Problem Set 3: Quasi-Normal modes, Part I **PHYS-F484**

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Black holes play a fundamental role in General Relativity and modern cosmology. Since Karl Schwarzschild first derived an exact solution to Einstein's equations in 1916—describing a non-rotating black hole—our theoretical understanding has greatly evolved. The later development of the Kerr solution, which describes rotating black holes, further expanded our knowledge of these fascinating objects.

A crucial aspect of black hole studies involves analysing their linear perturbations and associated *quasi-normal modes* (QNMs). These oscillation modes characterize how a black hole dynamically responds to external perturbations. Often referred to as the "fingerprints" of black holes, QNMs provide direct insights into their fundamental properties, such as mass, charge, and angular momentum.

With the groundbreaking detection of gravitational waves by LIGO and Virgo, the study of QNMs has gained immense practical significance. In particular, the *ringdown phase*—the final stage of a black hole merger dominated by QNMs—offers a powerful way to test General Relativity in the regime of strong gravitational fields and confirm the existence of black holes.

Within the two last exercise sessions, we will explore QNMs in both Schwarzschild and Kerr black holes. We will start by reviewing the theory of black hole perturbations, including the Regge-Wheeler and Zerilli equations, which describe axial and polar perturbations of Schwarzschild black holes. We will then analyse scalar and vector perturbations and investigate the key parameters influencing QNMs. Finally, we will briefly discuss observational aspects of these modes, particularly in the context of Kerr black holes, which provide an essential test of General Relativity.

1 Linear Perturbations of the Schwarzschild Black Hole

We will start by studying the perturbations of a static and spherical symmetric black hole, the well-known Schwarzschild black hole. You have seen that

$$ds^2 = g^{(0)}_{\mu\nu} dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{1}$$

where $g^{(0)}_{\mu\nu}$ is the background metric. We make the assumption that we are in an empty space such that Einstein equations take the simple form

$$R^{(0)}_{\mu\nu} = 0. \tag{2}$$

We will then perturb this metric through small $|h_{\mu\nu}| \ll 1$ perturbations such that

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \tag{3}$$

for which Einstein equations resume to

$$R_{\mu\nu} = 0. \tag{4}$$

Problem 1.1. *Express this equation with respect to the Christoffel symbols. To this end, we introduce*

$$h^{\mu\nu} = g^{(0)\mu\alpha} g^{(0)\beta\nu} h_{\alpha\beta} \tag{5}$$

and

$$g^{\mu\nu} = g^{(0)^{\mu\nu}} - h^{\mu\nu} + \mathcal{O}(h^2). \tag{6}$$

You should find that the perturb Einstein equations are given by

$$\delta \widetilde{R}_{\mu\nu} = 0 \quad \Leftrightarrow \quad \nabla_\nu \delta \widetilde{\Gamma}^\alpha_{\mu\alpha} = \nabla_\alpha \delta \widetilde{\Gamma}^\alpha_{\mu\nu}. \tag{7}$$

By Birkhoff theorem, one knows that the unique solution with spherical symmetry of Einstein equations in the vacuum is the Schwarzschild metric. This means that we can only perturb our black hole with non radial perturbations. This will constrain the form of $h_{\mu\nu} \equiv h_{\mu\nu}(t, r, \theta, \phi)$. This is where spherical harmonics are going to enter our analysis. You have seen that a scalar function depending only on spatial coordinates can be extended in spherical harmonics series

$$f(r, \theta, \phi) = \sum_{l,m} a_{lm}(r) Y_{lm}(\theta, \phi). \tag{8}$$

While considering QNMs, one cannot limit oneself to scalar function, and need to generalise these spherical harmonics. We know that

$$D^2 Y_{lm}(\theta, \phi) = -l(l+1) Y_{lm}(\theta, \phi), \tag{9}$$

with $D^2 = \Omega^{AB} D_A D_B$ is the Laplacian on the sphere, Ω^{AB} being the associated metric. The idea leading to its generalisation is to look for a tensor $Y_{a_1 \ldots a_n}$ such that

$$D^2 Y^{A \ldots D}_{lm} = -\lambda Y^{A \ldots D}_{lm}. \tag{10}$$

Problem 1.2. *Find the possible tensors that can be constructed from these scalar harmonics, satisfying Eq.* (10).

We can then use those tensorial harmonics to extend the perturb metric in the appropriate basis. Letting $x^a = (t, r)$, we have

$$h_{ab} = \sum_{l,m} a^{lm}_{ab}(x^a) Y_{lm}(x^A), \tag{11}$$

$$h_{aA} = \sum_{l,m} b^{lm}_a(x^a) Y^{lm}_A(x^A) + c^{lm}_a(x^a) X^{lm}_A(x^A), \tag{12}$$

$$h_{AB} = \sum_{l,m} d^{lm}(x^a) \Omega_{AB} Y_{lm}(x^A) + e_{lm}(x^a) Y^{lm}_{AB}(x^A) + f_{lm}(x^a) X^{lm}_{AB}(x^A). \tag{13}$$

Notice that this harmonic basis can be divided into two categories. Indeed, the parity operator P on the sphere send respectively θ and ϕ to $-\theta$ and $\pi + \phi$. Hence, one has

$$P\left(X^A_{lm}\right) = (-1)^{l+1} X^A_{lm}, \quad P\left(X^{AB}_{lm}\right) = (-1)^{l+1} X^{AB}_{lm} \tag{14}$$

meaning odd parity, giving rise to the *axial* modes c^a_{lm} and f_{lm}. As for the others, we have

$$P\left(Y^{AB}_{lm}\right) = (-1)^l Y^{AB}_{lm} \tag{15}$$

that has even parity and gives rise to *polar* modes. Thereafter, it is useful to consider them separately.

