_0$; i.e., it is a coordinate system in which

$$g_{\alpha\beta}(\mathscr{P}_0) = \eta_{\alpha\beta} \text{ (flat-spacetime metric)}, \tag{13.15a}$$

$$g_{\alpha\beta,\gamma}(\mathscr{P}_0) = 0, \tag{13.15b}$$

$$g_{\alpha\beta,\gamma\delta}(\mathscr{P}_0) \neq 0 \text{ except in special cases, such as flat space.]} \tag{13.15c}$$

The world line in that frame has zero acceleration,

$$d^2x^\alpha/d\tau^2 = 0 \text{ at } \mathscr{P}_0 \text{ ("straight-line equation")}, \tag{13.16}$$

where τ is proper time as measured by the particle's clock."

The geodesic representation says

Geodesic description of straight lines

"In the local Lorentz frame, as in any coordinate frame, the world line satisfies the geodesic equation

$$d^2x^\alpha/d\tau^2 + \Gamma^\alpha{}_{\beta\gamma}(dx^\beta/d\tau)(dx^\gamma/d\tau) = 0 \tag{13.17}$$

(τ is an affine parameter because it is time as measured by the test particle's clock)."

Condition of consistency: $\Gamma^\alpha{}_{\beta\gamma} = 0$ in local Lorentz frame

Consistency of the two representations for any and every choice of test particle (any and every choice of $dx^\alpha/d\tau$ at $\mathscr{P}_0$) demands

$$\Gamma^\alpha{}_{\beta\gamma}(\mathscr{P}_0) = 0 \text{ in any local Lorentz frame [coordinate system satisfying equations (13.15) at } \mathscr{P}_0]; \tag{13.18}$$

i.e., it demands that *every local Lorentz frame is a local inertial frame.* (On local inertial frames see §11.6.) In such a frame, all local effects of "gravitation" disappear. That is the physical shorthand for (13.18).

One does not have to speak in the language of a specific coordinate system when one demands identity between the geodesic (derived from the $\Gamma^\alpha{}_{\beta\gamma}$) and the straight line of the local Lorentz geometry ($g_{\mu\nu}$). The local Lorentz specialization of coordinates may be the most immediate way to see the physics ("no local effects of gravitation"), but it is not the right way to formulate the basic mathematical requirement in its full generality and power. The right way is to demand

Consistency reformulated: $\boldsymbol{\nabla g} = 0$.

$$\boldsymbol{\nabla g} = 0 \text{ ("compatibility of } \boldsymbol{g} \text{ and } \boldsymbol{\nabla}\text{")}. \tag{13.19}$$

Stated in the language of an arbitrary coordinate system, this requirement reads

$$g_{\alpha\beta;\gamma} \equiv \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} - \Gamma^\mu{}_{\alpha\gamma} g_{\mu\beta} - \Gamma^\mu{}_{\beta\gamma} g_{\alpha\mu} = 0. \tag{13.19$'$}$$

That this covariant requirement is fulfilled in every coordinate system follows from its validity in one coordinate system: a local Lorentz frame. (The first term in this equation, and the last two terms, are separately required to vanish in the local Lorentz frame at point $\mathscr{P}_0$—and required to vanish by the *physics*.) From $\boldsymbol{\nabla} \boldsymbol{g} = 0$, one can derive both the abstract chain rule

$$\boldsymbol{\nabla}_{\boldsymbol{u}} (\boldsymbol{v} \cdot \boldsymbol{w}) = (\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{v}) \cdot \boldsymbol{w} + \boldsymbol{v} \cdot (\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{w}) \tag{13.20}$$

$\Gamma^\alpha{}_{\beta\gamma}$ expressed in terms of metric

(Exercise 13.4) and the following equations for the connection coefficients in *any* frame in terms of (1) the metric coefficients, $g_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta$, and (2) the covariant commutation coefficients

$$c_{\alpha\beta\gamma} \equiv c_{\alpha\beta}{}^\mu g_{\mu\gamma} \equiv \langle \boldsymbol{\omega}^\mu, [\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] \rangle g_{\mu\gamma} \tag{13.21}$$

of that frame:

$$\Gamma^\alpha{}_{\beta\gamma} = g^{\alpha\mu} \Gamma_{\mu\beta\gamma} \qquad (\text{definition of } \Gamma_{\mu\beta\gamma}), \tag{13.22}$$

$$\begin{aligned} \Gamma_{\mu\beta\gamma} &= \frac{1}{2}(g_{\mu\beta,\gamma} + c_{\mu\beta\gamma} + g_{\mu\gamma,\beta} + c_{\mu\gamma\beta} - g_{\beta\gamma,\mu} - c_{\beta\gamma\mu}) \\ &= \frac{1}{2}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) \text{ in any coordinate frame.} \end{aligned} \tag{13.23}$$

(See Exercise 13.4).

Equations (13.23) are the connection coefficients required to make the geodesics of curved spacetime coincide with the straight lines of the local Lorentz geometry. And they are fixed uniquely; no other choice of connection coefficients will do the job!

Summary: in curved spacetime with a local Lorentz metric, the following seemingly different statements are actually equivalent: (1) the geodesics of curved spacetime coincide with the straight lines of the local Lorentz geometry; (2) every local Lorentz frame [coordinates with $g_{\alpha\beta}(\mathscr{P}_0) = \eta_{\alpha\beta}$, $g_{\alpha\beta,\gamma}(\mathscr{P}_0) = 0$] is a local inertial frame [$\Gamma^\alpha{}_{\beta\gamma}(\mathscr{P}_0) = 0$]; (3) the metric and covariant derivative satisfy the compatibility condition $\boldsymbol{\nabla} \boldsymbol{g} = 0$; (4) the covariant derivative obeys the chain rule (13.20); (5) the connection coefficients are determined by the metric in the manner of equations (13.23). A sixth equivalent statement, derived in the next section, says (6) the geodesics of curved spacetime coincide with world lines of extremal proper time.

EXERCISES

Exercise 13.3. MATHEMATICAL REPRESENTATION OF LOCAL LORENTZ FRAME

By definition, a local Lorentz frame at a given event $\mathscr{P}_0$ is the closest thing there to a global Lorentz frame. Thus, it should be a coordinate system with $g_{\mu\nu}(\mathscr{P}_0) = \eta_{\mu\nu}$, and with as many derivatives of $g_{\mu\nu}$ as possible vanishing at $\mathscr{P}_0$. Prove that there exist coordinates in which $g_{\mu\nu}(\mathscr{P}_0) = \eta_{\mu\nu}$ and $g_{\mu\nu,\rho}(\mathscr{P}_0) = 0$, but that $g_{\mu\nu,\rho\sigma}(\mathscr{P}_0)$ cannot vanish in general. Hence, such coordinates are the mathematical representation of a local Lorentz frame. [Hint: Let $\{x^{\alpha'}(\mathscr{P})\}$ be an arbitrary but specific coordinate system, and $\{x^\mu(\mathscr{P})\}$ be a local Lorentz frame, both with origins at $\mathscr{P}_0$. Expand the coordinate transformation between the two in powers of x^μ

$$x^{\alpha'} = M^\alpha{}_\mu x^\mu + \frac{1}{2} N^\alpha{}_{\mu\nu} x^\mu x^\nu + \frac{1}{6} P^\alpha{}_{\mu\nu\rho} x^\mu x^\nu x^\rho + \ldots;$$

and use the transformation matrix $L^{\alpha'}{}_\mu \equiv \partial x^{\alpha'}/\partial x^\mu$ to get $g_{\mu\nu}(\mathscr{P}_0)$, $g_{\mu\nu,\rho}(\mathscr{P}_0)$, and $g_{\mu\nu,\rho\sigma}(\mathscr{P}_0)$ in terms of $g_{\alpha'\beta'}$ and its derivatives and the constants $M^\alpha{}_\mu$, $N^\alpha{}_{\mu\nu}$, $P^\alpha{}_{\mu\nu\rho}$. Show that whatever $g_{\alpha'\beta'}$ may be (so long as it is nonsingular, so $g^{\alpha'\beta'}$ exists!), one can choose the 16 constants $M^\alpha{}_\mu$ to make $g_{\mu\nu} = \eta_{\mu\nu}$ (ten conditions); one can choose the $4 \times 10 = 40$ constants $N^\gamma{}_{\mu\nu}$ to make the $10 \times 4 = 40$ $g_{\mu\nu,\rho}(\mathscr{P}_0)$ vanish; but one cannot in general choose the $4 \times 20 = 80$ $P^\alpha{}_{\mu\nu\rho}$ to make the $10 \times 10 = 100$ $g_{\mu\nu,\rho\sigma}$ vanish.]

Exercise 13.4. CONSEQUENCES OF COMPATIBILITY BETWEEN $\boldsymbol{g}$ AND $\boldsymbol{\nabla}$

(a) From the condition of compatibility $\boldsymbol{\nabla g} = 0$, derive the chain rule (13.20).

(b) From the condition of compatibility $\boldsymbol{\nabla g} = 0$ and definitions (13.21) and (13.22), derive equation (13.23) for the connection coefficients. [Answer: See exercise 8.15, p. 216.]

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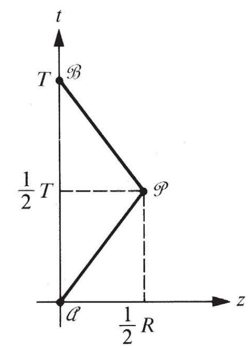
§13.4. GEODESICS AS WORLD LINES OF EXTREMAL PROPER TIME



In a local Lorentz frame, it is easy to distinguish a world line that is straight from one that is not. Position the Lorentz frame and so orient it that the starting point of the world line, $\mathscr{A}$, lies at the origin and the end point, $\mathscr{B}$, lies at $x = 0$, $y = 0$, $z = 0$, $t = T$. As an example of a nonstraight world line, consider passage at uniform velocity from $\mathscr{A}$ to point $\mathscr{P}$ with coordinates $(\frac{1}{2}T; 0, 0, \frac{1}{2}R)$ and from there again with uniform velocity to point $\mathscr{B}$. The lapse of proper time from start to finish ("length of world line") is

$$\tau = (T^2 - R^2)^{1/2}.$$

Thus the lapse of proper time is diminished from its straight-line value, and diminished moreover for any choice of R whatsoever, except for the zero or straight-line value $R = 0$. As for this simple nonstraight curve, so also for any other nonstraight curve: the lapse of proper time between $\mathscr{A}$ and $\mathscr{B}$ is less than the straight-line lapse (Exercise 6.3). Thus, in flat spacetime, extremal length of world line is an indicator of straightness.

In flat spacetime, straight lines have extremal length

Any local region of the curved spacetime of the real, physical world is Lorentz in character. In this local Lorentz geometry, it is easy to set up Lorentz coordinates and carry out the extremal-length analysis just sketched to distinguish between a straight line and a nonstraight line:

Extremal length in curved spacetime

$$\tau = \int_{\mathscr{A}}^{\mathscr{B}} d\tau = \int_{\mathscr{A}}^{\mathscr{B}} (-\eta_{\mu\nu}\, dx^\mu\, dx^\nu)^{1/2}$$

$$= \begin{pmatrix} \text{a maximum for straight line} \\ \text{as compared to any variant of} \\ \text{the straight line} \end{pmatrix}. \tag{13.24}$$

2

Such a test for straightness can be carried out separately in each local Lorentz region along the world line, or, with greater efficiency, it can be carried out over many local Lorentz regions simultaneously, i.e., over a region with endpoints $\mathscr{A}$ and $\mathscr{B}$ so widely separated that no single Lorentz frame can possibly contain them both. To carry out the analysis, one must abandon local Lorentz coordinates. Therefore introduce a general curvilinear coordinate system and find

$$\tau = \int_{\mathscr{A}}^{\mathscr{B}} d\tau = \int_{\mathscr{A}}^{\mathscr{B}} (-g_{\mu\nu}\, dx^{\mu}\, dx^{\nu})^{1/2}$$
$$= \begin{pmatrix} \text{an extremum for timelike world line that} \\ \text{is straight in each local Lorentz frame} \\ \text{along its path, as compared to any "nearby"} \\ \text{variant of this world line} \end{pmatrix}. \tag{13.25}$$

In the real world, the path of extremal τ, being straight in every local Lorentz frame, must be a geodesic of spacetime.

Notice that the word "maximum" in equation (13.24) has been replaced by "extremum" in the statement (13.25). When $\mathscr{A}$ and $\mathscr{B}$ are widely separated, they may be connected by several different geodesics with differing lapses of proper time (Figure 13.2). Each timelike geodesic extremizes τ with respect to nearby deformations of itself, but the extremum need not be a maximum. When several distinct geodesics connect two events, the typical one is not a local maximum ("mountain peak") but a saddle point ("mountain pass") in such a diagram as Figure 13.2 or 13.3.

Concord between locally straight lines (lines of extremal τ) and geodesics of curved spacetime demands that timelike geodesics have extremal proper length. If so, then any curve $x^{\mu}(\lambda)$ between $\mathscr{A}$ (where $\lambda = 0$) and $\mathscr{B}$ (where $\lambda = 1$) that extremizes τ should satisfy the geodesic equation. To test for an extremal by comparing times, pick a curve suspected to be a geodesic, and deform it slightly but arbitrarily:

Proof that curves of extremal length are geodesics

$$\begin{aligned} &\text{original curve, } x^{\mu} = a^{\mu}(\lambda); \\ &\text{deformed curve, } x^{\mu} = a^{\mu}(\lambda) + \delta a^{\mu}(\lambda). \end{aligned} \tag{13.26}$$

Along either curve the lapse of proper time is

$$\tau = \int_{\mathscr{A}}^{\mathscr{B}} d\tau = \int_0^1 \left(-g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}\right)^{1/2} d\lambda. \tag{13.27}$$

At fixed λ the metric coefficient $g_{\mu\nu}[x^{\alpha}(\lambda)]$ differs from one curve to the other by

$$\delta g_{\mu\nu} \equiv g_{\mu\nu}[a^{\alpha}(\lambda) + \delta a^{\alpha}(\lambda)] - g_{\mu\nu}[a^{\alpha}(\lambda)] = \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\, \delta a^{\sigma}(\lambda); \tag{13.28}$$

and the components $dx^{\nu}/d\lambda$ of the tangent vector differ by

$$\delta\left(\frac{dx^{\nu}}{d\lambda}\right) \equiv \frac{d(a^{\nu} + \delta a^{\nu})}{d\lambda} - \frac{da^{\nu}}{d\lambda} = \frac{d}{d\lambda}(\delta a^{\nu}). \tag{13.29}$$

These changes in $g_{\mu\nu}$ and $dx^\nu/d\lambda$, at fixed λ, produce corresponding changes in the lapse of proper time in equation (13.27):

$$\delta\tau = \int_0^1 \left\{ \frac{-g_{\mu\nu}(da^\mu/d\lambda)d(\delta a^\nu)/d\lambda - \dfrac{1}{2}(g_{\mu\nu,\sigma}\delta a^\sigma)(da^\mu/d\lambda)(da^\nu/d\lambda)}{[-g_{\gamma\delta}(da^\gamma/d\lambda)(da^\delta/d\lambda)]^{1/2}} \right\} d\lambda.$$

Integrate the first term by parts. Strike out the end-point terms, because both paths must pass through $\mathscr{A}$ and $\mathscr{B}$ ($\delta a^\mu = 0$ at $\lambda = 0$ and $\lambda = 1$). Thus find

$$\delta\tau = \int_{\lambda=0}^{\lambda=1} f_\sigma(\lambda)\,\delta a^\sigma \left[-g_{\gamma\delta}\frac{da^\gamma}{d\lambda}\frac{da^\delta}{d\lambda}\right]^{1/2} d\lambda. \tag{13.30}$$

Here the f_σ ("force terms") in the integrand are abbreviations for the four expressions

$$f_\sigma(\lambda) = \frac{1}{\left[-g_{\gamma\delta}\dfrac{da^\gamma}{d\lambda}\dfrac{da^\delta}{d\lambda}\right]^{1/2}} \frac{d}{d\lambda} \frac{g_{\sigma\nu}\dfrac{da^\nu}{d\lambda}}{\left[-g_{\gamma\delta}\dfrac{da^\gamma}{d\lambda}\dfrac{da^\delta}{d\lambda}\right]^{1/2}} - \frac{\dfrac{1}{2}\dfrac{\partial g_{\mu\nu}}{\partial x^\sigma}\dfrac{da^\mu}{d\lambda}\dfrac{da^\nu}{d\lambda}}{\left[-g_{\gamma\delta}\dfrac{da^\gamma}{d\lambda}\dfrac{da^\delta}{d\lambda}\right]}. \tag{13.31}$$

An extremum is achieved, and the first-order change $\delta\tau$ vanishes for every first-order deformation $\delta a^\sigma(\lambda)$ from an optimal path $x^\sigma = a^\sigma(\lambda)$, when the four quantities f_σ that multiply the δa^σ all vanish. Thus one arrives at the four conditions

$$f_\sigma(\lambda) = 0 \tag{13.32}$$

for the determination of an extremal world line. (An alternative viewpoint on the extremization is spelled out in Figure 13.3.)

Sufficient these four equations are, but *independent* they are not, by reason of a "bead argument" (automatic vanishing of $\delta\tau$ for any set of changes that merely slide points, like beads, along an existing world line). The operation of mere "sliding of beads" implies the trivial change

$$\delta a^\sigma(\lambda) = h(\lambda)\frac{da^\sigma}{d\lambda}, \tag{13.33}$$

where $h(\lambda)$ is an arbitrary function of position along the world line ("more sliding here than there"). Already knowing that this operation cannot change τ, one is guaranteed that the integrand in (13.30) must vanish when one inserts (13.33) for δa^σ; and must vanish, moreover, whatever choice is made for the arbitrary "magnitude of slide" factor $h(\lambda)$. This requirement implies and demands that the scalar product $f_\sigma\, da^\sigma/d\lambda$ must automatically vanish; or, otherwise stated,

$$f_\sigma \frac{da^\sigma}{d\tau} = 0. \tag{13.34}$$

The argument applies, and this equation holds, whether one is or is not dealing with an optimal world line. An equation of this type, valid whether or not the world line is an allowable track for a free test particle (track of extremal lapse of proper time), is known as an *identity*. Equation (13.34), an important identity in the realm of spacetime geodesics, is an appropriate forerunner for the Bianchi identities of Chapter 15: the most important identities in the realm of spacetime curvature.

The freedom that exists to "slide λ-values along the world line" can be exploited to replace the arbitrary parameter λ by the physically more interesting parameter of proper time itself,

$$d\tau = \left[-g_{\gamma\delta} \frac{da^\gamma}{d\lambda} \frac{da^\delta}{d\lambda} \right]^{1/2} d\lambda. \tag{13.35}$$

Figure 13.2.
Star oscillating back and forth through the plane of a disc galaxy, as an example of a situation where two events $\mathscr{A}$ and $\mathscr{B}$ can be connected by more than one geodesic. Upper left: The galaxy seen edge-on, showing (dashed line) the path of the star in question, referred to a local frame partaking of and comoving with the general revolution of the nearby "disc stars." Upper right: The effective potential sensed by the star, according to Newtonian gravitation theory, is like that experienced by a ball which rolls down one inclined plane and up another ("free fall toward galactic plane" with acceleration $g = \frac{1}{2}$ in the units used here). The three central frames: Possible and impossible world lines for the star connecting two given events $\mathscr{A}$ (plane of galaxy at $t = 0$) and $\mathscr{B}$ (plane of galaxy at $t = 2$). Right: Throw star up from the galactic plane with enough velocity so that it just gets back to the plane at $t = 2$. Left: Throw it up with half the velocity and it will come back in half the time (very contrary to behavior of a simple harmonic oscillation, but in accord with galaxy's v-shaped potential!), thus being able to make two excursions in the allotted time between $\mathscr{A}$ and $\mathscr{B}$. Center: A conceivable world line (conceivable with rocket propulsion!) but not a geodesic. Bottom: Comparison of these and any other paths that allow themselves to be approximated in the form

$$z = a_1 \sin(\pi t/2) + a_2 \sin(2\pi t/2).$$

Here the two adjustable parameters, a_1 and a_2, provide the coordinates in a two-dimensional "function space" (approximation to the infinite-dimensional function space required to depict all conceivable world lines connecting $\mathscr{A}$ and $\mathscr{B}$; note comparison in right center frame between one-term Fourier approximation and exact, parabolic law of free fall; similarly in left center frame, where the two curves agree too closely to be shown separate on the diagram). Details: In the context of general relativity, take an arbitrary world line that connects $\mathscr{A}$ and $\mathscr{B}$, evaluate lapse of proper time, repeat for other world lines, and say that a given world line represents a possible motion ("geodesic") when for it the proper time is an extremum with respect to all nearby world lines. In the Newtonian approximation, the difference between the lapse of proper time and the lapse $(t_{\mathscr{B}} - t_{\mathscr{A}})$ of coordinate time is all that comes to attention, in the form of the "action integral" (on a "per-unit-mass basis")

$$\begin{aligned} I &= \int_{\mathscr{A}}^{\mathscr{B}} \left[\begin{pmatrix} \text{kinetic} \\ \text{energy} \end{pmatrix} - \begin{pmatrix} \text{potential} \\ \text{energy} \end{pmatrix} \right] dt \\ &= \int \left[\frac{1}{2} \left(\frac{dz}{dt} \right)^2 - |z| \right] dt \end{aligned}$$

(maximum, or other extremum, in the proper time implies minimum, or corresponding other extremum, in the action I). The integration gives

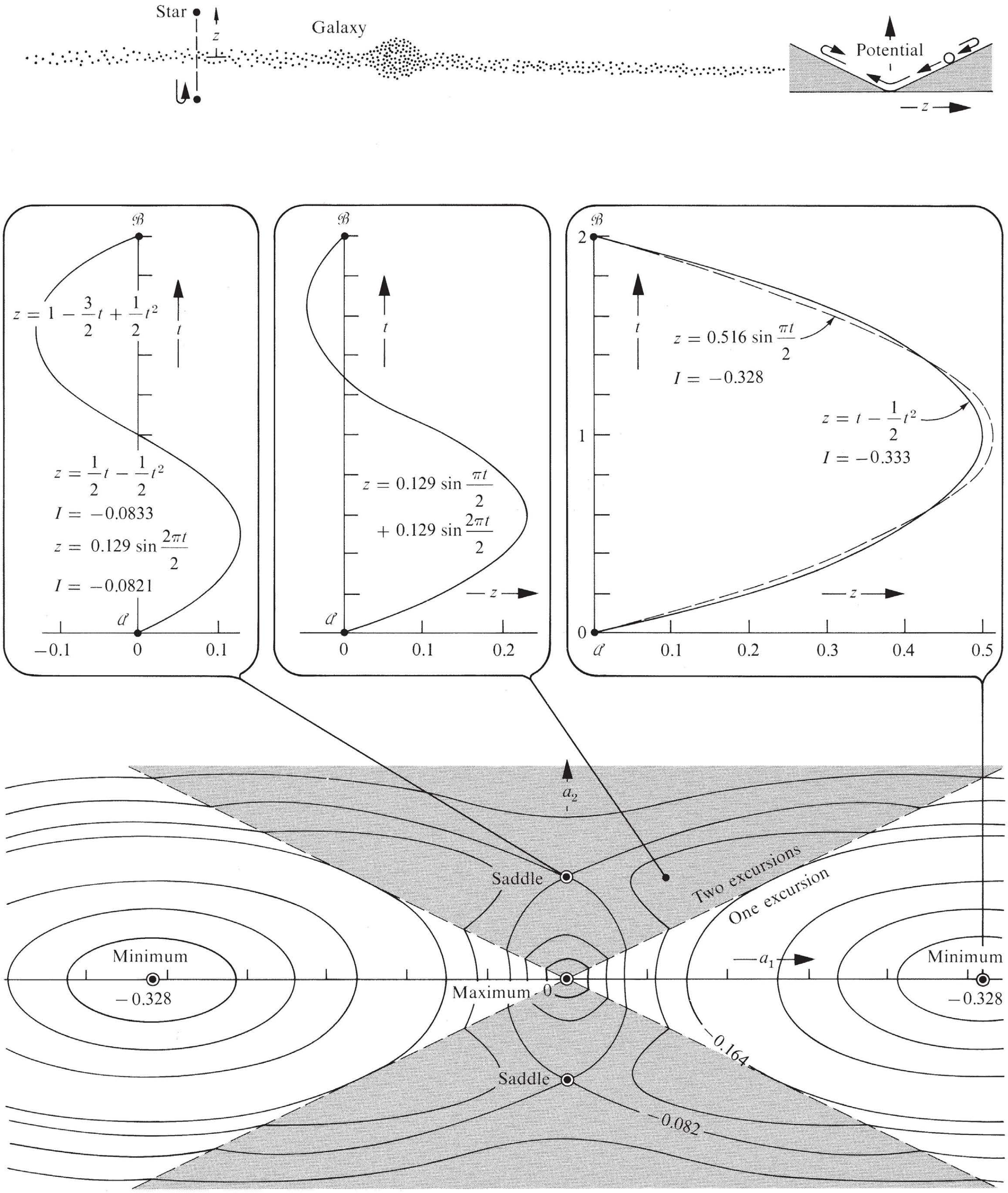
$$I = (\pi^2 a_1^2/8) - (4|a_1|/\pi) + (\pi^2 a_2^2/2)$$

for $|a_2| < \frac{1}{2}|a_1|$ (one-excursion motions), and for $|a_2| > \frac{1}{2}|a_1|$ (two-excursion motions),

$$I = (\pi^2 a_1^2/8) + (\pi^2 a_2^2/2) - (4|a_2|/\pi) - (a_1^2/\pi|a_2|).$$

The one-excursion motion minimizes the action (maximizes the lapse of proper time). The two-excursion motion extremizes the action but does not minimize it ("saddle point"; "mountain pass" in the topography). Choquard (1955) gives other examples of problems of mechanics where there is more than one extremum. Morse (1934) and Morse and Cairns (1969) give a theorem connecting the number of saddles of various types with the numbers of maxima and minima ("critical-point theorem of the calculus of variations in the large").

2



Star
Galaxy
Potential
z
$\mathscr{B}$
$\mathscr{A}$
t
$z = 1 - \frac{3}{2}t + \frac{1}{2}t^2$
$z = \frac{1}{2}t - \frac{1}{2}t^2$
$I = -0.0833$
$z = 0.129 \sin \frac{2\pi t}{2}$
$I = -0.0821$
$z = 0.129 \sin \frac{\pi t}{2} + 0.129 \sin \frac{2\pi t}{2}$
$z = 0.516 \sin \frac{\pi t}{2}$
$I = -0.328$
$z = t - \frac{1}{2}t^2$
$I = -0.333$
−0.1 0 0.1
0 0.1 0.2
0 0.1 0.2 0.3 0.4 0.5
0 1 2
a_2
a_1
Saddle
Two excursions
One excursion
Minimum
−0.328
Maximum 0
−0.164
−0.082

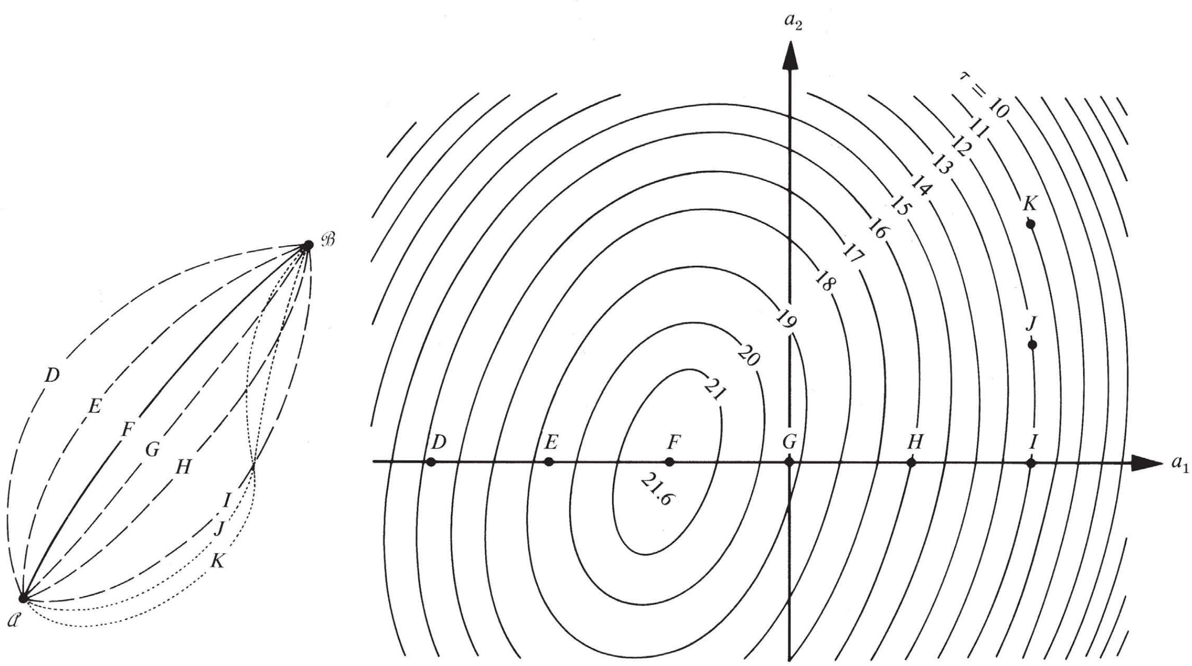


Figure 13.3.
Extremizing lapse of proper time by suitable choice of world line. Left: Spacetime; and world line F that extremizes the lapse of proper time τ from $\mathscr{A}$ to $\mathscr{B}$ compared to other world lines. The specific world lines depicted in the diagram happen to be distinguished from fiducial world line G by two "Fourier amplitudes" a_1 and a_2:

$$\delta a^\mu(\lambda) = a_1 \sin(\pi\lambda) + a_2 \sin(2\pi\lambda),$$

where the arbitrary scaling of λ, and its zero, are so adjusted that $\lambda(\mathscr{A}) = 0$, $\lambda(\mathscr{B}) = 1$.

Right: "Path space." The coordinates in this space are the Fourier amplitudes a_1 and a_2. Only these two amplitudes ("two dimensions") are shown out of what in principle are infinitely many amplitudes ("infinite-dimensional path space") required to represent the general timelike world line connecting $\mathscr{A}$ and $\mathscr{B}$. Any given contour curve runs through all those points (in path space) for which the corresponding world lines (in spacetime) rack up the indicated lapse of proper time τ. Foregoing description is classical; according to quantum mechanics, all the timelike world lines connecting $\mathscr{A}$ and $\mathscr{B}$ occur with the same probability amplitude ("principle of democracy of histories") with the only difference from one to another being the phase of this complex probability amplitude $\exp(-im\tau/\hbar)$ (m = mass of particle, $\hbar$ = quantum of angular momentum). In the sum over these probability amplitudes, however, destructive interference wipes out the contributions from all those histories which differ too much from the optimal or classical history ("Fresnel wave zone"; "Feynman's principle of sum over histories"; see Feynman and Hibbs, 1965). Capitalizing on this wave-mechanical background to show how the machinery of the physical world works, Box 25.3 spells out the Hamilton-Jacobi method ("short-wavelength limit of quantum mechanics") for determining geodesics, a method considerably more convenient for most applications than the usual "second-order differential equations for geodesics" (equation 10.27).

Focus on a specific world line, $x^\mu = a^\mu(\lambda)$, with all deformations of it gone from view; one may replace $a^\mu(\lambda)$ by $x^\mu(\lambda)$ everywhere. Then the differential equations (13.32) for an extremal world line reduce to

$$g_{\sigma\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \qquad (13.36)$$

As an aside, note that the identity (13.34) now follows by one differentiation (with respect to τ) of the equation

$$g_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} + 1 = 0. \tag{13.37}$$

Thus the identity is to be interpreted as saying that 4-velocity and 4-acceleration are orthogonal for any world line, extremal or not. Now return to (13.36), raise an index with $g^{\beta\sigma}$, and thereby bring the equation for a straight line of local Lorentz geometry into the form

$$\frac{d^2x^\beta}{d\tau^2} + g^{\beta\sigma}\frac{1}{2}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{13.38}$$

Compare with the standard form of the equation for a geodesic in "premetric geometry,"

$$\frac{d^2x^\beta}{d\lambda^2} + \Gamma^\beta{}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{13.39}$$

Conclude that the geodesics of the premetric geometry will agree with the straight lines of the local Lorentz geometry if and only if two conditions are satisfied: (1) the 40 connection coefficients $\Gamma^\beta{}_{\mu\nu}$ that define geodesics, covariant derivatives, and parallel transport must be given in terms of the 10 metric coefficients $g_{\mu\nu}$ ("Einstein gravitation potentials") by the equations (13.22) and (13.23) previously derived; and (2) the geodesic parameter λ must agree with the proper time τ up to an arbitrary normalization of zero point and an arbitrary but constant scale factor; thus

$$\lambda = a\tau + b.$$

(Nothing in the formalism has any resemblance whatsoever to the universal time t of Newton "flowing everywhere uniformly"; rather, there is a separate proper time τ for each geodesic). See Box 13.3 for another variational principle, which gives in one step both the extremal world line and the right parametrization on that line.

With this step, one has completed the transfer of the ideas of curved-space geometry from a foundation based on geodesics to a foundation based on metric. The resulting geometry always and everywhere anchors itself to the principle of "local Lorentz character," as the geometry of Newton-Cartan never did and never could.

EXERCISES

Exercise 13.5. ONCE TIMELIKE, ALWAYS TIMELIKE

Show that a geodesic of spacetime which is timelike at one event is everywhere timelike. Similarly, show that a geodesic initially spacelike is everywhere spacelike, and a geodesic initially null is everywhere null. [*Hint:* This is the easiest exercise in the book!]

