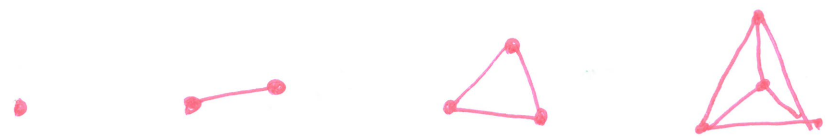
11/04/2024

Recap

- cliques or complete graphs.

Given $G=(V,E)$

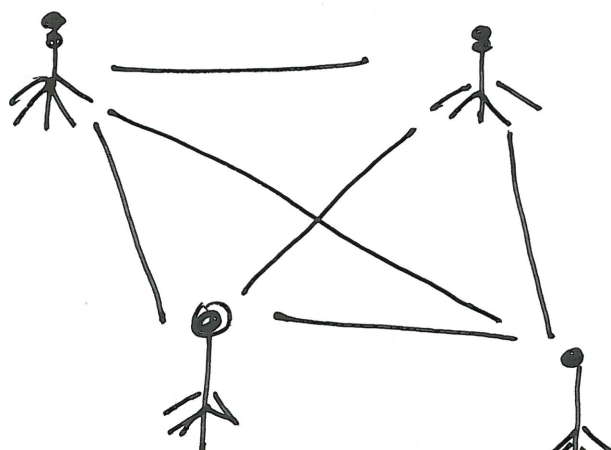
$\forall u,v: u \neq v \Rightarrow \{u,v\} \in E$



K_1 K_2 K_3 K_4

- Every node is connected to every other node

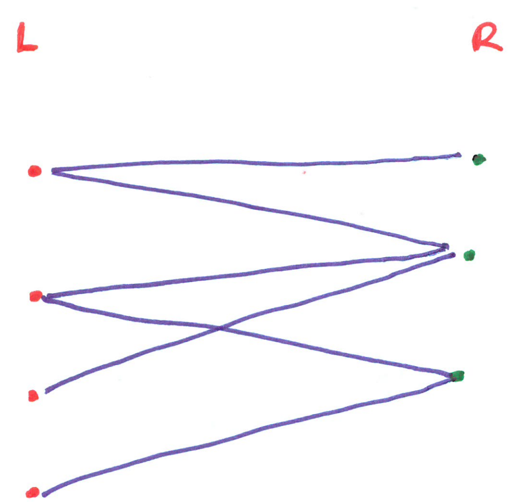
Ex:



Telecommunication towers.

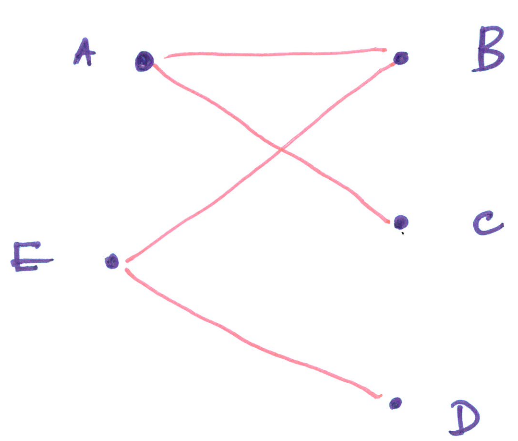
Def: Bi-partite graphs.

A graph G is called a Bi-partite graph, if $G = (L \cup R, E)$ s.t $L \cap R = \emptyset$ and $E \subseteq \{\{l, r\} : l \in L \wedge r \in R\}$



Basically, graph vertices should be able to partition into two sets such that these two sets are disjoint & all edges should go between nodes of L & nodes of R.

