PART 1

THE SM IS A CHIRAL THEORY. BY THIS WE MEAN THAT THE FERMIONIC BUILDING BLOCKS ARE WEYL FERMIONS OR CHIRAL FERMIONS

LET US START WITH THE DIRAC EQUATION FOR FREE FERMIONS $(i \not{\partial} - m)\psi = 0$

WITH $\not{\partial} = \gamma^\mu \partial_\mu = \gamma^0 \partial_0 + \gamma^i \partial_i \equiv \gamma^0 \partial_t + \vec{\gamma}\cdot\vec{\nabla}$

WITH $\vec{\nabla}$ THE SPATIAL GRADIENT.

THE γ^μ SATISFY $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ WITH $\eta^{\mu\nu}$ THE MINKOWSKI METRIC IN D DIMENSION.

THIS IS REQUIRED TO HAVE THAT ψ SATISFIES THE KLEIN-GORDON EQUATION $(\Box + m^2)\psi = 0$ AS REQUIRED FOR A MASSIVE RELATIVISTIC PARTICLE, HERE $\Box = \partial_\mu \partial^\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu$

EX: • SHOW THAT

$$\gamma^0 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{AND} \quad \gamma^1 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

SATISFY $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ IN $D=2$

• SHOW THAT $\gamma^\mu \tilde{\gamma}_5 = -\epsilon^{\mu\nu}\gamma_\nu$ WITH $\epsilon^{01} = +1$ AND $\tilde{\gamma}_5 \equiv \gamma^0\gamma^1$ THE $D=2$ VERSION OF THE $D=4$ γ_5 MATRIX.

CHECK THAT $\{\tilde{\gamma}_5, \gamma^\mu\} = 0$

ES: • SHOW THAT $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ (CHIRAL OR WEYL BASIS)

WITH $\overset{(-)}{\sigma}{}^\mu = (\mathbb{1}_2, (-)\sigma^1, \underbrace{(-)\sigma^2, -\sigma^3}_{\text{PAULI MATRICES}})$

SATISFY $\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}$ IN $D=4$

• SHOW THAT $\{\gamma_5, \gamma^\mu\} = 0$ FOR $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

• WRITE γ_5 IN THE WEYL BASIS.

• SINCE THE γ^μ ARE 4x4 MATRICES IN $D=4$, THE DIRAC FIELD ψ ARE OF THE FORM

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$ A COLLECTION OF 4 COMPLEX NUMBERS

IN THE WEYL BASIS, THE FORM OF THE γ^μ SUGGEST TO WRITE

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$ WITH $\chi_{L,R}$ 2 COMPONENTS OBJECTS.

• THEN $i\gamma^\mu \partial_\mu \psi = m\psi$ CAN BE WRITTEN AS

$$\begin{cases} i\bar{\sigma}^{\mu}\partial_{\mu}\chi_L = m\chi_R \\ i\sigma^{\mu}\partial_{\mu}\chi_R = m\chi_L \end{cases}$$

INTERESTINGLY, THE MASS TERM COUPLES χ_L AND χ_R. IF $m=0$ (MASSLESS PARTICLE)

- THEN WE HAVE TWO INDEPENDENT EQUATIONS

$$i\sigma^{\mu}\partial_{\mu}\chi_R = 0 \quad \text{AND} \quad i\bar{\sigma}^{\mu}\partial_{\mu}\chi_L = 0$$

- THE FIELD χ_L AND χ_R ARE CALLED WEYL OR CHIRAL FERMIONS. THEY CORRESPOND TO MASSLESS PARTICLES = THEY TRAVEL AT THE SPEED OF LIGHT.
- THESE ARE THE TRUE BUILDING BLOCKS OF THE SM!
- TO GAIN FURTHER INSIGHTS, CONSIDER THE γ_5 MATRIX (SEE EX.)

$$\gamma_5^2 = 1 \text{ AND ALSO } \mathrm{Tr}[\gamma_5] = 0 \iff \gamma_5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$

$\Updownarrow$

EIGENVALUES ARE ± 1

(IN WEYL BASIS)

- THE DIRAC EQUATION CAN BE WRITTEN AS

$$i\frac{\partial}{\partial t}\psi = H_D\psi \quad \text{WITH HAMILTONIAN}$$

$$H_D = -i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + m\gamma^0$$

- IT IS EASY TO CHECK THAT $[\gamma_5, H_D] = 2m\gamma_5\gamma^0$

- SO, IN THE MASSLESS LIMIT, γ_5 AND H_D COMMUTE! THIS MEANS THAT WE CAN LABEL STATES AS EIGENSTATES OF BOTH H_D AND γ_5. LET

$$\gamma_5 \psi_R = + \psi_R \quad \text{AND} \quad \gamma_5 \psi_L = - \psi_L$$

CLEARLY

$$\psi_R = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} \quad \text{AND} \quad \psi_L = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}$$

IN THE WEYL BASIS. THIS IS THE GOOD BASIS TO DISCUSS RELATIVISTIC ~ MASSLESS FERMIONS.

