1

CH 1 CANONICAL QUANTIZATION OF FREE FIELDS

1.1 Canonical quantization of the free E-M field

→ Using units where $c = 1 = \epsilon_0$, Maxwell's equations reads:

$$\begin{cases} \bar{\nabla}\cdot\bar{E} = \rho \ \text{(gauss)} & \bar{\nabla}\cdot\bar{B} = 0 \\ \bar{\nabla}\times\bar{E} + \partial_0 \bar{B} = 0 & \bar{\nabla}\times\bar{B} - \partial_0\bar{E} = \vec{j} \end{cases}$$

with $\bar{E}(x^\mu)$ the electric field ; $\rho(x^\mu)$ the electric charge density
$\bar{B}(x^\mu)$ the magnetic field $\bar{j}(x^\mu)$ the electric current density.

PROP The continuity equation reads $\partial_0 \rho + \bar{\nabla}\cdot\bar{j} = 0$

DEMO Indeed, $\bar{\nabla}(\bar{\nabla}\times\bar{B} - \partial_0\bar{E} = \bar{j}) = -\partial_0 \bar{\nabla}\cdot\bar{E} = \bar{\nabla}\cdot\bar{j}$

$\Leftrightarrow -\partial_0 \rho = \bar{\nabla}\cdot\bar{j}$ where we used $\bar{\nabla}\cdot(\bar{\nabla}\times\bar{B}) = \partial_i \epsilon_{ijk}\partial_j B_k = -\partial_j \epsilon_{jik}\partial_i B_k$ ▨

DEF We introduce the electric 4-vector j^μ and the Maxwell tensor $F^{\mu\nu}$:

$$j^\mu \equiv \begin{pmatrix} \rho \\ j^1 \\ j^2 \\ j^3 \end{pmatrix} \text{ and } F^{\mu\nu} \equiv \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

→ Notice $F^{\mu\nu} = F^{[\mu\nu]} \Leftrightarrow F^{\mu\nu} = -F^{\nu\mu}$

PROP The Maxwell eq. are now manifestly Lorentz covariant:

$$\partial_\nu F^{\mu\nu} = j^\mu \ (1) \quad \text{and} \quad \epsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = 0 \Leftrightarrow \partial_{[\nu} F_{\lambda\sigma]} = 0 \ (2)$$

DEMO Indeed, $F^{0i} = E^i$, $F^{ij} = \epsilon^{ijk} B_k$

(1): $\partial_i F^{0i} = j^0 \Leftrightarrow \partial_i E^i = \rho$ and with $\mu = i$, $\nu \neq i$, we get

$\partial_\nu F^{i\nu} = \partial_0 F^{i0} + \partial_j F^{ij} = -\partial_0 E^i + \partial_j \epsilon^{ijk} B_k = j^i$

(2) For $\mu = 0$, $0 = \epsilon^{ijk}\partial_i F_{jk} = \epsilon^{ijk}\partial_i(\epsilon_{jkl} B^l) = 2\delta_l^i \partial_i B^l = \partial_l B^l$

Now, for $\mu = i$: $0 = \epsilon^{i0jk}\partial_0 F_{jk} + \epsilon^{ij0k}\partial_j F_{0k} + \epsilon^{ijk0}\partial_j F_{k0}$

$= \epsilon^{0ijk}(-\partial_0 F_{jk} + \partial_j F_{0k} - \partial_j F_{k0}) = \epsilon^{ijk}(-\partial_0 \epsilon_{jkl} B^l - 2\partial_j E_k)$

$= -2(-\partial_0 \delta_l^i B^l - \epsilon^{ijk}\partial_j E_k)$ ▨

PROP The continuity eq. in its invariant form is $\partial_\mu j^\mu = 0$

PROP (Helmoltz decomposition)
On $\mathbb{R}^3$ with suitable fall-off conditions, every vector field $\vec{v}$ admits a unique decomposition into a longitudinal and a transverse part: $\bar{v} = \bar{\nabla}\psi + \bar{\nabla} \times \vec{u}$

DEMO We consider a field $\vec{v}$ such that $v \sim 1/r$ when $r \to \infty$. Then the Laplacian Δ is invertible.

