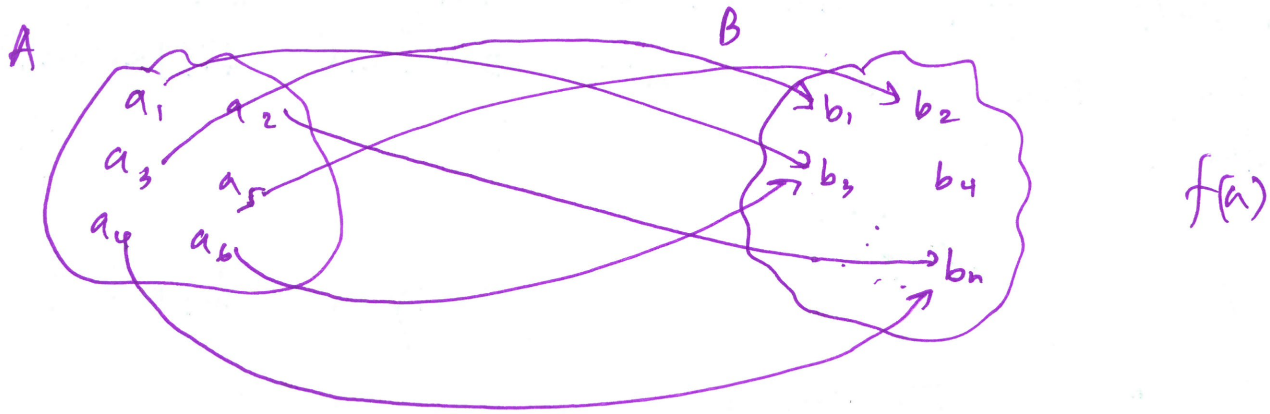
09/25/2024

Functions

Def Let A, B be sets. A function f from A to B, written as $f: A \to B$ assigns to each input value $a \in A$ a unique output value $b \in B$; the unique value assigned to a is denoted as $f(a)$. We sometime say that f maps a to $f(a)$.



A, B can be the same set.

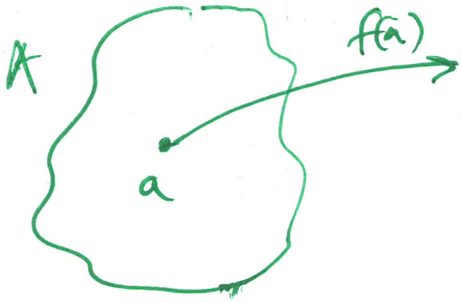
- predicates are functions
  - $isEven: \mathbb{Z} \longrightarrow \{T, F\}$

- $g: \mathbb{Z} \longrightarrow \mathbb{Z}$, g is defined $g = n^2$

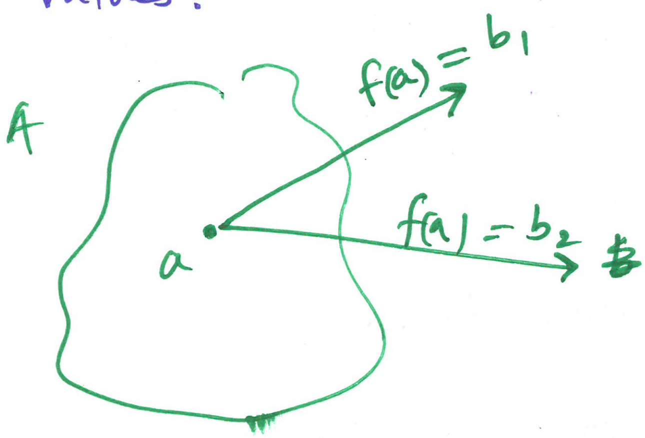
Equivalently we can define 3 properties of a function.

suppose $f: A \longrightarrow B$

① $\forall a \in A$, $f(a)$ is defined.



② $\forall a \in A$, $f(a)$ has to be unique, In other words, $f(a)$ should not produce different values.



only if $b_1 = b_2$

③ for each $a \in A$, $f(a) \in B$.