(*continued on page 324*)

2

Box 13.2 "GEODESIC" VERSUS "EXTREMAL WORLD LINE"

Once the connection coefficients $\Gamma^{\alpha}{}_{\mu\nu}$ have been expressed in terms of Einstein's gravitational potentials $g_{\mu\nu}$ by the equations (13.22) and (13.23), as they are now and hereafter will be in this book ("Riemannian or metric geometry"), it is permissible and appropriate to subsume under the one word "geodesic" two previously distinct ideas: (1) a parametrized world line that satisfies the geodesic equation

$$\frac{d^2x^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}{}_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0;$$

and (2) a world line that extremizes the proper time (or, if spacelike, a curve that extremizes the proper distance) between two events $\mathcal{A}$ and $\mathcal{B}$. The one possible source of confusion is this, that (1) presupposes a properly parametrized curve (as was essential, for example, in the Schild's ladder construction employed for parallel transport in Chapter 10), whereas (2) cares only about the course of the world line through spacetime, being indifferent to what parametrization is used or whether any parametrization at all is introduced. This is not to deny the possibility of "marking in afterward" along the extremal curve the most natural and easily evaluated of all parameters, the proper time itself, whereupon the extremal curve of (2) satisfies the geodesic equation of (1). Ambiguity is avoided by insisting on proper parametrization: henceforth the word "curve" means a parametrized curve, the word "geodesic" means a properly parametrized geodesic.

Box 13.3 "DYNAMIC" VARIATIONAL PRINCIPLE FOR GEODESICS

If the principle of extremal length

$$\tau = \int_{\mathcal{A}}^{\mathcal{B}} \left[-g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}\right]^{1/2} d\lambda = \text{extremum} \qquad (1)$$

is indifferent to choice of parametrization ["$d\lambda$" canceling out in (1)] and if the geodesic equation finds the proper parametrization a matter of concern, it is appropriate to search for another extremal principle that yields in one package both the right curve and the right parameter. By analogy with elementary mechanics, one expects that an equation of motion [the geodesic equation

$$d^2x^{\mu}/d\lambda^2 + \Gamma^{\mu}{}_{\alpha\beta}\,(dx^{\alpha}/d\lambda)(dx^{\beta}/d\lambda) = 0]$$

whose leading term has the form "$\ddot{x}$" can be derived from a Lagrangian with leading term "$\frac{1}{2}\dot{x}^2$" ("kinetic energy"; "dynamic" term). The simplest coordinate invariant generalization of $\frac{1}{2}\dot{x}^2$ is

$$\frac{1}{2}\,g_{\mu\nu}\,(dx^{\mu}/d\lambda)(dx^{\nu}/d\lambda).$$

Thus one is led to try, in place of the "geometric" principle of extremal length, a new "dynamic" extremal principle:

$$
\begin{aligned}
I &= \frac{1}{2}\int_{\mathscr{A}}^{\mathscr{B}} g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\,d\lambda \\
&= \int_{\mathscr{A}}^{\mathscr{B}} L\left(x^\sigma, \frac{dx^\sigma}{d\lambda}\right) d\lambda = \text{extremum}
\end{aligned} \tag{2}
$$

(replacement of square root in previous variational principle by first power). The condition for an extremum, here as before [equations (13.30) to (13.32)] is annulment of the so-called Euler-Lagrange "functional derivative"

$$
\begin{aligned}
0 &= \frac{\delta I}{\delta x^\sigma} \equiv \begin{pmatrix}\text{coefficient of } \delta x^\sigma \text{ in} \\ \text{the integrand of } \delta I\end{pmatrix} \\
&= \frac{\partial L}{\partial x^\sigma} - \frac{d}{d\lambda}\frac{\partial L}{\partial\left(\dfrac{dx^\sigma}{d\lambda}\right)};
\end{aligned} \tag{3}
$$

or, written out in full detail,

$$
g_{\sigma\nu}\frac{d^2x^\nu}{d\lambda^2} + \frac{1}{2}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0; \tag{4}
$$

or, after multiplication by the reciprocal metric,

$$
\frac{d^2x^\alpha}{d\lambda^2} + g^{\alpha\sigma}\frac{1}{2}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0; \tag{5}
$$

which translates into the geodesic equation

$$
\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{6}
$$

Thus, the new "dynamic" expression (2) is indeed extremal for geodesic curves—and, by contrast with proper length, (1), it is extremal when and only when the geodesic is affinely parametrized. [Its "Euler-Lagrange equations" (6) remain satisfied only under parameter changes $\lambda_{\text{new}} = a\lambda_{\text{old}} + b$, which keep the parameter affine; by contrast, the Euler-Lagrange equations (13.31) and (13.32) for the "principle of extremal length" (1) remain satisfied for any change of parameter whatsoever.]

Exercise 13.6. SPACELIKE GEODESICS HAVE EXTREMAL LENGTH

Show that any spacelike curve linking two events $\mathscr{A}$ and $\mathscr{B}$ is a geodesic if and only if it extremizes the proper length

$$s = \int_{\mathscr{A}}^{\mathscr{B}} (g_{\mu\nu}\, dx^{\mu}\, dx^{\nu})^{1/2}.$$

[*Hint:* This is almost as easy as exercise 13.5 if one has already proved the analogous theorem for timelike geodesics.]

Exercise 13.7. METRIC TENSOR MEASURED BY LIGHT SIGNALS AND FREE PARTICLES [Kuchař]

(a) Instead of parametrizing a timelike geodesic by the proper time τ, parametrize it by an arbitrary parameter μ,

$$\tau = F(\mu).$$

Write the geodesic equation in the μ-parametrization.

(b) Use now the coordinate time t as a parameter. Throw out a cloud of free particles with different "velocities" $v^i = dx^i/dt$ and observe their "accelerations" $a^i = d^2x^i/dt^2$. Discuss what combinations of the components of the affine connection $\Gamma^{\iota}{}_{\kappa\lambda}$ one can measure in this way. (Assume that no standard clocks measuring τ are available!)

(c) Show that one can measure the conformal metric $\bar{g}_{\iota\kappa}$, i.e., the ratios of the components of the metric tensor $g_{\iota\kappa}$ to a given component (say, g_{00})

$$\bar{g}_{\iota\kappa} = \Lambda g_{\iota\kappa}, \qquad \Lambda \equiv (-g_{00})^{-1},$$

using only the light signals moving along the null geodesics $g_{\iota\kappa}\, dx^{\iota}\, dx^{\kappa} = 0$.

(d) Combine now the results of (b) and (c). Assume that $\Gamma^{\iota}{}_{\kappa\lambda}$ is generated by the metric tensor by (13.22), (13.23), in the coordinate frame x^{ι}. Show that one can determine Λ everywhere, if one prescribes it at one event (equivalent to fixing the unit of time).

---

§13.5. METRIC-INDUCED PROPERTIES OF ***RIEMANN***

Symmetries of ***Riemann*** in absence of metric

In Newtonian spacetime, in the real, physical spacetime of Einstein—indeed, in any manifold with covariant derivative—the Riemann curvature tensor has these symmetries (exercise 11.6):

$$R^{\alpha}{}_{\beta\gamma\delta} \equiv R^{\alpha}{}_{\beta[\gamma\delta]} \qquad \text{(antisymmetry on last two indices)} \tag{13.40}$$

$$R^{\alpha}{}_{[\beta\gamma\delta]} \equiv 0 \qquad \text{(vanishing of completely antisymmetric part).} \tag{13.41}$$

In addition, it satisfies a differential identity (exercise 11.10):

$$R^{\alpha}{}_{\beta[\gamma\delta;\epsilon]} \equiv 0 \qquad \text{("Bianchi identity")} \tag{13.42}$$

(see Chapter 15 for geometric significance).

New symmetries imposed by metric

When metric is brought onto the scene, whether in Einstein spacetime or elsewhere, it impresses on ***Riemann*** the additional symmetry (exercise 13.8)

$$R_{\alpha\beta\gamma\delta} \equiv R_{[\alpha\beta]\gamma\delta} \text{ (antisymmetry on first two indices).} \tag{13.43}$$

This, together with (13.40) and (13.41), forms a complete set of symmetries for ***Riemann***; other symmetries that follow from these (exercise 13.10) are

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \qquad \text{(symmetry under pair exchange),} \tag{13.44}$$

and

$$R_{[\alpha\beta\gamma\delta]} = 0 \qquad \text{(vanishing of completely antisymmetric part).} \tag{13.45}$$

These symmetries reduce the number of independent components of ***Riemann*** from $4 \times 4 \times 4 \times 4 = 256$ to 20 (exercise 13.9).

With metric present, one can construct a variety of new curvature tensors from ***Riemann***. Some that will play important roles later are as follows.

The curvature tensor $\boldsymbol{\mathcal{G}}$

(1) The *double dual of* ***Riemann***, $\boldsymbol{\mathcal{G}} \equiv$ ****Riemann**** (analog of ***Maxwell*** $\equiv$ ****Faraday***), which has components

$$\mathcal{G}^{\alpha\beta}{}_{\gamma\delta} \equiv \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} R_{\mu\nu}{}^{\rho\sigma} \frac{1}{2}\epsilon_{\rho\sigma\gamma\delta} = -\frac{1}{4}\delta^{\alpha\beta\mu\nu}_{\rho\sigma\gamma\delta} R_{\mu\nu}{}^{\rho\sigma} \tag{13.46}$$

(exercise 13.11).

Einstein tensor

(2) The *Einstein curvature tensor,* which is symmetric (exercise 13.11)

$$G^{\beta}{}_{\delta} \equiv \mathcal{G}^{\mu\beta}{}_{\mu\delta}; \qquad G_{\beta\delta} = G_{\delta\beta}. \tag{13.47}$$

Ricci tensor

Curvature scalar

(3) The *Ricci curvature tensor,* which is symmetric, and the *curvature scalar*

$$R^{\beta}{}_{\delta} \equiv R^{\mu\beta}{}_{\mu\delta}, \qquad R_{\beta\delta} = R_{\delta\beta}; \qquad R \equiv R^{\beta}{}_{\beta}; \tag{13.48}$$

which are related to the Einstein tensor by (exercise 13.12)

$$R^{\beta}{}_{\delta} = G^{\beta}{}_{\delta} + \frac{1}{2}R\delta^{\beta}{}_{\delta}. \tag{13.49}$$

Weyl conformal tensor

(4) *The Weyl conformal tensor* (exercise 13.13)

$$C^{\alpha\beta}{}_{\gamma\delta} = R^{\alpha\beta}{}_{\gamma\delta} - 2\delta^{[\alpha}{}_{[\gamma}R^{\beta]}{}_{\delta]} + \frac{1}{3}\delta^{[\alpha}{}_{[\gamma}\delta^{\beta]}{}_{\delta]}R. \tag{13.50}$$

Bianchi identities

The Bianchi identity (13.42) takes a particularly simple form when rewritten in terms of the double dual $\boldsymbol{\mathcal{G}}$:

$$\mathcal{G}_{\alpha\beta\gamma}{}^{\delta}{}_{;\delta} \equiv 0 \qquad \text{("Bianchi identity")} \tag{13.51}$$

(exercise 13.11); and it has the obvious consequence

$$G_{\alpha}{}^{\beta}{}_{;\beta} \equiv 0 \qquad \text{("contracted Bianchi identity").} \tag{13.52}$$

Chapter 15 will be devoted to the deep geometric significance of these Bianchi identities.

EXERCISES

Exercise 13.8. *RIEMANN* ANTISYMMETRIC IN FIRST TWO INDICES

(a) Derive the antisymmetry condition (13.43). [Hint: Prove by abstract calculations that any vector fields $\boldsymbol{s}$, $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ satisfy $0 = \mathcal{R}(\boldsymbol{u}, \boldsymbol{v})(\boldsymbol{s} \cdot \boldsymbol{w}) = \boldsymbol{s} \cdot [\mathcal{R}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{w}] + \boldsymbol{w} \cdot [\mathcal{R}(\boldsymbol{u}, \boldsymbol{v})\boldsymbol{s}]$. Then from this infer (13.43).]

(b) Explain in geometric terms the meaning of this antisymmetry.

Exercise 13.9. NUMBER OF INDEPENDENT COMPONENTS OF *RIEMANN*

(a) In the absence of metric, a complete set of symmetry conditions for ***Riemann*** is $R^{\alpha}{}_{\beta\gamma\delta} = R^{\alpha}{}_{\beta[\gamma\delta]}$ and $R^{\alpha}{}_{[\beta\gamma\delta]} = 0$. Show that in four-dimensional spacetime these reduce the number of independent components from $4 \times 4 \times 4 \times 4 = 256$ to $4 \times 4 \times 6 - 4 \times 4 = 96 - 16 = 80$.

(b) Show that in a manifold of n dimensions without metric, the number of independent components is

$$\frac{n^3(n-1)}{2} - \frac{n^2(n-1)(n-2)}{6} = \frac{n^2(n^2-1)}{3}. \tag{13.53}$$

(c) In the presence of metric, a complete set of symmetries is $R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]}$, and $R_{\alpha[\beta\gamma\delta]} = 0$. Show that in four-dimensional spacetime, these reduce the number of independent components to $6 \times 6 - 4 \times 4 = 36 - 16 = 20$.

(d) Show that in a manifold of n dimensions with metric, the number of independent components is

$$\left[\frac{n(n-1)}{2}\right]^2 - \frac{n^2(n-1)(n-2)}{6} = \frac{n^2(n^2-1)}{12}. \tag{13.54}$$

Exercise 13.10. *RIEMANN* SYMMETRIC IN EXCHANGE OF PAIRS; COMPLETELY ANTISYMMETRIC PART VANISHES

From the complete set of symmetries in the presence of a metric, $R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]}$ and $R_{\alpha[\beta\gamma\delta]} = 0$, derive: (a) symmetry under pair exchange, $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$, and (b) vanishing of completely antisymmetric part, $R_{[\alpha\beta\gamma\delta]} = 0$. Then (c) show that the following form a complete set of symmetries:

$$R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]} = R_{\gamma\delta\alpha\beta}, \qquad R_{[\alpha\beta\gamma\delta]} = 0. \tag{13.55}$$

Exercise 13.11. DOUBLE DUAL OF *RIEMANN*; *EINSTEIN*

(a) Show that $\boldsymbol{G} \equiv$ ****Riemann**** contains precisely the same amount of information as ***Riemann***, and satisfies precisely the same set of symmetries [(13.40), (13.41), (13.43) to (13.45)].

(b) From the symmetries of $\boldsymbol{G}$, show that ***Einstein*** [defined in (13.47)] is symmetric ($G_{[\beta\delta]} = 0$).

(c) Show that the Bianchi identities (13.42), when written in terms of $\boldsymbol{G}$, take the form (13.51) ("vanishing divergence," $\boldsymbol{\nabla} \cdot \boldsymbol{G} = 0$).

(d) By contracting the Bianchi identities $\boldsymbol{\nabla} \cdot \boldsymbol{G} = 0$, show that $\boldsymbol{G} \equiv$ ***Einstein*** has vanishing divergence [equation (13.52)].

Exercise 13.12. *RICCI* AND *EINSTEIN* RELATED

(a) From the symmetries of ***Riemann***, show that ***Ricci*** is symmetric ($R_{[\beta\delta]} = 0$).

(b) Show that ***Ricci*** is related to ***Einstein*** by equation (13.49).

Exercise 13.13. THE WEYL CONFORMAL TENSOR

(a) Show that the Weyl conformal tensor (13.50) possesses the same symmetries [(13.40), (13.41), (13.43) to (13.45)] as the Riemann tensor.

(b) Show that the Weyl tensor is completely "trace-free"; i.e., that

$$\text{contraction of } C_{\alpha\beta\gamma\delta} \text{ on any pair of slots vanishes.} \tag{13.56}$$

Thus, $C_{\alpha\beta\gamma\delta}$ can be regarded as the trace-free part of ***Riemann***, and $R_{\alpha\beta}$ can be regarded as the trace of ***Riemann***. ***Riemann*** is determined entirely by its trace-free part $C_{\alpha\beta\gamma\delta}$ and its trace $R_{\alpha\beta}$ [see equation (13.50), and recall $R = R^{\alpha}{}_{\alpha}$].

(c) Show that in spacetime the Weyl tensor has 10 independent components.

(d) Show that in an n-dimensional manifold the number of independent components of ***Weyl*** [defined by a modification of (13.50) that maintains (13.56)] is

$$\frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2} \text{ for } n \geq 3, \qquad 0 \text{ for } n \leq 3. \tag{13.57}$$

Thus, in manifolds of 1, 2, or 3 dimensions, the Weyl tensor is identically zero, and the Ricci tensor completely determines the Riemann tensor.

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§13.6. THE PROPER REFERENCE FRAME OF AN ACCELERATED OBSERVER

Proper reference frame described physically

A physicist performing an experiment in a jet airplane (e.g., an infrared astronomy experiment) may use several different coordinate systems at once. But a coordinate system of special utility is one at rest relative to all the apparatus bolted into the floor and walls of the airplane cabin. This "proper reference frame" has a rectangular "$\hat{x}$, $\hat{y}$, $\hat{z}$" grid attached to the walls of the cabin, and one or more clocks at rest in the grid. That this proper reference frame is accelerated relative to the local Lorentz frames, the physicist knows from his own failure to float freely in the cabin, or, with greater precision, from accelerometer measurements. That his proper reference frame is rotating relative to local Lorentz frames he knows from the Coriolis forces he feels, or, with greater precision, from the rotation of inertial-guidance gyroscopes relative to the cabin walls.

Exercise 6.8 gave a mathematical treatment of such an accelerated, rotating, but locally orthonormal reference frame in flat spacetime. This section does the same in curved spacetime. In the immediate vicinity of the spatial grid's origin $x^{\hat{j}} = 0$ (region of spatial extent so small that curvature effects are negligible), no aspect of the coordinate system can possibly reveal whether spacetime is curved or flat. Hence, all the details of exercise 6.8 must remain valid in curved spacetime. Nevertheless, it is instructive to rediscuss those details, and some new ones, using the powerful mathematics of the last few chapters.

Six-step construction of coordinate grid for proper frame

Begin by making more precise the coordinate grid to be used. The following is perhaps the most natural way to set up the grid.

(1) Let τ be proper time as measured by the accelerated observer's clock (clock at center of airplane cabin in above example). Let $\mathcal{P} = \mathcal{P}_0(\tau)$ be the observer's world line, as shown in Figure 13.4,a.

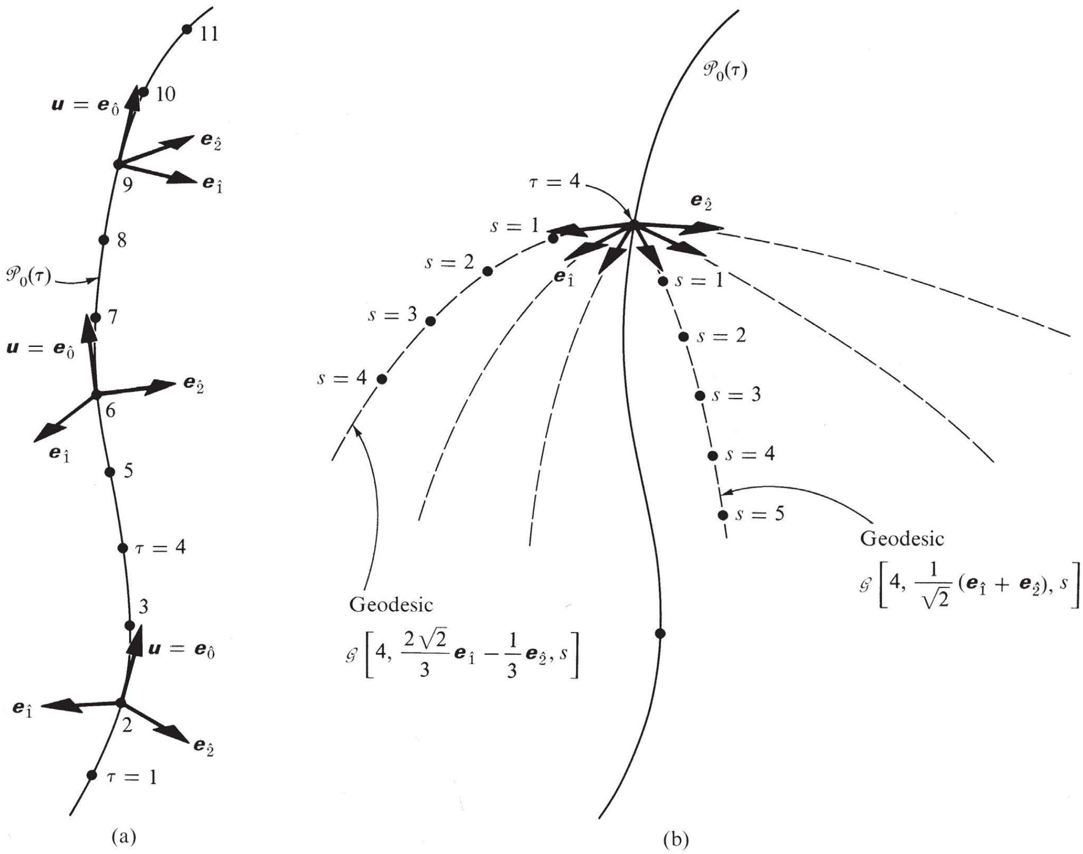


Figure 13.4.
The proper reference frame of an accelerated observer. Diagram (a) shows the observer's orthonormal tetrad $\{\boldsymbol{e}_{\hat\alpha}\}$ being transported along his world line $\mathscr{P}_0(\tau)$ [transport law (13.60)]. Diagram (b) shows geodesics bristling out perpendicularly from an arbitrary event $\mathscr{P}_0(4)$ on the observer's world line. Each geodesic is specified uniquely by (1) the proper time τ at which it originates, and (2) the direction (unit tangent vector $\boldsymbol{n} = d/ds = n^{\hat j}\boldsymbol{e}_{\hat j}$ along which it emanates). A given event on the geodesic is specified by τ, $\boldsymbol{n}$, and proper distance s from the geodesic's emanation point; hence the notation

$$\mathscr{P} = \mathscr{G}[\tau, \boldsymbol{n}, s]$$

for the given event. The observer's proper reference frame attributes to this given event the coordinates

$$x^{\hat 0}(\mathscr{G}[\tau, \boldsymbol{n}, s]) = \tau,$$
$$x^{\hat j}(\mathscr{G}[\tau, \boldsymbol{n}, s]) = sn^{\hat j}.$$

(2) The observer carries with himself an orthonormal tetrad $\{\boldsymbol{e}_{\hat\alpha}\}$ (Figure 13.4,a), with

$$\boldsymbol{e}_{\hat 0} = \boldsymbol{u} = d\mathscr{P}_0/d\tau = (\text{4-velocity of observer}) \tag{13.58}$$

($\boldsymbol{e}_{\hat 0}$ points along observer's "time direction"), and with

$$\boldsymbol{e}_{\hat\alpha} \cdot \boldsymbol{e}_{\hat\beta} = \eta_{\alpha\beta} \tag{13.59}$$

(orthonormality).

Transport law for observer's tetrad

(3) The tetrad changes from point to point along the observer's world line, relative to parallel transport:

$$\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{e}_{\hat{\alpha}} = -\boldsymbol{\Omega} \cdot \boldsymbol{e}_{\hat{\alpha}}, \tag{13.60}$$

$$\begin{aligned} \Omega^{\mu\nu} &= a^{\mu}u^{\nu} - u^{\mu}a^{\nu} + u_{\alpha}\omega_{\beta}\epsilon^{\alpha\beta\mu\nu} \\ &= \text{“generator of infinitesimal Lorentz transformation.”} \end{aligned} \tag{13.61}$$

This transport law has the same form in curved spacetime as in flat (§6.5 and exercise 6.8) because curvature can only be felt over finite distances, not over the infinitesimal distance involved in the "first time-rate of change of a vector" (equivalence principle). As in exercise 6.8,

$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{u} = \text{(4-acceleration of observer)}, \\ \boldsymbol{\omega} &= \begin{pmatrix} \text{angular velocity of rotation of spatial} \\ \text{basis vectors } \boldsymbol{e}_{\hat{j}} \text{ relative to Fermi-} \\ \text{Walker-transported vectors, i.e.,} \\ \text{relative to inertial-guidance gyroscopes} \end{pmatrix}, \\ \boldsymbol{u} \cdot \boldsymbol{a} &= \boldsymbol{u} \cdot \boldsymbol{\omega} = 0. \end{aligned} \tag{13.62}$$

If $\boldsymbol{\omega}$ were zero, the observer would be Fermi-Walker-transporting his tetrad (gyroscope-type transport). If both $\boldsymbol{a}$ and $\boldsymbol{\omega}$ were zero, he would be freely falling (geodesic motion) and would be parallel-transporting his tetrad, $\boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{e}_{\hat{\alpha}} = 0$.

(4) The observer constructs his proper reference frame (local coordinate system) in a manner analogous to the Riemann-normal construction of §11.6. From each event $\mathscr{P}_0(\tau)$ on his world line, he sends out purely spatial geodesics (geodesics orthogonal to $\boldsymbol{u} = d\mathscr{P}_0/d\tau$), with affine parameter equal to proper length,

$$\mathscr{P} = \mathscr{G}[\tau, \boldsymbol{n}, s]. \tag{13.63}$$

[proper time; tells "starting point" of geodesic] [tangent vector to geodesic at starting point; tells "which" geodesic] [proper length along geodesic from starting point; tells "where" on geodesic]

(See Figure 13.4,b.) The tangent vector has unit length, because the chosen affine parameter is proper length:

$$\begin{aligned} \boldsymbol{n} &= (\partial\mathscr{G}/\partial s)_{s=0}; \qquad n^{\mu} = (dx^{\mu}/ds) \text{ along geodesic}, \\ \boldsymbol{n} \cdot \boldsymbol{n} &= g_{\mu\nu}\left(\frac{dx^{\mu}}{ds}\right)\left(\frac{dx^{\nu}}{ds}\right) = \frac{ds^2}{ds^2} = 1. \end{aligned} \tag{13.64}$$

(5) Each event near the observer's world line is intersected by precisely one of the geodesics $\mathscr{G}[\tau, \boldsymbol{n}, s]$. [Far away, this is not true; the geodesics may cross, either because of the observer's acceleration, as in Figure 6.3, or because of the curvature of spacetime ("geodesic deviation").]

2

(6) Pick an event $\mathscr{P}$ near the observer's world line. The geodesic through it originated on the observer's world line at a specific time τ, had original direction $\boldsymbol{n} = n^{\hat{j}}\boldsymbol{e}_{\hat{j}}$, and needed to extend a distance s before reaching $\mathscr{P}$. Hence, the four numbers

$$(x^{\hat{0}}, x^{\hat{1}}, x^{\hat{2}}, x^{\hat{3}}) \equiv (\tau, sn^{\hat{1}}, sn^{\hat{2}}, sn^{\hat{3}}) \tag{13.65}$$

are a natural way of identifying the event $\mathscr{P}$. These are the coordinates of $\mathscr{P}$ in the observer's proper reference frame.

(7) Restated more abstractly,

$$\begin{aligned} x^{\hat{0}}(\mathscr{G}[\tau, \boldsymbol{n}, s]) &= \tau, \\ x^{\hat{j}}(\mathscr{G}[\tau, \boldsymbol{n}, s]) &= sn^{\hat{j}} = sn_{\hat{j}} = s\boldsymbol{n} \cdot \boldsymbol{e}_{\hat{j}}. \end{aligned} \tag{13.65$'$}$$

In flat spacetime this construction process and the resulting coordinates $x^{\hat{\alpha}}(\mathscr{P})$ are identical to the process and resulting coordinates $\xi^{\alpha'}(\mathscr{P})$ of exercise 6.8.

For use in calculations one wants not only the coordinate system, but also its metric coefficients and connection coefficients. Fortunately, $g_{\hat{\alpha}\hat{\beta}}$ and $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}}$ are needed only along the observer's world line, where they are especially simple. Only a foolish observer would try to use his own proper reference frame far from his world line, where its grid ceases to be orthonormal and its geodesic grid lines may even cross! (See §6.3.)

All along the observer's world line $\mathscr{P}_0(\tau)$, the basis vectors of his coordinate grid are identical (by construction) to his orthonormal tetrad

$$\partial/\partial x^{\hat{\alpha}} = \boldsymbol{e}_{\hat{\alpha}}, \tag{13.66}$$

and therefore its metric coefficients are

$$g_{\hat{\alpha}\hat{\beta}} = \boldsymbol{e}_{\hat{\alpha}} \cdot \boldsymbol{e}_{\hat{\beta}} = \eta_{\alpha\beta} \text{ all along } \mathscr{P}_0(\tau). \tag{13.67}$$

Connection coefficients along observer's world line

Some of the connection coefficients are determined by the transport law (13.60) for the observer's orthonormal tetrad:

$$\begin{aligned} \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{e}_{\hat{\alpha}} = \boldsymbol{\nabla}_{\hat{0}}\boldsymbol{e}_{\hat{\alpha}} &= \boldsymbol{e}_{\hat{\beta}}\Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{0}} \\ &= -\boldsymbol{\Omega} \cdot \boldsymbol{e}_{\hat{\alpha}} = -\boldsymbol{e}_{\hat{\beta}}\Omega^{\hat{\beta}}{}_{\hat{\alpha}}. \end{aligned}$$

Thus

$$\Gamma^{\hat{\beta}}{}_{\hat{\alpha}\hat{0}} = -\Omega^{\hat{\beta}}{}_{\hat{\alpha}} \text{ all along } \mathscr{P}_0(\tau). \tag{13.68}$$

Since $\boldsymbol{\Omega}$ has the form (13.61) and the observer's 4-velocity and 4-acceleration have components $u_{\hat{0}} = -1$, $u_{\hat{j}} = 0$, $a_{\hat{0}} = 0$ in the observer's own proper frame, these connection coefficients are

$$\left.\begin{aligned} &\Gamma^{\hat{0}}{}_{\hat{0}\hat{0}} = \Gamma_{\hat{0}\hat{0}\hat{0}} = 0, \\ &\Gamma^{\hat{0}}{}_{\hat{j}\hat{0}} = -\Gamma_{\hat{0}\hat{j}\hat{0}} = +\Gamma_{\hat{j}\hat{0}\hat{0}} = +\Gamma^{\hat{j}}{}_{\hat{0}\hat{0}} = a^{\hat{j}}, \\ &\Gamma^{\hat{j}}{}_{\hat{k}\hat{0}} = \Gamma_{\hat{j}\hat{k}\hat{0}} = -\omega^{\hat{i}}\epsilon_{\hat{0}\hat{i}\hat{j}\hat{k}}, \end{aligned}\right\} \text{ all along } \mathscr{P}_0(\tau). \tag{13.69a}$$

The remaining connection coefficients can be read from the geodesic equation for the geodesics $\mathscr{G}[\tau, \boldsymbol{n}, s]$ that emanate from the observer's world line. According to equation (13.65), the coordinate representation of each such geodesic is

$$x^{\hat{0}}(s) = \tau = \text{constant}, \qquad x^{\hat{j}}(s) = n^{\hat{j}}s;$$

hence, $d^2x^{\hat{\alpha}}/ds^2 = 0$ all along the geodesic, and the geodesic equation reads

$$0 = \frac{d^2x^{\hat{\alpha}}}{ds^2} + \Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}} \frac{dx^{\hat{\beta}}}{ds}\frac{dx^{\hat{\gamma}}}{ds} = \Gamma^{\hat{\alpha}}{}_{\hat{j}\hat{k}} n^{\hat{j}} n^{\hat{k}}.$$

This equation is satisfied on the observer's world line for all spatial geodesics (all $n^{\hat{j}}$) if and only if

$$\Gamma^{\hat{\alpha}}{}_{\hat{j}\hat{k}} = \Gamma_{\hat{\alpha}\hat{j}\hat{k}} = 0 \text{ all along } \mathscr{P}_0(\tau). \tag{13.69b}$$

The values (13.69) of the connection coefficients determine uniquely the partial derivatives of the metric coefficients [see equation (13.19′)]:

$$\left.\begin{aligned} g_{\hat{\alpha}\hat{\beta},\hat{0}} &= 0, & g_{\hat{j}\hat{k},\hat{l}} &= 0, \\ g_{\hat{0}\hat{0},\hat{j}} &= -2a_{\hat{j}}, & g_{\hat{0}\hat{j},\hat{k}} &= -\epsilon_{\hat{0}\hat{j}\hat{k}\hat{l}}\,\omega^{\hat{l}} \end{aligned}\right\} \text{all along } \mathscr{P}_0(\tau); \tag{13.70}$$

and these, plus the orthonormality condition $g_{\hat{\alpha}\hat{\beta}}[\mathscr{P}_0(\tau)] = \eta_{\alpha\beta}$, imply that the line element near the observer's world line is

Metric of proper reference frame, and its physical interpretation

$$\begin{aligned} ds^2 = {} & -(1 + 2a_{\hat{j}}x^{\hat{j}})\,dx^{\hat{0}^2} - 2(\epsilon_{\hat{j}\hat{k}\hat{l}}\,x^{\hat{k}}\omega^{\hat{l}})\,dx^{\hat{0}}\,dx^{\hat{j}} \\ & + \delta_{\hat{j}\hat{k}}\,dx^{\hat{j}}\,dx^{\hat{k}} + O(|x^{\hat{j}}|^2)\,dx^{\hat{\alpha}}\,dx^{\hat{\beta}}. \end{aligned} \tag{13.71}$$

Several features of this line element deserve notice, as follows.

(1) On the observer's world line $\mathscr{P}_0(\tau)$—i.e., $x^{\hat{j}} = 0$—$ds^2 = \eta_{\alpha\beta}\,dx^{\hat{\alpha}}\,dx^{\hat{\beta}}$.

(2) The observer's acceleration shows up in a correction term to $g_{\hat{0}\hat{0}}$,

$$\delta g_{\hat{0}\hat{0}} = -2\boldsymbol{a}\cdot\boldsymbol{x}, \tag{13.72a}$$

which is proportional to distance along the acceleration direction. For the flat-spacetime derivation of this correction term, see §6.6.

(3) The observer's rotation relative to inertial-guidance gyroscopes shows up in a correction term to $g_{\hat{0}\hat{j}}$, which can be rewritten in 3-vector notation

$$\delta g_{\hat{0}\hat{j}}\boldsymbol{e}_{\hat{j}} = -\boldsymbol{x}\times\boldsymbol{\omega} = +\boldsymbol{\omega}\times\boldsymbol{x}. \tag{13.72b}$$

(4) These first-order corrections to the line element are unaffected by spacetime curvature and contain no information about curvature. Only at second order, $O(|x^{\hat{j}}|^2)$, will curvature begin to show up.

