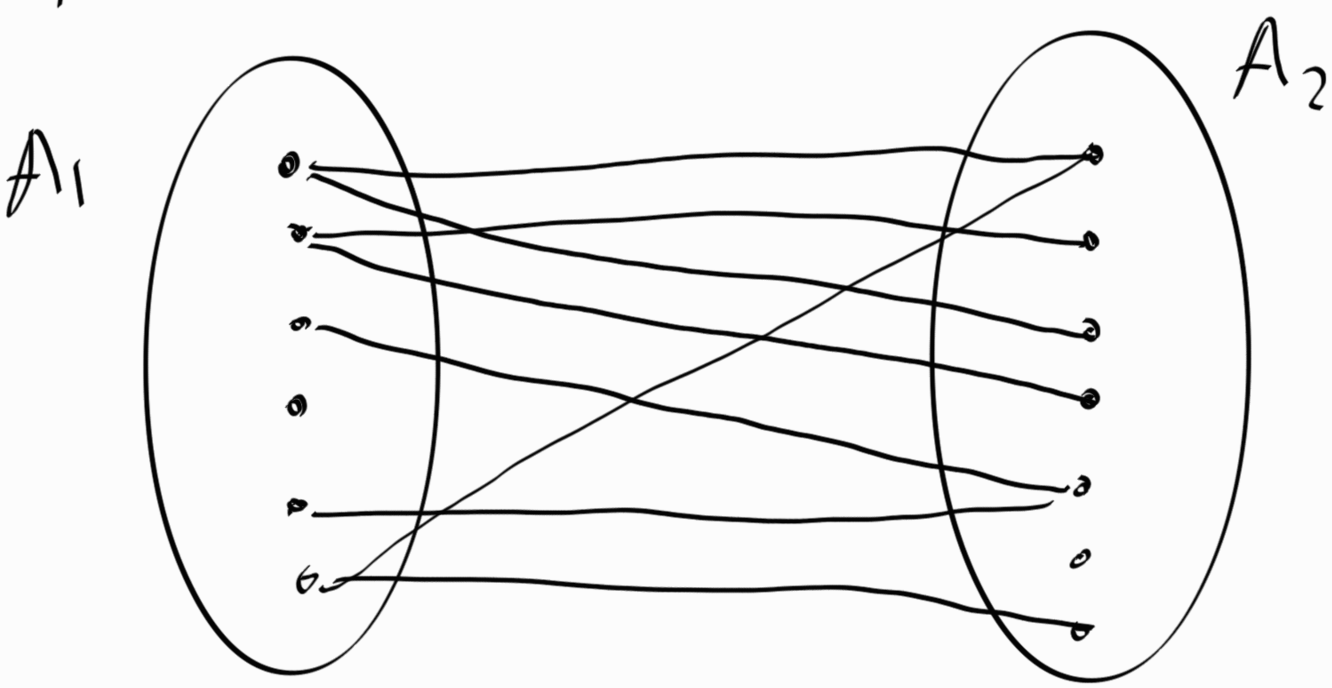
08/28/2025

Relations

K-ary relation R on K sets $A_1, A_2, A_3, \ldots, A_K$ as follows:

$$R \subseteq A_1 \times A_2 \times A_3 \times \cdots \times A_K$$

$K = 2$



In practice, we usually look at $K=2$

Binary relations

special binary relations defined on a single set:

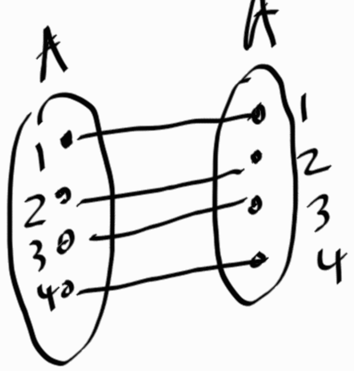
Let A be a set

Let R be a subset of $A \times A$

$R \subseteq A \times A \leftarrow$ special binary relation

Equivalence relation

Given a set A and binary relation $R \subseteq A \times A$ we say R is an equivalence relation.

