Ex: $L = \{a^n b^n c^n \mid n \geq 0\}$ is not a CFL.

Assume L is a CFL.

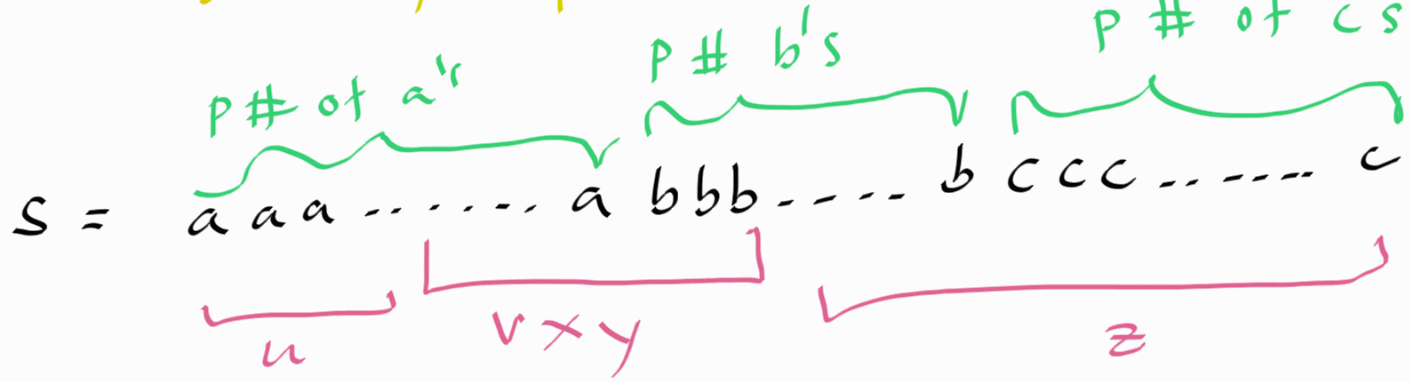
Then L must follow pumping lemma.

Suppose pumping length is p.

$s = a^p b^p c^p$, $|s| = 3p \geq p$

Then by pumping lemma, s must be able to be decomposed into 5 components where $s = uvxyz$, s.t

1. $uv^i x y^i z \in L$, $\forall i \geq 0$
2. $|vy| > 0$
3. $|vxy| \leq p$



case 1: vxy contains the same set of characters. Then v and y can only be allocated to the same type of character. Therefore, when we pump uv^ixy^iz, the # of one type of characters increases compared to other two types of characters.

$\therefore uv^ixy^iz \notin L$

case 2: vxy contains mix of characters.

subcase 2.1, v and y contain the same character. We handled this in case 1.

ex: $vxy = aaa \ldots a\, bbb \ldots b$

$v = aaa \ldots a \quad y = aaa \ldots a$

subcase 2.2 v and y contain two different character types.

ex: $vxy = bbb \ldots b\, ccc \ldots c$

$v = bbb \ldots b \quad y = ccc \ldots c$

Now, when we pump up, uv^ixy^iz contains different number of type 3 character.

$\therefore uv^ixy^iz \notin L$

Subcase 2.3 either v or y contains a mix of characters.

$vxy = aaa \cdots a\, bbb \cdots b$

$v = aa \cdots a b \cdots b \qquad y = bb \cdots b$

Now, when we pump up, $uv^i xy^i z$ contains different order of a, b, and c's.

$\therefore uv^i xy^i z \notin L$

Now, we considered all the possible decompositions for s, and none of them are pumpable.

$\therefore L$ is not a CFL.

$C = \{ ww \mid w \in \{0,1\}^* \}$ is not a CFL.

Assume C is a CFL.
Then C must follow pumping lemma.
Suppose pumping length is p.

Let $s = 0^p 1 0^p 1$ ← not good because it is pumpable

$$s = \overbrace{0000\ldots000}^{p \text{ \# of 0's}} 1 \overbrace{0000\ldots\ldots 0}^{p \text{ \# of 0's}} 1$$

u v x y z

$$uv^i x y^i z = 0^{p-3} 0^{3i} 1 0^{3i} 0^{p-3} 1$$

0.. 010...0|0...010...0

Pick $s = 0^p 1^p 0^p 1^p$

$$s = \underbrace{000\ldots\ldots 0111}_{vxy}\ldots 1 \mid 000\ldots 0111\ldots 1$$

$$|uv^2xy^2z| = 4p + |vy|$$

middle point would be $2p + \frac{|vy|}{2}$

This is pointing to a 1.

Since $|s| \geq p$, s must be able to be decomposed into $s = uvxyz$ s.t

1. $uv^ixy^iz \in C$, $\forall i \geq 0$
2. $|vy| > 0$
3. $|vxy| \leq p$

case 1: vxy does not straddle the mid point of s.

In this case $uv^2xy^2z \notin C$.

