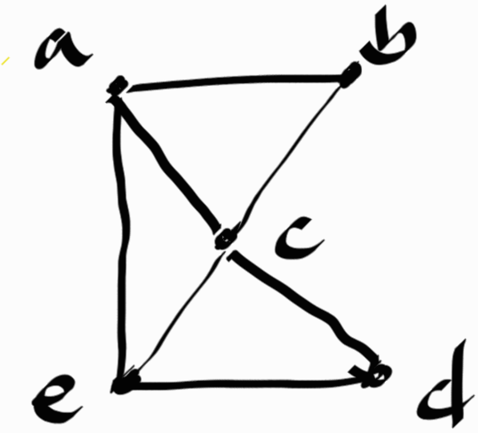
Example:

HC $\leq_p$ TSP (on complete weighted graphs)



G

$\langle a, b, c, d, e, a \rangle$ HC

$\downarrow$

G'

Given an 'input graph $G=(V,E)$, we construct graph $G'=(V',E',\omega)$, which is a complete weighted graph as follows.

- we keep the original vertices.
- we add new edges between vertices which we did not have edges.
- we give weights to edges as follows.

  $\forall e \in E'$

  If $e \in E$, $w(e)=1$

  If $e \notin E$, $w(e)=+\infty$
- This transformation takes $O(n^2)$

- Claim! There is a HC in G iff there is a TSP with weight n in G'

only if $\Rightarrow$

If G has a HC, then use the edges in HC to form a TSP in G'.
As each edge in TSP has weight 1.
The total weight is n.

If part $\Leftarrow$

If G' has a TSP of size n,
we know that all the edges with weight two cannot be part of it.
Therefore, all edges in TSP must have weight 1, which in turn means are in G.
Then HC gives us a HC in G.

SET-PARTITION $\le_p$ RECTANGLE PACKING

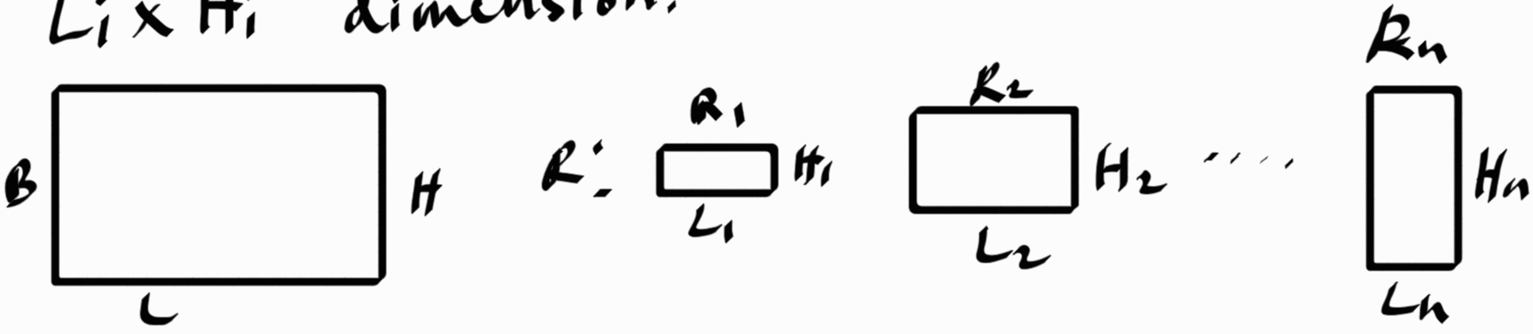
set partition problem is given a set of integers $S = \{a_1, a_2, \ldots, a_n\}$, find whether there exists a subset $S' \subseteq S$ s.t $\sum_{x \in S'} x = \sum_{x \in S \setminus S'} x$

$S = \{8, 1, 1, 9, 11, 2, 3, 5\}$

$\underbrace{\{8, 1, 11\}}_{20}$ $\underbrace{\{1, 9, 2, 3, 5\}}_{20}$

Rectangle packing

Given a container B of size $L \times H$. You are given set of rectangles R, $R = \{R_1, R_2, \ldots, R_n\}$, where R_i is of $L_i \times H_i$ dimension.



Reduction:

Let the input to the set partition be $S = \{a_1, a_2, a_3, \ldots, a_n\}$

$\forall a_i$, construct R_i with length a_i and height ε

- B is constructed to have length $\frac{\sum a_i}{2}$ and hieght 2ε

* There is a solution for set partition instance iff all the R_i's can be packed into B

$S = \{8, 1, 1, 9, 11, 2, 3, 5\}$

reduction takes $O(n)$

Recap

What is NP-complete?

