10/11/2024

Recap on Relations

A binary Relation R on Set A, B is a subset $R \subseteq A \times B$.

$\langle x, y \rangle \in R$ $\qquad$ xRy

$\langle x, y \rangle \notin R$ $\qquad$ $x \not R y$

Ex: Let A be the set of months, Let B be the days in a month.

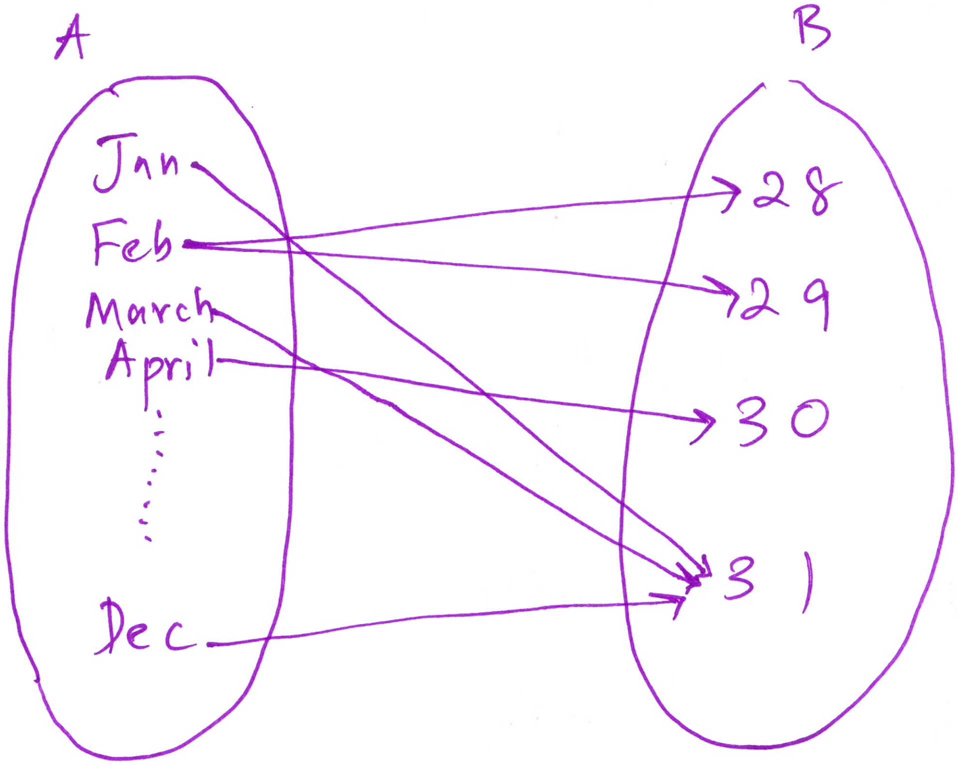
$A = \{$Jan, Feb, March, April, May, ..., Dec$\}$

$B = \{28, 29, 30, 31\}$

R_1: ~~A~~ The pairs that contain the month and its # of days.

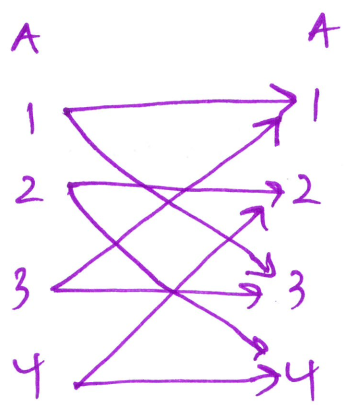
$R_1 \subseteq A \times B$

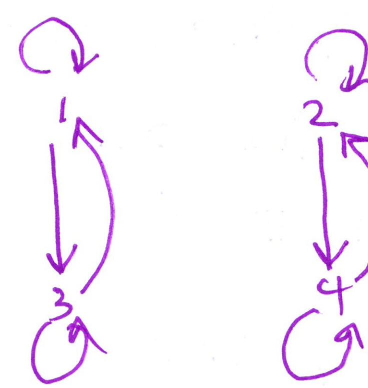
$R_1 = \{\langle \text{Jan}, 31 \rangle, \langle \text{Feb}, 28 \rangle, \langle \text{Feb}, 29 \rangle, \langle \text{March}, 31 \rangle, \langle \text{April}, 30 \rangle, \text{......}, \langle \text{Dec}, 31 \rangle\}$



05 $A = \{1, 2, 3, 4\}$

$R_2 \subseteq A \times A$





$(1,1) \in R_2$

$(2,4) \in R_2$

$(3,2) \notin R_2$

Relations on a Single Set

Let A be a set & let R be a subset defined on $A \times A$

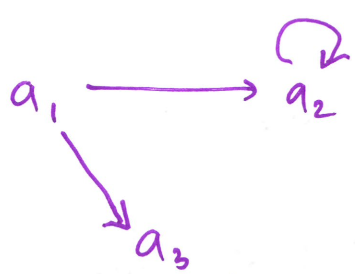
$$R \subseteq A \times A$$

Properties of relation (when relation is defined on a single set)

1. R is reflexive, if $\forall a \in A$: aRa

Ex1:

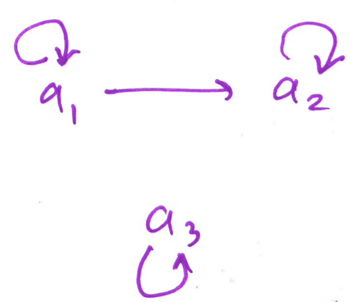
$A = \{a_1, a_2, a_3\}$



This is <u>not</u> reflexive.

