

09/16/2024

Recap:

\neg, \sim

- Propositions
- Logical connectors
 $\wedge, \vee, \sim, \oplus, \Rightarrow, \Leftrightarrow$
- Truth tables
- Tautology

Def. Inverse, converse and contrapositive statements of an implication.

Given two propositions p, q and the implication $p \Rightarrow q$

Converse of the implication: $q \Rightarrow p$

Inverse of the implication: $\neg p \Rightarrow \neg q$

Contrapositive statement of the implication: $\neg q \Rightarrow \neg p$

p	q	$p \Rightarrow q$	$\neg p$	$\neg q$	$q \Rightarrow p$	$\neg p \Rightarrow \neg q$	$\neg q \Rightarrow \neg p$
0	0	1	1	1	1	1	1
0	1	1	1	0	0	0	1
1	0	0	0	1	1	1	0
1	1	1	0	0	1	1	1

\equiv (they are equivalent)

If $\underbrace{\text{it rains in Bozeman}}_P$, then $\underbrace{\text{Bozeman is wet.}}_Q$

$$P \Rightarrow Q$$

$$\neg Q \Rightarrow \neg P$$

If Bozeman is not wet, then It does not rain in Bozeman.

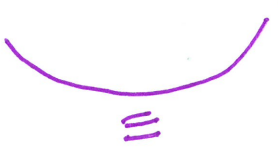
Def. Logical Equivalence

Two propositions φ and ψ are logically equivalent, written as $\varphi \equiv \psi$, if they have exactly identical truth tables.

(In other words, their truth values are same under every truth assignment)

Ex: consider $\neg(P \wedge Q)$, ~~and~~ $(\neg P \vee \neg Q)$

P	Q	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$(\neg P \vee \neg Q)$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0



$$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$$

$$\begin{aligned} \overline{A \cup B} &= \bar{A} \cap \bar{B} \\ \overline{A \cap B} &= \bar{A} \cup \bar{B} \end{aligned} \quad \left. \vphantom{\begin{aligned} \overline{A \cup B} &= \bar{A} \cap \bar{B} \\ \overline{A \cap B} &= \bar{A} \cup \bar{B} \end{aligned}} \right\} \text{De Morgan's law}$$

$$\neg(P \vee Q) \equiv (\neg P \wedge \neg Q) \quad \neg(P \wedge Q) \equiv (\neg P \vee \neg Q) \quad \text{De Morgan Law}$$

Proof by contradiction

P

$\neg P$

The idea: Suppose we want to prove that proposition φ is true. Instead of proving φ is true; we show that φ can not be false.

Def. Proof by contradiction

To prove φ using proof by contradiction; we assume the negation of φ and derive a contradiction; that is we assume $\neg \varphi$ and prove $\neg \varphi$ is false.

Theorem: Let $n \in \mathbb{Z}$, If $\underbrace{n^2 \text{ is even}}_P$, then $\underbrace{n \text{ is even}}_Q$

$$P \Rightarrow Q \equiv \neg P \vee Q$$

$$\neg(P \Rightarrow Q)$$

P	Q	$P \Rightarrow Q$	$\neg(P \Rightarrow Q)$	$P \wedge \neg Q$
0	0	1	0	0
0	1	1	0	0
1	0	0	1	1
1	1	1	0	0

$$\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$$

n^2 is even and n is odd

Trying to use proof by contradiction.

Assume the negation of ~~the~~ the initial claim is true,

$n \in \mathbb{Z}$ and n^2 is even and n is odd

$$n = 2k+1, k \in \mathbb{Z}$$

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$

$$n^2 = 2(\underbrace{2k^2 + 2k}_C) + 1$$

$$n^2 = 2 \cdot C + 1, C \in \mathbb{Z}$$

n^2 is odd

by def of odd

by algebra

by factoring

sum of product of ints is a int.

by def of odd.

This is ~~a~~ a contradiction, we assumed that n^2 is even and n is odd, but we got a contradiction that n^2 is odd. Therefore, our initial assumption is incorrect. Hence, the original claim is true. \square .

Ex! claim: $\sqrt{2}$ is not rational.

Try to prove this using proof by contradiction.

Assume the negation of the claim is true.

$\sqrt{2}$ is a rational number by assumption.

$$\sqrt{2} = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0$$

by def of rational numbers.

$$\sqrt{2}^2 = \left(\frac{p}{q}\right)^2$$

by algebra

$$2 = \frac{p^2}{q^2}$$

by algebra

$$p^2 = 2q^2$$

by algebra

p^2 is even

by def of even

p is even

by previous theorem

$$p = 2k, k \in \mathbb{Z}$$

by def of even.

$$(2k)^2 = 2q^2$$

by algebra

$$q^2 = 2k^2$$

by the def of even.

q^2 is even

q is even

$q \in \mathbb{Z}$, by previous theorem.

p, q has a common factor 2.

because p and q are even.

This is a contradiction

Therefore, the assumption is incorrect.

Hence $\sqrt{2}$ is irrational. \square