CH4 RENORMALIZATION AND ASYMPTOTIC BEHAVIOR

4.1 Casimir effect

4.1.1 Scalar Casimir effect

→ We consider a massless scalar field in 1+1 dimensions:
$S = \int dt \int dx \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2\right)$ and $H = \int dx \left(\frac{1}{2}\pi^2 + \frac{1}{2}\phi'^2\right)$

We quantize the field by going to momentum space:

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega_k}} \left(a(k)\, e^{-i\omega t + ikx} + a^*(k)\, e^{+i\omega t - ikx}\right)$$

$$\pi(x,t) = \partial_t \phi = -i \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \sqrt{\frac{\omega_k}{2}} \left(a(k)\, e^{-i\omega t + ikx} - h.c.\right)$$ where $\omega_k = |k|$

The canonical quantization $\Leftrightarrow [\hat{a}(k), \hat{a}^\dagger(k')] = \delta(k-k')$

→ The hamiltonian in the symmetric ordering prescription is

$$\hat{H} = \int dk\, \omega_k \left(\hat{a}^\dagger(k)\hat{a}(k) + 1/2\right)$$

↳ The vacuum energy, or the zero-point energy, is infinite,

$$E_{0\text{-}M} = \langle 0|\hat{H}|0\rangle = \frac{1}{2}\int dk\, |k| = \int_0^\infty dk\, k = \infty$$

→ 0 Minkowski

↳ To avoid it, we quantize the field in a box $(-L/2, L/2)$ with periodic boundary conditions.

→ Integral ↦ Fourier serie, with $k = \frac{2\pi}{L} n$, $\omega_k = \frac{2\pi}{L}|n|$, $n \in \mathbb{Z}$.

Thus

$$\phi(x,t) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2\omega_k L}} \left(e^{-i\omega_k t + ikx}\, a(k) + h.c.\right)$$

and $H = \sum_{n\in\mathbb{Z}} \omega_k \left(\hat{a}^\dagger(k)\hat{a}(k) + 1/2\right)$

Going to the continuum: $\sum_n \mapsto \int_{-\infty}^{\infty} dn = \frac{L}{2\pi}\int_{-\infty}^{\infty} dk$

→ The 0 mode energy becomes

Σ note

$$E_{0M} = \frac{L}{2\pi}\int_{\infty}^{\infty} dk\, k$$

→ Imposing Dirichlet conditions $\phi(0) = 0 = \phi(L)$, the field becomes

$\phi(x,t) = \sum_{n>0} e^{-i\omega_k t}\, \phi_k \sin(kx)$ with $k = \frac{\pi}{L} n$

and

$H = \sum_{n>0} \omega_k \left(\hat{a}^\dagger(k)\hat{a}(k) + 1/2\right)$ with $a(k) = \sqrt{\frac{\omega_k L}{2}}\left(\phi(x) + i\pi(x)\right)$

the vacuum energy is $E_{0D}(L) = \langle 0|\hat{H}|0\rangle = \frac{\pi}{2L}\sum_{n>0} n \; (*)$
↳ Dirichlet

⊙ Regularization:

→ Let us regularize E_{0M} and E_{0D} by multiplying by $e^{-\delta k}$ and then take $\delta \to 0$.

↳ $E_{0M} = \frac{L}{2\pi}\int_0^\infty dk \; k \; e^{-\delta k} = \frac{L}{2\pi}\left(\left[k(-1/\delta)e^{-\delta k}\right] + \int_0^\infty \frac{1}{\delta} e^{-\delta k}\right)$

$= \frac{L}{2\pi\delta}\left[-\frac{1}{\delta}e^{-\delta k}\right]_0^\infty \Rightarrow E_{0M} = L/2\pi\delta^2$

↳ $E_{0D} = \sum_{n>0} \frac{1}{2}\frac{\pi}{L} n \, e^{-\frac{\pi}{L}n\delta}$

Now, $\sum_{n>0} e^{-nx} = (1-e^{-x})^{-1}$
and $-\partial_x \sum e^{-nx} = \sum n e^{-nx} = \frac{e^{-x}}{(1-e^{-x})^2} = (2\,sh(x/2))^{-2}$