Ex:



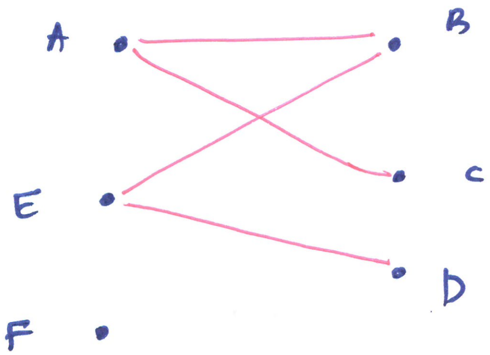
$V = \{A, B, C, D, E\}$

$L = \{A, E\}$

$R = \{B, C, D\}$

$L \cap R = \emptyset$ ✓

$E = \{\{A, B\}, \{A, C\}, \{E, B\}, \{E, D\}\}$



$L = \{A, E, F\}$

$R = \{B, C, D\}$

$L \cap R = \emptyset$

$L' = \{A, E\}$

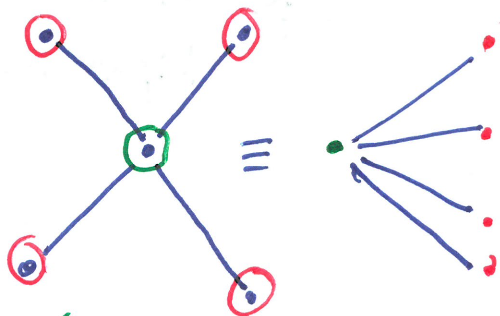
$R' = \{B, C, D, F\}$

$L' \cap R' = \emptyset$

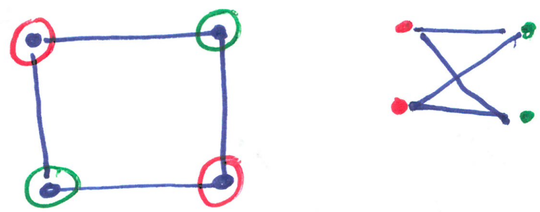
There no edges between the vertices in L' or R'

Popup test 08

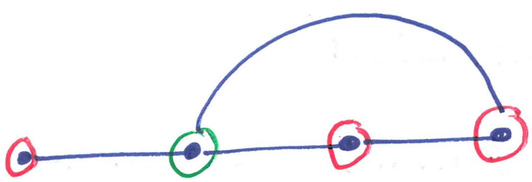
Determine the bipartite graphs.



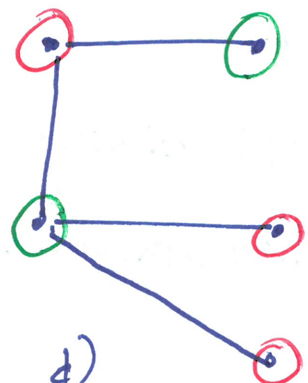
a)



b)

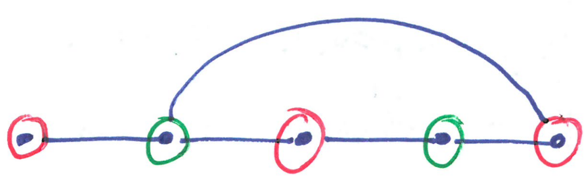


c) not-bipartite



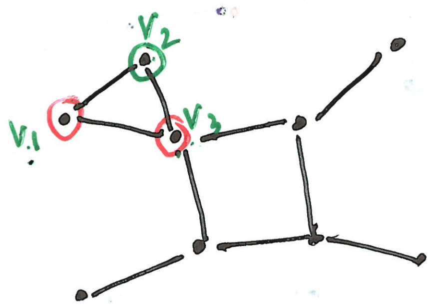
d)

e)



Claim: Let G be an undirected graph. If G contains a triangle, then it is not bipartite graph.

Ex:



G contains a traingle $\Longrightarrow$ G is <u>not</u> bipartite.

$p \Rightarrow q$

$\neg(p \Rightarrow q) \equiv p \wedge \neg q$

Proof: Aiming for contradiction.