$\rightarrow \{\bar{\nabla}\times(\bar{\nabla}\times\bar{v})\}_i = \epsilon_{ijk}\partial_j \epsilon_{klm}\partial_l v_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l v_m$
$= \partial_m \partial_i v_m - \partial_l\partial_l v_i = \bar{\nabla}(\bar{\nabla}.\bar{v}) - \Delta\bar{v}$

$\rightarrow \Delta\bar{v} = \bar{\nabla}(\bar{\nabla}.\bar{v}) - \bar{\nabla}\times(\bar{\nabla}\times\bar{v}) \Leftrightarrow \bar{v} = \bar{\nabla}.\Delta^{-1}(\bar{\nabla}.\bar{v}) - \bar{\nabla}\times\Delta^{-1}(\bar{\nabla}\times\bar{v})$ $\boxtimes$

↳ Explicitly, $\psi = \Delta^{-1}(\bar{\nabla}.\bar{v})$ and $\bar{u} = -\Delta^{-1}(\bar{\nabla}\times\bar{v})$

→ For $\bar{v}$ such that $\bar{\nabla}.\bar{v} = 0$, we have $\bar{v} = \bar{\nabla}\times\bar{u}$
For $\bar{v}$ such that $\bar{\nabla}\times\bar{v} = 0$, we have $\bar{v} = \bar{\nabla}\psi$

→ We can build Δ^{-1} explicitly using green function, namely resolving

$$\Delta\phi(\bar{x}) = -\delta^{(3)}(\bar{x}-\bar{y}) \Leftrightarrow \phi(\bar{x}) = \frac{1}{4\pi}\frac{1}{|\bar{x}-\bar{y}|}$$ so that if

$$\Delta\phi(x) = j(x),\ \phi(\bar{x}) = \frac{-1}{4\pi}\int d^3y\, \frac{j(y)}{|\bar{x}-\bar{y}|} \sim \Delta^{-1} j(\bar{x})$$

→ Since $\bar{\nabla}.\bar{B} = 0$, we can write $\bar{B} = \bar{\nabla}\times\bar{A}$ with $\vec{A}$ a vector potential.
Using $\bar{\nabla}\times\bar{E} + \partial_0\bar{B} = 0$: $\bar{\nabla}\times(E + \partial_0 A) = 0 \Leftrightarrow \bar{E} = -\partial_0 A - \bar{\nabla}\phi$ for some ϕ, a scalar potential.
↳ ϕ and $\bar{A}$ are not uniquely defined. Let's consider ϕ', A' such that $\bar{B} = \bar{\nabla}\times\bar{A}'$ and $\bar{E} = -\partial_0 A' - \bar{\nabla}\phi' \Rightarrow \bar{\nabla}\times(\bar{A}'-\bar{A}) = 0$
$\Rightarrow \bar{A}' = \bar{A} + \bar{\nabla}\chi$ and from $0 = \bar{\nabla}(\partial_0\chi + \phi' - \phi)$, we see that
$\phi' = \phi - \partial_0\chi + f(t)$ such that $\lim_{r\to\infty} f(t) = 0 \Leftrightarrow f(t) = 0\ \forall r$

DEF Defining $A_\mu \equiv (-\phi, A_1, A_2, A_3)$, we get $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the gauge transformations of A_μ read:

$$A'_\mu = A_\mu + \partial_\mu \chi$$

→ The 2 quantization methods (canonical and path integral) require an action or a hamiltonian. We rewrite Maxwell eq. such that it comes from a variational principle.

DEF | The Maxwell action reads

$$S[A_\mu ; j^\mu] \equiv \int d^4x \left\{ \frac{-1}{4} F_{\mu\nu} F^{\mu\nu} + j^\mu A_\mu \right\}$$

→ Its variation reads

$$\delta_A S \sim \frac{-1}{2} (\delta F_{\mu\nu}) F^{\mu\nu} + \delta A_\mu . j^\mu$$

$$\sim \partial_\nu \delta A_\mu F^{\mu\nu} + \delta A_\mu . j^\mu \sim (-\partial_\nu F^{\mu\nu} + j^\mu) \delta A_\mu \overset{!}{=} 0 \Leftrightarrow \partial_\nu F^{\mu\nu} = j^\mu$$

⊙ Hamiltonian formulation of EM

→ To go to the hamiltonian formalism, we need to compute the conjugate momenta $p_i = \dfrac{\partial L}{\partial \dot{q}^i} \longleftrightarrow \Pi_\mu = \dfrac{\partial L}{\partial \dot{A}_\mu}$

We write $S = \int dt \int d^3x \left\{ \frac{-1}{2} F_{0i} F^{0i} - \frac{1}{4} F_{ij} F^{ij} + A_\mu j^\mu \right\}$ $\qquad \epsilon_{ijk} \epsilon^{ijl} = 2 \delta_k^l$

$$= \int dt \int d^3x \left\{ \frac{1}{2} \partial_0 A_i \partial_0 A^i - \frac{1}{2} B_i B^i + A_i j^i + A_0 j^0 + A_0 \partial_0 \partial_j A^j - \frac{1}{2} A_0 \Delta A_0 \right\}$$

↳ The conjugate momentum $\dfrac{\partial L}{\partial [\partial_0 A_0]} = 0$ doesn't appear in the action!