(5) *In the special case of zero acceleration and zero rotation* ($\boldsymbol{a} = \boldsymbol{\omega} = 0$), *the observer's proper reference frame reduces to a local Lorentz frame* ($g_{\hat{\alpha}\hat{\beta}} = \eta_{\alpha\beta}$, $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\hat{\gamma}} = 0$) *all along his geodesic world line!* By contrast, the local Lorentz coordinate

systems constructed earlier in the book ("general" local Lorentz coordinates of §8.6, "Riemann normal coordinates" of §11.6) are local Lorentz only at a single event.

In the case of zero rotation and zero acceleration, one can derive the following expression for the metric, accurate to second order in $|x^{j}|$:

$$\begin{aligned} ds^2 = {} & (-1 - R_{\hat{0}\hat{l}\hat{0}\hat{m}} x^{\hat{l}} x^{\hat{m}})\, dt^2 - \left(\frac{4}{3} R_{\hat{0}\hat{l}\hat{j}\hat{m}} x^{\hat{l}} x^{\hat{m}}\right) dt\, dx^{\hat{j}} \\ & + \left(\delta_{\hat{i}\hat{j}} - \frac{1}{3} R_{\hat{i}\hat{l}\hat{j}\hat{m}} x^{\hat{l}} x^{\hat{m}}\right) dx^{\hat{i}}\, dx^{\hat{j}} + O(|x^{\hat{j}}|^3)\, dx^{\hat{\alpha}}\, dx^{\hat{\beta}} \end{aligned} \tag{13.73}$$

[see, e.g., Manasse and Misner (1963)]. Here $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}$ are the components of the Riemann tensor along the world line $x^{\hat{j}} = 0$. Such coordinates are called "Fermi Normal Coordinates."

EXERCISES

Exercise 13.14. INERTIAL AND CORIOLIS FORCES

An accelerated observer studies the path of a freely falling particle as it passes through the origin of his proper reference frame. If

$$\boldsymbol{v} \equiv (dx^{\hat{j}}/dx^{\hat{0}})\boldsymbol{e}_{\hat{j}} \tag{13.74}$$

is the particle's ordinary velocity, show that its ordinary acceleration relative to the observer's proper reference frame is

$$\frac{d^2x^{\hat{j}}}{dx^{\hat{0}2}}\boldsymbol{e}_{\hat{j}} = \underbrace{-\boldsymbol{a}}_{\text{[inertial acceleration]}} - \underbrace{2\boldsymbol{\omega} \times \boldsymbol{v}}_{\text{[Coriolis acceleration]}} + \underbrace{2(\boldsymbol{a}\cdot\boldsymbol{v})\boldsymbol{v}}_{\text{[relativistic correction to inertial acceleration]}}. \tag{13.75}$$

Here $\boldsymbol{a}$ is the observer's own 4-acceleration, and $\boldsymbol{\omega}$ is the angular velocity with which his spatial basis vectors $\boldsymbol{e}_{\hat{j}}$ are rotating [see equations (13.62)]. [*Hint:* Use the geodesic equation at the point $x^{\hat{j}} = 0$ of the particle's trajectory. *Note:* This result was derived in flat spacetime in exercise 6.8 using a different method.]

Exercise 13.15. ROTATION GROUP: METRIC

(Continuation of exercises 9.13, 9.14, 10.17 and 11.12). Show that for the manifold $SO(3)$ of the rotation group, there exists a metric $\boldsymbol{g}$ that is compatible with the covariant derivative $\boldsymbol{\nabla}$. Prove existence by exhibiting the metric components explicitly in the noncoordinate basis of generators $\{\boldsymbol{e}_\alpha\}$. [*Answer:*

$$g_{\alpha\beta} = \delta_{\alpha\beta}. \tag{13.76}$$

Restated in words: If one postulates that: (1) the manifold of the rotation group is locally Euclidean; (2) the generators of infinitesimal rotations $\{\boldsymbol{e}_\alpha\}$ are orthonormal, $\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta = \delta_{\alpha\beta}$; and (3) $\{\boldsymbol{e}_\alpha\}$ obey the standard rotation-group commutation relations

$$[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = -\epsilon_{\alpha\beta\gamma}\boldsymbol{e}_\gamma; \tag{13.77}$$

then the resulting geodesics of $SO(3)$ agree with the geodesics chosen in exercise 10.17.]

CHAPTER 14

CALCULATION OF CURVATURE

§14.1. CURVATURE AS A TOOL FOR UNDERSTANDING PHYSICS

This chapter is entirely Track 2.
Chapter 4 (differential forms) and Chapter 10, 11, and 13 (differential geometry) are necessary preparation for §§14.5–14.6.
This chapter is needed as preparation for Chapter 15 (Bianchi identities).
It will be helpful in many applications of gravitation theory (Chapters 23–40).

Elementary physics sometimes allows one to shortcircuit any systematized calculation of curvature (frequency of oscillation of test particle; tide-producing acceleration near a center of attraction; curvature of a closed 3-sphere model universe; effect of parallel transport on gyroscope or vector; see Figures 1.1, 1.10, and 1.12, and Boxes 1.6 and 1.7); but on other occasions a calculation of curvature is the quickest way into the physics. This chapter is designed for such occasions. It describes three ways to calculate curvature and gives the components of the Einstein curvature tensor for a plane gravitational wave (Box 14.4, equation 5), for the Friedmann geometry of the universe (Box 14.5), and for Schwarzschild geometry, both static (exercise 14.13) and dynamic (exercise 14.16). These and other calculations of curvature elsewhere are indexed under "curvature tensors."

It is enough to look at an expression for a 4-geometry as complicated as

$$
\begin{aligned}
ds^2 = &-(x/3^{1/2}L + y^2/12L^2)^{-3^{1/2}}\left(\int \frac{v\,dz}{z}\right)^{-1}(-z/L)^{3^{-1/2}}\,dt^2\\
&+(x/3^{1/2}L + y^2/12L^2)^{1+3^{1/2}}\left(\int \frac{v\,dz}{z}\right)^{1+2/3^{1/2}}(-z/L)^{-1+3^{-1/2}}\,dx^2\\
&+(x/3^{1/2}L + y^2/12L^2)^{2+3^{1/2}}\left(\int \frac{v\,dz}{z}\right)^{1+2/3^{1/2}}(-z/L)^{-3^{-1/2}}\,dy^2\\
&+(x/3^{1/2}L + y^2/12L^2)^{3+3^{1/2}}\left(\int \frac{v\,dz}{z}\right)^{1+2/3^{1/2}}(-z/L)^{-2-3^{-1/2}}\times\\
&\times\left(\frac{v^2-1}{-1-z/L}\right)dz^2 \qquad (14.1)
\end{aligned}
$$

Situations in which one must compute curvature

[Harrison (1959)] to realize that one might understand the physical situation better if one knew what the curvature is; similarly with any other complicated expressions for metrics that arise from solving Einstein's equations or that appear undigested in the literature. In any such case, the appropriate method often is: curvature first, understanding second.

Curvature is the simplest local measure of geometric properties (see Box 14.1). Curvature is therefore a good first step toward a more comprehensive picture of the spacetime in question.

One sometimes has an expression for a spacetime metric first, and then makes calculations of curvature to understand it. But more often one makes calculations of curvature, subject to specified conditions of symmetry in space and time, as an aid in arriving at an expression for a physically interesting metric (stars, Chapters 23 to 26; model cosmologies, Chapters 27 to 30; collapse and black holes, Chapters 31 to 34; and gravitational waves, Chapters 35 to 37).

"Standard procedure" for computing curvature

The basic "standard procedure for computing curvature" is illustrated in Box 14.2. Two formulas in Box 14.2, derived previously, are used in succession. The first (equations 1 and 2) has the form $\Gamma \sim g\,\partial g$ and provides the $\Gamma^{\mu}{}_{\alpha\beta}$. The other (equation 3) has the form $R \sim \partial\Gamma + \Gamma^2$ and gives the curvature components $R^{\mu}{}_{\nu\alpha\beta}$.

Methods of displaying curvature formulas

After the curvature components have been computed, there are helpful ways to present the results. (1) Form the Ricci tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ and the scalar curvature $R = R^{\mu}{}_{\mu}$. (2) Form other invariants such as $R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\mu\nu}$. (3) Form components $R^{\hat{\mu}\hat{\nu}}{}_{\hat{\alpha}\hat{\beta}}$ in a judiciously chosen orthonormal frame $\boldsymbol{\omega}^{\hat{\alpha}} = L^{\hat{\alpha}}{}_{\beta}\,\boldsymbol{d}x^{\beta}$, and (4) display $R^{[\hat{\mu}\hat{\nu}]}{}_{[\hat{\alpha}\hat{\beta}]}$ as a 6×6 matrix (in four dimensions; a 3×3 matrix in three dimensions) where $[\hat{\mu}\hat{\nu}] = [\hat{0}\hat{1}], [\hat{0}\hat{2}], [\hat{0}\hat{3}], [\hat{2}\hat{3}], [\hat{3}\hat{1}], [\hat{1}\hat{2}]$ labels the rows and $[\hat{\alpha}\hat{\beta}]$ labels the columns (exercises 14.14 and 14.15). (5) Last, but by far the most important for general relativity, form the Einstein tensor $G^{\hat{\mu}}{}_{\hat{\nu}}$ as described in §14.2.

The method of computation outlined above and described in more detail in Box 14.2 is used wherever it is quicker to employ a standard method than to learn or invent a better method. The standard method is always preferable for the student in a short course where physical insight has higher priority than technical facility.

Computation of curvature using a computer

It is, however, a dull method, better suited to computers than to people. Even the algebra can be handled by a computer (see Box 14.3).

EXERCISES

Exercise 14.1. CURVATURE OF A TWO-DIMENSIONAL HYPERBOLOID

Compute the curvature of the hyperboloid $t^2 - x^2 - y^2 = T^2 = \text{const}$ in 2 + 1 Minkowski spacetime with $ds_3{}^2 = -dt^2 + dx^2 + dy^2$. First show that intervals within this two-dimensional surface can be expressed in the form $ds^2 = T^2(d\alpha^2 + \sinh^2\alpha\, d\phi^2)$ by a suitable choice of coordinates α, ϕ, on the hyperboloid.

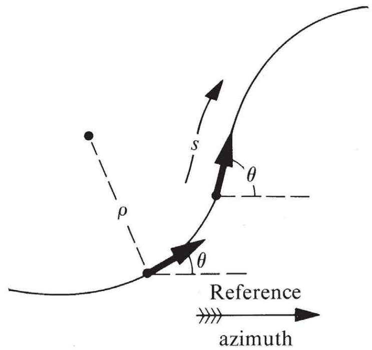
Exercise 14.2. RIEMANNIAN CURVATURE EXPRESSIBLE IN TERMS OF RICCI CURVATURE IN TWO AND THREE DIMENSIONS

In two dimensions, there is only one independent curvature component, R_{1212}. Evidently the single scalar quantity R must carry the same information. The two-dimensional identity $R_{\mu\nu\alpha\beta} = \frac{1}{2}R(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})$ is established by noting that it is the only tensor formula giving

(continued on page 343)

Box 14.1 PERSPECTIVES ON CURVATURE

1. Historical point of departure: a curved line on a plane. There is no way to define the curvature of a line by measurements confined to ("intrinsic to") the line itself. One needs, for example, the azimuthal bearing θ of the tangent vector relative to a fixed direction in the plane, as a function of proper distance s measured along the curve; thus, $\theta = \theta(s)$. Then curvature κ and its reciprocal, the radius of curvature ρ, are given by $\kappa(s) = 1/\rho(s) = d\theta(s)/ds$. Alternatively, one can examine departure, y, measured normally off from the tangent line as a function of distance x measured along that tangent line; then $\kappa = 1/\rho = d^2y/dx^2$.



2. This concept was later extended to a curved surface embedded in flat (Euclidean) 3-space. Departure, z, of the smooth curved surface from the flat surface tangent to it at a given point is described in the neighborhood of that point by the quadratic expression

$$z = \frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2.$$

Rotation of the axes by an appropriate angle α,

$$x = \xi \cos\alpha + \eta \sin\alpha,$$
$$y = -\xi \sin\alpha + \eta \cos\alpha,$$

reduces this expression to

$$z = \frac{1}{2}\kappa_1\xi^2 + \frac{1}{2}\kappa_2\eta^2,$$

with

$$\kappa_1 = 1/\rho_1$$

and

$$\kappa_2 = 1/\rho_2$$

representing the two "principal curvatures" of the surface.

3. Gauss (1827) conceived the idea of defining curvature by measurements confined entirely to the surface ("society of ants"). From a given point $\mathscr{P}$ on the surface, proceed on a geodesic on the surface for a proper distance ϵ measured entirely within the surface. Repeat, starting at the original point but proceeding in other directions.

Box 14.1 ***(continued)***

Obtain an infinity of points. They define a "circle". Determine its proper circumference, again by measurements confined entirely to the surface. Using the metric corresponding to the embedding viewpoint

$$\begin{aligned} ds^2 &= dz^2 + d\xi^2 + d\eta^2 \qquad \text{(Euclidean 3-space)} \\ &= [(\kappa_1\xi\, d\xi + \kappa_2\eta\, d\eta)^2 + (d\xi^2 + d\eta^2)] \qquad \begin{pmatrix}\text{metric intrinsic} \\ \text{to the curved} \\ \text{2-geometry}\end{pmatrix}, \end{aligned}$$

one can calculate the result of such an "intrinsic measurement." One calculates that the circumference differs from the Euclidean value, $2\pi\epsilon$, by a fractional correction that is proportional to the square of ϵ; specifically,

$$\lim_{\epsilon\to 0} \frac{6}{\epsilon^2}\left(1 - \frac{\text{circumference}}{2\pi\epsilon}\right) = \kappa_1\kappa_2 = \frac{1}{\rho_1\rho_2} = \det\begin{pmatrix} a & b \\ b & c\end{pmatrix}.$$

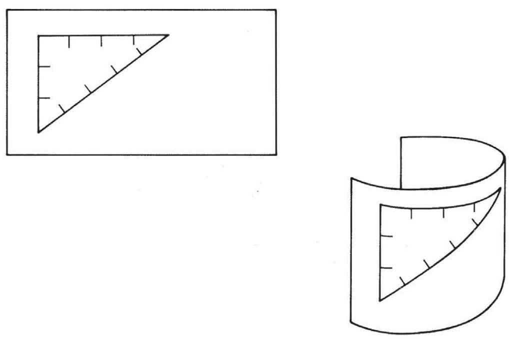
Note especially the first equality sign. Gauss did not conceal the elation he felt on discovering that something defined by measurements entirely within the surface agrees with the product of two quantities, κ_1 and κ_2, that individually demand for their definition measurements extrinsic to the surface.

4. The contrast between "extrinsic" and "intrinsic" curvature is summarized in the terms,

$$(\text{extrinsic curvature}) = \kappa = (\kappa_1 + \kappa_2)(\text{cm}^{-1}),$$

$$\begin{pmatrix}\text{intrinsic or Gaussian} \\ \text{curvature}\end{pmatrix} = \kappa_1\kappa_2(\text{cm}^{-2})$$

(the latter being identical with half the scalar curvature invariant, R, of the 2-geometry). Draw a 3 : 4 : 5 triangle on a flat piece of paper; then curl up the paper. The Euclidean 2-geometry intrinsic to the piece of paper is preserved by this bending. The Gaussian curvature intrinsic to the surface remains unaltered; it keeps the Euclidean value of zero (κ_2, non-zero; κ_1, zero; product, $\kappa_1\kappa_2 =$ zero). However, the extrinsic curvature is changed from $\kappa_1 + \kappa_2 = 0$ to a non-zero value, $\kappa_1 + \kappa_2 \neq 0$.

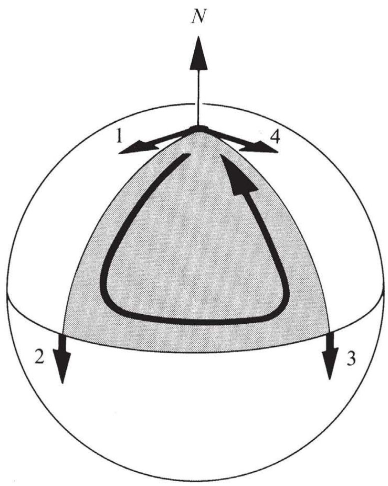


5. The curvature dealt with in this chapter is curvature intrinsic to spacetime; that is, curvature defined without any use of, and repelling every thought of, any embedding in any hypothetical higher-dimensional flat manifold (concept of Riemann,

Clifford, and Einstein that geometry is a dynamic participant in physics, not some God-given perfection above the battles of matter and energy).

6. The curvature of the geometry of spacetime imposes curvature on any spacelike slice (3-geometry; "initial-value hypersurface") through that spacetime (see "relations of Gauss and Codazzi" in Chapter 21, on the initial-value problem of geometrodynamics).

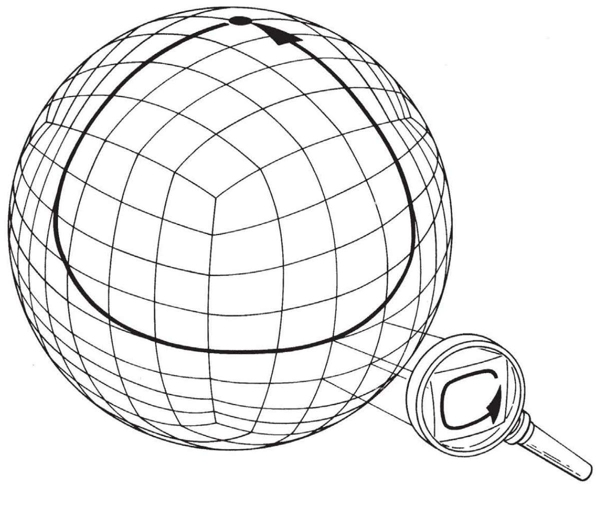
7. Rotation of a vector transported parallel to itself around a closed loop provides a definition of curvature as useful in four and three as in two dimensions. (In a curved two-dimensional geometry, at a point there is only one plane. Consequently only one number is required to describe the Gaussian curvature there. In three and four dimensions, there are more independent planes through a point and therefore more numbers are required to describe the curvature.) In the diagram, start with a vector at position 1 (North Pole). Transport it parallel to itself (position $2, 3, \ldots$) around a $90°$–$90°$–$90°$ spherical triangle. It arrives back at the starting point (position 4) turned through $90°$:



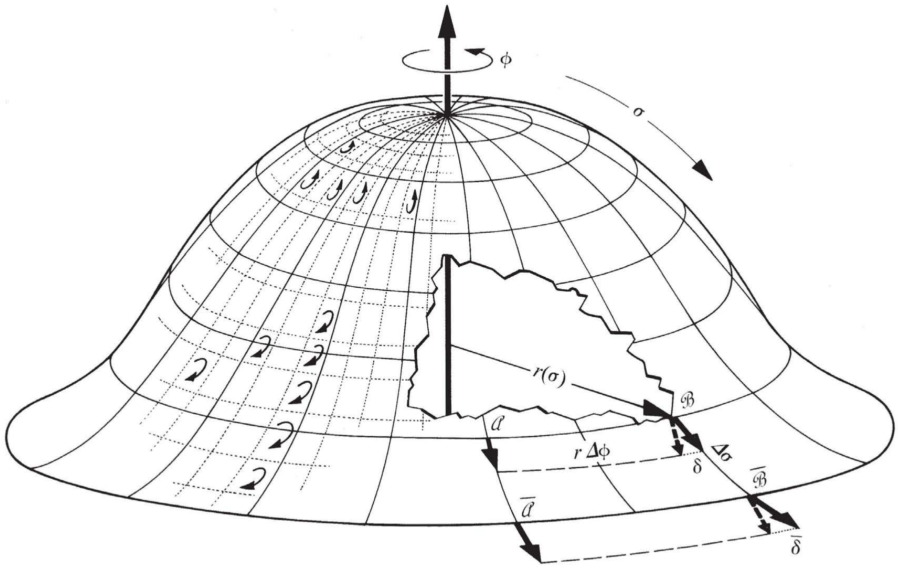
$$\begin{pmatrix}\text{Gaussian}\\ \text{curvature}\end{pmatrix} = \frac{\begin{pmatrix}\text{angle turned}\\ \text{through}\end{pmatrix}}{\begin{pmatrix}\text{area circum-}\\ \text{navigated}\end{pmatrix}} = \frac{(\pi/2)}{(1/8)(4\pi a^2)} = \frac{1}{a^2}$$

(positive; sense of rotation same as sense of circumnavigation).

8. Still staying for simplicity with a curved two-dimensional manifold, describe the curvature of the 2-surface as a 2-form ("box-like structure") defined over the entire surface. The number of boxes enclosed by any given route gives immediately the angle in radians (or tenths or hundredths of a radian, etc., depending on chosen fineness of subdivision) turned through by a vector carried parallel to itself around that route. The contribution of a given box is counted as positive or negative depending on whether the sense of the arrow marked on it (see magnified view) agrees or disagrees with the sense of circumnavigation of the route.



Box 14.1 (*continued*)



9. Curvature 2-form for the illustrated surface of rotational symmetry ("pith helmet") with metric $ds^2 = d\sigma^2 + r^2(\sigma)\,d\phi^2$ is

$$\textbf{\textit{curvature}} = -\frac{1}{r}\frac{d^2r}{d\sigma^2}\,\mathbf{d}\sigma \wedge r\,\mathbf{d}\phi \tag{1}$$

(positive on crown of helmet, negative around brim, as indicated by sense of arrows in the "boxes of the 2-form" shown at left). "Meaning" of r is illustrated by imbedding the surface in Euclidean 3-space, a convenience for visualization; but more important is the idea of a 2-geometry defined by measurements intrinsic to it, with no embedding.

10. How lengths ("metric") determine curvature in quantitative detail is shown nowhere more clearly than in this two-dimensional example, a model for "what is going on behind the scene" in the mathematical calculations done in this chapter with 1-forms and 2-forms in four-dimensional spacetime.

a. Net rotation in going around element of surface $\mathcal{A}\mathcal{B}\bar{\mathcal{B}}\bar{\mathcal{A}}\mathcal{A}$ is $\delta - \bar{\delta}$ (no turn of vector to left or to right in its transport along a meridian $\mathcal{A}\bar{\mathcal{A}}$ or $\mathcal{B}\bar{\mathcal{B}}$).

b. Rotation of vector in going from $\mathcal{A}$ to $\mathcal{B}$, relative to coordinate system (directions of meridians), is

$$(\text{angle } \delta) = \frac{\text{arc}}{\text{length}} = \frac{r(\sigma + d\sigma)\,\Delta\phi - r(\sigma)\,\Delta\phi}{d\sigma} = \left(\frac{dr}{d\sigma}\right)_\sigma \Delta\phi.$$

c. Rotation of vector in going from $\bar{\mathscr{A}}$ to $\bar{\mathscr{B}}$ is similarly

$$(\text{angle } \bar{\delta}) = \left(\frac{dr}{d\sigma}\right)_{\sigma + \Delta\sigma} \Delta\phi.$$

d. Thus net rotation is:

$$\delta - \bar{\delta} = -\left(\frac{d^2r}{d\sigma^2}\right)_{\sigma} \Delta\sigma\, \Delta\phi.$$

e. Expressed as a form, this gives immediately equation (1).

f. Ideas and calculations are more complicated in four dimensions, primarily because one has to deal with different choices for the orientation of the surface to be studied at the point in question.

11. Translation of these geometric ideas into the language of forms is most immediate when one stays with this example of two dimensions. A sample vector $A^i = (A^1, A^2)$ carried around the boundary of an element of surface comes back to its starting point slightly changed in direction:

$$-\begin{pmatrix}\text{change} \\ \text{in } A^i\end{pmatrix} = \mathscr{R}^i{}_j A^j. \tag{2}$$

a. To be more specific, it is convenient to adopt as the basis 1-forms $\boldsymbol{\omega}^{\hat{1}} = \boldsymbol{d}\sigma$ and $\boldsymbol{\omega}^{\hat{2}} = r\,\boldsymbol{d}\phi$, and have $A^{\hat{1}}$ as the component of $\boldsymbol{A}$ along the direction of increasing σ, $A^{\hat{2}}$ as the component of $\boldsymbol{A}$ along the direction of increasing ϕ. The matrix $\mathscr{R}^{\hat{i}}{}_{\hat{j}}$ is a rotation matrix, which produces a change in direction but no change in length (zero diagonal components); thus here

$$||\mathscr{R}^{\hat{i}}{}_{\hat{j}}|| = \begin{Vmatrix} 0 & \mathscr{R}^{\hat{1}}{}_{\hat{2}} \\ -\mathscr{R}^{\hat{1}}{}_{\hat{2}} & 0 \end{Vmatrix}. \tag{3}$$

In this example, $\mathscr{R}^{\hat{1}}{}_{\hat{2}}$ evidently represents the angle through which the vector $\boldsymbol{A}$ turns on transport parallel to itself around the element of surface.

b. So far the rotation is "indefinite" because the size of the element of surface has not yet been specified. It is most conveniently conceived as an elementary parallelogram, defined by two vectors ("bivector"). Thus $\mathscr{R}^{\hat{i}}{}_{\hat{j}}$, or, specifically, the one element that counts, $\mathscr{R}^{\hat{1}}{}_{\hat{2}}$ (the "angle of rotation"), has to be envisaged as a mathematical object ("2-form") endowed with two slots, into which these two vectors are inserted to get a definite number (angle in radians). In the example of the pith helmet, one has, from equation (1)

$$\mathscr{R}^{\hat{1}}{}_{\hat{2}} = -\frac{1}{r}\frac{d^2r}{d\sigma^2}\,\boldsymbol{\omega}^{\hat{1}} \wedge \boldsymbol{\omega}^{\hat{2}}. \tag{4}$$

Thus the $\mathscr{R}^{\mu}{}_{\nu}$ in the text are called "curvature 2-forms."

Box 14.1 *(continued)*

c. The text tells one how to read out of such expressions the components of the Riemann curvature tensor; for example here,

$$R^{\hat{1}}{}_{\hat{2}\hat{1}\hat{2}} = -R^{\hat{1}}{}_{\hat{2}\hat{2}\hat{1}} = (-1/r)(d^2r/d\sigma^2) \text{ (coefficients of } \boldsymbol{\omega}^{\hat{1}} \wedge \boldsymbol{\omega}^{\hat{2}} \text{ or } \boldsymbol{\omega}^{\hat{2}} \wedge \boldsymbol{\omega}^{\hat{1}}).$$

d. Generalizing to four dimensions, one understands by $R^{\alpha}{}_{\beta\mu\nu}$ the factor that one has to multiply by three numbers to obtain a fourth. The number obtained is the change (with reversed sign) that takes place in the αth component of a vector when that vector is transported parallel to itself around a closed path, defined, for example, by a parallelogram built from two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. The factors that multiply $R^{\alpha}{}_{\beta\mu\nu}$ are (1) the component of the vector $\boldsymbol{A}$ in the βth direction and (2, 3) the $\mu\nu$ component of the extension of the parallelogram, $(u^{\mu}v^{\nu} - u^{\nu}v^{\mu})$. Thus

$$\delta A^{\alpha} = -R^{\alpha}{}_{\beta|\mu\nu|}A^{\beta}(u^{\mu}v^{\nu} - u^{\nu}v^{\mu}).$$

Box 14.2 STRAIGHTFORWARD CURVATURE COMPUTATION (Illustrated for a Globe)

The elementary and universally applicable method for computing the components $R^{\mu}{}_{\nu\alpha\beta}$ of the Riemann curvature tensor starts from the metric components $g_{\mu\nu}$ in a coordinate basis, and proceeds by the following scheme:

$$g_{\mu\nu} \xrightarrow{\Gamma \,\dot{\sim}\, \partial g} \Gamma_{\mu\alpha\beta} \longrightarrow \Gamma^{\mu}{}_{\alpha\beta} \xrightarrow{R \sim \partial\Gamma + \Gamma\Gamma} R^{\mu}{}_{\nu\alpha\beta}.$$

The formulas required for these three steps are

$$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\beta}} + \frac{\partial g_{\mu\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}\right), \tag{1}$$

$$\Gamma^{\mu}{}_{\alpha\beta} = g^{\mu\nu}\Gamma_{\nu\alpha\beta}, \tag{2}$$

and

$$R^{\mu}{}_{\nu\alpha\beta} = \frac{\partial \Gamma^{\mu}{}_{\nu\beta}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\mu}{}_{\nu\alpha}}{\partial x^{\beta}} + \Gamma^{\mu}{}_{\rho\alpha}\Gamma^{\rho}{}_{\nu\beta} - \Gamma^{\mu}{}_{\rho\beta}\Gamma^{\rho}{}_{\nu\alpha}. \tag{3}$$

The metric of the two-dimensional surface of a sphere of radius a is

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{4}$$

To compute the curvature by the standard method, use the formula for ds^2 as a table of $g_{k\ell}$ values. It shows that $g_{\theta\theta} = a^2$, $g_{\theta\phi} = 0$, $g_{\phi\phi} = a^2\sin^2\theta$. Compute the six possible different $\Gamma_{jk\ell} = \Gamma_{j\ell k}$ (there will be 40 in four dimensions) from formula

(1). Thus

$$\begin{aligned}\Gamma_{\theta\phi\phi} &= -a^2 \sin\theta\cos\theta = -\Gamma_{\phi\phi\theta},\\ \Gamma_{\theta\theta\theta} &= \Gamma_{\phi\phi\phi} = 0,\\ \Gamma_{\theta\theta\phi} &= \Gamma_{\phi\theta\theta} = 0.\end{aligned} \tag{5}$$

Raise the first index:

$$\begin{aligned}\Gamma^{\theta}{}_{\phi\phi} &= -\sin\theta\cos\theta,\\ \Gamma^{\phi}{}_{\phi\theta} &= \cot\theta,\\ \Gamma^{\theta}{}_{\theta\theta} &= \Gamma^{\theta}{}_{\theta\phi} = 0 = \Gamma^{\phi}{}_{\theta\theta} = \Gamma^{\phi}{}_{\phi\phi}.\end{aligned} \tag{6}$$

Choose a suitable curvature component (one that is not automatically zero by reason of the elementary symmetry $R_{\mu\nu\alpha\beta} = R_{[\mu\nu][\alpha\beta]}$, nor previously computed in another form, as by $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$). In this two-dimensional example, there is only one choice (compared to 21 such computations in four dimensions); it is

$$\begin{aligned}R^{\theta}{}_{\phi\theta\phi} &= \frac{\partial \Gamma^{\theta}{}_{\phi\phi}}{\partial\theta} - \frac{\partial \Gamma^{\theta}{}_{\phi\theta}}{\partial\phi} + \Gamma^{\theta}{}_{k\theta}\Gamma^{k}{}_{\phi\phi} - \Gamma^{\theta}{}_{k\phi}\Gamma^{k}{}_{\phi\theta}\\ &= \frac{\partial \Gamma^{\theta}{}_{\phi\phi}}{\partial\theta} - 0 + 0 - \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\phi\theta}\\ &= \sin^2\theta - \cos^2\theta + \sin\theta\cos\theta\cot\theta;\end{aligned}$$

so

$$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta \tag{7}$$

or

$$R^{\theta\phi}{}_{\theta\phi} = \frac{1}{a^2}. \tag{8}$$

Contraction gives the components of the Ricci tensor,

$$R^{\theta}{}_{\theta} = R^{\phi}{}_{\phi} = \frac{1}{a^2}, \qquad R^{\theta}{}_{\phi} = 0, \tag{9}$$

and further contraction gives the curvature scalar

$$R = 2/a^2. \tag{10}$$

A convenient orthonormal frame in this manifold is

$$\boldsymbol{\omega}^{\hat\theta} = a\,\boldsymbol{d}\theta, \qquad \boldsymbol{\omega}^{\hat\phi} = a\sin\theta\,\boldsymbol{d}\phi. \tag{11}$$

More generally one writes $\boldsymbol{\omega}^{\hat\alpha} = L^{\hat\alpha}{}_{\beta}\,\boldsymbol{d}x^{\beta}$. To transform the curvature tensor to orthonormal components in this simple but illuminating example of a diagonal metric requires a single normalization factor for each index on a tensor. Thus $v^{\hat\theta} = av^{\theta}$, $v^{\hat\phi} = a\sin\theta\, v^{\phi}$, $v_{\hat\theta} = a^{-1}v_{\theta}$, $v_{\hat\phi} = (a\sin\theta)^{-1}v_{\phi}$. Similarly, from $R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta$ one finds the components of the curvature tensor,

$$R^{\hat\theta}{}_{\hat\phi\hat\theta\hat\phi} = \frac{1}{a^2} = R^{\hat\theta\hat\phi}{}_{\hat\theta\hat\phi}, \tag{12}$$

in the orthonormal frame.

Box 14.3 ANALYTICAL CALCULATIONS ON A COMPUTER

Research in gravitation physics and general relativity is sometimes beset by long calculations, requiring meticulous care, of such quantities as the Einstein and Riemann curvature tensors for a given metric, or the divergence of a given stress-energy tensor, or the Newman-Penrose tetrad equations under given algebraic assumptions. Such calculations are sufficiently straightforward and deductive in logical structure that they can be handled by a computer. Since 1966, computers have been generally taking over such tasks.

There are several computer languages in which the investigator can program his analytic calculations. The computer expert may wish to work in a machine-oriented language such as LISP [see, e.g., the work of Fletcher (1966) and of Hearn (1970)]. However, most appliers of relativity will prefer user-oriented languages such as REDUCE [created by Hearn (1970) and available for the IBM 360 and 370, and the PDP 10 computers], ALAM [created by D'Inverno (1969) and available on Atlas computers], CAMAL [created by Barton, Bourne, and Fitch (1970) and available on Atlas computers], and FORMAC [created by Tobey *et al.* (1967) and available on IBM 7090, 7094, 360, and 370]. For a review of activity in this area, see Barton and Fitch (1971). Here we describe only FORMAC. It is the most widely available and widely used of the languages; but it is probably *not* the most powerful [see, e.g., D'Inverno (1969)]. FORMAC is to analytic work what the earliest and most primitive versions of FORTRAN were to numerical work.