$= \frac{\pi}{2L}\sum_{n>0} \left(n e^{-\frac{\pi}{L}\delta}\right)^n \Rightarrow E_{0D} = \frac{\pi}{8L}\left(\sinh\left(\frac{\pi}{2L}\delta\right)\right)^{-2}$

$x = \frac{\pi}{L}\delta$

Developping E_{0D} around $\delta = 0$, $\quad \sinh x = x + \frac{x^3}{3!} + \mathcal{O}(x^5)$

$E_{0D} \simeq \frac{\pi}{8L}\left(\frac{\pi}{2L}\delta + \frac{1}{3!}\left(\frac{\pi}{2L}\delta\right)^3\right)^{-2}$

$\simeq \frac{\pi}{8L}\left(\frac{\pi}{2L}\delta\right)^{-2}\left(1 - \frac{2}{3!}\left(\frac{\pi}{2L}\delta\right)^2\right) \simeq \frac{L}{2\pi}\delta^2 - \frac{\pi}{24L}$

PROP The renormalized vacuum energy E_R reads

$$E_R \equiv \lim_{\delta \to 0}\left(E_{0D} - E_{0M}\right) = \frac{-\pi}{24L}$$

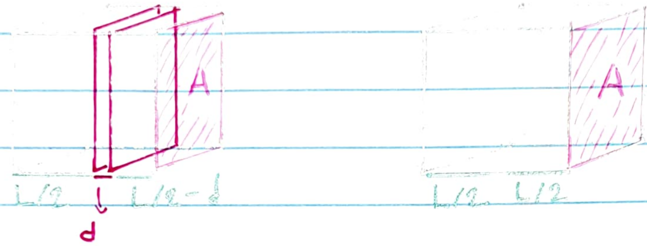
The Casimir force is therefore $F = -\frac{\partial E}{\partial L} = \frac{-\pi}{24L^2}$

→ For 2 perfectly conducting parallel plates of area A, one finds an attracting force $F = -\frac{\pi^2}{240}\frac{\hbar c}{L^4}$

→ The Casimir effect is purely quantum.

4.1.2 Electromagnetic Casimir effect:

→ For the E-M effect, one considers the zero-point energy between 2 perfectly conducting metallic plates separated by a distance d.
In order to compare and substract the result for empty Minkowski spacetime, we consider the following setup:
→ A finite box of size $L\times L\times L$
→ A situation with the plates
→ A situation without the plates
The Casimir energy $E_c(d)$ is the difference in vacuum energy between these 2 setup, taking $L\to\infty$.



↳ $E_c(d) = \lim_{L\to\infty}\left\{E(L/2)+E(d)+E(L/2-d)-E(L/2)-E(L/2)\right\}$

$= \lim_{L\to\infty}\left\{E(d)+E(\tfrac{L}{2}-d)-E(L/2)\right\}$

→ Perfectly conducting plates mean that $\begin{cases}\vec{n}\cdot\vec{B}\big|_{\text{plates}} = \vec{n}\cdot(\vec{\nabla}\times\vec{A})=0 \\ \vec{n}\times\vec{E}\big|_{\text{plates}} = \vec{n}\times(-\vec{\pi})=0\end{cases}$ at $z=0,\ z=d$

$\Leftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}\pi_x\\\pi_y\\\pi_z\end{pmatrix}=\begin{pmatrix}-\pi_y\\\pi_x\\0\end{pmatrix}$, $\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}\partial_x\\\partial_y\\\partial_z\end{pmatrix}\times\begin{pmatrix}A_x\\A_y\\A_z\end{pmatrix}$ $\begin{cases}\pi_x=0=\pi_y \text{ on the plates} \\ \partial_x A_y-\partial_y A_x=0\end{cases}$

→ We denote the tangential components $V^a\equiv(A^a,\pi^a)$, $a=1,2$
Then, $A^a=(A^x,A^y)$ and $\pi^a=(\pi^x,\pi^y)$ satisfy Dirichlet conditions

?
only $\partial A_3=0$ not $A=0$

→ To impose the Coulomb gauge $\vec{\nabla}\cdot\vec{A}=0$ and Gauss's law $\vec{\nabla}\cdot\vec{\pi}=0$, we need to impose von Neumann B.C. on π^3 and A^3: $\partial_3\pi^3|_p=0=\partial_3 A^3|_p$

