1) MAKE A LIST OF LEFT AND RIGHT SM FERMIONS AND THEIR REPRESENTATION UNDER $SU(3) \times SU(2) \times U(1)$

e.g. $e_L \in (1, 2, -1)$

SINGLET OF $SU(3)$ — DOUBLET OF $SU(2)$ — HYPERCHARGE $= -1$

WRITE ALL THE RIGHT-HANDED FERMIONS AS LEFT-HANDED AND, FOR THEM, DO THE SAME EXERCISE.

REM

3 of $SU(3)$ $\rightarrow$ $\bar{3}$ of $SU(3)$ (COMPLEX REP.)

2 of $SU(3)$ $\rightarrow$ 2 of $SU(2)$ (PSEUDOREAL)

ALSO $Q = T_3 + Y/2$ (STANDARD CHOICE)

① REPRESENTATIONS OF $SU(3)$

- IT IS A GOOD WARMUP TO WORK OUT BASICS OF $SU(3)$ REPS $\sim SU(N)$ AND THUS $SU(5)$

- THERE ARE 2 DISTINCT OBJECTS WITH 3 ENTRIES

$$N^i \sim 3 \quad = \text{TRIPLET} \sim \text{FUNDAMENTAL}$$

$$u_i \sim \bar{3} \quad = \text{CONJUGATE}$$

WITH $i = 1, 2, 3$

- LET US WRITE U IN THE FUNDAMENTAL REPS WITH $U^\dagger U = 1$ AND $\det U = 1$

THEN $N' = U N$ OR $N'^i = U^i_j N^j \qquad N \sim 3$

- TAKING THE COMPLEX CONJUGATE GIVES

$$N'^{i*} = (U^i_j)^* (N^j)^*$$

$$= (U^\dagger)^j_i (N^j)^*$$

$\Rightarrow$ THAT MOTIVATES TO INTRODUCE 3-COMPONENTS OBJECTS WITH LOWER INDICES SO THAT

$$u'_i = (U^\dagger)^j_i u_j \quad \text{AND} \quad u \sim N^* \sim \bar{3}$$

CLEARLY, THIS ALL APPLY TO ANY FUNDAMENTAL OF $SU(N)$, AND SO $SU(5)$ FOR WHICH

$$v^i \sim 5 \quad \text{AND} \quad u_i \sim \bar{5} \qquad i=1\dots5$$

FROM THIS POINT, WE CAN BUILT OBJECTS THAT TRANSFORM AS TENSORS UNDER $SU(3)$ (AND SO $SU(N)$).

- FOR INSTANCE $T^{ij} \to T'^{ij} = U^i_k U^j_e T^{ke}$

TRANSFORMS AS $v^i v^j$ OBJECTS. $\sim N \otimes N$

OR

$$T^i_j \to T'^i_j = U^i_k U^{+e}_j T^k_e$$

WHICH TRANSFORMS $v^i u_j \sim N \otimes \bar{u}$ ← I PUT A BAR TO MEAN u_i ←

- THESE ARE HOWEVER REDUCIBLE, IN THE SENSE OF TENSORS WHOSE PROPERTIES DON'T CHANGE UNDER U, U^+ TRANSFORMATIONS.

- FOR INSTANCE

$S^{ij} = \frac{T^{ij} + T^{ji}}{2}$ IS SYMMETRIC UNDER $i \leftrightarrow j$

AND $A^{ij} = \frac{T^{ij} - T^{ji}}{2}$ IS ANTISYMMETRIC.

<u>Ex</u>: - SHOW THAT S' IS SYMMETRIC
A' IS ANTISYMMETRIC

- HOW MANY COMPONENTS DO YOU COUNT FOR
S AND A
DO IT FOR $SU(3)$ AND THEN $SU(N)$
AND SO FOR $SU(5)$.

- NOTICE THAT $U^{+}U = \mathbb{1}$ CAN BE WRITTEN
AS $(U^{+})^{k}{}_{i}\, U^{i}{}_{e} = \delta^{k}_{e}$

SO THE KRONECKER δ HAS UPPER AND LOWER INDICES.

MOREOVER, THIS CAN BE WRITTEN AS

$$(U^{+})^{k}{}_{j}\, \delta^{j}_{i}\, U^{i}{}_{e} = \delta^{k}_{e}$$

WHICH MEANS THAT THAT δ^i_j IS AN <u>INVARIANT</u> TENSOR OF $SU(N)$.

- THERE IS ANOTHER ONE.

$\det U = 1$ CAN BE WRITTEN AS ($SU(3)$)

$$\varepsilon_{ijk}\, U^i_m\, U^j_n\, U^k_l = \varepsilon_{mnl}$$

OR

$$\varepsilon^{ijk}\, U^m_i\, U^n_j\, U^l_k = \varepsilon^{mnl}$$

Ex: CHECK THIS!