- FROM γ_5 WE CAN WRITE PROJECTORS ON THE SUBSPACES SPAN BY χ_L AND χ_R:

$$L = \frac{1 - \gamma_5}{2} \qquad R = \frac{1 + \gamma_5}{2}$$

SHOW THAT $L^2 = L$; $R^2 = R$; $RL = LR = 0$

AND $L + R = \mathbb{I}_4$ AND SO ARE PROJECTORS.

WITH $\psi_L = L \psi$ AND $\psi_R = R \psi$

WEYL SPINORS AS REPRESENTATIONS OF THE LORENTZ GROUP

- ROTATIONS ACT ON VECTORS WHILE PRESERVING THEIR LENGTH: $R\vec{x} = \vec{x}' \Rightarrow \vec{x}^2 = \vec{x}'^2$
- SIMILARLY, LORENTZ TRANSFORMATIONS ACT ON 4-VECTORS SO THAT

$$\Lambda x = x' \Rightarrow x^2 = x'^2$$

- IN COMPONENTS, WE WRITE THIS AS

$$* \quad \Lambda^\mu{}_\nu x^\nu = x'^\mu \Rightarrow x_\mu x^\mu = x'_\mu x'^\mu = s^2$$

WITH $x_\mu = \eta_{\mu\nu} x^\nu$

x^μ IS CONTRAVARIANT
x_μ IS COVARIANT

- LET US WRITE $*$ $\Lambda^\mu{}_\nu = \dfrac{\partial x'^\mu}{\partial x^\nu}$

Λ IS A CONSTANT 4-MATRIX BUT THIS HANDY.

- IN PARTICULAR WE HAVE THAT

① $$\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\nu} = \delta^\mu_\nu$$ SO THAT

$$(\Lambda^{-1})^\alpha{}_\nu = \frac{\partial x^\alpha}{\partial x'^\nu}$$

AND $x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu \iff x^\nu = \frac{\partial x^\nu}{\partial x'^\mu} x'^\mu$

② $$\begin{aligned} s^2 = x'_\mu x'^\mu &= \eta_{\mu\nu} x'^\mu x'^\nu \\ &= \eta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} x^\alpha x^\beta \\ &= \eta_{\alpha\beta} x^\alpha x^\beta \end{aligned}$$

SO $$\eta_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} = \eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta$$

OR $$\eta = \Lambda^T \eta \Lambda$$

$\Rightarrow$ LORENTZ TRANSFORMATIONS PRESERVE THE MINKOWSKI METRIC.

- THIS IS LIKE SAYING THAT $R^T R = \mathbb{1}$ FOR ROTATIONS: THEY PRESERVE THE EUCLIDIAN METRIC.

THIS IS THE GEOMETRIC WAY TO SPECIAL RELATIVITY.

$$\boxed{\begin{array}{l}\text{SPECIAL RELATIVITY} \Rightarrow \text{LORENTZ TRANSFORMATIONS} \\ \Rightarrow \text{GEOMETRIC OBJECTS} \Rightarrow \text{LORENTZ INVARIANTS}\end{array}}$$

- ONE SUCH IMPORTANT IS THE LAGRANGIAN

  = LORENTZ SCALAR

- IF WE KNOW HOW LORENTZ TRANSFORMATIONS ACT ON GEOMETRICAL OBJECTS, THEN WE KNOW HOW TO BUILT A LAGRANGIAN. FROM THAT LAGRANGIAN WE CAN BUILT AMPLITUDES THAT ARE ALSO LORENTZ SCALARS $\Longleftrightarrow$ THE SAME FOR ALL OBSERVERS!

<u>WHICH GEOMETRIC OBJECTS DO WE KNOW</u>?

- x^μ IS CONTRAVARIANT, MEANING OBJECTS THAT

  TRANSFORM AS $\quad x'^\nu = \dfrac{\partial x'^\nu}{\partial x^\mu} x^\mu \quad$ (UPPER INDEX)

- HOW ABOUT x_μ?

$$x_\mu x^\mu \to x'_\mu x'^\mu = x_\mu x^\mu$$

$$= x'_\mu \frac{\partial x'^\mu}{\partial x^\alpha} x^\alpha$$

$$\Rightarrow \quad x'_\mu \frac{\partial x'^\mu}{\partial x^\alpha} = x_\alpha$$

OR $$\boxed{x'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu} x_\alpha}$$

SUCH OBJECTS ARE COVARIANT (LOWER INDEX)