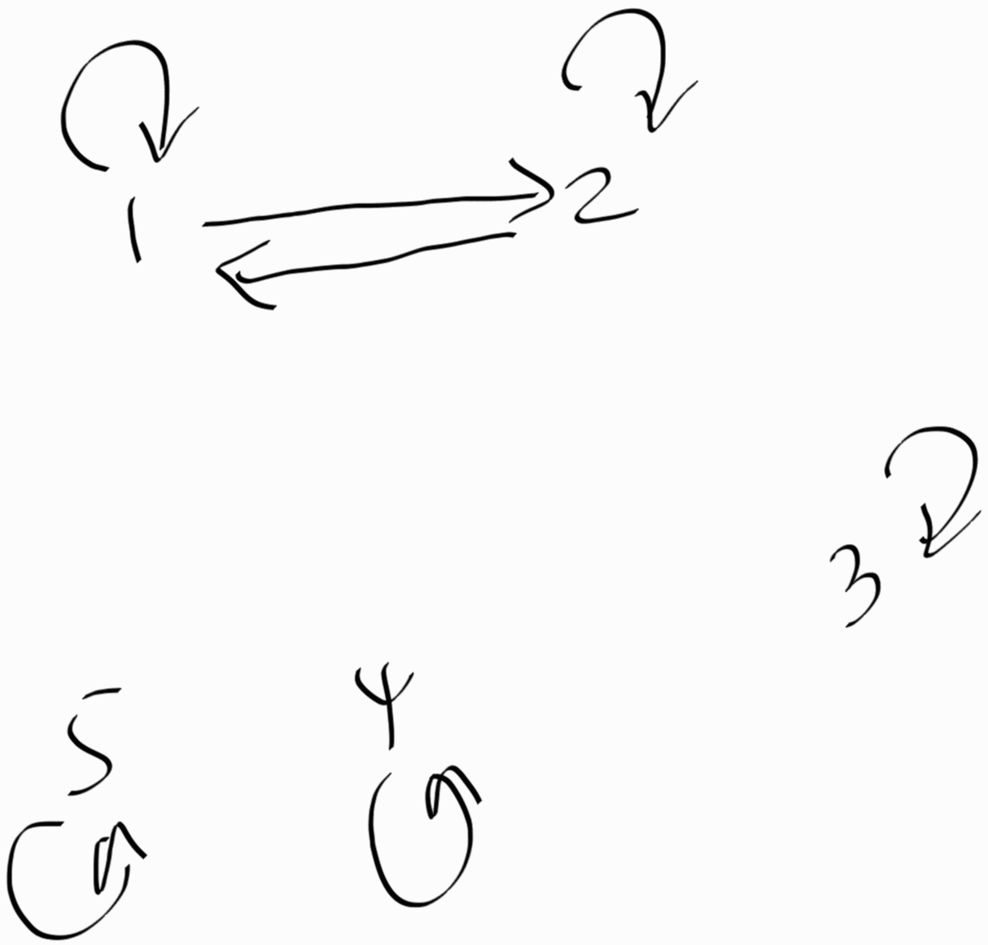


1. Reflexive
   $\forall x \in A: xRx$

2. Symmetric
   $\forall x, y \in A: xRy \Rightarrow yRx$

3. Transitive
   $\forall x, y, z \in A: (xRy) \wedge (yRz) \Rightarrow xRz$

$A = \{1, 2, 3, 4, 5\}$



Graphs

Simple graph

A graph with no self loops or parallel edges is called a simple graph.

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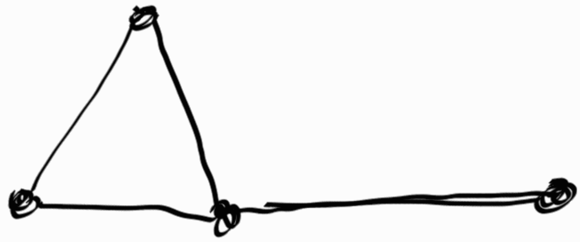
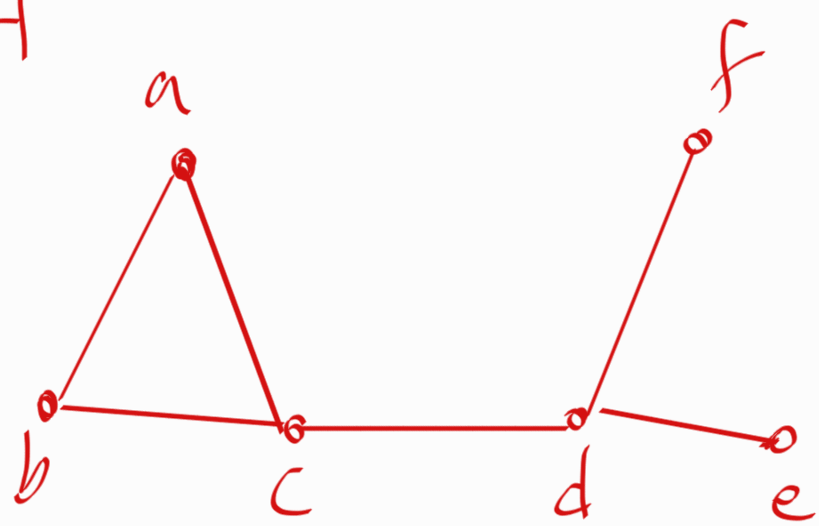