2 Regge-Wheeler Equation

First, let us focus on axial perturbations. A convenient gauge choice is to go to Regge-Wheeler gauge, letting $f_{lm} = 0$.

Problem 2.1. *Express the perturb metric in this gauge.*

We need the explicit expression of the vectoral harmonics X_A. You found that

$$X^{lm}_A = -\varepsilon_A{}^B D_B Y_{lm}(\theta, \phi). \tag{16}$$

Problem 2.2. *Give the expression of* (16) *in terms of θ and ϕ, as well as the associated non-vanishing components of $h_{\mu\nu}$. To simplify them further, one should use*

$$Y_{lm}(\theta, \phi) = N_l e^{im\phi} P^m_l(\cos\theta), \tag{17}$$

with N_l a normalisation constant and P^m_l the associated Legendre polynomials.

From this, you can solve the perturbed Einstein equations Eq. (7), giving rise to

$$\begin{cases} \dfrac{\partial^2 Q(t,r)}{\partial t^2} - \dfrac{\partial^2 Q(t,r)}{\partial r_*^2} + \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right] Q(t,r) = 0 \\ \dfrac{\partial h_0(t,r)}{\partial t} = \dfrac{\partial}{\partial r_*}(r_* Q(t,r)), \end{cases} \tag{18}$$

with

$$Q(t,r) = \frac{h_1(t,r)}{r}\left(1 - \frac{2M}{r}\right) \tag{19}$$

and

$$r_*(r) = r + 2M \ln\left(\frac{r}{2M} - 1\right), \tag{20}$$

the tortoise coordinate. Eq. (18) is known as the *Regge-Wheeler equation* and can be written as a wave equation when considering the *Regge-Wheeler potential*

$$V^-_{lm} = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right]. \tag{21}$$

Problem 2.3. *Analyse briefly the resulting wave equation.*

3 Zerilli Equation

Now that we mastered the odd metric perturbations, we can easily applied what we did to even metric perturbations. To spice the computations up, we notice that, because we are faced with a higher number of modes, the Regge-Wheeler gauge does not allow to cancel out every contribution from rank 2 tensorial harmonic perturbations. This allows only to set g_a^{lm} and e_{lm} coefficients to zero.

Problem 3.1. *Write the resulting perturbed metric.*

For convenience, one can introduce $H_0(t,r), H_1(t,r), H_2(t,r)$ and $K(t,r)$ functions sucht that

$$h_{\mu\nu} = \begin{pmatrix} H_0(t,r)\left(1-\frac{2M}{r}\right) & H_1(t,r) & 0 & 0 \\ H_1(t,r) & H_2(t,r)\left(1-\frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2K(t,r) & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta K(t,r) \end{pmatrix} Y_{lm}(x^A). \tag{22}$$

Solving again Einstein equations imposes that $H_1(t,r) = H_2(t,r)$. Moreover, introducing the *Zerilli function*

$$Z(t,r) = \frac{2r}{l(l+1)}\left(K(t,r) + \frac{2f(r)}{\Lambda(r)}\left(H_1(t,r) - r\partial_r K(t,r)\right)\right), \tag{23}$$

with $f(r) = 1 - \frac{2M}{r}$ and

$$\Lambda(r) = (l-1)(l+2) + \frac{6M}{r} \tag{24}$$

allow us to rewrite H_0 and H_1.

Problem 3.2 (Mathematica)**.** *Going to the tortoise coordinate, we can obtain the associated wave equation, called the Zerilli equation, while considering the Zerilli potential*

$$V_{lm}^{+} = \left(1-\frac{2M}{r}\right)\frac{2q(q+1)r^3 + 6q^2Mr^2 + 18qM^2r + 18M^3}{r^3(qr+3M)^2}, \tag{25}$$

with $q = (l-1)(l+2)/2$.

It's interesting to note that the Regge-Wheeler and Zerilli potentials are completely equivalent. Indeed, both potentials can be expressed from

$$V^{\pm} = W(r)^2 \mp \frac{dW(r)}{dr_*} - \beta^2, \tag{26}$$

with

$$W(r) = \frac{6M(2M-r)}{r^2(6M+\mu^2 r)} - \beta, \quad \beta = \frac{\mu^2(\mu^2+2)}{12M} \tag{27}$$

and $\mu^2 = (l-1)(l+2)$. An important consequence of this relationship between those two potentials is the *isospectrality* of even and odd perturbations. This isospectrality is the equivalence of the spectrum of quasi-normal modes for these two parities. We will demonstrate this property in Problem Set 4.