- THE ARCHETYPE OF A COVARIANT 4-VECTOR IS THE 4-GRADIENT $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$

NOTICE THAT $\partial_\mu = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i}\right) = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$

- FROM CONTRAVARIANT AND COVARIANT 4-VECTORS WE CAN WRITE DOWN TENSORS

• EG

$$A^\mu B^\nu \longrightarrow A'^\mu B'^\nu = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} A^\alpha B^\beta$$

DEFINES THE TRANSFORMATION PROPERTIES OF A TENSOR WITH TWO UPPER INDICES

$$T^{\mu\nu} \sim A^\mu B^\nu$$

$$\Longleftrightarrow \quad T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}$$

EXAMPLES ARE THE STRESS-ENERGY TENSOR OF A FIELD AND THE FIELD STRENGTH OF A GAUGE FIELD

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

• FROM THE LATTER, WE CAN WRITE

$F_{\mu\nu} F^{\mu\nu}$ AS A LORENTZ SCALAR.

$$\Longrightarrow \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \text{KINETIC TERM OF A GAUGE FIELD}$$

- ANOTHER EXAMPLE OF GEOMETRIC OBJECTS WITH SPECIFIC TRANSFORMATION PROPERTIES UNDER THE LORENTZ GROUP IS A DIRAC SPINOR.

- CONSIDER THE FOLLOWING 6 MATRICES

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

<u>EX:</u>

- WHY 6 MATRICES?

- CHECK THAT

$$S^{ij} = \varepsilon^{ijk} \begin{pmatrix} S^k & 0 \\ 0 & S^k \end{pmatrix} \quad \text{WITH} \quad S^k = \frac{1}{2}\sigma^k$$

AND

$$S^{0k} = \begin{pmatrix} K^k & 0 \\ 0 & -K^k \end{pmatrix} \quad \text{WITH} \quad K^k = -\frac{i}{2}\sigma^k$$

THERE ARE 6 KIND OF LORENTZ TRANSFORMATIONS

- 3 ROTATIONS AROUND $\{\hat{x}, \hat{y}, \hat{z}\}$ DIRECTIONS
- 3 BOOST ALONG $\{\hat{x}, \hat{y}, \hat{z}\}$ DIRECTIONS

THESE CAN BE PARAMETERIZED BY 6 NUMBERS

- 3 ANGLES $(\theta_x, \theta_y, \theta_z)$
- 3 RAPIDITIES (η_x, η_y, η_z)

EX: WRITE DOWN A LORENTZ TRANSFORMATION MATRIX

- AS A ROTATION AROUND $\hat{x}$ IN TERMS OF θ_x
- AS A BOOST ALONG $\hat{x}$ IN TERMS OF η_x

FOR THE LATTER, USE $\sinh \eta_x$ AND $\cosh \eta_x$

WITH $\sinh \eta_x = \dfrac{v_x}{\sqrt{1-v_x^2}}$ $\cosh \eta_x = \dfrac{1}{\sqrt{1-v_x^2}}$

AND SO $\tanh \eta_x = v_x \iff \underbrace{\eta_x = \sqrt{\dfrac{1+v_x}{1-v_x}}}_{= \text{RAPIDITY}}$

THESE SIX PARAMETERS CAN BE GROUPED INTO A $\omega_{\mu\nu} = -\omega_{\nu\mu}$ ANTISYMMETRIC MATRIX.

THEN A LORENTZ TRANSFORMATION ACTING ON A DIRAC SPINOR CAN BE WRITTEN AS

$$\psi \longrightarrow \psi' = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\psi \equiv \Lambda_{\frac{1}{2}}\psi$$

Ex: VERIFY THAT THIS TRANSFORMATION LEAVES THE DIRAC EQUATION INVARIANT

IE $\psi \longrightarrow \psi'$

$$\Longrightarrow \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0$$

$$\longrightarrow \left(i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m\right)\psi'(x') = 0$$

SEE PESKIN & SCHROEDER PAGE

REM: γ^μ IS A CONTRAVARIANT 4-VECTOR BUT OF A SPECIAL KIND SINCE

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad \text{AND} \quad \eta^{\mu\nu} \text{ IS INVARIANT}$$

$$\eta'_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} = \eta_{\mu\nu}$$

THIS IS CAPTURED BY THE FACT THAT

$$\Lambda^\mu{}_\nu \gamma^\nu = \Lambda^{-1}_{\frac{1}{2}} \gamma^\mu \Lambda_{\frac{1}{2}}$$

AS A CONSEQUENCE

$$\bar\psi \gamma^\mu \psi \longrightarrow \bar\psi' \gamma^\mu \psi' = \Lambda^\mu{}_\nu \bar\psi \gamma^\nu \psi$$

TRANSFORMS AS A (CONTROVARIANT) 4-VECTOR

IN $\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}}$

THE SPATIAL COMPONENTS CAN BE WRITTEN AS

$$\Lambda_{\frac{1}{2}}\Big|_{ROT} = e^{-\frac{i}{2}\omega_{ij}\,\varepsilon^{ijk} S^{k}}$$

$$\equiv e^{-i\vec{\theta}\cdot\vec{S}}$$

WITH $\theta^{k} = \omega_{ij}\,\varepsilon^{ijk}$ THE ANGLE OF ROTATION AROUND THE AXIS $\hat{x}_k$.

