Recap continued . . .

01/22/2025

Theorems and proofs

Theorem – A theorem is a true mathematical statement.

Ex: If x, y is rational, then $x \cdot y$ is rational.

– A proof is convincing logical arguement that the statement is true.

– given a statement, we can prove it with a proof
– given a statement, we can disprove it (usually with a counterexample)

Ex: If $x \in \mathbb{N} \wedge x \bmod 2 = 0$, then x is a power of 2.

$(x \in \mathbb{N}) \wedge (x \bmod 2 = 0) \Rightarrow$ x is a power of 2

$x = 6$

$x \in \mathbb{N}$, $6 \bmod 2 = 0$

6 is not a power of 2

disproof by counter example

$\boxed{p \Rightarrow q}$

whenever p is true q must be true.

$p \Leftrightarrow q$

when p is true q is true

when p is false q is false as well

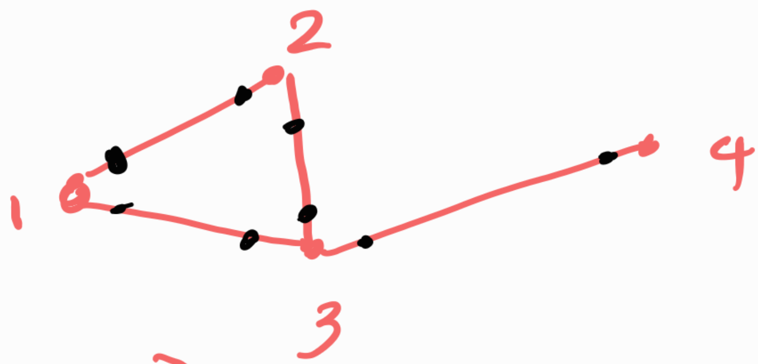
Proof methods:

1. Direct proofs
2. Proof by contradiction
3. Proof by induction
4. Proof by construction ($\neq$ by example)

Direct proof!

Ex: Given a graph $G = (V, E)$, Let $\deg(v)$ be the # of edges incident to v. Then

$$\sum_{v \in V} \deg(v) \text{ is even}$$



$V = \{1, 2, 3, 4\}$ $\quad E = \{(1,2), (1,3), (2,3), (3,4)\}$

$\deg(1) = 2 \quad \deg(2) = 2$

$\deg(3) = 3 \quad \deg(4) = 1$

$$\sum_{v \in V} \deg(v) = 2 + 2 + 3 + 1 = 8$$

Proof: when counting degrees, each edge (u,v) is counted twice and once for $\deg(v)$

$$\therefore \sum_{v \in V} \deg(v) = 2|E| \quad \text{(which is even)}$$

Ex 2: Given sets A and B,

$$x \rightarrow \overline{A \cup B} = \bar{A} \cap \bar{B} \leftarrow y$$

1

$$\left(\overline{A \cup B} = \bar{A} \cap \bar{B}\right) \equiv \left(\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}\right) \wedge \left(\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}\right)$$

1 WTS $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$ 2

If $x \in \overline{A \cup B}$, then $x \in \bar{A} \cap \bar{B}$

Let $x \in \overline{A \cup B}$

$x \notin A \cup B$

$(x \notin A) \land (x \notin B)$

$(x \in \overline{A}) \land (x \in \overline{B})$

$x \in \overline{A} \cap \overline{B}$

We have to prove part 2

$\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

Relations & functions

A function is a mapping between two sets, with set of rules.

formally function f from set A to B is written as

$$f : A \longrightarrow B$$

with following conditions.