The reason is that, in uv^2xy^2z, the first character of string 1 and first character of string 2 does not match.

case 2: If vxy cross mid point, consider $uv^0xy^0z = uxz = 0^p1^i0^j1^p$, where $i, j \neq p$.

$\therefore uv^0xy^0z \notin C$.

$\therefore C$ is not a CFL.

Prove $L = \{a^n b^j \mid n = j^2\}$ is not a CFL.

Assume L is a CFL.

Then, let p be the pumping length.

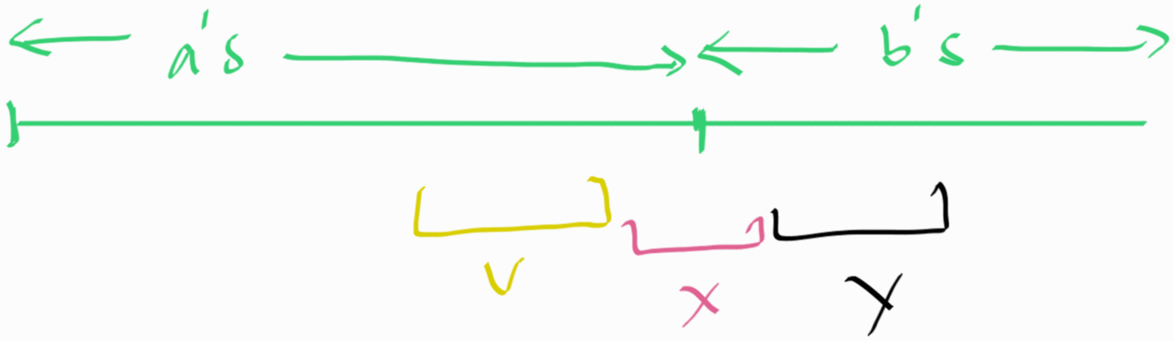
Let $s = a^{p^2} b^p$

$$S = \underbrace{aaaa \ldots\ldots a}_{p^2 \# a's} \underbrace{bbbb \ldots\ldots b}_{p \# b's}$$

Case 1: Either v or y crosses the breakpoint. Then, when pumping up, we will have out of order a's and b's.

$\therefore uv^ixy^iz \notin L$

case 2: Both v and y does not cross the break point.



k_1 # of a's to v

k_2 # of b's to y

Consider uv^ixy^iz.

$|uv^ixy^iz| = p^2+p+(i-1)k_1+(i-1)k_2$

subcase 2.1 $(k_1=0) \wedge (k_2=0)$

This violates condition 2.

subcase 2.2 $(k_1=0) \vee (k_2=0)$

then $uv^0xy^0z \notin L$ since this would imbalance the # of a's and # b's.

$\therefore uv^0xy^0z \notin L$.

subcase 2.3 $(k_1 \neq 0) \wedge (k_2 \neq 0)$

consider uv^ixy^iz

Now pumpdown, $i = 0$

$$|uv^0xy^0z| = p^2+p-k_1-k_2$$

suppose $uxz \in L$

$$p^2-k_1 = (p-k_2)^2$$

$$\cancel{p^2}-k_1 = \cancel{p^2}-2pk_2+k_2^2$$

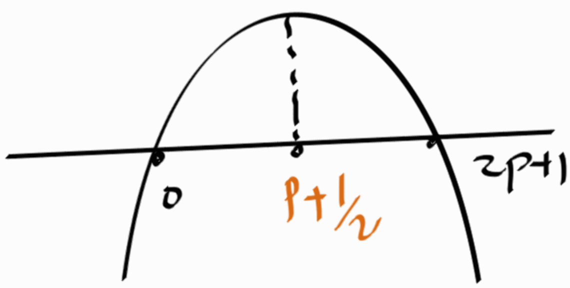
$$k_1 = -k_2^2+2pk_2$$

I am going to consider $|vxy|$

$$|vxy| = k_1+k_2+|x|$$

$$= \underbrace{-k_2^2+2pk_2+k_2}_{f(k_2)}+|x|$$

$$f(k_2) = -k_2^2+k_2(2p+1) = -k_2(k_2-(2p+1))$$



$$f'(k_2) = -2k_2 + (2p+1)$$

$$f'(k_2) = 0 = -2k_2 + 2p + 1 \quad , \quad k_2 = \frac{2p+1}{2} = p + 1/2$$

This means from $k_2 = 1$ to p $f(k_2)$ increases.

$$f(k_2) = -k_2^2 + k_2(2p+1)$$

$$f(1) = \cancel{-1} + 2p + \cancel{1} = 2p$$

$$f(p) = -p^2 + p\,(2p+1) = -p^2 + 2p^2 + p = p^2 + p$$

Here, we show $f(k_2)$ which is the length of $|vxy|$

$\therefore \forall k_2 \in [1 \ldots p] : f(k_2) > p$

$\therefore$ There is no decomposition of $uvxyz$ which is pumpable.

$\therefore$ $\mathcal{L}$ is not a CFL.