CSCI 347 Data Mining

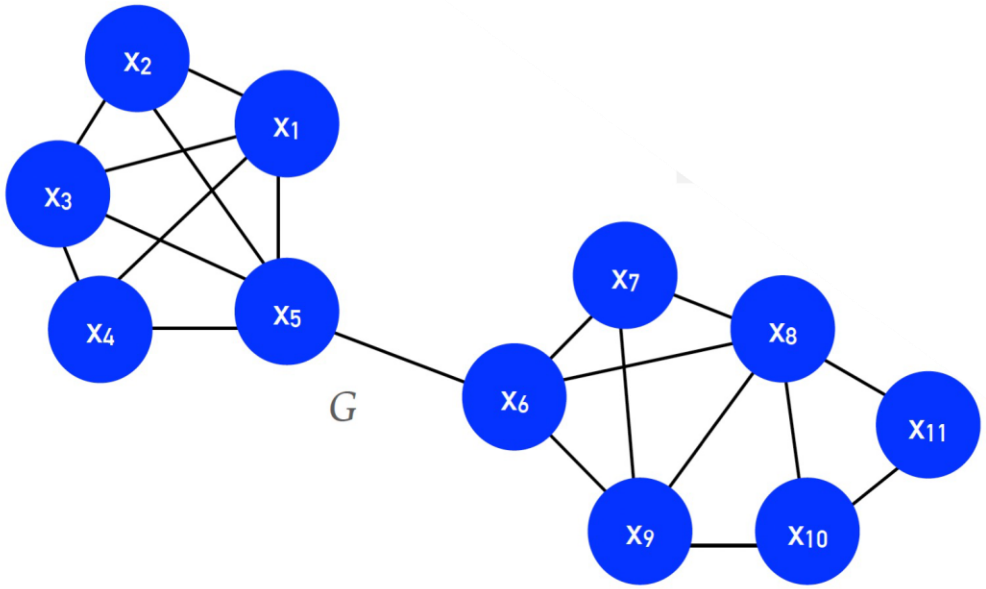
Graph Data

Graph Data

- Data instances are often not entirely independent
- they can be interconnected through various types of relationships.
- Graph data or networks are a data structure where instances are depicted as nodes, and the connections between these instances are represented by edges.

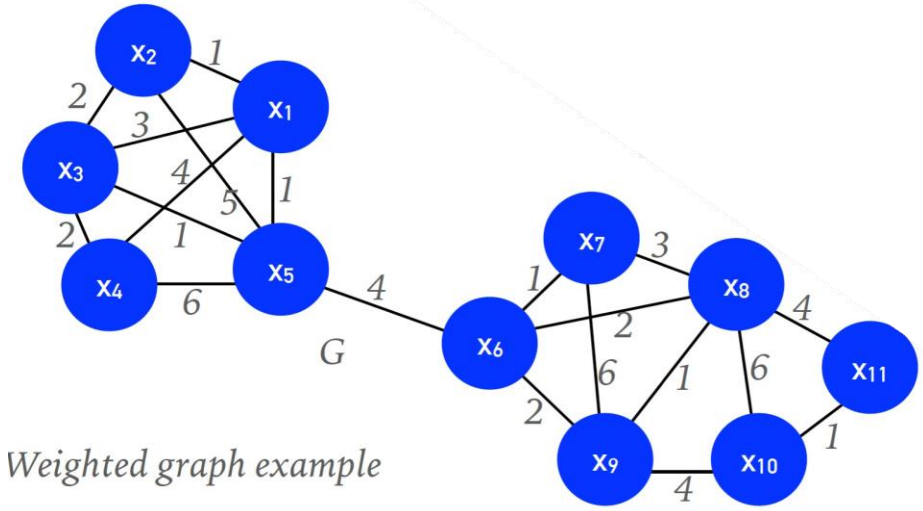
Graph Data

$$G = (V, E)$$
$$V = set\ of\ vertices$$
$$E \subseteq V \times V, is\ the\ set\ of\ vertices\ in\ the\ graph$$



Graph Data (Weighted graph)

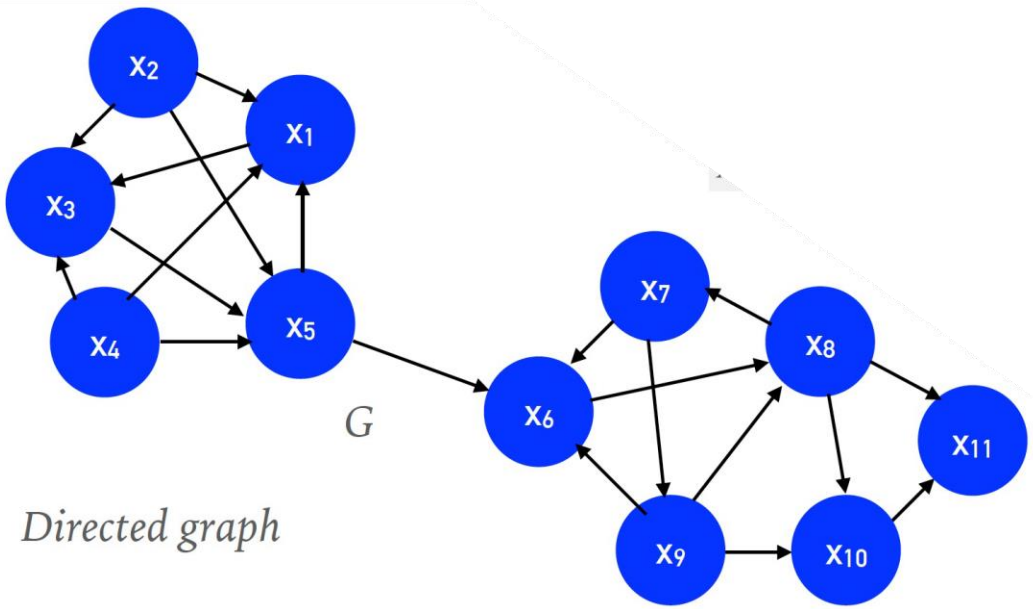
- G = (V, E)
- V = Vertices or Nodes
- E = Unordered pairs of vertices with different weights (w_{ij})



Weighted graph example

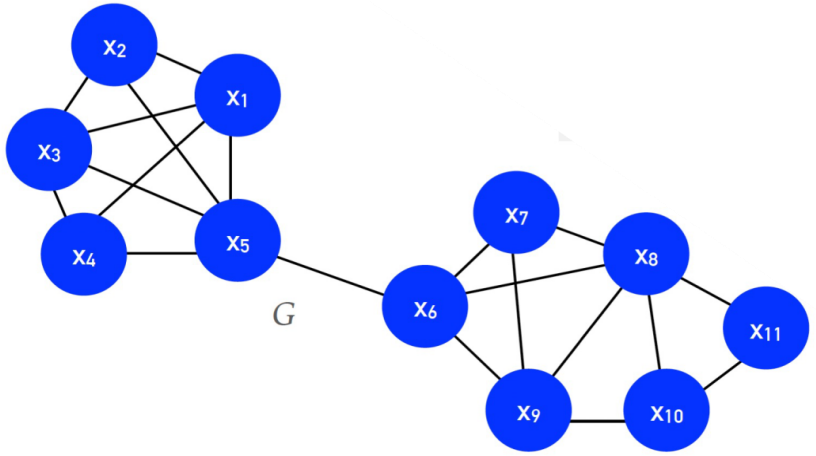
Graph Data (Directed Graph)

- G = (V, E)
- V = Vertices or Nodes
- E = **ordered** pairs of vertices.



Graph Data

- $G = (V, E)$
- $V = Vertices\ or\ Nodes$
- $E = Unordered\ pairs\ of\ vertices$
- Simple graph = Undirected graph without loops
- Edge, $e = (v_i, v_j)$, $v_i\ and\ v_j$ are adjacent or neighbors.
- Order: $|V| = n$, Size: $|E| = m$



Graph Data

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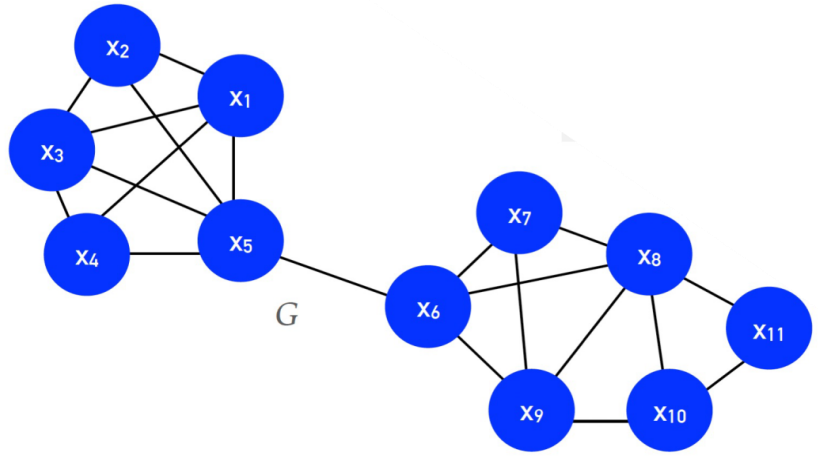
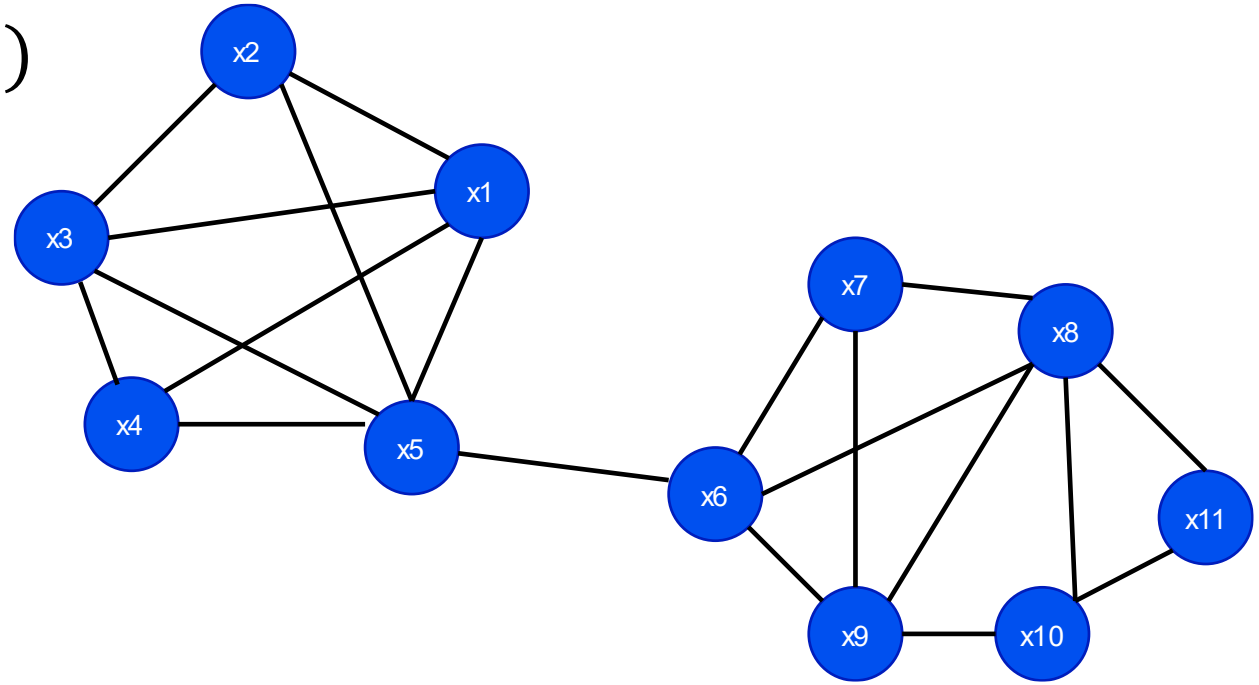
- $V = Vertices\ or\ Nodes$
- $E = Unordered\ pairs\ of\ vertices$
- Simple graph = Undirected graph without loops
- Edge, $e = (v_i, v_j)$, $v_i\ and\ v_j$ are adjacent or neighbors.
- Order: $|V| = n$, Size: $|E| = m$
- A graph $H = (V_H, E_H)$ is called a subgraph of $G = (V, E)$, if $V_H \subseteq V$ and $E_H \subseteq E$.



Degree of a node

- The degree of a node $v_i \in V$ is the number of edges incident with it and is denoted as $d(v_i)$ or just d_i .

