Example: prove $A = \{0^n 1^n \mid n \geq 0\}$ is not regular.

Assume A is regular.

Suppose pumping length of language A is p.

(Remember this is a fancy way of saying the DFA that recognize A has p states)

Now, the goal is to pick a string s in A of length at least p such that it violates the pumping lemma.

$A = \{0^n 1^n \mid n \geq 0\}$

$s = 0^p 1^p \quad |s| = 2p$

$$s = \underbrace{0\ 0\ 0\ 0\ 0\ 0 \cdots 0}_{p\ \#\ 0's}\ \underbrace{1\ 1\ 1\ 1 \cdots\cdots 1}_{p\ \#\ 1's}$$

$s = x\,y\,z$

$$s = \underbrace{0\ 0\ 0}_{x}\ \underbrace{0\ 0 \cdots 0\ 1\ 1}_{y}\ \underbrace{1\ 1 \cdots 1}_{z}$$

# of ways I can divide this string into 3 consecutive substrings is $\binom{2p}{2}$.

$xy = 0^p$

However, by condition ③ of pumping lemma, $|xy| \leq p$.
So whatever the decomposition for $s = xyz$, y could only consist

of 0's.

Since, in any decomposition y can only consist of 0's, then $xy^2z$ cannot be a part of language A.

$s = 0^p 1^p$ $\quad x = 0^a \quad y = 0^b \quad z = 0^c 1^p$

where $a + b + c = p$

and $b > 0$

$xy^2z = 0^a 0^{2b} 0^c 1^p$

$a + 2b + c \neq p$ since $b > 0$

$\therefore xy^2z \notin A$

Clearly $s = 0^p 1^p$ violates pumping lemma for all possible decompositions.

$\therefore$ A cannot be regular.

How about picking $s = 0^{p/2} 1^{p/2}$ to show the contradiction?

$$s = \underbrace{000 \cdots 0}_{p/2}\underbrace{1111 \cdots 1}_{p/2}$$

$$\underbrace{\qquad\qquad}_{x}\ \underbrace{\quad}_{y}\ \underbrace{\quad}_{z}$$

Example 02: Let $\Sigma = \{0,1,2\}$

Show that $B = \{0^n 1^n 2^n : n \geq 0\}$ is not regular.

Proof by contradiction.

Assume $B$ is regular

Then $B$ must follow pumping lemma.

Suppose pumping length is $p$.

Let us pick $s \in B$, s.t $|s| \geq p$.

$$s = 0^p 1^p 2^p, \quad |s| = 3p \qquad 3p \geq p$$

By pumping lemma, $s$ should be able to be decomposed into $s = xyz$,

s.t
1. $xy^i z \in B$, $\forall i \geq 0$
2. $|y| > 0$
3. $|xy| \leq p$.

By ③ of pumping lemma, for $s = 0^p 1^p 2^p$, in any decomposition $y$ could only consist of 0's.

Also, by ② of pumping lemma $y$ must consist at least 1 0.

Hence, $xy^iz \notin B$, as when you pump $y$, the # of 0's would change.

This is a contradiction

$\therefore$ B is not regular

Example 03:

$$E = \{ 0^i 1^j : i > j \}$$ $$\Sigma = \{0, 1\}$$

is not regular.

$001 \in E$

$01 \notin E$

$10 \notin E$

What would be a good string to pick?

$$s = 0^{p+1} 1^p$$

Proof: Assume E is regular.

Thus, E follows pumping lemma.

Pick $s = 0^{p+1} 1^{p}$, $s \in E$, $|s| = 2p+1 \geq p$

Since, E is regular and follows pumping lemma, s must be decomposible to $s = xyz$ s.t

1. $xy^i z \in E, \forall i \geq 0$
2. $|y| > 0$
3. $|xy| \leq p$

$s = \underbrace{0000\ldots0}_{\# p+1} \underbrace{111\ldots}_{\# p}$

By 2 and 3, In any decomposition of $S = 0^{p+1} 1^{p}$, y should only consist of 0's and it should at least consist 1 0.

$\therefore$ if you consider $xy^0 z$, it should contain either same number of 0's as 1's or less number of 0's compared to number of 1's.

Hence, $xy^0 z \notin E$.

$s = 0^{p+1} 1^{p}$, $x = 0^{a}$ $y = 0^{b}$ $z = 0^{c} 1^{p}$

$a + b + c = p+1$, $b > 0$

$xy^0 z = 0^{a} 0^{c} 1^{p}$ $a + c \leq p$

$xy^0z \notin E$

$a+b+c = p+1$

$b = \underline{p+1-(a+c)}$

$b \geq 1$

$p+\cancel{1}-(a+c) \geq \cancel{1}$

$p \geq a+c$

Therefore, $E$ is not regular.

$F = \{ ww \mid w \in \{0,1\}^* \}$ show that $F$ is not regular.

what if we pick $s = 0^p 0^p$ as the string s?

you cannot use this string for this language. Since s is pumpable.

$s = 0^p 0^p$, let $a \in \mathbb{N}$ s.t a is a even number.

now $s = 0^{p-a} 0^a 0^p$

$xy^i z = 0^{p-a} 0^{ai} 0^p$

$|xy^i z| = (p-a) + ai + p = 2p + a(i-1)$

$2p + a(i-1)$ is an even number

$0^{2p+a(i-1)} \in F$

This not a good string to pick.

$s = 0^p 1 0^p 1$

Proof: Assume F is regular. Thus, it must follow pumping lemma. Suppose pumping length is p.

Let us pick $s = 0^p 1 0^p 1$, where $s \in F$ and $|s| \geq p$ $(|s| = 2p+2)$

Then, s must be able to be decomposed into $s = xyz$ s.t

1. $xy^iz \in F$
2. $|y| > 0$
3. $|xy| \leq p$

So, by ③, in any decomposition of s, y must only contain 0's.

And By ② y must at least contain one 0.

Then, suppose $xy^iz = \overbrace{0^{p-(a+b)}}^{x} \overbrace{0^a}^{y} \overbrace{0^b 1 0^p 1}^{z}$, where $a > 0$, and $\cancel{p} - (a+b) + a \leq \cancel{p} \Rightarrow -b \leq 0$

$0 \leq b$.

$xy^0z = 0^{p-a-b} 0^b 1 0^p 1 = 0^{p-a} 1 0^p 1$

$xy^0z \notin F$

Hence, $F$ is not regular.