→ We expand the mode periodically in x, y:

$$V^a = \sum_{n_a \in \mathbb{Z}} \sum_{n>0} V^a_{k_a, k_3} \sin(k_3 x^3) e^{i k_a x^a} \; ; \; k_a = \frac{2\pi}{L} n_a, \; k_3 = \frac{\pi}{d} n$$

$$V^3 = \sum_{n_a \in \mathbb{Z}} \Big(\underbrace{V^3_{k_a, 0}}_{\text{0-mode, } z\text{-independent}} + \sum_{n>0} V^3_{k_a, k_3} \underbrace{\cos(k_3 x^3)}_{\hookrightarrow \text{ ensure } \partial_3 V^3 = 0} \Big) e^{i k_a x^a}$$

The frequencies are given by $\omega_{\vec{k}} = \sqrt{k_a k^a + (k_3)^2} =$

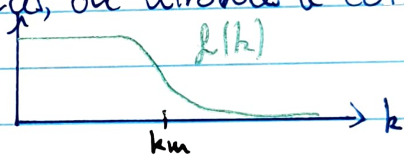
$$= \sqrt{\left(\frac{2\pi}{L} n_1\right)^2 + \left(\frac{2\pi}{L} n_2\right)^2 + \left(\frac{2\pi}{d} n_3\right)^2}$$

→ The vacuum energy between the plates $E_c(d)$ is

$$\frac{1}{2} \hbar \sum_{k} \omega_{k} = \frac{\hbar c}{2} \int L^2 \frac{d^2 k}{(2\pi)^2} \Big(\sqrt{k_a k^a} + \underset{\downarrow \text{ 2 independent polarizations}}{2} \sum_{n=1}^{\infty} \sqrt{k_a k^a + \pi^2 n^2 / d^2} \Big)$$

⊙ Regularization:

In order to get rid of the UV divergences, one introduces a cut-off function $f(k) = \begin{cases} 0 & \text{for } k \gg k_m \\ 1 & \text{for } k \lesssim k_m \end{cases}$



with $f^{(n)}(k)\big|_{k=0} = 0 \quad \forall n$

→ Going the sperical coordinates $k_\perp \equiv \sqrt{k^a k_a}$; $k_z \equiv \pi n / d$, one has

$$E_c(d) = \frac{\hbar c L^2}{\underset{\int d\Omega}{2\pi}} \int_0^\infty k_\perp dk_\perp \left(\frac{1}{2} k_\perp + \sum_{n=1}^{\infty} \sqrt{k_\perp^2 + k_z^2} \right)$$

? so that $E\left(\frac{L}{2} - d\right) - E\left(\frac{L}{2}\right) = \frac{\hbar c L^2}{2} \int \frac{dk^2}{(2\pi)^2} \left(\frac{L}{2} - d - \frac{L}{2} \right) \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \cdot 2 \cdot \sqrt{k_\perp^2 + k_z^2}$

$$= -d \frac{\hbar c L^2}{(2\pi)^2} \int_0^\infty dk_\perp \, k_\perp \cdot \frac{\pi}{d} \int_0^\infty dn \cdot 2 \cdot 2 \cdot \sqrt{k_\perp^2 + \frac{\pi^2 n^2}{d^2}}$$

$$= -\frac{\hbar c L^2}{2\pi} \int_0^\infty dk_\perp \cdot k_\perp \int_0^\infty dn \sqrt{k_\perp^2 + \pi^2 n^2 / d^2} \quad \text{(add } E(d) \text{ to get } E_c(d))$$

The Casimir energy becomes:

$$\frac{E_c(d)}{L^2} = \frac{\hbar c}{2\pi} \int_0^\infty dk_\perp \cdot k_\perp \left(\frac{1}{2} k_\perp + \sum_{n=1}^{\infty} \sqrt{k_\perp^2 + \frac{\pi^2 n^2}{d^2}} - \int_0^\infty dn \sqrt{k_\perp^2 + \pi^2 n^2 / d^2} \right)$$

→ Change of variable: $u = \frac{d^2 k_\perp^2}{\pi^2}$; $k_\perp = \frac{\pi}{d}\sqrt{u}$ so that $dk_\perp = \frac{\pi}{d}\frac{1}{2\sqrt{u}}du$

