11/08/2024

Randomness + probability in CS

- Randomized algorithms
- Data structures using randomness
- modeling real-world phenomena.
- machine learning

we first learn how to count a set.

Def Given two sets A, B

1. Sum Rule:

   If $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$

2. Product Rule:

   The number of pairs (x, y) with $x \in A$, $y \in B$ is $|A| \cdot |B|$

   $$|A \times B| = |A| \cdot |B|$$

Ex: Suppose you have a closet with

Shirts = {Red shirt, Blue shirt, Black shirt}

Pants = {Jeans, Khakis}

① what is the total number of unique outfits (shirt & a pant) that you can create

$$|\text{Shirts} \times \text{Pants}| = |\text{shirts}| \cdot |\text{pants}| = 3 \times 2 = 6$$

② what is the total number of independent clothing options?

$$|\text{shirts} \cup \text{Pants}| = |\text{shirts}| + |\text{Pants}|$$

$$= 3 + 2$$

The only reason we could do this is

$\text{shirts} \cap \text{pants} = \{\ \}$

More generalized product rule.

$$|A_1 \times A_2 \times A_3 \times \cdots \times A_k| = |A_1| \cdot |A_2| \cdot |A_3| \cdot \cdots \cdot |A_k|$$

Inclusion - Exclusion Rule.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

If $|A \cap B| = \phi$, this rule becomes the sum rule.

Ex: $ODD = \{1, 3, 5, 7, 9\}$ $\quad$ $Primes = \{2, 3, 5, 7\}$

$$|ODD \cup Primes| = |ODD| + |Primes| - |ODD \cap Primes|$$

$$= 5 + 4 - 3 = 6$$

$ODD \cap Primes = \{3, 5, 7\}$

$$|ODD \cup Primes| = |\{1, 2, 3, 5, 7, 9\}| = 6$$

Ex: How many length 32-bit binary strings are there?

example binary string: 01100111

example 32-bit binary string must have 32 bits

{0,1}

$\_ \ \_ \ \_ \ \_ \ \_ \ \_ \ \_ \ \ldots \ \_$

{0,1}

$\{0,1\} \times \{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}$

$|\{0,1\} \times \{0,1\} \times \{0,1\} \times \cdots \times \{0,1\}|$

$$\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{32 \text{ multiplications}}$$

$= 2^{32}$

Popup test 09

1) Mac Address (Media Access Control Address) is a unique identifier assigned to network devices. This address consists of 12 digits. Each of these 12 digits ~~are~~ ~~are~~ is a hexa-decimal digit.

What is the total number of unique MAC addresses we can have?

Ex: $\underbrace{0A:12:3B:AC:9F:11}_{12\text{-digits}}$ & each of them are hexa-decimal digits.

H = Hexa Decimal Digits $= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$

$$MAC\_ADDRESSES = H \times H \times H \times H \times H \times H \times H \times H \times H \times H \times H \times H$$

$$|MAC\_ADDRESSES| = |H| \cdot |H| \cdot |H| \cdots\cdots |H|$$

$$= |H|^{12}$$

$$= 16^{12}$$

Q: We define a PIN to be string with 4 digits.
Each of these digits are decimal digits.
An invalid PIN would be a PIN with starting with 3-repeated digits or ending with 3-repeated digits.

How many invalid ~~digits~~ Pins are there?

ex: 0002 is invalid
3222 is invalid
4444 is invalid.

S to be the invalid pins with 3-repeated starting digits.

$$|S| = 10 \cdot 10 = 100$$

E to be the invalid pins with 3-repeated ending digits.

$$|E| = 10 \cdot 10 = 100$$

$$|S \cap E| = 10$$

$$|S \cup E| = |S| + |E| - |S \cap E| = 100 + 100 - 10 = \underline{\underline{190}}$$

Def. ~~G~~ Given some random process, the **sample space** S is the set of all possible outcomes.

A probability function $Pr : S \longrightarrow \mathbb{R}$ and describes the fraction of the time that $s \in S$ occurs.

we have 2 rules

1. $$\sum_{s \in S} Pr[s] = 1$$

2. $\forall s \in S : Pr[s] \geq 0$

Example: Random process of flipping a fair coin.

$S = \{H, T\}$

$Pr[H] = 0.5 \geq 0$

$Pr[T] = 0.5 \geq 0$

$$\sum_{s \in S} Pr[s] = 0.5 + 0.5 = 1$$

Example: Drawing a card from a deck

$S = \{A\clubsuit, 2\clubsuit \ldots\ldots, A\heartsuit, \ldots\ldots, A\spadesuit \ldots\ldots \}$

$\forall s \in S : Pr[s] = 1/52$