FORMAC manipulates algebraic expressions involving: numerical constants, such as 1/3; symbolic constants, such as x or u; specific elementary functions, such as $\sin(u)$ or $\exp(x)$; and symbolic functions of several variables, such as $f(x, u)$ or $g(u)$. For example, it can add $ax + bx^2$ to $2x + (3 + b)x^2$ and get $(a + 2)x + (3 + 2b)x^2$; it can take the partial derivative of $x^2uf(x, u) + \cos(x)$ with respect to x and get

$$2xuf(x, u) + x^2u\, \partial f(x, u)/\partial x - \sin(x).$$

It can do any algebraic or differential-calculus computation that a human can do—but without making mistakes! Unfortunately, it cannot integrate analytically; integration requires inductive logic rather than deductive logic.

PL/1 is a language that can be used simultaneously with FORMAC or independently of it. PL/1 manipulates strings of characters—e.g., "Z/1×29−+/." It knows symbolic logic; it can tell whether two strings are identical; it can insert new characters into a string or remove old ones; but it does not know the rules of algebra or differential calculus. Thus, its primary use is as an adjunct to FORMAC (though from the viewpoint of the computer system FORMAC is an adjunct of PL/1).

FORMAC programs for evaluating Einstein's tensor in terms of given metric components and for doing other calculations are available from many past users [see, e.g., Fletcher, Clemens, Matzner, Thorne, and Zimmerman (1967); Ernst (1968); Harrison (1970)]. However, programming in FORMAC is sufficiently simple that one ordinarily does not have difficulty creating one's own program to do a given task. If a difficulty does arise, it may be because the analytic computation exhausts the core of the computer. It is easy to create an expression too large to fit in the core of any existing computer by several differentiations of an expression half a page long!

Users of FORMAC, confronted by core-exhaustion, have devised several ways to solve their problems. One is to remove unneeded parts of the program and of the FORMAC system from the core. Routines called PURGE and KILL have been developed for this purpose by Clemens and Matzner (1967). Another is to create the answer to a given calculation in manageable-sized pieces and output those pieces from the computer's core onto its disk. One must then add all the pieces together—a task that is impossible using FORMAC alone, or even FORMAC plus PL/1, but a task that James Hartle has solved [see Hartle and Thorne (1974)] by using a combination of FORMAC, PL/1, and IBM data-manipulation routines called SORT.

$R_{\mu\nu\alpha\beta}$ as a linear function of R, constructed from R and the metric alone, and with the correct contracted value $R^{\mu\nu}{}_{\mu\nu} = R$. Establish a corresponding three-dimensional identity expressing $R_{ijk\ell}$ in terms of the Ricci tensor R_{jk} and the metric.

Exercise 14.3. CURVATURE OF 3-SPHERE IN ORTHONORMAL FRAME

Compute the curvature tensor for a 3-sphere

$$ds^2 = a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{14.2}$$

or for a 3-hyperboloid

$$ds^2 = a^2[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)]. \tag{14.3}$$

Convert the coordinate-based components $R^i{}_{jk\ell}$ to a corresponding orthonormal basis, $R^{\hat{i}}{}_{\hat{j}\hat{k}\hat{\ell}}$. Display $R^{\hat{i}\hat{j}}{}_{\hat{k}\hat{\ell}} = R^{[\hat{i}\hat{j}]}{}_{[\hat{k}\hat{\ell}]}$ as a 3×3 matrix with appropriately labeled rows and columns.

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§14.2. FORMING THE EINSTEIN TENSOR

Three ways to compute the Einstein tensor from the Riemann tensor

The distribution of matter in space does not immediately tell all details of the local curvature of space, according to Einstein. The stress-energy tensor provides information only about a certain combination of components of the Riemann curvature tensor, the combination that makes up the Einstein tensor. Chapter 13 described two equivalent ways to calculate the Einstein tensor: (1) by successive contractions of the Riemann tensor

$$\begin{aligned} R_{\mu\nu} &= R^{\alpha}{}_{\mu\alpha\nu}, \qquad R = g^{\mu\nu}R_{\mu\nu}, \\ G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \end{aligned} \tag{14.4}$$

[equations (13.48) and (13.49)]; (2) by forming the dual of the Riemann tensor and then contracting:

$$\begin{aligned} \mathcal{G}_{\alpha\beta}{}^{\gamma\delta} \equiv ({}^*R^*)_{\alpha\beta}{}^{\gamma\delta} &= \epsilon_{\alpha\beta\mu\nu}R^{|\mu\nu|}{}_{|\rho\sigma|}\epsilon^{\rho\sigma\gamma\delta} \\ &= -\delta^{\rho\sigma\gamma\delta}{}_{\alpha\beta\mu\nu}R^{|\mu\nu|}{}_{|\rho\sigma|}, \end{aligned} \tag{14.5a}$$

$$G_\beta{}^\delta = \mathcal{G}_{\alpha\beta}{}^{\alpha\delta} \tag{14.5b}$$

[equations (13.46) and (13.47)]. A third method, usually superior to either of these, is discovered by combining equations (14.5a,b):

$$G_\beta{}^\delta = G^\delta{}_\beta = -\delta^{\delta\rho\sigma}{}_{\beta\mu\nu}R^{|\mu\nu|}{}_{|\rho\sigma|}. \tag{14.6}$$

[Note: in any frame, orthonormal or not, the permutation tensor $\delta^{\delta\rho\sigma}{}_{\beta\mu\nu}$ has components

$$\delta^{\delta\rho\sigma}{}_{\beta\mu\nu} = \delta_{\beta\mu\nu}{}^{\delta\rho\sigma} = \begin{cases} +1 \text{ if } \delta\rho\sigma \text{ is an even permutation of } \beta\mu\nu, \\ -1 \text{ if } \delta\rho\sigma \text{ is an odd permutation of } \beta\mu\nu, \\ 0 \text{ otherwise;} \end{cases}$$

to see this, simply evaluate $\delta^{\delta\rho\sigma}{}_{\beta\mu\nu}$ using definition (3.50h) and using the components (8.10) of $\epsilon_{\alpha\beta\mu\nu}$ and $\epsilon^{\rho\sigma\gamma\delta}$.] Equation (14.6) for the Einstein tensor, written out explicitly, reads

$$\begin{aligned} G^0{}_0 &= -(R^{12}{}_{12} + R^{23}{}_{23} + R^{31}{}_{31}), \\ G^1{}_1 &= -(R^{02}{}_{02} + R^{03}{}_{03} + R^{23}{}_{23}), \\ G^0{}_1 &= R^{02}{}_{12} + R^{03}{}_{13}, \\ G^1{}_2 &= R^{10}{}_{20} + R^{13}{}_{23}, \end{aligned} \tag{14.7}$$

and every other component is given by a similar formula, obtainable by obvious permutations of indices.

§14.3. MORE EFFICIENT COMPUTATION

Standard method of computing curvature is wasteful

If the answer to a problem or the result of a computation is not simple, then there is no simple way to obtain it. But when a long computation gives a short answer, *then* one looks for a better method. Many of the best-known applications of general relativity present one with metric forms in which many of the components $g_{\mu\nu}$, $\Gamma^\mu{}_{\alpha\beta}$, and $R^\mu{}_{\nu\alpha\beta}$ are zero; for them the standard computation of the curvature (Box 14.2) involves much "wasted" effort. One computes many $\Gamma^\mu{}_{\alpha\beta}$ that turn out to be zero. One checks off many terms in a sum like $-\Gamma^\mu{}_{\rho\beta}\Gamma^\rho{}_{\alpha\mu}$ that are zero, or cancel with others to give zero. Two alternative procedures are available to eliminate some of this "waste." The "geodesic Lagrangian" method provides an economical way to tabulate the $\Gamma^\mu{}_{\alpha\beta}$. The method of "curvature 2-forms" reorganizes the description from beginning to end, and computes both the connection and the curvature.

Ways to avoid "waste":

(1) geodesic Lagrangian method

(2) method of curvature 2-forms

The geodesic Lagrangian method is only a moderate improvement over the standard method, but it also demands only a modest investment in the calculus of variations, an investment that pays off in any case in other contexts in the world of mathematics and physics. In contrast, the method of curvature 2-forms is efficient, but demands a heavier investment in the mathematics of 1-forms and 2-forms than anyone would normally find needful for any introductory survey of relativity. Anyone facing several days' work at computing curvatures, however, would do well to learn the algorithm of the curvature 2-forms.

§14.4. THE GEODESIC LAGRANGIAN METHOD

One normally thinks that the connection coefficients $\Gamma^\mu{}_{\alpha\beta}$ must be known before one can write the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0. \tag{14.8}$$

(Here and below dots denote derivative with respect to the affine parameter, λ.) However, the argument can be reversed. Once the geodesic equations have been written down, the connection coefficients can be read out of them. For instance, on the 2-sphere as treated in Box 14.2, the geodesic equations are

$$\ddot{\theta} - \sin\theta \cos\theta\, \dot{\phi}^2 = 0, \tag{14.9θ}$$

$$\ddot{\phi} + 2\cot\theta\, \dot{\phi}\dot{\theta} = 0. \tag{14.9ϕ}$$

The first equation here shows that $\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$; the second equation shows that $\Gamma^{\phi}{}_{\phi\theta} = \Gamma^{\phi}{}_{\theta\phi} = \cot\theta$; and the absence of any further terms shows that all other $\Gamma^{i}{}_{jk}$ are zero.

The first essential principle is thus clear: an explicit writing out of the geodesic equation is equivalent to a tabulation of all the connection coefficients $\Gamma^{\mu}{}_{\alpha\beta}$.

The second principle says more: one can write out the geodesic equation without ever having computed the $\Gamma^{\mu}{}_{\alpha\beta}$. In order to arrive at the equations for a geodesic (see Box 13.3), one need only recall that a geodesic is a parametrized curve that extremizes the integral

$$I = \frac{1}{2}\int g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\, d\lambda \tag{14.10}$$

in the sense

$$\delta I = 0.$$

Geodesic Lagrangian method in 4 steps:

(1) write I in simple form

In practical applications of this variational principle, *the first step is to rewrite equation (14.10) in the simplest possible form, inserting the specific values of $g_{\mu\nu}$ for the problem at hand.* If one's interest attaches to the geodesics themselves, one can recognize many constants of motion even without carrying out any variations (see Chapter 25 on geodesic motion in Schwarzschild geometry, especially §25.2 on conservation laws and constants of motion). For the purpose of computing the $\Gamma^{\mu}{}_{\alpha\beta}$, one proceeds to vary each coordinate in turn, obtaining four equations. Next these equations are rearranged so that their leading terms are $\ddot{x}^{\mu}$. In this form they must be precisely the geodesic equations (14.8). Consequently, the $\Gamma^{\mu}{}_{\alpha\beta}$ are immediately available as the coefficients in these four equations. For the final step in computing curvature by this method, one returns to the standard method and to formulas of the type $R \sim \partial\Gamma + \Gamma\Gamma$, treated in the standard way (Box 14.2); and as the need arises for each Γ in turn, one scans the geodesic equation to find it. The procedure is best understood by following an example: Box 14.4 provides one.

(2) vary I to get geodesic equation

(3) read off $\Gamma^{\alpha}{}_{\beta\gamma}$

(4) compute $R^{\alpha}{}_{\beta\gamma\delta}$ etc. by standard method

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EXERCISE

Exercise 14.4. EINSTEIN EQUATIONS FOR THE CLOSED FRIEDMANN UNIVERSE CALCULATED BY USING THE GEODESIC LAGRANGIAN METHOD

The line element of interest here is (see Chapter 27)

$$ds^2 = -dt^2 + a^2(t)\,[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)].$$

(continued on page 348)

Box 14.4 GEODESIC LAGRANGIAN METHOD SHORTENS SOME CURVATURE COMPUTATIONS

Aim: Compute the curvature for the line element

$$ds^2 = L^2(e^{2\beta}\, dx^2 + e^{-2\beta}\, dy^2) - 2\, du\, dv \tag{1}$$

where L and β are functions of u only. [This metric is discussed as an example of a gravitational wave in §§35.9–35.12.]

Method: Obtain the $\Gamma^{\mu}{}_{\alpha\beta}$ from the geodesic equations as inferred from the variational principle (14.10), then compute $R^{\mu}{}_{\nu\alpha\beta} \sim \partial\Gamma + \Gamma^2$ as in Box 14.2.

Step 1. State the variational integral. For the metric under consideration, equation (14.10) requires $\delta I = 0$ for

$$I = \int \left[\frac{1}{2} L^2(e^{2\beta}\dot{x}^2 + e^{-2\beta}\dot{y}^2) - \dot{u}\dot{v}\right] d\lambda. \tag{2}$$

A world line that extremizes this integral is a geodesic.

Step 2: Vary the coordinates of the world line, one at a time, in their dependence on λ. First vary $x(\lambda)$, keeping fixed the functions $y(\lambda)$, $u(\lambda)$, and $v(\lambda)$. Then

$$\delta I = \int (L^2 e^{2\beta}\dot{x})\, \delta\dot{x}\, d\lambda = -\int (L^2 e^{2\beta}\dot{x})^{\cdot}\, \delta x\, d\lambda.$$

The requirement that $\delta I = 0$ for this variation (among others) gives

$$0 = (L^2 e^{2\beta}\dot{x})^{\cdot} = L^2 e^{2\beta}\ddot{x} + \dot{x}\dot{u}\frac{\partial}{\partial u}(L^2 e^{2\beta}).$$

Varying y, u, v, in the same way gives

$$0 = (L^2 e^{-2\beta}\dot{y})^{\cdot} = L^2 e^{-2\beta}\ddot{y} + \dot{y}\dot{u}\frac{\partial}{\partial u}(L^2 e^{-2\beta}),$$

$$0 = \ddot{v} + \frac{1}{2}\dot{x}^2\frac{\partial}{\partial u}(L^2 e^{2\beta}) + \frac{1}{2}\dot{y}^2\frac{\partial}{\partial u}(L^2 e^{-2\beta}),$$

$$0 = \ddot{u}.$$

Step 3: Rearrange to get $\ddot{x}^{\mu}$ leading terms. If this step is not straightforward, this method will not save time, and the technique of either Box 14.2 or Box 14.5 will be more suitable. In the example here, one quickly writes, using a prime for $\partial/\partial u$,

$$0 = \ddot{x} + 2(L^{-1}L' + \beta')\dot{x}\dot{u}, \tag{3x}$$

$$0 = \ddot{y} + 2(L^{-1}L' - \beta')\dot{y}\dot{u}, \tag{3y}$$

$$0 = \ddot{v} + (L^2 e^{2\beta})(L^{-1}L' + \beta')\dot{x}^2 + (L^2 e^{-2\beta})(L^{-1}L' - \beta')\dot{y}^2, \tag{3v}$$

$$0 = \ddot{u}. \tag{3u}$$

Step 3′: Interpret these equations as a tabulation of $\Gamma^{\mu}{}_{\alpha\beta}$. Equations (3) are the standard equations for a geodesic,

$$\ddot{x}^{\mu} + \Gamma^{\mu}{}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = 0.$$

Therefore it is enough to scan them to find the value of any desired Γ. For instance $\Gamma^{x}{}_{yu}$ must appear in the coefficient $(\Gamma^{x}{}_{yu} + \Gamma^{x}{}_{uy}) = 2\Gamma^{x}{}_{yu}$ of the $\dot{y}\dot{u}$ term in the equation for $\ddot{x}$. But no $\dot{y}\dot{u}$ term appears in equation (3x). Therefore $\Gamma^{x}{}_{yu}$ is zero in this example. Note that equations (3) are simple, in the sense that they contain few terms; therefore most of the $\Gamma^{\mu}{}_{\alpha\beta}$ must be zero. For instance, it follows from equation (3u) that all ten $\Gamma^{u}{}_{\alpha\beta}$ are zero. The only non-zero Γ's are $\Gamma^{x}{}_{xu} = \Gamma^{x}{}_{ux} = (L^{-1}L' + \beta')$ from equation (3x), $\Gamma^{y}{}_{yu} = \Gamma^{y}{}_{uy} = (L^{-1}L' - \beta')$ from equation (3y), and $\Gamma^{v}{}_{xx}$ and $\Gamma^{v}{}_{yy}$ from equation (3v).

Step 4: Compute each $R^{\mu}{}_{\nu\alpha\beta}$, etc. There is little relief from routine in systematically applying equation (3) from Box 14.2. One must list 21 components $R^{\mu}{}_{\nu\alpha\beta}$ that are not related by any of the symmetries $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu} = -R_{\mu\nu\beta\alpha}$, and compute each. In the example here, one notes that $\Gamma^{u}{}_{\alpha\beta} = 0$ implies $R^{u}{}_{\alpha\beta\gamma} = -R_{v\alpha\beta\gamma} = 0$. Therefore 15 of the list of 21 vanish at one swat. The list then is:

$$\begin{aligned}
R_{v\alpha\beta\gamma} &= -R^{u}{}_{\alpha\beta\gamma} = 0,\\
R_{uxux} &= -R^{v}{}_{xux} = -(\Gamma^{v}{}_{xx})' + \Gamma^{v}{}_{xx}\Gamma^{x}{}_{xu}\\
&\qquad\qquad = -(L^{2}e^{2\beta})\left(\frac{L''}{L} + \beta'' + 2\frac{L'}{L}\beta' + \beta'^{2}\right),\\
R_{uxxy} &= -R^{v}{}_{xxy} = 0,\\
R_{uxyu} &= -R^{v}{}_{xyu} = 0,\\
R_{uyuy} &= -R^{v}{}_{yuy} = -(\Gamma^{v}{}_{yy})' + \Gamma^{v}{}_{yy}\Gamma^{y}{}_{yu}\\
&\qquad\qquad = -(L^{2}e^{-2\beta})\left(\frac{L''}{L} - \beta'' - 2\frac{L'}{L}\beta' + \beta'^{2}\right),\\
R_{uyxy} &= -R^{v}{}_{yxy} = 0,\\
R_{xyxy} &= (L^{2}e^{2\beta})R^{x}{}_{yxy} = 0.
\end{aligned} \tag{4}$$

One can now calculate the Einstein tensor via equation (14.7). In the example here, however, it is equally simple to form first the Ricci tensor by the straightforward contraction $R^{\mu}{}_{\alpha\mu\beta}$. Only $\mu = x$ and $\mu = y$ give any contribution, because no superscript index can be a u, and no subscript a v. Thus one finds

$$\begin{gathered}
R_{uu} = -2[L^{-1}L'' + \beta'^{2}],\\
\text{all other } R_{\alpha\beta} = 0,
\end{gathered} \tag{5}$$

and

$$R = 0. \tag{6}$$

From this last result, it follows that here the desired Einstein tensor is identical with the Ricci tensor.

(a) Set up the variational integral (14.10) for a geodesic in this metric, then successively vary t, χ, θ, and ϕ to obtain, after some rearrangement, four equations $0 = \ddot{t} + \cdots$, $0 = \ddot{\chi} + \cdots$, etc. displaying the Γ's in the form of equation (14.8).

(b) Use this display as a table of Γ's to compute $R^{t}{}_{\chi\mu\nu}$ and $R^{\chi}{}_{\theta\mu\nu}$, of which only $R^{t}{}_{\chi t\chi}$ and $R^{\chi}{}_{\theta\chi\theta}$ are non-zero (consequence of the complete equivalence of all directions tangent to the $\chi\theta\phi$ sphere).

(c) Convert to an orthonormal frame with $\boldsymbol{\omega}^{\hat{t}} = \boldsymbol{d}t$, $\boldsymbol{\omega}^{\hat{\chi}} = a\,\boldsymbol{d}\chi$, $\boldsymbol{\omega}^{\hat{\theta}} = ?$, $\boldsymbol{\omega}^{\hat{\phi}} = ?$, and list $R^{\hat{t}\hat{\chi}}{}_{\hat{t}\hat{\chi}}$ and $R^{\hat{\chi}\hat{\theta}}{}_{\hat{\chi}\hat{\theta}}$. Explain why all other components are known by symmetry in terms of these two.

(d) Calculate, using equations (14.7), all independent components of the Einstein tensor $G^{\hat{\mu}}{}_{\hat{\nu}}$. [*Answer:* See Box 14.5.]

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§14.5. CURVATURE 2-FORMS

In electrodynamics the abstract notation

$$\boldsymbol{F} = \boldsymbol{dA}$$

saves space compared to the explicit notation

$$F_{31} = \frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1},$$

$$F_{12} = \frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2},$$

$$\ldots, \text{etc. (six equations);}$$

Concepts needed for method of curvature 2-forms

there is no reason to shun similar economies in dealing with the dynamics of geometry. Cartan introduced the decisive ideas, seen above, of differential forms (where a simple object replaces a listing of four components; thus, $\boldsymbol{\sigma} = \sigma_\mu\,\boldsymbol{d}x^\mu$), and of the exterior derivative $\boldsymbol{d}$. He went on (1928, 1946) to package the 21 components $R_{\mu\nu\alpha\beta}$ of the curvature tensor into six curvature 2-forms,

$$\mathcal{R}^{\mu\nu} = -\mathcal{R}^{\nu\mu}.$$

Regarded purely as notation, these 2-forms automatically produce a profit. They cut down the weight of paper work required to list one's answer after one has it. They also provide a route into deeper insight on "curvature as a geometric object," although that is not the objective of immediate concern in this chapter.

Cartan's exterior derivative $\boldsymbol{d}$ automatically effects many cancelations in the calculation of curvature. It often cancels terms before they ever need to be evaluated.

Extension of Cartan's calculus from electromagnetism and other applications (Chapter 4) to the analysis of curvature (this chapter) requires two minor additions to the armament of forms and exterior derivative: (1) the idea of a vector-valued (or tensor-valued) exterior differential form; and (2) a corresponding generalization of the exterior derivative $\boldsymbol{d}$. This section uses both these tools in deriving the key formulas (14.18), (14.25), (14.31), and (14.32). Once derived, however, these formulas demand no more than the standard exterior derivative for all applications and for all calculations of curvature (§14.6 and Box 14.5).

Extended exterior derivative:

(1) acting on a scalar

The extended exterior derivative leads to nothing new in the first two contexts to which one applies it: a scalar function ("0-form") and a vector field ("vector-valued 0-form"). Thus, take any function f. Its derivative in an unspecified direction is a 1-form; or, to make a new distinction that will soon become meaningful, a "scalar-valued 1-form." Specify the direction in which differentiation is to occur ("fill in the slot in the 1-form"). Thereby obtain the ordinary derivative as it applies to a function

$$\langle \boldsymbol{d}f, \boldsymbol{u}\rangle = \partial_{\boldsymbol{u}} f. \tag{14.11}$$

(2) acting on a vector

Next, take any vector field $\boldsymbol{v}$. Its covariant derivative in an unspecified direction is a "vector-valued 1-form." Specify the direction $\boldsymbol{u}$ in which differentiation is to occur ("fill in the slot in the 1-form"). Thereby obtain the covariant derivative

$$\langle \boldsymbol{dv}, \boldsymbol{u}\rangle \equiv \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{v}. \tag{14.12a}$$

This object too is not new; it is the covariant derivative of the vector $\boldsymbol{v}$ taken in the direction of the vector $\boldsymbol{u}$. When one abstracts away from any special choice of the direction of differentiation $\boldsymbol{u}$, one finds an expression that one has encountered before, though not under its new name of "vector-valued 1-form." This expression measures the covariant derivative of the vector $\boldsymbol{v}$ in an unspecified direction ("slot for direction not yet filled in"). From a look at (14.12a), one sees that this extended exterior derivative is applied to $\boldsymbol{v}$, without reference to $\boldsymbol{u}$, is

$$\boldsymbol{dv} = \boldsymbol{\nabla v}. \tag{14.12b}$$

Similarly, for any "tensor-valued 0-form" [i.e. $\binom{n}{0}$ tensor] $\boldsymbol{S}$, $\boldsymbol{dS} \equiv \boldsymbol{\nabla S}$.

Before proceeding further with the exterior (soon to be marked as "antisymmetric") differentiation of tensors, write down a formula (see exercise 14.5) for the exterior (antisymmetric) derivative of a product of forms:

$$\boldsymbol{d}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (\boldsymbol{d\alpha}) \wedge \boldsymbol{\beta} + (-1)^p \boldsymbol{\alpha} \wedge \boldsymbol{d\beta}, \tag{14.13a}$$

where $\boldsymbol{\alpha}$ is a p-form and $\boldsymbol{\beta}$ is a q-form.

(3) defined in general

Now extend the exterior derivative from elementary forms to the exterior product of a tensor-valued p-form $\boldsymbol{S}$ with any ordinary q-form, $\boldsymbol{\beta}$; thus,

$$\boldsymbol{d}(\boldsymbol{S} \wedge \boldsymbol{\beta}) = \boldsymbol{dS} \wedge \boldsymbol{\beta} + (-1)^p \boldsymbol{S} \wedge \boldsymbol{d\beta}. \tag{14.13b}$$

This equation can be regarded as a general definition of the extended exterior derivative. For example, if $\boldsymbol{S}$ is a tensor-valued 2-form, $\boldsymbol{S} = S^{\alpha\beta}{}_{|\gamma\delta|} \boldsymbol{e}_\alpha \boldsymbol{e}_\beta\, \boldsymbol{d}x^\gamma \wedge \boldsymbol{d}x^\delta$, then equation (14.13b) says

$$\boldsymbol{dS} = \boldsymbol{d}[(\boldsymbol{e}_\alpha \boldsymbol{e}_\beta S^{\alpha\beta}{}_{|\gamma\delta|})(\boldsymbol{d}x^\gamma \wedge \boldsymbol{d}x^\delta)] = \boldsymbol{d}(\boldsymbol{e}_\alpha \boldsymbol{e}_\beta S^{\alpha\beta}{}_{|\gamma\delta|}) \wedge (\boldsymbol{d}x^\gamma \wedge \boldsymbol{d}x^\delta).$$

As another example, use (14.13b) to calculate $\boldsymbol{d}(\boldsymbol{u\sigma})$, where $\boldsymbol{u}$ is a vector-valued 0-form (vector) and $\boldsymbol{\sigma}$ is a scalar-valued 1-form (1-form):

$$\boldsymbol{d}(\boldsymbol{u\sigma}) = (\boldsymbol{du}) \wedge \boldsymbol{\sigma} + \boldsymbol{u}\, \boldsymbol{d\sigma}.$$

If one were following the practice of earlier chapters, one would have written $\boldsymbol{u} \otimes \boldsymbol{\sigma}$ where $\boldsymbol{u\sigma}$ appears here, $\boldsymbol{u} \otimes \boldsymbol{d\sigma}$ instead of $\boldsymbol{u}\, \boldsymbol{d\sigma}$, and $\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta$ instead of $\boldsymbol{e}_\alpha \boldsymbol{e}_\beta$. However, to avoid overcomplication in the notation, all such tensor product symbols are omitted here and hereafter.

Equations (14.12) and (14.13) do more than define the (extended) exterior derivative $\boldsymbol{d}$ and provide a way to use it in computations. They also allow one to define and calculate the antisymmetrized second derivatives, e.g., $\boldsymbol{d}^2\boldsymbol{v}$. The relation

$$\boldsymbol{d}^2\boldsymbol{v} = \mathcal{R}\boldsymbol{v}$$

where $\boldsymbol{v}$ is a vector will then introduce the "operator-valued" or "$\binom{1}{1}$-tensor valued" curvature 2-form $\mathcal{R}$. The notation of the extended exterior derivative puts a new look on the old apparatus of base vectors and parallel transport, and opens a way to calculate the curvature 2-form $\mathcal{R}$.

Let the vector field $\boldsymbol{v}$ be expanded in terms of some field of basis vectors $\boldsymbol{e}_\mu$; thus

$$\boldsymbol{v} = \boldsymbol{e}_\mu v^\mu.$$

Then the exterior derivative of this vector is

$$\boldsymbol{dv} = \boldsymbol{de}_\mu v^\mu + \boldsymbol{e}_\mu\, \boldsymbol{d}v^\mu.$$

Expand the typical vector-valued 1-form $\boldsymbol{de}_\mu$ in the form

Definition of $\boldsymbol{\omega}^\mu{}_\nu$

$$\boldsymbol{de}_\mu = \boldsymbol{e}_\nu \boldsymbol{\omega}^\nu{}_\mu. \tag{14.14}$$

Here the "components" $\boldsymbol{\omega}^\nu{}_\mu$ in the expansion of $\boldsymbol{de}_\mu$ are 1-forms. Recall from equation (10.13) that the typical $\boldsymbol{\omega}^\nu{}_\mu$ is related to the connection coefficients by

$$\boldsymbol{\omega}^\nu{}_\mu = \Gamma^\nu{}_{\mu\lambda} \boldsymbol{\omega}^\lambda. \tag{14.15}$$

Therefore the expansion of the "vector" (really, "vector-valued 1-form") is

$$\boldsymbol{dv} = \boldsymbol{e}_\mu(\boldsymbol{d}v^\mu + \boldsymbol{\omega}^\mu{}_\nu v^\nu). \tag{14.16}$$

Now differentiate once again to find

$$\begin{aligned} \boldsymbol{d}^2\boldsymbol{v} &= \boldsymbol{de}_\alpha \wedge (\boldsymbol{d}v^\alpha + \boldsymbol{\omega}^\alpha{}_\nu v^\nu) \\ &\quad + \boldsymbol{e}_\mu(\boldsymbol{d}^2 v^\mu + \boldsymbol{d\omega}^\mu{}_\nu v^\nu - \boldsymbol{\omega}^\mu{}_\nu \wedge \boldsymbol{d}v^\nu) \\ &= \boldsymbol{e}_\mu(\boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{d}v^\alpha + \boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{\omega}^\alpha{}_\nu v^\nu \\ &\quad + \boldsymbol{d}^2 v^\mu + \boldsymbol{d\omega}^\mu{}_\nu v^\nu - \boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{d}v^\alpha). \end{aligned}$$

The simplifications made here use (1) the equation (14.14), for a second time; and (2) the product rule (14.13a), which introduced the minus sign in the last term, ready

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to cancel the first term. Now consider the term $\boldsymbol{d}^2 v^\mu$. Recall that any given component, for example, v^3, is an ordinary scalar function of position (as contrasted to $\boldsymbol{v}$ or $\boldsymbol{e}_3$ or $\boldsymbol{e}_3 v^3$). Therefore the standard exterior derivative (Chapter 4) as applied to a scalar function is all that $\boldsymbol{d}$ can mean in $\boldsymbol{d}^2 v^\mu$. But for the standard exterior derivative applied twice, one has automatically $\boldsymbol{d}^2 v^\mu = 0$ (Box 4.1, B; Box 4.4). This circumstance reduces the expansion for $\boldsymbol{d}^2\boldsymbol{v}$ to the form

Curvature 2-forms $\mathcal{R}^\mu{}_\nu$:

$$\boldsymbol{d}^2\boldsymbol{v} = \boldsymbol{e}_\mu \mathcal{R}^\mu{}_\nu v^\nu, \tag{14.17}$$

(1) in terms of $\boldsymbol{d}^2\boldsymbol{v}$

where the $\mathcal{R}^\mu{}_\nu$ are abbreviations for *the curvature 2-forms*

$$\mathcal{R}^\mu{}_\nu \equiv \boldsymbol{d}\boldsymbol{\omega}^\mu{}_\nu + \boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{\omega}^\alpha{}_\nu. \tag{14.18}$$

(2) in terms of $\boldsymbol{\omega}^\mu{}_\nu$

Ordinarily, equation (14.18) surpasses in efficiency every other known method for calculating the curvature 2-forms.

