Physics beyond the Standard model

ULB MA | 2024–2025 | Prof. Michel TYTGAT & Steven LOWETTE

Chapter 1: Basic Elementary Fields

Handwritten notes (scanned)

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Heads up: only Chapter 1 here. This DocHub upload contains **only the first chapter**. The full set of chapters, personal notes, exercise corrections, and a reference-book list are on my website.

— **All chapters:** see the course page

— **Exercise corrections & personal work:** see the main page.

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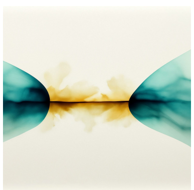


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CH1 BASIC ELEMENTARY FIELDS

→ Fundamental constituents are described by relativistic quantum mechanics laws: Quantum Field Theory
These field can be classified in terms of their transformation properties; each field corresponds to an irrep of the Lorentz group.

⊙ A real scalar field $\phi(x)$:

→ Spin 0: $\phi(x) \mapsto \phi'(x') \stackrel{!}{=} \phi(\Lambda^{-1}x')$ with $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$
→ It has 1 dof
→ The possible Lorentz invariant bilinears are: $\partial_\mu \phi \partial^\mu \phi$ and ϕ^2
→ The free scalar Lagrangian is $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m \phi^2$
↳ The EOM is $(\Box + m^2)\phi = 0$, the Klein-Gordon equation. Its solution is $\phi(x) = \int d^3k \{ a(k)\, e^{ikx} + a^\dagger(k)\, e^{-ikx} \}$ ⇝ $\phi^\dagger = \phi$

⊙ A complex scalar field:

→ Spin 0: $\phi(x) \mapsto \phi'(x') \stackrel{!}{=} \phi(\Lambda^{-1}x')$, with $\begin{cases} \phi(x) = \mathrm{Re}\{\phi\} + \mathrm{Im}\{\phi\} \\ \phi^\dagger = \mathrm{Re}\{\phi\} - \mathrm{Im}\{\phi\} \end{cases}$
→ It has 2 dof
→ The possible Lorentz invariant bilinears are: $\partial_\mu \phi^\dagger \partial^\mu \phi$, $\phi^\dagger \phi$.
They're hermitian to conserve probability
→ The free field Lagrangian is: $\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - \underbrace{m^2 \phi^\dagger \phi}$
↳ The EOM is $(\Box + m^2)\phi = 0$ & $(\Box + m^2)\phi^\dagger = 0$ degenerate mass
↳ Solution:

$$\phi(x) = \int d^3k \{ a(k)\, e^{-ikx} + b^\dagger(k)\, e^{ikx} \}$$

$$\phi^\dagger(x) = \int d^3k \{ a^\dagger(k)\, e^{ikx} + b(k)\, e^{-ikx} \}$$ ⇝ $\phi^\dagger \neq \phi$

⊙ A vector field $A^\mu(x)$:

→ Spin 1 : transforms as $A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x')$

→ The Lorentz invariant bilinears are $\partial_\mu A^\nu \partial^\mu A_\nu$, $\partial_\mu A^\nu \partial_\nu A^\mu$, $A_\mu A^\mu$.
↳ Only $F_{\mu\nu}F^{\mu\nu}$ allows for a positive defined Hamiltonian, with

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

→ The free field Lagrangian is: $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$
↳ EOM : $\partial_\mu F_{\mu\nu} + m^2 A_\nu = 0$
→ For $m \neq 0$, EOM $\Rightarrow$ $(\Box + m^2) A_\nu = 0$
→ For $m = 0$, we can place ourself in the Lorentz gauge $\partial_\mu A^\mu = 0$ to recover the K-G EOM.
↳ Solution:

$$A_\mu(x) = \int d^3k \sum_\lambda \left\{ a^\lambda(k)\, \varepsilon^\lambda_\mu(k)\, e^{-ikx} + a^{\lambda\dagger}(k)\, \varepsilon^{\lambda *}_\mu(k)\, e^{ikx} \right\}$$

Since $\partial_\mu A^\mu = 0$, $\varepsilon^\lambda_\mu k^\mu = 0 \Rightarrow \lambda \in \{1, 2, 3\}$
→ We have 2 transverse polarization mode
$\varepsilon^{1,2}_\mu = (0, \vec{e}_{1,2}(k))$ such that $\vec{e}_{1,2}(k) \cdot \vec{k} = 0$
and 1 longitudinal polarization mode
$\varepsilon^3_\mu = \left(\frac{|\vec{k}|}{m}, \frac{E}{m} \cdot \frac{\vec{k}}{|\vec{k}|} \right)$, unphysical for $m = 0$

→ ε^3_μ doesn't couple to J^μ throught $A_\mu J^\mu \in \mathcal{L}_I$. Indeed,
$\partial_\mu J^\mu = 0 \Rightarrow k_\mu J^\mu = 0 \Leftrightarrow E J^0 = \vec{k} \cdot \vec{J}$ so that

$$\varepsilon^3_\mu J^\mu = \frac{|k|}{m} J^0 - \frac{E}{m} \frac{k \cdot J}{|k|} = k \cdot J \left(\frac{|k|}{E \cdot m} - \frac{E}{m} \cdot \frac{1}{|k|} \right)$$

$$= J^0 \left(\frac{|k|}{m} - \frac{E^2}{m|k|} \right) = \frac{J^0}{|k|} \left(\frac{k^2 - E^2}{m} \right) = \frac{J^0 m}{|k|} \xrightarrow{m \to 0} 0$$

③ Dirac spinor $\psi(x)$:

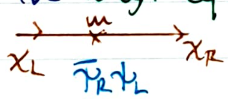
→ Spin 1/2 : $\psi(x) \mapsto \psi'(x') = e^{-\frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}} \cdot \psi(\Lambda^{-1}x')$
where $S^{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta]$ are the Lorentz algebra generators.

