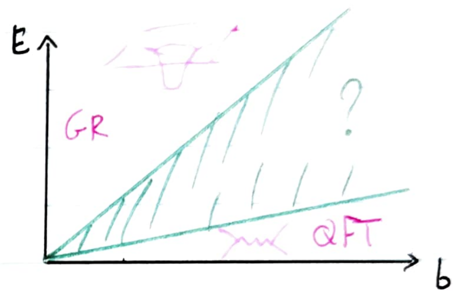
PART 1 : FOUNDATIONS

CH 1 SPACETIME AS A QUANTUM OBJECT

1.1 The problem

→ What we know :
1) Quantum mechanics : general framework for describing dynamics
2) The $SU(3) \times SU(2) \times U(1)$ standard model of particle physics : describes all matter and its non gravitational interactions
3) General relativity : describes gravity, space and time

→ Main problems :
1) Dark matter
2) Unification
3) Quantum gravity



→ What we know based on contradictory hypothesis

1.2 The end of space and time

→ (Einstein) spacetime = gravitational field
Since all field we know exhibit quantum properties, we expect the latter to do so.
↳ no (pseudo-)Riemanian manifold in QG.

→ Let's measure some field value at a location x, with precision L.
Let a particle be at x. QM ⇒ $\Delta x \Delta p > \hbar$
→ We want $\Delta x < L \to \Delta p > \hbar / L \rightsquigarrow p^2 > (\hbar/L)^2$
→ At large momentum, $E \simeq cp$
→ In GR, any energy acts as a gravitational mass : $M \sim E/c^2$
The Schwarzschild radius is $R = 2MG/c^2$
↳ The minimum value for L is $L = R$

Then; $L = R \sim \frac{MG}{c^2} = \frac{E}{c^2} \cdot \frac{G}{c^2} = \frac{p}{c} \cdot \frac{G}{c^2} = \frac{\hbar}{Lc} \cdot \frac{G}{c^2} = \frac{\hbar G}{Lc^3}$

$\Leftrightarrow L^2 = \hbar G / c^3$

DEF We found the Planck length L_{pe} defined as

$$L_{pe} \equiv \sqrt{\frac{\hbar G}{c^3}}$$

→ We need a genuinely new way of doing physics, where space and time come after the quantum states, and are a semiclassical approximations to quantum configurations.

1.3 Geometry quantized

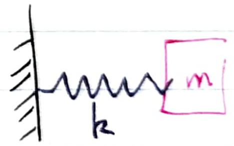
→ The quantum nature of a physical quantity is manifest in 3 forms:

1) possible discretization of the quantity (quantization)
2) short-scale fuzziness ⇐ uncertainty relations
3) probabilistic nature of its evolution (transition amplitudes)

↳ We only focus on ① and ② for now.

→ Let's review some basic QM through 3 examples.

- **Harmonic oscillator:**



→ We consider a mass attached to a spring.
The energy is given by $E = \frac{1}{2} m v^2 + \frac{1}{2} k q^2$
The Lagrangian by $\mathcal{L} = \frac{1}{2} m v^2 - \frac{1}{2} k q^2$ and the conjugated momentum by $\frac{\partial \mathcal{L}}{\partial v} = m v \equiv p$. The Hamiltonian is then $H = \frac{p^2}{2m} + \frac{1}{2} k q^2$

DEF The quantization postulate states that ∃ a Hilbert space $\mathcal{H}$ where (p, q) are non commuting self adjoint operators satisfying

$$[q, p] = i\hbar$$

↳ the energy operator has then a discrete spectrum of eigenvalues
$E \psi^{(n)} = E_n \psi^{(n)}$: $E_n = \hbar\omega(n + 1/2)$ with $\omega = \sqrt{k/m}$

Indeed, let's consider the following Hamiltonian: $H = \frac{1}{2}(p^2 + \omega^2 q^2)$

We introduce the Ladder operators
$$\begin{cases} a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \\ a^\dagger = \frac{1}{\sqrt{2\omega}}(\omega q - ip) \end{cases}$$

Then, $[a, a^\dagger] = \frac{1}{2\omega}(\omega[q, -ip] + \omega[ip, q]) = \frac{1}{2}(-i^2 + i(-i)) = 1$

Conversely, $q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$ and $p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$

We denote $\hat{a}|0\rangle = 0$ and $|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$. Rewriting H, we get

$H = \frac{1}{2}(p^2 + \omega^2 q^2) = \dots = \omega(a^\dagger a + 1/2) = \omega(n + 1/2)$

Indeed: $a^\dagger a |n\rangle \sim a^\dagger a (a^\dagger)^n = (a^\dagger)^1 [\,,\,](a^\dagger)^{n-1} + (a^\dagger)^2 [\,,\,](a^\dagger)^{n-2} + \dots + (a^\dagger)^n a$
$= n (a^\dagger)^n |0\rangle = n|n\rangle$ (eigenfunction)

→ Since a free field is a collection of oscillators (one per mode), a quantum field is a collection of discrete quanta.