A problem B is NP-complete

1. $B \in NP$

2. Every problem in NP can be polynomially reducible to B.

Only the first NP-complete problem (SAT) is proved using the above definition.

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we say B is NP-complete

1. $B \in NP$

2. $C \leq_p B$, where C is NP-complete

Suppose A is any NP problem

$A \leq_p C \leq_p B$

What is NP-hard?

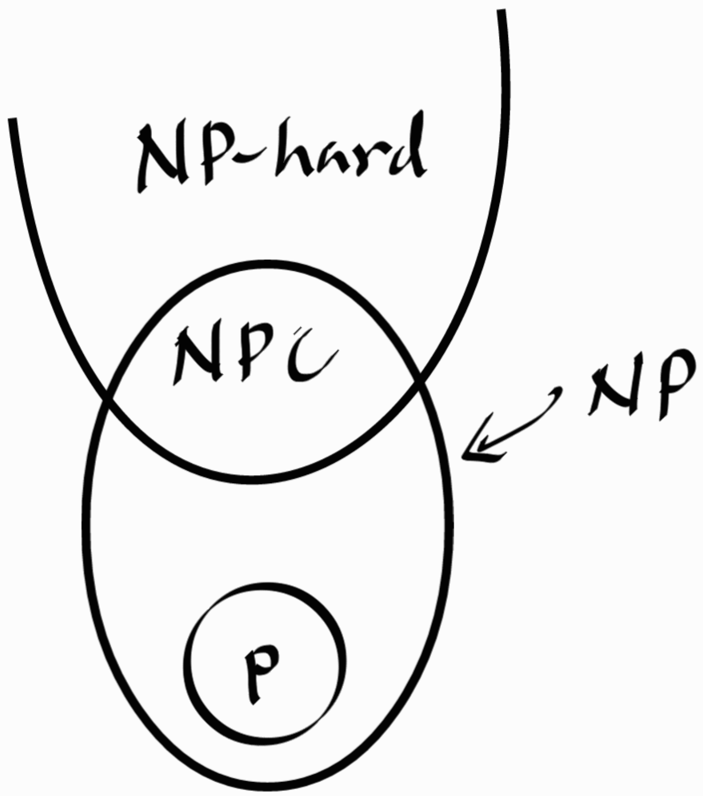
Suppose C is NPC.

Also suppose $C \leq_p A$.

If we donot show that A is in NP, then we cannot say A is NPC

But we can say A is NP-hard

usually optimization problems are NP-hard



SAT (satisfiability problem) is NP-complete.

Input $\phi = (x_1 \vee \boxed{x_2} \vee \bar{x}_3 \vee \boxed{x_4}) \wedge \overbrace{(x_2 \vee \bar{x}_3 \vee x_4 \vee \bar{x}_1)}^{\text{clauses}}$

$\wedge\ (\underbrace{x_7 \vee \bar{x}_1}_{\text{clauses}}) \ldots\ldots$

Q: Is there a truth assignment that satisfies the ϕ

3-SAT: A special case of SAT problem, where every clause has 3 literals.

Ex: $\phi = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (x_4 \vee \bar{x}_5 \vee x_6) \wedge \ldots$

