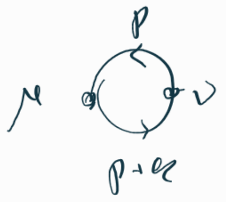
(I) LOOP CORRECTIONS & RUNNING COUPLINGS

1) USE THE FEYNMAN RULES (FR) FROM SP TO CHECK THAT

$$i\Pi^{\mu\nu}(k) = -(-i)^2 \int \frac{d^4p}{(2\pi)^4} \, \mathrm{Tr}\left(\gamma^\mu \frac{i}{(\not{p}-m)} \gamma^\nu \frac{i}{(\not{p}+\not{k}-m)} \right)$$

$=$ 

REMARK: IN PS, THE COUPLING ENTER THE FR

$\hookrightarrow -ie\gamma^\mu$

WE PUT INSTEAD e IN THE PROPAGATOR, SO THAT

$\hookrightarrow -i\gamma^\mu$

2) USE DIM. REG. TO CHECK THAT

$$i\Pi^{\mu\nu}(k) = i\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right)\Pi(k^2)$$

WITH $$\Pi(k^2) = -\frac{8}{16\pi^2}\int_0^1 dx \, x(1-x)\left[\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log\left(m^2 - k^2x(1-x)\right)\right]$$

3) THE COULOMB POTENTIAL BETWEEN TO OPPOSITE CHARGE PARTICLES (EG ELECTRON AND PROTON) IS GIVEN, AT LEADING ORDER, BY

$$V(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\vec{k}\cdot\vec{x}} \left(-\frac{e^2}{\vec{k}^2}\right)$$

THE ELECTRIC IS MODIFIED BY QUANTUM FLUCTUATIONS

$$e^2 \to \frac{e^2}{\left(1 - e^2\Pi_R(-\vec{k}^2)\right)}$$

REM: $k^2 = \omega_k^2 - \vec{k}^2$

$\neq -\vec{k}^2$

WITH $\Pi_R(k^2) = \Pi(k^2) - \Pi(0)$

AND $\Pi(k^2)$ GIVEN IN 2)

SHOW THAT FOR $\vec{k}^2 \ll m^2$

$$V(\vec{x}) \simeq -\frac{\alpha}{2} - \frac{4\alpha^2}{15 m^2} \delta^3(\vec{x})$$

- MORE ATTRACTIVE AT SHORT DISTANCE!
- IMPORTANT FOR FINE STRUCTURE OF THE HYDROGEN SPECTRUM (DARWIN TERM)

HINTS
- EXPAND $\frac{1}{1-e^2\Pi_R} \simeq 1 + e^2 \Pi_R$
- FOCUS FIRST ON HOW TO REDERIVE THE COULOMB POTENTIAL USING COMPLEX PLANE INTEGRATION
- LOOK AT THE DEFINITION OF $\delta^3(\vec{x})$ IN FOURIER SPACE

4) FOR APPLICATION TO NON-ABELIAN THEORIES, CHECK FOR $SU(2)$ THAT

- $\mathrm{Tr}\left[T^A T^B\right] = \frac{1}{2}\delta^{AB}$ WITH T^A GENERATORS IN THE FUNDAMENTAL REPRESENTATION
- DEFINE $(T_G^A)_{ij} = i f^{iAj}$

  AND CHECK THAT
  $$[T_G^A, T_G^B] = i f^{ABC} T_G^C$$

  FOR THIS, WRITE DOWN EXPLICITLY THE 3×3 MATRICES T_G^A

- CHECK THAT (AGAIN FOR $SU(2)$ SO $N=2$)
  $$f^{ACD} f^{BCD} = N \delta^{AB}$$

5) RUNNING OF GAUGE COUPLINGS IN SM

FROM $\mu \frac{d}{d\mu} g = \beta(g)$ IN $SU(N)$ THEORIES, WE GET

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu_0)} + \frac{2}{(4\pi)^2}\left[\frac{11}{3} N - \frac{2}{3} n_F\right] \log \frac{\mu}{\mu_0}$$

THE (1-LOOP) RUNNING OF THE GAUGE COUPLING FOR A $SU(N)$ GAUGE THEORY COUPLED TO n_F DIRAC FERMIONS.

DEFINE

$\alpha_3 = \frac{g_3^2}{4\pi}$ WITH g_3 THE COUPLING OF $SU(3)_c$

$\alpha_2 = \frac{g^2}{4\pi}$ WITH g THE " $SU(2)_L$

$\alpha_1 = \frac{5}{3} \frac{g'^2}{4\pi}$ WITH g' THE " $U(1)_Y$

(THE FUNNY $5/3$ WILL BE EXPLAINED LATER)

WRITE DOWN

$$\frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_i(\mu_0)} - \frac{1}{2\pi} b_i \log\left(\frac{\mu}{\mu_0}\right)$$

AND SHOW THAT

$$\begin{cases} b_3 = -11 + \frac{4}{3} N_F \\ b_2 = -\frac{22}{3} + \frac{4}{3} N_F \\ b_1 = \frac{4}{3} N_F \end{cases}$$

WITH N_F = # OF FERMION FAMILIES IN THE SM

REMARKS

- BEWARE, THE MEANING OF n_F = # OF FERMIONS CHANGES FROM GAUGE GROUP TO GAUGE GROUP
  - FOR $SU(3)$, n_F = # OF $SU(3)$ TRIPLETS
  - $SU(2)$, n_F = # OF $SU(2)$ DOUBLETS

+ FOR $SU(2)_L$, $n_F = \frac{\#}{2}$ OF $SU(2)$ DOUBLETS

↳ DIVIDED BY 2 BECAUSE ONLY $\frac{1}{2}$ OF A DIRAC (THE LEFT-HANDED ONES)