One gets:

$$\frac{E_c(d)}{L^2} = \frac{\hbar c}{2.2\pi}\cdot\frac{\pi^3}{d^3}\int_0^\infty du\left(\frac{\sqrt{u}}{2}f\left(\frac{\pi}{d}\sqrt{u}\right) + \sum_{n=1}^{\infty}\sqrt{u+n^2}\, f\left(\frac{\pi}{d}\sqrt{u+n^2}\right)\right.$$
$$\left. - \int_0^\infty du\sqrt{u+n^2}\, f\left(\frac{\pi}{d}\sqrt{u+n^2}\right)\right)$$
$$= \frac{\hbar c \pi^2}{4d^3}\left(\frac{1}{2}F(0) + \sum_{n=1}^{\infty}F(n) - \int_0^\infty dn\, F(n)\right)$$

with $F(n) = \int_0^\infty du\sqrt{u+n^2}\, f\left(\frac{\pi}{d}\sqrt{u+n^2}\right) = \int_{n^2}^\infty du\sqrt{u}\, f\left(\frac{\pi}{d}\sqrt{u}\right)$

Now, $\frac{1}{2}F(0) + \sum_{n=1}^{\infty}F(n) - \int_0^\infty du\, F(u) = -\frac{1}{2!}B_2 F'(0) - \frac{1}{4!}B_4 F'''(0) + \ldots$

where B_i are the Bernouilli numbers:

$$\frac{y}{e^y - 1} = \sum_{\nu=0}^{\infty} B_\nu \frac{y^\nu}{\nu!}.$$ Here, $B_2 = 1/6$, $B_4 = -1/30$

So $F'(n) = -2n^2 f(\pi n/d)$

$F''(n) = -4n\, f(\pi n/d) - 2n^2 f'(\pi/d n)\pi/2$

$F'''(n) = -4f(\pi n/d) - 4\frac{\pi n}{d}f'(\pi n/d) - 4\frac{\pi n}{d}f'(\pi n/d) - 2n^2(\pi/d)^2 f''(\pi n/d)$

$= -4f(\pi n/d) - \frac{8\pi n}{d}f'(\pi n/d) - 2n^2(\pi/d)^2 f''(\pi n/d)$

and $F'(0) = 0$, $F''(0) = 0$, $F'''(0) = -4f(0) = -4$, $F^{(n>3)}(0) = 0$

↳ We find $\frac{E_c(d)}{L^2} = \frac{\hbar c\pi^2}{4d^3}\left(\frac{-1}{4!}\right)\left(\frac{-1}{30}\right)(-4) = \frac{-\pi^2}{720}\frac{\hbar c}{d^3}$

So that the force is $F_c(d) = -\frac{\partial E_c(d)}{\partial d} = \frac{-\pi^2}{240}\frac{\hbar c L^2}{d^4}$

⊙ Discussion:

→ At $\begin{cases} T \to 0 \\ \beta \to \infty \end{cases}$, $\ln Z(\beta) = \beta E_0 + \ldots \Leftrightarrow Z(\beta) = e^{-\beta F(\beta)}$ // $Z[J] = e^{\frac{i}{\hbar}W[J]}$

The pressure (if isotropy) in the direction of d is $P = -\frac{\partial F}{\partial d}$

→ At $\beta \to 0$ (or, dimensionlessly $\beta/d \ll 1$),

$$Z(\beta) \xrightarrow[\beta\to 0]{} \frac{\pi^2}{45}\frac{L^2 d}{\beta^3} = \frac{\pi^2}{45}\frac{V}{\beta^3}$$

→ The way the field interacts with the boundaries in the Casimir effect is analogous to how fields interact with spacetime geometry in curved spacetime.

4.2 1-loop effective action for scalar field

4.2.1 Effective potential:

→ Consider the action

$$S[\phi] = \int d^4x \, -\left(\nu + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4\right)$$

where ν is a constant (no change in the dynamics)

→ As we've seen before, the effective action is

$$\Gamma[\phi] = S[\phi] - \frac{\hbar}{2i}\,\mathrm{Tr}\left[\ln\left\{\delta^A_B + (\mathcal{D}^{-1})^{AC}\, V''_{CB}(\phi)\right\}\right] + \mathcal{O}(\hbar^2)$$

$$= S[\phi] - \frac{\hbar}{2i}\,\mathrm{Tr}\left[\ln\left\{\delta^4(x,y) + \mathcal{D}^{-1}(x,y).\frac{g}{2}.\phi^2\right\}\right]$$