$\ldots \wedge (x_2 \vee \bar{x}_3 \vee \bar{x}_5)$

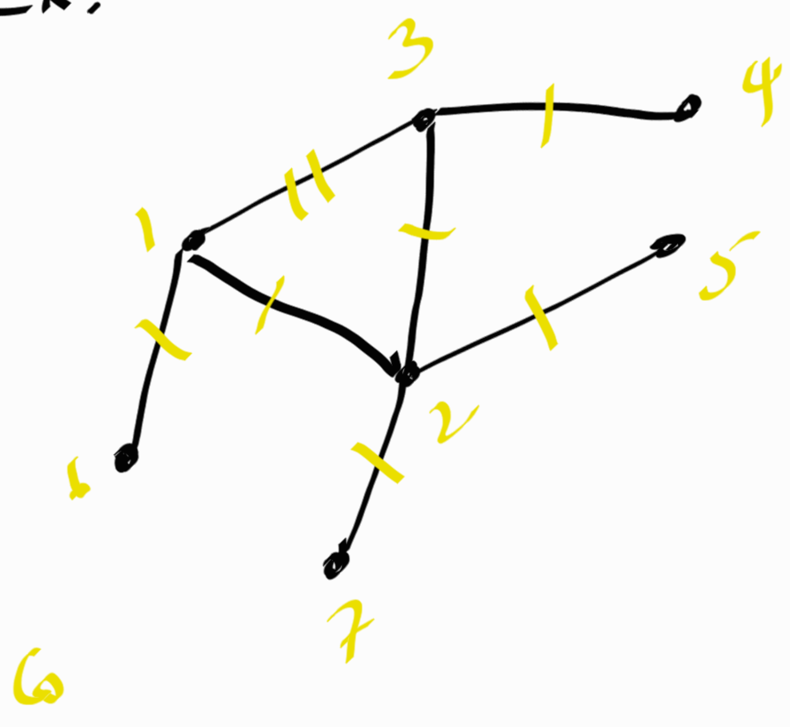
Let's try to learn how to show a problem is NP-complete.

We want to show vertex-cover problem is NP-complete.

What is vertex-cover?

Given a graph $G = (V, E)$, find the minimum number of vertices that covers all edges.

Ex:



This is an optimization problem. Let's convert this to a decision problem.

Given Graph $G = (V, E)$ and $k \in \mathbb{N}$, is there a vertex cover of G at most k?

Think about the definition of NP-completeness.

1. The problem is in NP.
2. Every problem in NP is polynomially reducible to the problem.

Proof: Let's prove VC ∈ NP.

Sketch:

1. VC ∈ NP
2. $3SAT \leq_p VC$

Let's create an NTM that decides VC.

N = on input ⟨G, k⟩

1. Nondeterministically select k nodes from G.
2. Delete the k nodes and edges incident to it. If there are edges left, reject; otherwise accept.

Clearly deletion of the nodes and edges can be done in $O(n^2)$

Second we need to show any problem is NP can be polynomially reducible to VC.

$$3SAT \leq_p VC$$

Proof! suppose ϕ has m variables and l clauses.

