Question: Can we use pumping Lemma to show that a particular language is regular?

You do not use pumping Lemma to prove a language is regular.

But we use this to prove a language is not regular.

Applications of pumping Lemma

Pumping lemma is used to show that a particular language is not regular.

This is the procedure we can follow to show a particular language is <u>not</u> <u>regular</u>.

Suppose you are given a language A and you need to show that the language is <u>not regular</u>.

1. Language A should be infinite becuse if it is finite then obviously A would be regular.

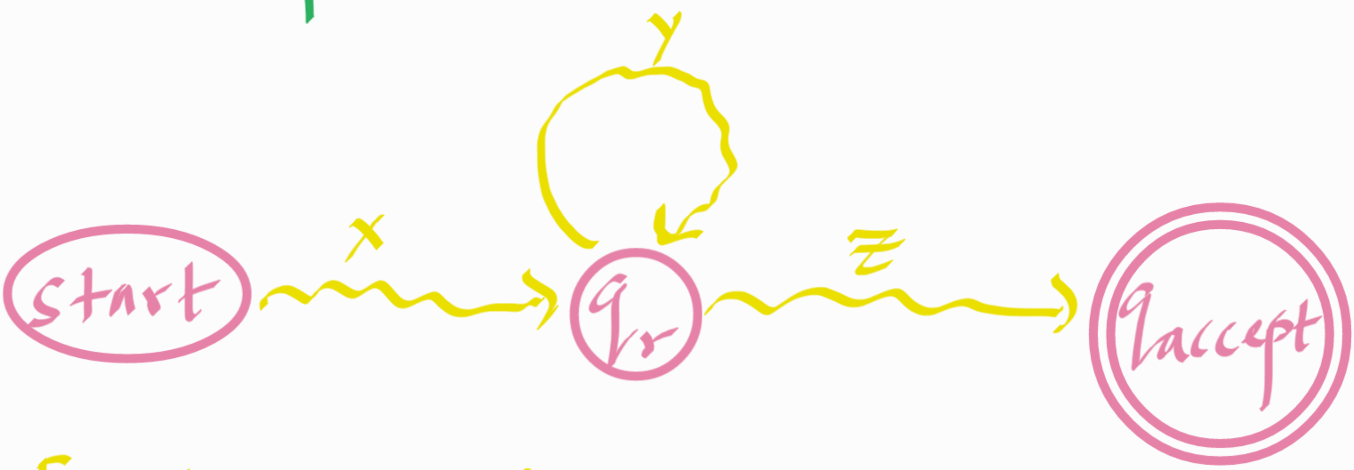
2. Now let's assume that A is regular.

3. Then, there should be a DFA that recognize this language. Let's call this machine M.

4. Since M is a DFA, the number of states in M should be finite. Let # of states in M be p.

5. Since A is an infinite language, there must be at least one string in A that is of length larger than p.

6. let's pick s s.t $|s| \geq p$.



Existence of s in A implies that there is a path to a accepting state that reuses a state in the machine.

then any string that uses this path must be accepted by this machine, e.g.,

xy

xyz

xy^2z

xy^3z

$\vdots$

xy^iz

All of these string uses the path to the accepting state.

The trick is to pick a string s from A s.t, once it is decomposed into s=xyz, the strings that we create by pumping y are no longer in language A.

* Remember there could be multiple ways to decompose s into xyz, therefore, we need to show that every decomposition of s leads to creating new strings that are not in A.

-If you create strings that are accepted by M but not in A, then we have a contradiction.

Example

Prove that $A = \{0^n 1^n \mid n \geq 0, n \in \mathbb{Z}\}$ is not regular.

Proof by contradiction.

Assume A is regular.

Suppose pumping length of language A is p.

(Remember this is a fancy way of saying the DFA that recognize A has p states.)

Now my goal is to pick a string s in A, length of which is greater than or equal to p such that it violates the pumping lemma.

$$\underbrace{s = 0^p}\,1^p \qquad \underline{0^{p/2}1^{p/2}}$$

$$|s| = 2p$$

$$s = \underbrace{000\ldots\ldots0}_{p \text{ \# of } 0\text{'s}}\,\underbrace{1\ldots\ldots1}_{p \text{ \# of } 1\text{'s}}$$

$$s = xyz$$

$$\overbrace{\underbrace{000}_{x}\ldots\ldots0}\,\underbrace{\ldots111}_{y}\ldots\underbrace{\ldots1}_{z}$$

$$\underbrace{}_{x}\underbrace{}_{y}\underbrace{}_{z}$$

\# of ways to split s into 3 parts.

$$\binom{2p}{2} \qquad |xy| \leq p$$

$$|y| > 0$$

* How about picking something like $S = 0^{p/2} 1^{p/2}$?