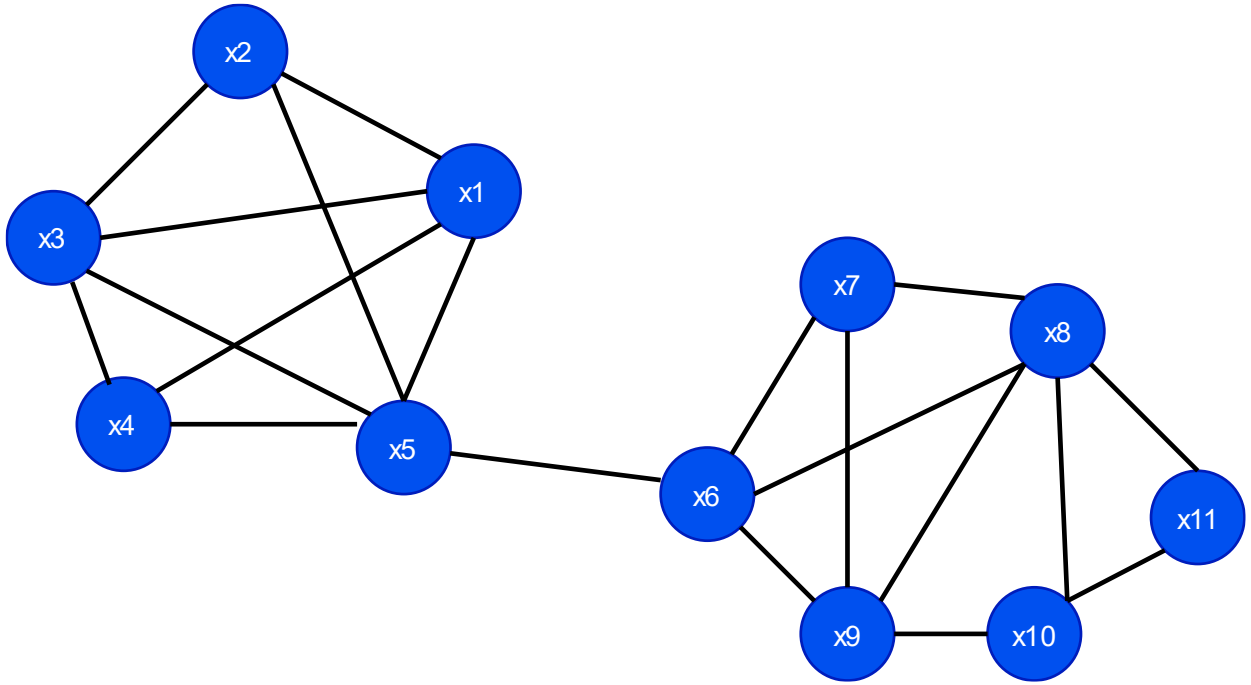


Degree of a node

- The degree of a node $v_i \in V$ is the number of edges incident with it and is denoted as $d(v_i)$ or just d_i .

What is the degree of $x9$?

1. 3
2. 1
3. 4
4. 8



Degree Distribution

- Let N_k denote the number of vertices with degree k. The degree frequency distribution of a graph is given as $(N_0, N_1, \ldots, N_t)$
  - t is the maximum degree of a node in the graph.

Degree Distribution

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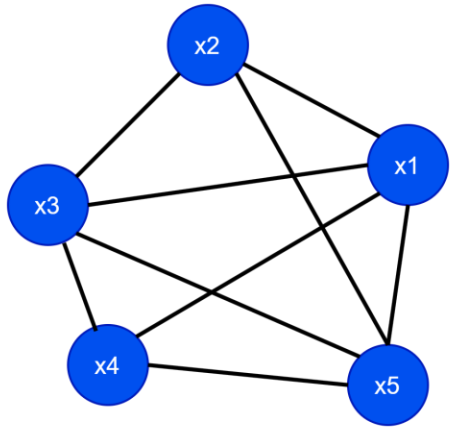
  - t is the maximum degree of a node in the graph.

What is the degree distribution of this graph?

1. $(4, 3, 4, 3, 4)$
2. $(3, 3, 4, 4, 4)$
3. $(0, 0, 0, 2, 3)$
4. $(2, 3)$



Degree Distribution

- The probability that a given node is of degree k is $\frac{N_k}{n}$.
- Suppose you have a random process of picking a node in a graph, and random variable X that assigns the degree of the picked node.

$$P(X = k) = \frac{N_k}{n}$$

Degree Distribution

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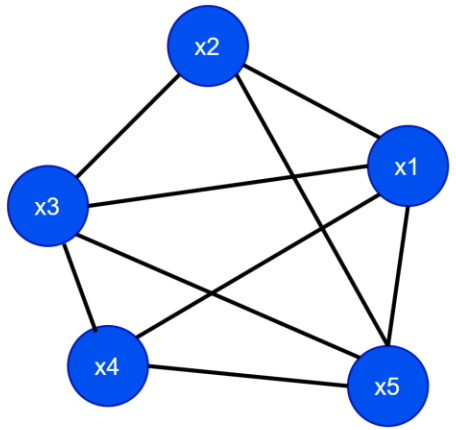
- Suppose you have a random process of picking a node in a graph, and random variable X that assigns the degree of the picked node.

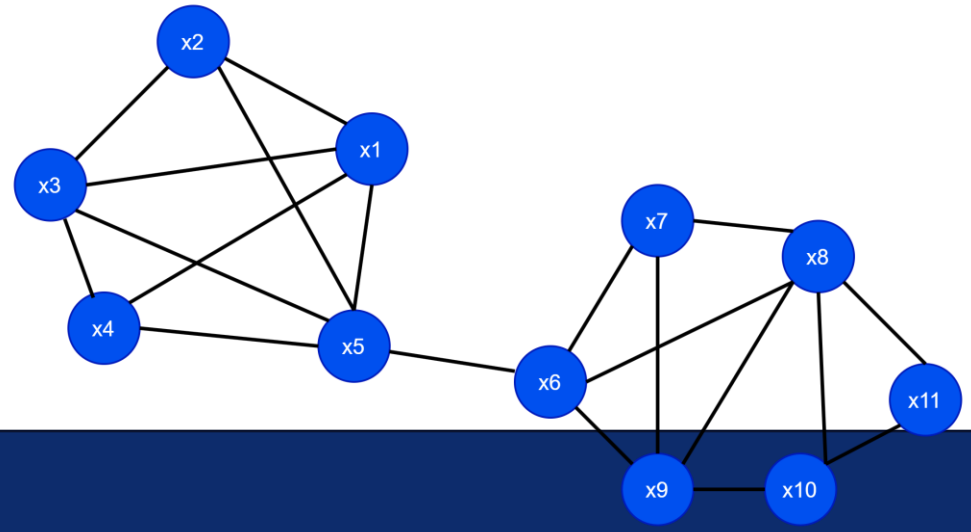
$$P(X = k) = \frac{N_k}{n}$$

- Given the node distribution $(0,0,0,2,3)$ what is the probability that a node is of degree 3?
  1. 0
  2. 2
  3. 3/5
  4. 3
  5. 2/5
  6. None of the above



Walk, Path, shortest path

- A walk in a graph G between nodes x and y is an ordered sequence of vertices, starting at x and ending at y.
  $Walk := < v_0, v_1, \ldots, v_t >, v_0 = x, v_t = y, \forall i \in [0..t-1]: (v_i, v_{i+1})\ exists$
- The length of the walk t, is the number of edges along the walk.
- A path is a walk with distinct vertices.
- A path of minimum length between nodes x and y is called a shortest path.

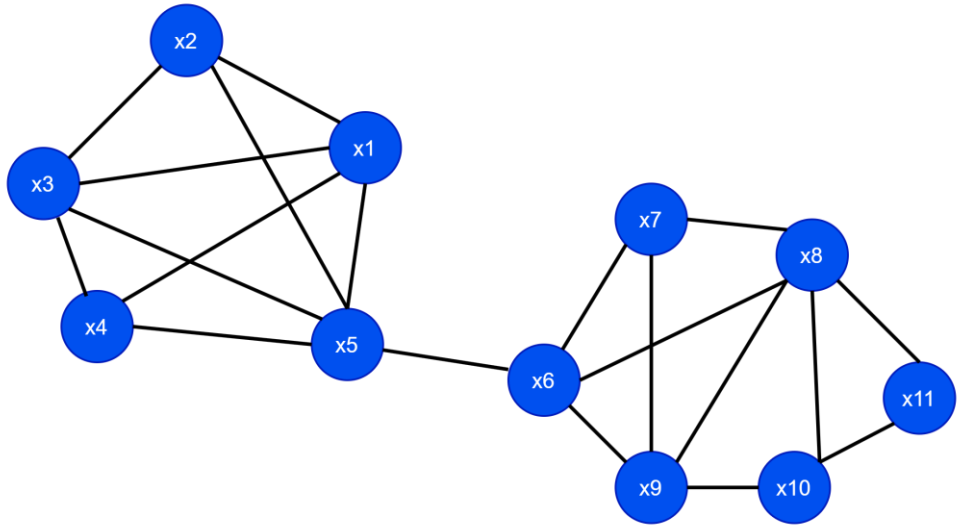


Walk, Path, shortest path

- A path of minimum length between nodes x and y is called a shortest path.

What is the length of the shortest path between $x2$ and $x10$?

1. 6
2. 3
3. 4
4. 1
5. 0

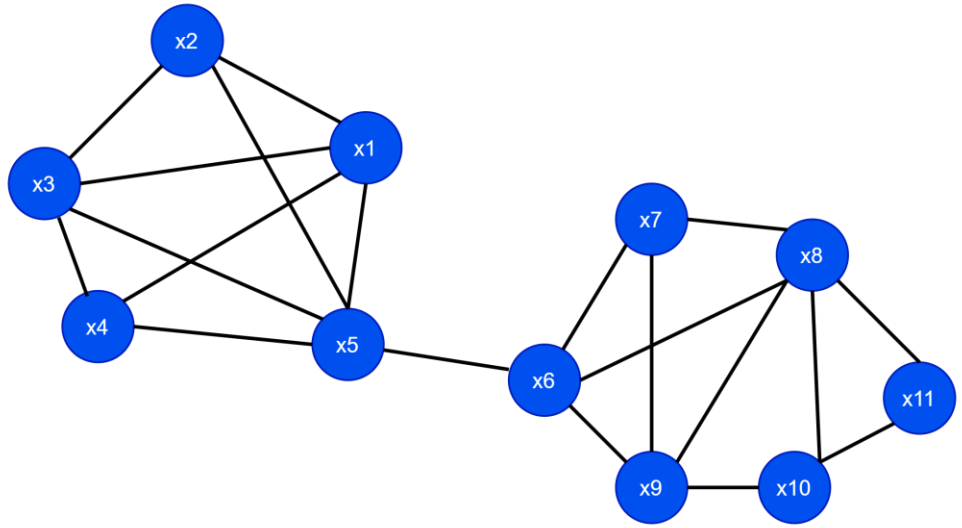


Connectedness

- Two nodes v_i and v_j are said to be connected if there exists a path between them.
- A graph is connected if there is a path between all pairs of vertices.
- A connected component, or just component, of a graph is a maximal connected subgraph.
  - **maximal** means that the subgraph cannot be extended any further while still maintaining the property of being connected.

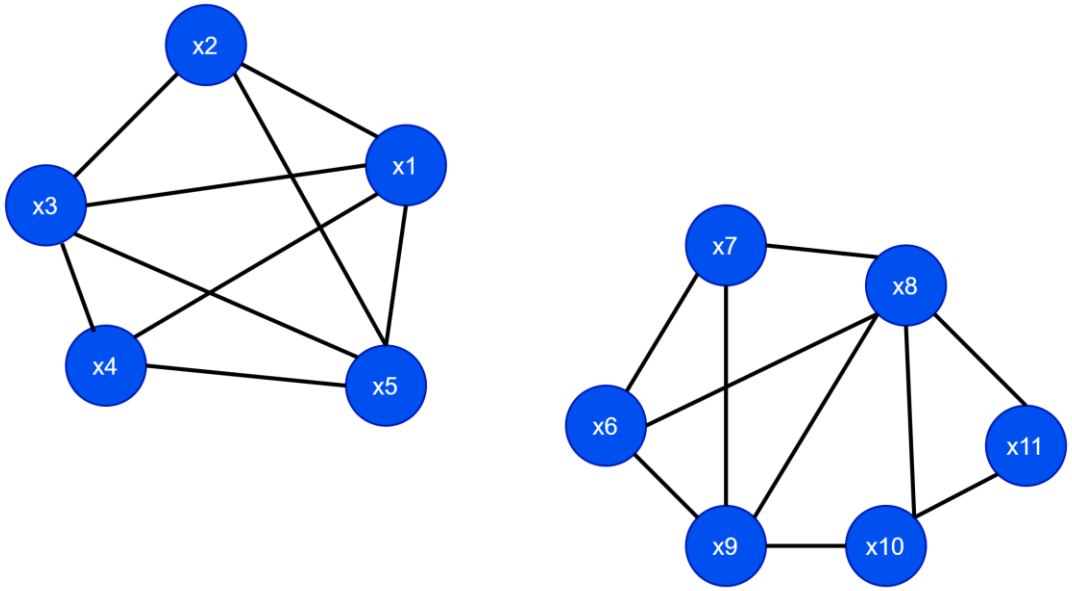
Connectedness

- Is this graph connected?
  - Yes
  - No



Connectedness

- Is this graph connected?
  - Yes
  - No



Adjacency Matrix

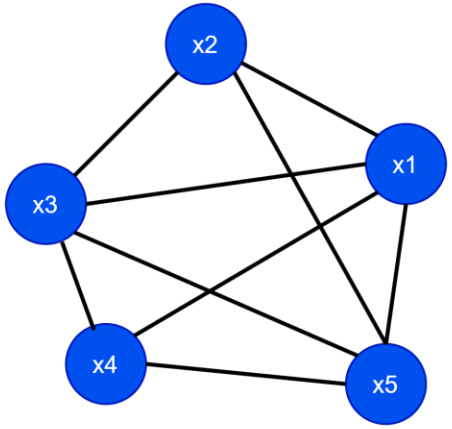
- A graph $G = (V, E)$, with $|V| = n$ vertices, can be conveniently represented in the form of an $n \times n$, symmetric binary adjacency matrix, A, defined as:

$$A(i,j) = \begin{cases} 1 & if\ v_i\ is\ adjacent\ to\ v_j \\ 0 & otherwise \end{cases}$$

- A weighted graph can be represented by $n \times n$ weighted adjacency matrix.