Ex! $\phi = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3)$

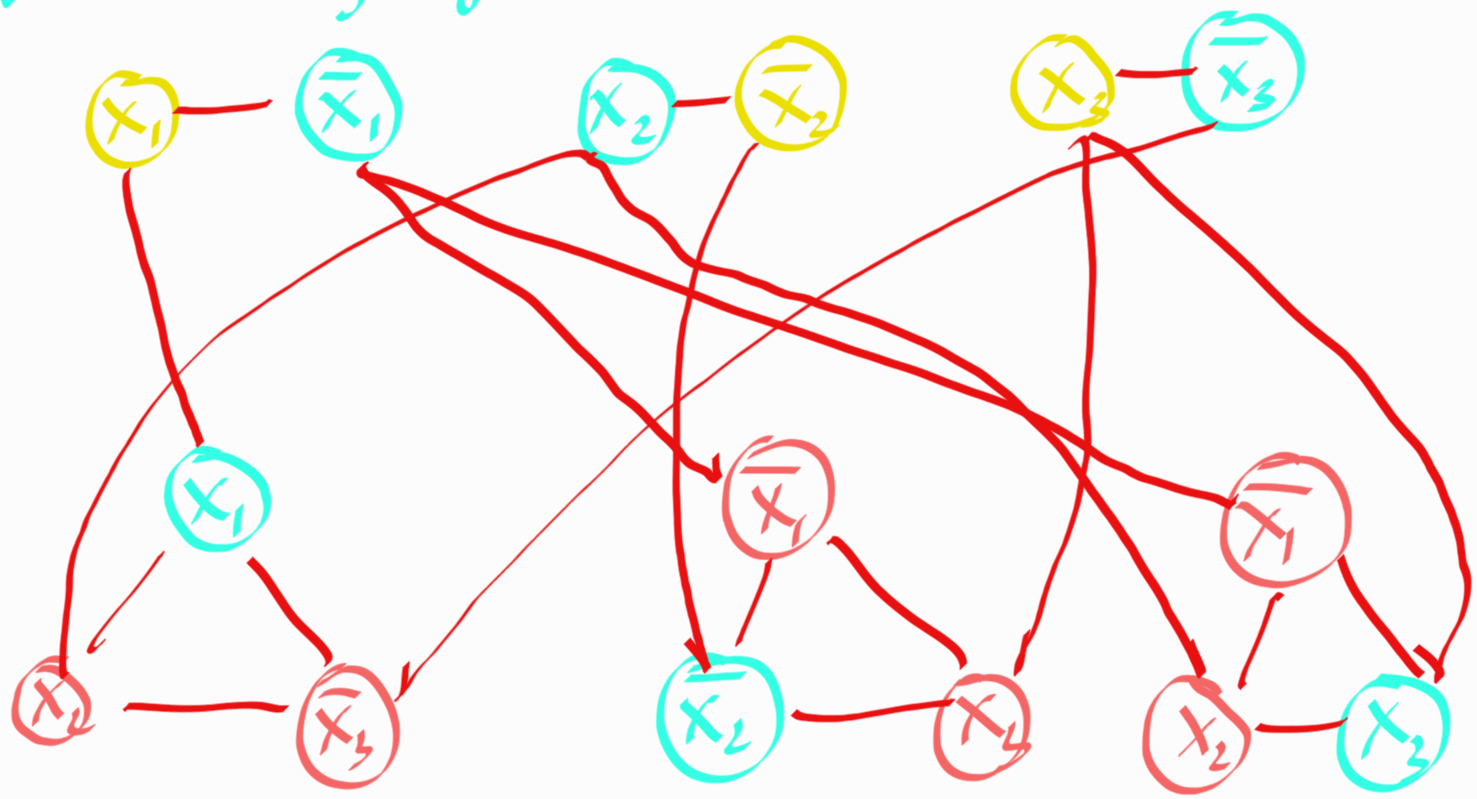
m-variables

l- clauses,

Truth assignment

$x_1 = T \quad x_2 = F \quad x_3 = T$

variable gadgets



Clause gadgets.

$x_1 = T \quad x_2 = F \quad x_3 = T$

The size of the graph G?

$2m+3l$ vertices, and $O(m+l)$ edges.

- $\underline{\phi}$ has a truth assignment iff G has a VC of size $k = m+2l$ nodes.

(i) $\Rightarrow$ If ϕ has a truth assignment G has a VC of size $m+2l$

(ii) $\Leftarrow$ If G has a VC of size $m+2l$, then ϕ has a truth assignment.

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Example 03

CLIQUE is NP-complete.