The remarkable form of equation (14.17) deserves comment. On the left appear two $\boldsymbol{d}$'s, reminders that one has twice differentiated the vector field $\boldsymbol{v}$. But on the right, as the result of the differentiation, one has only the vector field $\boldsymbol{v}$ at the point in question, undifferentiated. How $\boldsymbol{v}$ varies from place to place enters not one whit in the answer. All that matters is how the geometry varies from place to place. Here is curvature coming into evidence. It comes into evidence free of any special features of the vector field $\boldsymbol{v}$, because the operation $\boldsymbol{d}^2$ is an antisymmetrized covariant derivative [compare equation (11.8) for this antisymmetrized covariant derivative in the previously developed abstract language, and see Boxes 11.2 and 11.6 for what is going on behind the scene expressed in the form of pictures]. In brief, the result of operating on $\boldsymbol{v}$ twice with $\boldsymbol{d}$ is an algebraic linear operation on $\boldsymbol{v}$; thus,

$$\boldsymbol{d}^2\boldsymbol{v} = \mathcal{R}\boldsymbol{v}. \tag{14.19}$$

Tensor-valued curvature 2-form $\mathcal{R}$

Here $\mathcal{R}$ is an abbreviation for the "$\binom{1}{1}$-tensor valued 2-form,"

$$\mathcal{R} = \boldsymbol{e}_\mu \otimes \boldsymbol{\omega}^\nu \mathcal{R}^\mu{}_\nu. \tag{14.20}$$

If $\boldsymbol{d}$ is a derivative with a "slot in it" in which to insert the vector saying in what direction the differentiation is to proceed, then the $\boldsymbol{d}^2\boldsymbol{w}$ of $\boldsymbol{d}^2\boldsymbol{w} = \mathcal{R}\boldsymbol{w}$ has two slots and calls for two vectors, say, $\boldsymbol{u}$ and $\boldsymbol{v}$. These two vectors define the plane in which the antisymmetrized exterior derivative of (14.19) is to be evaluated (change in $\boldsymbol{w}$ upon going around the elementary route defined by $\boldsymbol{u}$ and $\boldsymbol{v}$ and coming back to its starting point; Boxes 11.6 and 11.7). To spell out explicitly this insertion of vectors into slots, return first to a simpler context, and see the exterior derivative of a 1-form (itself a 2-form) "evaluated" for a bivector $\boldsymbol{u} \wedge \boldsymbol{v}$ ("count of honeycomblike cells of the 2-form over the parallelogram-shaped domain defined by the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$"), and see the result of the evaluation (exercise 14.6) expressed as a commutator,

$$\langle \boldsymbol{d}\boldsymbol{\alpha}, \boldsymbol{u} \wedge \boldsymbol{v}\rangle = \partial_{\boldsymbol{u}}\langle \boldsymbol{\alpha}, \boldsymbol{v}\rangle - \partial_{\boldsymbol{v}}\langle \boldsymbol{\alpha}, \boldsymbol{u}\rangle - \langle \boldsymbol{\alpha}, [\boldsymbol{u}, \boldsymbol{v}]\rangle. \tag{14.21}$$

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This result generalizes itself to a tensor-valued 1-form $\boldsymbol{S}$ of any rank in an obvious way; thus,

$$\langle \boldsymbol{dS}, \boldsymbol{u} \wedge \boldsymbol{v} \rangle = \boldsymbol{\nabla}_{\boldsymbol{u}} \langle \boldsymbol{S}, \boldsymbol{v} \rangle - \boldsymbol{\nabla}_{\boldsymbol{v}} \langle \boldsymbol{S}, \boldsymbol{u} \rangle - \langle \boldsymbol{S}, [\boldsymbol{u}, \boldsymbol{v}] \rangle. \tag{14.22}$$

Apply this result to the vector-valued 1-form $\boldsymbol{S} = \boldsymbol{dw}$. Recall the expression for a directional derivative, $\langle \boldsymbol{dw}, \boldsymbol{u} \rangle = \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{w}$. Thus find the result

$$\begin{aligned} \langle \boldsymbol{d}^2 \boldsymbol{w}, \boldsymbol{u} \wedge \boldsymbol{v} \rangle &= \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{w} - \boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{\nabla}_{\boldsymbol{u}} \boldsymbol{w} - \boldsymbol{\nabla}_{[\boldsymbol{u},\boldsymbol{v}]} \boldsymbol{w} \\ &= \mathcal{R}(\boldsymbol{u}, \boldsymbol{v}) \boldsymbol{w}, \end{aligned} \tag{14.23}$$

Relation of curvature 2-form $\mathcal{R}$ to curvature operator $\mathcal{R}$

where $\mathcal{R}(\boldsymbol{u}, \boldsymbol{v})$ is the curvature operator defined already in Chapter 11 [equation (11.8)]. The conclusion is simple: the $\binom{1}{1}$-tensor-valued 2-form $\mathcal{R}$ of (14.19), evaluated on the bivector ("parallelogram") $\boldsymbol{u} \wedge \boldsymbol{v}$, is identical with the curvature operator $\mathcal{R}(\boldsymbol{u}, \boldsymbol{v})$ introduced previously; thus

$$\langle \mathcal{R}, \boldsymbol{u} \wedge \boldsymbol{v} \rangle = \mathcal{R}(\boldsymbol{u}, \boldsymbol{v}). \tag{14.24}$$

Now go from the language of abstract operators to a language that begins to make components show up. Substitute on the left the expression (14.20) and on the right the value of the curvature operator from (11.11); and rewrite (14.24) in the form

$$\boldsymbol{e}_\mu \otimes \boldsymbol{\omega}^\nu \langle \mathcal{R}^\mu{}_\nu, \boldsymbol{u} \wedge \boldsymbol{v} \rangle = \boldsymbol{e}_\mu \otimes \boldsymbol{\omega}^\nu R^\mu{}_{\nu\alpha\beta} u^\alpha v^\beta.$$

Compare and conclude that the typical individual curvature 2-form is given by the formula

Relation of $\mathcal{R}$ to components of ***Riemann***

$$\mathcal{R}^\mu{}_\nu = R^\mu{}_{\nu|\alpha\beta|} \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta \tag{14.25}$$

(sum over α, β, restricted to $\alpha < \beta$; so each index pair occurs only once).

Equation (14.25) provides the promised packaging of 21 curvature components into six curvature 2-forms; and equation (14.18) provides the quick means to calculate these curvature 2-forms. It is not necessary to take the key calculational equations (14.18) on faith, or to master the extended exterior derivative to prove or use them. Not one mention of any $\boldsymbol{d}$ do they make except the standard exterior $\boldsymbol{d}$ of Chapter 4. These key equations, moreover, can be verified in detail (exercise 14.8) by working in a coordinate frame. One adopts basis 1-forms $\boldsymbol{\omega}^\alpha = \boldsymbol{d}x^\alpha$. One goes on to use $\boldsymbol{\omega}^\mu{}_\nu = \Gamma^\mu{}_{\nu\lambda} \boldsymbol{d}x^\lambda$ from equation (14.15). In this way one obtains the "standard formula for the curvature" [equation (11.12) and equation (3) of Box 14.2] by standard methods.

In summary, the calculus of forms and exterior derivatives reduces the

$$\Gamma^\mu{}_{\alpha\beta} \longrightarrow R^\mu{}_{\nu\alpha\beta}$$

calculation to the

$$\boldsymbol{\omega}^\mu{}_\nu \longrightarrow \mathcal{R}^\mu{}_\nu$$

computation. Now look at the other link in the chain that leads from metric to curvature. It used to be

$$g_{\mu\nu} \longrightarrow \Gamma^\mu{}_{\alpha\beta}.$$

It now reduces to the calculation of "connection 1-forms"; thus

$$g_{\mu\nu} \longrightarrow \boldsymbol{\omega}^{\mu}{}_{\nu}.$$

Two principles master this first step in the curvature computation: (1) the symmetry of the covariant derivative; and (2) its compatibility with the metric. Condition (1), symmetry, appears in hidden guise in the principle

Symmetry of covariant derivative:

$$\boldsymbol{d}^2\mathscr{P} = 0. \tag{14.26}$$

(1) expressed as $\boldsymbol{d}^2\mathscr{P} = 0$

Here the notation "$\mathscr{P}$ for point" comes straight out of Cartan. He thought of a vector as defined by the movement of one point to another point infinitesimally close to it. To write $\boldsymbol{d}\mathscr{P}$ was therefore to take the "derivative of a point" [make a construction with a "point deleted" (tail of vector) and "point reinserted nearby" (tip of vector)]. The direction of the derivative $\boldsymbol{d}$ in $\boldsymbol{d}\mathscr{P}$ is indefinite. In other words, $\boldsymbol{d}\mathscr{P}$ contains a "slot." Only when one inserts into this slot a definite vector $\boldsymbol{v}$ does $\boldsymbol{d}\mathscr{P}$ give a definite answer for Cartan's vector. What is that vector that $\boldsymbol{d}\mathscr{P}$ then gives? It is $\boldsymbol{v}$ itself. "The movement that is $\boldsymbol{v}$ tells the point $\mathscr{P}$ to reproduce the movement that is $\boldsymbol{v}$"; or in concrete notation,

$$\langle \boldsymbol{d}\mathscr{P}, \boldsymbol{v}\rangle = \boldsymbol{v}. \tag{14.27}$$

Put the content of this equation into more formalistic terms. The quantity $\boldsymbol{d}\mathscr{P}$ is a $\binom{1}{1}$-tensor

$$\boldsymbol{d}\mathscr{P} = \boldsymbol{e}_{\mu}\boldsymbol{\omega}^{\mu}. \tag{14.28}$$

It is distinguished from the generic $\binom{1}{1}$-tensor

$$\boldsymbol{T} = \boldsymbol{e}_{\mu}T^{\mu}{}_{\nu}\boldsymbol{\omega}^{\nu}$$

by the special value of its components

$$T^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}.$$

In this sense it deserves the name of "unit tensor." Insert this tensor in place of $\boldsymbol{S}$ into equation (14.22) and obtain the result

$$\langle \boldsymbol{d}^2\mathscr{P}, \boldsymbol{u}\wedge\boldsymbol{v}\rangle = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} - \boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{u} - [\boldsymbol{u}, \boldsymbol{v}] = 0. \tag{14.29}$$

The zero on the right is a restatement of equation (10.2a) or of "the closing of the vector diagram" in the picture called "symmetry of covariant differentiation" in Box 10.2. The vanishing of the righthand side for arbitrary $\boldsymbol{u}$ and $\boldsymbol{v}$ demands the vanishing of $\boldsymbol{d}^2\mathscr{P}$ on the left; and conversely, the vanishing of $\boldsymbol{d}^2\mathscr{P}$ demands the symmetry of the covariant derivative. The other principle basic to the forthcoming computations is "compatibility of covariant derivative with metric," as expressed in the form

$$\boldsymbol{d}(\boldsymbol{u}\cdot\boldsymbol{v}) = (\boldsymbol{d}\boldsymbol{u})\cdot\boldsymbol{v} + \boldsymbol{u}\cdot(\boldsymbol{d}\boldsymbol{v}). \tag{14.30}$$

It is essential here to ascribe to the metric (the "dot") a vanishing covariant derivative; thus

$$\boldsymbol{d}(\cdot) = 0.$$

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Capitalize on the symmetry and compatibility of the covariant derivative by using basis vectors (and where appropriate the basis 1-forms dual to these basis vectors) in equations (14.26) and (14.30). Thus from

$$\boldsymbol{d}\mathscr{P} = \boldsymbol{e}_\mu \boldsymbol{\omega}^\mu$$

compute

$$\begin{aligned} 0 = \boldsymbol{d}^2\mathscr{P} &= \boldsymbol{de}_\mu \wedge \boldsymbol{\omega}^\mu + \boldsymbol{e}_\mu \boldsymbol{d\omega}^\mu \\ &= \boldsymbol{e}_\mu(\boldsymbol{\omega}^\mu{}_\nu \wedge \boldsymbol{\omega}^\nu + \boldsymbol{d\omega}^\mu), \end{aligned}$$

and conclude that the coefficient of $\boldsymbol{e}_\mu$ must vanish; or

(2) expressed as $\boldsymbol{d\omega}^\mu + \boldsymbol{\omega}^\mu{}_\nu \wedge \boldsymbol{\omega}^\nu = 0$

$$0 = \boldsymbol{d\omega}^\mu + \boldsymbol{\omega}^\mu{}_\nu \wedge \boldsymbol{\omega}^\nu \qquad (\text{"symmetry"}). \tag{14.31a}$$

Next, into (14.30) in place of the general $\boldsymbol{u}$ and $\boldsymbol{v}$ insert the specific $\boldsymbol{e}_\mu$ and $\boldsymbol{e}_\nu$, respectively, and find

Compatibility of $\boldsymbol{g}$ and $\boldsymbol{\nabla}$ expressed as $\boldsymbol{d}g_{\mu\nu} = \boldsymbol{\omega}_{\mu\nu} + \boldsymbol{\omega}_{\nu\mu}$

$$\boldsymbol{d}g_{\mu\nu} = \boldsymbol{\omega}_{\mu\nu} + \boldsymbol{\omega}_{\nu\mu} \qquad (\text{"compatibility"}), \tag{14.31b}$$

where

$$\boldsymbol{\omega}_{\mu\nu} \equiv g_{\mu\alpha}\boldsymbol{\omega}^\alpha{}_\nu = \Gamma_{\mu\nu\alpha}\boldsymbol{\omega}^\alpha. \tag{14.31c}$$

In equations (14.31) one has the connection between metric and connection forms expressed in the most compact way.

§14.6. COMPUTATION OF CURVATURE USING EXTERIOR DIFFERENTIAL FORMS

Method of curvature 2-forms in 4 steps:

(1) select metric and frame

(2) calculate connection 1-forms $\boldsymbol{\omega}^\mu{}_\nu$

The use of differential forms for the computation of curvature is illustrated in Box 14.5. This section outlines the method. There are three main steps: compute $\boldsymbol{\omega}^\mu{}_\nu$; compute $\mathscr{R}^\mu{}_\nu$; and compute $G^\mu{}_\nu$. More particularly, first select a metric and a frame. Thereby fix the basis forms $\boldsymbol{\omega}^\mu = L^\mu{}_{\alpha'}\boldsymbol{d}x^{\alpha'}$ and the metric components $g_{\mu\nu}$ in $\boldsymbol{ds}^2 = g_{\mu\nu}\boldsymbol{\omega}^\mu \otimes \boldsymbol{\omega}^\nu$. Then determine the connection forms $\boldsymbol{\omega}^\mu{}_\nu$, and determine them uniquely, as solutions of the equations

$$0 = \boldsymbol{d\omega}^\mu + \boldsymbol{\omega}^\mu{}_\nu \wedge \boldsymbol{\omega}^\nu, \tag{14.31a}$$

$$\boldsymbol{d}g_{\mu\nu} = \boldsymbol{\omega}_{\mu\nu} + \boldsymbol{\omega}_{\nu\mu}. \tag{14.31b}$$

The "guess and check" method of finding a solution to these equations (described and illustrated in Box 14.5) is often quick and easy. [Exercise (14.7) shows that a solution always exists by showing that the Christoffel formula (14.36) is the unique solution in coordinate frames.] It is usually most convenient to use an orthonormal frame with $g_{\mu\nu} = \eta_{\mu\nu}$ (or some other simple frame where $g_{\mu\nu} = \text{const}$, e.g., a null frame). Then $\boldsymbol{d}g_{\mu\nu} = 0$ and equation (14.31b) shows that $\boldsymbol{\omega}_{\mu\nu} = -\boldsymbol{\omega}_{\nu\mu}$. Therefore there are only six $\boldsymbol{\omega}_{\mu\nu}$ for which to solve in four dimensions.

(*continued on page 358*)

Box 14.5 CURVATURE COMPUTED USING EXTERIOR DIFFERENTIAL FORMS (METRIC FOR FRIEDMANN COSMOLOGY)

The Friedmann metric

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)]$$

(Box 27.1) represents a spacetime where each constant-t hypersurface is a three-dimensional hypersphere of proper circumference $2\pi a(t)$. An orthonormal basis is easily found in this spacetime; thus,

$$\boldsymbol{ds}^2 = -(\boldsymbol{\omega}^{\hat{t}})^2 + (\boldsymbol{\omega}^{\hat{\chi}})^2 + (\boldsymbol{\omega}^{\hat{\theta}})^2 + (\boldsymbol{\omega}^{\hat{\phi}})^2,$$

where

$$\begin{aligned} \boldsymbol{\omega}^{\hat{t}} &= \boldsymbol{d}t, \\ \boldsymbol{\omega}^{\hat{\chi}} &= a\, \boldsymbol{d}\chi, \\ \boldsymbol{\omega}^{\hat{\theta}} &= a \sin\chi\, \boldsymbol{d}\theta, \\ \boldsymbol{\omega}^{\hat{\phi}} &= a \sin\chi \sin\theta\, \boldsymbol{d}\phi. \end{aligned} \tag{1}$$

A. Connection Computation

Equation (14.31b) gives, since $\boldsymbol{d}g_{\mu\nu} = \boldsymbol{d}\eta_{\mu\nu} = 0$, just

$$\boldsymbol{\omega}_{\mu\nu} = -\boldsymbol{\omega}_{\nu\mu}; \tag{2}$$

so there are only six 1-forms $\boldsymbol{\omega}_{\mu\nu}$ to be found. Turn to the second basic equation (14.31a). The game now is to *guess* a solution (because this is so often quicker than using systematic methods) to the equations $0 = \boldsymbol{d\omega}^\mu + \boldsymbol{\omega}^\mu{}_\nu \wedge \boldsymbol{\omega}^\nu$ in which the $\boldsymbol{\omega}^\nu$ and thus also $\boldsymbol{d\omega}^\mu$ are known, and $\boldsymbol{\omega}^\mu{}_\nu$ are unknown. The solution $\boldsymbol{\omega}^\mu{}_\nu$ is known to be unique; so guessing (if it leads to any answer) can only give the right answer.

Proceed from the simplest such equation. From $\boldsymbol{\omega}^{\hat{t}} = \boldsymbol{d}t$, compute

$$\boldsymbol{d\omega}^{\hat{t}} = 0.$$

Compare this with $\boldsymbol{d\omega}^{\hat{t}} = -\boldsymbol{\omega}^{\hat{t}}{}_\mu \wedge \boldsymbol{\omega}^\mu$ or (since $\boldsymbol{\omega}^{\hat{t}}{}_{\hat{t}} = -\boldsymbol{\omega}_{\hat{t}\hat{t}} = 0$, by $\boldsymbol{\omega}_{\mu\nu} = -\boldsymbol{\omega}_{\nu\mu}$)

$$\boldsymbol{d\omega}^{\hat{t}} = -\boldsymbol{\omega}^{\hat{t}}{}_k \wedge \boldsymbol{\omega}^k = 0.$$

This equation could be satisfied by having $\boldsymbol{\omega}^{\hat{t}}{}_k \propto \boldsymbol{\omega}^k$, or in more complicated ways with cancelations among different terms, or more simply by $\boldsymbol{\omega}^{\hat{t}}{}_k = 0$. Proceed, not

Box 14.5 (*continued*)

looking for trouble, until some non-zero $\boldsymbol{\omega}^\mu{}_\nu$ is required. From $\boldsymbol{\omega}^{\hat{\chi}} = a\,\boldsymbol{d}\chi$, find

$$\begin{aligned}\boldsymbol{d\omega}^{\hat{\chi}} &= \dot{a}\,\boldsymbol{d}t \wedge \boldsymbol{d}\chi \\ &= (\dot{a}/a)\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\chi}} = -(\dot{a}/a)\boldsymbol{\omega}^{\hat{\chi}} \wedge \boldsymbol{\omega}^{\hat{t}}.\end{aligned}$$

Compare this with

$$\begin{aligned}\boldsymbol{d\omega}^{\hat{\chi}} &= -\boldsymbol{\omega}^{\hat{\chi}}{}_\mu \wedge \boldsymbol{\omega}^\mu \\ &= -\boldsymbol{\omega}^{\hat{\chi}}{}_{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{t}} - \boldsymbol{\omega}^{\hat{\chi}}{}_{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\theta}} - \boldsymbol{\omega}^{\hat{\chi}}{}_{\hat{\phi}} \wedge \boldsymbol{\omega}^{\hat{\phi}}.\end{aligned}$$

Guess that $\boldsymbol{\omega}^{\hat{\chi}}{}_{\hat{t}} = (\dot{a}/a)\boldsymbol{\omega}^{\hat{\chi}}$ from the first term; and hope the other terms vanish. (Note that this allows $\boldsymbol{\omega}^{\hat{t}}{}_{\hat{\chi}} \wedge \boldsymbol{\omega}^{\hat{\chi}} = -\boldsymbol{\omega}_{\hat{t}\hat{\chi}} \wedge \boldsymbol{\omega}^{\hat{\chi}} = \boldsymbol{\omega}_{\hat{\chi}\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\chi}} = 0$ in the $\boldsymbol{d\omega}^{\hat{t}}$ equation.) Look at $\boldsymbol{\omega}^{\hat{\theta}} = a \sin\chi\,\boldsymbol{d}\theta$, and write

$$\begin{aligned}\boldsymbol{d\omega}^{\hat{\theta}} &= (\dot{a}/a)\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\theta}} + a^{-1}\cot\chi\,\boldsymbol{\omega}^{\hat{\chi}} \wedge \boldsymbol{\omega}^{\hat{\theta}} \\ &= -\boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{t}} - \boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{\chi}} \wedge \boldsymbol{\omega}^{\hat{\chi}} - \boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{\phi}} \wedge \boldsymbol{\omega}^{\hat{\phi}}.\end{aligned}$$

Guess, consistent with previously written equations, that

$$\begin{aligned}\boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{t}} &= \boldsymbol{\omega}^{\hat{t}}{}_{\hat{\theta}} = (\dot{a}/a)\boldsymbol{\omega}^{\hat{\theta}}, \\ \boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{\chi}} &= -\boldsymbol{\omega}^{\hat{\chi}}{}_{\hat{\theta}} = a^{-1}\cot\chi\,\boldsymbol{\omega}^{\hat{\theta}}.\end{aligned}$$

Finally from

$$\begin{aligned}\boldsymbol{d\omega}^{\hat{\phi}} &= (\dot{a}/a)\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\phi}} + a^{-1}\cot\chi\,\boldsymbol{\omega}^{\hat{\chi}} \wedge \boldsymbol{\omega}^{\hat{\phi}} \\ &\quad + (a\sin\chi)^{-1}\cot\theta\,\boldsymbol{\omega}^{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\phi}} \\ &= -\boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{t}} - \boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{\chi}} \wedge \boldsymbol{\omega}^{\hat{\chi}} - \boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\theta}},\end{aligned}$$

deduce values of $\boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{t}}$, $\boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{\chi}}$, and $\boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{\theta}}$. These are not inconsistent with previous assumptions that terms like $\boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{\phi}} \wedge \boldsymbol{\omega}^{\hat{\phi}}$ vanish (in the $\boldsymbol{d\omega}^{\hat{\theta}}$ equation); so one has in fact solved $\boldsymbol{d\omega}^\mu = -\boldsymbol{\omega}^\mu{}_\nu \wedge \boldsymbol{\omega}^\nu$ for a set of connection forms $\boldsymbol{\omega}^\mu{}_\nu$, as follows:

$$\begin{aligned}\boldsymbol{\omega}^k{}_{\hat{t}} &= \boldsymbol{\omega}^{\hat{t}}{}_k = (\dot{a}/a)\boldsymbol{\omega}^k, \\ \boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{\chi}} &= -\boldsymbol{\omega}^{\hat{\chi}}{}_{\hat{\theta}} = a^{-1}\cot\chi\,\boldsymbol{\omega}^{\hat{\theta}} \\ &= \cos\chi\,\boldsymbol{d}\theta, \\ \boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{\chi}} &= -\boldsymbol{\omega}^{\hat{\chi}}{}_{\hat{\phi}} = a^{-1}\cot\chi\,\boldsymbol{\omega}^{\hat{\phi}} \\ &= \cos\chi\sin\theta\,\boldsymbol{d}\phi, \\ \boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{\theta}} &= -\boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{\phi}} = (a\sin\chi)^{-1}\cot\theta\,\boldsymbol{\omega}^{\hat{\phi}} \\ &= \cos\theta\,\boldsymbol{d}\phi.\end{aligned} \tag{3}$$

Of course, if these hit-or-miss methods of finding $\boldsymbol{\omega}^\mu{}_\nu$ do not work easily in some problem, one may simply use equations (14.32) and (14.33).

2

B. Curvature Computation

The curvature computation is a straightforward substitution of $\boldsymbol{\omega}^{\mu}{}_{\nu}$ from equations (3) above into equation (14.34), which is

$$\mathcal{R}^{\mu}{}_{\nu} = \boldsymbol{d\omega}^{\mu}{}_{\nu} + \boldsymbol{\omega}^{\mu}{}_{\alpha} \wedge \boldsymbol{\omega}^{\alpha}{}_{\nu}.$$

This equation is short enough that one can write out the sum

$$\mathcal{R}^{\hat{t}}{}_{\hat{\chi}} = \boldsymbol{d\omega}^{\hat{t}}{}_{\hat{\chi}} + \boldsymbol{\omega}^{\hat{t}}{}_{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\theta}}{}_{\hat{\chi}} + \boldsymbol{\omega}^{\hat{t}}{}_{\hat{\phi}} \wedge \boldsymbol{\omega}^{\hat{\phi}}{}_{\hat{\chi}}$$

in contrast to the ten terms in the corresponding $R = \partial\Gamma + \Gamma^2$ equation [equation (3) of Box 14.2]. *Warning!:* From $\boldsymbol{\omega}^{\hat{t}}{}_{\hat{\chi}} = (\dot{a}/a)\boldsymbol{\omega}^{\hat{\chi}}$, do not compute $\boldsymbol{d\omega}^{\hat{t}}{}_{\hat{\chi}} = (\dot{a}/a)^{\boldsymbol{\cdot}}\, \boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\chi}}$. Missing is the term $(\dot{a}/a)\, \boldsymbol{d\omega}^{\hat{\chi}}$. Instead write $\boldsymbol{\omega}^{\hat{t}}{}_{\hat{\chi}} = (\dot{a}/a)\boldsymbol{\omega}^{\hat{\chi}} = \dot{a}\, \boldsymbol{d}\chi$, and then find $\boldsymbol{d\omega}^{\hat{t}}{}_{\hat{\chi}} = \ddot{a}\, \boldsymbol{d}t \wedge \boldsymbol{d}\chi = (\ddot{a}/a)\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\chi}}$. With elementary care, then, in correctly substituting from (3) for the $\boldsymbol{\omega}^{\mu}{}_{\nu}$ in the formula for $\mathcal{R}^{\mu}{}_{\nu}$, one finds

$$\mathcal{R}^{\hat{t}}{}_{\hat{\chi}} = (\ddot{a}/a)\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{\hat{\chi}},$$

and

$$\mathcal{R}^{\hat{\chi}}{}_{\hat{\theta}} = (1 + \dot{a}^2)a^{-2}\boldsymbol{\omega}^{\hat{\chi}} \wedge \boldsymbol{\omega}^{\hat{\theta}}.$$

This completes the computation of the $R^{\mu}{}_{\nu\alpha\beta}$, since in this isotopic model universe, all space directions in the orthonormal frame $\boldsymbol{\omega}^{\mu}$ are algebraically equivalent. One can therefore write

$$\begin{aligned} \mathcal{R}^{\hat{t}}{}_{k} &= (\ddot{a}/a)\boldsymbol{\omega}^{\hat{t}} \wedge \boldsymbol{\omega}^{k}, \\ \mathcal{R}^{k}{}_{\ell} &= a^{-2}(1 + \dot{a}^2)\boldsymbol{\omega}^{k} \wedge \boldsymbol{\omega}^{\ell}, \end{aligned} \tag{4}$$

for the complete list of $\mathcal{R}^{\mu}{}_{\nu}$. Specific components, such as

$$R^{\hat{t}}{}_{\hat{\chi}\hat{t}\hat{\chi}} = \ddot{a}/a, \qquad R^{\hat{t}}{}_{\hat{\chi}\hat{t}\hat{\theta}} = 0, \text{ etc.},$$

or

$$R^{\hat{t}}{}_{\hat{\theta}\hat{\chi}\hat{\phi}} = 0, \qquad R^{\hat{\theta}}{}_{\hat{\phi}\hat{\theta}\hat{\phi}} = a^{-2}(1 + \dot{a}^2),$$

are easily read out of this display of $\mathcal{R}^{\mu}{}_{\nu}$.

C. Contraction

From equations (14.7), find

$$G^{\hat{t}\hat{t}} = +3a^{-2}(1 + \dot{a}^2), \tag{5a}$$

$$G^{\hat{t}\hat{\chi}} = G^{\hat{t}\hat{\theta}} = G^{\hat{t}\hat{\phi}} = 0 = G^{\hat{\chi}\hat{\theta}} = G^{\hat{\theta}\hat{\phi}} = G^{\hat{\phi}\hat{\chi}}, \tag{5b}$$

$$G^{\hat{\chi}\hat{\chi}} = G^{\hat{\theta}\hat{\theta}} = G^{\hat{\phi}\hat{\phi}} = -[2a^{-1}\ddot{a} + a^{-2}(1 + \dot{a}^2)], \tag{5c}$$

and

$$R = -G^{\mu}{}_{\mu} = 6[a^{-1}\ddot{a} + a^{-2}(1 + \dot{a}^2)]. \tag{6}$$

2

If guessing is not easy, there is a systematic way to solve equations (14.31) in an orthonormal frame or in any other frame in which $\boldsymbol{d}g_{\mu\nu} = 0$. Compute the $\boldsymbol{d\omega}^{\mu}$ and arrange them in the format

$$\boldsymbol{d\omega}^{\alpha} = -c_{|\mu\nu|}{}^{\alpha}\boldsymbol{\omega}^{\mu} \wedge \boldsymbol{\omega}^{\nu}. \tag{14.32}$$

In this way display the 24 "commutation coefficients" $c_{\mu\nu}{}^{\alpha}$. These quantities enter into the formula

$$\boldsymbol{\omega}_{\mu\nu} = \frac{1}{2}(c_{\mu\nu\alpha} + c_{\mu\alpha\nu} - c_{\nu\alpha\mu})\boldsymbol{\omega}^{\alpha} \tag{14.33}$$

to provide the six $\boldsymbol{\omega}^{\mu}{}_{\nu}$ (exercise 14.12).

(3) calculate curvature 2-forms $\mathcal{R}^{\mu}{}_{\nu}$

Once the $\boldsymbol{\omega}_{\mu\nu}$ are known, one computes the curvature forms $\mathcal{R}^{\mu}{}_{\nu}$ (again only six in four dimensions, since $\mathcal{R}^{\mu\nu} = -\mathcal{R}^{\nu\mu}$) by use of the formula

$$\mathcal{R}^{\mu}{}_{\nu} = \boldsymbol{d\omega}^{\mu}{}_{\nu} + \boldsymbol{\omega}^{\mu}{}_{\alpha} \wedge \boldsymbol{\omega}^{\alpha}{}_{\nu}. \tag{14.34}$$

(4) calculate components of curvature tensors

Out of this tabulation, one reads the individual components of the curvature tensor by using the identification scheme

$$\mathcal{R}^{\mu\nu} = R^{\mu\nu}{}_{|\alpha\beta|}\boldsymbol{\omega}^{\alpha} \wedge \boldsymbol{\omega}^{\beta}. \tag{14.35}$$

The Einstein tensor $G^{\mu}{}_{\nu}$ is computed by scanning the $\mathcal{R}^{\mu\nu}$ display to find the appropriate $R^{\mu\nu}{}_{\alpha\beta}$ components for use in formulas (14.7).

EXERCISES

Exercise 14.5. EXTERIOR DERIVATIVE OF A PRODUCT OF FORMS

Establish equation (14.13a) by working up recursively from forms of lower order to forms of higher order. [*Hints:* Recall from equation (4.27) that for a p-form

$$\boldsymbol{\alpha} = \alpha_{|\mu_1 \ldots \mu_p|}\boldsymbol{d}x^{\mu_1} \wedge \cdots \wedge \boldsymbol{d}x^{\mu_p},$$

the exterior derivative is defined by

$$\boldsymbol{d\alpha} = \frac{\partial \alpha_{|\mu_1 \ldots \mu_p|}}{\partial x^{\mu_0}}\boldsymbol{d}x^{\mu_0} \wedge \boldsymbol{d}x^{\mu_1} \wedge \cdots \wedge \boldsymbol{d}x^{\mu_p}.$$

Applied to the product $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ of two 1-forms, this formula gives

$$\begin{aligned}
\boldsymbol{d}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) &= \boldsymbol{d}[(\alpha_{\lambda}\,\boldsymbol{d}x^{\lambda}) \wedge (\beta_{\mu}\,\boldsymbol{d}x^{\mu})] \\
&= \boldsymbol{d}[(\alpha_{\lambda}\beta_{\mu})(\boldsymbol{d}x^{\lambda} \wedge \boldsymbol{d}x^{\mu})] \\
&= \frac{\partial(\alpha_{\lambda}\beta_{\mu})}{\partial x^{\kappa}}\boldsymbol{d}x^{\kappa} \wedge \boldsymbol{d}x^{\lambda} \wedge \boldsymbol{d}x^{\mu} \\
&= \left(\frac{\partial \alpha_{\lambda}}{\partial x^{\kappa}}\boldsymbol{d}x^{\kappa} \wedge \boldsymbol{d}x^{\lambda}\right) \wedge \beta_{\mu}\,\boldsymbol{d}x^{\mu} - (a_{\lambda}\,\boldsymbol{d}x^{\lambda}) \wedge \left(\frac{\partial \beta_{\mu}}{\partial x^{\kappa}}\boldsymbol{d}x^{\kappa} \wedge \boldsymbol{d}x^{\mu}\right) \\
&= (\boldsymbol{d\alpha}) \wedge \boldsymbol{\beta} - \boldsymbol{\alpha} \wedge \boldsymbol{d\beta}.
\end{aligned}$$

Extend the reasoning to forms of higher order.]

Exercise 14.6. RELATIONSHIP BETWEEN EXTERIOR DERIVATIVE AND COMMUTATOR

Establish formula (14.21) by showing (a) that the righthand side is an *algebraic* linear function of $\boldsymbol{u}$ and an algebraic linear function of $\boldsymbol{v}$, and (b) that the equation holds when $\boldsymbol{u}$ and $\boldsymbol{v}$ are coordinate basis vectors $\boldsymbol{u} = \partial/\partial x^k$, $\boldsymbol{v} = \partial/\partial x^l$.

Exercise 14.7. CHRISTOFFEL FORMULA DERIVED FROM CONNECTION FORMS

In a coordinate frame $\boldsymbol{\omega}^\mu = \boldsymbol{d}x^\mu$, show that equation (14.31a) requires $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$, and that, with this symmetry established, equation (14.31b) gives an expression for $\partial g_{\mu\nu}/\partial x^\alpha$ which can be solved to give the Christoffel formula

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right). \tag{14.36}$$

Exercise 14.8. RIEMANN-CHRISTOFFEL CURVATURE FORMULA RELATED TO CURVATURE FORMS

Substitute $\boldsymbol{\omega}^\mu{}_\nu = \Gamma^\mu{}_{\nu\lambda}\, \boldsymbol{d}x^\lambda$ into equation (14.18), and from the result read out, according to equation (14.25), the classical formula (3) of Box 14.2 for the components $R^\mu{}_{\nu\alpha\beta}$.

Exercise 14.9. MATRIX NOTATION FOR REVIEW OF CARTAN STRUCTURE EQUATIONS

Let $e \equiv (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ be a row matrix whose entries are the basis vectors, and let ω be a column of basis 1-forms $\boldsymbol{\omega}^\mu$. Similarly let $\Omega = \|\boldsymbol{\omega}^\mu{}_\nu\|$ and $\mathcal{R} = \|\mathcal{R}^\mu{}_\nu\|$ be square matrices with 1-form and 2-form entries. This gives a compact notation in which $\boldsymbol{d}\boldsymbol{e}_\mu = \boldsymbol{e}_\nu \boldsymbol{\omega}^\nu{}_\mu$ and $\boldsymbol{d}\mathcal{P} = \boldsymbol{e}_\mu \boldsymbol{\omega}^\mu$ read

$$\boldsymbol{d}e = e\Omega \text{ and } \boldsymbol{d}\mathcal{P} = e\omega, \tag{14.37}$$

respectively.

(a) From equations (14.37) and $\boldsymbol{d}^2\mathcal{P} = 0$, derive equation (14.31a) in the form

$$0 = \boldsymbol{d}\omega + \Omega \wedge \omega. \tag{14.38}$$

[Solution: $\boldsymbol{d}^2\mathcal{P} = \boldsymbol{d}e \wedge \omega + e\, \boldsymbol{d}\omega = e(\Omega \wedge \omega + \boldsymbol{d}\omega)$.]