We cannot perform a Legendre transform. But since $S \sim \int A_0 (\partial_0 \partial_j A^j + j^0 - \frac{1}{2} \Delta A_0)$, imposing $\delta_{A_0} S \overset{!}{=} 0 \Rightarrow A_0 = \frac{1}{\Delta} (\partial_0 \partial_j A^j + j^0)$

↳ The EOM for A_0 can be solved algebraically for A_0 without invoking initial conditions ⇒ we can inject the solution in the action.

This gives to a **reduced action principle**. The reduced action reads:

$$S = \int d^4x \left\{ \frac{1}{2} \partial_0 A^i \partial_0 A^i + \frac{1}{2} \frac{1}{\Delta} (\partial_0 \partial_j A^j + j^0)(\partial_0 \partial_k A^k + j^0) - \frac{1}{2} B^i B_i + A_i j^i \right\}$$

$$= S[A^i ; j^\mu]$$

→ Using Helmoltz decomposition, we write for A_i:

$\vec{v} = \vec{\nabla} \psi + \vec{\nabla} \times \vec{\omega} \equiv \vec{\nabla} \psi + \vec{v}_\perp$ with $\vec{\nabla} . \vec{v}_\perp = 0$ and $\psi \equiv \Delta^{-1} (\vec{\nabla} . \vec{v})$

$A_i = \partial_i (\Delta^{-1} \partial_j A^j) + A_i^\perp$ such that $\vec{\nabla} . \vec{A}^\perp = 0$

→ We also have that $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}^\perp$

↳ We also have that $\int d^3x \, v^i \omega_i = \int d^3x \, (v^{i\perp} \omega_i^\perp + v^{iL} \omega_i^L)$

with $v^i \equiv v^{i\perp} + v^{iL}$

→ The reduced action becomes

$$S = \int d^4x \left\{ \frac{1}{2} \partial_0 A_i^\perp \partial A_\perp^i - \frac{1}{2} B^i B_i + \frac{1}{2} j^0 \frac{1}{\Delta} j^0 + A_i^\perp j_\perp^i \right\} \quad \text{with } B^i = B^i(A_\perp^i)$$

→ Computing $\frac{\delta L}{\delta A_i^\perp} = 0$, we get $\partial_0^2 \vec{A}_\perp = -\vec{\nabla} \times (\vec{\nabla} \times \vec{A}_T) + \vec{j}_T$

$$\partial_0^2 \vec{A}_\perp = \Delta \vec{A}_\perp + \vec{j}_\perp$$

↳ We can now write the conjugate momenta to $A_i^\perp$:

$$\pi_\perp^i(\bar{x}, t) = \frac{\delta L}{\delta \partial_0 A_i^\perp(\bar{x}, t)} = \partial_0 A_i^\perp(t, \bar{x})$$

→ The above formulation doesn't depend on $A_0 = -\phi$ nor $\vec{A}_L$, we are in the Coulomb gauge. We're left with the physical dof only.

DEF | Eliminate the non physical dof from a system before quantifying it is called reduction before quantization.

→ The hamiltonian now reads:

$$H = \int d^3x \left\{ \frac{1}{2} \pi_\perp^i \pi_i^\perp + \frac{1}{2} B^i B_i - \underbrace{\frac{1}{2} j^0 \frac{1}{\Delta} j_0}_{\text{pure source}} - A_i^\perp j_\perp^i \right\}$$

Since $\bar{E} = -\partial_0 \bar{A} + \bar{\nabla} A_0$, $\vec{E}_T = -\vec{\pi}_T$ and $\vec{E}_L = \bar{\nabla}\left(\frac{1}{\Delta} j^0\right)$, we can rewrite

$$H = \int d^3x \left\{ \frac{1}{2}\left(E^i E_i + B^i B_i\right) - A_i^\perp j_\perp^i \right\}$$

⊙ Electromagnetic radiation in a box:

→ We obtained wave equations: $\begin{cases} \dot{A}_{i\perp} = \pi_{i\perp} \\ \dot{\pi}_\perp^i = \Delta A_i^\perp \end{cases}$

Indeed, if $j^\mu = 0$, the E-M theory reduces on the classical level to the free wave eq. for $\vec{A}_\perp$. To see it, we compute

$$\delta H = \int d^3x \left\{ \delta \pi_i^\perp \pi_\perp^i - \delta A_k^\perp \left(\Delta A_\perp^k - \partial^k(\cancel{\vec{\nabla} \cdot \vec{A}_T})\right) \right\}$$

so that the Hamiltonian EOM are

$$\dot{A}_T^i = \{A_T^i, H\} = \frac{\delta H}{\delta \pi_i^\perp} = \pi_T^i \quad \text{and} \quad \dot{\pi}_i^T = \{\pi_i^T, H\} = \frac{-\delta H}{\delta A_T^i} = \Delta A_i^T$$

Then, $\dot{\pi}_\perp^i = \ddot{A}_T^i = \Delta A_T^i \Leftrightarrow \partial_\mu \partial^\mu A_T^i = 0$

→ In a box of size of length L with periodic boundary conditions, we can write $A_i(x)$ in a Fourier space:

$$A_i(x) = \bar{A}_i(t) + \sum_{\vec{k}\neq\vec{0}} \sqrt{\frac{\hbar}{2\omega L^3}}\, \tilde{A}_i(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}$$

with $k^i = \frac{2\pi n^i}{L}$, $n^i \in \mathbb{Z}$, $\omega(k) = \sqrt{k^2} = |k|$ and the factor $\sqrt{\hbar/2\omega L^3}$ is chosen for convenience.

→ by going in Fourier space, resolving the wave equation becomes simple.

↳ The general solution follows from $(\partial_t^2 + \omega^2)\tilde{A}_i(\vec{k},t) = 0$

$\Rightarrow \tilde{A}_i(\vec{k},t) = c_i(\vec{k}) e^{-i\omega t} + c_i^* e^{i\omega t}$, $c_i(\vec{k}) \in \mathbb{C}$ and $\bar{A}_i(t) = \bar{A}_i + \bar{\pi}_i t$ with $\bar{A}_i, \bar{\pi}_i \in \mathbb{R}$. In what follow, we discard the 0-mode : $\bar{A}_i = \bar{\pi}_i = 0$

→ Since A_i^T must be transverse, $\vec{\nabla}\cdot\vec{A}_\perp = 0 \Rightarrow k_i \tilde{A}^i(\vec{k},t) = 0$

DEF We introduce polarization vectors $\vec{e}^{\,m}(k)$ such that $e_i^3 = \frac{k_i}{|\vec{k}|}$ and $k_i e_i^{1,2} = 0$. They furnish an orthonormal frame: $\sum_m e_i^m e_j^m = \delta_{ij}$

→ We can then write $c_i(\vec{k}) \equiv a_1(\vec{k}) e_i^1(\vec{k}) + a_2(\vec{k}) e_i^2(\vec{k}) \equiv a_s(\vec{k}) e_i^s(\vec{k})$

↳ 2 dof corresponding to the 2 Fourier coeff.

→ Explicitly, we could pick:

$$\vec{e}^{\,1} = \frac{1}{k_\perp}(k_2, -k_1, 0)\,;\quad \vec{e}^{\,2} = \frac{1}{k k_\perp}(k_1 k_3, k_2 k_3, -k_\perp^2) \quad \text{with } k_\perp \equiv \sqrt{k_1^2 + k_2^2}$$

→ Discarding the 0-mode (subdominant in the calculation of the partition function), the general solution (with transverse condition) becomes:

$$A_i(x) = \sum_{\vec{k}\neq\vec{0}} \sqrt{\frac{\hbar}{2\omega(\vec{k})V}} \left(a_s(\vec{k})\, e_i^s(\vec{k})\, e^{ikx} + a_s^*(\vec{k})\, e_i^s(\vec{k})\, e^{-ikx} \right)$$

with $kx = k_\mu x^\mu = -\omega t + \vec{k}\cdot\vec{x}$

→ In the box, the hamiltonian reduces to:

$$H(t) = \frac{1}{2}\int_{box} d^3x \left(E_\perp^i(\vec{x}) E_i^\perp(\vec{x}) + B^i(\vec{x}) B_i(\vec{x}) \right)$$

