# Sets

Def of sets in Naive set theory.

**Def**

A set is an <u>unordered</u> collection of <u>distinct</u> objects (elements)

we denote sets using capital letters.

$D = \{0, 1, 2, 3, 5, 4, 6, 7, 8, 9\}$ has 10 elements.

$bits = \{0, 1\}$

finite sets

$\mathbb{Z}$ = set of all integers ← infinite set.

$\mathbb{Z} = \{ \ldots\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\ldots \}$

$\mathbb{Q}$ = set of all rational numbers

$\mathbb{R}$ = set of all real numbers

$\mathbb{N} = \{1, 2, 3, 4, \ldots\ldots\ldots\}$

$T = \{$"Apple", $\square$, $2$, $\pi$, $1.234\}$

$C = \{0, 1, 0, 2\}$ ← not a set since, 0 is repeated

**Def** Two sets A and B are equal (denoted as $A=B$), if A and B contains exactly the same elements.

$A = \{0,1,2\}$ $B = \{1,2,3\}$

$A \neq B$

$C = \{3,2,1\}$

$B = C$

**Def** Set Membership

For a set S and an object x, the expression $x \in S$ is true when x is one of the objects contained in set S.

$x \in S$ ← "x is an element of set S"

$x \notin S$ ← "x is not an element of set S"

$0 \in \{0,1\}$ ✓

$2 \in \{0,1\}$ ✗

$2 \notin \{0,1\}$ ✓

$\pi \notin \mathbb{Z}$

**Def** Cardinality or size of the set $S$, is the number of distinct elements in $S$.

$$|S|$$

$bits = \{0, 1\}$

$|bits| = 2$

$$\Big|\underbrace{\{2, \text{"Apple"}, \{2,3\}\}}_{S}\Big| = 3$$

$3 \in S$ this is not true

$\{2,3\} \in S$

Question: can we have a set $S$, such that $|S| = 0$? yes, this set is called empty set

The empty set is represented by $\{\,\}$, $\emptyset$

$$\{\,\} = \emptyset$$

Question

$$|\emptyset| = 0$$

$$|\{\,\}| = 0$$

$$|\{\emptyset\}| = 1$$

$$|\{\{\,\}\}| = 1$$

Questions! If $A=B$, can I say $|A|=|B|$ ✓ true

Question If $|A|=|B|$, can we say $A=B$?

$A=\{0,-1\}$ $B=\{0,1\}$ ← False counter example

## Def Set builder notation

Set builder notation defines a set as follows:

The set $S = \{x : \text{a rule about } x\}$

Examples

$$EVEN = \{x : x \in \mathbb{Z} \wedge 2|x\}$$
$$= \{x : x \in \mathbb{Z} \wedge x = 2k \wedge k \in \mathbb{Z}\}$$
$$= \{x \in \mathbb{Z} : 2|x\}$$

$$\mathbb{Q} = \{x : x = p/q, p, q \in \mathbb{Z}, q \neq 0\}.$$

$$\mathbb{Z}^+ = \{y : y \text{ is an integer and } y > 0\}$$

$$\mathbb{Z}^* = \{a : a \in \mathbb{Z} \text{ and } a \geq 0\}$$

**Def** A set A is a subset of set B (denoted as $A \subseteq B$), if every element of A is also an element of B.

$A = \{0,1,2,3\}$ $B = \{1,3\}$

$A \not\subseteq B$ $B \subseteq A$

"A is not a subset of B" "B is a subset of A"

$EVENS \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

$\mathbb{R} \not\subseteq \mathbb{Q}$

Question: $\emptyset \subseteq \mathbb{R}$ True, because there are no elements in $\emptyset$. Therefore all elements in empty set appears in $\mathbb{R}$

$\emptyset \subseteq S$ for all sets $S$

$S \subseteq S$ for all sets.

## Def proper subsets.

A set $A$ is a proper subset of set $B$ (denoted as $A \subset B$), If $A \subseteq B$ and $A \neq B$.

$S \subset S$ (for any set) (This is wrong)

Ex: $S = \{0, 1, 2\}$ $T = \{0, 1\}$

$T \subset S$ ✓

$<$ , $\leq$

$1 \leq 2$, $2 \leq 2$

$2 < 2$

Question

1. If $A \subseteq B$, can you say $|A| \leq |B|$? True.

2. If $|A| \leq |B|$, can you say $A \subseteq B$?

$A = \{1\}$ $B = \{2, 3\}$

$|A| = 1$ $|B| = 2$ $|A| \leq |B|$ but $A \not\subseteq B$

Claim:

$$\underbrace{\{x : x \in \mathbb{Z} \text{ and } 18 \mid x\}}_{A} \subseteq \underbrace{\{x : x \in \mathbb{Z} \text{ and } 6 \mid x\}}_{B}$$

$A \subseteq B$

$A = \{x : x \in \mathbb{Z} \text{ and } 18 \mid x\}$

$B = \{y : y \in \mathbb{Z} \text{ and } 6 \mid y\}$

set A

| | | |
|---|---|---|
| $-18$ | $6 \mid -18$ | ✓ |
| $0$ | $6 \mid 0$ | ✓ |

Proof strategy: I am going to pick any arbitrary element in A and show that it appears in set B.

Proof:

Let us assume "a" is an arbitrary element in set A.

| Statements | Reasoning |
|---|---|
| $a \in A$ | by assumption |
| $a \in \mathbb{Z}$ and $a = 18 \cdot c$, $c \in \mathbb{Z}$ | by def of set A. |
| $a = 18 \cdot c = 6 \cdot (3c)$ | by rearranging. |
| $3 \cdot c \in \mathbb{Z}$ | product of ints is an int. |

$6 \mid a$ by def of divisibility,

$a = 6k,\ k = 3 \cdot c,\ k \in \mathbb{Z}$

$a \in B$ because $a \in \mathbb{Z}$ and $6 \mid a$

$A \subseteq B$ by the def of subset.

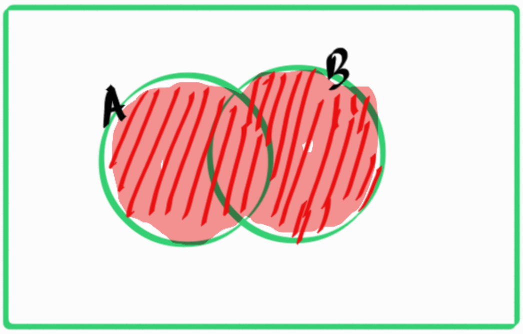
Basically, we showed that any element in set A has the property of an element in set B.

$\boxtimes$.

## Def

$A \cup B$, "A union B", is the set that contains all elements in set A and all elements in set B.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$




$A = \{2, 4, 6\}$

$B = \{2, 3, 4\}$

$A \cup B = \{2, 4, 6, 3\}$

$$\text{EVENS} \cup \text{ODDS} = \mathbb{Z}$$

$$\mathbb{R}^{\geq 0} \cup \mathbb{R}^{\leq 0} = \mathbb{R}$$

$$\mathbb{R}^{>0} \cup \mathbb{R}^{<0} = \mathbb{R}^{\neq 0}$$

$$A \cup \emptyset = A \cup \{\} = A$$

$$A \cup A = A$$

Question:

can I say $2 \cup \{1, 3\}$ ← This does not make sense since 2 is not a set