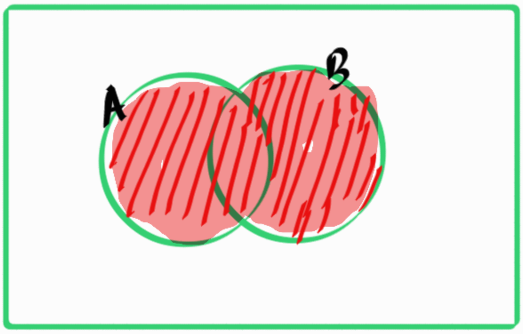
Def

$A \cup B$, "A union B", is the set that contains all elements in set A and all elements in set B.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$



$A = \{2, 4, 6\}$

$B = \{2, 3, 4\}$

$A \cup B = \{2, 4, 6, 3\}$

$\text{EVENS} \cup \text{ODDS} = \mathbb{Z}$

$\mathbb{R}^{\geq 0} \cup \mathbb{R}^{\leq 0} = \mathbb{R}$

$\mathbb{R}^{>0} \cup \mathbb{R}^{<0} = \mathbb{R}^{\neq 0}$

$A \cup \emptyset = A \cup \{\} = A$

$A \cup A = A$

Question:

can I say $2 \cup \{1, 3\}$ $\leftarrow$ This does not make sense since 2 is not a set

Recap

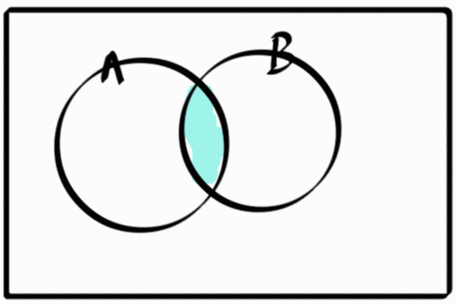
- Def set
- $\in$ ✓
- $\subseteq$
- union $\cup$ $\quad A \cup B$

Def Set Intersection $\cap$

A intersection B, "$A \cap B$", is the set that contains elements that appear in both A and B.

Ex: $A = \{2, 3, 4\}$ $\qquad B = \{1.5, 2, 3\}$

$$A \cap B = \{2, 3, 4\} \cap \{1.5, 2, 3\} = \{2, 3\}$$



$$\text{EVENS} \cap \text{ODDS} = \{\ \} = \phi$$

$$A \cap A = A$$

$$\mathbb{R}^{\geq 0} \cap \mathbb{R}^{\leq 0} = \{0\}$$

Def Disjoint Sets

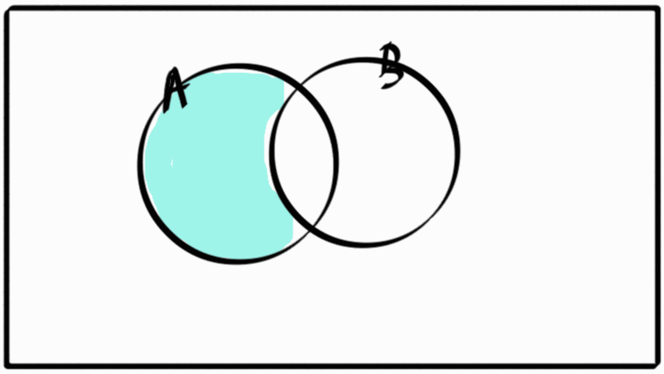
Sets A and B are called disjoint, if $A \cap B = \emptyset = \{\ \}$

are $\mathbb{R}^{\geq 0}$ and $\mathbb{R}^{\leq 0}$ disjoint? No

are EVENS and ODDS disjoint? Yes

Def

$A - B$ or $A \setminus B$, "A minus B" is the set of elements which are in set A but not in set B.



$A - B$

$$A - B = \{x : x \in A \wedge x \notin B\}$$

$$\underbrace{\{2,4,6\}}_{A} - \underbrace{\{2,3,4\}}_{B} = \{6\}$$

$$\{2,3,4\} - \{2,4,6\} = \{3\}$$

EVENS − ODDS = EVENS

$A - B \subseteq A$

$A - \phi = A$

$A - \{\phi\} = ?$ ← Answer depends on A.

