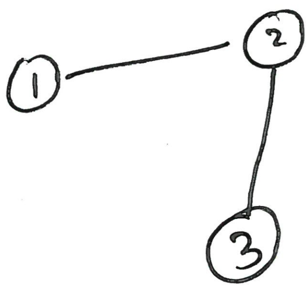
11/01/2024

Def A graph G is called a complete graph or "Kleek" clique if
$G = (V, E)$ s.t
$\forall u, v \in V : u \neq v \Rightarrow \{u, v\} \in E$



is this a clique?

No: why?

$\exists u, v$ s.t $\{u, v\} \notin E$

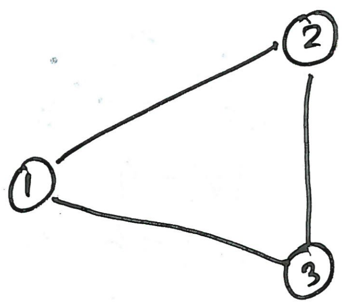
Let $u = 1$ $v = 3$

$\{1, 3\} \notin E$

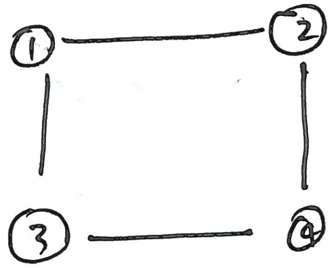
is this a clique?

Yes, because we cannot pick two distinct nodes, therefore the implication is vacuously true.

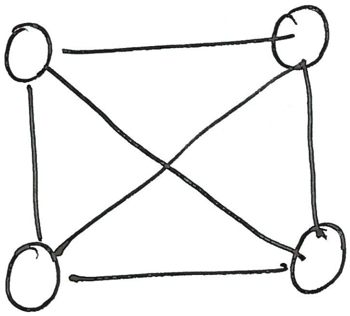
Yes a clique

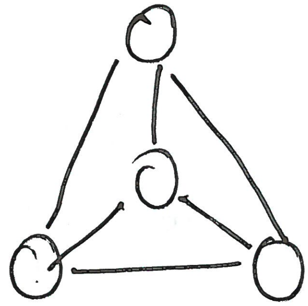


Yes a clique.



No why the edge $\{2,3\}$ does not exist





cliques are so important in CS/maths we give a special names to them.

we say:

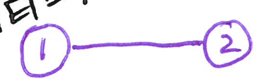
A clique/completegraph on n nodes is denotes as K_n

K_1

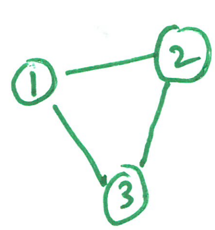
$|V|=1$
$|E|=0$

K_2

$|V|=2$
$|E|=1$

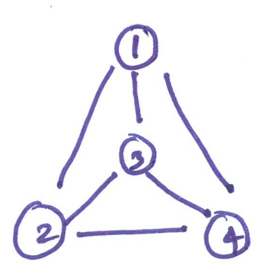


K_3



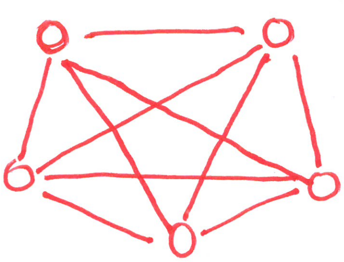
$|V|=3$
$|E|=3$

K_4



$|V|=4$
$|E|=6$

K_5



$|V|=5$
$|E|=10$

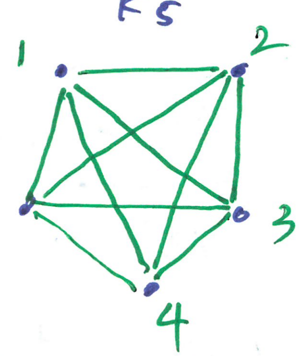
we are going to prove ~~this~~ following claim using different techniques.

claim: K_n has $\frac{n\cdot(n-1)}{2}$ edges.