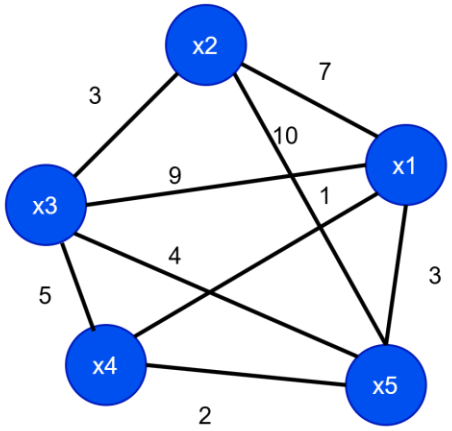
$$A(i,j) = \begin{cases} w_{ij} & if\ v_i\ is\ adjacent\ to\ v_j \\ 0 & otherwise \end{cases}$$

Adjacency matrix example.



	x_1	x_2	x_3	x_4	x_5
x_1	0	1	1	1	1
x_2	1	0	1	0	1
x_3	1	1	0	1	1
x_4	1	0	1	0	1
x_5	1	1	1	1	0

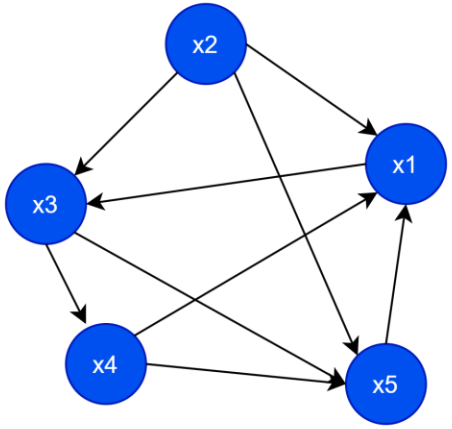
Adjacency matrix (weighted) example.



	x_1	x_2	x_3	x_4	x_5
x_1	0	7	9	1	3
x_2	7	0	3	0	10
x_3	9	3	0	3	4
x_4	1	7	5	0	2
x_5	3	10	4	2	0

Adjacency matrix: directed graph

- In a directed graph adjacency matrix is not symmetric.



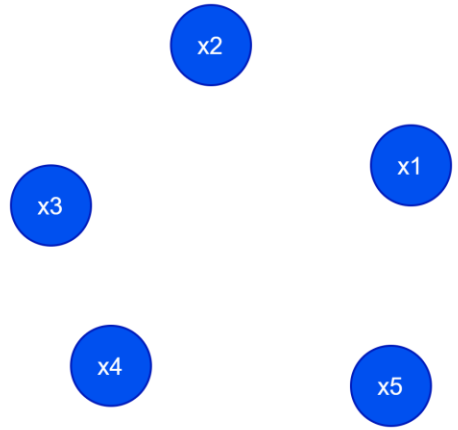
	x_1	x_2	x_3	x_4	x_5
x_1	0	0	1	0	0
x_2	1	0	1	0	1
x_3	0	0	0	1	1
x_4	1	0	0	0	1
x_5	1	0	0	0	0

Graphs from Data Matrix

- Given a dataset in the form of a matrix, can we create a graph?

	X_1	X_2	X_3
x_1	0.2	1	12.3
x_2	1.3	4	89.23
x_3	5.6	5	56.1
x_4	4.5	7	47.3
x_5	7.3	12	45.23

?



But what can we do about the edges?

Graphs from Data Matrix

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	X_1	X_2	X_3
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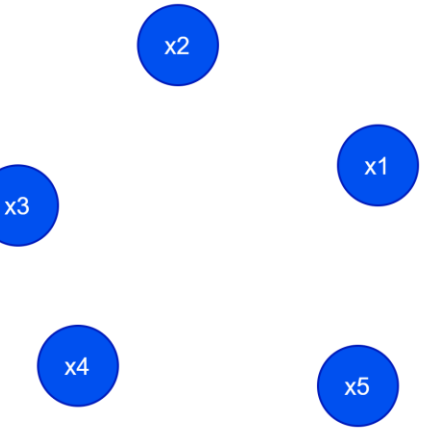
?



How about we using a similarity measure and then use the similarity measure as the edge weights?

How to create a graph from matrix?

- Define a weighted graph $G = (V, E)$.

$$V = \{v_i \mid v_i \, represents \, the \, entity \, x_i\}$$

$$w_{ij} = sim(x_i, x_j)$$

$$sim(x_i, x_j)$$

$$represents \, the \, similarity \, between \, points \, x_i and \, x_j$$

Gaussian similarity

$$w_{ij} = sim(x_i, x_j) = e^{-\frac{\|x_i - x_j\|^2}{2\sigma^2}}$$

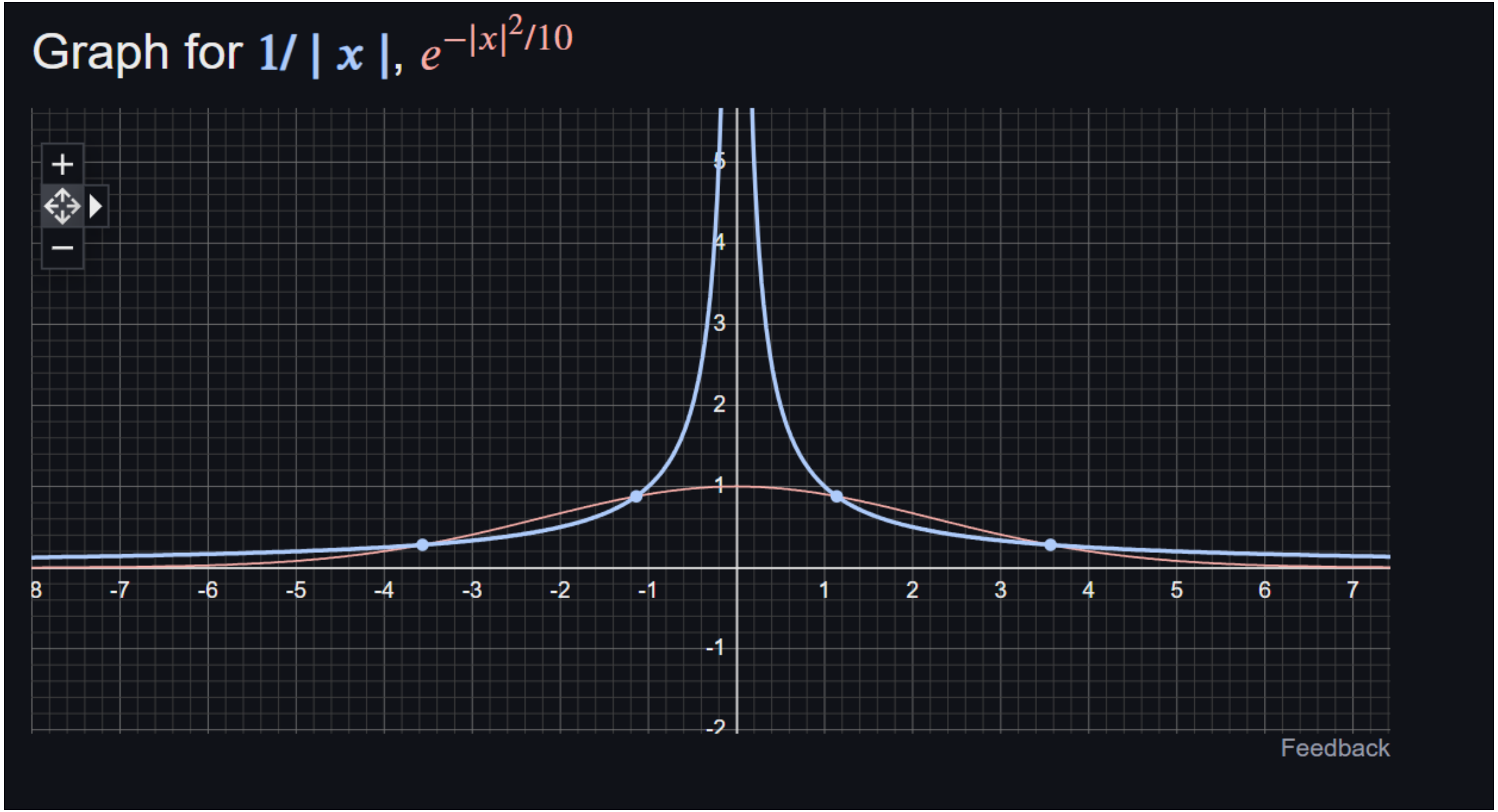
$$\sigma \, is \, the \, spread \, parameter.$$

Gaussian similarity

- Similarity is defined as being inversely related to the Euclidean distance.
- If two vectors are far apart, then we say it's less similar.
  - Therefore, we can put lower weight between them.
- But why do we use this?
  - We can use something like $\frac{1}{\|x_i - x_j\|}$

Gaussian similarity

- Exponential Decay
  - The similarity measure w_{ij} decays smoothly and asymptotically to 0 as the distance increases, ensuring that distant points contribute very little but not abruptly.
  - It is bounded between 0 and 1, which simplifies interpretation and normalization in algorithms.
- Inverse distance
  - $\frac{1}{\|x_i - x_j\|}$ decays too slowly as the distance increases, leading to non-negligible contributions from distant points.
  - It has an unbounded range $(0, \infty)$, which can create numerical instability and make it harder to interpret.


Graph for 1/ | x |, $e^{-|x|^2/10}$
5
4
3
2
1
-1
-2
8
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
7
Feedback

Gaussian similarity

- Handling zero distance:
  - When handling $\|x_i - x_j\| = 0$, w_{ij} simplifies to $e^0 = 1$, but if we use $\frac{1}{\|x_i - x_j\|}$, then mathematically this is not defined.
- Sensitivity control
  - The parameter σ allows you to control the sensitivity to distance.
    - Smaller σ, similarity decays quickly.
    - Larger σ, similarity decays slowly.
    - We can tweak the graph by changing this parameter.
  - $\frac{1}{\|x_i - x_j\|}$ does not have this property.

Gaussian similarity

- Robustness to outliers.
  - Gaussian similarity drops off quickly for large distances, effectively ignoring the outliers.
  - $\frac{1}{\|x_i - x_j\|}$, even if far away, can have disproportionately large effect to slow decay of $\frac{1}{d}$.