$A - \{0\} \subseteq A$

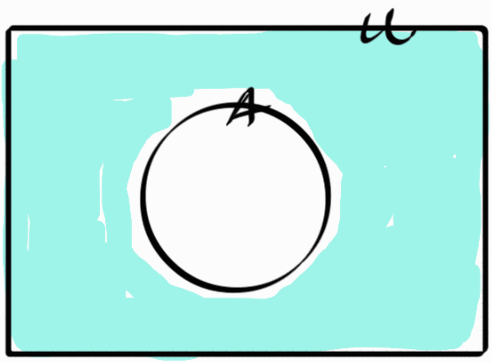
$A = \{\phi, \{2\}, \{3\}, 1, 2\}$

$A - \{\phi\} = \{\{2\}, \{3\}, 1, 2\}$

Def complement of set A. $(\bar{\cdot})$

The complement of set A, $\sim A$, $\bar{A}$, "A complement" is the set of all elements that are not in A.

$\bar{A} = \{x : x \notin A\}$



Ex:

$\mathcal{U} = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 4, 6\}$

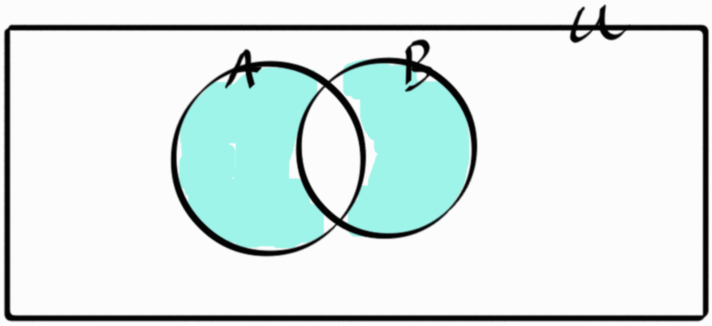
$\bar{A} = \{1, 3, 5\}$

$\overline{\text{EVENS}} = \text{ODDS}$

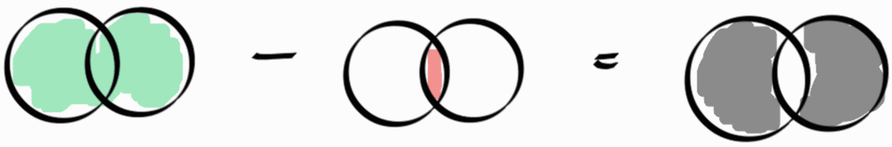
$\mathcal{U} = \mathbb{Z}$

Def Exclusive or $\oplus$

$A \oplus B$, "A exclusive or B" is the set of all elements that are either in A or B but not in both.



$A \cup B - A \cap B$



Claim:

$$\underbrace{\{x : x \in \mathbb{Z} \wedge 2 | x\}}_{A} \cap \overbrace{\{x : x \in \mathbb{Z} \wedge 9 | x\}}^{B} \subseteq \{\underbrace{a : a \in \mathbb{Z} \wedge 6 | a}_{C}\}$$

Any element that is divisible by 2 and 9, is divisible by 6.

$$A \cap B \subseteq C$$

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

$$A \cap B = \{x : x \in \mathbb{Z} \wedge 2 | x \wedge 9 | x\}$$

$$D = A \cap B$$

$$D \subseteq C$$

Proof strategy: Show any arbitrary element in D must also appear in C.

$$C = \{y : y \in \mathbb{Z} \wedge 6 | y\}$$

Proof

Statements	Reasoning
Let us take an arbitrary element d from D.	
$d \in D$	by assumption.
$d \in A \cap B$	by labeling def.

$d \in \mathbb{Z} \wedge 2 \mid d \wedge 9 \mid d$	by def. of $A \cap B$
$d = 18 \cdot k,\ k \in \mathbb{Z}$	because LCM of 9,2 is 18.
$d = 6 \cdot (3k)$	by factoring
$d = 6 \cdot c,\ c = 3k,\ c \in \mathbb{Z}$	by product of ints and labeling.
$6 \mid d$	by def of divisibility.
$d \in \mathbb{Z} \wedge 6 \mid d$	by previous statements.
$d \in C$	by def of set C.

we proved that any arbitrary element in $A \cap B$ exists in C.

Therefore, the claim is true $\boxtimes$.

Def Power set

Given a set S, the power set of S is the set of all subsets of S.