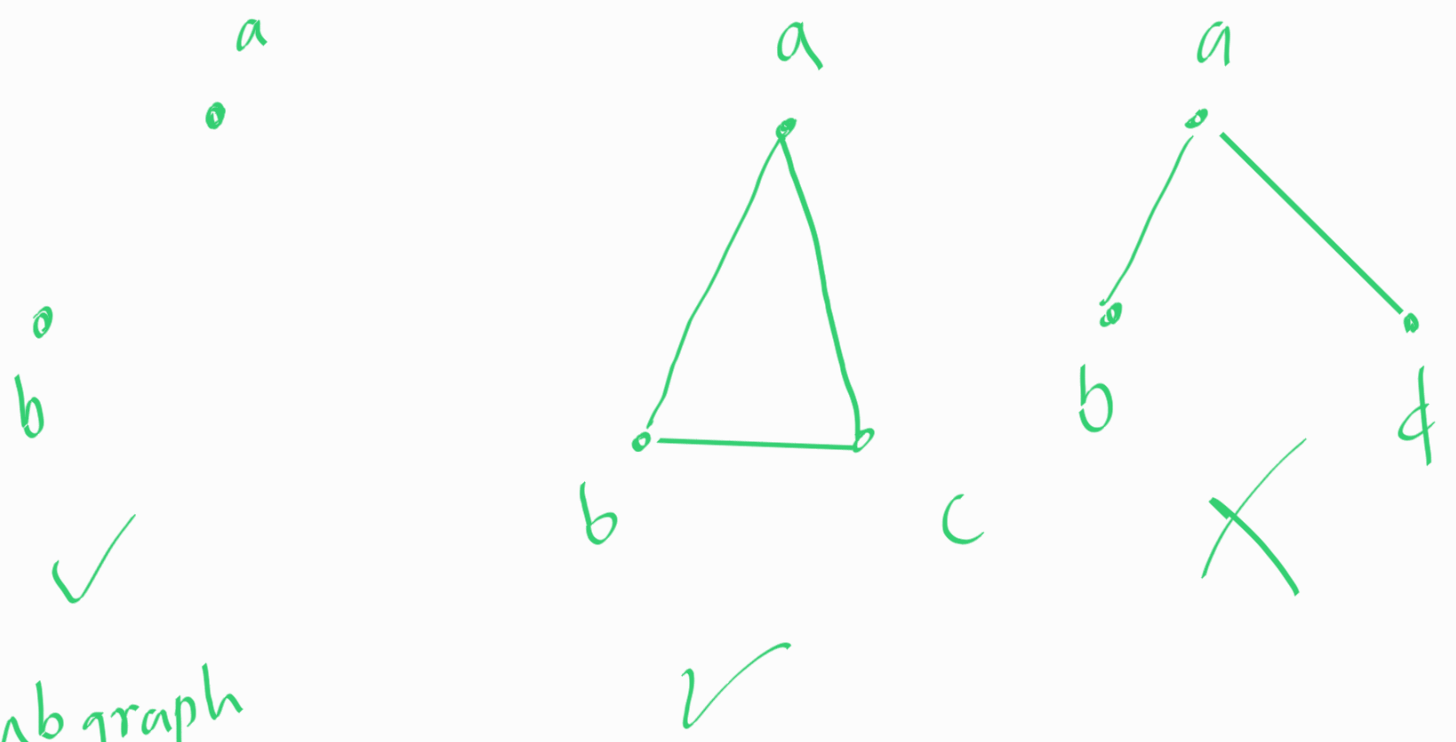
✓

Subgraphs

A graph G is a subgraph of graph H, if the nodes of G are a subset of nodes in H and the edges in G are the edges on H on the corresponding nodes of G in H. (If edges on G is a subset of edges in H)