Graphs from Data Matrix

- Gaussian similarity with $\sigma = 25$

	X_1	X_2	X_3
x_1	0.2	1	12.3
x_2	1.3	4	89.23
x_3	5.6	5	56.1
x_4	4.5	7	47.3
x_5	7.3	12	45.23

0	0.008709	0.207862	0.359302	0.366178
0.008709	0	0.409967	0.241611	0.196716
0.207862	0.409967	0	0.936019	0.87281
0.359302	0.241611	0.936019	0	0.970737
0.366178	0.196716	0.87281	0.970737	0

Graphs from Data Matrix

- Gaussian similarity with $\sigma = 25$

	X_1	X_2	X_3
x_1	0.2	1	12.3
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x_3	5.6	5	56.1
x_4	4.5	7	47.3
x_5	7.3	12	45.23

	x1	x2	x3	x4	x5
x1	0	0.008709	0.207862	0.359302	0.366178
x2	0.008709	0	0.409967	0.241611	0.196716
x3	0.207862	0.409967	0	0.936019	0.87281
x4	0.359302	0.241611	0.936019	0	0.970737
x5	0.366178	0.196716	0.87281	0.970737	0

Graphs from Data Matrix

- Gaussian similarity with $\sigma = 50$

	X_1	X_2	X_3
x_1	0.2	1	12.3
x_2	1.3	4	89.23
x_3	5.6	5	56.1
x_4	4.5	7	47.3
x_5	7.3	12	45.23

	x1	x2	x3	x4	x5
x1	0	0.30549	0.675218	0.774221	0.777899
x2	0.30549	0	0.800179	0.701098	0.665978
x3	0.675218	0.800179	0	0.983606	0.966562
x4	0.774221	0.701098	0.983606	0	0.992603
x5	0.777899	0.665978	0.966562	0.992603	0

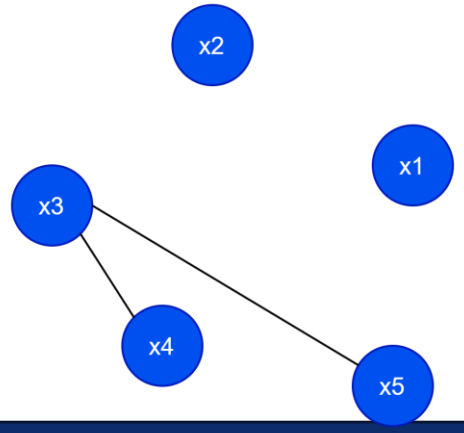
Creating a graph from matrix

	X_1	X_2	X_3
x_1	0.2	1	12.3
x_2	1.3	4	89.23
x_3	5.6	5	56.1
x_4	4.5	7	47.3
x_5	7.3	12	45.23

	x1	x2	x3	x4	x5
x1	0	0.30549	0.675218	0.774221	0.777899
x2	0.30549	0	0.800179	0.701098	0.665978
x3	0.675218	0.800179	0	0.983606	0.966562
x4	0.774221	0.701098	0.983606	0	0.992603
x5	0.777899	0.665978	0.966562	0.992603	0

$$\tau = 0.94$$

	x1	x2	x3	x4	x5
x1	0	0	0	0	0
x2	0	0	0	0	0
x3	0	0	0	1	1
x4	0	0	1	0	1
x5	0	0	1	1	0



$$sim(x_i, x_j) = e^{-\frac{\|x_i - x_j\|^2}{2\sigma^2}}$$

$$A(i,j) = \begin{cases} 1 & if sim(x_i, x_j) \geq \tau \\ 0 & otherwise \end{cases}$$

Iris Similarity Graph: Gaussian Similarity

$\sigma = \frac{1}{\sqrt{2}}$, edge exist if and only if $w_{ij} \geq 0.777$

Order: $|V| = n = 150$, size: $|E| = m = 753$

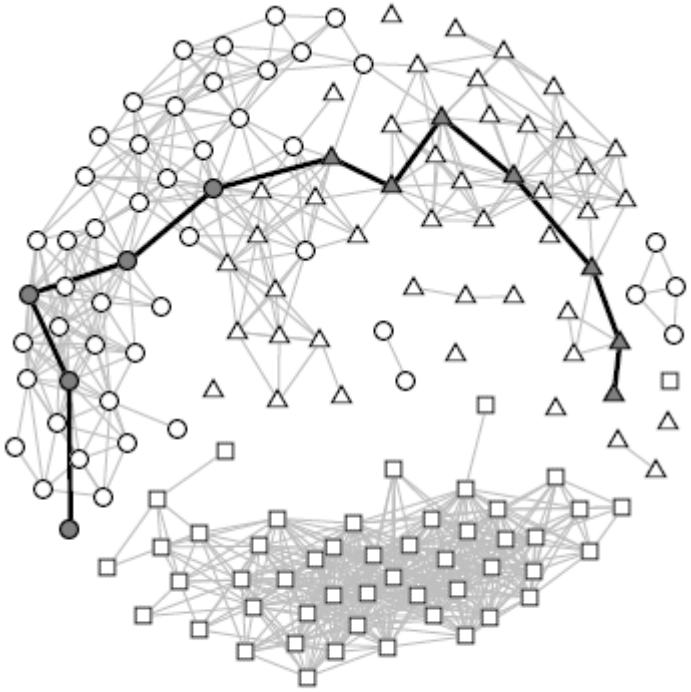


Figure 4.2: Iris Similarity Graph

Topological graph attributes

- **Topological graph attributes** refer to properties of a graph that describe its structure and connectivity without considering specific geometric or spatial embedding.
- We say a graph attribute is:
  - Local if they apply only to a single node.
  - Global if they refer to the entire graph.

- Degree
  - The degree of a node $v_i \in G$ is defined as:

$$d_i = \sum_j A_{ij}$$

  - Clearly, this is a local attribute.
  - The corresponding global attribute for the entire graph G is the average degree:

$$\mu_d = \frac{\sum_i d_i}{n}$$

- We can generalize this to weighted and directed graphs as well.
  - The in-degree of a node $v_i \in G$ is defined as:

$$id(v_i) = \sum_j A(j,i)$$

$$od(v_i) = \sum_j A(i,j)$$

  - The average indegree and average outdegree can be obtained by summing them up and dividing by n.

$$\mu_{indeg} = \frac{\sum_i id(v_i)}{n}, \mu_{outndeg} = \frac{\sum_i od(v_i)}{n}$$

Average path length

- The average path length, also called the characteristic path length, of a connected graph is given as:

$$\mu_L = \frac{\sum_i \sum_{j>i} d(v_i, v_j)}{\binom{n}{2}} = \frac{2}{n(n-1)} \sum_i \sum_{j>i} d(v_i, v_j)$$

- For directed graphs,

$$\mu_L = \frac{1}{n(n-1)} \sum_i \sum_{j>i} d(v_i, v_j)$$

For disconnected graphs , the average is taken over only the connected pairs of vertices.

Eccentricity

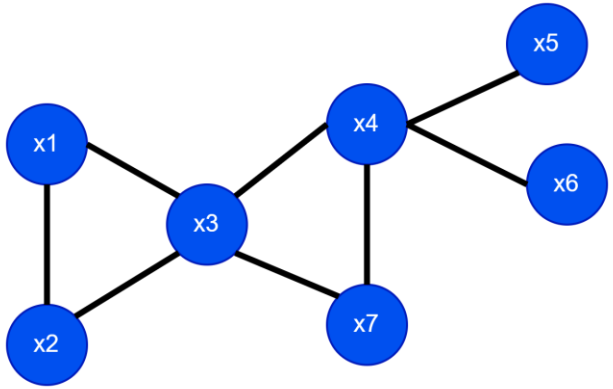
- The eccentricity of a node v_i is the maximum distance from v_i to any other node in the graph:

$$e(v_i) = \max_j \{d(v_i, v_j)\}$$

- The less eccentric a node is more central it is in the graph.

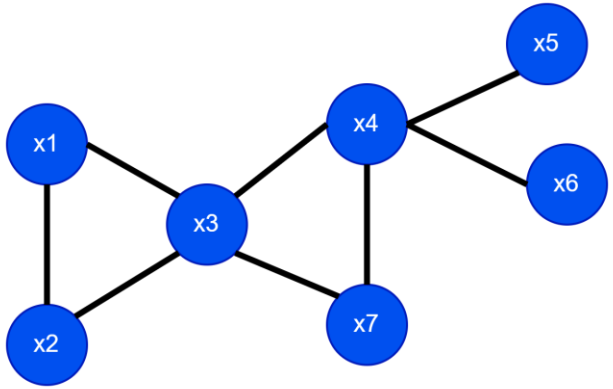
Example:

- What is the eccentricity of $x3$?
  A. 2
  B. 3
  C. 4
  D. 5



Example:

- What is the eccentricity of $x6$?
  A. 2
  B. 3
  C. 4
  D. 5



Radius and Diameter of a graph G

- The radius of a connected graph, denoted $r(G)$, is the minimum eccentricity of any node in the graph.

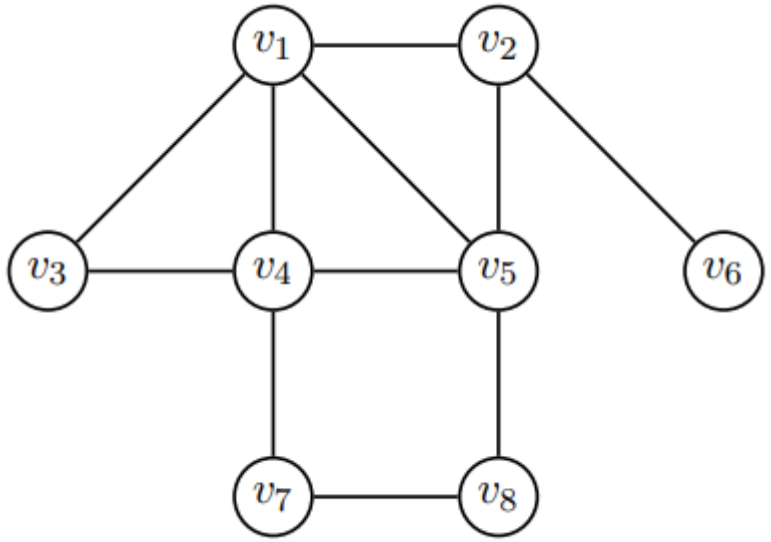
$$r(G) = \min_i\{e(v_i)\} \ = \min_i\left\{\max_j\{d(v_i, v_j)\}\right\}$$

- The diameter, denoted $d(G)$, is the maximum eccentricity of any vertex in the graph

$$d(G) = \max_i\{e(v_i)\} = \max_{ij}\{d(v_i, v_j)\}$$

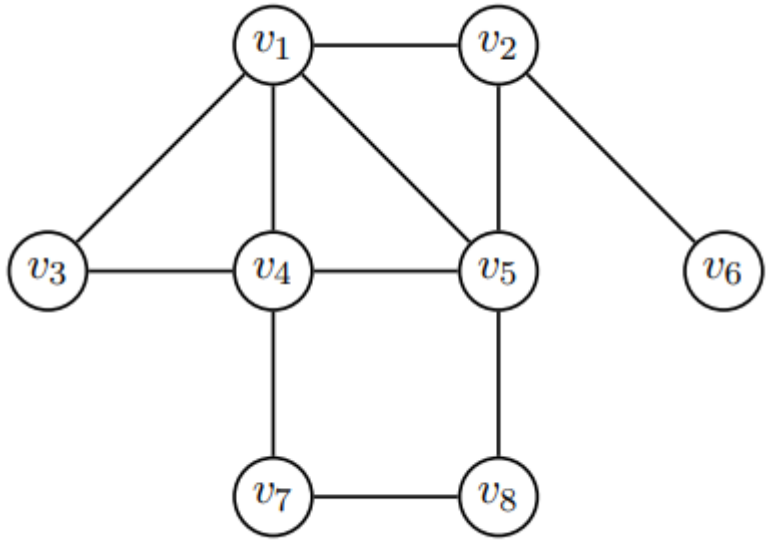
- For a disconnected graph, values are computed over the connected components of the graph.
- The diameter of a graph G is sensitive to outliers.

- What is the radius and diameter of the following graph?
  1. 2
  2. 3
  3. 4
  4. 5



(a)

- What is the radius and diameter of the following graph?
  1. 2
  2. 3
  3. 4
  4. 5



(a)

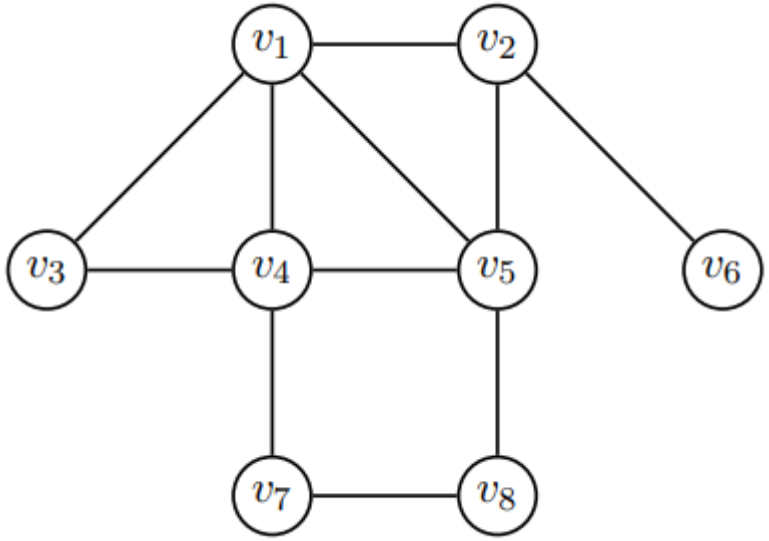
Clustering coefficient of a node

- The clustering coefficient of a node v_i is a measure of the density of edges in the neighborhood of v_i.
- Let $G_i = (V_i, E_i)$ be the subgraph induced by the neighbors of vertex v_i. Note that $v_i \notin V_i$, since we assume that G is simple. Let $|V_i| = n_i$ be the number of neighbors of v_i, and $|E_i| = m_i$ be the number of edges among the neighbors of v_i. The clustering coefficient of v_i is defined as:

$$C(v_i) = \frac{\#\ of\ edges\ in\ G_i}{maximum\ number\ of\ edges\ in\ G_i} = \frac{m_i}{\binom{n_i}{2}} = \frac{2 \cdot m_i}{n_i(n_i - 1)}$$

What is the clustering coefficient of v_3?