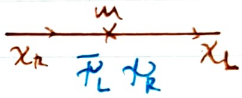
↳ In chiral components; in the Weyl representation:
$$\psi = \psi_L + \psi_R = P_L\psi + P_R\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$
$$\Rightarrow \chi_{L,R}(x) \mapsto \chi'_{L,R}(x') = \exp\Big\{\underbrace{\mp \tfrac{1}{2}\omega_{0i}\sigma^i}_{3\text{ boosts}} \underbrace{- \tfrac{i}{4}\omega_{ij}\epsilon^{ij}{}_k\sigma^k}_{3\text{ rotations}}\Big\} \chi_{L,R}(\Lambda^{-1}x')$$

→ The Lorentz invariant bilinears are $i\bar{\psi}\not{\partial}\psi$ and $\bar{\psi}\psi$.

→ The Dirac Lagrangian is $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$

↳ EoM : $(i\not{\partial} - m)\psi = 0$ the Dirac equation. In Weyl rep.,
$$\begin{cases} i(\partial_t - \sigma\cdot\nabla)\chi_L = m\,\chi_R \\ i(\partial_t + \sigma\cdot\nabla)\chi_R = m\,\chi_L \end{cases}$$
the Weyl equations





↳ Solution:
$$\psi(x) = \int d^3k \sum_{s=1}^{2} \left\{ a_k^s\, u^s(k)\, e^{-ikx} + b_k^{s\dagger}\, v^s(k)\, e^{ikx} \right\}$$
$$\bar{\psi}(x) = \int d^3k \sum_{s=1}^{2} \left\{ a_k^{s\dagger}\, \bar{u}^s(k)\, e^{ikx} + b_k^s\, \bar{v}^s(k)\, e^{-ikx} \right\}$$

with $u^s(k) = \begin{pmatrix} \sqrt{k\cdot\sigma}\,\xi^s \\ \sqrt{k\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}$; $v^s(k) = \begin{pmatrix} \sqrt{k\cdot\sigma}\,\xi^s \\ -\sqrt{k\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}$ and $\xi^\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ spin up, $\xi^\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ spin down

and $\sigma^\mu = (\mathbb{1}, \sigma^i)$; $\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$

Spin ↑, ↓ are physical states (⇐ conservation of spin). There are superposition of ψ_L and ψ_R, which are only physical if $m = 0$. Hence, helicity $\propto \vec{S}\cdot\vec{p}$ is ≠ from chirality if $m \neq 0$.

→ Weyl rep : $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$

→ Clifford algebra : $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

→ $\underset{\text{rotation}}{S^{ij}} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$ and $\underset{\text{boost}}{S^{0i}} = \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$

→ Since $\eta'_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu}$, $\Lambda^\mu{}_\nu \gamma^\nu = \Lambda_{1/2}^{-1}\gamma^\mu \Lambda_{1/2}$

⊙ Massless Weyl spinor:

→ If $m=0$, χ_L and χ_R do not mix. One could have just 2 d.o.f.:
χ_L or χ_L but not both.
↳ χ_L transforms as L-field ≡ particle
↳ $\chi_L^c \equiv -i\sigma^2 \chi_L^*$ as a R-field ≡ anti-particle

→ The Lorentz invariant, $\mathcal{L}$ are the same as for Dirac. The solutions too, reducing to
$u_\downarrow(\vec{k}) = \sqrt{2\omega_k}\begin{pmatrix}\xi_\downarrow \\ 0\end{pmatrix}$: anti-// to $\vec{k}$ ⇝ $\begin{pmatrix}\chi_L \\ 0\end{pmatrix}$
$v_\uparrow(\vec{k}) = -\sqrt{2\omega_k}\begin{pmatrix}0 \\ \xi_\downarrow\end{pmatrix}$: // to $\vec{k}$ ⇝ $\begin{pmatrix}0 \\ \chi_L^c\end{pmatrix}$

⊙ Massive Majorana Spinor:

→ A single ψ_L or ψ_R can have a mass: $\chi_L^{c\dagger}\chi_L$ is Lorentz inv.

↳ $\mathcal{L}^{Maj\text{-}Mass} = -\frac{1}{2}\left(\chi_L^{c\dagger}\chi_L + \chi_L^\dagger \chi_L^c\right) m_M$ is Lorentz inv and hermitian.

→ 2 types of mass: $\xrightarrow{\chi_L} \underset{m_D}{\times} \xrightarrow{\chi_R}$ $\xrightarrow{\chi_L^c} \underset{m_M}{\times} \xrightarrow{\chi_L}$

→ The physical state $\psi_M = \begin{pmatrix}\chi_L \\ \chi_L^c\end{pmatrix} = P_L\psi_M + P_R\psi_M = \psi_M^c$
where $\psi_M^c \equiv C\bar{\psi}_M^T$ with $C \equiv \begin{pmatrix}i\sigma_2 & 0 \\ 0 & -i\sigma_2\end{pmatrix} = i\gamma^2\gamma^0$

→ $\mathcal{L} = \bar{\psi}_L i\cancel{\partial}\psi_L - \frac{m_M}{2}\left(\bar{\psi}_L^c\psi + \bar{\psi}_L\psi_L^c\right)$
$= \frac{1}{2}\bar{\psi}_M i\cancel{\partial}\psi_M - \frac{m_M}{2}\bar{\psi}_M\psi_M$

→ EOM: $\psi_M = \int d^3\tilde{k} \sum_s \left\{ a_k^s u^s(k) e^{-ikx} + a_p^{s\dagger} v^s(p) e^{ikx} \right\}$