08/26/2025

$$S = \{x \in D : \text{rule about } x\}$$

Domain of discourse

Ex: $\mathbb{Q} = \left\{a : (p, q \in \mathbb{Z}) \wedge (q \neq 0) \wedge \left(a = \frac{p}{q}\right)\right\}$

$$\text{EVENS} = \{e \in \mathbb{Z} \mid (e \bmod 2 = 0)$$

# Set operations

1. Union ($\cup$)
2. Intersection ($\cap$)
3. Complement ($\overline{\phantom{A}}$)
4. Exclusive or ($\oplus$)

$A = \{1, 3, 5, 6\}$ $B = \{2, 4, 6, 8\}$

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$

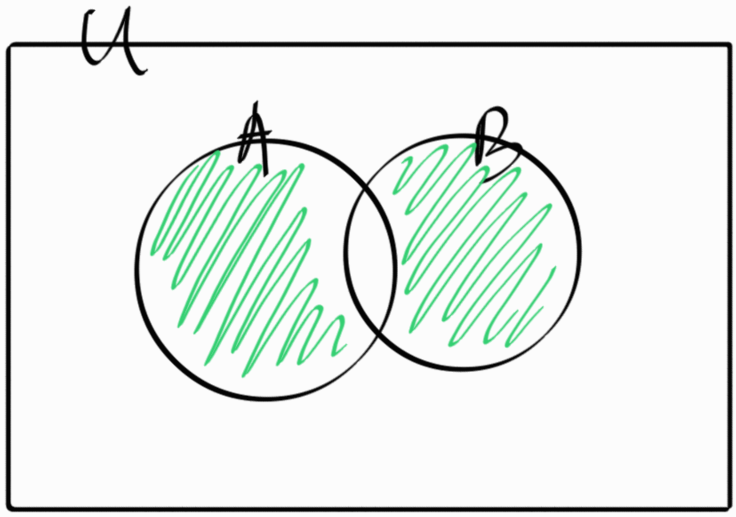
$A \cap B = \{6\}$

$\overline{A} = \{2, 4, 7, 8\}$

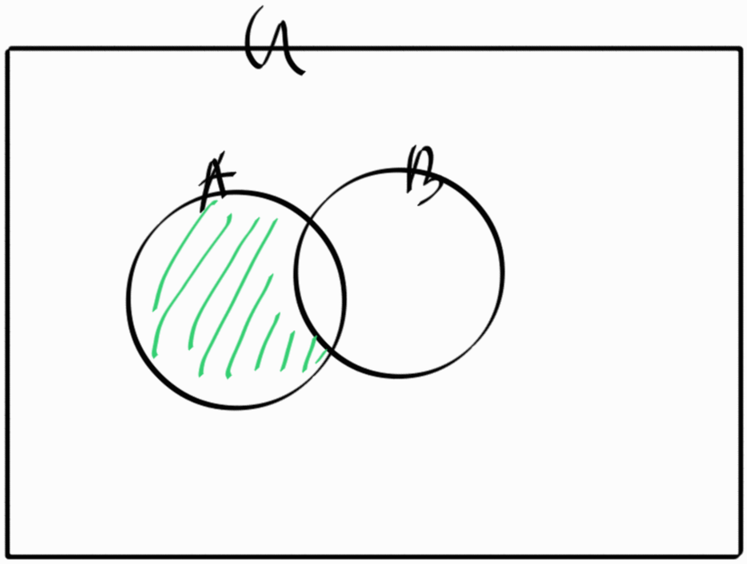
Note $A \cup \bar{A} = \mathcal{U}$

$A - B = A \setminus B = \{1, 3, 5\}$

$A \oplus B = \{1, 3, 5, 2, 4, 8\}$




$A \oplus B$



$$A - B = A - (A \cap B)$$

$$A \oplus B = (A \cup B) - (A \cap B)$$

# Sequences and Tuples.

- A sequence is a list of ordered elements.

$$\langle 1,5,6,7\rangle = (1,5,6,7)$$

$$\langle 1,5,6,7\rangle \neq \langle 1,5,7,6\rangle$$

$$\langle 1,1,5,6,7\rangle \ \checkmark$$

- finite sequences are called tuples
- 2-tuple is called a pair.

$$(a,b) \text{ or } \langle a,b\rangle$$

Ex: A graph $G=(V,E)$

- $V$ is a set of vertices
- $E$ is a set of edges.

Cartesian product of two sets

Given sets $A, B$, the cartesian product of $A$ and $B$ is denoted by $A \times B$

$$A \times B = \{ \langle a, b \rangle : a \in A \text{ and } b \in B \}$$

contains all ordered pairs where first element comes from $A$ and second element comes from $B$.