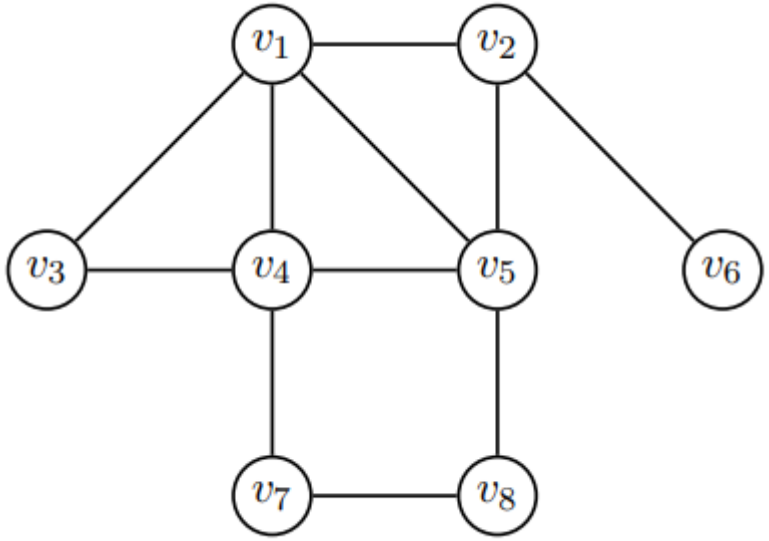
1. 1
2. 1/3
3. 1/2
4. 1/6
5. 6/10



(a)

What is the clustering coefficient of v_5?

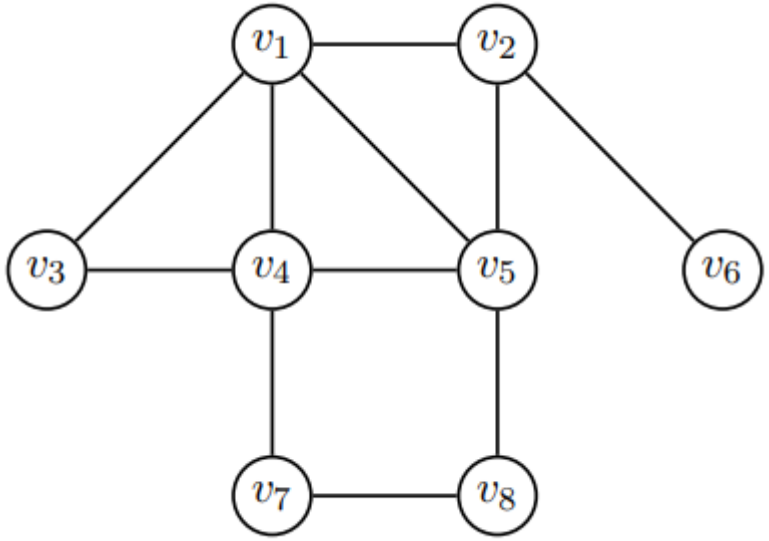
1. 1
2. 1/3
3. 1/2
4. 1/6
5. 6/10



(a)

What is the clustering coefficient of v_6?

1. 1
2. 1/3
3. 1/2
4. 1/6
5. 6/10
6. Not defined



(a)

Clustering coefficient of a node

- Note: Clustering coefficient of a node is not defined for nodes with degree less than 2.
- Therefore, if we need, we can consider 0 for the clustering coefficient for nodes with degree less than 2.

Clustering coefficient of a node

- How to interpret this value?
  - $C(v_i) = 1 \rightarrow$ All neighbors are connected to each other.
  - $C(v_i) = 0 \rightarrow$ None of the neighbors are connected to each other.
  - A higher clustering coefficient indicates a more tightly knit local community around the node.
- Example:
  - Fraud Detection in Financial Networks
    - In a bank transaction network, accounts are nodes, and transactions form edges.
    - Fraud rings often have a high clustering coefficient since fraudulent accounts transfer money within a small, well-connected group.

Clustering coefficient of a graph

- The clustering coefficient of a graph G is simply the average clustering coefficient over all the nodes, given as:

$$C(G) = \frac{1}{n} \cdot \sum_{i} C(v_i)$$

Clustering coefficient of a graph

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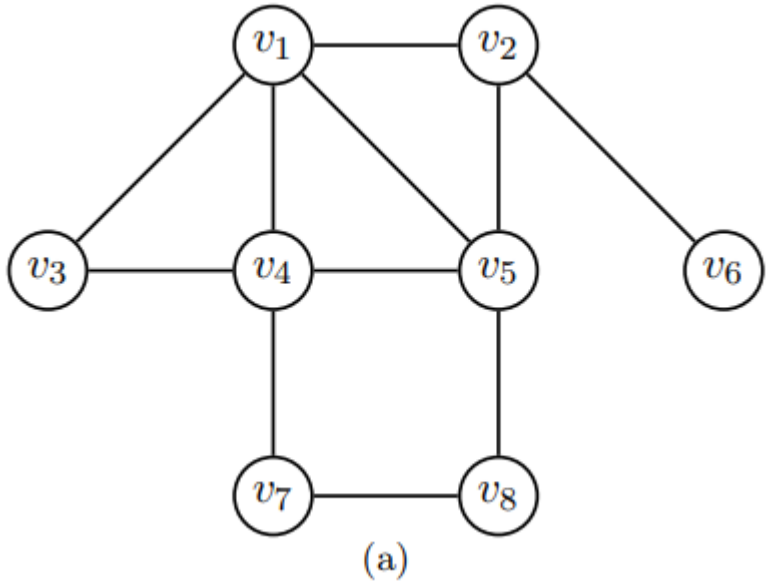
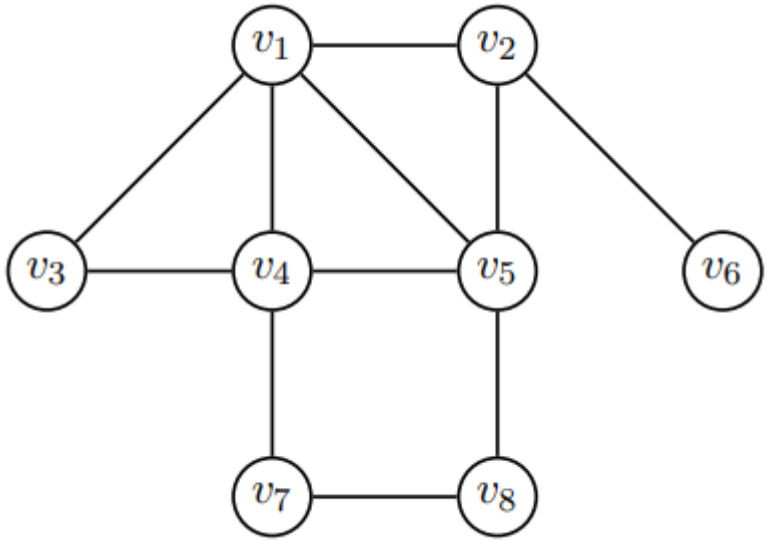


(a)

What is the clustering coefficient of this graph?

1. 3/8
2. 3/16
3. 5/16
4. 1/2



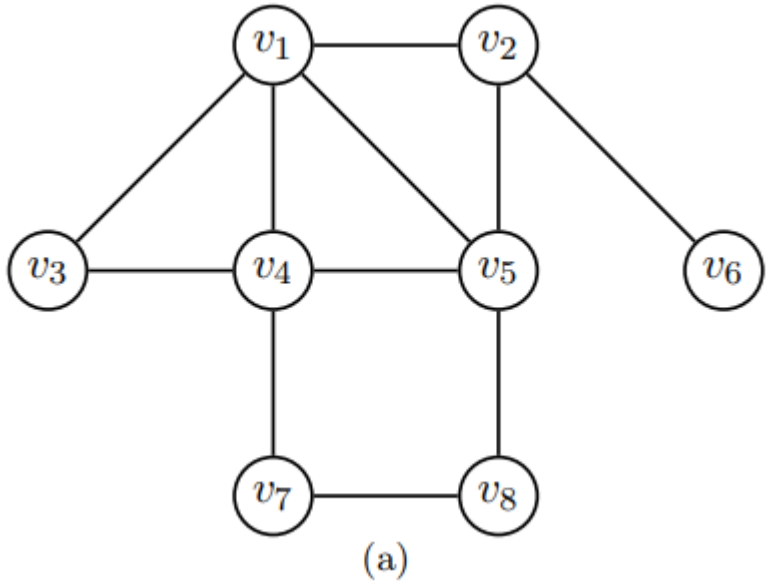
(a)

$$C(G) = \frac{1}{8} \cdot \left(\frac{3}{\binom{4}{2}} + \frac{1}{\binom{3}{2}} + 1 + \frac{2}{\binom{4}{2}} + \frac{2}{\binom{4}{2}} + 0 + 0 + 0 \right) = \frac{2.5}{8} = \frac{5}{16}$$

Clustering coefficient of a graph

- The clustering coefficient of a graph G is simply the average clustering coefficient over all the nodes, given as:

$$C(G) = \frac{1}{n} \cdot \sum_{i} C(v_i)$$



(a)

What is the clustering coefficient of this graph?

1. 3/8
2. 3/16
3. 5/16
4. 1/2

Efficiency

- The efficiency for a pair of nodes v_i and v_j is defined as $\frac{1}{d(v_i,v_j)}$
- If v_i and v_j are not connected, then efficiency between these two vertices is $\frac{1}{\infty} \approx 0$.
- Smaller the distance between two nodes, these nodes are efficient in communicating between them.
- The efficiency of a graph is defined as G is defined as the average efficiency of over all pairs of nodes.

$$\frac{2}{n \cdot (n-1)} \cdot \sum_{i} \sum_{j>i} \frac{1}{d(v_i, v_j)}$$

Measure of centrality

- The notion of centrality is used to rank the vertices of a graph in terms of how **“central”** or important they are.
- A centrality can be formally defined as a function:

$$c: V \rightarrow \mathbb{R}$$
$$that\ induces\ a\ total\ order\ on\ V$$

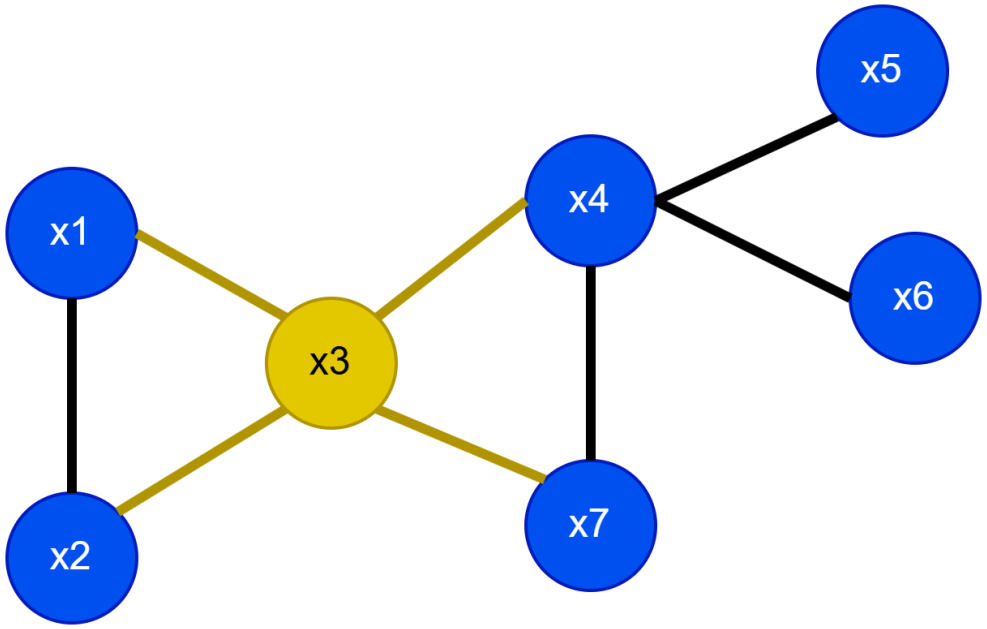
- We say that v_i is at least as central as v_j if $c(v_i) \geq c(v_j)$.
- A set has a total order if it has a partial order and every pair of elements in the set are comparable.