Ex 2:

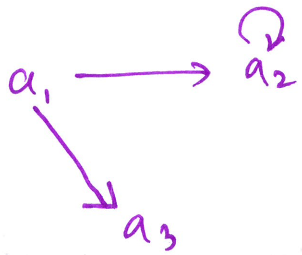
$A = \{a_1, a_2, a_3\}$



This is reflexive.

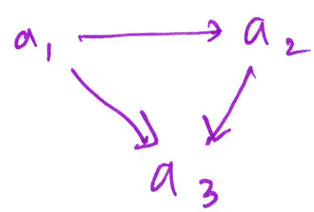
2. R is irreflexive, if $\forall a \in A : a \not R a$

Ex1:



This is <u>not</u> irreflexive

Ex2:



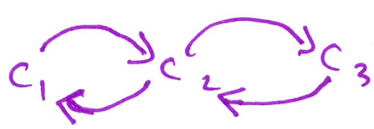
This is irreflexive

$\boxed{\text{Ex1 is ~~not~~ neither reflexive nor irreflexive}}$

3. R is symmetric, if $[\forall a_1, a_2 \in A : a_1 R a_2 \Rightarrow a_2 R a_1]$

Ex3:

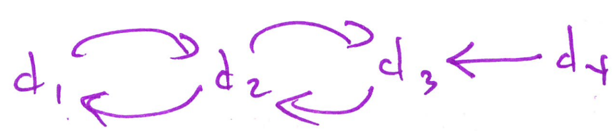
$C = \{c_1, c_2, c_3\}$



Symmetric

Ex4:

$D = \{d_1, d_2, d_3, d_4\}$



not-symmetric

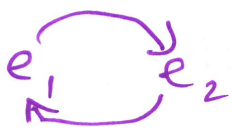
4. R is anti-symmetric, if $\forall a_1, a_2 \in A$:
$(a_1 R a_2) \wedge (a_2 R a_1) \Rightarrow a_1 = a_2$

$$\forall a_1, a_2 \in A : (a_1 R a_2) \wedge (a_2 R a_1) \Rightarrow (a_1 = a_2)$$

In the graph, we can never backward edges but self-loops are okay.

Ex5:

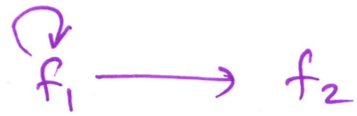
$E = \{e_1, e_2\}$



Not anti-symmetric

Ex6:

$F = \{f_1, f_2\}$



$a_1 = f_1 \qquad a_2 = f_1$

$\underline{a_1 R a_2 \wedge a_2 R a_1} \Rightarrow a_1 = a_2$

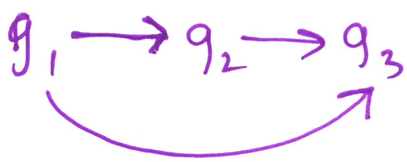
Anti-symmetric

5. R is transitive, ~~iff $\forall a,b,c \in A$:~~

if, $\forall a,b,c \in A : \left[(aRb) \wedge (bRc) \Rightarrow (aRc) \right]$

Ex7:

$G = \{g_1, g_2, g_3\}$



$a_1 = g_3 \qquad a_2 = g_2 \qquad a_3 = g_1$

$\underbrace{g_3 R g_2}_{F} \wedge \underbrace{g_2 R g_1}_{F} \Longrightarrow \underbrace{g_3 R g_1}_{F}$

$\boxed{\text{Transitive}}$

Q: Is
$$\begin{tikzcd} a_1 \arrow[r, bend left] & a_2 \arrow[l, bend left] \end{tikzcd}$$
transitive?

$$\underset{T}{(a_1 R a_2)} \wedge \underset{T}{(a_2 \wedge a_1)} \Rightarrow a_1 R a_1$$

Not transitive

Is
$$\begin{tikzcd} f_1 \arrow[r] \arrow[rr, bend right] & f_2 & f_3 \arrow[l] \end{tikzcd}$$
transitive

$$(f_1 R f_3) \wedge (f_3 \wedge f_2) \Rightarrow f_1 \wedge f_2$$

Transitive.

If
$$\begin{tikzcd} a_1 \arrow[loop above] \end{tikzcd}$$
transitive?

Yes, Transitive

Def of transitive: $\left[\forall a, b, c \in A : (a R b) \wedge (b R c) \Rightarrow (a R c)\right]$

$\left[\forall a_1, a_2, a_3 \in A : (a_1 R a_2) \wedge (a_2 R a_3) \Rightarrow a_1 R a_3\right]$