↳ We need to compute $E_i^\perp = -\partial_0 A_i^\perp$ and $\vec{B} = \vec{\nabla}\times\vec{A}$

$\rightarrow E_i^{\perp} = -\partial_0 A_i^{\perp} = i \sum_{k \neq 0} \sqrt{\frac{\hbar}{2\omega V}} \left(\omega(k) a_s(k) e_i^s(k) e^{ikx} - \omega a_s^* e_i^s e^{-ikx} \right)$

$\rightarrow B^i(x) = i \sum_{k \neq 0} \sqrt{\frac{\hbar}{2\omega V}} \left(\epsilon^{ijk} k_j a_s(k) e_k^s(k) e^{ikx} - \epsilon^{ijk} k_j a_s^* e_k^s e^{ikx} \right.$

PROP Using $\int_{box} d^3x \exp\{i(k-k')\cdot\vec{x}\} = V \delta^3(k-k')$, one gets

$$H = \sum_{k \neq 0} \hbar\omega\, a_s^*(k)\, a^s(k)$$

$\rightarrow$ The hamiltonian becomes a superposition of harmonic oscillators, degenerated in $s = 1, 2$, the transverse polarisations.

$\rightarrow$ By defining $a_s(k,t) = a_s(k) e^{-i\omega t}$, we get $\dot{a}_s(k,t) = -i\omega\, a_s(k,t)$ and $\frac{d}{dt} a_s(k,t) = \{a_s(k,t), H\}$ only if we have the

following Poisson brackets:

$\{a_s(k), a_{s'}^*(k')\} = \frac{-i}{\hbar} \delta_{k,k'} \delta_{s,s'}$ and $\{a_s(k), a_{s'}(k')\} = 0$

$\{a_s^*(k), a_{s'}^*(k')\} = 0$

$\hookrightarrow$ This is equivalent to $\{A_i^T(x), \Pi_\perp^j(y)\} = \delta_i^j \delta_{x,y}$

⊙ A digression on the harmonic oscillator:

$\rightarrow$ Consider a collection of n decoupled harmonic oscillators with frequencies ω_a, $a = 1, \dots, n$.

$\rightarrow$ The associated lagrangian is $L = \frac{1}{2} \dot{q}^a \dot{q}_a - \frac{1}{2} \omega_{ab}^2 q^a q^b$ with $\omega_{ab}^2 \equiv \omega_{(a)}^2 \delta_{ab}$

$\rightarrow$ Canonical momenta are $\partial L / \partial \dot{q}^a = \dot{q}_a \equiv p_a$ and the hamiltonian is

$$H = \frac{1}{2} p_a p^a + \frac{1}{2} \omega_{ab}^2 q^a q^b$$

$\rightarrow$ The Poisson brackets are canonical: $\{q^a, p_b\} = \delta^a{}_b$, $\{q^a, q^b\} = 0 = \{p_a, p_b\}$

$\rightarrow$ We perform a change of variables:

$$\hat{a}_a \equiv \frac{\sqrt{\omega_{(a)}}\, \hat{q}^a + i \hat{p}^a / \sqrt{\omega_{(a)}}}{\sqrt{2\hbar}} \quad ; \quad \hat{a}_b^* = \frac{\sqrt{\omega_{(b)}}\, \hat{q}^b - i \hat{p}^b / \sqrt{\omega_{(b)}}}{\sqrt{2\hbar}}$$

Inversing, we get:

$$\hat{q}^a = \sqrt{\frac{\hbar}{2\omega_{(a)}}} \left(\hat{a}^a + \hat{a}^{+a} \right) \quad \text{and} \quad \hat{p}_a = -i \sqrt{\frac{\hbar\omega_{(a)}}{2}} \left(\hat{a}_a - \hat{a}_a^{+} \right)$$

→ The canonical commutation relations become
$$[\hat{a}^a, \hat{a}^{\dagger b}] = \delta^{ab} \text{ and } [\hat{a}^a, \hat{a}^b] = 0 = [\hat{a}^{\dagger a}, \hat{a}^{\dagger b}]$$
and the hamiltonian is given by:
$$H = \hbar \omega_{ab} \left(\hat{a}^{\dagger a} \hat{a}^b + \frac{1}{2} \delta^{ab} \right)$$

→ We used the quantization rule: for $A(q,p)$, $B(q,p)$ two function on the phase space with Poisson bracket $\{A, B\}$, their equivalent quantum operator follows the following commutation relation:
$$[A, B] = i\hbar \widehat{\{A, B\}} + O(\hbar^2)$$

↳ The time evolution is given by $\dot{f} = \{f, H\}$ for any $f = f(q,p)$.

