Disproof by counterexample.

original claim: If x, y are rational, then xy is rational.

claim: If xy is rational, then x, y are rational.

$x = \frac{1}{\sqrt{2}}$ $y = \frac{1}{\sqrt{2}}$

$xy = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$

xy is rational, but x, y are not rational.

Therefore, this claim does not hold.

$x = 2$ $y = 6$ $xy = 12$

Def

A disproof by counter example constructs an example which the claim is false and explains why it is false.

Def

Let n, m be integers, then n is divisible by m, If there exists an integer k, such that $n = m \cdot k$

Q: is 10 divisibly by 2?

$10 = 2 \cdot k = 2 \cdot 5$ (5 is an integer)

Q: is 11 divisibly by 3?

$11 = 3 \cdot \left(\frac{11}{3}\right) \leftarrow$ is not an integer

Q: is 0 divisibly by 4?

$0 = 4 \cdot 0 \leftarrow$ 0 is an integer.

Def

we say, if n is divisibly by m, we say m divides n

$$m \mid n$$

$2 \mid 10$ $\qquad\qquad$ $3 \nmid 10$

2 divides 10 $\qquad\qquad$ 3 does not divide 10

propositions

claim: Let n be an integer, then $n \cdot (n+1)^2$ is even.

step 01:

What is even? If n is divisible by 2, then n is even.

Then I can write $n = 2 \cdot k$, where k is an integer.

Step 02: Let's do some examples.

n	$n\cdot(n+1)^2$	is $n\cdot(n+1)^2$ even?
0	$0\cdot 1^2=0$	✓
1	$1\cdot 2^2=4$	✓
2	$2\cdot 3^2=18$	✓
-2	$-2\cdot(1)^2=-2$	✓
3	$3\cdot 4^2=48$	✓

Observe that n can be any integer. Any integer has to be either odd integer or an even integer.

Therefore, If we show that when n is odd $n\cdot(n+1)^2$ is even and when n is even $n\cdot(n+1)^2$ is even, then we can claim that our proposition is true.

Step 03:

Case 01: n is even

statements	reasoning
$n = 2 \cdot c$, c is an integer	by def. of even
$n \cdot (n+1)^2 = 2 \cdot c\,(n+1)^2$	by substitution
$n \cdot (n+1)^2 = 2 \cdot (c \cdot (n+1)^2)$	by rearranging
$n \cdot (n+1)^2 = 2 \cdot K$, where $K = (c \cdot (n+1)^2)$	" "
K is an integer	product of integers is an integer
$n \cdot (n+1)^2 = 2 \cdot K$	
$n \cdot (n+1)^2$ is even	by def. of even.

case 2: n is odd

$n+1$ is even — because n is odd

$n+1 = 2 \cdot b$, b is an integer — by def. of even

$n \cdot (n+1)^2 = n \cdot (2b)^2 = n \cdot 4 \cdot b^2$ — by substitution

$n \cdot (n+1)^2 = 4 \cdot b^2 \cdot n = 2 \cdot (2 \cdot b^2 \cdot n)$ — by rearranging

$2 \cdot b^2 \cdot n$ is an integer — product of ints. is an int.

$n \cdot (n+1)^2$ is even — by def of even.

Since n is either even or odd and in either case $n(n+1)^2$ is even, therefore our claim is true.

Def

To give a proof by cases of a proposition φ, we identify set of cases, and then prove two different types of facts.

1. In any case φ holds
2. One of the cases must hold.

claim: Let x be a real value.
Then $-|x| \leq x \leq |x|$

step 01:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

$x = 3$ $\quad$ $x = -5$ $\quad$ $x = 0$

$|x| = 3$ $\quad$ $|-5| = 5$ $\quad$ $|0| = 0$

Step 02: $-|x| \leq x \leq |x|$

x	$\lvert x \rvert$	$-\lvert x \rvert$	is $-\lvert x \rvert \leq x \leq \lvert x \rvert$
2	2	-2	✓
0	0	0	✓
-3	3	-3	✓

Case 01: $x \geq 0$

$$|x| = \begin{cases} x & , \text{if } x \geq 0 \\ -x & x < 0 \end{cases}$$

$|x| = x$ by def of $|x|$

$-|x| = -x$ by algebra

$-|x| \leq x \leq |x|$ by substitution

case 02!