Writing $K(x,y) \equiv \mathcal{D}^{-1}(x,y).\, g\phi^2/2$ and using $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}x^n/n$,

$$\ln\left(\delta^4(x,y) + K(x,y)\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int dz_1\cdots dz_{n-1}\, K(x,z_1)K(z_1,z_2)\cdots K(z_{n-1},y)$$

$$= K(x,y) - \frac{1}{2}\int d^4z_1\, K(x,z_1)K(z_1,y) + \dots$$

DEF | The effective potential is defined as, writing $\bar\Phi = \mathrm{cst}$,

$$\Gamma[\bar\Phi] \equiv -\int d^4x\, V_{\mathrm{eff}}[\bar\Phi] = -V_{\mathrm{eff}}[\bar\Phi]\,(2\pi)^4\,\delta^4(0)$$

Indeed, $S[\bar\Phi]$ has its kinetic term vanishing.
Recall that $\delta^4(x-y) = (2\pi)^{-4}\int d^4p\, e^{ip(x-y)}$

→ For $\phi = \bar\Phi$, $K(x,y) = \displaystyle\int \frac{d^4p}{(2\pi)^4}\underbrace{\frac{g\bar\Phi^2/2}{p^2+m^2-i\epsilon}}_{\equiv\,\tilde K(p)} e^{ip(x-y)}$ since $\bar\Phi$ is a constant

The logarithm becomes

$$\ln(\delta^4 + K) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{1}{(2\pi)^4}\int d^4p\, e^{ip(x-y)}\,\tilde K^n(p) = \int\frac{d^4p}{(2\pi)^4}\ln\left\{1+\tilde K\right\}e^{+ip(x-y)}$$

Taking the trace consists in putting $x=y$ and integrating over x:

$$\Gamma[\bar\Phi] = S[\bar\Phi] - \frac{\hbar}{2i}\int d^4x\int\frac{d^4p}{(2\pi)^4}\ln\left\{1+\tilde K(p)\right\} + \mathcal{O}(\hbar^2)$$

$$= -\int d^4x\left[\nu + \frac{1}{2}m^2\bar\Phi^2 + \frac{g}{4!}\bar\Phi^4 + \frac{\hbar}{2i}\int\frac{d^4p}{(2\pi)^4}\left(\ln\left\{1+\frac{g\bar\Phi^2/2}{p^2+m^2-i\epsilon}\right\}\right)\right]$$

$$\dots = \ln\left(\frac{p^2+m^2-i\epsilon+g\bar\Phi^2/2}{p^2+m^2-i\epsilon}\right) = \ln\left(p^2+m^2-i\epsilon+g\bar\Phi^2/2\right) - \ln\left(p^2+m^2-i\epsilon\right)$$

We denote the contribution of quantum fluctuations to the potential as $J(\sigma^2) \equiv \int \frac{d^4p}{(2\pi)^4} \ln\{p^2+\sigma^2-i\epsilon\}$

and the effective mass $\mu^2 \equiv m^2 + g\Phi^2/2$

→ The effective now reads:
$$V_{eff}[\Phi] = \nu + \frac{1}{2}m^2\Phi^2 + \frac{g}{4!}\Phi^4 + \frac{h}{2i}\left(J(\mu^2) - J(m^2)\right) + \mathcal{O}(\hbar^2)$$

4.2.2 Computing the divergent integral:

→ Let's compute $J(\sigma^2)$. We compute a Wick rotation $p^0 \mapsto ip^4$
Writing $A, B, \ldots \in \{1,2,3,4\}$, one has:
$$J(\sigma^2) = \frac{i}{(2\pi)^4}\int d^4p_A \ln\left(p^B p_B + \sigma^2\right)$$

→ Going to spherical coord. (k, ϕ, θ, χ), $|\partial p^A/\partial(k,\phi,\theta,\chi)| = k^3 \sin^2\theta \sin\chi$;
$$J(\sigma^2) = \frac{i}{(2\pi)^4}\int_0^\infty dk.\, k^3 \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \sin^2\theta \int_0^\pi d\chi .\ln(k^2+\sigma^2)$$
$$= \frac{i}{(2\pi)^4} 2.\pi^2 \int_0^\infty dk.\, k^3 \ln(k^2+\sigma^2)$$