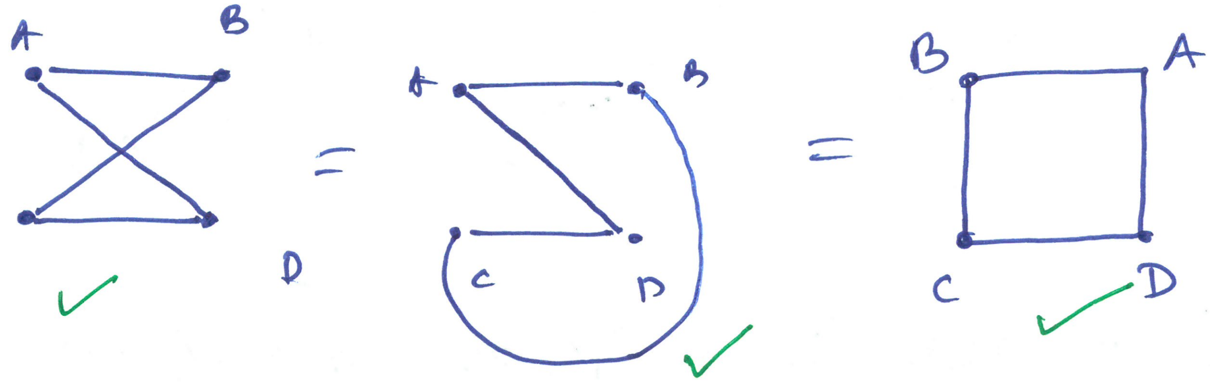
negation statement: G contains a triangle and G is bipartite.

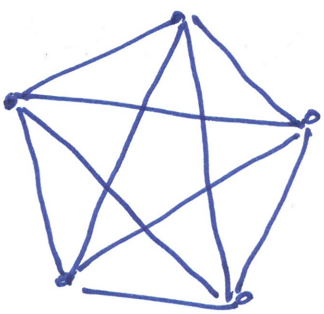
Let v_1, v_2, v_3 be the nodes of the triangle. Without loss of generality, suppose $v_1 \in L$, $v_2 \in R$. Since $v_2 \in R$, $v_3 \in L$. But there is an edge from v_1 to v_3 and ~~bothe~~ both of them are in L.

This is an contradiction. $\boxtimes$

Def.: A graph G is planar, if we can draw it in the 2D-plane without edge crossings.

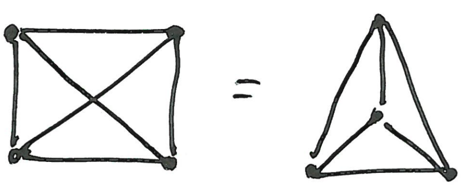
Ex:





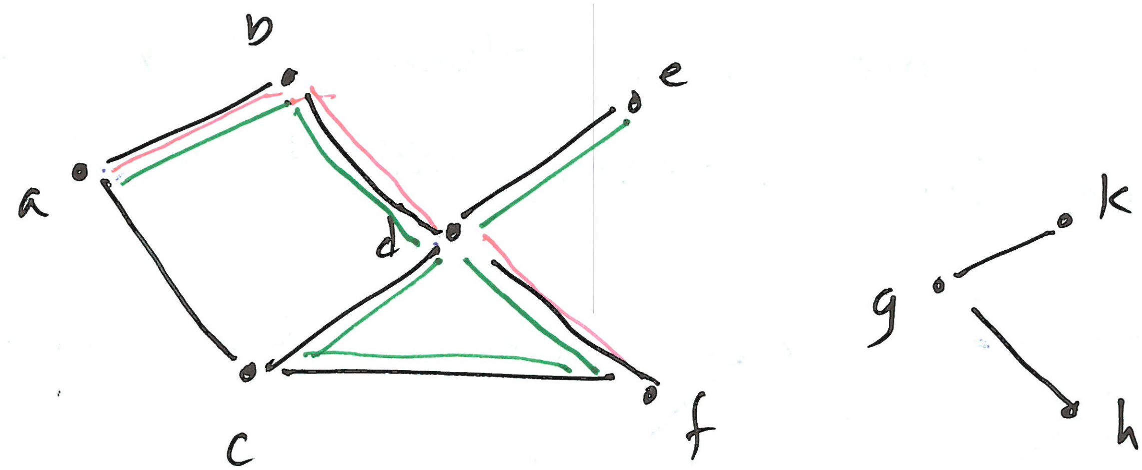
not planar.

