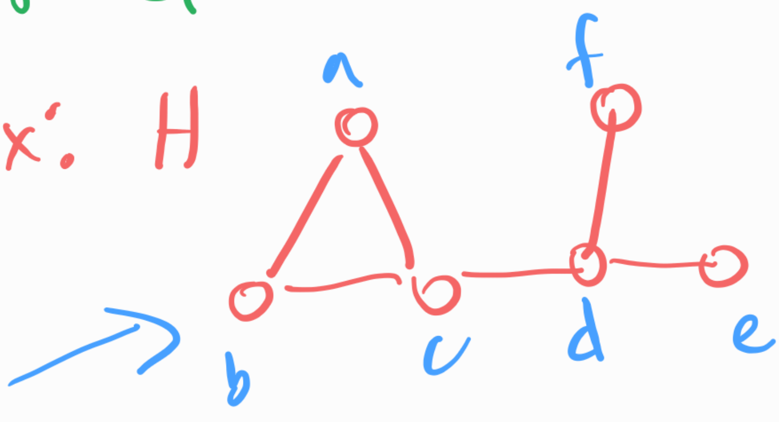
Graphs

Simple graphs

A graph with no self loops or parallel edges.

Subgraphs: A graph G is a subgraph of H if nodes/edges of G are subsets that of S.

Ex: H

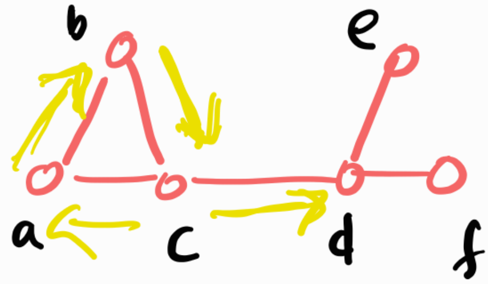


Ex:

Possible subgraphs

- Enumerating all subgraphs could be costly!
  - # of subgraphs is exponential.

connectedness: A path is a sequence of nodes connected by edges.



✓ $\langle a, b, c, d \rangle$

✗ $\langle a, b, c, e \rangle$

A simple path is a path that has no repeated vertices.

Example of a simple path

$\langle a, b, c \rangle$

Example of a path but not a simple path

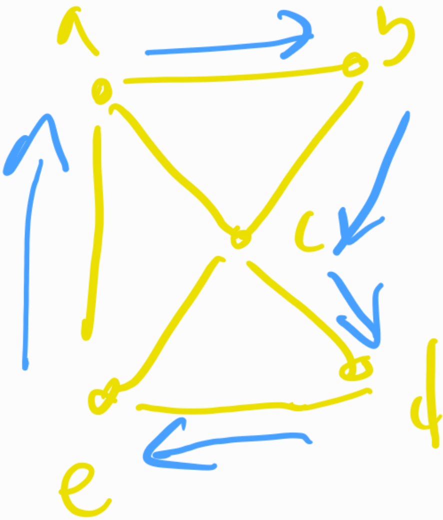
$\langle a, b, c, a, b, c, d \rangle$

A path is a cycle if it starts and ends at the same node.

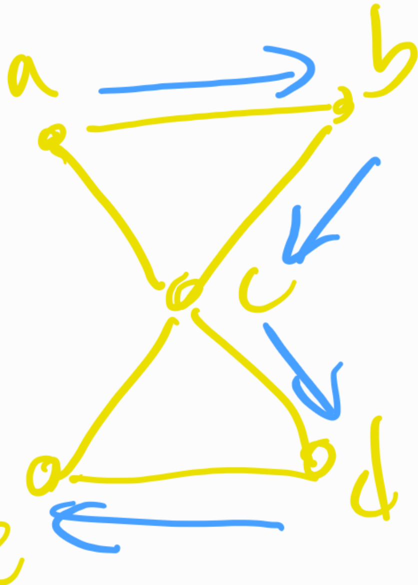
Hamiltonian Cycle

A cycle that visits all nodes of the graph exactly once

Ex 01:



Ex 02:



Example proof by contradiction

Claim $\sqrt{2}$ is irrational.

Proof:

Assume $\sqrt{2}$ is rational.

$\sqrt{2} = \frac{p}{q}$, where $p, q \in \mathbb{Z} \wedge q \neq 0$ and p, q can only be divisible by 1.