Degree centrality

- The simplest notion of centrality is the degree d_i of a vertex v_i.
- the higher the degree the more important or central the vertex.
- For directed graphs, one may further consider the indegree centrality and outdegree centrality of a vertex.

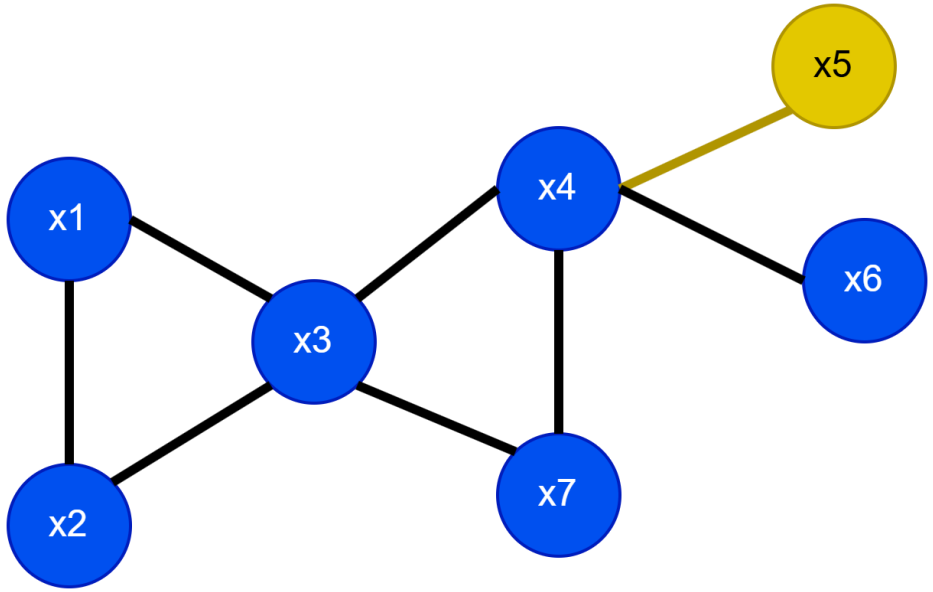
Degree centrality

$$d(v_i) = \sum_{j}^{n} A(i,j)$$



$$d(x_3) = \sum_{1}^{n} A(3,j) = 4$$

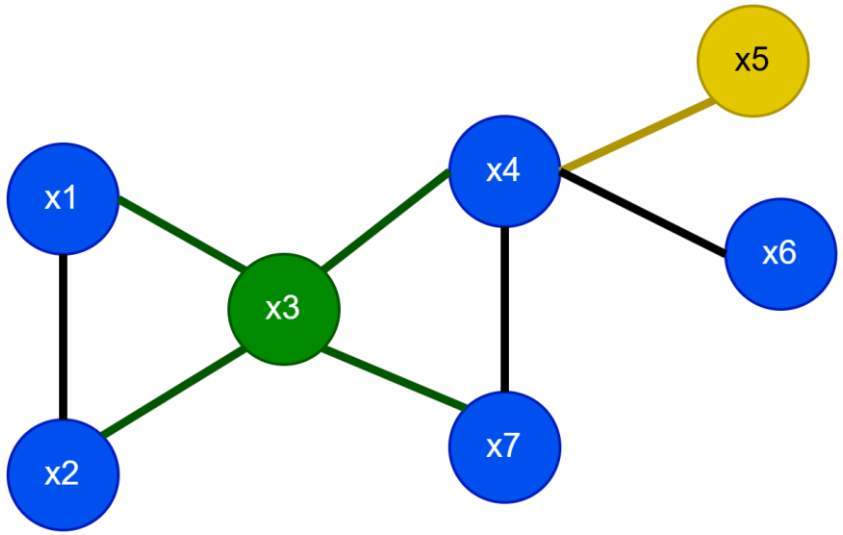
Degree centrality



$$d(v_i) = \sum_{j}^{n} A(i,j)$$

$$d(x_5) = \sum_{1}^{n} A(5,j) = 1$$

Degree centrality



$$d(x_3) = \sum_{1}^{n} A(3, j) = 4$$

$$d(x_5) = \sum_{1}^{n} A(5, j) = 1$$

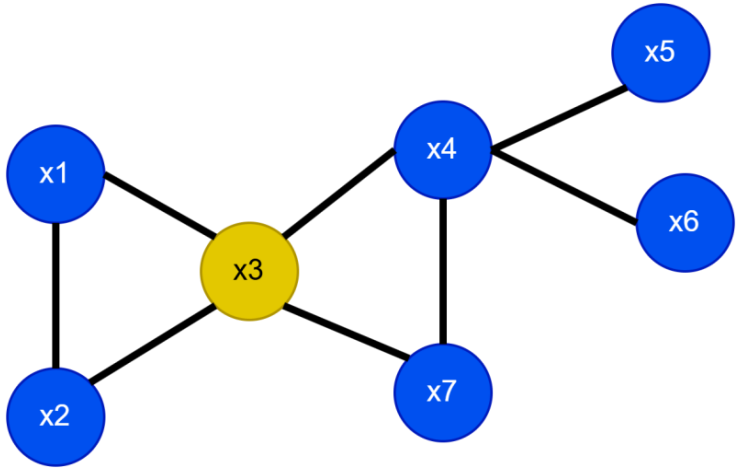
$x3$ has higher importance than $x5$ according to degree centrality.

Eccentricity Centrality

- Eccentricity centrality is thus defined as follows

$$c(v_i) = \frac{1}{e(v_i)} = \frac{1}{\max_j \{d(v_i, v_j)\}}$$

- The less eccentric a node is the more central it is.



What is the eccentric centrality of $x3$?

1. ½
2. 1/3
3. 4
4. 2
5. 3

Closedness centrality

- The closeness centrality uses the sum of all the distances to rank how central a node is.

$$c(v_i) = \frac{1}{\sum_j d(v_i, v_j)}$$

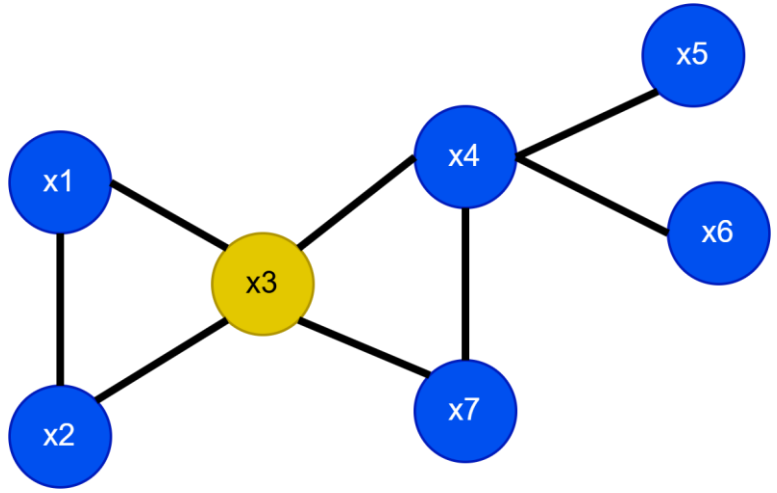
A node v_i with the smallest total distance, $\sum_j d(v_i, v_j)$is called the ***median node***.

- Why is this important?

Closedness centrality

- Closedness centrality tries to minimize the total distance over all the other nodes, and thus, a median node, which has the highest closedness centrality, is the optimal one to, say, locate a facility that minimizes the distance to all other points.

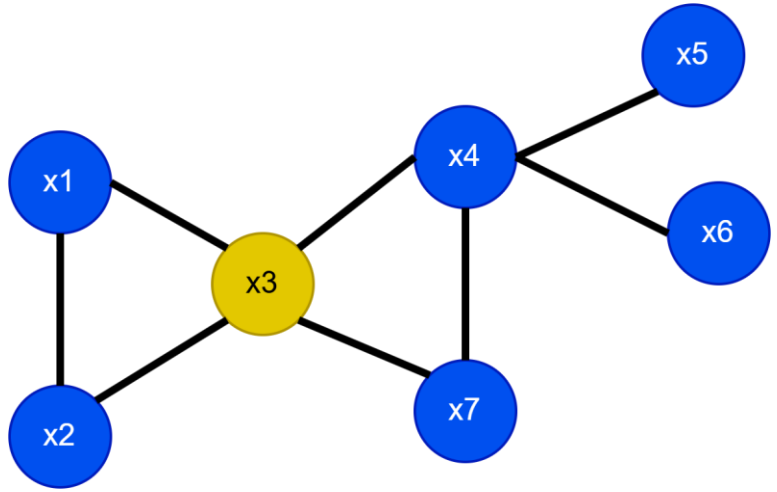
Closedness centrality



What is the **closedness centrality** of $x3$?

1. 1/8
2. 1/3
3. 3
4. 4
5. 8

Closedness centrality



What is the **eccentricity centrality** of $x3$?

1. ½
2. 1/3
3. 4
4. 2
5. 3

Betweenness centrality

- For a given vertex v_i the betweenness centrality measures how many shortest paths between all pairs of vertices include v_i.
- This gives an indication as to the central "**monitoring**" role played by v_i for various pairs of nodes.

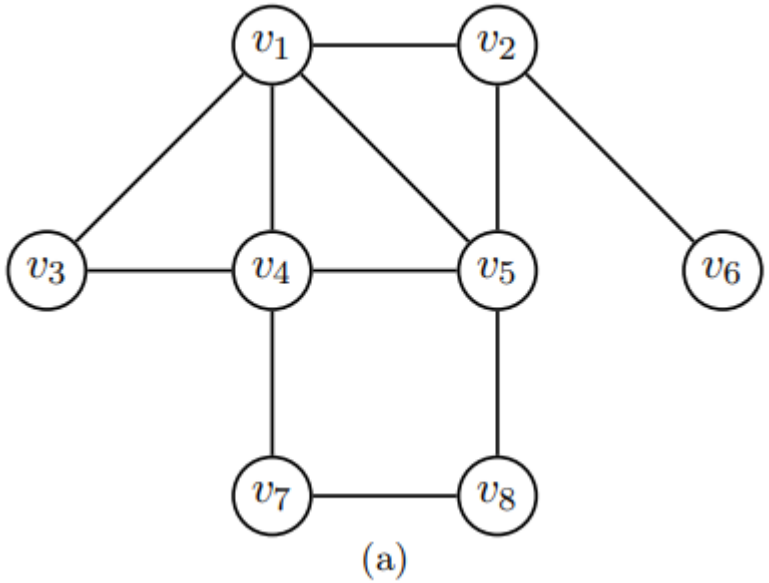
Betweenness centrality

- Let η_{jk} denote the number of shortest paths between vertices v_j and v_k, and let
- $\eta_{jk}(v_i)$ denote the number of such paths that include or contain v_i, then the fraction of paths through v_i is denoted as

$$\gamma_{jk}(v_i) = \frac{\eta_{jk}(v_i)}{\eta_{jk}}.$$

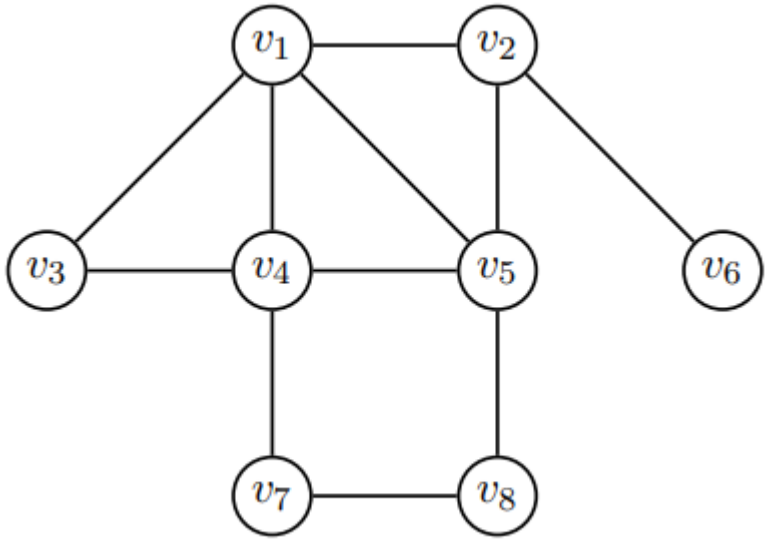
- If the two vertices v_j and v_k are not connected, we assume $\gamma_{jk} = 0$.

$$c(v_i) = \sum_{j \neq i} \sum_{\substack{k \neq i \\ k > j}} \gamma_{jk}$$



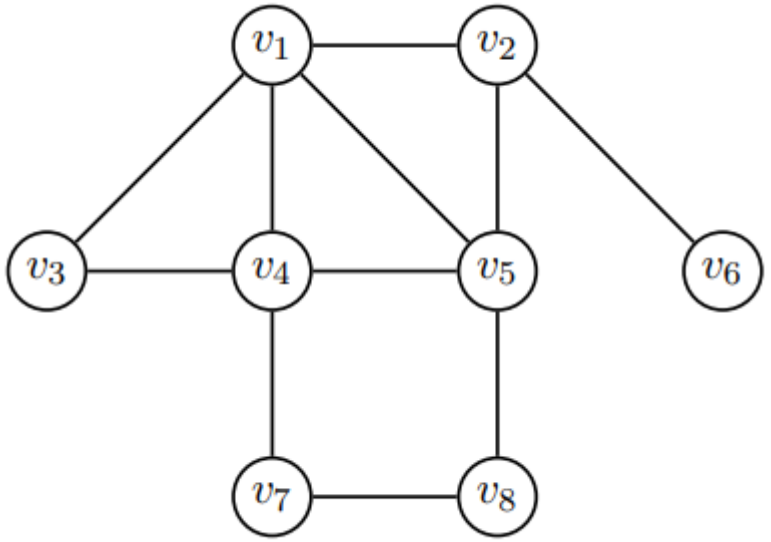
(a)

What is the betweenness centrality centrality of v_5?



