Applications of pumping Lemma.

Ex: $L = \{a^n b^n c^n \mid n \geq 0\}$ is not CFL.

Assume L is a CFL.

Then L must follow pumping lemma.

Suppose pumping length for L is p.

$s = a^p b^p c^p$ $\quad |s| = 3p \geq p$

Then by pumping lemma s must be able to be decomposed into 5 components where $s = u\underline{vxy}z$, s.t

1. $uv^ixy^iz \in L \quad \forall i \geq 0$
2. $|vxy| \leq p$
3. $|vy| > 0$

$$s = \overbrace{aaa \ldots a}^{p\ \#\ a's}\ \overbrace{bbb \ldots b}^{p\ \#\ b's}\ \overbrace{ccc \ldots c}^{p\ \#\ c's}$$

$vxy = aaa\ldots\ldots a$

$vxy = \underbrace{\overbrace{aaa\ldots\ldots}^{v}\overbrace{abbb\ldots\ldots b}^{y}}_{vxy}$

$v = aaa\ldots a \quad x = \varepsilon$

$y = abb\ldots\ldots b$

$v^i x y^i = aaa\ldots\ldots a\, abb\ldots b\, abb\ldots b$

case 1: vxy contains the same set of characters. Then v and y can only be allocated to the same type of character. Therefore, when we pump uv^ixy^iz the # of one type of characters increases.

$uv^ixy^iz \notin L$

case 2: vxy contains mix of characters.

subcase 1: v and y contains the same character. we handled this in case 1.

subcase 2: v and y contains two different types of characters.

ex: $vxy = aaa\ldots a\,bbb\ldots b$

$v = aaa\ldots a \quad y = bbb\ldots b$

now when we pump up, uv^ixy^iz contains different # of type 3 characters.

Therefor $uv^ixy^iz \notin L$

Subcase 3: either v or y contains mix of two characters.

$vxy = aaa\ldots a\,bbb\ldots b$

$v = aaa\ldots a \quad y = aa\ldots bb\ldots$

uv^ixy^iz contains characters in out

of order.

Hence $uv^i x y^i z \notin L$

we considered all possible decompositions. None of them are pumpable.

Therefore, L is not a CFL.

$C = \{ ww \mid w \in \{0,1\}^* \}$ is not CFL.

Assume C is a CFL.

Then C must follow pumping lemma.

Let p be the pumping length.

Let $s = 0^p 1^p 0^p 1^p$

$0^p 1 0^p 1$

not good

$000 \ldots\ldots 0111 \ldots\ldots 1 \mid 000 \ldots\ldots 0111 \ldots\ldots 1$

$u = 0^p 1^{p-k}$ $v = 1^k$ $x = \varepsilon$ $y = 0^k$

$z = 0^{p-k} 1^p$

$s = 0^p 1^{p-k} 1^k 0^k 0^{p-k} 1^p = 0^p 1^p 0^p 1^p$

$uv^i x y^i z = 0^p 1^{p-k} 1^{ki} \mid 0^{ki} 0^{p-k} 1^p$

$i=0$

$$uv^0xy^0z = 0^p 1^{p-k} 0^{p-k} 1^p \notin C$$

Case 1: If vxy does not straddle (cross) the mid point of s, $uv^ixy^iz \notin C$.

Case 2: If vxy does cross the mid point, the best possible allocation for v, x, y is $v = 1^k$ $x = \varepsilon$ $y = 0^k$, then consider $uv^0xy^0z = 0^p 1^{p-k} 0^{p-k} 1^p \notin C$.

Hence, C is not a CFL.

$L = \{ a^n b^j \mid n = j^2 \}$ is not a CFL.

Assume L is a CFL.

Then let p be pumping length

Let $s = a^{p^2} b^p$ $\quad |s| = p^2 + p \geq p$

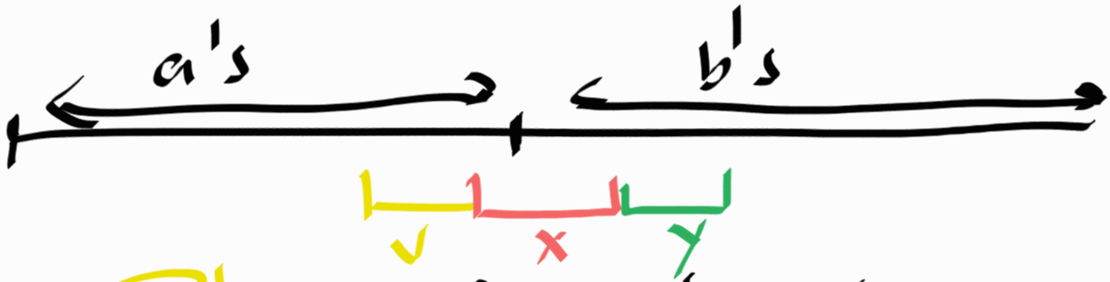
So s should follow pumping Lemma.

$s = \underbrace{aaaa \ldots\ldots a}_{p^2 \text{ \# a's}} \underbrace{bbb \ldots\ldots b}_{p \text{ \# of b's}}$

case 1 : Either v or y crosses the break point Then when pumping up we will have out of order characters.

$uv^ixy^iz \notin L$.

case 2: Both v and y does not cross the break point.



$\boxed{K_1}$ # of a's to v

$\boxed{K_2}$ # of b's to y

When we pump up or down

$\underline{p^2 + K_1(i-1)}$ a's and

$\underline{p + K_2(i-1)}$ b's

$u = a^{p^2 - k_1}$ $v = a^{k_1 i}$ $x = \varepsilon$

$y = b^{k_2 i}$ $z = b^{p - k_2}$

$p^2 + (i-1)k_1$ # of a's $p + (i-1)k_2$ # of b's

case 2.1 If $k_1 \neq 0$ $k_2 \neq 0$

we set $i = 0$

consider this inequality

$$\underbrace{(p-k_2)^2}_{\text{\# of b's}} \leq (p-1)^2 \leq p^2 - 2p + 1 < \underbrace{p^2 - k_1}_{\text{\# of a's}}$$

$k_2 \neq 0 \Rightarrow k_2 \geq 1$

$uv^i x y^i z \notin L$

case 2.2 If either $k_1 = 0$ or $k_2 = 0$, then again pick $i = 0$, $uv^0 x y^0 z$ will have some a's or b's deleted. In which $uv^0 x y^0 z \notin L$