Ex: H



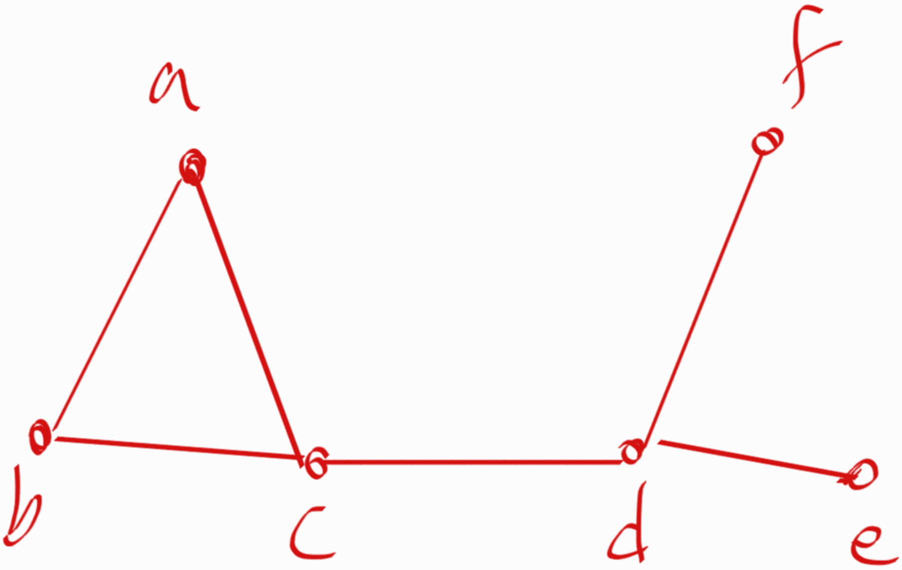


Subgraph

Enumerating all subgraphs could be costly

- # of subgraphs is exponential

Connectedness: A path is a sequence of nodes connected by edges



$\langle a, b, c, d \rangle$ ✓

$\langle a, b, c, e \rangle$ ✗ not a path

A simple path is a path that has no repeated vertices

$\langle a, b, c \rangle$ ✓

$\langle a, b, c, a, b, c, d \rangle$

not a simple path

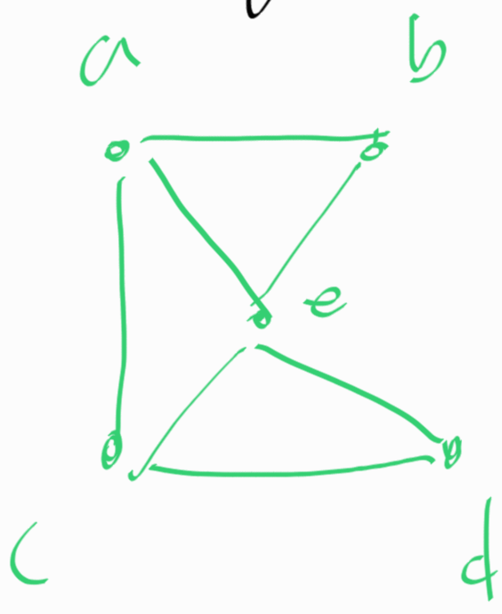
A path is a cycle if it starts and ends at the same vertex.

$\langle a, b, c, a \rangle$

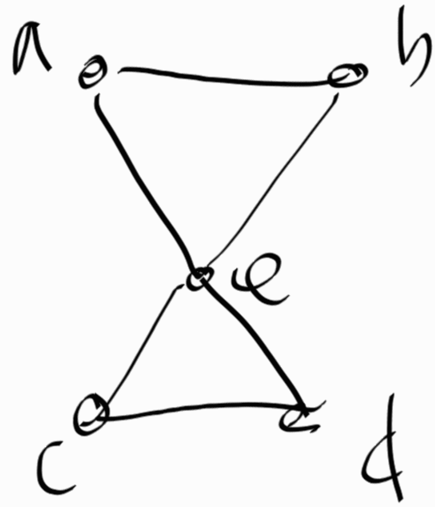
Hamiltonian cycle

A cycle that visits all nodes of the graph exactly once.