(a)

	Shortest paths	Shortest paths through v_5 $\eta_{jk}(v_5)$
η_{12}	1	0
η_{13}	1	0
η_{14}	1	0
η_{16}	1	0
η_{17}	1	0
η_{18}	1	1
η_{23}	1	0
η_{24}	2	1
η_{26}	1	0
η_{27}	3	2
η_{28}	1	1
η_{34}	1	0
…		

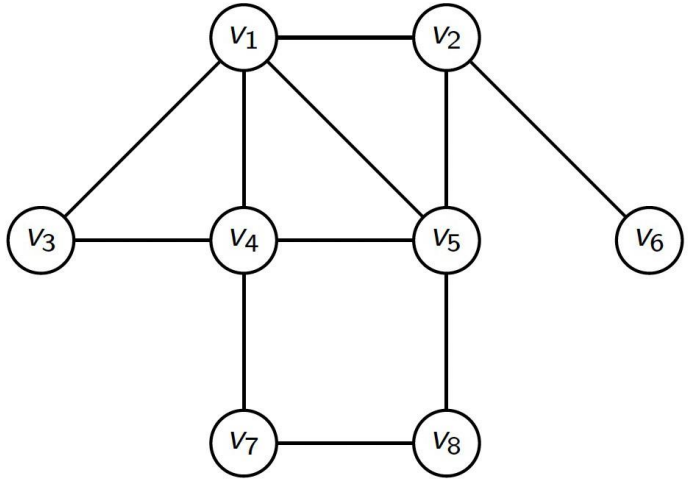


(a)

What is the betweenness centrality centrality of v_5?

$$c(v_5) = \gamma_{18} + \gamma_{24} + \gamma_{27} + \gamma_{28} + \gamma_{38} + \gamma_{46} + \gamma_{48} + \gamma_{67} + \gamma_{68}$$

$$c(v_5) = 1 + \frac{1}{2} + \frac{2}{3} + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} = 6.5$$



Centrality	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
Degree	4	3	2	4	4	1	2	2
Eccentricity $e(v_i)$	0.5 2	0.33 3	0.33 3	0.33 3	0.5 2	0.25 4	0.25 4	0.33 3
Closeness $\sum_j d(v_i, v_j)$	0.100 10	0.083 12	0.071 14	0.091 11	0.100 10	0.056 18	0.067 15	0.071 14
Betweenness	4.5	6	0	5	6.5	0	0.83	1.17

Centrality measures for directed graphs.

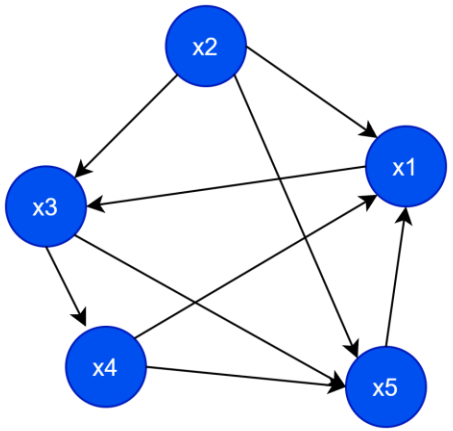
- Up until now we only looked at undirected graphs.
- Let’s look at how we can use centrality measures for directed graphs.
- Especially in the context of the web, like citation networks, and centrality measures that are best suited for large-scale web graphs.

Prestige (Eigenvector centrality)

- The idea is, the more the links that point to a given node, the higher its prestige.
- The prestige does not simply depend on the indegree, it also (recursively) depends on the prestige of the nodes that point to it.

Prestige (Eigenvector centrality)

- Recap:
- Let $G = (V, E)$, be a directed graph, with $|V| = n$. The adjacency matrix of G is a $n \times n$ asymmetric matrix A is given as:



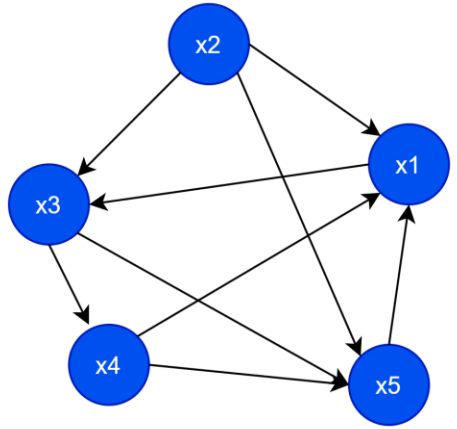
$$A(u,v) = \begin{cases} 1 & if\ (u,v) \in E \\ 0 & if\ (u,v) \notin E \end{cases}$$

	x_1	x_2	x_3	x_4	x_5
x_1	0	0	1	0	0
x_2	1	0	1	0	1
x_3	0	0	0	1	1
x_4	1	0	0	0	1
x_5	1	0	0	0	0

Prestige (Eigenvector centrality)

Idea: If there are many ***prestigious*** nodes pointing to a node, then the node is itself prestigious. (recursive definition)

$$p(x_5) = p(x_4) + p(x_3) + p(x_2)$$



	x_1	x_2	x_3	x_4	x_5
x_1	0	0	1	0	0
x_2	1	0	1	0	1
x_3	0	0	0	1	1
x_4	1	0	0	0	1
x_5	1	0	0	0	0

$$p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

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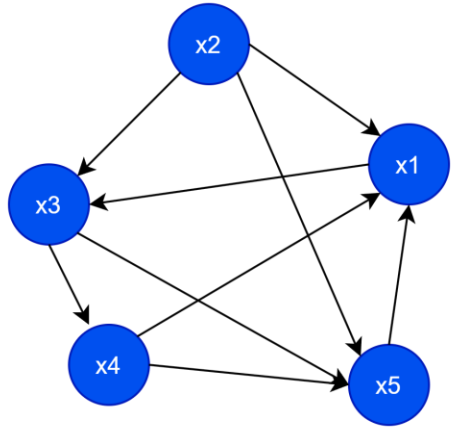
Prestige (Eigenvector centrality)

$$p(x_5) = p(x_4) + p(x_3) + p(x_2)$$
$$p(x_1) = p(x_2) + p(x_4) + p(x_5)$$
$$p(x_2) = 0$$
$$p(x_3) = p(x_1) + p(x_2)$$
$$p(x_4) = p(x_3)$$

	x_1	x_2	x_3	x_4	x_5
x_1	0	0	1	0	0
x_2	1	0	1	0	1
x_3	0	0	0	1	1
x_4	1	0	0	0	1
x_5	1	0	0	0	0

$$p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$



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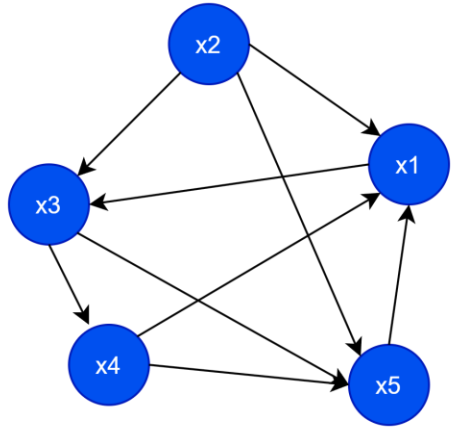
Prestige (Eigenvector centrality)

- **Circular Dependencies**: Prestige often relies on the prestige of the nodes pointing to you. Imagine a simple cycle: A -> B -> C -> A. To calculate A's prestige, we need B's. To calculate B's, we need C's. To calculate C's, we need A's! This creates a circular dependency. We can't just start calculating because we don't have the "starting points."



	x_1	x_2	x_3	x_4	x_5
x_1	0	0	1	0	0
x_2	1	0	1	0	1
x_3	0	0	0	1	1
x_4	1	0	0	0	1
x_5	1	0	0	0	0

$$p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

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Linear algebra review..

- We need some additional linear algebra knowledge.. Let's do a recap

- Transpose of a matrix:

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ -4 & 1 & 2 \end{pmatrix} \quad \mathrm{A} = (a_{ij}), \qquad then\ A_{ij}^{T} = (a_{ji})$$

The element in row i and column j of A^T is the element in row j and column i of A.

$$A^T = \begin{pmatrix} 1 & 0 & -4 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix}$$

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Matrix vector multiplication

- Matrix vector multiplication.

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ -4 & 1 & 2 \end{pmatrix} \qquad v = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

For $A \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^n, then\ the\ ith\ element\ of\ Av\ is: (Av)_i = \sum_{j=1}^{n} A_{ij} v_j$

Matrix vector multiplication

- Matrix vector multiplication.

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ -4 & 1 & 2 \end{pmatrix} \qquad v = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

For $A \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^n, then\ the\ ith\ element\ of\ Av\ is: (Av)_i = \sum_{j=1}^{n} A_{ij} v_j$

$$Av = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1(2) + 3(1) + 0(3) \\ 0(2) + 2(1) + 4(3) \\ -4(2) + 1(1) + 2(3) \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \\ -1 \end{pmatrix}$$

Matrix vector multiplication

- Find the product of Av

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ -4 & 1 & 2 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

For $A \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^n, then\ the\ ith\ element\ of\ Av\ is: (Av)_i = \sum_{j=1}^{n} A_{ij} v_j$

Matrix vector multiplication

- Find the product of Av

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ -4 & 1 & 2 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

For $A \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^n, then\ the\ ith\ element\ of\ Av\ is: (Av)_i = \sum_{j=1}^{n} A_{ij} v_j$

$$Av = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 4 \\ -4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1(1) + 3(1) + 0(2) \\ 0(1) + 2(1) + 4(2) \\ -4(1) + 1(1) + 2(2) \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 1 \end{pmatrix}$$

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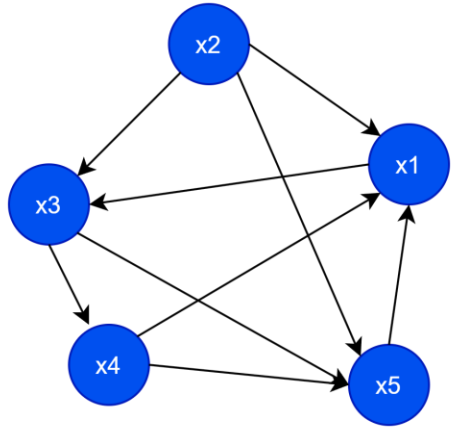
Prestige (Eigenvector centrality)

- **Idea: First map this calculation to matrix multiplication**

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$



We took a row of the transpose of the A

$$p(x_3) = \sum_u A(u, x_3) \cdot p(u)$$

$$p(x_3) = \sum_u A^T(x_3, u) \cdot p(u)$$

$$p(x_3) = A^T_{3*} p = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} = p(x_1) + p(x_2)$$

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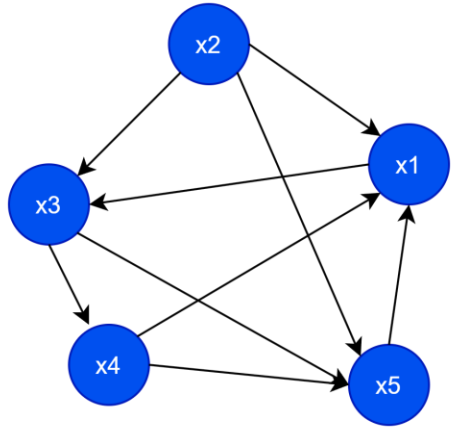
Prestige (Eigenvector centrality)