SO WE HAVE 2 INVARIANT TENSORS WITH WHICH WE LOWER OR RAISE INDICES (THINK OF $\eta_{\mu\nu}$)

eg $$u_{ij} = \varepsilon_{ijk}\, v^k$$

Ex: CHECK THAT $u_{ij} \sim u_i \sim \bar{3}$

(READ " THE TENSOR u_{ij} TRANSFORMS AS THE CONJUGATE OF A 3)

THIS GENERALIZES TO $SU(N)$ WITH THE OBVIOUS

$\epsilon_{ijk} \to \epsilon_{i_1 \ldots i_N}$

so if $v^i \sim N$ (FUNDAMENTAL, N COMPONENT OBJECTS)

$$u_{i_1 \ldots i_{N-1}} = \epsilon_{i_1 \ldots i_N} v^{i_N} \sim \bar{N} \quad \text{(THE CONJUGATE)}$$

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eg: CONSIDER SU(5) SO $\epsilon_{ijk\ell}$

CHECK THAT

$$u_{kpq} = \epsilon_{ijpq} v^{ij}_k \to (U^+)^s_p (U^+)^t_q (U^+)^m_k u_{stm}$$

NB THERE IS A WHOLE FORMALISM, BASED ON YOUNG TABLEAUX, TO WORK OUT THE NUMBERS OF COMPONENTS OF ARBITRARY SU(N) TENSORS.

FOR LOW DIMENSIONAL REPS, THIS IS REALLY NECESSARY.

MULTIPLYING REPRESENTATIONS

BACK TO SU(3). WITH δ^i_j WE CAN TAKE TRACES OVER INDICES,

eg $\delta^i_j u_i v^j$ OR $\delta^i_j w^j{}_{ik} \sim u_k$

ex: • SHOW THAT (I KNOW, IT IS OBVIOUS)

$$\delta^i_j u_i v^j = \delta^e_e u'_k v'^e$$

• TAKE $E^j_i = u_i v^j$

⊕ HOW MANY COMPONENTS?

⊕ WRITE DOWN $\bar{E}^j_i$ FROM E^j_i

SUCH THAT $\bar{E}^j_i \delta^i_j = 0$ (TRACELESS)

HOW MANY COMPONENTS?

⊕ CHECK THAT THE PROPERTIES OF $\bar{E}^j_i$ ARE PRESERVED BY $SU(3)$ TRANSFORMATIONS (IRREDUCIBLE REPRESENTATION)

⊕ CONVINCE YOURSELF THAT

$\bar{E}^j_i \sim$ ADJOINT OF $SU(3)$

(MORE GENERALLY OF $SU(N)$)

CONCLUDE THAT

$$3 \otimes \bar{3} = 8 \oplus 1$$

ex: CONVINCE YOURSELF THAT

$$3 \otimes 3 = 6 \oplus \bar{3}$$

Ex: JUMPING AHEAD, CHECK OR CONVINCE YOURSELF THAT

$$5 \otimes \bar{5} = 24 \oplus 1$$

$$5 \otimes 5 = 10 \oplus 15$$

$$\bar{5} \otimes 10 = 5 \oplus 45$$

WARMUP FOR SO(10)

- SO(10) BELONGS TO A FAMILY OF LIE GROUP NOTED AS $SO(2n)$ (WITH $n>1$ (WHY?)) WITH n THE RANK OF THE GROUP.
- THE RANK IS THE NUMBER OF LINEARLY INDEPENDENT OF GENERATORS THAT COMMUTE WITH EACH OTHERS

Ex USING THE PROPERTIES

$$O^T O = \mathbb{1} \quad \text{AND} \quad \det O = 1$$

OF $SO(2n)$: $2n \times 2n$ MATRICES

SHOW THAT THE RANK OF $SO(2n)$ IS n.

LIKE FOR $SU(N)$, WE CONSIDER $2n$ COMPONENTS OBJECTS $\longrightarrow$

$$\vec{v} = \{ v^i, i = 1 \dots 2n \} \text{ OR VECTORS}$$

THAT TRANSFORM AS

REAL NUMBERS

$$v'^i = O^{ij} v^j$$

SO NO DISTINCTION BETWEEN v^i AND u_i (CONJUGATE) AS FOR $SU(N)$

Ex: SHOW THAT δ^{ij} (KRONECKER) IS AN INVARIANT TENSOR OF $SO(2n)$

Ex: TAKE v^i AND w^j AND WRITE DOWN

$$A^{ij} = -A^{ji}$$

AND $\bar{S}^{ij} = \bar{S}^{ji}$ AND $\delta^{ij} \bar{S}^{ij} = 0$

WORK OUT THE NUMBER OF COMPONENTS

OF A^{ij} AND $\tilde{S}^{ij}$ FOR $SO(2n)$

CONCLUDE THAT (with $2n = N$)

$$N \otimes N = \mathbb{1} \oplus \frac{N(N-1)}{2} \oplus \frac{N(N+1)}{2}$$