$2 = \frac{p^2}{q^2}$

$2q^2 = p^2$

$2 \mid p^2$ (2 divides p^2)

$2 \mid p$ ← one of the theorems we learnt in CSCI-246

$p = 2k$, $k \in \mathbb{Z}$

$2q^2 = (2k)^2$ (substitution)

$q^2 = 2k^2$

$2 \mid q^2$

$2 \mid q$

but this means $\sqrt{2} = \frac{p}{q}$, where p, q are divisible by 2, which contradicts the original claim.

$\therefore \sqrt{2}$ irrational.

Example.

A simple graph with n verticy and n edges must contain a cycle.

Induction

Mathematical induction is a method of proving predicates over natural numbers.

we can prove claims in the form of $[\forall n \in \mathbb{N} : P(n)]$

$P(n)$ is a predicate.

1. The claim should be expressed in the form of predicate over natural numbers, typically with in the scope of a universal quantifier

Typically: $[\forall n \in \mathbb{N} : P(n)]$

2. Then we prove the predicate is true for the smallest natural number in the claim.
(base case)

3. then we prove the inductive step

$$\forall n \geq 1 : P(n) \Rightarrow P(n+1)$$

$a \Rightarrow b$ whenever a = True
b = True

$P(1) \Rightarrow P(2)$ $P(2) \Rightarrow P(3)$

4. we conclude that if the claim is true for base case and inductive step then $[\forall n \in \mathbb{N} : P(n)]$

Ex: $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$

$= \frac{1}{6} \cdot n(n+1) \cdot (2n+1)$

Proof: step: define the predicate:

Defining predicate $P: \mathbb{N} \longrightarrow \{T, F\}$

$$P(n) = \begin{cases} T, & \text{if } \sum_{i=1}^{n} i^2 = \frac{1}{6} n \cdot (n+1)(2n+1) \\ F & \text{otherwise} \end{cases}$$

Rewording the claim using Predicate and universal quantifie

$[\forall n \in \mathbb{N} : P(n)]$

$=$

1. Prove the basis

WTS: $\underbrace{\sum_{i=1}^{1} 1^2}_{LHS} = \underbrace{\frac{1}{6}(1)(1+1)(2\cdot 1+1)}_{RHS}$

$LHS = \sum_{i=1}^{1} i^2 = 1^2 = 1$

$RHS = \frac{1}{\cancel{6}} \cdot 1 \times \cancel{2} \times \cancel{3} = 1$

$LHS = RHS$

Base case is proven

2. Prove the inductive step

$\forall n \geq 1 : P(n) \Rightarrow P(n+1)$

Assume the premise (left hand side)

Assume $P(n)$

$\sum_{i=1}^{n} i^2 = \frac{1}{6}(n+1)(2n+1) \cdot n$

WTS $P(n+1)$ is true

$$\underbrace{\sum_{i=1}^{n+1} i^2}_{\text{LHS}} = \frac{1}{6}(n+1)(n+2)(2(n+1)+1)$$

$$\text{LHS} = \sum_{i=1}^{n+1} i^2 = \underbrace{\sum_{i=1}^{n} i^2}_{\frac{1}{6}(n+1)(2n+1)} + (n+1)^2$$

$$\sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n+1)(2n+1)\cdot n + (n+1)^2$$

$$= (n+1)\left[\frac{2n^2+n+6n+6}{6}\right]$$

$$= \frac{1}{6}(n+1)\left[2n^2+7n+6\right]$$

$$= \frac{1}{6}(n+1)\left[2n^2+4n+3n+6\right]$$

$$= \frac{1}{6}(n+1)\left[2n(n+2)+3(n+2)\right]$$

$$= \frac{1}{6}(n+1)\left[2n+3\right]\left[n+2\right]$$

$$= \frac{1}{6}(n+1)(n+2)(2(n+1)+1)$$

RHS

LHS = RHS

$$\therefore \left[\forall n \in \mathbb{N} : P(n)\right]$$

Directed graph

A graph where all edges are directed.

pair

$G = (V, E)$, $E = \{\langle u, v \rangle : u, v \in V\}$

$\langle u, v \rangle \neq \langle v, u \rangle$ $\{u, v\}$

Strings & Languages

Alphabet: Any <u>finite</u> set, each element is called a letter / symbol.

Ex:

$\Sigma_1 = \{0, 1\}$, $\Sigma_2 = \{☆, \triangle\}$

$\Sigma = \{A, C, G, T\}$

String: a finite sequence of symbols or letters

$s_1 = 0101$ $s_2 = 11110$ over alphabet $\Sigma = \{0, 1\}$

Language: A set of string (could be finite or infinite)

$L = \{s_1, s_2\}$

In class exercise

In class exercise 01

Answer: The claim is given a graph $G = (V, E)$. Then let R be a relation that there is a path between two vertices x and y in G.