THE S^{ij} ARE OBVIOUSLY THE GENERATORS OF ROTATION FOR SPIN $1/2$ PARTICLES ($S^{k} = \frac{\sigma^{k}}{2}$)

. SIMILARLY, THE S^{0k} ARE GENERATORS OF THE BOOSTS (IN THE DIRAC SPINOR REPRESENTATION)

EQ.: WRITE DOWN A BOOST ALONG DIRECTION $\hat{x}^1$ IN THE WEYL BASIS. SHOW THAT

$$\chi_L \to \Lambda_L\, \chi_L = e^{-\frac{\beta^1}{2}\sigma^1}\chi_L$$

AND $\chi_R \to \Lambda_R \chi_R = e^{+\frac{\eta}{2}' \sigma^1} \chi_R$

- SHOW THAT

$$e^{\mp \frac{\eta}{2}' \sigma^1} = \cosh \frac{\eta'}{2} \mp \sigma^1 \sinh \frac{\eta'}{2}$$

(COMPARE THE SERIES EXPANSIONS ON BOTH SIDES)

- SHOW THAT $\chi_L^+ \to \chi_L^+ e^{-\frac{\eta}{2}' \sigma^1}$

χ_L AND χ_R ARE $\neq$! $\chi_R^+ \to \chi_R^+ e^{+\frac{\eta}{2}' \sigma^1}$

(REMARK: UNLIKE ROTATIONS, WHICH ARE REPRESENTED BY UNITARY OPERATORS, BOOST ARE NOT UNITARY)

- VERIFY THAT

$$\chi_L^+ \bar{\sigma}^\mu \chi_L \quad \text{AND} \quad \chi_R^+ \sigma^\mu \chi_R$$

TRANSFORM AS 4-VECTORS UNDER THE BOOST ALONG $\hat{x}$.

- FROM THE TRANSFORMATION PROPERTIES OF χ_L AND χ_R WE CAN START BUILDING A LORENTZ INVARIANT LAGRANGIAN.

  ↳ GLOBAL BOOST/ROTATIONS

- THE FOLLOWING IS A LORENTZ SCALAR:

$$\chi_R^\dagger \sigma^\mu \partial_\mu \chi_R$$

TO MAKE IT REAL (HERMITIAN IN THE SENSE OF OPERATOR) WE JUST NEED A FACTOR OF i (WHY?)

- SO

$$\mathcal{L}_R = i \chi_R^\dagger \sigma^\mu \partial_\mu \chi_R$$

IS BOTH REAL AND LORENTZ INVARIANT. IT INVOLVES ONLY χ_R. ANOTHER INSTANCE IS

$$\mathcal{L}_L = i \chi_L^\dagger \bar{\sigma}^\mu \partial_\mu \chi_L$$

- WE CAN COMBINE THE TWO IN SOMETHING THAT INVOLVES BOTH χ_L AND χ_R:

$$\mathcal{L}_D = i\,\chi_R^\dagger \sigma^\mu \partial_\mu \chi_R + i\,\chi_L^\dagger \bar{\sigma}^\mu \partial_\mu \chi_L$$

$$\equiv i\,\bar{\psi}\,\gamma^\mu \partial_\mu \psi$$

WHICH IS THE LAGRANGIAN FOR A DIRAC FERMION.

WITH $\bar{\psi} = \psi^\dagger \gamma^0$

- WHAT IS THE POINT? WE DON'T NEED BOTH χ_L AND χ_R TO WRITE A LAGRANGIAN WITH BOTH χ_L AND χ_R. SO THE χ_L AND χ_R ARE THE TRUE BUILDING BLOCKS FOR A THEORY OF FERMIONS
- THE SM IS BASED ON THIS. BY EXTENSION, THIS IS TRUE OF GRAND UNIFIED THEORIES.
- SO FAR WE JUST HAVE KINETIC TERMS FOR χ_L AND χ_R.