$x < 0$

$$|x| = \begin{cases} x & , x \geq 0 \\ \underline{-x} & \underline{x < 0} \end{cases}$$

$|x| = -x$ by def of $|x|$

$-|x| = x$ by algebra

$-|x| \leq x \leq |x|$ by substitution

I have considered all possible cases for x, and we have proved that for each case the claim holds.

Therefore, $-|x| \leq x \leq |x|$

∎

Claim! Let x, y, z be real numbers, Then we say that

$$|x-y| \leq |x-z| + |y-z|$$

step 01:

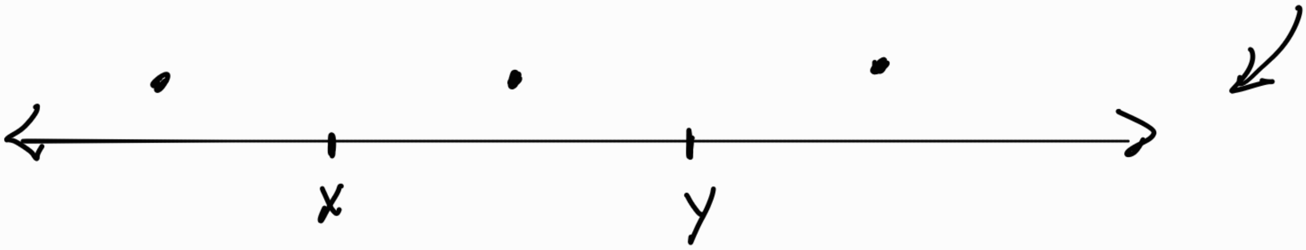
Step 02:

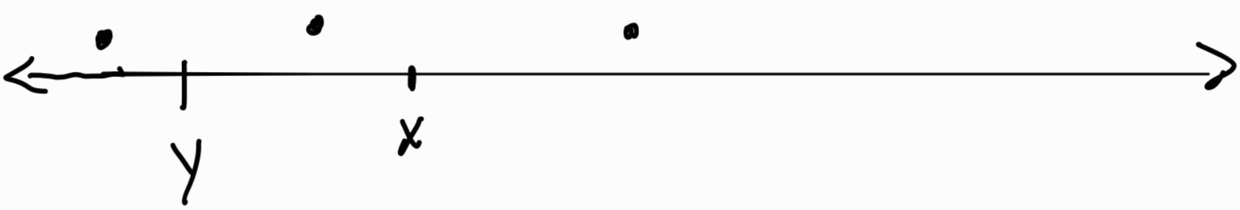
$x = 4 \qquad y = 5 \qquad z = 2$

$1 \leq 2 + 3$ ✓

$x = -2 \qquad y = -3 \qquad z = 1$

$1 \leq 3 + 4$

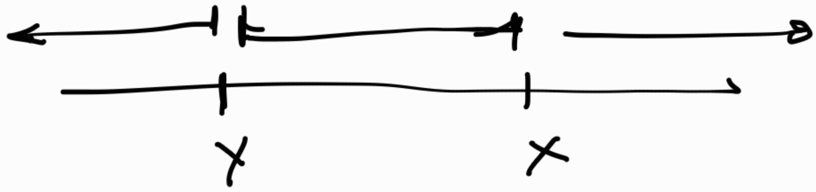




$x \le y$

1. $z \le x$
2. $x < z \le y$
3. $y < z$

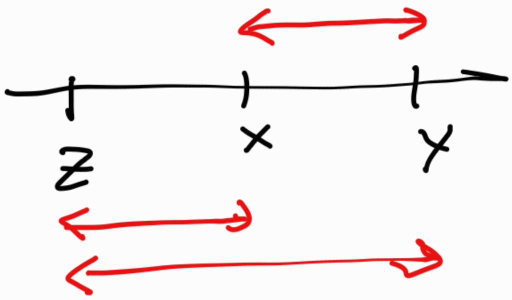
$x > y$



4. $z \le y$
5. $y < z \le x$
6. $x < z$

Proof:

case 01: $x \le y$, $z \le x$



$$|x-y| \le |x-z| + |y-z|$$

$$|y-z| = |y-z|$$

$$|y-z| + |x-z| \ge |y-z|$$

$$|y-z| + |x-z| \ge y-z$$

$$|y-z|+|x-z| \geqslant y-z \geqslant y-x$$

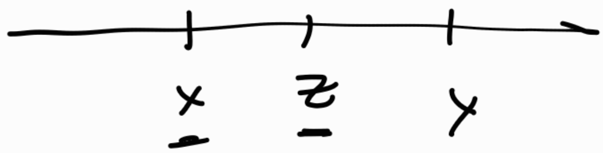
$$|y-z|+|x-z| \geqslant y-x$$

$$|y-z|+|x-z| \geqslant |y-x|$$

$$|y-z|+|x-z| \geqslant |-(x-y)|$$

$$|y-z|+|x-z| \geqslant |(x-y)|$$

Case 2! $x \leq y$, $x < z \leq y$



$$|x-y| \leq |x-z|+|y-z|$$

$$|x-z|+|y-z| = \cancel{z}-x+y-\cancel{z} \quad \text{(by def of } |\cdot| \text{)}$$

$$= y-x$$

$$= -(x-y)$$

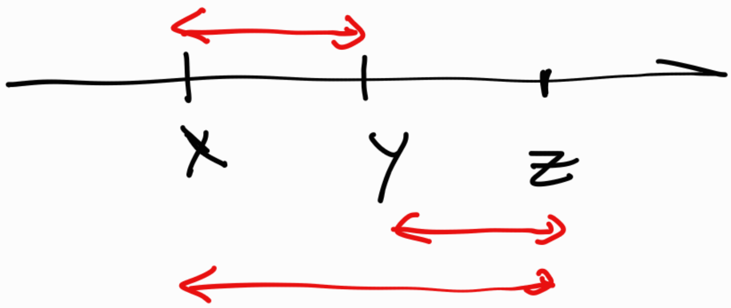
$$|x-z|+|y-z| = |x-y|$$

$$|x-z|+|y-z| \geqslant |x-y|$$

claim is true for case 2.

Case 3! $x \leq y$, $z > y$

$|x-z|+|y-z| \geq |x-y|$



$$|x-z| = |x-z|$$

$$|x-z|+|y-z| \geq \underbrace{|x-z|}_{z-x}$$

$$|x-z|+|y-z| \geq z-x$$

$$z > y$$

$$z-x > y-x$$

$$|x-z|+|y-z| \geq z-x > y-x$$

$$|x-z|+|y-z| \geq y-x$$

$$\text{"} \qquad \text{"} \quad \geq -(x-y)$$

$$|x-z|+|y-z| \geq |x-y|$$

case is proven.