The value generated by function f should be an element ~~B~~ in Set B.

$f: A \longrightarrow B$

A is called ~~as~~ the domain of the function f.

B is called the codomain of the function f.

The range of the function f is the set of all elements that is generated by the function f.

$$\{ f(a) : a \in A \}$$

ex: $f: \mathbb{Z} \longrightarrow \mathbb{Z}$, ~~g~~ f is defined as $2n$ ~~$g(n) = 2n$~~ $f(n) = 2n$

Domain: $\mathbb{Z}$

Codomain: $\mathbb{Z}$

range: $\cancel{\{x \in \mathbb{Z} : 2 \mid x\}}$ $\{ x \in \mathbb{Z} : 2 \mid x \}$

$$\text{range} \subseteq \text{codomain}$$

- functions does not necessarily need to have real life idea. As long as it maps every element in the domain to a unique value in codomain, we can call it a function.

$A = \{1, 2, 3\}$

$B = \{\Delta, \square\}$

1. $\forall a \in A$, $f(a)$ is defined ✓
2. $\forall a \in A$, $f(a)$ is uniq ✓
3. $\forall a \in A$, $f(a) \in B$ ✓

$a \in A$	$f(a)$
1	$\Delta = f(1)$
2	$\square = f(2)$
3	$\Delta = f(3)$

Ex: $f : \mathbb{R} \longrightarrow \mathbb{R}$, defined as $f(x) = x^2$

Domain : $\mathbb{R}$

coDomain : $\mathbb{R}$

Range : $\mathbb{R}^{\geq 0}$

Idea: we show that f follows 3 properties of a function.

Rough proof

1) $\forall x \in \mathbb{R}$, $f(x) = x^2$, $f(x)$ is defined for all $x \in \mathbb{R}$

2) $\forall x \in \mathbb{R}$, $f(x)$ is unique.

If $f(x) = a$ and $f(x) = b$, then $a = b$

$f(x) = a = x^2$ $\quad$ $f(x) = b = x^2$ $\quad$ by def.

$a = b$ $\quad$ by substitution

therefore $f(x)$ is unique.

For the following mappings, prove or disprove that they are functions. If they are functions, ~~g~~ write down the domain, codomain and range.

Q1. $g: \mathbb{Z} \longrightarrow \mathbb{Z}$ ~~g~~ defined as $g(a) = 6$

Q2. $E: \mathbb{Z} \longrightarrow \{T, F\}$, defined as $E(x) = \begin{cases} T & \text{if } x \text{ is even} \\ F & \text{if } x \text{ is odd} \end{cases}$

Q3. $p: \mathbb{Z}^{\geq 0} \longrightarrow \mathbb{Z}^{\geq 0}$ defined as $p(x) = x-1$

Answers

Q1:

① ~~eg~~ $\forall a \in \mathbb{Z}$ $g(a)$ is defined ✓

② $\forall a \in \mathbb{Z}$, $g(a)$ should be unique. ✓
$g(a)$ is always 6.

③ $\forall a \in \mathbb{Z}$, $g(a) \in \mathbb{Z}$, because $6 \in \mathbb{Z}$

Domain: $\mathbb{Z}$ Codomain: $\mathbb{Z}$ Range: $\{6\}$

Q2: Domain: $\mathbb{Z}$ codomain: $\{T, F\}$, Range: $\{T, F\}$

Q3: when $x = 0$ $p(0) = -1$ $-1 \notin \mathbb{Z}^{\geq 0}$
This violates property 3.

For the following mapping ~~determine~~ determine whether they are functions.
Then provide the domain, codomain and range of the functions.

Q1 $g: \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $g(a) = 5$

Q2 $E: \mathbb{Z} \longrightarrow \{T, F\}$, $E(x) = \begin{cases} T & x \text{ is even} \\ F & x \text{ is odd} \end{cases}$

Q3 $p: \mathbb{Z}^{\geq 0} \longrightarrow \mathbb{Z}^{\geq 0}$ defined by $p(x) = x - 1$

~~Q4 $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(x)$~~