⊙ Magic circle : discreteness is kinematics

→ Consider a particle moving on a circle, with potential $V(\alpha)$, $\alpha \in S^1 \sim [0, 2\pi]$. The associated Hamiltonian is $H = \frac{p^2}{2c} + V(\alpha)$ where $p = c\dot{\alpha}$ and $c \in \mathbb{R}$; $[c] = ML^2$.
The corresponding Hilbert space is $\mathcal{H} = L_2[S^1]$ and the momentum operator is $p = -i\hbar\partial_\alpha$, with eigenvalues $p_n = n\hbar$

DEF We call **kinematic** the properties of a system that depends only on its basic variables $(q, p, \dots)$ and **dynamic** the ones that depends on the evolution.

↳ Discreteness is a kinematic property.
↳ Notice that $[\alpha, p] \neq i\hbar$ since $\alpha = 0 \sim 2\pi$. A correct function would be $\sin(\alpha)$ or $\cos(\alpha)$, since they're continuous on S^1. In general, one could take $h \equiv e^{i\alpha}$.

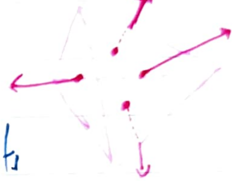
⊙ Angular momentum:

→ Let $\vec{L} \equiv (L^1, L^2, L^3)$ be the angular momentum of a system that can rotate. The total angular momentum is $L = \sqrt{L^i L^i}$.
From CM, $\vec{L}$ is the generator of ∞al rotations. Postulating that the corresponding quantum operator is also the generator of rotations in the Hilbert space, we have $[L^i, L^j] = i\hbar\, \varepsilon^{ij}{}_k L^k$

→ The SU(2) rep. theory gives the eigenvalues of L:

$$L_j = \hbar\sqrt{j(j+1)} \quad \text{with } j = 0, \tfrac{1}{2}, 1, 3/2, 2, \dots$$

⊙ Geometry:



→ Consider a tetrahedron τ. Its geometry is characterized by the lenght of its sides, the area of its faces...

→ We pick 4 vectors $\vec{L}_a$, normal to each face, with their norm equal to the area of the face

PROP | The $\{\vec{L}_a\}$ satisfy the closure relation:

$$\vec{C} \equiv \sum_{a=1}^{4} \vec{L}_a = 0$$

? PROP | The volume is $V^2 = \frac{2}{9}(\vec{L}_1 \times \vec{L}_2)\cdot\vec{L}_3 = \frac{2}{9}\varepsilon_{ijk} L_1^i L_2^j L_3^k = \frac{2}{9}\det L$

⊙ Quantization of geometry:

? → We postulate $[L_a^i, L_b^j] = i\,\delta_{ab}\, \ell_0^2\, \varepsilon^{ij}{}_k L^k_a$ Indices ??
with $[\ell_0] = L^2$ and must be related to $\ell_{P\ell}$. By dimensional analysis, we get $\ell_0^2 = 8\pi\gamma\, \ell_{P\ell}^2 = \gamma \frac{\hbar(8\pi G)}{c^3}$, with $\gamma \sim 1$

1.3.1 Quanta of area and volume:

→ One consequence of $[L^i_a, L^j_b] = i\,\delta_{ab}\,\frac{8\pi G\hbar}{c^3}\,\gamma\,\varepsilon^{ij}{}_k L^k_a$ is that $A_a = |\vec{L}_a|$ behaves as total angular momentum

PROP The area of the triangles bounding any tetrahedron is quantized with eigenvalues $A = l_0^2\sqrt{j(j+1)}$, with $j = 0, \frac{1}{2}, 1, 3/2, \ldots$