$A = \{1, 2, 3\}$ $\quad B = \{ \star, \Delta \}$

$$A \times B = \{ (1, \star), (1, \Delta), (2, \star), (2, \Delta), (3, \star), (3, \Delta) \}$$

$A \times B \neq B \times A$ when $A \neq B$

# Theorems and proofs

Theorem - A true mathematical statement.

Ex: If $x, y$ is rational, then $x \cdot y$ is rational.

- A proof is a convincing logical arguement that the statement is true.

- Given a statement, we can prove it using a proof.

- Given a universal statement, we can disprove it using counter example.

Ex: All swans are white

there exist at least one green swan

Ex! If $x \in \mathbb{N} \wedge x \bmod 2 = 0$, then $x$ is a power of 2.

Let $x = 6$

$x \in \mathbb{N} \wedge 6 \bmod 2 = 0$ but 6 is not a power of 2.

disproof by counter example

$p \Rightarrow q$ means whenever $p$ is true, $q$ is true

$p \Leftrightarrow q$

If moon is made out of cheese, then $2+2=5$

Proof methods:

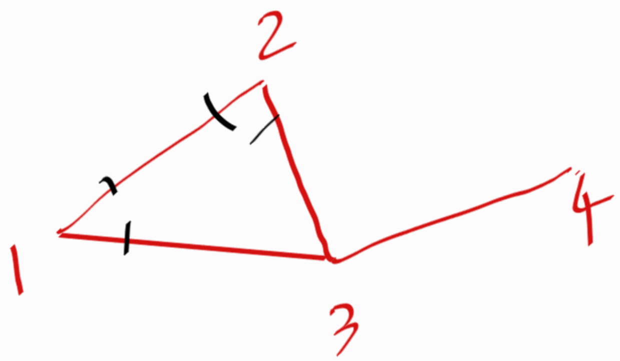
1. Direct proofs
2. proof by contradiction
3. proof by induction
4. proof by construction

   ($\neq$ by example)

## Direct proof

Given a graph $G=(V,E)$, let $\deg(v)$ be the # of edges incident to $v$.

$$\sum_{v\in V} \deg(v) \text{ is even}$$




$V=\{1,2,3,4\}$

$E=\{\{1,2\},\{1,3\},\{2,3\},\{3,4\}\}$

$\deg(1)=2 \quad \deg(2)=2$

$\deg(3)=3 \quad \deg(4)=1$

$$\sum_{v\in V} \deg(v) = 2+2+3+1 = 8$$

Proof: When counting degrees, each edge $\{u,v\}$ is counted twice and once for $\deg(v)$

$\therefore \sum_{v \in V} \deg(v) = 2 \cdot |E|$ which is even

Ex2: Given sets A and B,

$$\underbrace{\overline{A \cup B}}_{X} = \underbrace{\bar{A} \cap \overline{B}}_{Y}$$

$\underset{①}{X \subseteq Y} \wedge \underset{②}{Y \subseteq X}$

①: WTS $\overline{A \cup B} \subseteq \bar{A} \cap \overline{B}$

If $a \in \overline{A \cup B}$, then $x \in \bar{A} \cap \overline{B}$

Let $x \in \overline{A \cup B}$

$$x \notin A \cup B$$

$$(x \notin A) \wedge (x \notin B)$$

$$(x \in \overline{A}) \wedge (x \in \overline{B})$$

$$x \in \overline{A} \cap \overline{B}$$

# Relations and functions

A function is a mapping between two sets, with set of rules.

formally function $f$ from set $A$ to $B$ is written as

$f: A \longrightarrow B$ with following conditions.

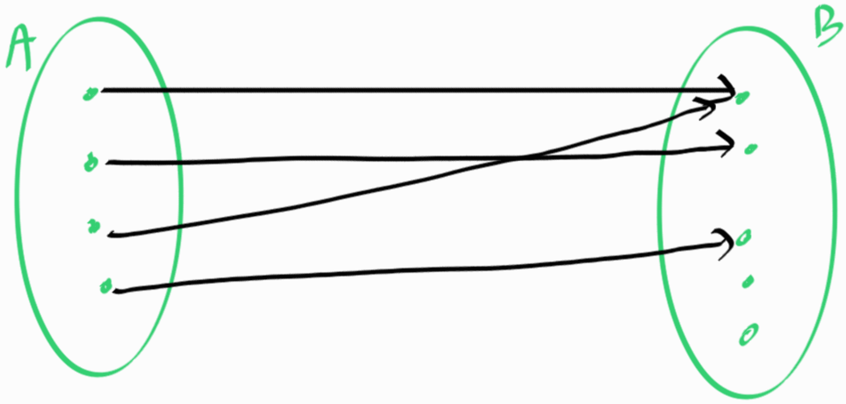
1. $\forall a \in A$ : $f(a)$ must be defined
2. $\forall a \in A$ : $f(a)$ should produce unique element.
3. $\forall a \in A$ : $f(a) \in B$