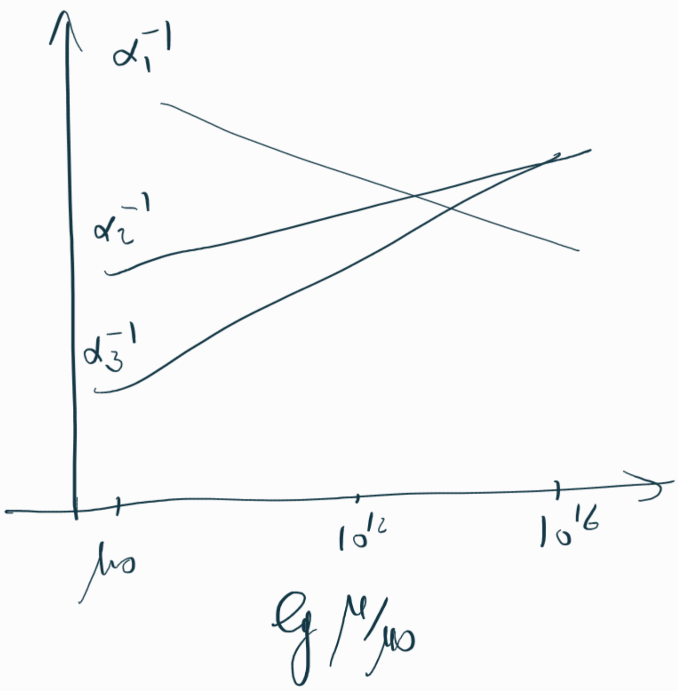
+ FOR $U(1)_Y$, $n_F = \sum_i \left(\frac{Y_i}{2}\right)^2$ THE SUM OF LEFT-HANDED AND RIGHT-HANDED

THERE IS NO $\frac{1}{2}$ HERE. THIS FACTOR FOR $SU(3)$ AND $SU(2)$ COMES FROM $\mathrm{Tr}[T^A T^B] = \frac{1}{2}\delta^{AB}$

FOR $U(1)$ IT IS JUST 1.

LOOK FOR VALUES OF THE $\alpha_i(\mu_0)$ ABOVE AND PLOT $1/\alpha_i$ VS $\log \frac{\mu}{\mu_0}$

WITH $\mu_0 \sim M_Z$ SCALE



(II) CHIRAL ANOMALIES

① CHECK THAT $\gamma^0 = \sigma_1$

$\gamma^1 = i\sigma_2$

AND $\gamma_5 = -\sigma_3$

SATISFY $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

AND $\{\gamma^\mu, \gamma_5\} = 0$

IN $D=2$

(2) CHECK THAT $\gamma^\mu \gamma_5 = -\varepsilon^{\mu\nu}\gamma_\nu$

$\Rightarrow \quad \bar{\psi}\gamma^\mu\gamma_5\psi = -\varepsilon^{\mu\nu}\bar{\psi}\gamma_\nu\psi$

IN $D=2$

(3) LET $\phi \to (1 + i\alpha^A T_R^A)\phi$

WITH $[T_R^A, T_R^B] = i f^{ABC} T_R^C$ (f^{ABC} REAL AND COMPLETELY ANTISYMMETRIC)

SHOW THAT THE CONJUGATE FIELD ϕ^* TRANSFORM WITH GENERATORS $T_{\bar{R}}^A = -(T_R^A)^*$

$= -(T_R^A)^T$

SHOW THAT

$$[T_{\bar{R}}^A, T_{\bar{R}}^B] = i f^{ABC} T_{\bar{R}}^C$$

(4) CONSIDER A CHIRAL (WEYL) FERMION

χ_L OR χ_R

REMEMBER THAT UNDER A BOOST ALONG, SAY x, THEY TRANSFORM AS

$\chi_L \to e^{-\eta/2\,\sigma^1}\chi_L$

$$\chi_L \to e^{\ldots}\chi_L$$

$$\chi_R \to e^{\eta/2\,\sigma^2}\chi_R$$

So as inequivalent representations of Lorentz

Show that the conjugates transform as

$$\sigma^2\chi_R^* \to e^{-\eta/2\,\sigma^1}\sigma^2\chi_R^*$$

And $-\sigma^2\chi_L^* \to e^{\eta/2\,\sigma^1}(-\sigma^2\chi_L^*)$ (The minus sign is for later convenience)

So $\sigma^2\chi_R^* \sim \chi_L$

And $-\sigma^2\chi_L^* \sim \chi_R$

Show that this can be written as

$$\psi = \begin{pmatrix}\chi_L\\ \chi_R\end{pmatrix} \to \psi_c = \gamma^2\psi^* \equiv \gamma^2\gamma^0\bar{\psi}^T$$

For a Dirac spinors. ψ_c is the charge conjugate of ψ

(4) Consider $\chi_R^\dagger\, i\sigma^\mu(\partial_\mu - igA_\mu^A T_R^A)\chi_R$

with χ_R transforming according to representation R under some gauge group.

Show that this can rewritten as

$$\chi_R^\dagger i\sigma^\mu(\partial_\mu - igA_\mu^A T_R^A)\chi_R \equiv i\psi_L^\dagger\bar{\sigma}^\mu(\partial_\mu - igA_\mu^A T_{\bar{R}}^A)\psi_L$$

with $\psi_L = \sigma^2\chi_R^*$

Hints: . Take the complex conjugate of the LHS.

Use that $(\sigma^2)^2 = \mathbb{1}_2$

• TREAT THE χ_{iR} AS <u>ANTICOMMUTING</u> FIELDS!!!

$$\Rightarrow \chi_{R_i}\,\chi^*_{R_j} = -\,\chi^*_{R_j}\,\chi_{R_i}$$

WITH $i=1,2$ THE WEYL INDICES