K_4



✓ planar graph.

Ex!



Def: A path in a graph $G = (V, E)$ is a sequence of nodes $\langle u_1, u_2, u_3, \ldots, u_k \rangle$ s.t

- $\forall i \in \{1, 2, 3, \ldots, k\} : u_i \in V$
- $\forall i \in \{1, 2, 3, \ldots, k-1\} : \{u_i, u_{i+1}\} \in E$

Pink path from $a \to f$: $\langle a, b, d, f \rangle$

is $\langle a, b, e \rangle$ a path in this graph?

No

Def A path is simple if all of its nodes are unique.

$\langle a, b, \underline{d}, f, c, \underline{d}, e \rangle$ is a valid path but not a simple path

$\langle a, b, d, f \rangle$ this is a simple path.

11/06/2024

Def. Length of a path

- The length of a path is the number of edges in that path.

The length of $\langle a, b, d, f \rangle$ would be 3

In general the length of the path $\langle u_1, u_2, u_3 \ldots u_{k-1}, u_k \rangle$ is $k-1$.

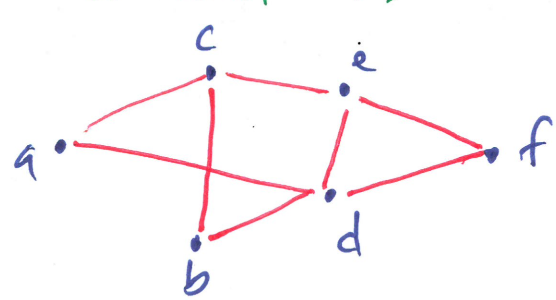
Def. shortest path between two nodes.

Let $G=(V,E)$ be a graph (undirected or directed), and let $s \in V$, $t \in V$ be two nodes. A path from s to t is the shortest path if its length is the smalles out of all the paths from s to t

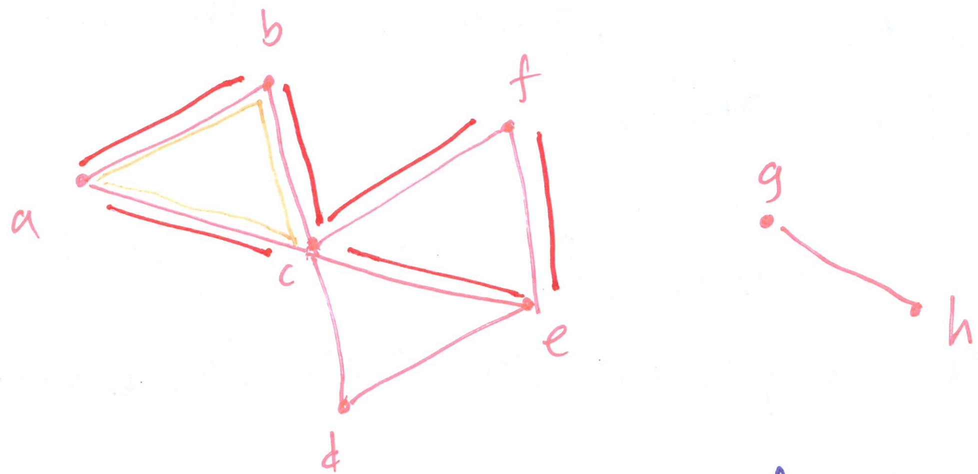
Shortest path between a to f would be 2

Def. The distance, $dist(u,v)$, $d(u,v)$ between u and v is the length of the shortest path between u and v.

$d(a,f) = 2$.



Def A graph G is connected if $\forall u, v \in V$, $\exists$ "a path" from u to v.



This is not connected because there is no path between e and g.