1. $\forall a \in A$: $f(a)$ must be defined
2. $\forall a \in A$: $f(a)$ should produce an unique value.
3. $\forall a \in A$: $f(a) \in B$

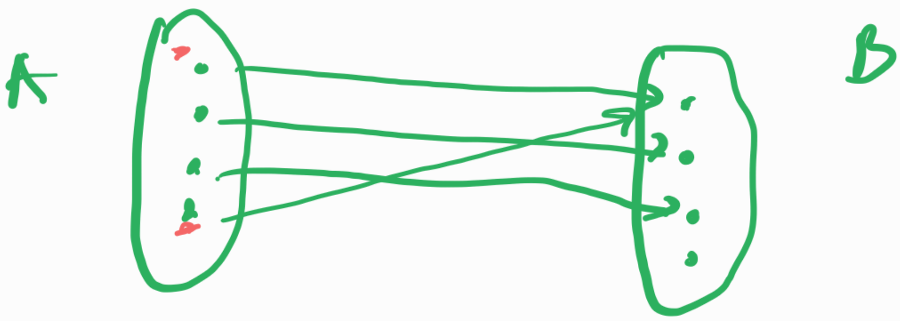
A – Domain

B – Codomain.

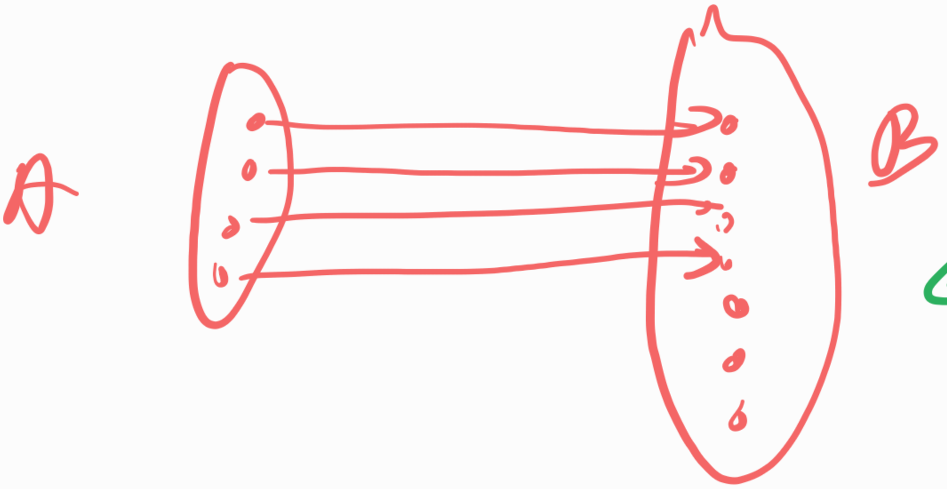
R – range is the set of all possible outputs.

Injective functions (one-to-one)

1. $\forall a_1, a_2 \in A : a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$



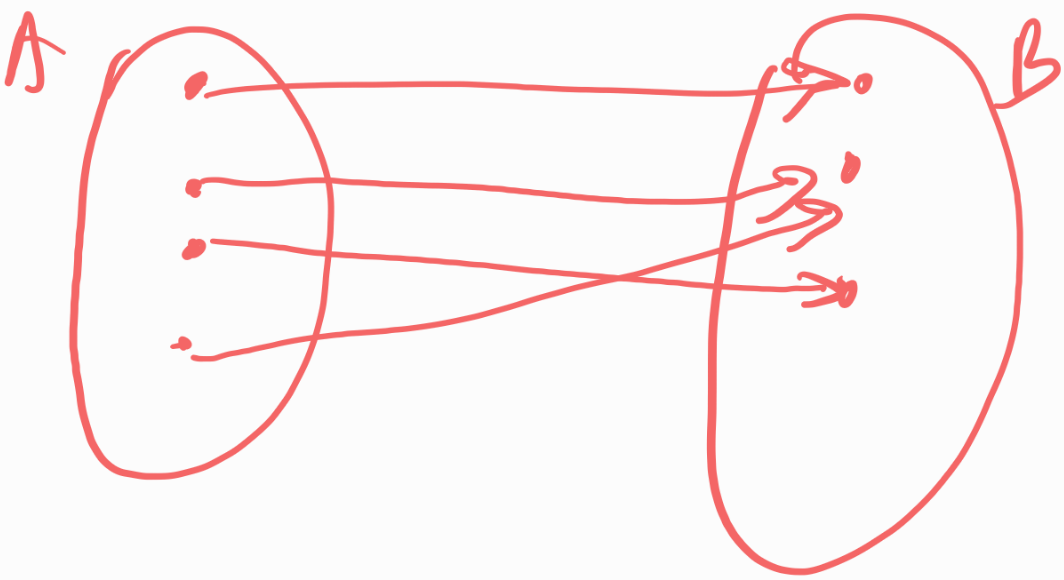
not one-to-on



one-to-one

Surjective function (onto)