A - Domain
B - Co-Domain

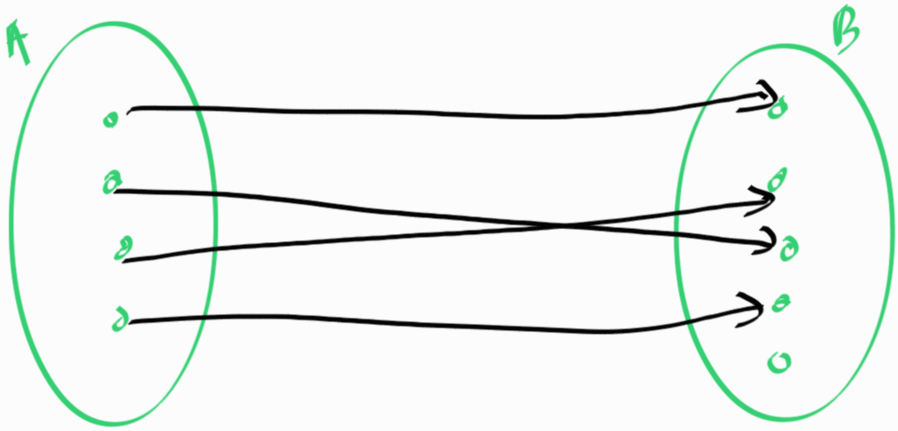
R - Range : range is the set of all outputs.

# Injective functions (one-to-one)

1. $\forall a_1, a_2 \in A : a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$



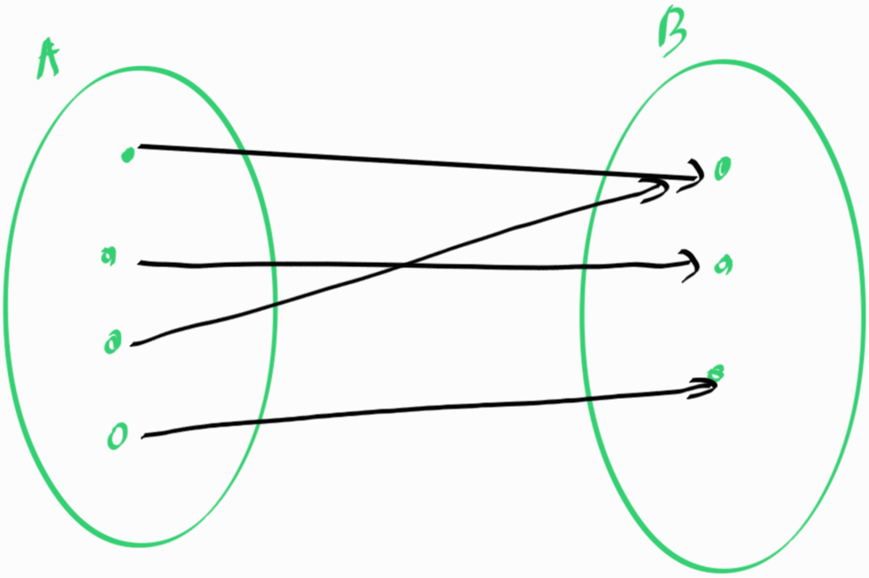

Not a one-to-one function




one-to-one

# Surjective function (onto)

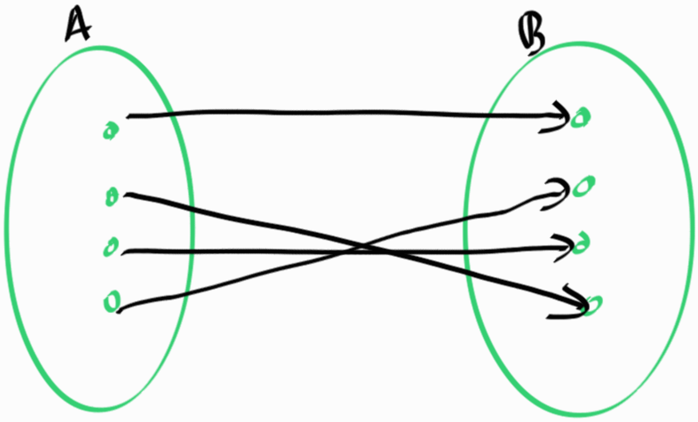
$$\left[\forall b \in B : \left[\exists a \in A : f(a) = b\right]\right]$$




onto function

## Bijection

Any function that is injective and surjective is called a bijection




Bijection

## Predicates

Predicates / property is a function whose range is $\{True, False\}$

Ex: $isEven : \mathbb{Z} \longrightarrow \{T, F\}$

$$isEven(x) = \begin{cases} True, & \text{if } x \text{ is even} \\ False, & \text{if } x \text{ is odd} \end{cases}$$