$$P(S) = \{A : A \subseteq S\}$$

$\emptyset \subseteq S$ ✓

$S \subseteq S$ ✓

Ex: $S = \{1, 2, 3\}$

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

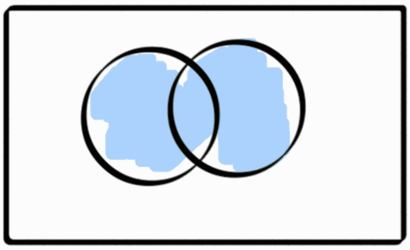
$|P(S)| = 8 = 2^3$ $\quad |S| = 3$

Fact: for a given set S,

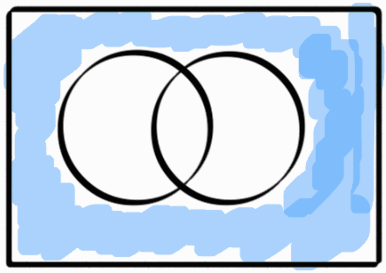
$$|P(S)| = 2^{|S|}$$

Def De morgan's law 1

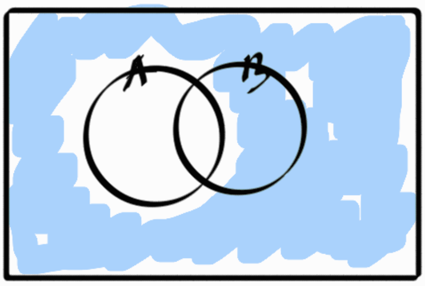
$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$



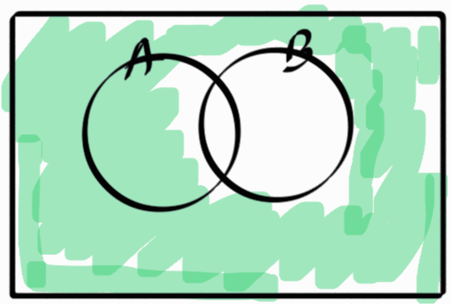
$A \cup B$



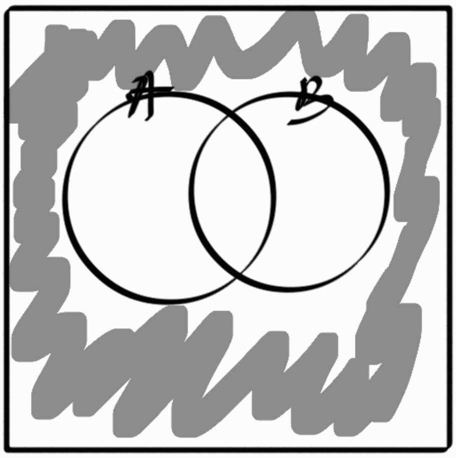
$\overline{A \cup B}$



$\bar{A}$



$\bar{B}$



$\bar{A} \cap \bar{B}$

Claim: $\overline{A \cup B} = \bar{A} \cap \bar{B}$

$X = Y$

$X \subseteq Y$ and $Y \subseteq X$

I am gonna show 2 things

$\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$ —— ①

and

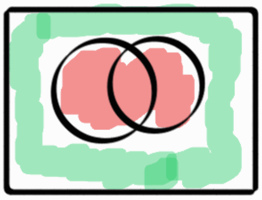
$\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$ —— ②

① $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

Assume x is an arbitrary element in $\overline{A \cup B}$

W.T.S $x \in \overline{A} \cap \overline{B}$

Statements	Reasoning
$x \in \overline{A \cup B}$	by assumption
$x \notin A \cup B$	by def of complement.



Statements	Reasoning
$x \notin A \wedge x \notin B$	because $x \notin A \cup B$
$x \in \overline{A} \wedge x \in \overline{B}$	by def of complement
$x \in \overline{A} \cap \overline{B}$	by def of $\cap$

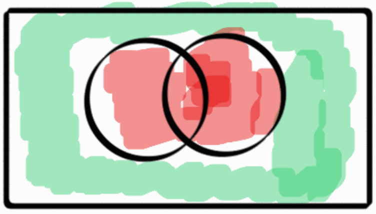
Proved case 1

Case 02: $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Assume y is an arbitrary element in $\overline{A} \cap \overline{B}$.

W.T.S $y \in \overline{A \cup B}$

Statements	Reasoning
$y \in \overline{A} \cap \overline{B}$	by assumption
$y \in \overline{A} \wedge y \in \overline{B}$	by def of $\cap$
$y \notin A \wedge y \notin B$	by def of complement.



$y \notin A \cup B$	because $y \notin A$ and $y \notin B$
$y \in \overline{A \cup B}$	by def. complement.