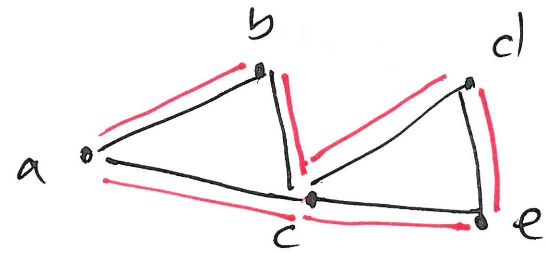
Def A cycle $\langle u_1, u_2, u_3, \ldots u_k, u_1 \rangle$ is a path of length ≥ 2 from u_1 to u_1 that does not traverse the same edge twice.

The path $\langle a, b, c, a \rangle$ is a cycle.

The path $\langle a, b, c, f, e, c, a \rangle$ is a cycle

The path $\langle a, b, c, f, e, c, b, a \rangle$ is not a cycle

Def. A cycle is simple if its nodes are distinct.

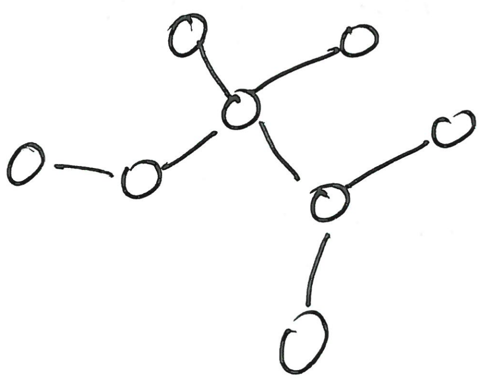


<a, b, c, d, e, c, a> is a cycle but not a simple cycle

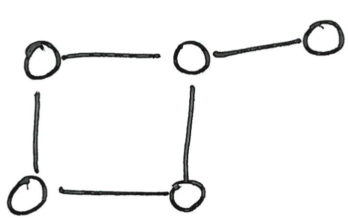
<a, b, c, a> is a simple cycle.

Def A graph is acyclic if it contains no cycles.

acyclic

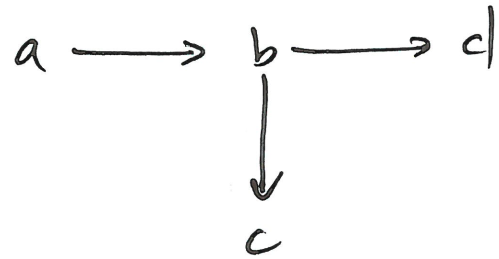


acyclic

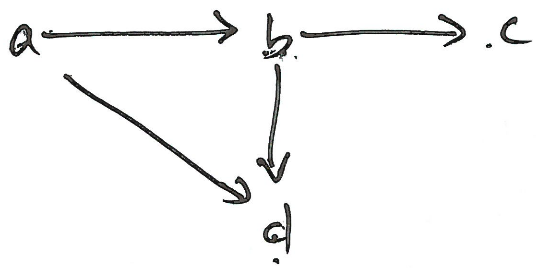


not acyclic

In directed graphs

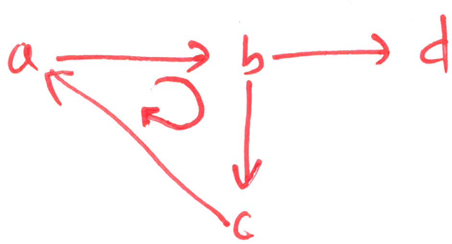


acyclic



acyclic graph

$\langle a, b, d, a \rangle$ is not a path in this directed graph.



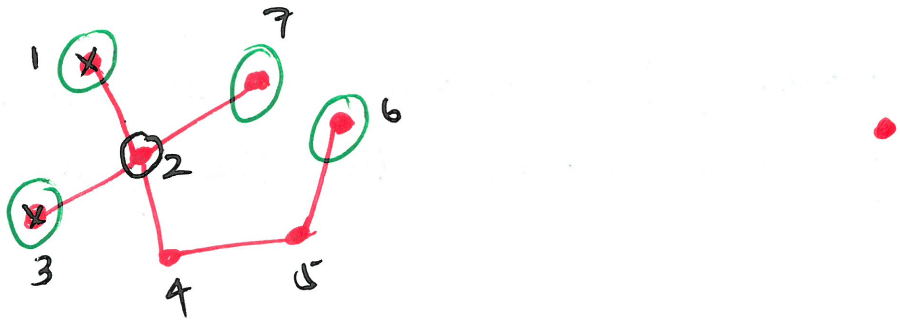
not acyclic graph.

