Introduction to predicate Logic

- proposition is either ~~true~~ always true or always false.

Ex: 2 is even } proposition
2 is odd

n is even not a proposition

- predicates: predicate logic is ~~a~~ more general type of logic that allows us to write function like logical expressions called predicates.

- Informally, predicate is a property that a particular entity may or may not have.

Ex: Being a vowel is a property of some letters (A, E, I, O, U), some letters do not have this property (C, D, F, Q, ...,)

- A proposition with a blank

ex: The integer ______ is a prime

* Once the blanks of a predicate is filled, then it becomes a proposition.

The integer 57 is a prime. (false)

Def: A predicate P is a function that assigns value True or False to each element of a Set $\mathcal{U}$. (predicate P is a function $P : \mathcal{U} \longrightarrow \{\text{True}, \text{False}\}$.

The Set $\mathcal{U}$ is called the universe, or the domain of discourse, and we say that P is a predicate over $\mathcal{U}$.

Ex:

1. $$\text{IsPrime}(n) = \begin{cases} \text{True} & n \text{ is a prime} \\ \text{False} & n \text{ is not a prime} \end{cases}$$

$\mathcal{U} = \mathbb{Z}$

$\text{isPrime} : \mathbb{Z} \longrightarrow \{\text{True}, \text{False}\}$

Proposition

$\text{isPrime}(2) = \text{True}$

$\text{isPrime}(4) = \text{False}$

2. $$\text{IsEven}(n) = \begin{cases} \text{True}, & n \text{ is even} \\ \text{False}, & n \text{ is odd} \end{cases}$$

$\mathcal{U} = \mathbb{Z}$

3. $\text{IsRational}(x) = \begin{cases} \text{True} & x \in \mathbb{Q} \\ \text{False} & x \notin \mathbb{Q} \end{cases}$

$\mathbb{Q}$

$\mathcal{U} = \mathbb{R}$

~~on its own P(x)~~

$P : \mathcal{U} \longrightarrow \{\text{True}, \text{False}\}$

$x \in \mathcal{U}$, then $P(x)$ has no value of its own, the reason is x is an arbitrary element.

How can we make a predicate a proposition?

1. Apply the predicate to a specific element of the universe.
   Suppose we have the predicate isEven(x), then let $x = 5$,
   isEven(5) is a proposition.
   let $x = 6$
   isEven(6) is a proposition.

For all intergers n isEven(n) is true.

This is a proposition and it is false.

There exists an integer n such that isEven(n) is true.

Proposition, and it is true.

2. Using quantifiers

a) Universal quantifier $\forall$ "for all"

$[\forall n \in \mathbb{Z} : \text{isEven}(n)]$ ← proposition

$[\forall n \in \mathbb{Z} : \neg\text{isEven}(n)]$.

$\forall x \in U : P(x)$ is true

"for all x in U, P(x) is true"

"True if P(x) evaluates to True for all $x \in U$"

Remember the claim:

Given $n \in \mathbb{Z}$

If $\underbrace{n^2 \text{ is even}}_{p}$ then $\underbrace{n \text{ is even}}$

$\forall n \in \mathbb{Z} : [\text{If } n^2 \text{ is even, then } n \text{ is even}]$

In natural languages, sometimes we omit the explicit mention of quantifiers.

But from here onwards, we will ~~s~~ use quantifiers along with predicates to write logically sound propositions.

b) Existential quanifier $\exists$ "there exists"

$$\exists x \in U : P(x)$$

" there exists an element x in U, such that $P(x)$ is true.

" True if $P(x)$ evaluates to true for some $x \in U$.

$$\exists x \in \mathbb{Z} : \text{isEven}(x)$$

$$\exists x \in \mathbb{Z} : \neg \text{isEven}(x)$$

1. for all integers n, $2n$ is Even

   $\forall n \in \mathbb{Z} : \text{isEven}(2n)$ True

2. for all integers n, $2n+1$ is not even.

   $\forall n \in \mathbb{Z} : \neg \text{isEven}(2n+1)$ True

3. for all integers n, If n^2 is even, then n is even

   $\forall n \in \mathbb{Z} : [\text{isEven}(n^2) \Rightarrow \text{isEven}(n)]$ True

4 . There exists real numbers x, ~~and~~ y such that xy is rational and not (both x and y are rational.)

$\exists x, y \in \mathbb{R} : xy \in \mathbb{Q} \wedge \neg(x \in \mathbb{Q} \wedge y \in \mathbb{Q})$