Recap:

- proof by cases
  - We looked at a claim, and we divided the 算 input space into several cases.
  - Then for each case, we tried to prove that the claim is true,
  - Then we argued that cases are exhaustive.

Def

To give a proof by cases of a proposition φ, we identify a set of cases and then we prove two different types of facts.

1. In every case, φ holds
2. One of the cases has to hold.

Claim: Let x be a real value. Then

$$-|x| \leq x \leq |x|$$

Step 01:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$x = 3$ $\quad$ $x = 0$ $\quad$ $x = -2$

$|3| = 3$ $\quad$ $|0| = 0$ $\quad$ $|-2| = -(-2) = 2$

step 02: $-|x| \le x \le |x|$.

x	$\lvert x\rvert$	$-\lvert x\rvert$	is $-\lvert x\rvert \le x \le \lvert x\rvert$
2	2	-2	$-2 \le 2 \le 2$ ✓
0	0	0	$0 \le 0 \le 0$ ✓
-1	1	-1	$-1 \le -1 \le 1$ ✓

(case 1) $x \geq 10$, $x < 10$ (case 2)

case 1: $x \geq 0$ case 2 $x < 0$

case 1 $x > 0$, case 2: $x < 0$ case 3: $x = 0$

$\longrightarrow$ proof

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

case 1: $x \geq 0$

$-x \le x \le x$ because $x \geq 0$

$|x| = x$ by def. of $|x|$

$-|x| = -x$ by algebra

$-|x| \le x \le |x|$ by substitution.

case 2: $x < 0$ $\qquad |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$x \leq x \leq -x$ because $x < 0$

$|x| = -x$ by def. of $|x|$

$-|x| = x$ by algebra

$-|x| \leq x \leq |x|$ by substitution

Then, I have considered all the possible cases for x, and I have proved that for each case the claim holds.

Therefore, $-|x| \leq x \leq |x|$ $\square$

claim: if x, y, z are real numbers, then

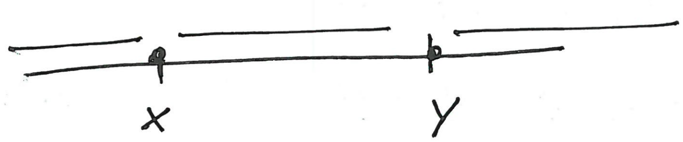
$$|x-y| \leq |x-z| + |y-z|$$

case 1: $z \leq x$

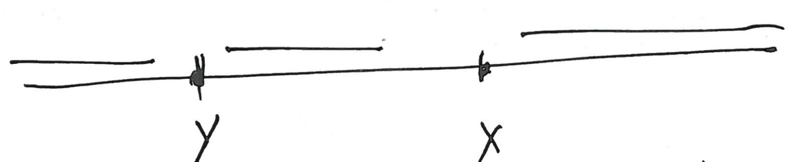
case 2: $x < z \leq y$

case 3: $z > y$

$y \geq x$



$y < x$

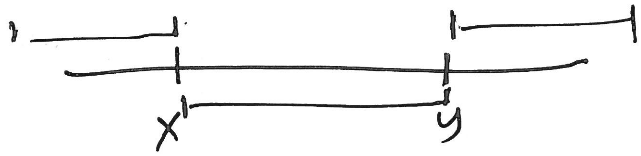


$|y-x| \leq |y-z| + |x-z|$

$|y-x| = |-(x-y)|$
$= |(x-y)|$

~~I would consider $y \geq x$, and consider 3 cases:~~

without any loss of generality I can consider $y \geq x$, then I would consider 3 cases.



case 1: $z \leq x$, case 2: $x < z \leq y$

case 3: $y < z$

$$|x-y| \leq |x-z| + |y-z|$$

$$y \geq x$$

$$|x-y| = y-x$$

$$= |-1(y-x)| = y-x$$

$$|x-z| + |y-z| \geq y-x$$ — I rewrote the claim.

case 1: $z \leq x$

$$|y-z| = |y-z|$$ by algebra

$$|y-z| + |x-z| \geq |y-z|$$ $|x-z| \geq 0$

$$|y-z| + |x-z| \geq y-z$$ $x \leq y$, $z \leq x$

$$|y-z| + |x-z| \geq y-x.$$ $x \leq y$

claim is true.