09/27/2024

Recap

- Def of functions
- A function maps all the elements in the domain to ~~want~~ a unique value in codomain.
- $f : A \longrightarrow B$
- 1. $\forall a \in A$, $f(a)$ is defined
  2. $\forall a \in A$, $f(a)$ is unique.
  3. $\forall a \in A$, $f(a) \in B$

Ex: $S : \mathbb{Z} \longrightarrow \mathbb{Z}$, defined as $S(x) = x+1$

Prove S is a function.

Proof: We are trying to show that S has properties of a function.

1. ~~W~~ Want to show (WTS) $\forall a \in \mathbb{Z}$, ~~$f(a)$~~ $S(a)$ is defined.

   $S(a) = a+1$ by the def of S.

2. WTS, ~~$\forall a \in A$~~ $\forall a \in \mathbb{Z}$, $S(a)$ is unique.

   $\equiv$ If $S(a) = b_1$ and $S(a) = b_2$, then $b_1 = b_2$

Assume $S(a) = b_1$ and $S(a) = b_2$

$S(a) = b_1 = a+1$ by def of S.

$S(a) = b_2 = a+1$ by def of S.

$b_1 = b_2$ by substitution.

therefore, $\forall a \in A$, $f(a)$ is unique.

3. $\forall a \in \mathbb{Z}$, $S(a) \in \mathbb{Z}$ (WTS)

$S(a) = a+1$ by def of S.

$a+1 \in \mathbb{Z}$ $a \in \mathbb{Z}$, $1 \in \mathbb{Z}$, sum of ints is an int.

$S(a) \in \mathbb{Z}$

property 3 is satisfied.

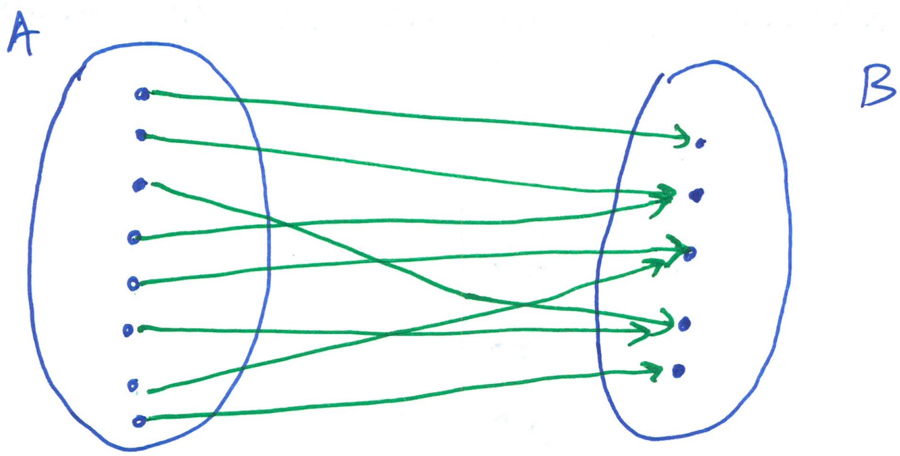
S satisfy properties of a function, therefore S is a function.

Def onto/surjective functions.

A function $f: A \longrightarrow B$ is called onto (surjective) if $\forall b \in B \ \exists a \in A : f(a) = b$

$\equiv \forall b \in B$, something in A maps to it.

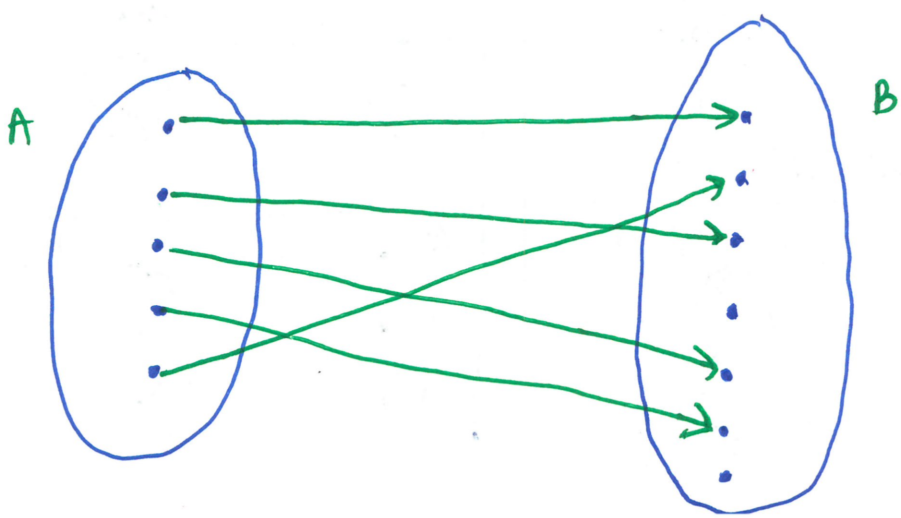
$\boxed{\text{Range} = \text{codomain}}$



Def one-to-one/injective functions.

A function $f: A \longrightarrow B$ is called one-to-one/injective if $\forall a_1, a_2 \in A$, $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

$\equiv \forall b \in B$, at most 1 element in A maps to it.