→ Let a quantum geometry be in state $j_1, \ldots, j_4$. The 4 vector operators $\vec{L}_a$ act on the tensor product $\mathcal{H}$ of 4 rep. of $SU(2)$, with respective spin $j_1, \ldots, j_4$ ⇒ the Hilbert space of the quantum states of the geometry of the $\triangle$ at fixed value of the area of its face is

$$\mathcal{H} = \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}$$

→ Taking into account the closure relation, the states that satisfy $\vec{C}\psi = 0$ are the states invariant under a global, diag. action of $SU(2)$: $\psi \in \mathcal{K} \equiv \mathrm{Inv}_{SU(2)}\{\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}\}$

Physical states invariant under common rotations

→ Notice that $[V^2, \vec{C}] = 0$ for $\psi \in \mathcal{K}$. We have a well-posed eigenvalue problem for the self-adjoint volume operator on $\mathcal{K}$. Since $\dim(\mathcal{K}) < \infty$, the volume has discrete eigenvalue

⊙ **Shape of the quanta of space and the fuzziness of the geometry**

→ $(\{A_a\}, V)$ form a maximally commuting set of operators ⇒ the quantum states of the $\triangle$ are unically characterized by their eigenvalues $|j_a, v\rangle$

→ Classical: $\{L_a\} + \vec{C}\psi = 0$ ⇒ $4 \times 3 - 3 - 3 = 6$ numbers

Quantum: $|j_a, v\rangle$ ⇒ 5 numbers

Analogous to angular momentum: $\vec{L}$ → 3 # classically, but (L^2, L_z) → 2 #

→ Two characteristic features of quantum geometry:

1) Areas and volumes have discrete eigenvalues
2) Geometry is spread quantum mechanically at the Planck scale.

1.4 Physical consequences of the existence of the Planck scale

1.4.1 Discreteness: scaling is finite

→ Existence of L_{Pl} sets quantum GR aside from standard QFT for two reasons:

1) QG cannot be described by local QFT. Local QFT requires quantum fields to be described by observables at arbitrarily small regions in a continuous manifold.

2) QFT from the standard model are defined in terms of an ∞ renormalization group. $\exists\, L_{Pl} \Rightarrow$ not the case for QG. The Planck-scale cut-off is a genuine physical feature of the system formed by quantum spacetime.

→ The existence of a minimal length gives QG a universal character.

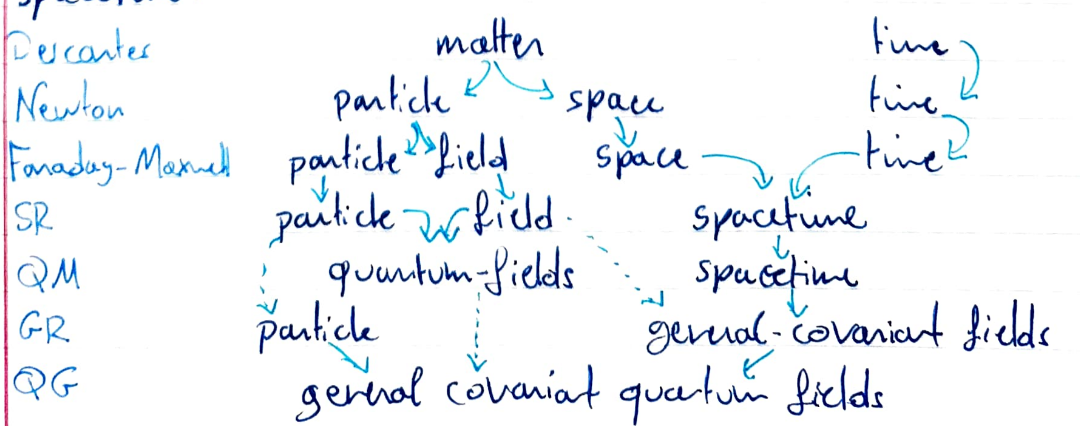
- → SR $\Leftarrow \exists$ maximal local physical velocity
- → QM $\Leftarrow$ compact region of phase space contains only a finite # of distinguishable quantum states $\Rightarrow$ there is a minimal amount of information in the state of the system.

?

→ In the part on, we'll use Planck unit: $c = \hbar = 8\pi G = 1$

1.4.2 Fuzziness: disappearance of classical space and time

→ In QFT, time evolution is captured by the unitary rep. of Poincaré group. The physic of QG is the physic of quantum fields that build up spacetime.