(b) Compute $\boldsymbol{d}^2 e$ as motivation for definition (14.18), which reads

$$\mathcal{R} = \boldsymbol{d}\Omega + \Omega \wedge \Omega. \tag{14.39}$$

(c) From $\boldsymbol{d}^2\omega = 0$, deduce $\mathcal{R} \wedge \omega = 0$ and then decompress the notation to get the antisymmetry relation $R^\mu{}_{[\alpha\beta\gamma]} = 0$.

(d) Compute $\boldsymbol{d}\mathcal{R}$ from equation (14.39), and relate it to the Bianchi identity $R^\mu{}_{\nu[\alpha\beta;\gamma]} = 0$.

(e) Let $v = \{v^\mu\}$ be a column of functions; so $\boldsymbol{v} = ev = \boldsymbol{e}_\mu v^\mu$ is a vector field. Compute, in compact notation, $\boldsymbol{d}\boldsymbol{v}$ and $\boldsymbol{d}^2\boldsymbol{v}$ to show $\boldsymbol{d}^2\boldsymbol{v} = e\mathcal{R}v$ (which is equation 14.17).

Exercise 14.10. TRANSFORMATION RULES FOR CONNECTION FORMS IN COMPACT NOTATION

Using the notation of the previous exercise, write $e' = eA$ in place of $\boldsymbol{e}_{\mu'} = \boldsymbol{e}_\nu A^\nu{}_{\mu'}$, and similarly $\omega' = A^{-1}\omega$, to represent a change of frame. Show that $\boldsymbol{d}\mathcal{P} \equiv e\omega = e'\omega'$. Substitute $e' = eA$ in $\boldsymbol{d}e' = e'\Omega'$ to deduce the transformation law

$$\Omega' = A^{-1}\Omega A + A^{-1}\, \boldsymbol{d}A. \tag{14.40}$$

Rewrite this in decompressed notation for coordinate frames with $A^\nu{}_{\mu'} = \partial x^\nu/\partial x^{\mu'}$ as a formula of the form $\Gamma^{\mu'}{}_{\alpha'\beta'} = (?)$.

Exercise 14.11. SPACE IS FLAT IF THE CURVATURE VANISHES (see §11.5)

If coordinates exist in which all straight lines ($d^2x^\mu/d\lambda^2 = 0$) are geodesics, then one says the space is flat. Evidently all $\Gamma^\mu{}_{\alpha\beta}$ and $R^\mu{}_{\nu\alpha\beta}$ vanish in this case, by equation (14.8) and equation (3) in Box 14.2. Show conversely that, if $\mathcal{R} = 0$, then such coordinates exist. Use the results of the previous problem to find differential equations for a transformation A to a basis e' where $\Omega' = 0$. What are the conditions for complete integrability of these equations? [Note that $\boldsymbol{d}f_K = F_K(x, f)$ is completely integrable if $\boldsymbol{d}^2 f_K = 0$ modulo the original equations.] Why will the basis forms $\boldsymbol{\omega}^{\mu'}$ in this new frame be coordinate differentials $\boldsymbol{\omega}^{\mu'} = \boldsymbol{d}x^{\mu'}$?

Exercise 14.12. SYSTEMATIC COMPUTATION OF CONNECTION FORMS IN ORTHONORMAL FRAMES

Deduce equation (14.32) by applying equation (14.21) to basis vectors, using equations (8.14) to define $c_{\mu\nu}{}^\alpha$. Then show that, in an orthonormal frame (or any frames with $g_{\mu\nu} = \text{const}$), equation (14.33) provides a solution of equations (14.31), which define $\boldsymbol{\omega}^\mu{}_\nu$. [Compare also equation (8.24b).]

Exercise 14.13. SCHWARZSCHILD CURVATURE FORMS

Use the obvious orthonormal frame $\boldsymbol{\omega}^{\hat t} = e^\Phi\, \boldsymbol{d}t$, $\boldsymbol{\omega}^{\hat r} = e^\Lambda\, \boldsymbol{d}r$, $\boldsymbol{\omega}^{\hat\theta} = r\, \boldsymbol{d}\theta$, $\boldsymbol{\omega}^{\hat\phi} = r \sin\theta\, \boldsymbol{d}\phi$ for the Schwarzschild metric

$$ds^2 = -e^{2\Phi}\, dt^2 + e^{2\Lambda}\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{14.41}$$

in which Φ and Λ are functions of r only; and compute the curvature forms $\mathcal{R}^{\hat\mu}{}_{\hat\nu}$ and the Einstein tensor $G^{\hat\mu}{}_{\hat\nu}$ by the methods of Box 14.5. [*Answer:* $\mathcal{R}^{\hat t\hat r} = E\boldsymbol{\omega}^{\hat t} \wedge \boldsymbol{\omega}^{\hat r}$, $\mathcal{R}^{\hat t\hat\theta} = \bar E\boldsymbol{\omega}^{\hat t} \wedge \boldsymbol{\omega}^{\hat\theta}$, $\mathcal{R}^{\hat t\hat\phi} = \bar E\boldsymbol{\omega}^{\hat t} \wedge \boldsymbol{\omega}^{\hat\phi}$, $\mathcal{R}^{\hat\theta\hat\phi} = F\boldsymbol{\omega}^{\hat\theta} \wedge \boldsymbol{\omega}^{\hat\phi}$, $\mathcal{R}^{\hat\phi\hat r} = \bar F\boldsymbol{\omega}^{\hat\phi} \wedge \boldsymbol{\omega}^{\hat r}$, $\mathcal{R}^{\hat r\hat\theta} = \bar F\boldsymbol{\omega}^{\hat r} \wedge \boldsymbol{\omega}^{\hat\theta}$, with

$$\begin{aligned}
E &= -e^{-2\Lambda}(\Phi'' + \Phi'^2 - \Phi'\Lambda'),\\
\bar E &= -\frac{1}{r}\, e^{-2\Lambda}\Phi',\\
F &= \frac{1}{r^2}(1 - e^{-2\Lambda}),\\
\bar F &= \frac{1}{r}\, e^{-2\Lambda}\Lambda';
\end{aligned} \tag{14.42}$$

and then

$$\begin{aligned}
G^{\hat t}_{\hat t} &= -(F + 2\bar F),\\
G^{\hat r}_{\hat r} &= -(F + 2\bar E),\\
G^{\hat\theta}_{\hat\theta} = G^{\hat\phi}_{\hat\phi} &= -(E + \bar E + \bar F),\\
G^{\hat t}_{\hat r} = G^{\hat t}_{\hat\theta} = G^{\hat t}_{\hat\phi} &= 0 = G^{\hat r}_{\hat\theta} = G^{\hat r}_{\hat\phi} = G^{\hat\theta}_{\hat\phi}.]
\end{aligned} \tag{14.43}$$

Exercise 14.14. MATRIX DISPLAY OF THE RIEMANN-TENSOR COMPONENTS

Use the symmetries of the Riemann tensor to justify displaying its components in an orthonormal frame in the form

$$R^{\hat\mu\hat\nu}{}_{\hat\alpha\hat\beta} = \begin{array}{c} 01\\ 02\\ 03\\ 23\\ 31\\ 12 \end{array} \left(\begin{array}{c|c} E & H \\ \hline -H^T & F \end{array}\right), \tag{14.44}$$

where the rows are labeled by index pairs $\hat{\mu}\hat{\nu} = 01, 02$, etc., as shown; and the columns $\hat{\alpha}\hat{\beta}$, similarly. Here E, F, and H are each 3×3 matrices with (why?)

$$E = E^T, \qquad F = F^T, \qquad \text{trace } H = 0, \tag{14.45}$$

where E^T means the transpose of E.

Exercise 14.15. RIEMANN MATRIX WITH VANISHING EINSTEIN TENSOR

Show that the empty-space Einstein equations $G^{\hat{\mu}}{}_{\hat{\nu}} = 0$ allow the matrix in equation (14.44) to be simplified to the form

$$R^{\hat{\mu}\hat{\nu}}{}_{\hat{\alpha}\hat{\beta}} = \left(\begin{array}{c|c} E & H \\ \hline -H & E \end{array}\right), \tag{14.46}$$

where now, in addition to the equality $E = F$ that this form implies, the further conditions

$$\text{trace } E = 0, \; H = H^T \tag{14.47}$$

hold.

Exercise 14.16. COMPUTATION OF CURVATURE FOR A PULSATING OR COLLAPSING STAR

Spherically symmetric motions of self-gravitating bodies are discussed in Chapters 26 and 32. A metric form often adopted in this situation is

$$ds^2 = -e^{2\Phi}\, dT^2 + e^{2\Lambda}\, dR^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{14.48}$$

where now Φ, Λ, and r are each functions of the two coordinates R and T. Compute the curvature 2-forms and the Einstein tensor for this metric, using the methods of Box 14.5. In the guessing of the $\boldsymbol{\omega}^{\mu}{}_{\nu}$, most of the terms will already be evident from the corresponding calculation in exercise 14.13. [*Answer,* in the obvious orthonormal frame $\boldsymbol{\omega}^{\hat{T}} = e^{\Phi}\, \boldsymbol{d}T$, $\boldsymbol{\omega}^{\hat{R}} = e^{\Lambda}\, \boldsymbol{d}R$, $\boldsymbol{\omega}^{\hat{\theta}} = r\, \boldsymbol{d}\theta$, $\boldsymbol{\omega}^{\hat{\phi}} = r \sin\theta\, \boldsymbol{d}\phi$:

$$\begin{aligned}
\mathcal{R}^{\hat{T}}{}_{\hat{R}} &= E\boldsymbol{\omega}^{\hat{T}} \wedge \boldsymbol{\omega}^{\hat{R}}, \\
\mathcal{R}^{\hat{T}}{}_{\hat{\theta}} &= \bar{E}\boldsymbol{\omega}^{\hat{T}} \wedge \boldsymbol{\omega}^{\hat{\theta}} + H\boldsymbol{\omega}^{\hat{R}} \wedge \boldsymbol{\omega}^{\hat{\theta}}, \\
\mathcal{R}^{\hat{T}}{}_{\hat{\phi}} &= \bar{E}\boldsymbol{\omega}^{\hat{T}} \wedge \boldsymbol{\omega}^{\hat{\phi}} + H\boldsymbol{\omega}^{\hat{R}} \wedge \boldsymbol{\omega}^{\hat{\phi}}, \\
\mathcal{R}^{\hat{\theta}}{}_{\hat{\phi}} &= F\boldsymbol{\omega}^{\hat{\theta}} \wedge \boldsymbol{\omega}^{\hat{\phi}}, \\
\mathcal{R}^{\hat{R}}{}_{\hat{\theta}} &= \bar{F}\boldsymbol{\omega}^{\hat{R}} \wedge \boldsymbol{\omega}^{\hat{\theta}} - H\boldsymbol{\omega}^{\hat{T}} \wedge \boldsymbol{\omega}^{\hat{\theta}}, \\
\mathcal{R}^{\hat{R}}{}_{\hat{\phi}} &= \bar{F}\boldsymbol{\omega}^{\hat{R}} \wedge \boldsymbol{\omega}^{\hat{\phi}} - H\boldsymbol{\omega}^{\hat{T}} \wedge \boldsymbol{\omega}^{\hat{\phi}},
\end{aligned} \tag{14.49}$$

which, in the matrix display of exercise 14.14, gives

$$R^{\mu\nu}{}_{\alpha\beta} = \begin{array}{c} \left(\begin{array}{ccc|ccc}
E & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \bar{E} & \cdot & \cdot & \cdot & H \\
\cdot & \cdot & \bar{E} & \cdot & -H & \cdot \\
\hline
\cdot & \cdot & \cdot & F & \cdot & \cdot \\
\cdot & \cdot & H & \cdot & \bar{F} & \cdot \\
\cdot & -H & \cdot & \cdot & \cdot & \bar{F}
\end{array}\right) \end{array}
\begin{array}{l} \hat{T}\hat{R} \\ \hat{T}\hat{\theta} \\ \hat{T}\hat{\phi} \\ \hat{\theta}\hat{\phi} \\ \hat{\phi}\hat{R} \\ \hat{R}\hat{\theta} \end{array}. \tag{14.50}$$

Here

$$\begin{aligned}
E &= e^{-2\Phi}(\ddot{\Lambda} + \dot{\Lambda}^2 - \dot{\Lambda}\dot{\Phi}) - e^{-2\Lambda}(\Phi'' + \Phi'^2 - \Phi'\Lambda'),\\
\bar{E} &= \frac{1}{r}e^{-2\Phi}(\ddot{r} - \dot{r}\dot{\Phi}) - \frac{1}{r}e^{-2\Lambda}r'\Phi',\\
H &= \frac{1}{r}e^{-\Phi-\Lambda}(\dot{r}' - \dot{r}\Phi' - r'\dot{\Lambda}),\\
F &= \frac{1}{r^2}(1 - r'^2e^{-2\Lambda} + \dot{r}^2e^{-2\Phi}),\\
\bar{F} &= \frac{1}{r}e^{-2\Phi}\dot{r}\dot{\Lambda} + \frac{1}{r}e^{-2\Lambda}(r'\Lambda' - r'').
\end{aligned} \tag{14.51}$$

The Einstein tensor is

$$\begin{aligned}
G^{\hat{T}\hat{T}} &= -G^{\hat{T}}{}_{\hat{T}} = F + 2\bar{F},\\
G^{\hat{T}\hat{R}} &= G^{\hat{T}}{}_{\hat{R}} = 2H,\\
G^{\hat{T}}{}_{\hat{\theta}} &= G^{\hat{T}}{}_{\hat{\phi}} = 0,\\
G^{\hat{R}}{}_{\hat{R}} &= -(2\bar{E} + F),\\
G^{\hat{\theta}}{}_{\hat{\theta}} &= G^{\hat{\phi}}{}_{\hat{\phi}} = -(E + \bar{E} + \bar{F}),\\
G^{\hat{R}}{}_{\hat{\theta}} &= G^{\hat{R}}{}_{\hat{\phi}} = G^{\hat{\theta}}{}_{\hat{\phi}} = 0.]
\end{aligned} \tag{14.52}$$

Exercise 14.17. BIANCHI IDENTITY IN $\boldsymbol{d}\mathcal{R} = 0$ FORM

Define the Riemann tensor as a bivector-valued 2-form,

$$\mathcal{R} = \frac{1}{2}\boldsymbol{e}_\mu \wedge \boldsymbol{e}_\nu \mathcal{R}^{\mu\nu}, \tag{14.53}$$

and evaluate $\boldsymbol{d}\mathcal{R}$ to make it manifest that $\boldsymbol{d}\mathcal{R} = 0$. Use

$$\mathcal{R}^{\mu\nu} = \boldsymbol{d\omega}^{\mu\nu} - \boldsymbol{\omega}^\mu{}_\alpha \wedge \boldsymbol{\omega}^{\nu\alpha}, \tag{14.54}$$

which is derived easily in an orthonormal frame (adequate for proving $\boldsymbol{d}\mathcal{R} = 0$), or (as a test of skill) in a general frame where $\mathcal{R}^{\mu\nu} = \mathcal{R}^\mu{}_\alpha g^{\alpha\nu}$ and (why?) $\boldsymbol{d}g^{\mu\nu} = -g^{\mu\alpha}(\boldsymbol{d}g_{\alpha\beta})g^{\beta\nu}$. [*Note:* only wedge products between forms (not those between vectors) count in fixing signs in the product rule (14.13) for $\boldsymbol{d}$.]

Exercise 14.18. LOCAL CONSERVATION OF ENERGY AND MOMENTUM: $\boldsymbol{d}{*}\boldsymbol{T} = 0$ MEANS $\boldsymbol{\nabla}\cdot\boldsymbol{T} = 0$

Let the duality operator *, as defined for exterior differential forms in Box 4.1, act on the forms, *but not on the contravariant vectors,* which appear when the stress-energy tensor $\boldsymbol{T}$ or the Einstein tensor $\boldsymbol{G}$ is written as a mixed $\binom{1}{1}$ tensor:

$$\boldsymbol{T} = \boldsymbol{e}_\mu T^\mu{}_\nu \boldsymbol{\omega}^\nu$$

or

$$\boldsymbol{G} = \boldsymbol{e}_\mu G^\mu{}_\nu \boldsymbol{\omega}^\nu.$$

(a) Give an expression for ${*}\boldsymbol{T}$ (or ${*}\boldsymbol{G}$) expanded in terms of basis vectors and forms.

(b) Show that

$$*\boldsymbol{T} = \boldsymbol{e}_\mu T^{\mu\nu}\, d^3\Sigma_\nu,$$

where $d^3\Sigma_\nu = \epsilon_{\nu|\alpha\beta\gamma|}\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma$ [see Box 5.4 and equations (8.10)].

(c) Compute $\boldsymbol{d}*\boldsymbol{T}$ using the generalized exterior derivative $\boldsymbol{d}$; find that

$$\boldsymbol{d}*\boldsymbol{T} = \boldsymbol{e}_\mu T^{\mu\nu}{}_{;\nu}\sqrt{|g|}\,\boldsymbol{\omega}^0 \wedge \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3.$$

2

CHAPTER 15

BIANCHI IDENTITIES AND THE BOUNDARY OF A BOUNDARY

This chapter is entirely Track 2.
As preparation, one needs to have covered (1) Chapter 4 (differential forms) and (2) Chapter 14 (computation of curvature).
In reading it, one will be helped by Chapters 9–11 and 13.
It is not needed as preparation for any later chapter, but it will be helpful in Chapter 17 (Einstein field equations).

§15.1. BIANCHI IDENTITIES IN BRIEF

Geometry gives instructions to matter, but how does matter manage to give instructions to geometry? Geometry conveys its instructions to matter by a simple handle: "pursue a world line of extremal lapse of proper time (geodesic)." What is the handle by which matter can act back on geometry? How can one identify the right handle when the metric geometry of Riemann and Einstein has scores of interesting features? Physics tells one what to look for: *a machinery of coupling between gravitation (spacetime curvature) and source* (matter; stress-energy tensor $\boldsymbol{T}$) *that will guarantee the automatic conservation of the source* ($\boldsymbol{\nabla} \cdot \boldsymbol{T} = 0$). Physics therefore asks mathematics: "What tensor-like feature of the geometry is automatically conserved?" Mathematics comes back with the answer: "The Einstein tensor." Physics queries, "How does this conservation come about?" Mathematics, in the person of Élie Cartan, replies, "Through the principle that 'the boundary of a boundary is zero'" (Box 15.1).

Actually, two features of the curvature are automatically conserved; or, otherwise stated, the curvature satisfies two Bianchi identities, the subject of this chapter. Both features of the curvature, both "geometric objects," lend themselves to representation in diagrams, moreover, diagrams that show in action the principle that "the boundary of a boundary is zero." In this respect, the geometry of spacetime shows a striking analogy to the field of Maxwell electrodynamics.

Identities and conservation of the source: electromagnetism and gravitation compared:

In electrodynamics there are four potentials that are united in the 1-form $\boldsymbol{A} \equiv A_\mu\, \boldsymbol{d}x^\mu$. Out of this quantity by differentiation follows the ***Faraday***, $\boldsymbol{F} = \boldsymbol{d}\boldsymbol{A}$. This field satisfies the identity $\boldsymbol{dF} = 0$ (identity, yes; identity lending itself to the definition of a conserved source, no).

$\boldsymbol{dF} \equiv 0$

In gravitation there are ten potentials (metric coefficients $g_{\mu\nu}$) that are united in the metric tensor $\boldsymbol{g} = g_{\mu\nu}\, \boldsymbol{d}x^{\mu} \otimes \boldsymbol{d}x^{\nu}$. Out of this quantity by two differentiations follows the curvature operator

$$\mathcal{R} = \frac{1}{4} \boldsymbol{e}_{\mu} \wedge \boldsymbol{e}_{\nu} R^{\mu\nu}{}_{\alpha\beta}\, \boldsymbol{d}x^{\alpha} \wedge \boldsymbol{d}x^{\beta}.$$

This curvature operator satisfies the Bianchi identity $\boldsymbol{d}\mathcal{R} = 0$, where now "$\boldsymbol{d}$" is a generalization of Cartan's exterior derivative, described more fully in Chapter 14 (again an identity, but again one that does not lend itself to the definition of a conserved source).

$\boldsymbol{d}\mathcal{R} \equiv 0$

In electromagnetism, one has to go to the dual, $\boldsymbol{{}^*F}$, to have any feature of the field that offers a handle to the source, $\boldsymbol{d{}^*F} = 4\pi\, \boldsymbol{{}^*J}$. The conservation of the source, $\boldsymbol{d{}^*J} = 0$, appears as a consequence of the identity $\boldsymbol{dd{}^*F} = 0$; or, by a rewording of the reasoning (Box 15.1), as a consequence of the vanishing of the boundary of a boundary.

$\boldsymbol{dd{}^*F} \equiv 0$ plus Maxwell equations $\Longrightarrow \boldsymbol{d{}^*J} = 0$

(*continued on page 370*)

Box 15.1 THE BOUNDARY OF A BOUNDARY IS ZERO

A. The Idea in Its 1-2-3-Dimensional Form

Begin with an oriented cube or approximation to a cube (3-dimensional).

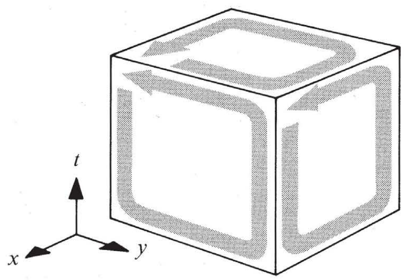
Its boundary is composed of six oriented faces, each two-dimensional. Orientation of each face is indicated by an arrow.

Boundary of any one oriented face consists of four oriented edges or arrows, each one-dimensional.

Every edge unites one face with another. No edge stands by itself in isolation.

"Sum" over all these edges, with due regard to sign. Find that any given edge is counted twice, once going one way, once going the other.

Conclude that the one-dimensional boundary of the two-dimensional boundary of the three-dimensional cube is identically zero.

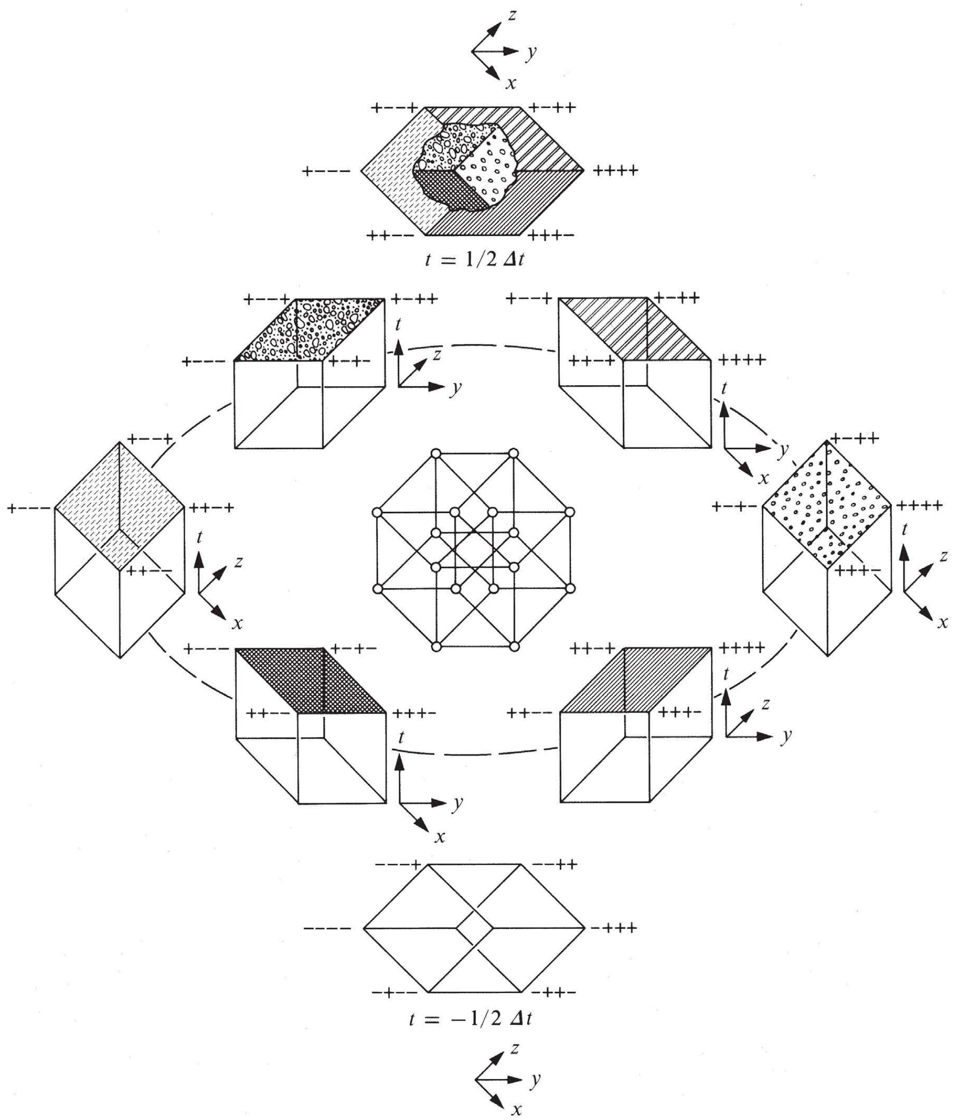


Box 15.1 ***(continued)***

B. The Idea in Its 2-3-4-Dimensional Form

Begin with an oriented four-dimensional cube or approximation thereto. The coordinates of the typical corner of the four-cube may be taken to be $(t_0 \pm \frac{1}{2}\Delta t,\ x_0 \pm \frac{1}{2}\Delta x,\ y_0 \pm \frac{1}{2}\Delta y,\ z_0 \pm \frac{1}{2}\Delta z)$; and, accordingly, a sample corner itself, in an obvious abbreviation, is conveniently abbreviated $+--+$. There are 16 of these corners. Less complicated in appearance than the 4-cube itself are its three-dimensional faces, which are "exploded off of it" into the surrounding area of the diagram, where they can be inspected in detail.

The boundary of the 4-cube is composed of eight oriented hyperfaces, each of them three-dimensional (top hyperface with extension $\Delta x\,\Delta y\,\Delta z$, for example; a "front" hyperface with extension $\Delta t\,\Delta y\,\Delta z$; etc.)



Boundary of any one hyperface ("cube") consists of six oriented faces, each two-dimensional.

Every face (for example, the hatched face $\Delta x\ \Delta y$ in the lower lefthand corner) unites one hypersurface with another (the "3-cube side face" $\Delta t\ \Delta x\ \Delta y$ in the lower lefthand corner with the "3-cube top face" $\Delta x\ \Delta y\ \Delta z$, in this example). No face stands by itself in isolation. The three-dimensional boundary of the 4-cube exposes no 2-surface to the outside world. It is *faceless.*

"Sum" over all these faces, with due regard to orientation. Find any given face is counted twice, once with one orientation, once with the opposite orientation.

Conclude that the two-dimensional boundary of the three-dimensional boundary of the four-dimensional cube is identically zero.

C. The Idea in Its General Abstract Form

$\partial\partial = 0$ (the boundary of a boundary is zero).

D. Idea Behind Application to Gravitation and Electromagnetism

The one central point is a law of conservation (conservation of charge; conservation of momentum-energy).

The other central point is "automatic fulfillment" of this conservation law.

"Automatic conservation" requires that source not be an agent free to vary arbitrarily from place to place and instant to instant.

Source needs a tie to something that, while having degrees of freedom of its own, will cut down the otherwise arbitrary degrees of freedom of the source sufficiently to guarantee that the source automatically fulfills the conservation law. Give the name "field" to this something.

Define this field and "wire it up" to the source in such a way that the conservation of the source shall be an automatic consequence of the "zero boundary of a boundary." Or, more explicitly: Conservation demands no creation or destruction of source inside the four-dimensional cube shown in the diagram. Equivalently, integral of "creation events" (integral of $\boldsymbol{d}{}^*\boldsymbol{J}$ for electric charge; integral of $\boldsymbol{d}{}^*\boldsymbol{T}$ for energy-momentum) over this four-dimensional region is required to be zero.

Integral of creation over this four-dimensional region translates into integral of source density-current (${}^*\boldsymbol{J}$ or ${}^*\boldsymbol{T}$) over three-dimensional boundary of this region. This boundary consists of eight hyperfaces, each taken with due regard to orientation. Integral over upper hyperface ("$\Delta x\ \Delta y\ \Delta z$)" gives amount of source present at later moment; over lower hyperface gives amount of souce present at earlier moment; over such hyperfaces as "$\Delta t\ \Delta x\ \Delta y$" gives outflow of source over intervening period of time. Conservation demands that sum of these eight three-dimensional integrals shall be zero (details in Chapter 5).

Box 15.1 (*continued*)

Vanishing of this sum of three-dimensional integrals states the conservation requirement, but does not provide the machinery for "automatically" (or, in mathematical terms, "identically") meeting this requirement. For that, turn to principle that "boundary of a boundary is zero."

Demand that integral of source density-current over any oriented hyperface $\mathcal{V}$ (three-dimensional region; "cube") shall equal integral of field over faces of this "cube" (each face being taken with the appropriate orientation and the cube being infinitesimal):

$$4\pi \int_{\mathcal{V}} {}^{*}\boldsymbol{J} = \int_{\partial\mathcal{V}} {}^{*}\boldsymbol{F}; \qquad 8\pi \int_{\mathcal{V}} {}^{*}\boldsymbol{T} = \int_{\partial\mathcal{V}} \begin{pmatrix}\text{moment of}\\ \text{rotation}\end{pmatrix}.$$

Sum over the six faces of this cube and continue summing until the faces of all eight cubes are covered. Find that any given face (as, for example, the hatched face in the diagram) is counted twice, once with one orientation, once with the other ("boundary of a boundary is zero"). Thus is guaranteed the conservation of source: integral of source density-current over three-dimensional boundary of four-dimensional region is automatically zero, making integral of creation over interior of that four-dimensional region also identically zero.

Repeat calculation with boundary of that four-dimensional region slightly displaced in one locality [the "bubble differentiation" of Tomonaga (1946) and Schwinger (1948)], and conclude that conservation is guaranteed, not only in the four-dimensional region as a whole, but at every point within it, and, by extension, everywhere in spacetime.

E. Relation of Source to Field

One view: Source is primary. Field may have other duties, but its prime duty is to serve as "slave" of source. Conservation of source comes first; field has to adjust itself accordingly.

Alternative view: Field is primary. Field takes the responsibility of seeing to it that the source obeys the conservation law. Source would not know what to do in absence of the field, and would not even exist. Source is "built" from field. Conservation of source is consequence of this construction.

One model illustrating this view in an elementary context: Concept of "classical" electric charge as nothing but "electric lines of force trapped in the topology of a multiply connected space" [Weyl (1924b); Wheeler (1955); Misner and Wheeler (1957)].

On any view: Integral of source density-current over any three-dimensional region (a "cube" in simplified analysis above) equals integral of field over boundary of this region (the six faces of the cube above). No one has ever found any other way to understand the correlation between field law and conservation law.

2

F. Electromagnetism as a Model: How to "Wire Up" Source to Field to Give Automatic Conservation of Source Via "$\partial\partial = 0$" in Its 2-3-4-Dimensional Form

Conservation means zero creation of charge (zero creation in four-dimensional region Ω).

Conservation therefore demands zero value for integral of charge density-current over three-dimensional boundary of this volume; thus,

$$0 = \int_{\Omega} \frac{\partial J^{\mu}}{\partial x^{\mu}}\, d^4\Omega = \int_{\partial\Omega} J^{\mu}\, d^3\Sigma_{\mu}$$

in the Track-1 language of Chapters 3 and 5. Equivalently, in the coordinate-free abstract language of §§4.3-4.6, one has

$$0 = \int_{\Omega} \boldsymbol{d}{}^*\boldsymbol{J} = \int_{\partial\Omega} {}^*\boldsymbol{J},$$

where

$$\begin{aligned} {}^*\boldsymbol{J} = {}&{}^*J_{123}\, \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 + {}^*J_{023}\, \boldsymbol{d}x^0 \wedge \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 \\ &+ {}^*J_{031}\, \boldsymbol{d}x^0 \wedge \boldsymbol{d}x^3 \wedge \boldsymbol{d}x^1 + {}^*J_{012}\, \boldsymbol{d}x^0 \wedge \boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 \end{aligned}$$

("eggcrate-like structure" of the 3-form of charge-density and current-density).

Fulfill this conservation requirement automatically ("identically") through the principle that "the boundary of a boundary is zero" by writing $4\pi\, {}^*\boldsymbol{J} = \boldsymbol{d}{}^*\boldsymbol{F}$; thus,

$$4\pi \int_{\partial\Omega} {}^*\boldsymbol{J} = \int_{\partial\Omega} \boldsymbol{d}{}^*\boldsymbol{F} = \int_{\partial\partial\Omega(\text{zero!})} {}^*\boldsymbol{F} \equiv 0$$

or, in Track-1 language, write $4\pi J^{\mu} = F^{\mu\nu}{}_{;\nu}$, and have

$$4\pi \int_{\partial\Omega} J^{\mu}\, d^3\Sigma_{\mu} = \int_{\partial\Omega} F^{\mu\nu}{}_{;\nu}\, d^3\Sigma_{\mu} = \int_{\partial\partial\Omega(\text{zero!})} F^{\mu\alpha}\, d^2\Sigma_{\mu\alpha} \equiv 0.$$

In other words, half of Maxwell's equations in their familiar flat-space form,

$$\operatorname{div} \boldsymbol{E} = \boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\pi\rho, \qquad \operatorname{curl} \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{B} = \dot{\boldsymbol{E}} + 4\pi\boldsymbol{J},$$

"wire up" the source to the field in such a way that the law of conservation of source follows directly from "$\partial\partial\Omega = 0$."