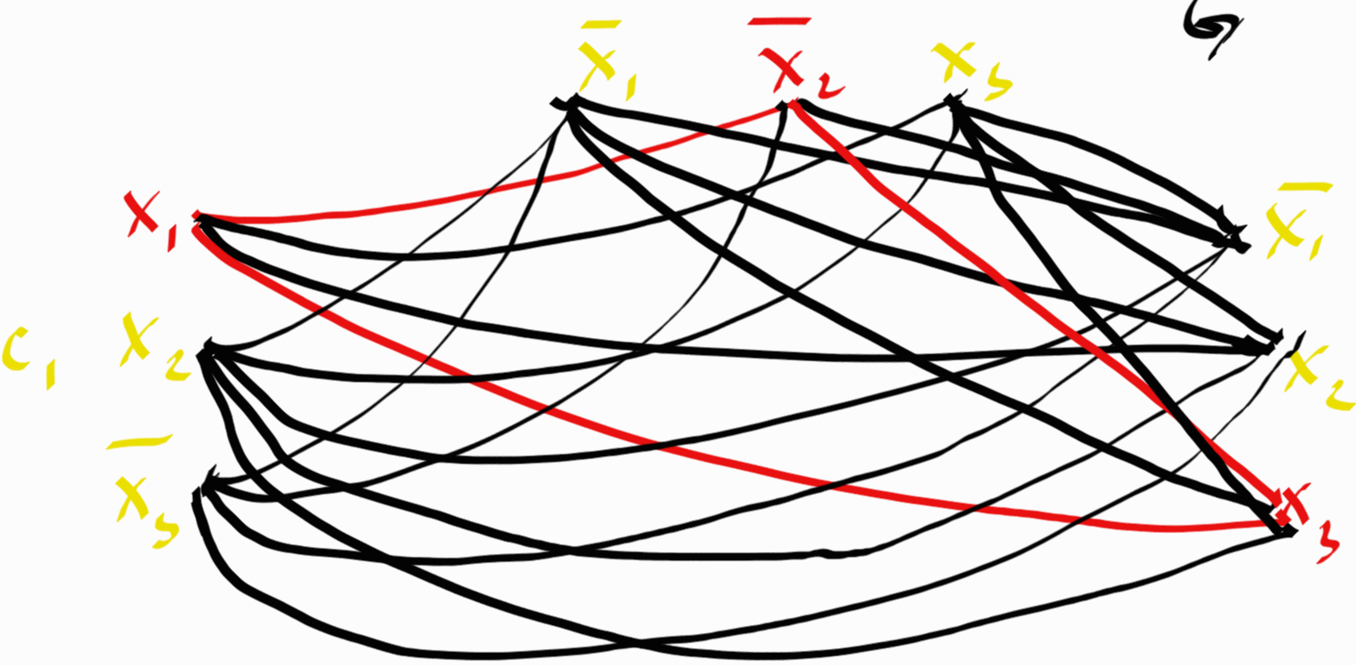
$$3SAT \leq_p CLIQUE$$

Proof sketch:

1. Show $CLIQUE \in NP$
2. $3SAT \leq_p CLIQUE$

$$\phi = \overset{C_1}{(x_1 \vee x_2 \vee \overline{x_3})} \wedge \overset{C_2}{(\overline{x_1} \vee \overline{x_2} \vee x_3)} \wedge$$
$$C_3 \; (\overline{x_1} \vee x_2 \vee x_3)$$

G



m = variables

$O(m^2)$

$x_1 = T$ $x_2 = F$

$x_3 = T$

ϕ has a truth assignment iff G has a k-clique.

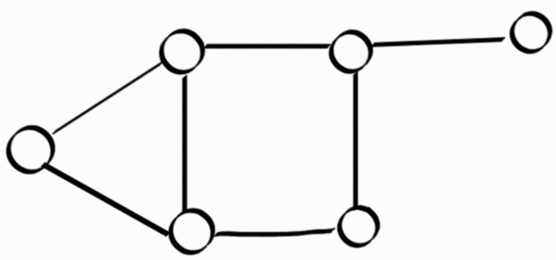
P(302-303)

$DS = \{ \langle G, k \rangle \mid G$ is a graph and G has a dominating set of size at least $k \}$

Dominating Set $D \subseteq V$ of $G = (V, E)$ is a set D s.t

$$\forall v \in V : (v \in D) \vee (\exists (u,v) \in E : u \in D)$$

every node is either in the dominating set or there exist a node in the dominating set that is adjacent to it.



Problem 04.

$VC \leq_p DS$

Part 1.

show $DS \in NP$ (easy)

guess k nodes and check whether these k nodes cover all other nodes.

Part 2:

(input to VC)

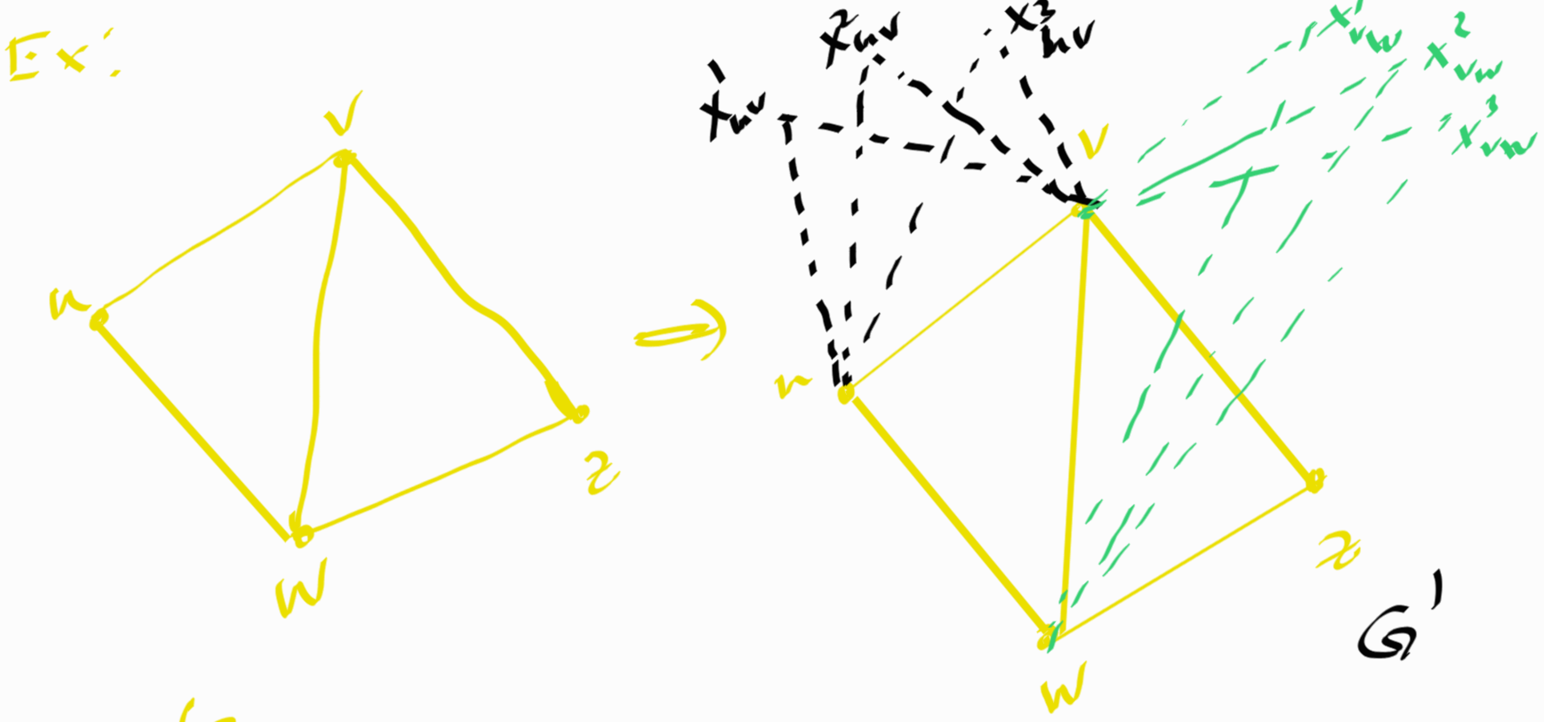
Given a graph $G=(V,E)$, we do the following to create the graph $G'=(V',E')$.