1.5 Graphs, loops, and quantum Faraday lines

→ A region of (curved) spacetime can be described by a set of interconnected grains of space ≡ graph, where the nodes are the grains, and the links relate adjacent grains.

→ The existence of fermions → the use of tetrads. This introduces a local Lorentz gauge invariance (freedom of choosing an independent Lorentz frame at each point of spacetime), that implies the existence of a connection field $\omega(x)$ (governs the //-transport).
The path ordered exponential of the connection (holonomy) $U_e = P\exp\{\int_e \omega\}$ along any curve e in the manifold, is a group element (U_e is in the Lie group, and ω in the Lie algebra).
↳ $U_e = \mathbb{1} + \omega(\dot{e}) + \mathcal{O}(|e|^2)$. If space is discrete, U_e remains well defined, but $\omega(x)$ is not.
→ We seek a quantisation using the group variables U_e: the loops.

1.6 The landscape

→ 2 main theories with UV finiteness ⇐ ∃ L_{pe}: LQG and string theory.

→ ST attempts to quantize gravity by splitting $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}(x)$, then look for a larger renormalizable or finite theory. This lead to modif. of GR (add R^2 in S), Kaluza-Klein-like theory, supergravity, ... All of this led to string theory, a finite quantum theory of all interactions, defined in 10-D.

→ LQG was born by the discovery of "loop" solutions of the Wheeler-deWitt equation (quantization of canonical GR in Ashtekar variables). This led to the spin network description of Q geometry. The spinfoam theory merged with the canonical LQG kinematics and evolved into the current covariant theory.

1.7 Complements

→ Recall some basic SU(2) rep. theory.

1.7.1 SU(2) representations and spinors

DEF $SU(2) \equiv \{ U \in \mathrm{Mat}(2\times 2, \mathbb{C}) \,/\, U^{-1} = U^{\dagger},\ \det U = 1 \}$

→ A SU(2) matrix is of the form $U = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$.
We write the matrix elements as $U^A{}_B$, with $A, B, C = 0, 1$.

→ SU(2) being an universal cover of SO(3), they have the same algebra.

→ SU(2) can be seen as a minimal building block of quantum spacetime.

⊙ Measure

→ $|a|^2 + |b|^2 = 1$ defines a unit sphere in $\mathbb{C}^2 \sim \mathbb{R}^4$. The topology of the group is that of the 3-sphere S^3. The euclidean metric of $\mathbb{R}^4$ restricted to S^3 defines an invariant measure on the group. Normalizing it as $\int_{SU(2)} dU = 1$, we get the Haar measure, invariant under $dU = d(UV) = d(VU)$, $\forall V \in SU(2)$

⊙ Spinors

→ The space of the fundamental rep. of SU(2) is the space of spinors: $z \equiv \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \in \mathbb{C}^2$. Notation: we use $z^A \equiv z$ and $v^i \equiv v$ (denote the full object, not its components).

⊙ Representations and spin

→ The vector space of the completely symmetric n-index spinors $z^{A_1 \dots A_n} = z^{(A_1 \dots A_n)}$, transforms under the action of SU(2) as

$$z^{A_1' \dots A_n'} = U^{A_1'}{}_{A_1} \cdots U^{A_n'}{}_{A_n} z^{A_1 \cdots A_n}$$

and therefore is a rep. of the group. This space is denoted $\mathcal{H}_j$, where $j = n/2$ and the rep it defines is called the spin-j rep.
Each $\mathcal{H}_j$ is an irrep, of dim $\dim(\mathcal{H}_j) = 2j + 1$

→ Consider $\epsilon^{AB} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ and $\epsilon_{AB} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$. They can be used to raise or lower index of spinors, using the $_A/^A$ rule:

$$\xi_A = \epsilon_{AB}\,\xi^B \quad \text{and} \quad \xi^A = \xi_B\,\epsilon^{BA}$$

↳ For example: $\xi^A{}_A = \epsilon_{AB}\,\xi^{AB} = -\epsilon_{AB}\,\xi^{BA} = -\xi_A{}^A$

→ $\epsilon_{AC}\,\epsilon^{CB} = -\delta_A{}^B$, $\epsilon_{BA}\,\epsilon^{AB} = -2$, $\epsilon_{AB}\,\epsilon^{AB} = 2$