BUT WE CAN COUPLE THEM. SINCE

$$\chi_L \to e^{-\eta/2\,\sigma^1}\chi_L$$

$$\chi_R \to e^{+\eta/2\,\sigma^1}\chi_R$$

AND SO $\chi_R^\dagger \to \chi_R^\dagger e^{+\eta/2\,\sigma^1}$

WE SEE THAT

$$\chi_R^\dagger \chi_L \longrightarrow \chi_R^\dagger e^{\eta/2\,\sigma^1} e^{-\eta/2\,\sigma^1} \chi_L$$

$$= \chi_R^\dagger \chi_L$$

- THIS IS NOT HERMITIAN BUT

$$\chi_R^\dagger \chi_L + \chi_L^\dagger \chi_R \quad \text{IS.}$$

- THIS IS NOTHING BUT

$\bar{\psi}\psi = \psi^\dagger \gamma^0 \psi$ WRITTEN IN TERMS OF

$$\psi = \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$$

- NOW $[\mathcal{L}] = E^4$, MEANING THAT A LAGRANGIAN DENSITY HAS ENGINEERING DIMENSION $(\text{ENERGY})^4$ → FOURTH POWER

THIS IS BECAUSE

$$S = \int d^4x \, \mathcal{L} = \int dt \, L$$

ACTION: $[S] = E \times T$

$= \text{ENERGY} \times \text{TIME}$

$= [\hbar]$

AND $\hbar = 1$ (NATURAL UNITS)

- SINCE $c = 1$ (" ") TOO

$$[d^4x] = E^{-4}$$

- THUS $[\mathcal{L}] = E^4$.

- BY CONVENTION, A KINETIC TERM IS NORMALIZED WITH NO PARAMETERS BUT THE FACTOR OF FIELDS (QUADRATIC FOR USUAL THEORIES) AND DERIVATIVES.

- THIS IMPLIES THAT

$$[\psi] = E^{3/2} \text{ (in } D=4 \text{ SPACETIME DIMENSIONS)}$$

SINCE

$$\mathcal{L}_D = i\bar{\psi}\gamma^\mu\partial_\mu\psi$$

- THUS $[\bar{\psi}\psi] = E^3$. TO MAKE A TERM FOR A LAGRANGIAN, WITH NEED A PARAMETER WITH DIMENSION

$$\text{ENERGY} = \text{MASS} = m$$

- SO

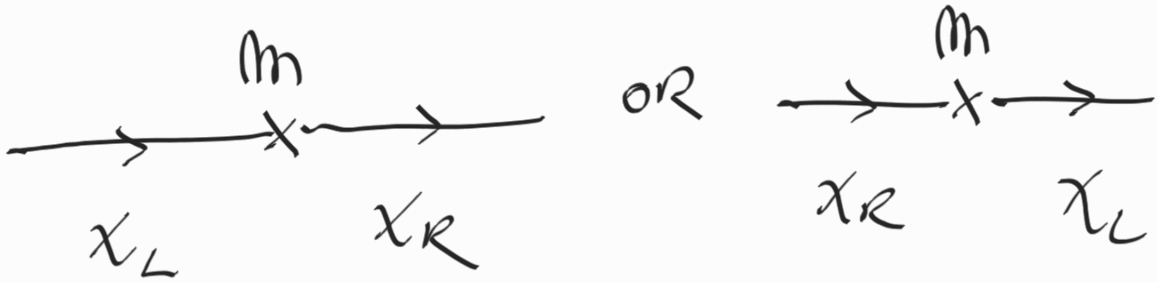
$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi$$

$$= i\chi_R^\dagger\sigma^\mu\partial_\mu\chi_R + i\chi_L^\dagger\bar{\sigma}^\mu\partial_\mu\chi_L$$

$$- m(\chi_R^\dagger\chi_L + \chi_L^\dagger\chi_R)$$

IS A GOOD LAGRANGIAN!

• NOTICE AGAIN THAT THE MASS TERM COUPLES χ_R AND χ_L, DIFFERENT BUILDING BLOCKS. IT IS GOOD TO THINK OF A MASS AS A SORT OF INTERACTION. GRAPHICALLY (IE A FEYNMAN DIAGRAM) WE HAVE



• THE ARROW REPRESENT FERMIONIC CURRENTS.

• INDEED, FROM THE LAGRANGIAN $\mathcal{L}_R$ AND $\mathcal{L}_L$ WE HAVE THE FOLLOWING CONSERVED CURRENTS

$$\partial_\mu \left(\chi_R^\dagger \sigma^\mu \chi_R \right) = 0$$

AND $\partial_\mu \left(\chi_L^\dagger \bar{\sigma}^\mu \chi_L \right) = 0$

• THESE CAN BE WRITTEN EQUIVALENTLY IN TERMS OF A DIRAC FIELD:

$$\psi_L = P_L \psi \quad ; \quad \psi_R = P_R \psi$$

$$\Rightarrow \quad \chi_R^\dagger \sigma^\mu \chi_R \equiv \bar{\psi}_R \gamma^\mu \psi_R$$

$$\equiv \bar{\psi} \gamma^\mu P_R \psi = J_R^\mu$$

$$\chi_L^\dagger \bar{\sigma}^\mu \chi_L = \bar{\psi}_L \gamma^\mu \psi_L$$

$$= \bar{\psi} \gamma^\mu P_L \psi = J_L^\mu$$

Ex. CHECK THE RELATIONS ABOVE.

