CSCI-246 Discrete Structures HW 9

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Objective

- Understanding mathematical induction.
- Understanding graph definitions.
- Understanding the problem solving process.
- Understanding summation notation.

Submission requirements

- ***Type or clearly hand-write*** your solutions into a ***PDF FORMAT.***
- ***DO NOT UPLOAD images.***
- ***non-pdf or emailed solutions will not be graded.***
- **If you take pictures of your handwritten homework, put it into pdf format.**
- ***Start each problem on a new page.***
- Follow the model that you have learned during the lectures for proofs.
- Do not wait until the last minute to submit the assignment.
- You can submit any number of times before the deadline.

- If you are using latex, and you do not know how to type a symbol, use the following website. You can draw the symbol here and it will give you the latex code and the packages that you have to import. `https://detexify.kirelabs.org/classify.html`
- If you are using latex to write the answer, you can use overleaf to make your life easier. **Overleaf is a free, online platform that helps users create and publish scientific and technical documents using LaTeX, a markup-based document preparation system**
- If you do not understand a problem, ask questions during/after the lectures, or during office hours or via discord.
- Go to TA office hours and talk with them and ask for help.
- ***Do not use generative AI to write answers.***

Homework 02 contains **3 questions**.

1 Q1

Write the following sequences using summation notation. (This is an easy question, don't try to complicate this. I am not asking to give an expression, I am asking you to write the sequences using summation notation.)

- $0 + 3 + 6 + 9 + \cdots + 3 \cdot i + \cdots + 3 \cdot n$. (Sum of the first $n + 1$ natural numbers.)
- $0^3 + 1^3 + 2^3 + \cdots + i^3 + \cdots + n^3$. (Sum of the first $n + 1$ cubes of natural numbers).
- $2^2 + 4^2 + 6^2 + \cdots + (2i)^2 + \cdots + (2 \cdot n)^2$. (Sum of the squares of first n even numbers starting at 2.)

2 Q2

Draw a graph with the nodes $V = \{1, 2, \ldots, 9, 10\}$, and edges between $x \in V$ and $y \in V$ if x divides y. Does it make sense to use a directed or undirected graph? Is the graph that you have drawn simple?

3 Q3

In the class we looked at a property of an undirected graph $G = (V, E)$.

$$\sum_{v \in V} deg(v) = 2 \cdot |E|$$

This property is known as **Handshaking Lemma**.

Here we will try to prove this using induction. The task for you is to fill in the missing pieces of this proof.

Proof: Here, I define the graph G as a recursively defined structure.

Undirected Graph G can be defined as follows:

- Base case: Start with an graph with no edges and **any number of isolated vertices** (each with degree zero).
- Recursive case: Add an edge between two vertices in the graph, updating the degrees of those two vertices.

Now that we have defined the undirected graph as a recursively defined structure, we can move to induction steps as follows:

For any natural number $m \geq 0$, let predicate $P(m)$ be true if any graph $G = (V, E)$ with m edges has the following property: $\sum_{v \in V} deg(v) = 2 \cdot m$– false otherwise. We show that $\forall m \geq 0 : P(m)$ using mathematical induction over m.

Base case:

You have to fill the base case and the proof of base case here.

Inductive case:

We want to show that $\forall m \geq 1 : P(m-1) \implies P(m)$. For the inductive hypothesis we assume that $P(m-1)$ is true; that is, we assume that **any** undirected graph $G = (V, E)$ with $m-1$ edges has $\sum_{v \in V} deg(v) = 2 \cdot (m-1)$. Now let $H = (W, F)$ be any graph with m edges. We want to prove that $P(m)$ holds; that is, $\sum_{v \in W} deg(v) = 2 \cdot m$.

Let $\{u, w\}$ be any edge of H. Let us consider the graph G constructed from H by removing that one edge $\{u, w\}$ from H; that is, $G = (W, F \backslash \{\{u, w\}\})$ (basically G is the graph that is obtained by removing the edge $\{u, w\}$). Note that we keep the same set of vertices; we only remove a single edge. For any vertex $v \in W$, let $deg_G(v)$ denote the degree of v in graph G, and $deg_H(v)$ denote the degree in graph H, since these could be different once you remove the edge $\{u, w\}$.

Fill the rest of the proof of the inductive case here.

Since we showed that $P(0)$, and $\forall m \geq 1 : P(m-1) \implies P(m)$, we can conclude that $\forall m \geq 0 : P(m)$ using the principle of mathematical induction.