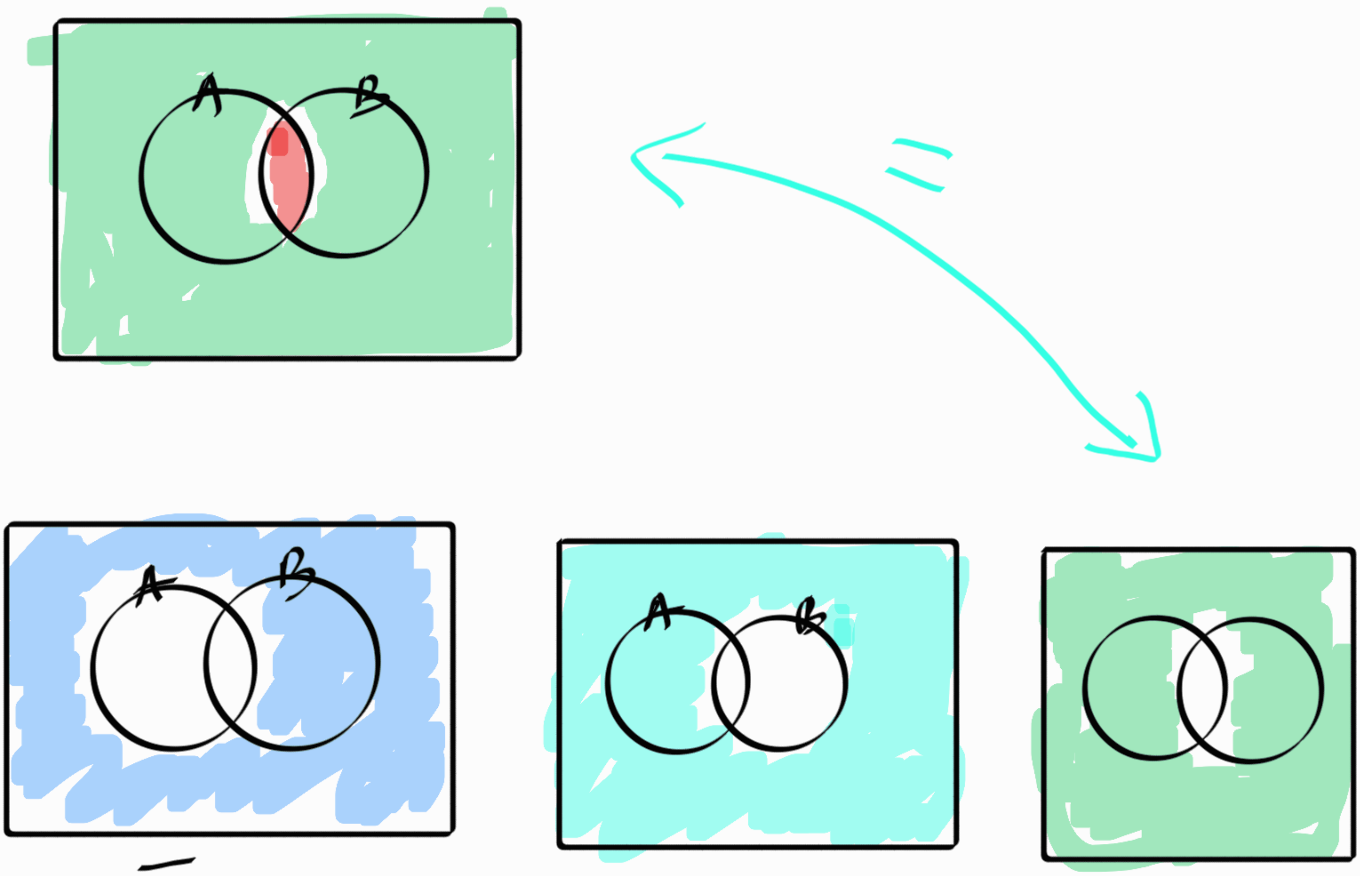
Proved the case 2.

$\therefore$, $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$, then

$\underline{\overline{A \cup B} = \overline{A} \cap \overline{B}}$ $\square$

De morgan's law 2

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$



$\overline{A}$ $\overline{B}$ $\overline{A} \cup \overline{B}$

claim: If $P(A) \subseteq P(B)$, then
$A \subseteq B$

$A = \{1\}$ $B = \{2\}$

$P(A) = \{\{\}, \{1\}\}$ $P(B) = \{\{\}, \{2\}\}$

is $P(A) \subseteq P(B)$ No!

$A = \{1\}$ $B = \{1, 2\}$

$P(A) = \{\{\}, \{1\}\}$ $P(B) = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$

$P(A) \subseteq P(B)$? yes

So now is $A \subseteq B$? Yes!

Proof: Given $P(A) \subseteq P(B)$, assume x is an arbitrary element of A.

W.T.S. $x \in B$

Statements	Reasoning
$x \in A$	by assumption
$\{x\} \subseteq A$	because $x \in A$
$\{x\} \in P(A)$	by def of power set.

$\{x\} \in P(B)$

we are given that $P(A) \subseteq P(B)$

$x \in B$

by def of powerset and subset

we showed that any arbitrary element $x \in A$ appears in B, when $P(A) \subseteq P(B)$

$\therefore A \subseteq B$ // $\blacksquare$