→ $U^A{}_C\, U^B{}_D\, \epsilon^{CD} = \epsilon^{AB}$ invariant under $SU(2)$

→ $\det U = \epsilon^{BD}\, U^0{}_B\, U^1{}_D = \frac{1}{2}\,\epsilon_{AC}\,\epsilon^{BD}\, U^A{}_B\, U^C{}_D = 1$

→ $U^{-1} = -\epsilon U \epsilon \Leftrightarrow (U^{-1})^A{}_B = -\epsilon_{BD}\, U^D{}_C\, \epsilon^{CA}$

→ There are 2 invariant quadratic forms defined on $\mathbb{C}^2$:

1) scalar product $\langle \xi | y \rangle \equiv \sum_A \bar{\xi}^A y^A = \bar{\xi}^0 y^0 + \bar{\xi}^1 y^1$

2) antisym quadratic form $(\xi, y) \equiv \epsilon_{AB}\,\xi^A y^B = \xi^0 y^1 - \xi^1 y^0$

↳ Defining the antilinear map $J: \mathbb{C} \to \mathbb{C}$, $\xi^A \mapsto (J\xi)^A \equiv \begin{pmatrix}\bar{\xi}^1 \\ -\bar{\xi}^0\end{pmatrix}$, we get

$$\langle \xi, y \rangle = (J\xi, y)$$

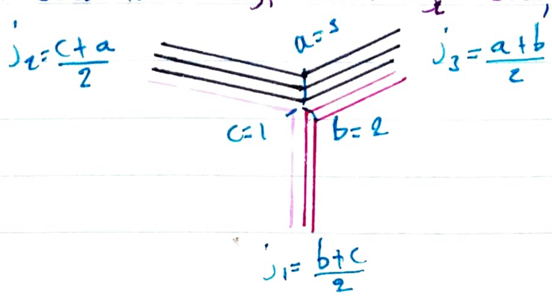
↳ $\langle , \rangle$ and $(,)$ are $SU(2)$ invariant, but $(,)$ is also $SL(2, \mathbb{C})$ invariant

→ Consider 2 fundamental ($j = 1/2$) rep. of $SU(2)$, ξ and y. Then $(\xi \otimes y)^{AB} = \xi^A y^B$ can be decomposed in

$$z^{AB} = \frac{1}{2}\, z^C{}_C\, \epsilon^{AB} + z^{(AB)}$$, so we see that $1/2 \otimes 1/2 = 0 \oplus 1$

→ In general, if we tensor 2 rep. of spin j_1 and j_2, we obtain space of spinors with $(2j_1 + 2j_2)$ indices, symmetric in the 1st $2j_1$ and the last $2j_2$ indices. By symmetrizing all indices, we obtain an invariant subspace transforming in the $(j_1 + j_2)$ rep.

Alternatively, we can contract k indices of the 1st group with k of the 2nd, using k time ϵ_{AB}, and then symmetrize the remaining $2(j_1 + j_2 - k)$ indices to obtain the spin j_3 rep. Then, we have the Clebsh-Gordan conditions $j_1 + j_2 + j_3 \in \mathbb{N}$ and $|j_1 - j_2| \leq j_3 \leq (j_1 + j_2)$, equivalent to the existence of $a, b, c \in \mathbb{N}$ such that $2j_1 = b + c$, $2j_2 = c + a$, $2j_3 = a + b$.



→ The spinor basis is not always the most convenient for SU(2) rep. If we diagonalize L_3 in $\mathcal{H}_j$, we get $|j,m\rangle$ where $m=-j,\dots,j$. The rep. matrices are then the Wigner matrices $D^j_{nm}(U)$

→ The 2 bilinear form of the fund. rep generalize to all irrep. Given $\xi, y \in \mathcal{H}_j$, we define:
→ the invariant contraction $\mathcal{H}_j \otimes \mathcal{H}_j \to \mathcal{H}_0$ given by

$$(\xi, y) = \xi^{A_1 \cdots A_{2j}} y^{B_1 \cdots B_{2j}} \epsilon_{A_1 B_1} \cdots \epsilon_{A_{2j} B_{2j}}$$

→ The scalar product

$$\langle \xi, y \rangle = \overline{\xi^{A_1 \cdots A_{2j}}} y^{B_1 \cdots B_{2j}} \delta_{A_1 B_1} \cdots \delta_{A_{2j} B_{2j}}$$

1.7.2 Pauli matrices:

→ The Pauli matrices are $\sigma_i{}^A{}_B = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$
Any SU(2) group element can be written as $U = e^{i\alpha \hat{n}\cdot\hat{\sigma}}$