Case 4, 5, 6

$$x > y$$

$$|x-y| \leq |x-z| + |y-z|$$

$x = t$ $\quad y = r \quad t > r$

$$|t-r| \leq |t-z| + |r-z|$$

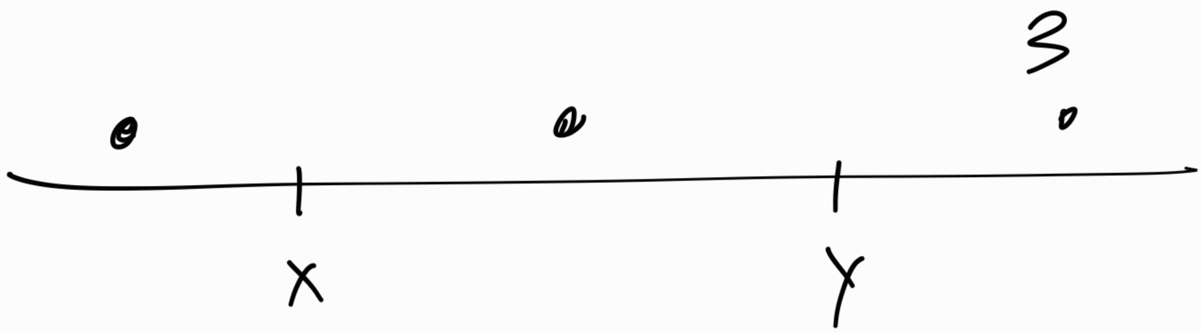
$$|t-r| \leq |r-z| + |t-z|$$

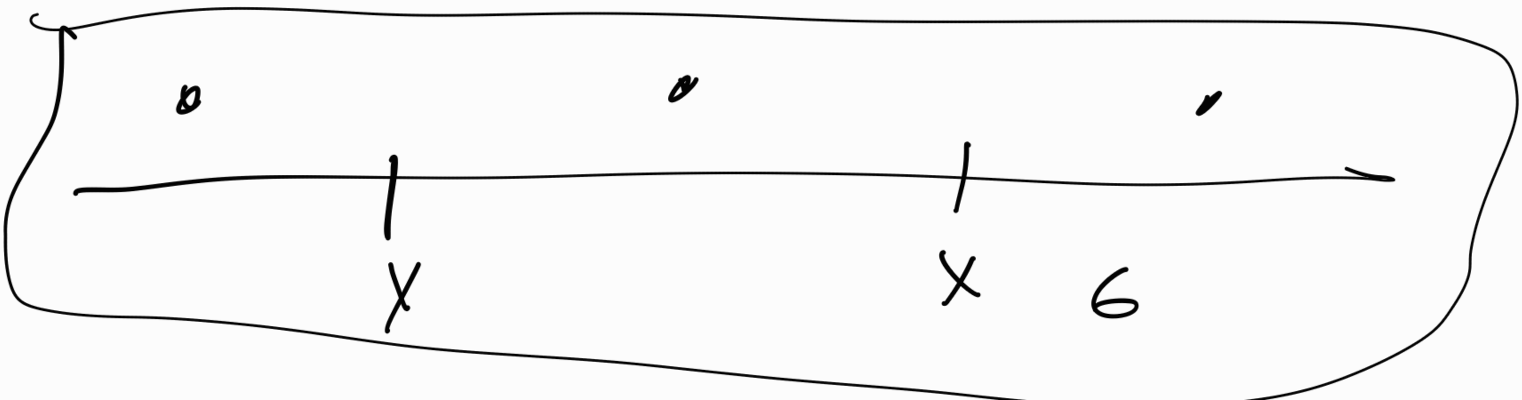
$t = y$ $\quad r = x \quad y > x$

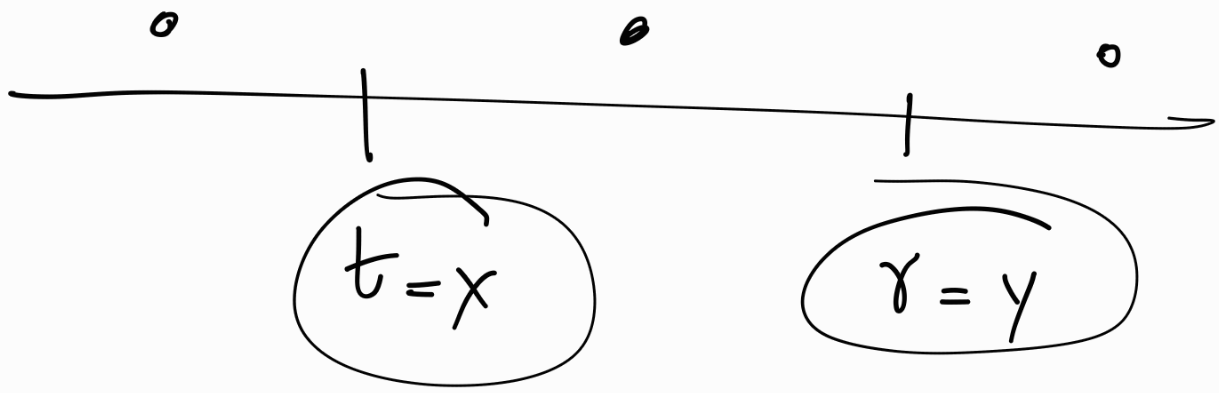
$$|y-x| \leq |x-z| + |y-z|$$

we proved $x \leq y$ case earlier.

$$x \leq y$$







case 4, 5, 6 are identical to cases 1, 2, 3. Therefore, the claim holds for cases 1, 2, 3, 4, 5, 6.

Hence, the claim is true.

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