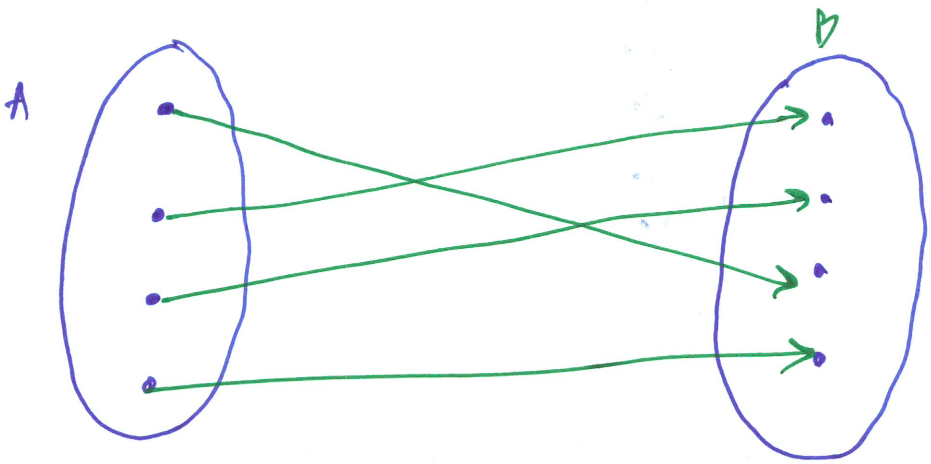


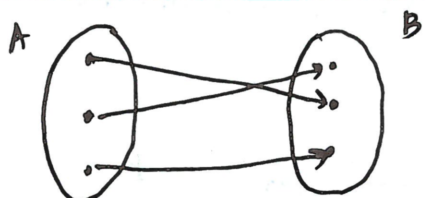
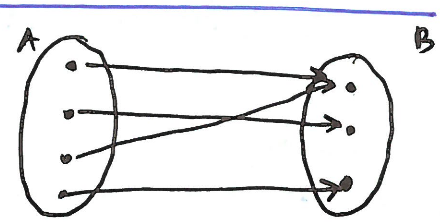
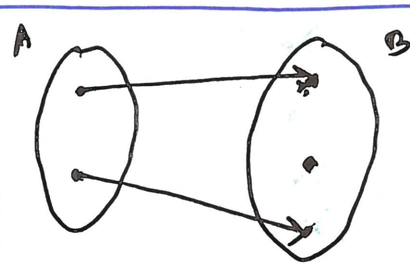
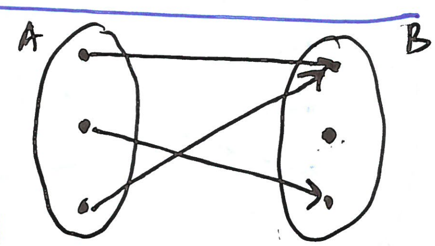
~~I~~ Injective and surjective are <u>NOT</u> the exact opposite of each other.

<u>Def</u> Bijection function.

A function $f: A \longrightarrow B$ is called a bijection if f is onto and one-to-one.

$\equiv \forall b \in B$, exactly 1 element in A maps to it.



	1:1 (one-to-one)	not 1:1
onto	 (Bijection)	
not onto		

How can we prove a function f is onto or 1:1 ?

onto

WTS $\left[\forall b \in B : \left[\exists a \in A : f(a) = b\right]\right]$

~~$\equiv \forall b \in B$~~

$\equiv$ If $b \in B$, then $\exists a \in A : f(a) = b$

Step 1: Assume $b \in B$

Step 2: construct "a" s.t $f(a) = b$

Ex: $S : \mathbb{Z} \to \mathbb{Z}$, $S(x) = x + 1$

Let's prove that S is an onto function

WTS If $b \in \mathbb{Z}$, then $\exists a \in \mathbb{Z} : S(a) = b$

Proof:

Assume $b \in \mathbb{Z}$ by assumption

Then let's construct $a \in \mathbb{Z}$ s.t $S(a) = b$

consider $a = b - 1$, $a \in \mathbb{Z}$ int-int is an int.

$S(a) = (b-1) + 1 = b$ by def of S.

we found an $a \in \mathbb{Z}$ that maps to b.

$\therefore$ S is an onto function. ▨

Therefore

How can we prove a function is one-to-one.

WTS $\forall a_1, a_2 \in A, : a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$.

$$a_1 \longrightarrow f(a_1)$$

$$a_2 \longrightarrow f(a_2)$$

If $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$ $\qquad p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

we consider the contrapositive statement and prove the claim is true.

$$\equiv \forall a_1, a_2 \in A : \left[f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \right]$$

ex: $S : \mathbb{Z} \longrightarrow \mathbb{Z}$, $\quad S(x) = x + 1$

Prove S is 1:1.

Proof:

Proof by contrapositive

WTS, $\forall a_1, a_2 \in \mathbb{Z} : S(a_1) = S(a_2) \Rightarrow a_1 = a_2$

Assume $S(a_1) = S(a_2)$

$a_1 + 1 = a_2 + 1$ $\qquad$ by def. of S and by substitution.

$a_1 = a_2$ $\qquad$ by algebra.

$\therefore S$ is 1:1 function $\boxtimes$.