Ex: $L = \{a^n b^n c^n \mid n \geq 0\}$ is not a CFL.

Assume L is a CFL.

Then L must follow pumping lemma.
Suppose pumping length is p.

$s = a^p b^p c^p$, $|s| = 3p \geq p$

Then by pumping lemma, s must be able to be decomposed into 5 components where $s = uvxyz$, s.t

1. $uv^ixy^iz \in L$, $\forall i \geq 0$
2. $|vy| > 0$
3. $|vxy| \leq p$

p # of a's   p # b's   p # of c's

$s = \overbrace{aaa \ldots\ldots a}^{p \# \text{ of a's}}\ \overbrace{bbb \ldots\ldots b}^{p \# \text{ b's}}\ \overbrace{ccc \ldots\ldots c}^{p \# \text{ of c's}}$

u   vxy   z

case 1: vxy contains the same set of characters. Then v and y can only be allocated to the same type of character. Therefore, when we pump uv^ixy^iz, the # of one type of characters increases compared to other two types of characters.

$\therefore uv^ixy^iz \notin L$

case 2: vxy contains mix of characters.

subcase 2.1, v and y contain the same character. We handled this in case 1.

ex: $vxy = aaa\ldots a\,bbb\ldots b$
$v = aaa\ldots a$ $y = aaa\ldots a$

subcase 2.2 v and y contain two different character types.

ex: $vxy = bbb\ldots b\,ccc\ldots c$
$v = bbb\ldots b$ $y = ccc\ldots c$

Now, when we pump up, uv^ixy^iz contains different number of type 3 character.

$\therefore uv^ixy^iz \notin L$

Subcase 2.3 either v or y contains a mix of characters.

$$vxy = aaa \cdots a\,bbb \cdots b$$

$$v = aa \cdots ab \cdots b \qquad y = bb \cdots b$$

Now, when we pump up, uv^ixy^iz contains different order of a, b, and c's.

$\therefore uv^ixy^iz \notin L$

Now, we considered all the possible decompositions for s, and none of them are pumpable.

$\therefore L$ is not a CFL.

$C = \{ ww \mid w \in \{0,1\}^* \}$ is not a CFL.

Assume C is a CFL.
Then C must follow pumping lemma.
Suppose pumping length is p.

Let $s = 0^p 1 0^p 1$ ← not good because it is pumpable

$s = \underbrace{0000 \ldots 000}_{p \# \text{ of } 0\text{'s}} 1 \underbrace{0000 \ldots 0}_{p \# \text{ of } 0\text{'s}} 1$

$s = \underbrace{0000\ldots}_{u}\underbrace{000}_{v}\underbrace{1}_{x}\underbrace{000}_{y}\underbrace{0\ldots01}_{z}$

$uv^ixy^iz = 0^{p-3} 0^{3i} 1 0^{3i} 0^{p-3} 1$

Pick $s = 0^p 1^p 0^p 1^p$

$0 \ldots 010 \ldots 0 \mid 0 \ldots 010 \ldots 0$

$s = 000 \ldots \underbrace{\ldots 0111}_{vxy} \ldots 1 \mid 000 \ldots 0111 \ldots 1$

$|uv^2xy^2z| = 4p + |vy|$

middle point would be $2p + \frac{|vy|}{2}$

This is pointing to a 1.

Since $|s| \geq p$, s must be able to be decomposed into $s = uvxyz$ s.t

1. $uv^ixy^iz \in C$, $\forall i \geq 0$
2. $|vy| > 0$
3. $|vxy| \leq p$

case 1: vxy does not straddle the mid point of s.

In this case $uv^2xy^2z \notin C$.

The reason is that, in uv^2xy^2z, the first character of string 1 and first character of string 2 does not match.

case 2: If vxy does not cross the mid point, consider $uv^0xy^0z = uxz = 0^p1^i0^j1^p$, where $i, j \neq p$.

$\therefore uv^0xy^0z \notin C$.

$\therefore C$ is not a CFL.

Prove $L = \{a^n b^j \mid n = j^2\}$ is not a CFL.