09/18/2024

Recap

- We looked at proof by contradiction.
- If we need to prove P is true.
    - Assume $\neg p$ is true.
    - Then, by deduction, we try to come up with a contradiction.
    - $\neg p$ is false
    - Hence, p is true.

- Direct proof, proof by cases, proof by contradictions.

Proof by ~~contradiction~~ contrapositive

This is a technique that we can use to prove implication propositions are true.

Let's consider the implication $p \Rightarrow q$

Then, its contrapositive statement: $\neg q \Rightarrow \neg p$

p	q	$\neg p$	$\neg q$	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	0	1	1

$$(p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p)$$

Def
To give a proof by contrapositive of an implication $A \Rightarrow B$, we instead give a proof for $\neg B \Rightarrow \neg A$

If It rains in Bozeman, then Bozeman is wet

P: It rains in Bozeman

Q: Bozeman is wet

~~contrap~~ Contrapositive Statement: $\neg Q \Rightarrow \neg P$

If Bozeman is not wet, then It does not rain in Bozeman

Given $n \in \mathbb{Z}$

Claim: If n^2 is even, then n is even.

- This is an implication, therefore we can use proof by contrapositive technique.

p(premise): n^2 is even

q(conclusion): n is even.

contrapositive statement:

If n is not even, then n^2 is not even

$$\equiv$$

If n is odd, then n^2 is odd

Assume n is odd

$n = 2k+1, \; k \in \mathbb{Z}$ by def of odd.

$n^2 = (2k+1)^2 = 4k^2+4k+1$ by algebra.

$n^2 = 2 \cdot (2k^2+2k) + 1$ by factoring

$n^2 = 2 \cdot c + 1, \quad c \in \mathbb{Z}$ sum of product of integers is an integer

n^2 is an odd number by def of odd.

we have shown that the contrapositive statement is true.

Therefore, our intial claim must be true. $\boxtimes$

Claim: All prime numbers are odd.

- This claim is not true.

~~B~~ Disproof by counterexample.

Consider the prime number 2, 2 is not odd. Therefore, the claim is false.

New claim: All prime numbers greater than 2 are odd.

Given $p \in \mathbb{Z}$ and $p \geq 3$,

If p is a prime number, then p is odd.

premise (p) : p is a prime number

conclusion (q) : p is odd.

contrapositive statement: $\neg q \Rightarrow \neg p$

If p is even, then p is not a prime number.

Assume p is even

$p = 2k$, $k \in \mathbb{Z}$ by def of even

p is not a prime number because $2 \mid p$ and ($2 \neq 1$ or $2 \neq p$)

The contrapositive statement is true, therefore the original claim is true.

claim: Given $x, y \in \mathbb{R}$,

If $|x| + |y| \neq |x+y|$, then $xy < 0$

contrapositive statement:

If $xy \geq 0$, then $|x| + |y| = |x+y|$

There are two cases in which $xy \geq 0$

case 1: $x, y \geq 0 \Rightarrow xy \geq 0$

case 2: $x, y \leq 0 \Rightarrow xy \geq 0$

Let's consider case 1: $x, y \geq 0$

obviously $xy \geq 0$ by case 1:

$|x| + |y| = x + y$ by the def of absolute value

$|x| + |y| = |x+y|$ because, $x, y \geq 0$ and $x + y \geq 0$

case 1 is proven.

Let's consider case 2: $x, y \leq 0$

$|x| + |y| = -x - y$ by def of absolute value.

$|x| + |y| = -(x+y)$ by factoring

$|x| + |y| = |x+y|$ because $x, y \leq 0$, $x + y \leq 0$ and by def of absolute value

case 2 is proven

The contrapositive statement is true for all the cases.

Therefore, the original claim is true. $\boxtimes$