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Recap: $\neg, \sim$

- Propositions
- Logical connectors
  $\wedge, \vee, \sim, \oplus, \Rightarrow, \Leftrightarrow$
- Truth tables
- Tautology

Def. Inverse, converse and contrapositive statements of an implication.

Given two propositions p, q and the implication $p \Rightarrow q$

Converse of the implication: $q \Rightarrow p$

Inverse of the implication: $\neg p \Rightarrow \neg q$

Contrapositive statement of the implication: $\neg q \Rightarrow \neg p$

p	q	$p \Rightarrow q$	$\neg p$	$\neg q$	$q \Rightarrow p$	$\neg p \Rightarrow \neg q$	$\neg q \Rightarrow \neg p$
0	0	1	1	1	1	1	1
0	1	1	1	0	0	0	1
1	0	0	0	1	1	1	0
1	1	1	0	0	1	1	1

$\equiv$ (they are equivalent)

If $\underbrace{\text{it rains in Bozeman}}_{p}$, then $\underbrace{\text{Bozeman is wet}}_{q}$.

$$p \Longrightarrow q$$

$$\neg q \Rightarrow \neg p$$

If Bozeman is not wet, then It does not rain in Bozeman.

Def Logical Equivalence

Two propositions φ and ψ are logically equivalent, written as $\varphi \equiv \psi$, If they have exactly identical truth tables.

(In other words, their truth values are same under every truth assignment)

Ex: Consider $\neg(p \wedge q)$, $(\neg p \vee \neg q)$

p	q	$\neg p$	$\neg q$	$\neg(p \wedge q)$	$(\neg p \vee \neg q)$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	1	0	0	0	0

$\equiv$

$$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$
$\overline{A \cap B} = \bar{A} \cup \bar{B}$ } De morgan's law

$\neg(p \vee q) \equiv (\neg p \wedge \neg q)$ $\quad$ $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$ De morgan Law

Proof by contradiction

p

$\neg p$

The idea: Suppose we want to prove that proposition φ is true. Instead of proving φ is true; we show that φ can not be false.

Def. Proof by contradiction

To prove φ using proof by contradiction; we assume the negation of φ and derive a contradiction; that is we assume $\neg\varphi$ and prove $\neg\varphi$ is false.

Theorem: Let $n \in \mathbb{Z}$, If $\underbrace{n^2 \text{ is even}}_{p}$, then $\underbrace{n \text{ is even}}_{q}$

$p \Rightarrow q \equiv \neg p \vee q$

$\neg(p \Rightarrow q)$

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$	$p \wedge \neg q$
0	0	1	0	0
0	1	1	0	0
1	0	0	1	1
1	1	1	0	0

$\neg(p \Rightarrow q) \equiv p \wedge \neg q$

n^2 is even and n is odd

Trying to use proof by contradiction.

Assume the negation of ~~the~~ the initial claim is true,

$n \in \mathbb{Z}$ and n^2 is even and n is odd

$n = 2k+1, \ k \in \mathbb{Z}$ by def of odd

$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$ by algebra

$n^2 = 2\underbrace{(2k^2+2k)} + 1$ by factoring

$n^2 = 2 \cdot c + 1, \ c \in \mathbb{Z}$ sum of product of ints is a int.

n^2 is odd by def of odd.

This is ~~a~~ a contradiction, we assumed that n^2 is even and n is odd, but we got a contradiction that n^2 is odd. Therefore, our initial assumption is incorrect. Hence, the original claim is true. $\blacksquare$.

Ex: claim: $\sqrt{2}$ is not rational.

Try to prove this using proof by contradiction.

Assume the negation of the claim is true.

$\sqrt{2}$ is a rational number	by assumption.
$\sqrt{2} = \frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$	by def of rational numbers.
$\sqrt{2}^2 = \left(\frac{p}{q}\right)^2$	by algebra
$2 = \frac{p^2}{q^2}$	by algebra
$p^2 = 2q^2$	by algebra
p^2 is even	by def of even
p is even	by previous theorem
$p = 2k$, $k \in \mathbb{Z}$	by def of even.
$(2k)^2 = 2q^2$	by algebra
$q^2 = 2k^2$	by the def of even.
q^2 is even	
q is even	$q \in \mathbb{Z}$, by previous theorem.

p, q has a common factor 2.

because p and q are even.

This is a contradiction

Therefore, the assumption is incorrect.

Hence $\sqrt{2}$ is irrational. $\blacksquare$.