→ It diverges ☹. Let's differentiate $(\partial_{\sigma^2})^n J$ until it converges

1) $\partial_{\sigma^2} J(\sigma^2) = \frac{i}{8\pi^2}\int_0^\infty dk\; k^3/(k^2+\sigma^2) \to \infty$

2) $\partial^2_{\sigma^2} J(\sigma^2) = \frac{-i}{8\pi}\int_0^\infty dk\; k^3/(k^2+\sigma^2)^2 \sim \ln(k) \sim \infty$

3) $\partial^3_{\sigma^2} J(\sigma^2) = \frac{+i}{4\pi^2}\int_0^\infty dk\, \overbrace{\frac{k}{(k^2+\sigma^2)^3}}^{g'} . \overbrace{k^2}^{g}$

$$= \frac{i}{4\pi^2}\left(\left[\frac{-1}{4(k^2+\sigma^2)^2}.k^2\right]_0^\infty + \int_0^\infty dk \frac{2k}{4(k^2+\sigma^2)^2}\right) = \frac{i}{4\pi^2}\left[\frac{-1}{4(k^2+\sigma^2)}\right]_0^\infty$$

$$\frac{\partial^3 J(\sigma^2)}{(\partial\sigma^2)^3} = \frac{i}{16\pi^2\sigma^2}$$

Now, we integrate it back: recall $(x\ln(x) - x)' = \ln x$
$J'(\sigma^2) = (i/16\pi^2)(\sigma^2\ln\sigma^2 - \sigma^2) + \overline{C}\sigma^2 + 2iB$ just a constant
Noticing that $(x^2/2 .\ln x - x^2/4)' = x\ln x$, we find
$$J(\sigma^2) = \frac{i}{32\pi^2}\sigma^4\ln\sigma^2 + 2iC\sigma^4 + 2iB\sigma^2 + 2i\overline{A}$$

PROP The effective potential is now a function of 4 diverging constants:

$$V_{eff}[\bar\Phi] = V + \frac{1}{2}m^2\bar\Phi^2 + \frac{g}{4!}\bar\Phi^4 + \frac{\hbar}{64\pi^2}\mu^4 \ln\mu^2 + \hbar C\mu^4 + \hbar B\mu^2 + \hbar A + \mathcal{O}(\hbar^2)$$

with $A \equiv \bar A - \frac{\hbar}{2i} J(m^2)$

DEF We introduce renormalized couplings in order to cancel all diverging constants: V_R, m_R^2, g:

$$\begin{cases} V_R \equiv V + \hbar A + m^2\hbar B + m^4 \hbar C \\ m_R \equiv m^2 + g\hbar B + 2m^2 g\hbar C \\ g_R \equiv g + 6g^2\hbar C \end{cases} \Leftrightarrow \begin{cases} V = V_R - \hbar A - m_R^2\hbar B - m_R^4 \hbar C + \mathcal{O}(\hbar^2) \\ m^2 = m_R^2 - g_R\hbar B - 2m_R^2 g_R \hbar C + \mathcal{O}(\hbar^2) \\ g = g_R - 6g_R^2\hbar C + \mathcal{O}(\hbar^2) \end{cases}$$

so that the potential reads:

$$V_{eff}[\bar\Phi] = V_R + \frac{1}{2}m_R^2\bar\Phi^2 + \frac{g_R}{4!}\bar\Phi^4 + \frac{\hbar}{64\pi^2}\mu_R^2 \ln\mu_R^2 + \mathcal{O}(\hbar^2)$$

→ $V_{eff}[\bar\Phi]$ can be made finite to order $\hbar$ if the renormalized coupling constants are allowed to be finite ⇔ the bare quantities are the one diverging ⇔ We add counterterms of order $\hbar$ to the lagrangian to extract physical quantities.

4.2.3 Renormalized coupling constant at 1-loop:

→ Let's derive explicit expression for g_R (thus for C).