G. Electromagnetism Also Employs "$\partial\partial = 0$" in its 1-2-3-Dimensional Form ("No Magnetic Charge")

Magnetic charge is linked with field via $4\pi\boldsymbol{J}_{\text{mag}} = \boldsymbol{dF}$ (see point **F** above for translation of this compact Track-2 language into equivalent Track-1 terms). Absence of

Box 15.1 ***(continued)***

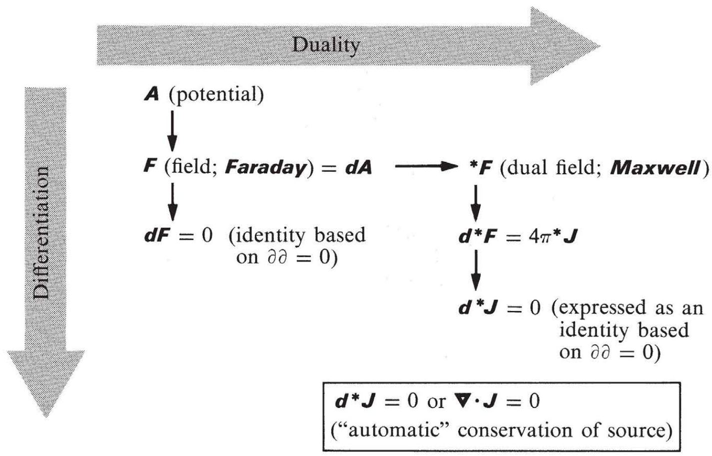
any magnetic charge says that integral of $\boldsymbol{J}_{\text{mag}}$ over any 3-volume $\mathcal{V}$ is necessarily zero; or ("integration by parts," generalized Stokes theorem)

$$0 = \int_{\mathcal{V}} \boldsymbol{dF} = \int_{\partial\mathcal{V}} \boldsymbol{F} = \begin{pmatrix}\text{total magnetic flux} \\ \text{exiting through } \partial\mathcal{V}\end{pmatrix}.$$

In order to satisfy this requirement "automatically," via principle that "the boundary of a boundary is zero," write $\boldsymbol{F} = \boldsymbol{dA}$ ("expression of field in terms of 4-potential"), and have

$$\int_{\partial\mathcal{V}} \boldsymbol{F} = \int_{\partial\mathcal{V}} \boldsymbol{dA} = \int_{\partial\partial\mathcal{V}\,(\text{zero!})} \boldsymbol{A} \equiv 0.$$

H. Structure of Electrodynamics in Outline Form

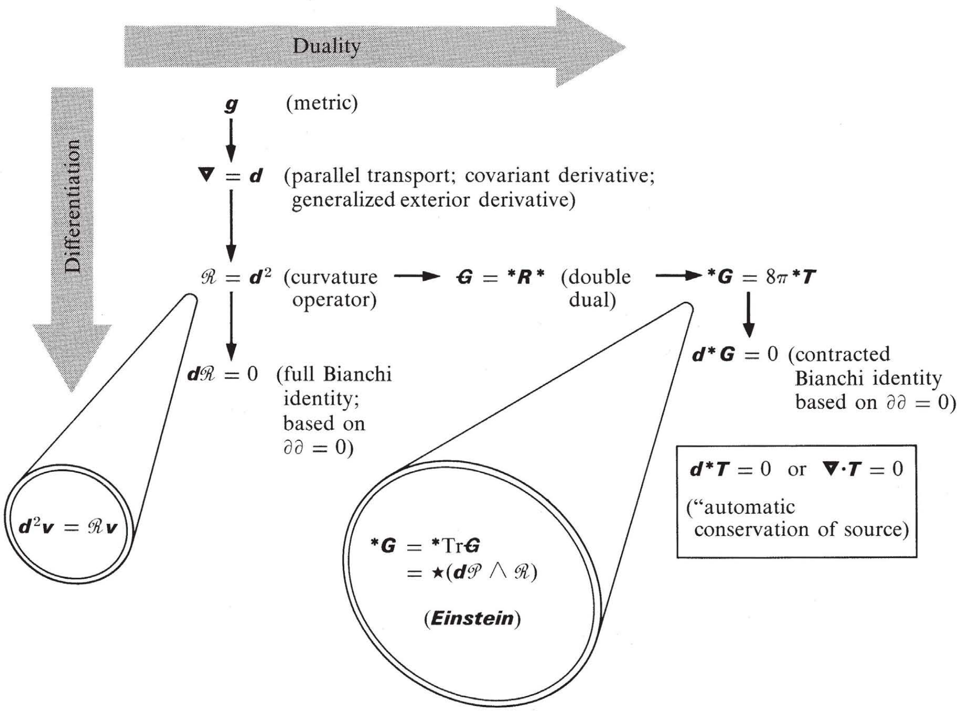


In gravitation physics, one has to go to the "double dual" (two pairs of alternating indices, two places to take the dual) $\boldsymbol{G} = {}^*\boldsymbol{R}^*$ of ***Riemann*** to have a feature of the field that offers a handle to the source:

$$\boldsymbol{G} = \mathrm{Tr}\boldsymbol{G} = \textbf{\textit{Einstein}} = 8\pi\boldsymbol{T} = 8\pi \times (\text{density of energy-momentum}).$$

The conservation of the source $\boldsymbol{T} \equiv \boldsymbol{e}_\mu T^\mu{}_\nu \boldsymbol{\omega}^\nu$ can be stated $\boldsymbol{\nabla}\cdot\boldsymbol{T} = 0$. But better suited for the present purpose is the form (see Chapter 14 and exercise 14.18)

I. Structure of Geometrodynamics in Outline Form



$$\boldsymbol{d}^{*}\boldsymbol{T} = 0,$$

$\boldsymbol{d}^{*}\boldsymbol{G} \equiv 0$ plus Einstein field equation $\Longrightarrow \boldsymbol{d}^{*}\boldsymbol{T} = 0$

where

$$^{*}\boldsymbol{T} \equiv \boldsymbol{e}_{\mu} T^{\mu}{}_{\nu}(^{*}\boldsymbol{\omega}^{\nu}) = \boldsymbol{e}_{\mu} T^{\mu\nu}\, d^{3}\Sigma_{\nu}.$$

This conservation law arises as a consequence of the "contracted Bianchi identity", $\boldsymbol{d}^{*}\boldsymbol{G} = 0$, again interpretable in terms of the vanishing of the boundary of a boundary.

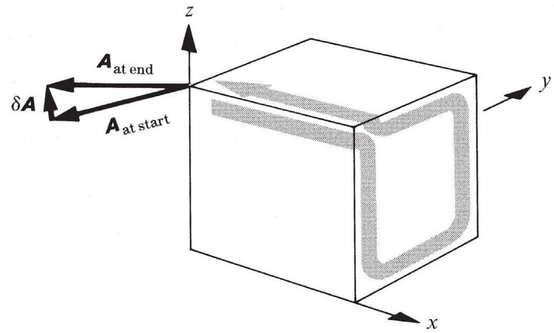


Figure 15.1.
Combine rotations associated with each of the six faces of the illustrated 3-volume and end up with zero net rotation ("full Bianchi identity"). Reason: Contribution of any face is measured by change in a test vector $\boldsymbol{A}$ carried in parallel transport around the perimeter of that face. Combine contributions of all faces and end up with each edge traversed twice, once in one direction, once in the other direction [boundary (here one-dimensional) of boundary (two-dimensional) of indicated three-dimensional figure is zero]. Detail: The vector $\boldsymbol{A}$, residing at the indicated site, is transported parallel to itself over to the indicated face, then carried around the perimeter of that face by parallel transport, experiencing in the process a rotation measured by the spacetime curvature associated with that face, then transported parallel to itself back to the original site. To the lowest relevant order of small quantities one can write

$$(\text{change in } \boldsymbol{A}) = -\Delta y\ \Delta z\ \mathcal{R}(\boldsymbol{e}_y, \boldsymbol{e}_z)\ \boldsymbol{A}$$

in operator notation; or in coordinate language,

$$-\delta A^\alpha = R^\alpha{}_{\beta yz}(\text{at } x + \Delta x)A^\beta\ \Delta y\ \Delta z.$$

§15.2. BIANCHI IDENTITY $d\mathcal{R} = 0$ AS A MANIFESTATION OF "BOUNDARY OF BOUNDARY = 0"

Bianchi identity, $\boldsymbol{d}\mathcal{R} \equiv 0$, interpreted in terms of parallel transport around the six faces of a cube.

Such is the story of the two Bianchi identities in outline form; it is now appropriate to fill in the details. Figure 15.1 illustrates the full Bianchi identity, $\boldsymbol{d}\mathcal{R} = 0$ (see exercise 14.17), saying in brief, "The sum of the curvature-induced rotations associated with the six faces of any elementary cube is zero." The change in a vector $\boldsymbol{A}$ associated with transport around the perimeter of the indicated face evaluated to the lowest relevant order of small quantities is given by

$$-\delta A^\alpha = R^\alpha{}_{\beta yz}\ (\text{at } x + \Delta x)A^\beta\ \Delta y\ \Delta z. \tag{15.1}$$

The opposite face gives a similar contribution, except that now the sign is reversed and the evaluation takes place at x rather than at $x + \Delta x$. The combination of the contributions from the two faces gives

$$\frac{\partial R^\alpha{}_{\beta yz}}{\partial x} A^\beta\ \Delta x\ \Delta y\ \Delta z, \tag{15.2}$$

when Riemann normal coordinates are in use. In such coordinates, the vanishing of the total $-\delta A^\alpha$ contributed by all six faces implies

$$R^\alpha{}_{\beta yz;x} + R^\alpha{}_{\beta zx;y} + R^\alpha{}_{\beta xy;z} = 0. \tag{15.3}$$

Here semicolons (covariant derivatives) can be and have been inserted instead of commas (ordinary derivatives), because the two are identical in the context of Riemann normal coordinates; and the covariant version (15.3) generalizes itself to arbitrary curvilinear coordinates. Turn from an xyz cube to a cube defined by any set of coordinate axes, and write Bianchi's identity in the form

$$R^\alpha{}_{\beta[\lambda\mu;\nu]} = 0. \tag{15.4}$$

(See exercise 14.17 for one reexpression of this identity in the abstract coordinate-independent form, $\boldsymbol{d}\mathscr{R} = 0$, and §15.3 for another.) This identity occupies much the same place in gravitation physics as that occupied by the identity $\boldsymbol{dF} = \boldsymbol{ddA} \equiv 0$ in electromagnetism:

$$F_{[\lambda\mu,\nu]} = F_{[\lambda\mu;\nu]} = 0. \tag{15.5}$$

§15.3. MOMENT OF ROTATION: KEY TO CONTRACTED BIANCHI IDENTITY

The contracted Bianchi identity, the identity that offers a "handle to couple to the source," was shown by Élie Cartan to deal with "moments of rotation" [Cartan (1928); Wheeler (1964b); Misner and Wheeler (1972)]. Moments are familiar in elementary mechanics. A rigid body will not remain at rest unless all the forces acting on it sum to zero:

$$\sum_i \boldsymbol{F}^{(i)} = 0. \tag{15.6}$$

Although necessary, this condition is not sufficient. The sum of the moments of these forces about some point $\mathscr{P}$ must also be zero:

$$\sum_i (\mathscr{P}^{(i)} - \mathscr{P}) \wedge \boldsymbol{F}^{(i)} = 0. \tag{15.7}$$

Exactly what point these moments are taken about happily does not matter, and this for a simple reason. The arbitrary point in the vector product (15.7) has for coefficient the quantity $\Sigma_i \boldsymbol{F}^{(i)}$, which already has been required to vanish. The situation is similar in the elementary cube of Figure 15.1. Here the rotation associated with a given face is the analog of the force $\boldsymbol{F}^{(i)}$ in mechanics. That the sum of these rotations vanishes when extended over all six faces of the cube is the analog of the vanishing of the sum of the forces $\boldsymbol{F}^{(i)}$.

What is the analog for curvature of the moment of the force that one encounters in mechanics? It is the *moment of the rotation associated with a given face of the*

Net moment of rotation over all six faces of a cube:

(1) described

cube. The value of any individual moment depends on the reference point $\mathcal{P}$. However, the sum of these moments taken over all six faces of the cube will have a value independent of the reference point $\mathcal{P}$, for the same reason as in mechanics. Therefore $\mathcal{P}$ can be taken where one pleases, inside the elementary cube or outside it. Moreover, the cube may be viewed as a bit of a hypersurface sliced through spacetime. Therefore $\mathcal{P}$ can as well be off the slice as on it. It is only required that all distances involved be short enough that one obtains the required precision by calculating the moments and the sum of moments in a local Riemann-normal coordinate system. One thus arrives at a $\mathcal{P}$-independent totalized moment of rotation (not necessarily zero; gravitation is not mechanics!) associated with the cube in question.

(2) equated to integral of source, $\int {}^*\boldsymbol{T}$, over interior of cube

Now comes the magic of "the boundary of the boundary is zero." Identify this net moment of rotation of the cube, evaluated by summing individual moments of rotation associated with individual faces, with the integral of the source density-current (energy-momentum tensor ${}^*\boldsymbol{T}$) over the interior of the 3-cube. Make this identification not only for the one 3-cube, but for all eight 3-cubes (hyperfaces) that bound the four-dimensional cube in Box 15.1. Sum the integrated source density-current ${}^*\boldsymbol{T}$ not only for the one hyperface of the 4-cube, but for all eight hyperfaces. Thus have

$$
\begin{aligned}
\int_{\text{4-cube}} \begin{pmatrix}\text{source}\\ \text{creation}\\ \boldsymbol{d}{}^*\boldsymbol{T}\end{pmatrix} &= \int_{\substack{\text{3-boundary}\\ \text{of this 4-cube}}} \begin{pmatrix}\text{source current-}\\ \text{density, } {}^*\boldsymbol{T}\end{pmatrix} \\
&= \sum_{\substack{\text{these eight}\\ \text{bounding}\\ \text{3-cubes}}} \begin{pmatrix}\text{net moment of rotation}\\ \text{associated with speci-}\\ \text{fied cube}\end{pmatrix} \\
&= \underbrace{\sum_{\substack{\text{eight}\\ \text{bounding}\\ \text{3-cubes}}} \sum_{\substack{\text{six faces}\\ \text{bounding}\\ \text{given 3-cube}}}}_{\text{(zero!)}} \begin{pmatrix}\text{moment of rotation}\\ \text{associated with specified}\\ \text{face of specified cube}\end{pmatrix}.
\end{aligned}
\tag{15.8}
$$

(3) conserved

Let the moments of rotation, not only for the six faces of one cube, but for all the faces of all the cubes, be taken with respect to one and the same point $\mathcal{P}$. Recall (Box 15.1) that any given face joins two cubes or hyperfaces. It therefore appears twice in the count of faces, once with one orientation ("sense of circumnavigation in parallel transport to evaluate rotation") and once with the opposite orientation. Therefore the double sum vanishes identically (boundary of a boundary is zero!) This identity establishes existence of a new geometric object, a feature of the curvature, that is conserved, and therefore provides a handle to which to couple a source. The desired result has been achieved. Now to translate it into standard mathematics!

2

§15.4. CALCULATION OF THE MOMENT OF ROTATION

It remains to find the tensorial character and value of this conserved Cartan moment of rotation that appertains to any elementary 3-volume. The rotation associated with the front face $\Delta y\, \Delta z \boldsymbol{e}_y \wedge \boldsymbol{e}_z$ of the cube in Figure 15.1 will be represented by the bivector

(4) evaluated

$$\begin{pmatrix}\text{rotation associated}\\ \text{with front } \Delta y\, \Delta z \text{ face}\end{pmatrix} = \boldsymbol{e}_\lambda \wedge \boldsymbol{e}_\mu R^{|\lambda\mu|}{}_{yz}\, \Delta y\, \Delta z \tag{15.9}$$

located at $\mathscr{P}_{\text{front}} = (t - \frac{1}{2}\Delta t,\ x + \Delta x,\ y + \frac{1}{2}\Delta y,\ z + \frac{1}{2}\Delta z)$. This equation uses Riemann normal coordinates; indices enclosed by strokes, as in $|\lambda\mu|$, are summed with the restriction $\lambda < \mu$. The moment of this rotation with respect to the point $\mathscr{P}$ will be represented by the trivector

$$\begin{pmatrix}\text{moment of rotation}\\ \text{associated with}\\ \text{front } \Delta y\, \Delta z \text{ face}\end{pmatrix} = (\mathscr{P}_{\substack{\text{center}\\ \text{of front}\\ \text{face}}} - \mathscr{P}) \wedge \boldsymbol{e}_\lambda \wedge \boldsymbol{e}_\mu R^{|\lambda\mu|}{}_{yz}\, \Delta y\, \Delta z. \tag{15.10}$$

Here neither $\mathscr{P}_{\text{center front}}$ nor $\mathscr{P}$ has any well-defined meaning whatsoever as a vector, but their difference is a vector in the limit of infinitesimal separation, $\Delta\mathscr{P} = \mathscr{P}_{\text{center front}} - \mathscr{P}$. With the back face a similar moment of rotation is associated, with the opposite sign, and with $\mathscr{P}_{\text{center front}}$ replaced by $\mathscr{P}_{\text{center back}}$. In the difference between the two terms, the factor $\mathscr{P}$ is of no interest, because one is already assured it will cancel out [Bianchi identity (15.4); analog of $\Sigma \boldsymbol{F}^{(i)} = 0$ in mechanics]. The difference $\mathscr{P}_{\text{center front}} - \mathscr{P}_{\text{center back}}$ has the value $\Delta x \boldsymbol{e}_x$. Summing over all six faces, one has

$$\begin{aligned}\begin{pmatrix}\text{net moment of}\\ \text{rotation associated}\\ \text{with cube or hyper-}\\ \text{face } \Delta x\, \Delta y\, \Delta z\end{pmatrix} = {}&\\ &\boldsymbol{e}_x \wedge \boldsymbol{e}_\lambda \wedge \boldsymbol{e}_\mu R^{|\lambda\mu|}{}_{yz}\, \Delta x\, \Delta y\, \Delta z \text{ (front and back)}\\ &+ \boldsymbol{e}_y \wedge \boldsymbol{e}_\lambda \wedge \boldsymbol{e}_\mu R^{|\lambda\mu|}{}_{zx}\, \Delta y\, \Delta z\, \Delta x \text{ (sides)}\\ &+ \boldsymbol{e}_z \wedge \boldsymbol{e}_\lambda \wedge \boldsymbol{e}_\mu R^{|\lambda\mu|}{}_{xy}\, \Delta z\, \Delta x\, \Delta y \text{ (top and bottom)}.\end{aligned} \tag{15.11}$$

This sum one recognizes as the value (on the volume element $\boldsymbol{e}_x \wedge \boldsymbol{e}_y \wedge \boldsymbol{e}_z\, \Delta x\, \Delta y\, \Delta z$) of the 3-form

$$\boldsymbol{e}_\nu \wedge \boldsymbol{e}_\lambda \wedge \boldsymbol{e}_\mu R^{|\lambda\mu|}{}_{|\alpha\beta|}\, \boldsymbol{d}x^\nu \wedge \boldsymbol{d}x^\alpha \wedge \boldsymbol{d}x^\beta.$$

Moreover this 3-form is defined, and precisely defined, at a point, whereas (15.11), applying as it does to an extended region, does not lend itself to an analysis that is at the same time brief and precise. Therefore forego (15.11) in favor of the 3-form. Only remember, when it comes down to interpretation, that this 3-form is to be evaluated for the "cube" $\boldsymbol{e}_x \wedge \boldsymbol{e}_y \wedge \boldsymbol{e}_z \,\Delta x\, \Delta y\, \Delta z$. Now note that the "trivector-valued moment-of-rotation 3-form" can also be written as

(5) abstracted to give $\boldsymbol{d}\mathscr{P} \wedge \mathscr{R}$

$$\begin{pmatrix}\text{moment of}\\ \text{rotation}\end{pmatrix} = \boldsymbol{d}\mathscr{P} \wedge \mathscr{R} = \boldsymbol{e}_\nu \wedge \boldsymbol{e}_\lambda \wedge \boldsymbol{e}_\mu R^{|\lambda\mu|}{}_{|\alpha\beta|}\, \boldsymbol{d}x^\nu \wedge \boldsymbol{d}x^\alpha \wedge \boldsymbol{d}x^\beta. \tag{15.12}$$

Here

$$\boldsymbol{d}\mathscr{P} = \boldsymbol{e}_\sigma\, \boldsymbol{d}x^\sigma \tag{15.13}$$

is Cartan's $\binom{1}{1}$ unit tensor. Also $\mathscr{R}$ is the curvature operator, treated as a bivector-valued 2-form:

$$\mathscr{R} = \boldsymbol{e}_\lambda \wedge \boldsymbol{e}_\mu R^{|\lambda\mu|}{}_{|\alpha\beta|}\, \boldsymbol{d}x^\alpha \wedge \boldsymbol{d}x^\beta. \tag{15.14}$$

Using the language of components as in (15.11), or the abstract language introduced in (15.12), one finds oneself dealing with a trivector. A trivector can be left a trivector, as, in quite another context, an element of 3-volume on a hypersurface in 4-space can be left as a trivector. However, there it is more convenient to take the dual representation, and speak of the element of volume as a vector. Denote by $\star$ a duality operation that acts only on contravariant vectors, trivectors, etc. (but *not* on forms). Then in a Lorentz frame one has $\star(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3) = \boldsymbol{e}_0$; but $\star(\boldsymbol{d}x^3) = \boldsymbol{d}x^3$. More generally,

$$\star(\boldsymbol{e}_\nu \wedge \boldsymbol{e}_\lambda \wedge \boldsymbol{e}_\mu) = \varepsilon_{\nu\lambda\mu}{}^\sigma \boldsymbol{e}_\sigma. \tag{15.15}$$

(6) abstracted to give $\star(\boldsymbol{d}\mathscr{P} \wedge \mathscr{R}) = \boldsymbol{e}_\sigma G^{\sigma\tau}\, d^3\Sigma_\tau$

In this notation, the "vector-valued moment-of-rotation 3-form" is

$$\begin{aligned}\begin{pmatrix}\text{moment}\\ \text{of rotation}\end{pmatrix} &= \star(\boldsymbol{d}\mathscr{P} \wedge \mathscr{R}) = \boldsymbol{e}_\sigma \varepsilon_{\nu\lambda\mu}{}^\sigma R^{|\lambda\mu|}{}_{|\alpha\beta|}\, \boldsymbol{d}x^\nu \wedge \boldsymbol{d}x^\alpha \wedge \boldsymbol{d}x^\beta \\ &= \boldsymbol{e}_\sigma ({}^*R)_\nu{}^\sigma{}_{|\alpha\beta|}\, \boldsymbol{d}x^\nu \wedge \boldsymbol{d}x^\alpha \wedge \boldsymbol{d}x^\beta,\end{aligned}$$

or, in one more step,

$$\begin{pmatrix}\text{moment}\\ \text{of rotation}\end{pmatrix} = \star(\boldsymbol{d}\mathscr{P} \wedge \mathscr{R}) = \boldsymbol{e}_\sigma ({}^*R^*)_\nu{}^{\sigma\nu\tau}\, d^3\Sigma_\tau. \tag{15.16}$$

Here $d^3\Sigma_\tau$ is a notation for basis 3-forms, as in Box 5.4; thus,

$$\boldsymbol{d}x^\nu \wedge \boldsymbol{d}x^\alpha \wedge \boldsymbol{d}x^\beta = \varepsilon^{\nu\alpha\beta\tau}\, d^3\Sigma_\tau. \tag{15.17}$$

(In a local Lorentz frame, $\boldsymbol{d}x^1 \wedge \boldsymbol{d}x^2 \wedge \boldsymbol{d}x^3 = d^3\Sigma_0$.)

Nothing is more central to the analysis of curvature than the formula (15.16). It starts with an element of 3-volume and ends up giving the moment of rotation in that 3-volume. The tensor that connects the starting volume with the final moment, the "contracted double-dual" of ***Riemann***, is so important that it deserves and receives a name of its own, $\boldsymbol{G} \equiv$ ***Einstein***; thus

$$(\text{Einstein})^{\sigma\tau} \equiv G^{\sigma\tau} = \mathcal{G}_\nu{}^{\sigma\nu\tau} = ({}^*R^*)_\nu{}^{\sigma\nu\tau}. \tag{15.18}$$

This tensor received attention in §§13.5 and 14.2, and also in the examples at the end of Chapter 14. In terms of ***Einstein***, the connection between element of 3-volume and "vector-valued moment of rotation" is

$$\begin{pmatrix}\text{moment}\\ \text{of rotation}\end{pmatrix} = \star(\boldsymbol{d}\mathscr{P} \wedge \mathscr{R}) = \boldsymbol{e}_\sigma G^{\sigma\tau}\, d^3\Sigma_\tau. \tag{15.19}$$

The amount of "vector-valued moment of rotation" contained in the element of 3-volume $d^3\Sigma_\mu$ is identified by general relativity with the amount of energy-momentum contained in that 3-volume. However, defer this identification for now. Concentrate instead on the conservation properties of this moment of rotation. See them once in the formulation of integral calculus, as a consequence of the principle "$\partial\partial \equiv 0$." See them then a second time, in differential formulation, as a consequence of "$\boldsymbol{dd} \equiv 0$."

§15.5. CONSERVATION OF MOMENT OF ROTATION SEEN FROM "BOUNDARY OF A BOUNDARY IS ZERO"

Conservation of net moment of rotation:

(1) derived from "$\partial\partial = 0$"

The moment of rotation defines an automatically conserved quantity. In other words, the value of the moment of rotation for an elementary 3-volume $\Delta x\, \Delta y\, \Delta z$ after the lapse of a time Δt is equal to the value of the moment of rotation for the same 3-volume at the beginning of that time, corrected by the inflow of moment of rotation over the six faces of the 3-volume in that time interval (quantities proportional to $\Delta y\, \Delta z\, \Delta t$, etc.) Now verify this conservation of moment of rotation in the language of "the boundary of a boundary." Follow the pattern of equation (15.8), but translate the words into formulas, item by item. Evaluate the amount of moment of rotation created in the elementary 4-cube Ω, and find

$$\text{"creation"} \underset{\substack{\uparrow\\ \text{definition}}}{\equiv} \int \begin{pmatrix}\text{"creation of moment of}\\ \text{rotation" in the elementary}\\ \text{4-cube of spacetime } \Omega\end{pmatrix} \underset{\substack{\uparrow\\ \text{definition}}}{=} \int_\Omega \boldsymbol{d}{}^*\boldsymbol{G};$$

$$\int_\Omega \boldsymbol{d}{}^*\boldsymbol{G} \underset{\substack{\uparrow\\ \text{step 1}}}{=} \int_{\partial\Omega} {}^*\boldsymbol{G} \underset{\substack{\uparrow\\ \text{step 2}}}{=} \int_{\partial\Omega} \star(\boldsymbol{d}\mathscr{P} \wedge \mathscr{R}) \underset{\substack{\uparrow\\ \text{step 3}}}{=} \sum_{\substack{\text{the eight}\\ \text{3-cubes}\\ \text{that bound } \Omega}} \star \begin{pmatrix}\text{moment of rotation}\\ \int_{\text{3-cube}} (\boldsymbol{d}\mathscr{P} \wedge \mathscr{R})\\ \text{associated with}\\ \text{specified 3-cube}\end{pmatrix} \underset{\substack{\uparrow\\ \text{step 4}}}{=}$$

$$\underset{\substack{\uparrow\\ \text{step 4}}}{=} \sum_{\substack{\text{eight bounding}\\ \text{3-cubes}}} \sum_{\substack{\text{six faces bounding}\\ \text{specified 3-cube}}} \star \begin{pmatrix}\text{moment of rotation}\\ \int_{\text{face}} (\mathscr{P} \wedge \mathscr{R})\\ \text{associated with}\\ \text{specified face of}\\ \text{specified cube}\end{pmatrix} \underset{\substack{\uparrow\\ \text{step 5}}}{\equiv} 0. \tag{15.20}$$

2

Here step 1 is the theorem of Stokes. Step 2 is the identification established by (15.19) between the Einstein tensor and the moment of rotation. Step 3 breaks down the integral over the entire boundary $\partial\Omega$ into integrals over the individual 3-cubes that constitute this boundary. Moreover, in all these integrals, the star $\star$ is treated as a constant and taken outside the sign of integration. The reason for such treatment is simple: the duality operation $\star$ involves only the metric, and the metric is locally constant throughout the infinitesimal 4-cube over the boundary of which the integration extends. Step 4 uses the formula

$$\boldsymbol{d}(\mathcal{P} \wedge \mathcal{R}) = \boldsymbol{d}\mathcal{P} \wedge \mathcal{R} + \mathcal{P} \wedge \boldsymbol{d}\mathcal{R} = \boldsymbol{d}\mathcal{P} \wedge \mathcal{R} \tag{15.21}$$

and the theorem of Stokes to express each 3-cube integral as an integral of $\mathcal{P} \wedge \mathcal{R}$ over the two-dimensional boundary of that cube. The culminating step is 5. It has nothing to do with the integrand. It depends solely on the principle $\partial\partial \equiv 0$.

In brief, the conservation of moment of rotation follows from two circumstances. (1) The moment of rotation associated with any elementary 3-cube is by definition a net value, obtained by adding the six moments of rotation associated with the six faces of that cube. (2) When one sums these net values for all eight 3-cubes in (15.20), which are the boundary of the elementary 4-cube Ω, one counts the contribution of a given 2-face twice, once with one sign and once with the opposite sign. In virtue of the principle that "the boundary of a boundary is zero," the conservation of moment of rotation is thus an identity.

§15.6. CONSERVATION OF MOMENT OF ROTATION EXPRESSED IN DIFFERENTIAL FORM

(2) derived from "$\boldsymbol{dd} = 0$"

Every conservation law stated in integral form lends itself to restatement in differential form, and conservation of moment of rotation is no exception. The calculation is brief. Evaluate the generalized exterior derivative of the moment of rotation in three steps, and find that it vanishes; thus:

$$\begin{aligned} \boldsymbol{d}{}^*\boldsymbol{G} &= \boldsymbol{d}[\star(\boldsymbol{d}\mathcal{P} \wedge \mathcal{R})] \\ &= \star[\boldsymbol{d}(\boldsymbol{d}\mathcal{P} \wedge \mathcal{R})] \\ &= \star[\boldsymbol{d}^2\mathcal{P} \wedge \mathcal{R} - \boldsymbol{d}\mathcal{P} \wedge \boldsymbol{d}\mathcal{R}] \\ &= 0 \end{aligned} \quad \begin{aligned} &\} \text{ step 1} \\ &\} \text{ step 2} \\ &\} \text{ step 3} \end{aligned}$$

Step 1 uses the relation $\boldsymbol{d}\star = \star\boldsymbol{d}$. The star duality and the generalized exterior derivative commute because when $\boldsymbol{d}$ is applied to a contravariant vector, it acts as a covariant derivative, and when $\star$ is applied to a covariant vector or 1-form, it is without effect. Step 2 applies the standard rule for the action of $\boldsymbol{d}$ on a product of tensor-valued forms [see equation (14.13b)]. Step 3 deals with two terms. The first term vanishes because the first factor in it vanishes; thus, $\boldsymbol{d}^2\mathcal{P} = 0$ [Cartan's equation of structure; expresses the "vanishing torsion" of the covariant derivative; see equation (14.26)]. The second term also vanishes, in this case, because the second factor in it vanishes; thus, $\boldsymbol{d}\mathcal{R} = 0$ (the full Bianchi identity). Thus briefly is conservation of moment of rotation established.

2

Box 15.2 THE SOURCE OF GRAVITATION AND THE MOMENT OF ROTATION: THE TWO KEY QUANTITIES AND THE MOST USEFUL MATHEMATICAL REPRESENTATIONS FOR THEM

	Energy-momentum as source of gravitation (curvature of space-time)	Moment of rotation as automatically conserved feature of the geometry
Representation as a vector-valued 3-form, a coordinate-independent geometric object	Machine to tell how much energy-momentum is contained in an elementary 3-volume: $\boldsymbol{*T} = \boldsymbol{e}_\sigma T^{\sigma\tau}\, d^3\Sigma_\tau$ ("dual of stress-energy tensor")	Machine to tell how much net moment of rotation—expressed as a vector—is obtained by adding the six moments of rotation associated with the six faces of the elementary 3-cube: $\star(\boldsymbol{d}\mathscr{P} \wedge \mathscr{R}) = \boldsymbol{*G} = \boldsymbol{e}_\sigma G^{\sigma\tau}\, d^3\Sigma_\tau$ ("dual of Einstein")
Representation as a $\binom{2}{0}$-tensor (also a coordinate independent geometric object)	Stress-energy tensor itself: $\boldsymbol{T} = \boldsymbol{e}_\sigma T^{\sigma\tau}\boldsymbol{e}_\tau$	Einstein itself: $\boldsymbol{G} = \boldsymbol{e}_\sigma G^{\sigma\tau}\boldsymbol{e}_\tau$
Representation in language of components (values depend on choice of coordinate system)	$T^{\sigma\tau}$	$G^{\sigma\tau}$
Conservation law in language of components	$T^{\sigma\tau}{}_{;\tau} = 0$	$G^{\sigma\tau}{}_{;\tau} \equiv 0$
Conservation in abstract language, for the $\binom{2}{0}$-tensor	$\boldsymbol{\nabla}\cdot\boldsymbol{T} = 0$	$\boldsymbol{\nabla}\cdot\boldsymbol{G} \equiv 0$
Conservation in abstract language, as translated into exterior derivative of the dual tensor (vector-valued 3-form)	$\boldsymbol{d*T} = 0$	$\boldsymbol{d*G} \equiv 0$ or $\boldsymbol{d}\star(\boldsymbol{d}\mathscr{P} \wedge \mathscr{R}) \equiv 0$
Same conservation law expressed in integral form for an element of 4-volume Ω	$\int_{\partial\Omega} \boldsymbol{*T} = 0$	$\int_{\partial\Omega} \boldsymbol{*G} \equiv 0$ or $\star\int_{\partial\Omega} (\boldsymbol{d}\mathscr{P} \wedge \mathscr{R}) \equiv 0$ or $\star\int_{\partial\partial\Omega} (\mathscr{P} \wedge \mathscr{R}) \equiv 0$

§15.7. FROM CONSERVATION OF MOMENT OF ROTATION TO EINSTEIN'S GEOMETRODYNAMICS: A PREVIEW

Einstein field equation "derived" from demand that (conservation of net moment of rotation) $\Longrightarrow$ (conservation of source)

Mass, or mass-energy, is the source of gravitation. Mass-energy is one component of the energy-momentum 4-vector. Energy and momentum are conserved. The amount of energy-momentum in the element of 3-volume $d^3\boldsymbol{\Sigma}$ is

$$\boldsymbol{*T} = \boldsymbol{e}_\sigma T^{\sigma\tau}\, d^3\Sigma_\tau \tag{15.22}$$

(see Box 15.2). Conservation of energy-momentum for an elementary 4-cube Ω expresses itself in the form

$$\int_{\partial\Omega} \boldsymbol{*T} = 0. \tag{15.23}$$

This conservation is not an accident. According to Einstein and Cartan, it is "automatic"; and automatic, moreover, as a consequence of exact equality between energy-momentum and an automatically conserved feature of the geometry. What is this feature? It is the moment of rotation, which satisfies the law of automatic conservation,

$$\int_{\partial\Omega} {}^*\boldsymbol{G} = 0. \tag{15.24}$$

In other words, the conservation of momentum-energy is to be made geometric in character and automatic in action by the following prescription: *Identify the stress-energy tensor* (up to a factor 8π, or $8\pi G/c^4$, or other factor that depends on choice of units) *with the moment of rotation;* thus,

$$\star(\boldsymbol{d}\mathcal{P} \wedge \mathcal{R}) = {}^*\boldsymbol{G} = 8\pi\, {}^*\boldsymbol{T}; \tag{15.25}$$

or equivalently (still in the language of vector-valued 3-forms)

$$\begin{pmatrix}\text{moment of}\\ \text{rotation}\end{pmatrix} = \star(\boldsymbol{d}\mathcal{P} \wedge \mathcal{R}) = \boldsymbol{e}_\sigma G^{\sigma\tau}\, d^3\Sigma_\tau = 8\pi \boldsymbol{e}_\sigma T^{\sigma\tau}\, d^3\Sigma_\tau; \tag{15.26}$$

or, in the language of tensors,

$$\boldsymbol{G} = \boldsymbol{e}_\sigma G^{\sigma\tau}\boldsymbol{e}_\tau = 8\pi\boldsymbol{e}_\sigma T^{\sigma\tau}\boldsymbol{e}_\tau = 8\pi\boldsymbol{T}; \tag{15.27}$$

or, in the language of components,

$$G^{\sigma\tau} = 8\pi T^{\sigma\tau} \tag{15.28}$$

(Einstein's field equation; more detail, and more on the question of uniqueness, will be found in Chapter 17; see also Box 15.3). Thus simply is all of general relativity tied to the principle that the boundary of a boundary is zero. No one has ever discovered a more compelling foundation for the principle of conservation of momentum and energy. No one has ever seen more deeply into that action of matter on space, and space on matter, which one calls gravitation.