- START FROM THE DIRAC LAGRANGIAN

$$\mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi$$

TO SHOW THAT

$J^\mu = \bar{\psi} \gamma^\mu \psi$ IS A CONSERVED CURRENT.

HINT: . WRITE THE EULER-LAGRANGE EQS FOR ψ AND FOR $\bar{\psi}$ AND COMBINE THEM TO SHOW THAT

$$\partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0$$

- THE DIRAC THEORY, WITH BOTH χ_L AND χ_R PLAYING A SYMMETRIC ROLE, THE CONSERVED CURRENT INVOLVES BOTH J^μ_L AND J^μ_R.
- WE CAN ALSO CONSIDER THEIR DIFFERENCE

IE

$$J^\mu_5 = J^\mu_R - J^\mu_L$$
$$= \bar{\psi}\gamma^\mu\left(\frac{1+\gamma_5}{2}\right)\psi - \bar{\psi}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\psi$$
$$= \bar{\psi}\gamma^\mu\gamma_5\psi = \text{CHIRAL CURRENT}$$

EX: CHECK THAT THE EL EQUATIONS OF THE DIRAC THEORY IMPLY THAT

$$\partial_\mu J^\mu_5 = 2im\bar{\psi}\gamma_5\psi$$

- THAT CURRENT, CALLED THE AXIAL CURRENT, IS ONLY CONSERVED IF $m=0$
- THIS IS EQUIVALENT TO SAY THAT BOTH J^μ_L AND J^μ_R ARE CONSERVED

INDEPENDENTLY IF $m=0$.

- THIS IS TRUE TO $O(\hbar^0)$ OR CLASSICALLY. WE WILL SEE THAT THERE MAY NOT BE CONSERVED TO $O(\hbar)$, A PHENOMENON CALLED "CHIRAL ANOMALY"

FINAL REMARKS - DISCRETE SYMMETRIES

- THIS IS ALL A BIT FORMAL. WE HAVE SEEN THAT THERE ARE TWO KINDS OF FERMIONS χ_L AND χ_R AND THAT THEY ARE DECOUPLED IN THE MASSLESS LIMIT.
- DISCRETE SYMMETRIES SHED SOME LIGHT IN THEIR MEANINGS.

① PARITY

- THIS IS A SYMMETRY THAT $\vec{x} \to -\vec{x}$ LIKE A MIRROR TRANSFORMATION.

- IT IS AN IMPORTANT SYMMETRY, BUT IT IS NOT A SYMMETRY OF THE SM, AS YOU KNOW WELL.
- FROM OUR PERSPECTIVE, WE HAVE

$$\sigma^\mu = (1, \vec{\sigma}) \quad \text{AND} \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

AND, ALSO

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \xrightarrow{\vec{x} \to -\vec{x}} \partial_\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

- FROM THIS IT IS TEMPTING TO

TAKE $\quad \chi_R(t, \vec{x}) \to \chi_L(t, -\vec{x})$

AND $\quad \chi_L(t, \vec{x}) \to \chi_R(t, -\vec{x})$

SO THAT

$$\chi_R^\dagger \sigma^\mu \partial_\mu \chi_R \overset{P}{\longleftrightarrow} \chi_L^\dagger \bar{\sigma}^\mu \partial_\mu \chi_L$$

- WE SEE THAT PARITY EXCHANGES

χ_R AND χ_L

- THIS IS THE REASON FOR THE R = RIGHT AND L = LEFT INDICES.
- PARITY EXCHANGES LEFT FOR RIGHT, ETC.
- IF THERE ARE BOTH L AND R FERMIONS WITH SAME OTHERWISE PROPERTIES (INTERACTIONS) PARITY IS A GOOD SYMMETRY. THIS IS THE CASE FOR THE DIRAC THEORY SINCE

$$\mathcal{L} = i\bar{\Psi}\not{\partial}\Psi - m\bar{\Psi}\Psi$$

$$= i\chi_R^\dagger \sigma^\mu \partial_\mu \chi_R + i\chi_L^\dagger \bar{\sigma}^\mu \partial_\mu \chi_L$$

$$- m(\chi_R^\dagger \chi_L + \chi_L^\dagger \chi_R)$$

WHICH IS SYMMETRIC FOR $\chi_L \leftrightarrow \chi_R$.

