10/14/2024

Recap Relations and properties of relations.

A binary relation R on set A, B is a subset of $A \times B$

$$R \subseteq A \times B$$

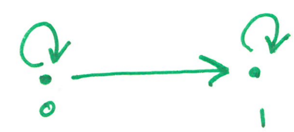
Properties of relation R defined on a single set A.

$$R \subseteq A \times A$$

- Reflexive

$A = \{0, 1\}$

$\forall a \in A : a R a$



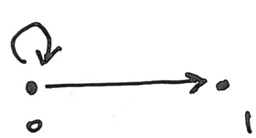
$R = \{\langle 0,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle\}$

All nodes have self-loops.

- Irreflexive

$\forall a \in A : a \not R a$

$R = \{\langle 0,1\rangle\}$



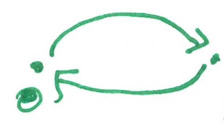
- There can be relation which is neither reflexive nor irreflexive.

- Symmetric

$$\forall a_1, a_2 \in A : a_1 R a_2 \Rightarrow a_2 R a_1$$

If you have a forward edge, then you need to have a backward edge.

Symmetric

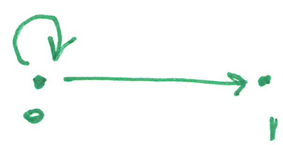


not symmetric

symmetric.

- Anti-Symmetric

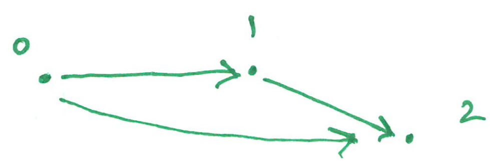
$$\forall a_1, a_2 : (a_1 R a_2) \wedge (a_2 R a_1) \Rightarrow a_1 = a_2$$



anti-symmetric

- Transitive

$$\forall a_1, a_2, a_3 \in A : (a_1 R a_2) \wedge (a_2 R a_3) \Rightarrow (a_1 R a_3)$$



transitive

transitive

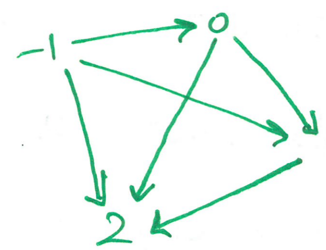
$a_1 = 0$	$a_1 = 0$
$a_2 = 1$	$a_2 = 2$
$a_3 = 2$	$a_3 = 1$

Ex: Relation $<$ on $\mathbb{Z}$

R_2

$R_2 \subseteq \mathbb{Z} \times \mathbb{Z}$

$(a,b) \in R_2$, if $a < b$



- is it reflexive?

WTS $\forall a \in \mathbb{Z} : aRa$: No.

$\forall a \in \mathbb{Z} : a \not< a \Rightarrow a \not R a$

- is it irreflexive? Yes

WTS : $\forall a \in \mathbb{Z} : a \not R a$

$\forall a \in \mathbb{Z} : a \not< a$

$\forall a \in \mathbb{Z} : a \not R a$

- is it symmetric?

WTS: $\forall a_1, a_2 \in \mathbb{Z} : a_1 R a_2 \Rightarrow a_2 R a_1$ No

Let $a_1, a_2 \in \mathbb{Z}$

Assume $a_1 < a_2$

but $a_2 \not< a_1$

$1, 2 \in \mathbb{Z}$

$1 < 2$ but $2 \not< 1$

is it anti-symmetric

WTS $\forall a_1, a_2 \in \mathbb{Z} : (a_1 R a_2) \wedge (a_2 R a_1) \Rightarrow a_1 = a_2$

~~Assume~~ Let $a_1, a_2 \in \mathbb{Z}$

Assume $a_1 < a_2$ and $a_2 < a_1$

However, we cannot pick a_1, a_2 to be both $(a_1 < a_2) \wedge (a_2 < 1)$. Therefore $(a_1 R a_2) \wedge (a_2 R a_1)$ is false.

Therefore the implication is vacuously tree

$\boxed{R \text{ is anti-symmetric}}$

is it transitive?

