11/20/2024

Expected values.

How many times do we have to flip a coin to get 100 heads?

<u>Def</u> A random variable X assigns a numerical value to every outcome of a sample space.

- X is actually a function

$$X : S \longrightarrow \mathbb{R}$$

ex! Suppose we flip a coin 3 times.

$$S = \{H, T\}^3 = \{\langle H,H,H\rangle, \langle H,H,T\rangle, \langle H,T,H\rangle, \langle H,T,T\rangle, \langle T,H,H\rangle, \langle T,H,T\rangle, \langle T,T,H\rangle, \langle T,T,T\rangle\}$$

$$\Pr[s] = \frac{1}{8} \quad \forall s \in S$$

Let X be the # of heads of an outcome.

Let Y be the # of consecutive tails from the beginning.

$X(\langle H,H,H\rangle) = 3 \qquad X(\langle H,H,T\rangle) = 2$

$Y(\langle H,H,H\rangle) = 0 \qquad Y(\langle T,H,T\rangle) = 1$

<u>Def</u> Expectation of a random variable X, denoted as E[X], is the average value of the random variable X.

$$E[X] = \sum_{s \in S} X(s) \cdot Pr[s] = \sum_{y} y \cdot Pr[X=y]$$

ex! counting heads in a 3 coin flips

$$S = \{H, T\}^3$$

X = # of heads in the outcome.

$$E[X] = \sum_{s \in S} X(s) \cdot Pr[s]$$

$$= X(\langle H,H,H \rangle) \cdot Pr[\langle H,H,H \rangle] + X(\langle H,H,T \rangle) \cdot Pr[\langle H,H,T \rangle]$$

$$+$$

$$\vdots$$

$$+$$

$$X(\langle T,T,T \rangle) \cdot Pr[\langle T,T,T \rangle]$$

$$= 3 \cdot \tfrac{1}{8} + 2 \cdot \tfrac{1}{8} + 2 \cdot \tfrac{1}{8} + 1 \cdot \tfrac{1}{8} + 2 \cdot \tfrac{1}{8} + 1 \cdot \tfrac{1}{8} + 1 \cdot \tfrac{1}{8} + 0 \cdot \tfrac{1}{8}$$

$$= \tfrac{12}{8} = 1.5$$

$$E[X] = \sum_{y} y \cdot Pr[X=y]$$

$$= 0\cdot\left(\tfrac{1}{8}\right) + 1\left(\tfrac{3}{8}\right) + 2\cdot\left(\tfrac{3}{8}\right) + 3\cdot\left(\tfrac{1}{8}\right)$$

$$= \tfrac{12}{8} = 1.5$$

N choose k examples.

Ex: How many different ways you can create 4-card hands from a 52-card deck? (without replacement)

meaning once we choose a card we do not put it back

I cannot pick A clubs, A clubs, A clubs, A clubs

$$\binom{52}{4} = \frac{52!}{4!\,(52-4)!}$$

"52 choose 4"

why is this not $52 \cdot 51 \cdot 50 \cdot 49$

$\boxed{52}$ $\boxed{51}$ $\boxed{50}$ $\boxed{49}$.

A club A hearts A spade A diamond

A hearts A clubs A spades A diamonds.

$$52 \cdot 51 \cdot 50 \cdot 49 = \frac{52!}{(52-4)!}$$

This is the # of ways you can pick 4 cards with unique permutation

Popup test 10

How many different 8-bit binary strings are there with exactly 2 ones.

11111110 00011111

10010000

__ __ __ __ __ __ __ __

1 1

1 1 1

1

$\binom{8}{2}$ //

ex: what is the expected # of Aces in a 13-card hand?

D = set of cards in 52-card deck

$|S| = \binom{52}{13}$

X = The # of Aces in the outcome

$X : S \longrightarrow \mathbb{R}$

Range of this function is $\{0, 1, 2, 3, 4\}$

$$E[X] = \sum_{s \in S} X(s) \cdot Pr[s] = \sum_{y} y \cdot Pr[X=y]$$

$$= 0 \cdot Pr[X=0] + 1 \cdot Pr[X=1] + 2 \cdot Pr[X=2] + 3 \cdot Pr[X=3] + 4 \cdot Pr[X=4]$$

$$Pr[X=0] = \frac{\#\ \text{13-card hands without any Ace}}{\#\ \text{of 13-card hands}}.$$

Pick
- 0 Ace

Then pick
- 13 cards

$$= \frac{\binom{48}{13}}{\binom{52}{13}}$$

remove 4 Aces from the card pack then pick 13-cards.

All the possible ways you can pick 13-cards

$$Pr[X=1] = \frac{\#\ \text{13-card hands with 1 Ace}}{\#\ \text{of 13-card hands}}.$$

- 1 Ace*
- 12 cards*.

$$*\ \binom{4}{1}\cdot\binom{1}{1}\cdot\binom{48}{12}$$

$$Pr[X=1] = \frac{\binom{4}{1}\binom{1}{1}\cdot\binom{48}{12}}{\binom{52}{13}}$$

$$\Pr[X=2] = \frac{\#\text{ of 13-card hands with 2 Aces}}{\#\text{ of 13-card hands}}.$$

- pick 2 Aces *
- Then pick 11 cards from 48-cards (without aces)

*

$$\binom{4}{2}\cdot\binom{1}{1}$$

ways you could choose 2 suits ~~from~~ out of 4

ways you can choose an Ace from Aces to kings

you can think of this as ~~picking~~ # of ways to pick 2 Aces from 4 Aces $= \binom{4}{2}$

or

$\binom{48}{11}$ ← # of ways to pick 11 cards from 48 cards (card pack without Aces)

$$\Pr[X=2] = \frac{\binom{4}{2}\cdot\binom{1}{1}\cdot\binom{48}{11}}{\binom{52}{13}} \quad //$$

using the same idea:

$$\Pr[X=3] = \frac{\binom{4}{3}\cdot\binom{1}{1}\cdot\binom{48}{10}}{\binom{52}{13}}$$

$$\Pr[X=4] = \frac{\binom{4}{4}\cdot\binom{1}{1}\cdot\binom{48}{9}}{\binom{52}{13}}$$

$$E[X] = 0\times\frac{\binom{48}{13}}{\binom{52}{13}} + 1\times\frac{\binom{4}{1}\binom{1}{1}\binom{48}{12}}{\binom{52}{13}} + 2\times\frac{\binom{4}{2}\binom{1}{1}\binom{48}{11}}{\binom{52}{13}}$$

$$+ 3\times\frac{\binom{4}{3}\binom{1}{1}\binom{48}{10}}{\binom{58}{13}} + 4\times\frac{\binom{4}{4}\binom{1}{1}\binom{48}{9}}{\binom{58}{13}}$$

//

Ex: What is the probability of drawing a full house?

→ In poker full house is a hand of 5-cards that has

- 3 cards of the same rank and
- 2 cards of the same rank.

→ Basically, 5-card hand with a three of a kind and a pair.

Poker language.

$$\frac{\#\text{ of ways to get a full house}}{\#\text{ of ways to pick 5-cards}} = \frac{\binom{13}{1}\binom{4}{3}\cdot\binom{12}{1}\cdot\binom{4}{2}}{\binom{52}{5}}$$