- QED AND QCD ARE PARITY SYMMETRIC (IN ABSENCE OF A THETA TERM)
- $SU(2)_L \times U(1)_Y$ IS NOT P SYMMETRIC

$\Rightarrow$ PARITY CANNOT BE DEFINED!

NB: WHY PARITY IS RESTORED AT LARGE DISTANCES IS A BIT OF A MYSTERY.

Ex: CHECK THAT $\chi_L \overset{P}{\longleftrightarrow} \chi_R$

IS REPRESENTED BY

$$\psi \longrightarrow \psi_P = \eta \gamma_0 \psi$$

IN THE DIRAC THEORY, WITH η A PHASE.

SOME AUTHORS TAKE $\eta^2 = 1$, SO THAT $\eta = \pm 1$, OTHERS $\eta^2 = -1$ ($P^2 =$ IDENTITY BUT $= 2\pi$ ROTATION TOO)

- NOW, L AND R ARE JUST LABELS. WHAT DO THEY MEAN?
- TO SEE THIS, IT IS IMPORTANT TO HAVE IN MIND THAT FERMIONS SPIN. IF $m = 0$, THERE IS NO POSSIBLE REST FRAME, SO THERE ARE TWO POSSIBILITIES FOR SPINNING, TURNING TO THE LEFT OR TO THE RIGHT.

- THINK OF LIGHT. A MONOCHROMATIC PLANE-WAVE CAN BE TAKEN TO BE CIRCULARLY POLARIZED TO THE LEFT OR TO THE RIGHT WITH RESPECT TO THE DIRECTION OF PROPAGATION, THIS IS CALLED HELICITY BY PARTICLE PHYSICISTS.
- IN MATH TERMS, $\epsilon^{\mu}_{\pm} = \left(0, \frac{1}{\sqrt{2}}, \frac{\pm i}{\sqrt{2}}, 0\right)$

FOR $\hat{k}^{\mu} = (1, 0, 0, 1)$ ($\hat{k}^2 = 0$ FOR LIGHT OF COURSE)

SO THAT $\hat{k} \cdot \varepsilon_{\pm} = 0$

- FOR LIGHT, BOTH POLARIZATIONS EXIST

$\Longrightarrow$ 2 DEGREES OF FREEDOM BUT PHOTON IS ITS OWN ANTIPARTICLE (IE REAL FIELD)

- FOR SPINORS, THERE IS SOMETHING SIMILAR BUT A χ_L OR A χ_R CAN DESCRIBE PARTICLES AND ANTIPARTICLES BECAUSE OF THE CONSERVED CURRENT $\chi_L^{+} \bar{\sigma}^{\mu} \chi_L$ AND $\chi_R^{\dagger} \sigma^{\mu} \chi_R$

So χ_L = 2 DEGREES OF FREEDOM (AND THE SAME FOR χ_R). So THEY CAN ONLY SPIN LEFT (RESP. RIGHT)

THIS IS OF COURSE CONVENTIONAL, THE SAME AS WHEN WE FIX THE ORIENTATION OF

WITH THE RIGHT HAND...

So

= χ_L MOVING TO THE RIGHT

= LEFT-HELICITY PARTICLE MOVING TO THE RIGHT.

. IT CAN ALSO REPRESENT AN ANTIPARTICLE MOVING TO THE LEFT. ITS HELICITY IS THEN TO THE RIGHT; CHIRALITY AND HELICITY HAVE THE SAME SIGN FOR PARTICLES AND OPPOSITE SIGN FOR ANTIPARTICLES.

FOR A χ_R, IT IS THE OPPOSITE

χ_R : RIGHT PARTICLE WITH RIGHT HELICITY MOVING TO THE RIGHT

OR

RIGHT ANTIPARTICLE WITH LEFT-HELICITY MOVING TO THE LEFT.

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ex: THIS CAN BE ALL DISCUSSED IN THE FRAMEWORK OF THE DIRAC THEORY

WITH HELICITY OPERATOR DEFINED

AS
$$\hat{h} = \hat{p}\cdot\vec{S} = \begin{pmatrix} \frac{1}{2}\hat{p}\cdot\vec{\sigma} & 0 \\ 0 & \frac{1}{2}\hat{p}\cdot\vec{\sigma} \end{pmatrix}$$

WITH $\hat{p} = \dfrac{\vec{p}}{p_0}$

(WEYL BASIS)

SHOW THAT

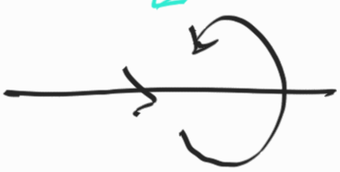
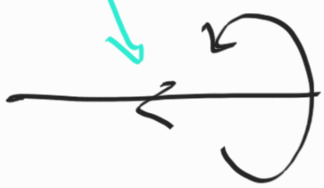
$\not{p}\,\psi = 0$ (DIRAC EQUATION MASSLESS LIMIT)