WTS $\forall a_1, a_2, a_3 \in \mathbb{Z} : (a_1 R a_2) \wedge (a_2 R a_3) \Rightarrow a_1 R a_3$

Let $a_1, a_2, a_3 \in \mathbb{Z}$

Assume $(a_1 R a_2) \wedge (a_2 R a_3)$

$(a_1 < a_2) \wedge (a_2 < a_3)$ by def of R

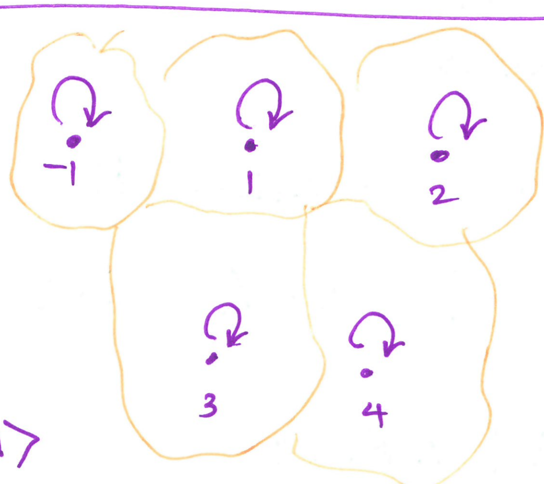
$a_1 < a_3$ by def of less than.

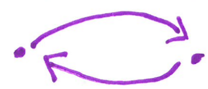
$a_1 R a_3$

R is transitive.

Def Relation R on set A, is an equivalence relation, if R is reflexive, symmetric, and transitive.

$A = \{-1, 1, 2, 3, 4\}$





reflexive ✓
symmetric ✓
transitive ✓

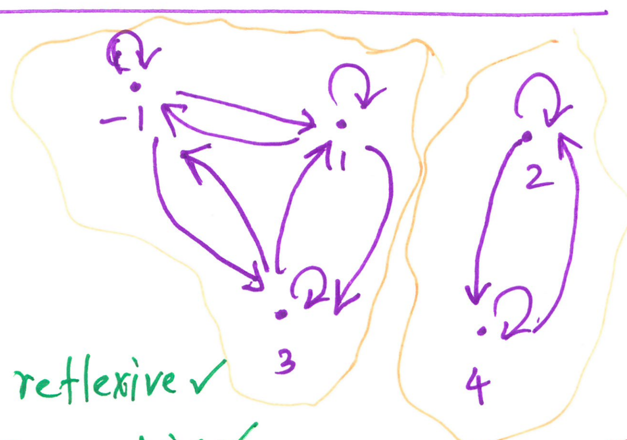
$\langle -1,-1 \rangle$
$\langle 1,1 \rangle$
$\langle 2,2 \rangle$
$\langle 3,3 \rangle$
$\langle 4,4 \rangle$

= relation

equivalence relation

$[3] = \{3\}$
$[-1] = \{-1\}$ $[2] = \{2\}$
$[1] = \{1\}$ $[4] = \{4\}$

Same parity relation

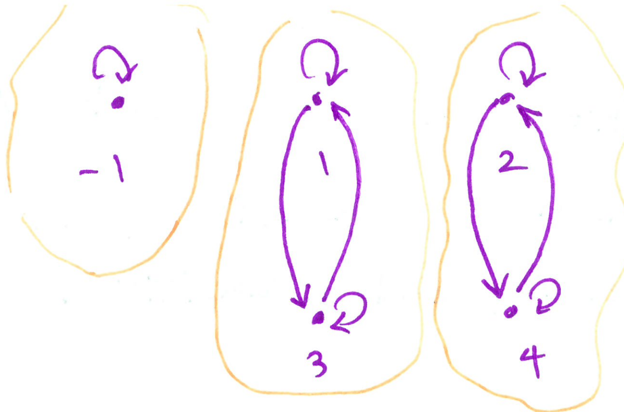


reflexive ✓
symmetric ✓
transitive ✓

equivalence relation

$[2] = \{2, 4\} = [4]$
$[-1] = \{-1, 1, 3\} = [1] = [3]$

same sign
and
same parity



equivalence relation

reflexive ✓
symmetric ✓
transitive ✓

$[-1] = \{-1\}$

$[1] = \{1, 3\} = [3]$

$[2] = \{2, 4\} = [4]$

Def Equivalance classes

For an equivalence relation R on set A the equivalence class of $a \in A$ is:

$$[a] = \{x \in A : xRa\}$$