→ Some properties: $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, and $\epsilon \sigma_i \epsilon = \sigma_i{}^T = \sigma_i{}^*$

→ $\{\tau_i\} = \{\frac{-i}{2}\sigma_i\}$ are the generator of SU(2) in its fundamental rep. They verify $[\tau_i, \tau_j] = \epsilon_{ij}{}^k \tau_k$ the algebra of su(2)

→ The Euler angle param. of SU(2) is $U(\psi, \theta, \phi) \equiv e^{\psi\tau_3} e^{\theta\tau_2} e^{\phi\tau_3}$ where $\psi \in [0, 2\pi[$, $\theta \in [0, \pi[$, $\phi = [0, 4\pi[$. The Haar measure reads $\int dU = \frac{1}{16\pi^2} \int_0^{2\pi} d\psi \int_0^{\pi} \sin\theta \, d\theta \int_0^{4\pi} d\phi$

1.7.3 Eigenvalues of the volume

→ Let's compute the volume for a quantum of space whose sides have a minimal area.
→ We had $V^2 = 2/9 \cdot \epsilon_{ijk} L_1^i L_2^j L_3^k$ with $[L_a^i, L_b^j] = i\delta_{ab} l_0^2 \epsilon^{ij}{}_k L_a^k$
→ Since the Casimir of the rep. of each L_a has to be minimal, $j_1 = j_2 = j_3 = j_4 = 1/2$. Therefore, $L_j^i \propto \frac{\sigma^i}{2}$. The proportion factor is given by the commutation relation:

$$L_j^i = \frac{l_0^2}{2} \sigma^i$$

→ The Hilbert space on which they act is therefore

$$\mathcal{H} = \mathcal{H}_{1/2}\otimes\mathcal{H}_{1/2}\otimes\mathcal{H}_{1/2}\otimes\mathcal{H}_{1/2} \ni \zeta^{ABCD}$$

The operator $\vec{t}_a$ acts on the a-th index. Therefore the volume acts as

$$(V^2\zeta)^{ABCD} = \frac{2}{9}\left(\frac{l_0^2}{2}\right)^3 \epsilon^{ijk}\,\sigma_i{}^{A}{}_{A'}\,\sigma_j{}^{B}{}_{B'}\,\sigma_k{}^{C}{}_{C'}\,\zeta^{A'B'C'D}$$

Using the closure relation, we define $K_{1/2\,1/2\,1/2\,1/2} = \mathrm{Inv}_{SU(2)}\{\mathcal{H}\}$.
We have to look for subspaces invariant under a common rotation for each $\mathcal{H}_{j_i}$: $\mathcal{H} = \frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = (0\oplus1)\otimes(0\oplus1) = 0\oplus1\oplus1\oplus(0\oplus1\oplus2)$
Since the trivial rep appears twice, $\dim(K_{\frac12\frac12\frac12\frac12}) = 2$: $\exists$ 2 independent invariant tensors with 4 indices. The only invariant objects available are ϵ^{AB} and $\sigma_i^{AB} \equiv (\sigma_i\epsilon)^{AB} = \sigma_i{}^{A}{}_{C}\,\epsilon^{CB}$. Then
$\langle K_{\frac12\frac12\frac12\frac12}\rangle = \{\zeta_1^{ABCD} \equiv \epsilon^{AB}\epsilon^{CD},\ \zeta_2^{ABCD} \equiv \sigma_i^{AB}\sigma_i^{CD}\}$. To find the eigenvalues of the volume, it suffices to diagonalize V^2_{nm}: $V^2\zeta_n = V^2_{nm}\zeta_m$

to do!

We find $V^2\zeta_1 = \frac{-il_0^6}{18}\zeta_2$ and $V^2\zeta_2 = \frac{il_0^6}{6}\zeta_1 \Rightarrow V^2 = \frac{-il_0^6}{18}\begin{pmatrix}0 & 1\\ -3 & 0\end{pmatrix}$
with eigenvalues:

$$V^2 = \pm\frac{l_0^{12}}{6\sqrt{3}}$$

The sign depends on the orientation of the volume. Inserting l_0, we find

$$V = \frac{1}{\sqrt{6\sqrt{3}}}\left(\frac{8\pi\gamma\hbar G}{c^3}\right)^{3/2}$$

↳ About 10^{100} quanta of volume of this size fit into a cm^3