Why?

Let's say you want to show that the language ALL_{CFG} is undecidable.

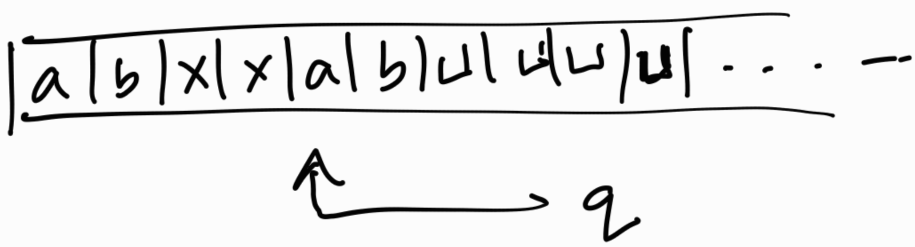
$$ALL_{CFG} = \{ \langle G \rangle \mid G \text{ is a CFG and } L(G) = \Sigma^* \}$$

$$A_{TM} \leq ALL_{CFG}$$

$\langle M, w \rangle$ $\qquad\qquad$ $\langle G \rangle$

TM $\qquad\qquad\qquad$ CFG

We are going to look at theories that would allow us to do reductions such as $A_{TM} \leq ALL_{CFG}$.



abxxqb

Definition

Let M be a Turing Machine and w an input string. An accepting computation history for M on w is a sequence of configurations $c_1, c_2, \ldots\ldots, c_l$ where c_1 is the start configuration of M on w, c_l is the accepting configuration of M on w, and each c_i legally follows from c_{i-1} according to rules of M.

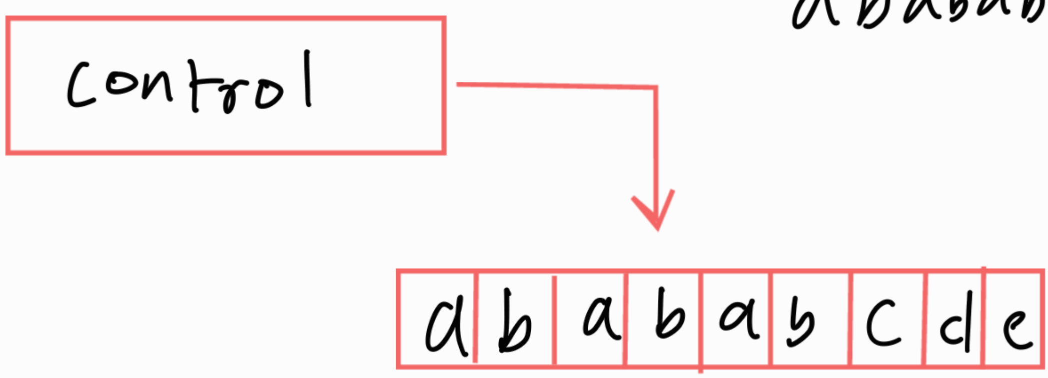
A rejecting configuration is defined similarly, except c_l is the rejection configuration.

* Computation histories are finite sequences.
* If M does not halt on w, then there are no accepting or rejecting computation history

Linear Bounded Automaton (LBA)

This is a restricted type of TM wherein the tapehead isn't permitted to move off the portion of the tape containing the input. If the machine tries to move its head off either end of the input, the head stays where it is.

abababcde

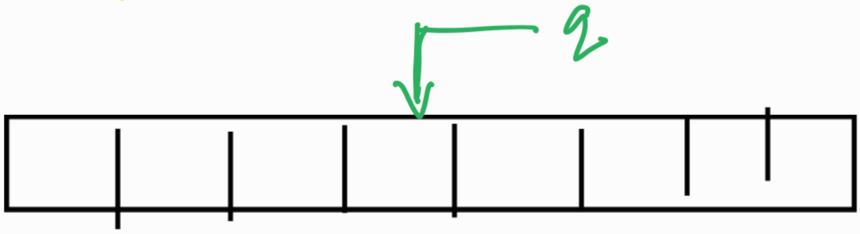


Now we define the following language

$$A_{LBA} = \{\langle M, w\rangle \mid M \text{ is a LBA and } M \text{ accepts } w\}$$

Theorem 5.9 A_{LBA} is decidable.

Lemma 5.8: Let M be a LBA with q states and g symbols in the tape alphabet. There are exactly qng^n distinct configurations of M on a tape of length n.



$$g \times g \times g \times g \times \cdots\cdots = g^n$$

$$qng^n$$

g - Size of the tape alphabet
n - Size of the input
q - # of states in the LBA.

A_{LBA} is decidable proof:

I am going to create a decider for A_{LBA} where the input is <M, w>
M is a LBA, w is a string.

I am going to create TM L that decides A_{LBA}.

L = on input <M, w>

1. Simulate M on w for qng^n steps or until it halts.
2. If it halts on an accept state, accept
3. If it halts on a reject state, reject
4. If M does not halt during

qng^n steps, reject.