IMPLIES THAT

$$\vec{p}\cdot\vec{S}\,\psi = \frac{1}{2}p^0\gamma_5\psi$$

AND COMPARE TO THE PREVIOUS STATMENTS ABOUT CHIRALITY VS HELICITY

NOTICE THAT (!) ARROWS

 TO 

CORRESPONDS TO GOING TO LEFT-HELICITY MOVING TO THE RIGHT TO RIGHT HELICITY MOVING TO THE LEFT OR χ_L TO χ_R

- THIS CAN BE ACHIEVED BY A CHANGE OF FRAME ONLY IF THERE $\exists$ A REST FRAME, SO FOR MASSIVE FERMIONS!

② CHARGE CONJUGATION

- NAIVELY, CHARGE CONJUGATION EXCHANGES PARTICLES WITH ANTIPARTICLES (ie $e^{-ip\cdot x}$ WITH $e^{ip\cdot x}$) SOLUTIONS

- THIS IS HOWEVER NOT A SYMMETRY OF THE SM BECAUSE, AGAIN, OF THE DISTINCTION BETWEEN χ_L AND χ_R.

- TO SEE THIS, GO TO DIRAC THEORY

---

ex: SHOW THAT

$$\psi \xrightarrow{C} \psi^c = \gamma^2 \psi^*$$

IS A SYMMETRY OF THE DIRAC THEORY

IE

$$(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow (i\partial\!\!\!/ - m)\psi^c = 0$$

- FOR THIS SHOW THAT $\gamma^2 \gamma^{\mu *} \gamma^2 = \gamma^\mu$

• IN THE DIRAC THEORY

$$\psi_c \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} \longrightarrow \begin{pmatrix} -i\sigma^2 \chi_R^* \\ i\sigma^2 \chi_L^* \end{pmatrix}$$

$$\equiv \begin{pmatrix} \chi_R^c \\ \chi_L^c \end{pmatrix}$$

SO $\chi_L \xrightarrow{C} \chi_R^c$

AND $\chi_R \xrightarrow{C} \chi_L^c$

• IF WE HAVE A THEORY WITH χ_L AND χ_R WITH DISTINCT PROPERTIES (A CHIRAL THEORY, LIKE THE SM), C IS NOT A GOOD SYMMETRY.

• HOWEVER, CP IS OK (AT THIS LEVEL) SINCE $\chi_L \xrightarrow{CP} \chi_L^c$

AND $\chi_R \xrightarrow{CP} \chi_R^c$

Ex: AS A CLOSING EXERCISE, WE CONSIDER A BSM APPLICATION.

WE WANT TO MODELIZE THE POSSIBILITY THAT THE PROTON CAN BE UNSTABLE.

- FOR INSTANCE $p \to e^+ \pi^0$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \text{PROTON} & \text{ANTI-ELECTRON} & \text{NEUTRAL} \\ \text{CHARGE } +1 & \text{or POSITRON} & \text{PION} \\ & \text{CHARGE } +1 & \end{array}$$

- PROTON AND POSITRON ARE FERMIONS
- PION IS A SCALAR PARTICLE
- A POSSIBILITY IS TO TAKE

$$\mathcal{L}_{BSM} = g \bar{\psi}_p \psi_e^c \phi_{\pi_0} + h.c.$$

AS A BSM LAGRANGIAN TO EXPRESS THE POSSIBILITY OF PROTON DECAY.

THE AMPLITUDE FOR DECAY IS $\propto g$ SOME COUPLING.

- WHAT IS THE ENGENEERING DIMENSION OF $[g] = E^?$

- THE AMPLITUDE FOR DECAY IS $\mathcal{M} \simeq g$ TIMES SOME SPINORS (u AND $\bar{v}$'S)

$$\Rightarrow \quad \Gamma \simeq g^2 m_p$$

THIS IS BECAUSE $\Gamma \propto |\mathcal{M}|^2$

AND $m_p \gg m_e, m_{\pi_0}$ IS THE ONLY DIMENSIONFULL PARAMETER THAT IS TRULY RELEVANT FOR THE DECAY RATE (HEAVY PARTICLES LIVE LESS)

- LOOK FOR EXPS BOUND ON THE LIFETIME OF THE PROTON AND GET AN ESTIMATE FOR g

(OF COURSE, USE NATURAL UNITS)

- SHOW THAT g IS VERY SMALL

$\Rightarrow$ THIS IS UNNATURAL (SEE LECTURE 2)