input to DS

$\forall (u,v) \in E$, we create few "lifted" new nodes, say $x^1_{uv}, x^2_{uv}, x^3_{uv}$.

* creating one lifted node for each edge is enough. But creating more makes it easier to prove the iff claim.

Ex:



only the lifted nodes for uv and vw edges are shown here.

Claim: G has a VC of size k iff G' has a DS of size k.

- for the above example {v, w} is the OPT VC for G and {v, w} is the OPT DS for G'.

- Some arguements for ⇒ and ⇐ direction needs to be made.

SUBSET-SUM is NP-complete

Given $S = \{a_1, a_2, \ldots\ldots, a_n\}$ and t, is there a subset $S' \subseteq S$, s.t $\sum_{x \in S'} x = t$?

Part 1: SUBSET-SUM $\in$ NP $\leftarrow$ easy.

Part 2: 3SAT $\leq_p$ SUBSET-SUM

Suppose you have

$(x_1 \vee \bar{x}_2 \vee x_5) \wedge (x_2 \vee x_3 \vee \ldots) \wedge \ldots \wedge (\bar{x}_3 \vee \ldots \vee \ldots)$

Say there are l variables and k clauses.

we need to create a SUBSET-SUM instance.

what are the integers in set S?

what is the target.

1. For every x_i in ϕ create two integers y_i and z_i

2. For every clause c_j in ϕ create two variables g_j and h_j

	1	2	3	4	.....	l	c_1	c_2	...	c_k
y_1	1	0	0	0	.....	0	1	0	....	0
z_1	1	0	0	0	....	0	0	0	....	0
y_2	0	1	0	0	...	0	0	1	....	0
z_2	0	1	0	0	...	0	1	0	....	0
y_3			1	0	...	0	1	1	....	0
z_3			1	0	...	0	0	0	.....	1
$\vdots$					$\ddots$	$\vdots$				
y_l						1	0	0	...	0
z_l						1	0	0	...	0
g_1							1	0	....	0
h_1							1	0	....	0
g_2								1	.....	0
h_2								1	.....	0

$\vdots$		$\ddots$ $\vdots$
g_k		$\cdots$ 1
h_k		$\cdots$ 1
t	1 1 1 1 $\cdots$ 1	3 3 $\cdots$ 3

$t = 111 \ldots\ldots 1\,33 \ldots\ldots 3$

ϕ has a truth assignment iff SUBSETSUM has a solution in which sum $= 1111 \ldots\ldots 1\,333 \ldots\ldots 3$

table size $(k+\ell)^2$

$O(n^2)$