DEF We introduce an UV cut-off Λ, the upper limit on the norm of the momentum space vector. This defines:

$$J_\Lambda(\sigma^2) = \frac{i}{8\pi^2}\int_0^\Lambda dk.\, k^3 \ln(k^2+\sigma^2)$$

→ Lets compute $J_\Lambda(\sigma^2)$. Let $k=\sqrt{x}$; $dk = dx/2\sqrt{x}$ so that

$$\frac{16\pi^2}{i}J_\Lambda(\sigma^2) = \int_0^{\Lambda^2} dx.\, x\ln(x+\sigma^2)$$

$$= \int_0^{\Lambda^2} dx.\,(x+\sigma^2)\ln(x+\sigma^2) - \sigma^2\int_0^{\Lambda^2} dx.\ln(x+\sigma^2)$$

$$= \frac{(\Lambda^2+\sigma^2)^2}{2}\ln(\Lambda^2+\sigma^2) - \frac{(\Lambda^2+\sigma^2)^2}{4} - \frac{\sigma^4}{2}\ln(\sigma^2) + \frac{\sigma^4}{4}$$

$$-\sigma^2\left((\Lambda^2+\sigma^2)\ln(\Lambda^2+\sigma^2) - (\Lambda^2+\sigma^2) - \sigma^2\ln(\sigma^2) + \sigma^2\right)$$

Since $J(\sigma^2) = 2iC\sigma^4 + \dots$, we look a terms $\propto \sigma^4$. We have

$$2iC\sigma^4 = \frac{-i}{32\pi^2}\ln(\Lambda^2+\sigma^2)\,\sigma^4$$

so that $C = \frac{-1}{64\pi^2} \ln\left\{\frac{\Lambda^2 + m^2}{m^2}\right\}$ and recall $g = g_R - 6 g_R^2 \hbar C + \mathcal{O}(\hbar^2)$

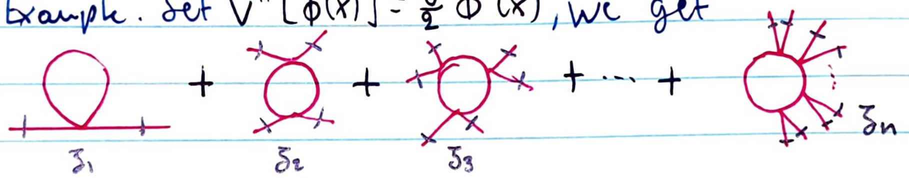
PROP | The bare quartic coupling reads
$$g = g_R + \hbar g_R^2 \frac{3}{32\pi^2} \ln\left\{\frac{\Lambda^2 + m_R^2}{m_R^2}\right\}$$

4.2.4 Structure of 1-loop divergences of effective action:

→ We've shown that $\Gamma[\phi] = S[\phi] + \hbar \Gamma^{(1)}[\phi] + \mathcal{O}(\hbar^2)$ with
$$\Gamma^{(1)}[\phi] = -1/2i \,.\, \mathrm{Tr}\left[\ln\left\{\delta^d(x,y) + D^{-1}(x,y) V''[\phi(y)]\right\}\right]$$
$$= \frac{-1}{2i} \int d^dx \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int d^dz_1 \cdots d^dz_{n-1}\, D^{-1}(x,z_1) V''[\phi(z_1)] \cdots D^{-1}(z_{n-1},x) V''[\phi(x)]$$

$z_n \equiv x$ →
$$= \frac{-1}{2i} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int d^dz_1 \cdots d^dz_n\, V''[\phi(z_1)] D^{-1}(z_1,z_2) \cdots D^{-1}(z_{n-1},z_n) V''[\phi(z_n)] D^{-1}(z_n,z_1)$$

→ Example. Set $V''[\phi(x)] = \frac{g}{2}\phi^2(x)$, we get



→ Expressing the propagator in Fourier space:
$$\Gamma^{(1)}[\phi] = \frac{-1}{2i} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int d^dz_1 \cdots d^dz_n \int \frac{d^dp_1}{(2\pi)^d} \cdots \frac{d^dp_n}{(2\pi)^d} e^{ip_1(z_1-z_2)} \cdots e^{ip_n(z_n-z_1)}$$
$$\times \frac{1}{p_1^2 + m^2 - i\epsilon} \cdots \frac{1}{p_n^2 + m^2 - i\epsilon} V''[\phi(z_1)] \cdots V''[\phi(z_n)]$$