Ex1:



Ex2:



Induction

Mathematical induction is a method of proving predicates over natural numbers.

We can prove claims in the form of

$$\left[\forall n \in \mathbb{N} : P(n) \right]$$

1. The claim should be expressed in the form of predicate over natural numbers.

2. Then we prove the predicate is true for the smallest natural number. (base case)

3. Then we prove the inductive step:

$$\forall n \geq 1: P(n) \Rightarrow P(n+1)$$

4. we conclude that the claim is true.

or the predicate is true for all natural numbers.

$P(1)$

$\forall n \geq 1 \ \dot{\circ}\ P(n) \Rightarrow P(n+1)$

When $n=1$

$$P(1) \Rightarrow P(2)$$

when $n=2$

$$P(2) \Rightarrow P(3)$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$$

$$= \frac{1}{6} \cdot n(n+1)(2n+1)$$

Define predicate $P : \mathbb{N} \to \{T, F\}$

$$P(n) = \begin{cases} T, & \text{if } \sum_{i=1}^{n} i^2 = \frac{1}{6} \cdot n(n+1) \cdot (2n+1) \\ F, & \text{otherwise} \end{cases}$$

$$\left[\forall n \in \mathbb{N} : P(n) \right]$$

$$\sum_{i=1}^{1} i^2 = \frac{1}{6} \cdot (1+1)(1)(2+1)$$

$$1^2 = \frac{1}{\cancel{6}} \cdot \cancel{2} \cdot \cancel{3} = 1$$

we want to prove

$$\left[\forall n \in \mathbb{N} : P(n)\right]$$

1. Show the base case is true.

$n = 1$

WTS: $P(1)$ is true.

$P(1)$ basically says $\underbrace{\sum_{i=1}^{1} 1^2}_{L.H.S} = \underbrace{\frac{1}{6} \cdot 1 \times 2 \times 3}_{R.H.S}$

$L.H.S = 1^2 = 1$

$R.H.S = \frac{1}{6} \times 2 \times 3 = 1$

$L.H.S = R.H.S$

$\therefore P(1)$ is true.

2. WTS: $\forall n \geq 1 : P(n) \Rightarrow P(n+1)$

$$\forall n \geq 1: \underbrace{\sum_{i=1}^{n} i^2 = \frac{1}{6} \cdot n \cdot (n+1)(2n+1)}_{P(n)} \Rightarrow P(n+1)$$

$$\underbrace{\sum_{i=1}^{n+1} i^2}_{L.H.S} = \frac{1}{6} \underbrace{(n+1)(n+2)(2\cdot(n+1)+1)}_{R.H.S}$$

Assume inductive hypothesis is true.

$P(n)$ is true.

WTS: $P(n+1)$

$$L.H.S = \sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2$$

$$= \frac{1}{6} \cdot n(n+1) \cdot (2n+1) + (n+1)^2$$

$$= \frac{1}{6}(n+1)\left[2n^2 + n + 6n + 6\right]$$

$$= \frac{1}{6}(n+1)\left[2n^2 + 7n + 6\right]$$

$$= \frac{1}{6}(n+1)(n+2)(2(n+1)+1)$$

$$\left[2n^2 + 4n + 3n + 6\right]$$

$$\left[2n(n+1) + 3(n+1)\right]$$

$$(n+1)(2n+3)$$

$$(n+1)(2n+2+1)$$

$$(n+1)(2(n+1)+1)$$

Alphabet: Any finite set.
Each element is called a letter.

Ex: $\Sigma = \{a, b, c, d, \dots, z\}$

$$\Sigma = \{A, C, G, T\}$$

String: a finite sequence of symbols.

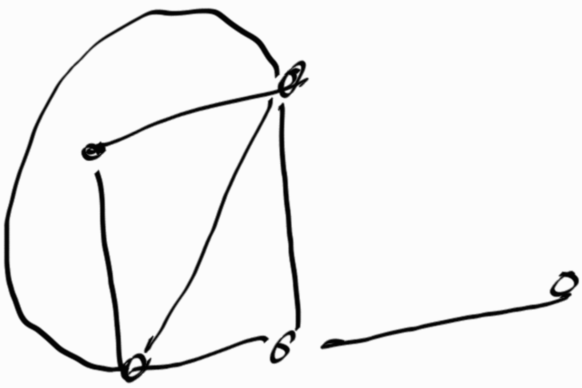
$S = AAGGTTT$ is a string over alphabet $\Sigma = \{A, C, G, T\}$

Empty string is denoted using ε.

epsilon

Planar graphs

A graph whose vertice and edges can be drawn on the plane, with no edges crossing.



