10/28/2024

Introduction to Graphs

Def: Undirected Graph

A Undirected graph is a pair $G = \langle V, E \rangle$ where V is a nonempty set of vertices or nodes, and $E \subseteq \{\{u,v\} : u,v \in V\}$ is a set of edges joining pairs of vertices.

Edges represent connection between vertices

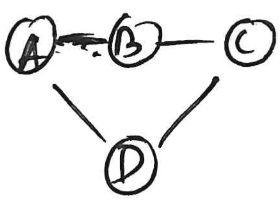
Ex!

$V = \{A\}$

$E = \{\} = \emptyset$

$V = \{A, B\}$

$E = \{\{A,B\}\}$

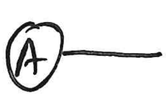


$V = \{A, B, C, D\}$

$E = \{\{A,B\}, \{B,C\}, \{C,D\}, \{A,D\}\}$

$V = \{A, B\}$

$E = \emptyset$



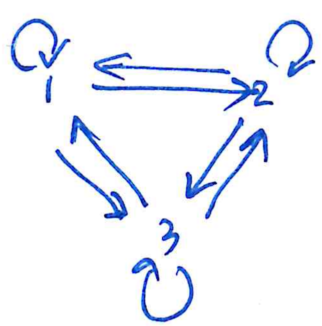
Not a valid graph

All edges needs endpoints.

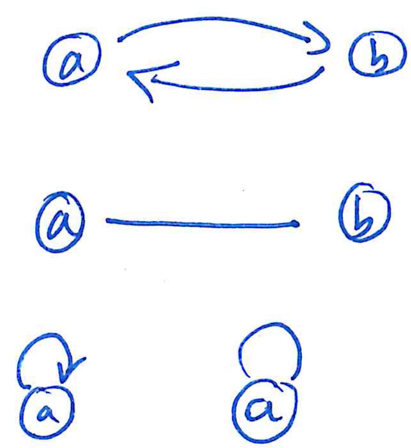
Real-world examples

- facebook friendships
    - nodes can be people on FB.
    - Edge between two people/nodes, if these two people are friends on FB.
- Blood relationships

Q: what properties would mathematical relations need to have to be represented by an undirected graph?

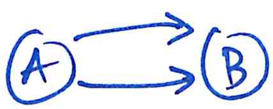


In a symmetric relation,



Def: A graph is simple if it contains no parallel edges or self-loops.

Parallel edge



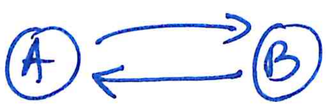
$V = \{A, B\}$

$E = \{\langle A, B\rangle, \langle A, B\rangle\}$

$V = \{A, B\}$

$E = \{\{A, B\}, \{A, B\}\}$

Def does not allow this.



This has no parallel edges.

Self-loops

If a graph contains any of the above, it is not considered as a simple graph.

Def Directed Graph.

A Directed Graph $G = (V, E)$, where V is a nonempty set of nodes, and $E \subseteq V \times V$ is a set of edges joining (ordered) pairs of vertices.

$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C & \end{tikzcd}$$

$V = \{A, B, C\}$

$V \times V = \{\langle A,A\rangle, \langle A,B\rangle, \langle A,C\rangle, \langle B,A\rangle, \langle B,B\rangle, \langle B,C\rangle, \langle C,A\rangle, \langle C,B\rangle, \langle C,C\rangle\}$

$E = \{\langle A,B\rangle, \langle B,C\rangle\}$

$$E \subseteq V \times V$$

A $\quad V = \{A\}$

$E = \emptyset$

$A \longrightarrow B$ $\quad V = \{A, B\}$

$E = \{\langle A, B\rangle\}$

$\equiv$

$B \longleftarrow A$ $\quad V = \{A, B\}$

$E = \{\langle A, B\rangle\}$

$\not\equiv$

$A \longleftarrow B$ $\quad V = \{A, B\}$

$E = \{\langle B, A\rangle\}$

real-world example

- Twitter follower relation
  - node - twitter user
  - Edge from vertex a to b would be a follows b

Example 11.3: Self-loops and parallel edges.