Notice that $e^{ip_1(z_1-z_2)} \cdots e^{ip_n(z_n-z_1)} = e^{iz_1(p_1-p_n)} e^{iz_2(p_2-p_1)} \cdots e^{iz_n(p_n-p_{n-1})}$. We perform a triangular change of variables: $p_1 = q, p_2 = q + q_2, \ldots, p_n = q + q_2 + \cdots + q_n$
$$\Gamma^{(1)}[\phi] = \frac{-1}{2i} \sum_{n=1} \frac{(-1)^{n-1}}{n} \int d^dz_1 \cdots d^dz_n\, V''[\phi(z_1)] \cdots V''[\phi(z_n)]$$
$$\times \int \frac{d^dq_2}{(2\pi)^d} \cdots \frac{d^dq_n}{(2\pi)^d} e^{-iz_1(q_2 + \cdots + q_n)} e^{iz_2 q_2} \cdots e^{iz_n q_n}$$
$$\times \int \frac{d^dq}{(2\pi)^d} \frac{1}{q^2 + m^2 - i\epsilon} \frac{1}{(q+q_2)^2 + m^2 - i\epsilon} \cdots \frac{1}{(q + q_2 + \cdots + q_n)^2 + m^2 - i\epsilon}$$

DEF | We denote $\mathcal{J}^{(n)}(q_2, \ldots, q_n) \equiv \int \frac{d^dq}{(2\pi)^d} \left(\frac{1}{q^2 + m^2 - i\epsilon} \frac{1}{(q+q_2)^2 + m^2 - i\epsilon} \cdots \frac{1}{(q + q_2 + \cdots + q_n)^2 + m^2 - i\epsilon} \right)$

→ We want to know for which n the integral $\mathcal{J}^{(n)}$ diverges.

→ Going to Euclidean, $q^0 \mapsto i q^d$, we have spherical coord. of radius K, with $\gamma^{(n)}(q_2, \dots, q_n) \sim \frac{K^{d-1}}{K^{2n}}$ for large K.

↳ The integral converges for $d-1-2n<-1 \Leftrightarrow n > d/2$.
In 4-d spacetime, only $\gamma^{(1)}$ and $\gamma^{(2)}$ are divergent.

→ Let us expand $\gamma^{(2)}$ in term of external momenta, around 0:

$$\gamma^2(q_2) = \gamma^2(0) + q_2^A \frac{\partial \gamma^{(2)}}{\partial q_2^A} + \dots$$

Now, $\frac{\partial \gamma^{(2)}}{\partial q_2^A} = \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2+m^2-i\epsilon} \frac{(-2(q^A+q_2^A))}{((q+q_2)^2+m^2-i\epsilon)^2} \sim \frac{K^4}{K^6}$

$< \infty$ It converges.

We can then write

$\gamma^{(1)} = A + \gamma^{(1)}_{finite}$ and $\gamma^{(2)} = B + \gamma^{(2)}_{finite}$ with A, B divergent constants

→ The diverging part of the propagator is then:

$$\Gamma^{(1)}_{div}[\phi] = \frac{-1}{2i} \int d^4 z_1 V''[\phi(z_1)] . A + \frac{1}{4i} \int \frac{d^4 q_2 \, d^4 z_1 \, d^4 z_2}{(2\pi)^4} e^{-i z_1 q_2} e^{i z_2 q_2} \times B V''[\phi(z_1)] V''[\phi(z_2)]$$

so that $\Gamma^{(1)}_{div} = \int d^4 z_1 \left(\frac{-A}{2i} V''[\phi(z_1)] + \frac{B}{4i} \left(V''[\phi(z_1)] \right)^2 \right)$

PROP At leading order in $\hbar$, in $d=4$, the effective action for the $\lambda \phi^4$ theory reads:

$$\Gamma[\phi] = S[\phi] + \hbar \int d^4 z_1 \left(\frac{-A}{2i} V''[\phi(z_1)] + \frac{B}{4i} \left(V''[\phi(z_1)] \right)^2 \right) + \hbar \Gamma^{(1)}_{finite}[\phi]$$

Corr For $V[\phi] = \frac{\lambda}{4!} \phi^4$ so that $V''[\phi] = \frac{\lambda}{2} \phi^2$, both $V''[\phi] \propto \phi^2$ and

the 1 loop divergences can be absorbed by a redefinition of the mass and the coupling constant.