K_2 :

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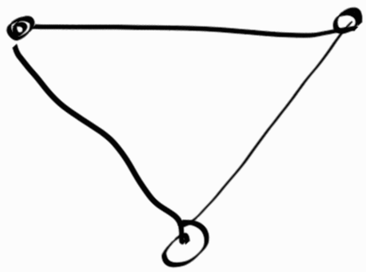
K_3 :



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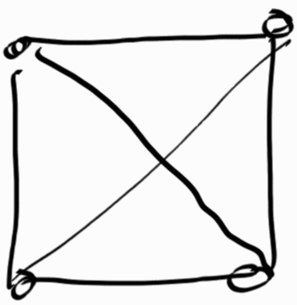
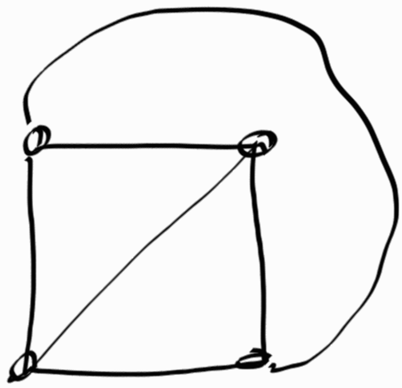
K_4 :



$\Rightarrow$



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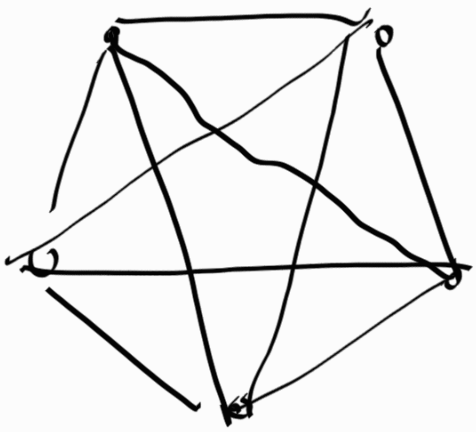
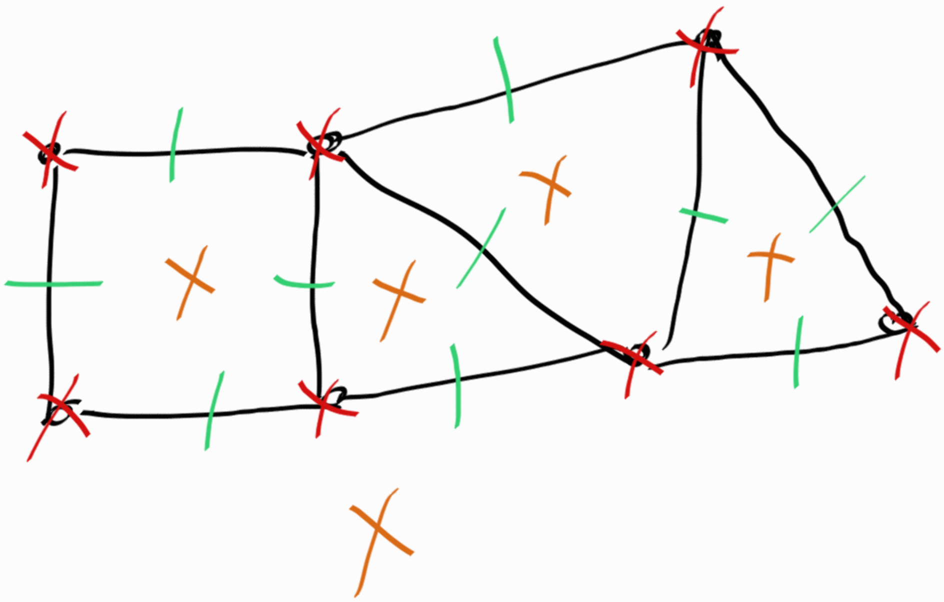
K_5 :



X

Euler's formula (for connected planar graphs)

$$|V| + |F| - |E| = 2$$



$|V| = 7$

$|E| = 10$

$|F| = 5$