Suppose that we construct a graph to model each of the following phenomena. In which settings do self-loops or parallel edges make sense?

1. A social network: nodes correspond to people; (undirected) edges represent friendships.
2. The web: nodes correspond to web pages; (directed) edges represent links.
3. The flight network for a commercial airline: nodes correspond to airports; (directed) edges denote flights scheduled by the airline in the next month.
4. The email network at a college: nodes correspond to students; there is a (directed) edge $\langle u, v \rangle$ if u has sent at least one email to v within the last year.

Solution. *A social network:* Neither self-loops nor parallel edges make sense. A self-loop would correspond to a person being a friend of himself, and parallel edges between two people would correspond to them being friends "twice." (But two people are either friends or not friends.)

The web: Both self-loops and parallel edges are reasonable. It is easy to imagine a web page p that contains a hyperlink to p itself. It is also easy to imagine a web page p that contains two separate links to another web page q. (For example, as of this writing, the "CNN" logo on `www.cnn.com` links to `www.cnn.com`. And, as of the end of this sentence, this page has three distinct references to `www.cnn.com`.)

A commercial flight network: In a flight network, many parallel edges will exist: there are generally many scheduled commercial flights from one airport to another—for example, there are dozens of flights every week from BOS (Boston, MA) to SFO (San Francisco, CA) on most major airlines. However, there are no self-loops: a commercial flight from an airport back to the same airport doesn't go anywhere!

A who-emailed-whom network: Self-loops are reasonable but parallel edges are not. A student u has either sent email to v in the last year or she has not, so parallel edges don't make sense in this network. However, self-loops exist if any student has sent an email to herself (as many people do to remind themselves to do something later).

	Self-loops	Parallel edges
Social Network	No	No
The Web	Yes	Yes
flight-network	No	Yes
Email network	Yes	No

Popup Test

Form Groups and attempt following question.
Write your name(s).

Prove following claim:

$\forall n \geq 0$: $n^3 + 2n$ is divisible by 3.

$n \in \mathbb{N}$, This is true for all natural #

Use induction.

Step 01: $P(n)$

$P: N \longrightarrow \{T, F\}$

$$P(n) = \begin{cases} T & \text{if } n^3+2n \text{ is divisble by } 3 \\ F & \text{otherwise} \end{cases}$$

Step 3, 4: $n = 0$

~~this~~ $0^3 + 2 \times 0$ is divisible by 3

0 is divisible by 3

Step ~~5~~,6: $\forall n \geq 1 : P(n-1) \Rightarrow P(n)$

~~n^3+2n~~ $(n-1)^3 + 2(n-1)$ is divisible by 3

Assume $P(n-1)$ is true.

WTS: n^3+2n is divisible by 3

$n^3 + 2n$

$(n-1+1)^3 + 2(n-1+1)$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(n-1)^3 + 3(n-1)^2 + 3(n-1) + 1 + 2(n-1) + 2$

$\underbrace{(n-1)^3 + 2(n-1)} + 3(n-1)^2 + 3(n-1) + 3$

$3 \cdot K + 3 \cdot \underbrace{((n-1)^2 + (n-1) + 1)}_{c}$, $K \in \mathbb{N}$

$3 \cdot K + 3 \cdot c$ $\quad c \in \mathbb{N}$

$3 \cdot (K + c)$

$3 \cdot r$ $\quad r = c + K$, $r \in \mathbb{N}$

n^3+2n is divisible by 3.

Inductive ~~ste~~ case is true.

Step 07: Since we have proven base case & inductive case by principle of mathematical induction, [$\forall n \geq 0 : n^3+2n$ is divisible by 3]