$\forall b \in B : \exists a \in A : f(a) = b$



onto

Bijection

Any function that is injective and surjective is called a bijection

- Predicates

Predicate / property is a function whose range is {True, False}

Ex: $\text{isEven} : \mathbb{Z} \to \{T, F\}$

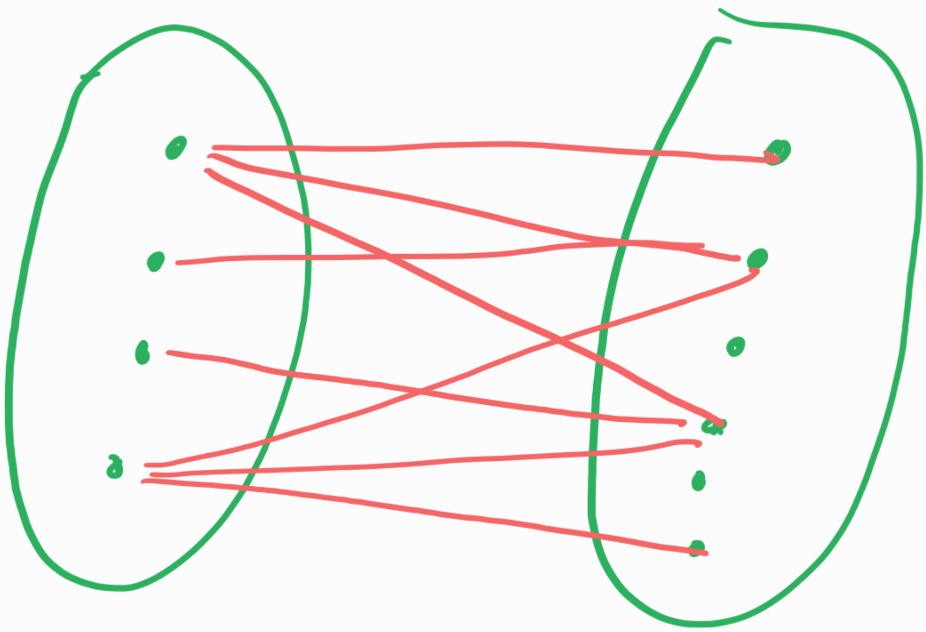
$$\text{isEven}(x) = \begin{cases} \text{True}, & \text{if } x \text{ is even} \\ \text{False}, & \text{if } x \text{ is odd} \end{cases}$$

Relations

k-ary relation R on k sets $A_1, A_2, A_3, \ldots, A_k$ as follows:

$$R \subseteq A_1 \times A_2 \times A_3 \times \cdots \times A_k$$

$k = 2$



In practice, usually $k=2$ (Binary relations) special binary relations defined on a single set

Let A be a set.

R be a subset of $\underline{A} \times \underline{A}$

$R \subseteq A \times A$ ← special binary relation.

* Equivalence relations

1. reflexive : $\forall x \in A : xRx$
2. symmetric : $\forall x, y \in A : xRy \Rightarrow yRx$
3. transitive : $\forall x, y, z : xRy \wedge yRz \Rightarrow xRz$

$\equiv_5$ (multiple of 5)

$\equiv_5 \subseteq \mathbb{N} \times \mathbb{N}$

$$\equiv_5 = \{\langle a,b\rangle, (a \in \mathbb{N}) \wedge (b \in \mathbb{N}) \wedge (a \bmod 5 = 0) \wedge (b \bmod 5 = 0)$$

$\langle 5,5\rangle, \langle 5,10\rangle, \langle 0,5\rangle, \langle 5,0\rangle$