Show that R is an equivalence relation.

$$R = \{ \langle a, b \rangle : (a, b \in V) \wedge (\text{there is a path from } a \text{ to } b \text{ in } G) \}$$

To show that this is an equivalence relation, we need to show 3 things about R.

1. R is reflexive
$[\forall x \in V : xRx]$

2. R is symmetric
$[\forall a,b \in V : aRb \Rightarrow bRa]$

3. R is transitive
$[\forall a,b,c \in V : aRb \wedge bRc \Rightarrow aRc]$

Proof:

1. R is reflexive

$\forall x \in V$: x is connected to itself

$\therefore \forall x \in V : xRx$

2. R is symmetric

Given $a, b \in V$,

Assume aRb.

Then there is a path from a to b. Since G is undirected there is a path from b to a

$\therefore bRa$

$\therefore \forall a, b \in V: aRb \Rightarrow bRa$

3. Transitive

Let $a, b, c \in V$

Assume $aRb \wedge bRc$

This means there is a path from a to b and b to c

therefore there is a path from a to c

$\therefore aRc$

$\therefore \forall a, b, c \in \mathbb{R}: aRb \wedge bRc \Rightarrow aRc$

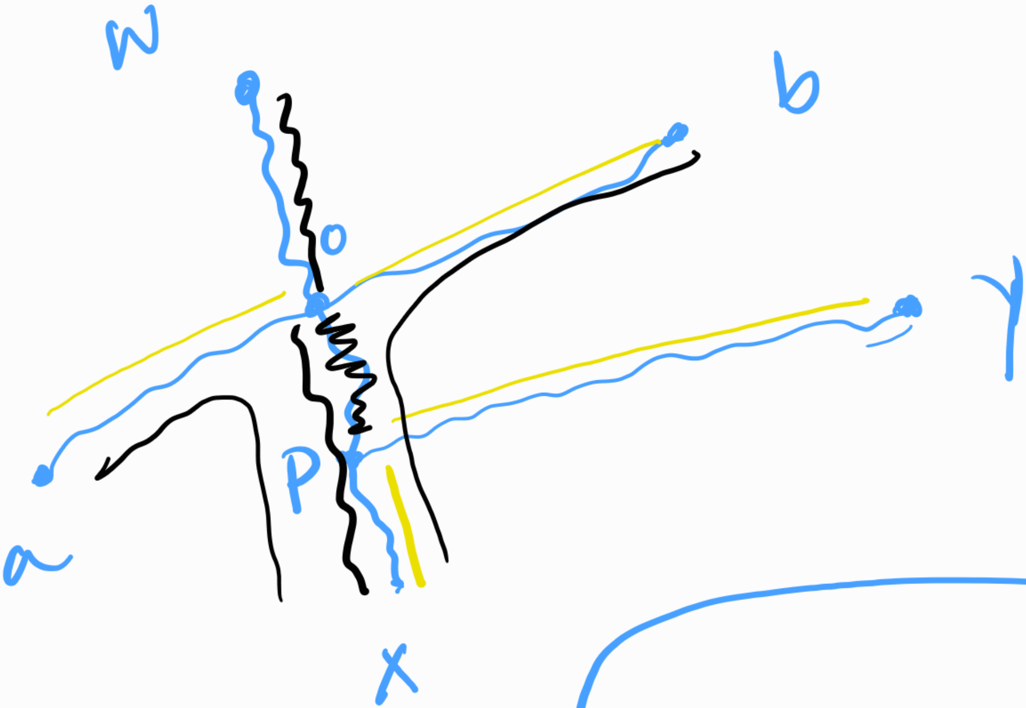
R is an equivalence relation.

Question 06 Homework

1. pick any vertex $w \in V$, Run BFS, find the furthest point x

2. Run BFS from x and find the furthest point y

3. Return (x,y) as the diameter.



Assume $\boxed{\delta(a,b) > \delta(x,y)}$

$\delta(w,x) \geq \delta(w,a) \rightarrow$ 1

$\delta(w,x) \geq \delta(w,b)$ — 2

$\delta(x,y) > \delta(x,a)$ — 3

$\delta(x,y) > \delta(x,b)$ — 4

$$\delta(a,b) < \delta(x,y)$$

using 1, 2, 3, 4

show this