⊙ Hilbert space:

→ For 1 HO, a complete set of orthonormal states is given by
$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle, \text{ with } \langle m|n\rangle = \delta_{mn}$$

→ For n HO, the Hilbert space $\mathcal{H}$ is the Fock space generated by the creation operators $\hat{a}_s^\dagger(\vec{k})$ for each mode $\vec{k}, s$:
$$\mathcal{H} = \bigotimes_{\vec{k},s} \mathcal{H}_{\vec{k},s} \text{ with } |n_{\vec{k},s}\rangle \equiv \frac{(\hat{a}_s^\dagger(\vec{k}))^{n_{\vec{k},s}}}{\sqrt{n_{\vec{k},s}}} |0\rangle \in \mathcal{H}_{\vec{k},s}$$

↳ The hamiltonian operator in the normal ordered form is given by
$$\hat{H} = \sum_{\vec{k},s} \hbar \omega(k) \hat{a}^\dagger_{(s)}(k) \hat{a}_{(s)}(k) \quad \text{(dropped the 1/2)}$$

1.2 Partition function and Thermodynamics

DEF] The partition function Z is defined as
$$Z \equiv \mathrm{Tr}\{\exp\{-\beta \hat{H}\}\} \text{ where } \beta \equiv (k_B T)^{-1}$$

→ For a sum of non interacting HO, the partition function factorizes:
$$Z = \mathrm{Tr}\, e^{-\beta \hat{H}} = \prod_{\vec{k},s} \sum_{n_{\vec{k},s}} \langle n_{k,s} | e^{-\beta \hat{H}_{k,s}} | n_{k,s} \rangle$$
$$= \prod_{k,s} \sum_{n_{k,s}} \exp\{-\beta \hbar \omega\, n_{k,s}\} \qquad \sum_{n=0}^{\infty} c^n = (1-c)^{-1}$$
$$= \prod_{k,s} (1 - e^{-\beta \hbar k})^{-1}$$

→ We get $\ln Z = -\sum_{k,s} (1 - e^{-\beta \hbar k}) = -2 \sum_{k} (1 - e^{-\beta \hbar k})$

→ Using the Euler-Maclaurin formula: $\sum_{n \in \mathbb{Z}} \rightsquigarrow \int_{-\infty}^{\infty} dn = \frac{L}{2\pi} \int_{-\infty}^{\infty} dk$ with $k = \frac{2\pi}{L} n$ and taking $V \to \infty$, we get:
$$\ln Z = -2 \cdot \left(\frac{L}{2\pi}\right)^3 \int d^3k \, \ln(1 - e^{-\beta \hbar k})$$
$$= -2 \cdot \frac{V}{(2\pi)^3} \cdot 4\pi \int_0^{\infty} dk \cdot k^2 \cdot \ln(1 - e^{-\beta \hbar k}) \qquad k \mapsto x/\beta\hbar \quad e^{-\beta \hbar k} \mapsto e^{-x}$$
$$= -\frac{V}{\pi^2} \cdot \frac{1}{\beta \hbar} \cdot \frac{1}{(\beta \hbar)^2} \int_0^{\infty} dx \cdot x^2 \cdot \ln(1 - e^{-x}) \qquad x^2 = f' ;\ \ln(1-e^{-x}) = g \text{ and } \int f'g = [fg] - \int f g'$$
$$= -\frac{\beta^{-3} V}{\hbar^3 \pi^2} \cdot (-1) \cdot \int_0^{\infty} dx \cdot \frac{x^3}{3} \cdot \frac{1}{1 - e^{-x}}$$
$$= \frac{\beta^{-3} V}{3 \hbar^3 \pi^2} \int_0^{\infty} \frac{x^3}{e^x - 1} = \frac{\beta^{-3} V}{3 \hbar^3 \pi^2} \cdot \Gamma(4) \cdot \zeta(4) \qquad \Gamma(4) = 3! ;\ \zeta(4) = \pi^4/90$$

We get: $\boxed{\ln Z = \frac{\beta^{-3} V \pi^2}{45 \hbar^3} = \frac{b}{3} \cdot V \cdot \beta^{-3}}$

→ $U = \langle H \rangle = -\partial_\beta \ln Z = b V \beta^{-4} \Leftrightarrow \beta = (U/bV)^{-1/4}$