$\langle a, b, c, a \rangle$ is a cycle.

lemma 11.3

If $G=(V,E)$ is an undirected acyclic graph,
then $\exists v \in V$ s.t $deg(v)=0$ or $deg(v)=1$

such that



$deg(1)=1$ $\quad$ $deg(2)=4$

Proof: we give a proof by construction

Note! when we have a claim with $\exists$, one way we can prove this claim is by showing we can alway construct the $\exists$ instance.

we give a proof by construction via an algorithm.

Algorithm: Let u_0 be any node in V.

let $i=0$

while current node u_i has unvisited neighbors:

Let u_{i+1} = any such unvisited neighbor

$i=i+1$

return u_i

Let t be the node returned by the algorithm, on the graph G.

WTS $\deg(t)=0$ or $\deg(t)=1$

case 1: let $t=u_0$, if $t=u_0$ that means there is only one node in the graph.

case 2: $t=u_k$ $k\geq 1$, we show that $\deg(t)=1$

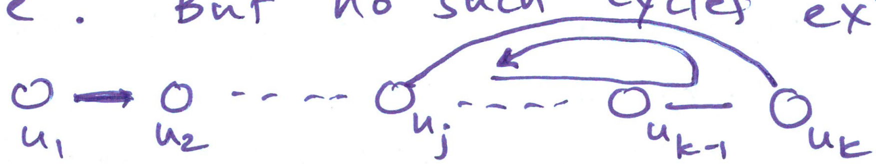
Since u_k is the last in the sequence

$\langle u_1, u_2, u_3, \ldots, u_{k-1}, u_k \rangle$

there is no edge from $t=u_k$ to any other unvisited nodes.

If $\exists$ some edge from t to any other node u_j other than u_{k-1}, then it has to be in $(u_0, u_1, u_2, \ldots, u_{k-2})$

But then it means $(u_j, \ldots u_{k-1}, u_k, u_j)$ is a cycle. But no such cycles exist in G.

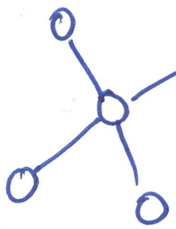


Therefore $\deg(u_k)=\deg(t)=1$

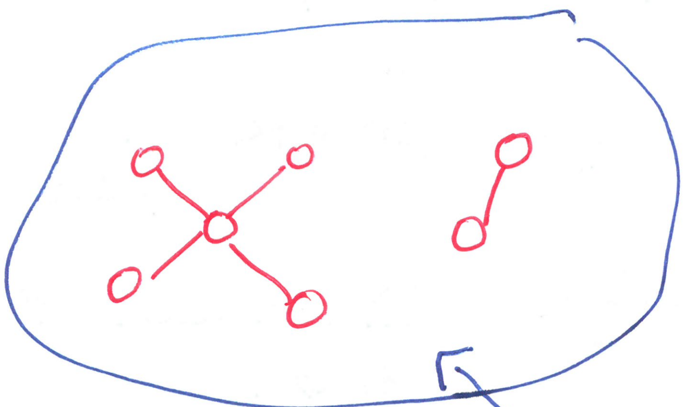
$\mathbb{Z}$

Def. Tree

A tree is a graph that is connected and acyclic.

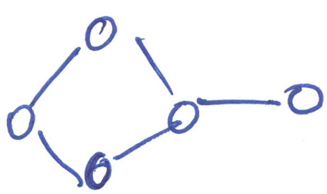


Tree



not a tree

A forrest (A set of trees)



not a tree

Thm

If $T=(V,E)$ is a tree, then

$$|E| = |V| - 1$$

Theorem

If $T=(V,E)$ is a tree, then

1) Adding an edge creates a cycle

2) removing an edge disconnects the graph.