In summary, *the Einstein theory realizes the conservation of energy-momentum as the identity, "the boundary of a boundary is zero."*

EXERCISES

Exercise 15.1. THE BOUNDARY OF THE BOUNDARY OF A 4-SIMPLEX

In the analysis of the development in time of a geometry lacking all symmetry, when one is compelled to resort to a computer, one can, as one option, break up the 4-geometry into simplexes [four-dimensional analog of two-dimensional triangle, three-dimensional tetrahedron; vertices of "central simplex" conveniently considered to be at $(t, x, y, z) = (0, 1, 1, 1)$, $(0, 1, -1, -1)$, $(0, -1, 1, -1)$, $(0, -1, -1, 1)$, $(5^{1/2}, 0, 0, 0)$, for example], sufficiently numerous, and each sufficiently small, that the geometry inside each can be idealized as flat (Lorentzian), with all the curvature concentrated at the join between simplices (see discussion of dynamics of geometry via Regge calculus in Chapter 42). Determine ("give a mathematical

Box 15.3 OTHER IDENTITIES SATISFIED BY THE CURVATURE

(1) The source of gravitation is energy-momentum. (2) Energy-momentum is expressed by stress-energy tensor (or by its dual) as a vector-valued 3-form ("energy-momentum per unit 3-volume"). (3) This source is conserved (no creation in an elementary spacetime 4-cube).

These principles form the background for the probe in this chapter of the Bianchi identities. That is why two otherwise most interesting identities [Allendoerfer and Weil (1943); Chern (1955, 1962)] are dropped from attention. One deals with the 4-form

$$\Pi = \frac{1}{24\pi^2} g^{\alpha\gamma} g^{\beta\delta} \mathscr{R}_{\alpha\beta} \wedge \mathscr{R}_{\gamma\delta}, \tag{1}$$

and the other with the 4-form

$$\Gamma = \frac{1}{8\pi^2 |\det g_{\mu\nu}|^{1/2}} (\mathscr{R}_{12} \wedge \mathscr{R}_{30} + \mathscr{R}_{13} \wedge \mathscr{R}_{02} + \mathscr{R}_{10} \wedge \mathscr{R}_{23}). \tag{2}$$

Both quantities are built from the tensorial "curvature 2-forms"

$$\mathscr{R}_{\alpha\gamma} = \frac{1}{2} R_{\alpha\gamma\beta\delta}\, \boldsymbol{dx}^\beta \wedge \boldsymbol{dx}^\delta. \tag{3}$$

The four-dimensional integral of either quantity over a four-dimensional region Ω has a value that (1) is a scalar, (2) is not identically equal to zero, (3) depends on the boundary of the region of spacetime over which the integral is extended, but (4) is independent of any changes made in the spacetime geometry interior to that surface (provided that these changes neither abandon the continuity nor change the connectivity of the 4-geometry in that region). Property (1) kills any possibility of identifying the integral, a scalar, with energy-momentum, a 4-vector. Property (2) kills it for the purpose of a conservation law, because it implies a non-zero creation in Ω.

Also omitted here is the Bel-Robinson tensor (see exercise 15.2), built bilinearly out of the curvature tensor, and other tensors for which see, e.g., Synge (1962).

One or all of these quantities may be found someday to have important physical content.

The integral of the 4-form Γ of equation (2) over the entire manifold gives a number, an integer, the so-called Euler-Poincaré characteristic of the manifold, whenever the integral and the integer are well-defined. This result is the four-dimensional generalization of the Gauss-Bonnet integral, widely known in the context of two-dimensional geometry:

$$\int \begin{pmatrix} \text{Riemannian scalar curvature} \\ \text{invariant (value } 2/a^2 \\ \text{for a sphere of radius } a) \end{pmatrix} g^{1/2}\, d^2x.$$

This integral has the value 8π for any closed, oriented, two-dimensional manifold with the topology of a 2-sphere, no matter how badly distorted; and the value 0 for any 2-torus, again no matter how rippled and twisted; and other equally specific values for other topologies.

description of") the boundary (three-dimensional) of such a simplex. Take one piece of this boundary and determine its boundary (two-dimensional). For one piece of this two-dimensional boundary, verify that there is at exactly one other place, and no more, in the book-keeping on the boundary of a boundary, another two-dimensional piece that cancels it ("facelessness" of the 3-boundary of the simplex).

Exercise 15.2. THE BEL-ROBINSON TENSOR [Bel (1958, 1959, 1962), Robinson (1959b), Sejnowski (1973); see also Pirani (1957) and Lichnerowicz (1962)].

Define the Bel-Robinson tensor by

$$T_{\alpha\beta\gamma\delta} = R_{\alpha\rho\gamma\sigma}R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} + {}^{*}R_{\alpha\rho\gamma\sigma}{}^{*}R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma}. \tag{15.29}$$

Show that in empty spacetime this tensor can be rewritten as

$$T_{\alpha\beta\gamma\delta} = R_{\alpha\rho\gamma\sigma}R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} + R_{\alpha\rho\delta\sigma}R_{\beta}{}^{\rho}{}_{\gamma}{}^{\sigma} - \frac{1}{8}g_{\alpha\beta}g_{\gamma\delta}R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda\mu}. \tag{15.30a}$$

Show also that in empty spacetime

$$T^{\alpha}{}_{\beta\gamma\delta;\alpha} = 0, \tag{15.30b}$$

$T_{\alpha\beta\gamma\delta}$ is symmetric and traceless on all pairs of indices. (15.30c)

Discussion: It turns out that Einstein's "canonical energy-momentum pseudotensor" (§20.3) for the gravitational field in empty spacetime has a second derivative which, in a Riemann-normal coordinate system, is

$$t_{\mathrm{E}\alpha\beta,\gamma\delta} = -\frac{4}{9}\left(T_{\alpha\beta\gamma\delta} - \frac{1}{4}S_{\alpha\beta\gamma\delta}\right). \tag{15.31a}$$

Here $T_{\alpha\beta\gamma\delta}$ is the completely symmetric Bel-Robinson tensor, and $S_{\alpha\beta\gamma\delta}$ is defined by

$$S_{\alpha\beta\gamma\delta} \equiv R_{\alpha\delta\rho\sigma}R_{\beta\gamma}{}^{\rho\sigma} + R_{\alpha\gamma\rho\sigma}R_{\beta\delta}{}^{\rho\sigma} + \frac{1}{4}g_{\alpha\beta}g_{\gamma\delta}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{15.31b}$$

$S_{\alpha\beta\gamma\delta}$ appears in the empty-space covariant wave equation

$$\Delta R_{\alpha\beta\gamma\delta} \equiv -R_{\alpha\beta\gamma\delta;\mu}{}^{;\mu} + R_{\alpha\beta\rho\sigma}R_{\gamma\delta}{}^{\rho\sigma} + 2(R_{\alpha\rho\gamma\sigma}R_{\beta}{}^{\rho}{}_{\delta}{}^{\sigma} - R_{\alpha\rho\delta\sigma}R_{\beta}{}^{\rho}{}_{\gamma}{}^{\sigma}) = 0, \tag{15.31c}$$

where Δ is a variant of the Lichnerowicz-de Rham wave operator [Lichnerowicz (1964)], when one rewrites this wave equation as

$$\Box R_{\alpha\beta}{}^{\gamma\delta} \equiv R_{\alpha\beta}{}^{\gamma\delta}{}_{;\mu}{}^{;\mu} = 2S_{[\alpha}{}^{[\gamma}{}_{\beta]}{}^{\delta]}. \tag{15.31d}$$

PART **IV**

EINSTEIN'S GEOMETRIC THEORY OF GRAVITY

Wherein the reader is seduced into marriage with the most elegant temptress of all—Geometrodynamics—and learns from her the magic potions and incantations that control the universe.

CHAPTER 16

EQUIVALENCE PRINCIPLE AND MEASUREMENT OF THE "GRAVITATIONAL FIELD"

Rather than have one global frame with gravitational forces we have many local frames without gravitational forces.

STEPHEN SCHUTZ (1966)

§16.1. OVERVIEW

With the mathematics of curved spacetime now firmly in hand, one is tempted to rush headlong into a detailed study of Einstein's field equations. But such temptation must be resisted for a short time more. To grasp the field equations fully, one must first understand how the classical laws of physics change, or do not change, in the transition from flat spacetime to curved (§§16.2 and 16.3); and one must understand how the "gravitational field" (metric; covariant derivative; spacetime curvature; . . .) can be "measured" (§§16.4 and 16.5).

Purpose of this chapter

§16.2. THE LAWS OF PHYSICS IN CURVED SPACETIME

Wherever one is and whenever one probes, one finds that then and there one can introduce a local inertial frame in which all test particles move along straight lines. Moreover, this local inertial frame is also locally Lorentz: in it the velocity of light has its standard value, and light rays, like world lines of test particles, are straight. But physics is more, and the analysis of physics demands more than an account solely of the motions of test particles and light rays. What happens to Maxwell's equations, the laws of hydrodynamics, the principles of atomic structure, and all the rest of physics under the influence of "powerful gravitational fields"?

Einstein's equivalence principle

The answer is simple: *in any and every local Lorentz frame, anywhere and anytime in the universe, all the (nongravitational) laws of physics must take on their familiar special-relativistic forms.* Equivalently: there is no way, by experiments confined to infinitesimally small regions of spacetime, to distinguish one local Lorentz frame in one region of spacetime from any other local Lorentz frame in the same or any other region. This is Einstein's principle of equivalence in its strongest form—a principle that is compelling both philosophically and experimentally. (For the relevant experimental tests, see §38.6.)

Equivalence principle as tool to mesh nongravitational laws with gravity

The principle of equivalence has great power. With it one can generalize all the special relativistic laws of physics to curved spacetime. And the curvature need not be small. It may be as large as that in the center of a neutron star; as large as that at the edge of a black hole; arbitrarily large, in fact—or almost so. Only at the endpoint of gravitational collapse and in the initial instant of the "big bang," i.e., only at "singularities of spacetime," will there be a breakdown in the conditions needed for direct application of the equivalence principle (see §§28.3, 34.6, 43.3, 43.4, and chapter 44). Everywhere else the equivalence principle acts as a tool to mesh all the nongravitational laws of physics with gravity.

Example: Mesh the "law of local energy-momentum conservation," $\boldsymbol{\nabla} \cdot \boldsymbol{T} = 0$, with gravity. *Solution:*

(1) The law in flat spacetime, written in abstract geometric form, reads

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} = 0. \tag{16.1a}$$

(2) Rewritten in a global Lorentz frame of flat spacetime, it reads

$$T^{\mu\nu}{}_{,\nu} = 0. \tag{16.1b}$$

(3) Application of equivalence principle gives same equation in local Lorentz frame of curved spacetime:

$$T^{\hat{\mu}\hat{\nu}}{}_{,\hat{\nu}} = 0 \text{ at origin of local Lorentz frame.} \tag{16.1c}$$

Because the connection coefficients vanish at the origin of the local Lorentz frame, this can be rewritten as

$$T^{\hat{\mu}\hat{\nu}}{}_{;\hat{\nu}} = 0 \text{ at origin of local Lorentz frame.} \tag{16.1d}$$

(4) The geometric law in curved spacetime, of which these are the local-Lorentz components, is

$$\boldsymbol{\nabla} \cdot \boldsymbol{T} = 0; \tag{16.1e}$$

and its component formulation in any reference frame reads

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{16.1f}$$

Compare the abstract geometric law (16.1e) in curved spacetime with the corresponding law (16.1a) in flat spacetime. They are identical! That this is not an accident one can readily see by tracing out the above four-step argument for any other law of physics (e.g., Maxwell's equation $\boldsymbol{\nabla} \cdot \boldsymbol{F} = 4\pi \boldsymbol{J}$). *The laws of physics, written in abstract geometric form, differ in no way whatsoever between curved spacetime and flat spacetime;* this is guaranteed by, and in fact is a mere rewording of, the equivalence principle.

Compare the component version of the law $\boldsymbol{\nabla} \cdot \boldsymbol{T} = 0$, as written in an arbitrary frame in curved spacetime [equation (16.1f)], with the component version in a global Lorentz frame of flat spacetime [equation (16.1b)]. They differ in only one way: the comma (partial derivative; flat-spacetime gradient) is replaced by a semicolon (covariant derivative; curved-spacetime gradient). This procedure for rewriting the equations has universal application. *The laws of physics, written in component form, change on passage from flat spacetime to curved spacetime by a mere replacement of all commas by semicolons* (no change at all physically or geometrically; change due only to switch in reference frame from Lorentz to non-Lorentz!). This statement, like the nonchanging of abstract geometric laws, is nothing but a rephrased version of the equivalence principle.

"Comma-goes-to-semicolon" rule

The transition in formalism from flat spacetime to curved spacetime is a trivial process when performed as outlined above. But it is nontrivial in its implications. It meshes gravity with all the laws of physics. Gravity enters in an essential way through the covariant derivative of curved spacetime, as one sees clearly in the following exercise.

EXERCISES

Exercise 16.1. HYDRODYNAMICS IN A WEAK GRAVITATIONAL FIELD

(a) In §18.4 it will be shown that for a nearly Newtonian system, analyzed in an appropriate nearly global Lorentz coordinate system, the metric has the form

$$ds^2 = -(1 + 2\Phi)\, dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) \tag{16.2a}$$

where Φ is the Newtonian potential ($-1 \ll \Phi < 0$). Consider a nearly Newtonian perfect fluid [stress-energy tensor

$$T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + p g^{\alpha\beta}, \qquad p \ll \rho; \tag{16.2b}$$

see Box 5.1 and §5.10] moving in such a spacetime with ordinary velocity

$$v^j \equiv dx^j/dt \ll 1. \tag{16.2c}$$

Show that the equations $T^{\mu\nu}{}_{;\nu} = 0$ for this system reduce to the familiar Newtonian law of mass conservation, and the Newtonian equation of motion for a fluid in a gravitational field:

$$\frac{d\rho}{dt} = -\rho \frac{\partial v^j}{\partial x^j}, \qquad \rho \frac{dv^j}{dt} = -\rho \frac{\partial \Phi}{\partial x^j} - \frac{\partial p}{\partial x^j}, \tag{16.3a}$$

where d/dt is the time derivative comoving with the matter

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v^j \frac{\partial}{\partial x^j}. \tag{16.3b}$$

(b) Use these equations to calculate the pressure gradient in the Earth's atmosphere as a function of temperature and pressure. In the calculation, use the nonrelativistic relation $\rho = n_M \mu_M$, where n_M is the number density of molecules and μ_M is the mean rest mass per molecule; use the ideal-gas equation of state

$$p = n_M kT \qquad (k = \text{Boltzmann's constant});$$

and use the spherically symmetric form, $\Phi = -M/r$, for the Earth's Newtonian potential. If the pressure at sea level is 1.01×10^6 dynes/cm^2, what, approximately, is the pressure on top of Mount Everest (altitude 8,840 meters)? (Make a reasonable assumption about the temperature distribution of the atmosphere.)

Exercise 16.2. WORLD LINES OF PHOTONS

Show that in flat spacetime the conservation law for the 4-momentum of a freely moving photon can be written

$$\nabla_{\boldsymbol{p}} \boldsymbol{p} = 0. \tag{16.4a}$$

According to the equivalence principle, this equation must be true also in curved spacetime. Show that this means photons move along null geodesics of curved spacetime with affine parameter λ related to 4-momentum by

$$\boldsymbol{p} = d/d\lambda \tag{16.4b}$$

In exercise 18.6 this result will be used to calculate the deflection of light by the sun.

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§16.3. FACTOR-ORDERING PROBLEMS IN THE EQUIVALENCE PRINCIPLE

Factor-ordering problems and coupling to curvature

On occasion in applying the equivalence principle to get from physics in flat spacetime to physics in curved spacetime one encounters "factor-ordering problems" analogous to those that beset the transition from classical mechanics to quantum mechanics.* *Example:* How is the equation (3.56) for the vector potential of electrodynamics to be translated into curved spacetime? If the flat-spacetime equation is written

$$-A^{\alpha,\mu}{}_{\mu} + A^{\mu}{}_{,\mu}{}^{\alpha} = 4\pi J^{\alpha},$$

then its transition ("comma goes to semicolon") reads

$$-A^{\alpha;\mu}{}_{\mu} + A^{\mu}{}_{;\mu}{}^{\alpha} = 4\pi J^{\alpha}. \tag{16.5}$$

However, if the flat-spacetime equation is written with two of its partial derivatives interchanged

$$-A^{\alpha,\mu}{}_{\mu} + A^{\mu,\alpha}{}_{\mu} = 4\pi J^{\alpha},$$

*For a discussion of quantum-mechanical factor-ordering problems, see, e.g., Merzbacher (1961), pp. 138–39 and 334–35; also Pauli (1934).

then its translation reads

$$-A^{\alpha;\mu}{}_{\mu} + A^{\mu;\alpha}{}_{\mu} = 4\pi J^{\alpha},$$

which can be rewritten

$$-A^{\alpha;\mu}{}_{\mu} + A^{\mu}{}_{;\mu}{}^{\alpha} + R^{\alpha}{}_{\mu}A^{\mu} = 4\pi J^{\alpha}. \tag{16.5$'$}$$

(Ricci tensor appears as result of interchanging covariant derivatives; see exercise 16.3.) Which equation is correct—(16.5) or (16.5′)? This question is nontrivial, just as the analogous factor-ordering problems of quantum theory are nontrivial. For rules-of-thumb that resolve this and most factor-ordering problems, see Box 16.1. These rules tell one that (16.5′) is correct and (16.5) is wrong (see Box 16.1 and §22.4).

EXERCISES

Exercise 16.3. NONCOMMUTATION OF COVARIANT DERIVATIVES

Let $\boldsymbol{B}$ be a vector field and $\boldsymbol{S}$ be a second-rank tensor field. Show that

$$B^{\mu}{}_{;\alpha\beta} = B^{\mu}{}_{;\beta\alpha} + R^{\mu}{}_{\nu\beta\alpha}B^{\nu} \tag{16.6a}$$

$$S^{\mu\nu}{}_{;\alpha\beta} = S^{\mu\nu}{}_{;\beta\alpha} + R^{\mu}{}_{\rho\beta\alpha}S^{\rho\nu} + R^{\nu}{}_{\rho\beta\alpha}S^{\mu\rho}. \tag{16.6b}$$

From equation (16.6a), show that

$$B^{\mu;\alpha}{}_{\mu} = B^{\mu}{}_{;\mu}{}^{\alpha} + R^{\alpha}{}_{\mu}B^{\mu}. \tag{16.6c}$$

[*Hint* for Track-1 calculation: Work in a local Lorentz frame, where $\Gamma^{\alpha}{}_{\beta\gamma} = 0$ but $\Gamma^{\alpha}{}_{\beta\gamma,\delta} \neq 0$; expand the lefthand side in terms of Christoffel symbols and partial derivatives; and use equation (8.44) for the Riemann tensor. An alternative Track-2 calculation notices that $\nabla_{\beta}\nabla_{\alpha}\boldsymbol{B}$ is *not* linear in $\boldsymbol{e}_{\alpha}$, and that $B^{\mu}{}_{;\alpha\beta}$ are *not* its components; but, rather, that

$$B^{\mu}{}_{;\alpha\beta} \equiv \underset{\uparrow\,\text{[Third-rank tensor]}}{\nabla\nabla\boldsymbol{B}}(\boldsymbol{\omega}^{\mu}, \boldsymbol{e}_{\alpha}, \boldsymbol{e}_{\beta}). \tag{16.7}$$

The calculation then proceeds as follows:

$$\begin{aligned}
\langle\boldsymbol{\omega}^{\mu}, \nabla_{\beta}\nabla_{\alpha}\boldsymbol{B}\rangle &= \langle\boldsymbol{\omega}^{\mu}, \nabla_{\beta}(\boldsymbol{e}_{\alpha}\cdot\nabla\boldsymbol{B})\rangle \\
&= \langle\boldsymbol{\omega}^{\mu}, (\nabla_{\beta}\boldsymbol{e}_{\alpha})\cdot\nabla\boldsymbol{B} + \boldsymbol{e}_{\alpha}\cdot(\nabla_{\beta}\nabla\boldsymbol{B})\rangle \\
&= \langle\boldsymbol{\omega}^{\mu}, \Gamma^{\nu}{}_{\alpha\beta}\boldsymbol{e}_{\nu}\cdot\nabla\boldsymbol{B} + \nabla\nabla\boldsymbol{B}(\ldots, \boldsymbol{e}_{\alpha}, \boldsymbol{e}_{\beta})\rangle \\
&= B^{\mu}{}_{;\nu}\Gamma^{\nu}{}_{\alpha\beta} + B^{\mu}{}_{;\alpha\beta}.
\end{aligned}$$

Consequently

$$\begin{aligned}
B^{\mu}{}_{;\alpha\beta} - B^{\mu}{}_{;\beta\alpha} &= \langle\boldsymbol{\omega}^{\mu}, [\nabla_{\beta}, \nabla_{\alpha}]\boldsymbol{B}\rangle - B^{\mu}{}_{;\nu}(\Gamma^{\nu}{}_{\alpha\beta} - \Gamma^{\nu}{}_{\beta\alpha}) \\
&= \langle\boldsymbol{\omega}^{\mu}, [\nabla_{\beta}, \nabla_{\alpha}]\boldsymbol{B}\rangle - \langle\boldsymbol{\omega}^{\mu}, \nabla_{(\nabla_{\beta}\boldsymbol{e}_{\alpha} - \nabla_{\alpha}\boldsymbol{e}_{\beta})}\boldsymbol{B}\rangle \\
&= \langle\boldsymbol{\omega}^{\mu}, ([\nabla_{\beta}, \nabla_{\alpha}] - \nabla_{[\boldsymbol{e}_{\beta},\boldsymbol{e}_{\alpha}]})\boldsymbol{B}\rangle = \langle\boldsymbol{\omega}^{\mu}, \mathcal{R}(\boldsymbol{e}_{\beta}, \boldsymbol{e}_{\alpha})\boldsymbol{B}\rangle \\
&= R^{\mu}{}_{\nu\beta\alpha}B^{\nu},
\end{aligned}$$

in agreement with (16.6a). Note: because of slight ambiguity in the abstract notation, one must think carefully about each step in the above calculation. Component notation, by contrast, is completely unambiguous.]

(*continued on page 392*)

Box 16.1 FACTOR ORDERING AND CURVATURE COUPLING IN APPLICATIONS OF THE EQUIVALENCE PRINCIPLE

The Problem

In what order should derivatives be written when applying the "comma-goes-to-semicolon rule"? Interchanging derivatives makes no difference in flat spacetime, but in curved spacetime it produces terms that couple to curvature, e.g., $2B^{\alpha}{}_{;[\gamma\beta]} \equiv B^{\alpha}{}_{;\gamma\beta} - B^{\alpha}{}_{;\beta\gamma} = R^{\alpha}{}_{\mu\beta\gamma}B^{\mu}$ for any vector field (see exercise 16.3). Hence, the problem can be restated: *When must the comma-goes-to-semicolon rule be augmented by terms that couple to curvature?*

The Solution

There is no solution in general, but in most cases the following types of mathematical and physical reasoning resolve the problem unambiguously.

A. *Mathematically, curvature terms almost always arise from the noncommutation of covariant derivatives.* Consequently, one needs to worry about curvature terms in any equation that contains a double covariant derivative (e.g., $-A^{\alpha,\mu}{}_{\mu} + A^{\mu}{}_{,\mu}{}^{\alpha} = 4\pi J^{\alpha}$); or in any equation whose derivation from more fundamental laws involves double covariant derivatives (e.g. $\boldsymbol{\nabla_u S} = 0$ in Example B(3) below). But one can ignore curvature coupling everywhere else (e.g., in Maxwell's first-order equations).

B. *Coupling to curvature can surely not occur without some physical reason.* Therefore, if one applies the comma-goes-to-semicolon rule only to physically measurable quantities (e.g., to the electromagnetic field, but not to the vector potential), one can "intuit" whether coupling to curvature is likely. *Examples:*

(1) *Local energy-momentum conservation.* A coupling to curvature in the equations $T^{\alpha\beta}{}_{;\beta} = 0$—e.g., replacing them by $T^{\alpha\beta}{}_{;\beta} = R^{\alpha}{}_{\beta\gamma\delta}T^{\beta\gamma}u^{\delta}$—would not make sense at all. In a local inertial frame such terms as $R^{\alpha}{}_{\beta\gamma\delta}T^{\beta\gamma}u^{\delta}$ would be interpreted as forces produced at a single point by curvature. But it should not be possible to feel curvature except over finite regions (geodesic deviation, etc.)! Put differently, the second derivatives of the gravitational potential (metric) can hardly produce net forces at a point; they should only produce tidal forces!

(2) *Maxwell's equations* for the electromagnetic field tensor. Here it would also be unnatural to introduce curvature terms. They would cause a breakdown in charge conservation, in the sense of termination of electric and magnetic field lines at points where there is curvature but no charge. To maintain charge conservation, one omits curvature coupling when one translates Maxwell's equations (3.32) and (3.36) into curved spacetime:

$$F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha, \qquad F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0.$$

Moreover, one continues to regard $F_{\mu\nu}$ as arising from a vector potential by the curved-spacetime translation of (3.54′)

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}.$$

These points granted, one can verify that the second of Maxwell's equations is automatically satisfied, and verify also that the first is satisfied if and only if

$$-A^{\alpha;\mu}{}_{\mu} + A^{\mu}{}_{;\mu}{}^{\alpha} + R^{\alpha}{}_{\mu}A^{\mu} = 4\pi J^\alpha.$$

(See §22.4 for fuller discussion and derivation.)

(3) *Transport law for Earth's angular-momentum vector.* If the Earth were in flat spacetime, like any other isolated body it would parallel-transport its angular-momentum vector $\boldsymbol{S}$ along the straight world line of its center of mass, $\nabla_{\boldsymbol{u}}\boldsymbol{S} = 0$ ("conservation of angular momentum"). When translating this transport law into curved spacetime (where the Earth actually resides!), can one ignore curvature coupling? No! Spacetime curvatures due to the moon and sun produce tidal gravitational forces in the Earth; and because the Earth has an equatorial bulge, the tidal forces produce a nonzero net torque about the Earth's center of mass. (In Newtonian language: the piece of bulge nearest the Moon gets pulled with greater force, and hence greater torque, than the piece of bulge farthest from the Moon.) Thus, in curved spacetime one expects a transport law of the form

$$\nabla_{\boldsymbol{u}}\boldsymbol{S} = (\text{Riemann tensor}) \times (\text{Earth's quadrupole moment}).$$

This curvature-coupling torque produces a precession of the Earth's rotation axis through a full circle in the plane of the ecliptic once every 26,000 years ("general precession"; "precession of the equinoxes"; discovered by Hipparchus about 150 B.C.). The precise form of the curvature-coupling term is derived in exercise 16.4.

Exercise 16.4. PRECESSION OF THE EQUINOXES

(a) Show that the transport law for the Earth's intrinsic angular momentum vector S^α in curved spacetime is

$$\frac{DS^\alpha}{D\tau} = \epsilon^{\alpha\beta\gamma\delta} \mathcal{I}_{\beta\mu} R^\mu{}_{\nu\gamma\zeta} u_\delta u^\nu u^\zeta. \tag{16.8}$$

Here $d/d\tau = \boldsymbol{u}$ is 4-velocity along the Earth's world line; $\mathcal{I}_{\beta\mu}$ is the Earth's "reduced quadrupole moment" (trace-free part of second moment of mass distribution), defined in the Earth's local Lorentz frame by

$$\mathcal{I}_{\hat{0}\hat{0}} = \mathcal{I}_{\hat{0}\hat{j}} = 0, \qquad \mathcal{I}_{\hat{j}\hat{k}} = \int \rho(x^{\hat{j}} x^{\hat{k}} - \frac{1}{3} \hat{r}^2 \delta_{jk})\, d^3\hat{x}; \tag{16.9}$$

and $R^\mu{}_{\nu\gamma\zeta}$ is the Riemann curvature produced at the Earth's location by the moon, sun, and planets. [*Hint:* Derive this result in the Earth's local Lorentz frame, ignoring the spacetime curvature due to the Earth. (In this essentially Newtonian situation, curvature components $R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}}$ due to the Earth, sun, moon, and planets superpose linearly; "gravity too weak to be nonlinear"). Integrate up the torque produced about the Earth's center of mass by tidal gravitational forces ("geodesic deviation"):

$$\left(\begin{array}{l}\text{acceleration at } x^{\hat{j}}, \text{ relative to center of mass } (x^{\hat{j}} = 0),\\ \text{produced by tidal gravitational forces but counterbalanced}\\ \text{in part by Earth's internal stresses}\end{array}\right)$$

$$= \left(\frac{d^2 x^{\hat{k}}}{d\hat{t}^2}\right)_{\substack{\text{geodesic}\\ \text{deviation}}} = -R^{\hat{k}}{}_{\hat{0}\hat{l}\hat{0}} x^{\hat{l}} \text{ [see equation (1.8}')\text{]};$$

$$\left(\begin{array}{l}\text{force per unit volume due to this}\\ \text{acceleration, relative to center}\\ \text{of mass}\end{array}\right)^{\hat{k}} = \underbrace{\rho}_{\text{mass density}} \frac{d^2 x^{\hat{k}}}{d\hat{t}^2} = -\rho R^{\hat{k}}{}_{\hat{0}\hat{l}\hat{0}} x^{\hat{l}};$$

$$\left(\begin{array}{l}\text{torque per unit volume relative}\\ \text{to center of mass}\end{array}\right)_{\hat{i}} = \epsilon_{\hat{0}\hat{i}\hat{j}\hat{k}} x^{\hat{j}} (-\rho R^{\hat{k}}{}_{\hat{0}\hat{l}\hat{0}} x^{\hat{l}});$$

$$\left(\begin{array}{l}\text{total torque about center}\\ \text{of mass}\end{array}\right)_{\hat{i}} = \int [\epsilon_{\hat{0}\hat{i}\hat{j}\hat{k}} x^{\hat{j}} (-\rho R^{\hat{k}}{}_{\hat{0}\hat{l}\hat{0}} x^{\hat{l}})]\, d^3\hat{x}.$$

Put this expression into a form involving $\mathcal{I}_{\hat{j}\hat{l}}$, equate it to $dS_{\hat{i}}/d\tau$, and then reexpress it in frame-independent, component notation. The result should be equation (16.8).]

(b) Rewrite equation (16.8) in the Earth's local Lorentz frame, using the equation

$$R^{\hat{j}}{}_{\hat{0}\hat{k}\hat{0}} = \partial^2 \Phi / \partial x^{\hat{j}}\, \partial x^{\hat{k}}$$

for the components of ***Riemann*** in terms of the Newtonian gravitational potential. (Newtonian approximation to Einstein theory. Track-2 readers have met this equation in Chapter 12; track-one readers will meet it in §17.4.)

(c) Calculate $dS^{\hat{j}}/d\hat{t}$ using Newton's theory of gravity from the beginning. The answer should be identical to that obtained in part (b) using Einstein's theory.

(d) Idealizing the moon and sun as point masses, calculate the long-term effect of the spacetime curvatures that they produce upon the Earth's rotation axis. Use the result of part (b), together with moderately accurate numerical values for the relevant solar-system parameters. [*Answer:* The Earth's rotation axis precesses relative to the axes of its local Lorentz frame ("precession of the equinoxes"; "general precession"); the precession period is 26,000 years. The details of the calculation will be found in any textbook on celestial mechanics.]