// M exhaust all configs

// There is a loop

Theorem 5.10 E_{LBA} is undecidable.

$$A_{TM} \leq E_{LBA}$$

$E_{LBA} = \{ \langle M_1 \rangle \mid M \text{ is a LBA and } L(M) = \emptyset \}$

Idea: I am going to create a LBA B such that the language of B recognizes all accepting computation histories for M on w. If M accepts w, L(B) contains exactly one string that is the computation history for M on w. If M does not accept w, $L(B) = \emptyset$.

we constonct LBA B as follows:
(M and w are encoded inside this)

B = on input <x>

1. If x is an accepting computation history for M on w, then accept x.

* B looks at following 3 conditions,

1. C_1 is the starting config for M on w.
2. Each C_{i+1} legally follows C_i
3. C_ℓ is the accepting config for M on w.

$\# C_1 \# C_2 \# C_3 \# \cdots \# C_\ell \#$

Assume R is the decider for E_{LBA}.

We construct TM S for A_{TM}

S = on $\langle M, w\rangle$, where M is a TM and w is a string

1. Construct LBA B with $\langle M, w\rangle$ encoded incide it
2. Run R on $\langle B\rangle$
3. If R accepts ; reject
4. If R rejects ; accept

$\therefore$ E_{LBA} is undecidable

$ALL_{CFG} = \{ \langle G \rangle \mid G \text{ is a CFG and }$
$L(G) = \Sigma^* \}$

Proof is simillar to theorem 5.10 but with a slight twist.

$$A_{TM} \leq ALL_{CFG}$$

Proof Idea: I am going to create a CFG G for a given $\langle M, w \rangle$ where $L(G) = \Sigma^*$ if and only if M does not accept w.

If M accepts w, we make sure that the computation history for M on w is not generated by the G.

To make my life easier we are going to create a PDA D that behaves as above.

- Let the computation history for M on w be

$\# C_1 \# C_2 \# C_3 \# \ldots \ldots \# C_l \#$

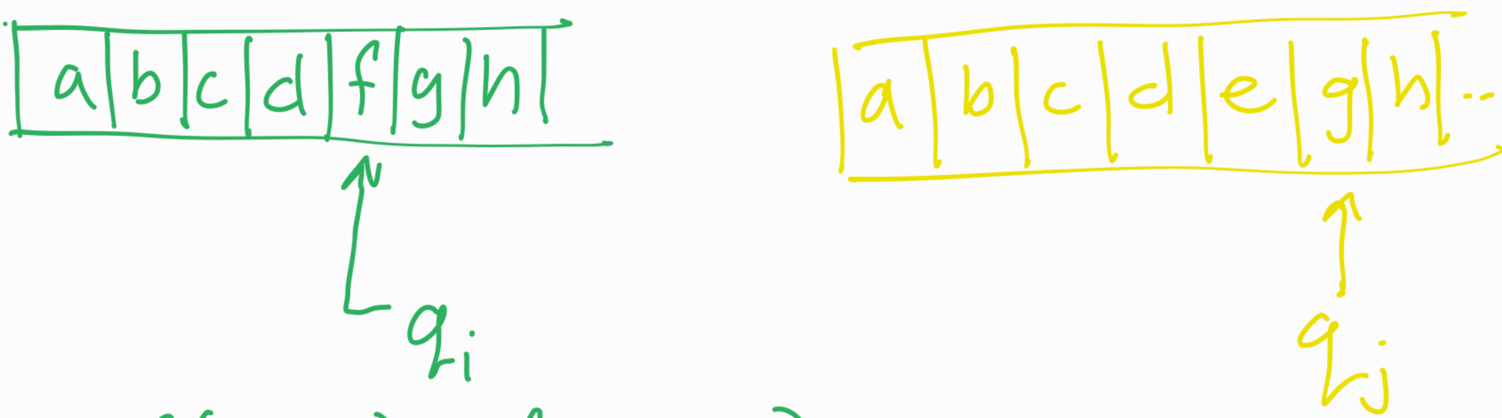
- D accepts all strings

1. do not start with C_1 * branch
2. do not end with C_l * branch
3. Some C_i does not yield C_{i+1} *

- Scan the input until it non-deterministically decides that it has reached C_i. It pushes C_i on the the stack until we reach #. Then D pops the stack to compare C_{i+1}.

They should match all except the head position.

$$abcdq_ifgh \Rightarrow abcdeq_jgh$$



$$\delta(q_i, f) = (q_j, e, R)$$

Proof: Assume R is the decider for ALL_{CFG}. Construct TM S that decides A_{TM}.

S = "on input <M,w> where M is a TM and w is a string

1. construct PDA D with <M, w> encoded inside.
2. convert D into CFG G.

3. Run R on $\langle G \rangle$

4. If R accepts, reject

If R rejects, accept

S is a decider for A_{TM}.

But A_{TM} is undecidable.

This is a contradiction.

$\therefore$ ALL_{CFG} is undecidable.