Proof #1: we give a way (algorithm) to count the edges and show that it is $\frac{n(n-1)}{2}$

Suppose we have a complete graph K_n. Label the vertices as $v_1, v_2, v_3, \ldots \ldots v_n$

starting with v_1 count the uncounted edges adjacent to V_1, and add them to the total.



V_1 has $n-1$ uncounted edges

V_2 has $n-2$ uncounted edges

V_3 has $n-3$ " "

⋮

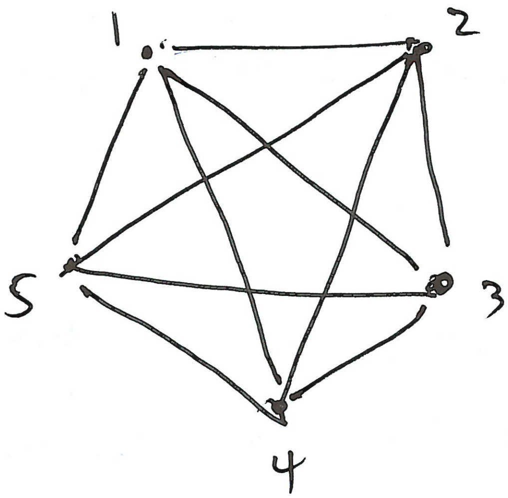
V_{n-1} has 1 uncounted edges

V_n has 0 " "

$$m = |E| = 0 + 1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$$

Proof #2: Using the handshaking lemma.

K_5:



$deg(3) = 4$
$deg(1) = 4$

Note that every node in K_n has $(n-1)$ degree.

$$\sum_{v \in V} deg(v) = \sum_{v \in V} (n-1) = n \cdot (n-1)$$

using the handshaking lemma;

$$\sum_{v \in V} deg(v) = 2 \cdot |E| = n \cdot (n-1)$$

$$|E| = \frac{n \cdot (n-1)}{2}$$ //

Proof #3: using induction

claim: K_n has $\frac{n.(n-1)}{2}$ edges.

First I am going to show that K_n is a recursively defined structure.

Base case: Let K_1 be a graph with 1 vertex and 0 edges

Recursive case: Defining K_n using K_{n-1} graph. Take any K_{n-1} graph and add a new vertex V_n and connect V_n to each of the existing vertices in K_{n-1}

let $P(n)$ denote that K_n has $\frac{n(n-1)}{2}$ edges.

$$P(n) = \begin{cases} T & \text{if } K_n \text{ has } \frac{n(n-1)}{2} \text{ edges} \\ F & \text{otherwise.} \end{cases}$$

We prove $\forall n \geq 1 : P(n)$ using induction over n.

Base case: Let K_1 be a graph with 1 vertex and 0 edges. We want to show that K_1 has 0 edges. Obviously this is true for K_1

Inductive case: $\forall n \geq 2 : P(n-1) \Rightarrow P(n)$

Let $n \geq 2$

Assume $P(n-1)$ is true (Inductive hypothesis)

$$K_{n-1} \text{ has } \frac{(n-1)(n-2)}{2} \text{ edges}$$

WTS: K_n has $\frac{n(n-1)}{2}$ edges.

Let K_n be a clique of n nodes.

Let K_n' be the graph created by removing 1 node from K_n and all its incident edges

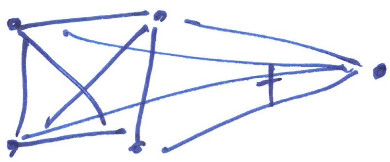
Note that $K_n' = K_{n-1}$

Using the inductive hypothesi

$$K_n' \text{ has } \frac{(n-1)(n-2)}{2} \text{ edges}$$

We add the ~~the~~ node that was removed previously to K_n' to create K_n again

$\#$ edges in K_n $= \dfrac{(n-1)(n-2)}{2} + (n-1)$

 $\Rightarrow$

$$= (n-1)\left[\frac{(n-2)+2}{2}\right]$$

$$= \frac{(n-1)\,n}{2}$$

We proved the inductive case $\boxtimes$.