08/30/2024

Q: Let x, y, z be real numbers, Then we say that

$$|x-y| \leq |x-z| + |y-z| \quad \text{--- } (1)$$

$x \leq y$

case 1: $z \leq x$

case 2: $x \leq z \leq y$

case 3: $y \leq z$

$x \geq y$

case 4: $z \leq y$

case 5: $y \leq z \leq x$

case 6: $x \leq z$

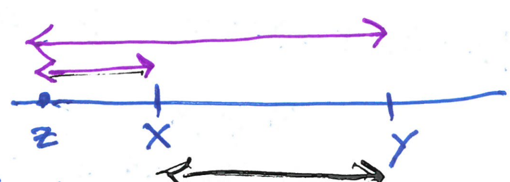
~~Ass~~ Assume $x \leq y$

(1) $\Rightarrow$

$$|x-y| \leq |x-z| + |y-z|$$

$$y-x \leq |x-z| + |y-z|$$

case 1: $x \leq y$, $z \leq x$



$|y-z| = |y-z|$ by algebra

$|y-z| + |x-z| \geq |y-z|$ $\quad |x-z| \geq 0$

$|y-z| + |x-z| \geq y-z$ $\quad x \geq z$ and $y \geq x$

$$|y-z| + |x-z| \geq y-z \geq y-x$$

$z \leq x$ — — — — — by assumption

$-x \leq -z$ — — — — — by algebra

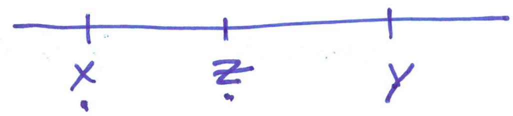
$(y-x) \leq (y-z)$ " "

$\cancel{(y-z) \geq (y-x)}$

$(\underline{y-z}) \geq (\underline{y-x})$ " " "

$$|y-z| + |x-z| \geq y-x$$

Case 2: $x \leq y$, $x \leq z \leq y$



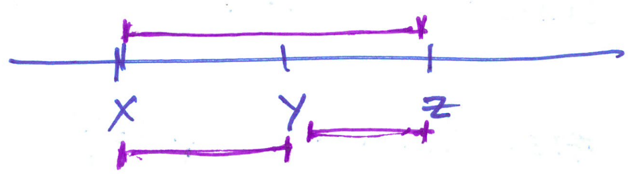
$|x-z| + |y-z| = \cancel{z} - x + y - \cancel{z}$ (because $z \geq x$ and $y \geq z$)

$|x-z| + |y-z| = y-x$

claim is true for case 2.

Case 3:

$x \leq y$, $y \leq z$



$|x-z| = |x-z|$

$|x-z| + |y-z| \geq |x-z|$ (because $|y-z| \geq 0$)

$|x-z| + |y-z| \geq z-x$ because $z \geq x$

$\cancel{z \geq x}$

$\cancel{-x \geq -z}$

$\cancel{y-x \geq y-z}$

$\cancel{|x-z| + |y-z| \geq z-x}$

$\cancel{y \geq x}$

$\cancel{-x \geq -y}$

$\cancel{z-x \geq}$

$$|x-z|+|y-z| \geqslant z-x \geqslant y-x$$

$$z \geqslant y.$$

$$z-x \geqslant y-x$$

$$|x-z|+|y-z| \geqslant y-x$$

case 3: claim is true.

$\longrightarrow$

case 4, case 5, case 6.

$$y \leq x.$$

$$|x-z|+|y-z| \geqslant |x-y|$$

$$|x-z|+|y-z| \geqslant (x-y)$$

$x=t \quad y=r$

$$|t-z|+|r-z| \geqslant (t-r)$$

$$|r-z|+|t-z| \geqslant (t-r)$$

case 4, 5, 6 are identical to case 1, 2, 3

Therefore, the claim holds for case 1, 2, 3, 4, 5, 6.

Hence, the claim is true.