1. This is problematic since p could be odd

2. Even if you pick this, it will still be very hard to show that $0^{p/2} 1^{p/2}$ violates pumping lemma.

How about $S = 0^p 1^p$?

Q1: what is the size of S ?

case 1: $\underline{x=\varepsilon}$ $\underline{y=0^p}$ $z=1^p$

$\underline{xy} = \varepsilon \cdot 0^p$ $z=1^p$

$\underline{xyz} = 0^p 1^p$ ✓

$\underline{x}y^0\underline{z} = 1^p \notin A$

$xy^2z = 0^p 0^p 1^p \notin A$

$\underline{\text{case 2}}$: $\underline{x=0^a}$ $\underline{y=0^b}$ where $\underline{a+b=p}$

$z=1^p$

$\underline{xy^0z} = 0^a 1^p$ $a<p$

$\notin A$

$xy^2z = 0^a 0^b 0^b 1^p = 0^{a+2b} 1^p \notin A$

$a+2b>p$

Basically $s=0^p1^p$ violates pumping lemma regardless of the decomposition of s to xyz.

Therefore A should not be regular.

Remember tricky part of these proofs is to pick a string that is easier to show that it violates the pumping lemma.

Example 02: Let $\Sigma = \{0,1,2\}$

Show that $B = \{0^n 1^n 2^n \mid n \geq 0\}$ is not regular, where

Proof by contradiction

Assume B is regular.

Then B must follow the pumping lemma.

Let us pick $s \in B$, s.t $|s| \geq p$.

$s = 0^p 1^p 2^p$ $\quad |s| = 3 \cdot p \geq p$

By pumping lemma, s should be able to be decomposed into $s = xyz$ s.t

1. for each $i \geq 0$, $xy^iz \in B$
2. $|y| > 0$
3. $|xy| \leq p$

$s = \underbrace{0000\cdots0}_{p}\underbrace{1\cdots1}_{p}\underbrace{2\cdots2}_{p}$

By ③ $|xy| \leq p$, therefore,
y must be composed of
entirely 0's, than 1's and 2's,

Hence $xy^2z \notin B$.

This is an contradiction.

$\therefore$ B is not regular

Example 03:

$$E = \{ 0^i 1^j \mid i > j \}$$

is not regular.

what would be a good string to pick?

$$s = 0^{p+1} 1^p$$

Assume E is regular.

Then E must follow pumping lemma.

Pick $s = 0^{p+1} 1^p$, $s \in E$

$|s| = 2p+1 \geq p$

Hence, we should be able to decompose $s = xyz$, s.t

1. $|y| > 0$
2. $|xy| \leq p$
3. $xy^i z \in E$

$xz \notin E$

$xy^0 z = 0^p 1^p$

$xy^2 z = 0^{p+2} 1^p$

$x = 0^p$

$y = 0$

$z = 1^p$

By ① & ② y should be entirely composed of 0's and $|y| > 0$.

$xy^2 z = 0^a 1^p$ $a > p$

Therefore, consider $xy^0 z = xz$, where at least one 0 is deleted, then $|0's| \leq |1's|$, therefore $xy^0 z \notin E$

Then E is not regular.

$$s = 0^{p+1} 1^{p}$$

$$x = 0^{a} \quad \underline{y = 0^{b}} \quad z = 0 1^{p}$$

$$xyz = 0^{a} 0^{b} 0 1^{p} = 0^{a+b+1} 1^{p};$$

$$xy^{2}z = 0^{a} 0^{2b} 0 1^{p} = 0^{a+2b+1} 1^{p} \in E$$

$$xy^{0}z = 0^{a} 0 1^{p} = 0^{a+1} 1^{p} \notin E$$

$$a+1 \le p$$

Example 04: $F = \{ww \mid w \in \{0,1\}^*\}$

Show that F is not regular.

$$S = 0^p 1^p$$

$$S = 0^{p+1} 1^p$$

$|xy| \le p$ y must be entirely consisting of 0

$$S = \underbrace{0^a}_{x} \underbrace{0^b}_{y} \underbrace{1^p}_{z}, \text{ where } a+b = p+1, b>0$$

$$xy^2z = 0^a 0^{2b} 1^p = 0^{a+2b} 1^p \in E$$

$$xy^3z = 0^a 0^{3b} 1^p = 0^{a+3b} 1^p \in E$$

$$xz = 0^a 1^p \notin E$$ Pumping down.