- **Idea: First map this calculation to matrix multiplication**

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad A^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$



$$p(x_2) = A^T_{2*} p = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} = 0$$

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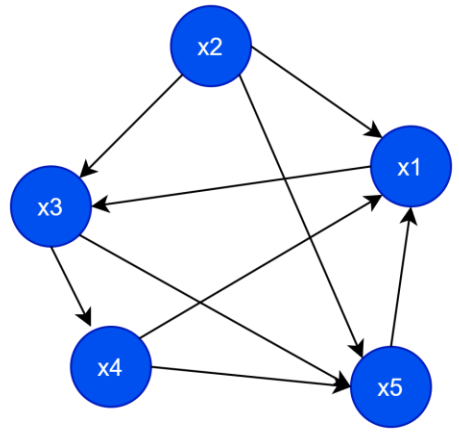
Prestige (Eigenvector centrality)

- **Idea: First map this calculation to matrix multiplication.**
- **Consider the matrix $A^T p$.**
- **Since each row of this column vector be the prestige of the corresponding node:**

$$\boldsymbol{p = A^T p}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$



$$\begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

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Prestige (Eigenvector centrality)

- **Idea: First map this calculation to matrix multiplication.**
- **Consider the matrix $A^T p$.**
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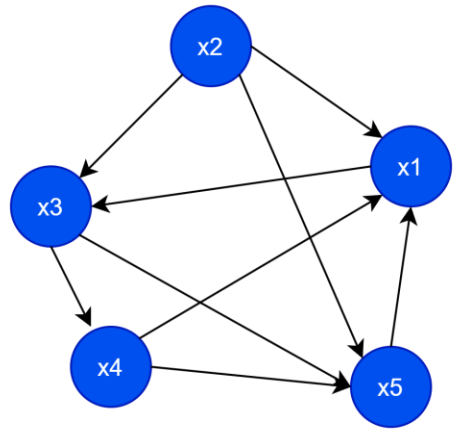
$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$



$$\begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

How do we solve this? (basically, prestige vector is an eigenvector of the A^T matrix.)

We use an algorithm called power iteration method to retrieve the dominant eigenvector of A^T.

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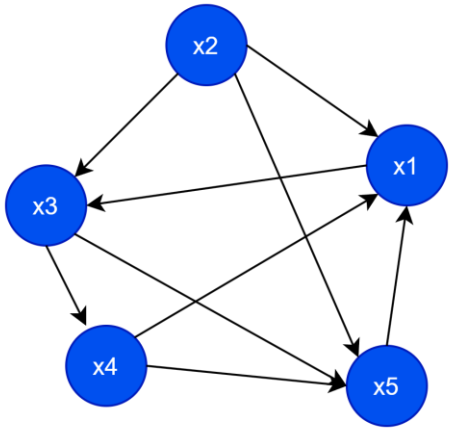
Prestige (Eigenvector centrality)

- **Idea: First map this calculation to matrix multiplication.**
- **Consider the matrix $A^T p$.**
- **Since each row of this column vector be the prestige of the corresponding node:**

$$\boldsymbol{p = A^T p}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$



$$\begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

How do we solve this?

We start with an initial prestige vector (**this is a guess**—we'll talk about this in next slides) p_0 and iterate:

$$p_1 = A^T p_0$$

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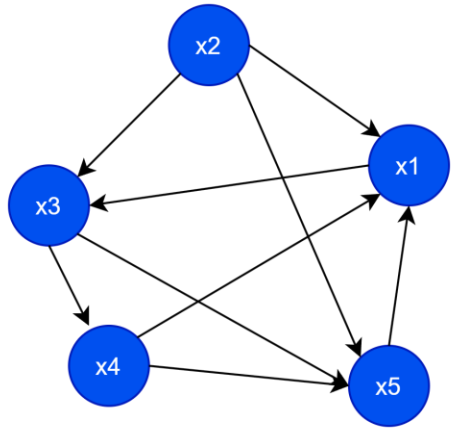
Prestige (Eigenvector centrality)

$$\boldsymbol{p} = \boldsymbol{A^T p}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad A^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$



We start with an initial prestige vector (a guess) p_0 and iterate:

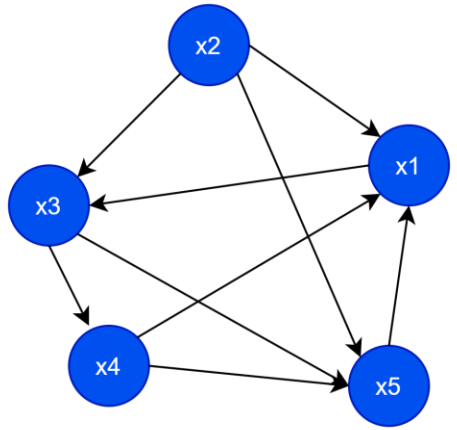
$$p_1 = A^T p_0$$

$$p_2 = A^T p_1 = A^T (A^T p_0) = (A^T)^2 p_0$$

Prestige (Eigenvector centrality)

$$\boldsymbol{p} = \boldsymbol{A^T p}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad A^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$



We start with an initial prestige vector (a guess) p_0 and iterate:

$$p_1 = A^T p_0$$

$$p_2 = A^T p_1 = A^T (A^T p_0) = (A^T)^2 p_0$$

$$p_3 = A^T p_2 = A^T (A^T p_1) = A^T \left((A^T)^2 p_0\right) = (A^T)^3 p_0$$

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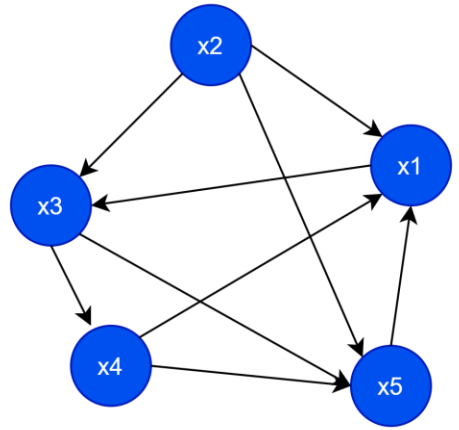
Prestige (Eigenvector centrality)

$$\boldsymbol{p} = \boldsymbol{A^T p}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad A^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$



We start with an initial prestige vector (a guess) p_0 and iterate:

$$p_1 = A^T p_0$$

$$p_2 = A^T p_1 = A^T (A^T p_0) = (A^T)^2 p_0$$

$$p_3 = A^T p_2 = A^T (A^T p_1) = A^T \left((A^T)^2 p_0\right) = (A^T)^3 p_0$$

$$\vdots$$

$$p_k = A^T p_{k-1} = A^T (A^T p_{k-1}) = (A^T)^k p_0$$

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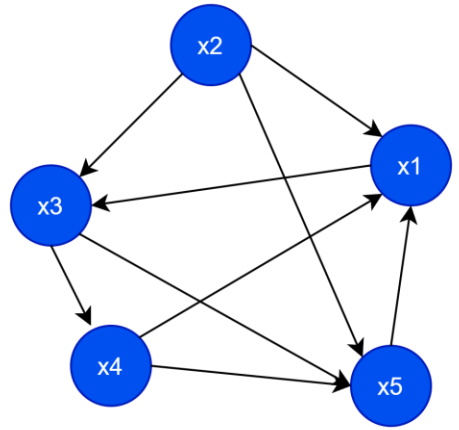
Prestige (Eigenvector centrality)

$$\boldsymbol{p} = \boldsymbol{A^T p}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad A^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$



We start with an initial prestige vector (a guess) p_0 and iterate:

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$$p_3 = A^T p_2 = A^T (A^T p_1) = A^T \left((A^T)^2 p_0\right) = (A^T)^3 p_0$$

$$\vdots$$

$$p_k = A^T p_{k-1} = A^T (A^T p_{k-1}) = (A^T)^k p_0$$

Basically, this algorithm converges to the ***dominant eigenvector*** of A^T, and the iterative process used to find p is called **power iteration**.

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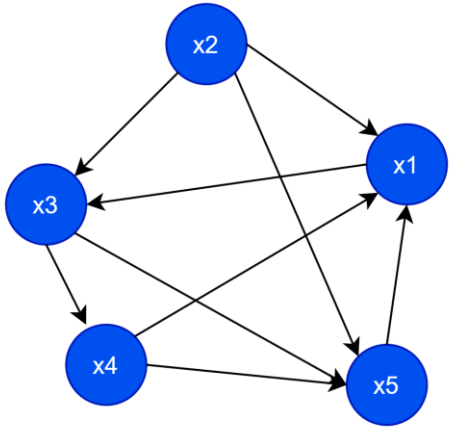
Prestige (Eigenvector centrality)



$$\boldsymbol{p} = \boldsymbol{A^T p}$$

Let's do an example

Let's start with a guess: $p_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

$$p_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

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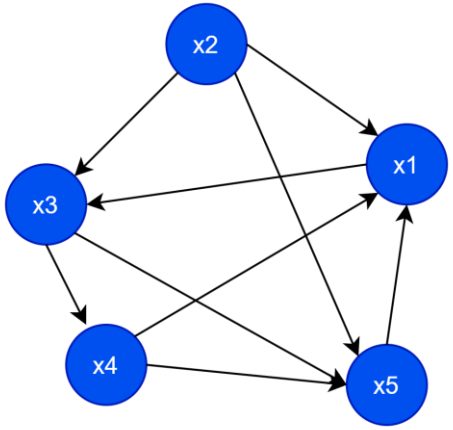
Prestige (Eigenvector centrality)



$$\boldsymbol{p} = \boldsymbol{A^T p}$$

Let's do an example

Let's start with a guess: $p_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

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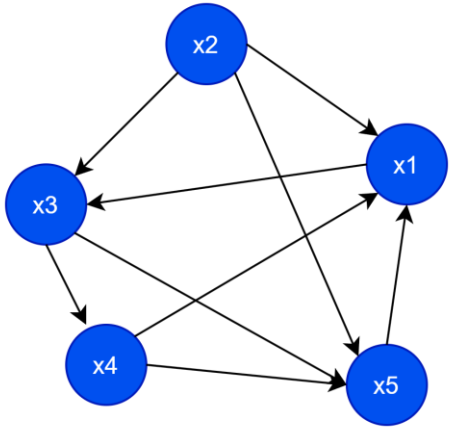
$$p_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

$$p_2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \\ 2 \\ 3 \end{pmatrix}$$

Prestige (Eigenvector centrality)

$$\boldsymbol{p = A^T p}$$



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$$p_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

$$p_2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \\ 2 \\ 3 \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 4 \\ 3 \\ 5 \end{pmatrix}$$

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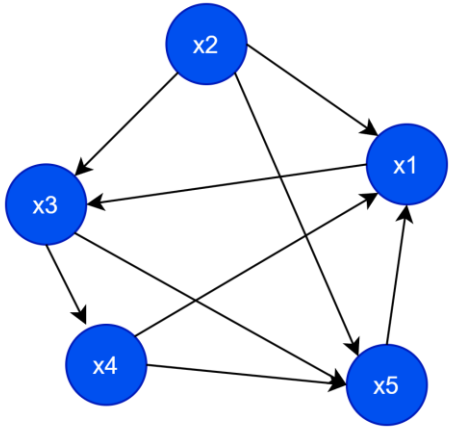
Prestige (Eigenvector centrality)

$$\boldsymbol{p = A^T p}$$



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$$p_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

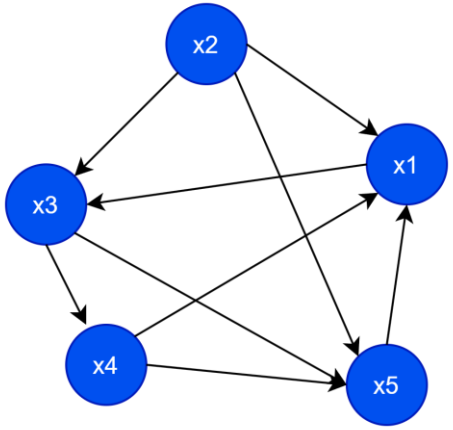
$$p_2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \\ 2 \\ 3 \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 4 \\ 3 \\ 5 \end{pmatrix}$$

$$p_4 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 5 \\ 4 \\ 7 \end{pmatrix}$$

Prestige (Eigenvector centrality)

$$\boldsymbol{p = A^T p}$$



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$$p_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

$$p_2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 3 \\ 2 \\ 3 \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 4 \\ 3 \\ 5 \end{pmatrix}$$

$$p_4 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 5 \\ 4 \\ 7 \end{pmatrix}$$

It does not look like the vector p converges.... ????

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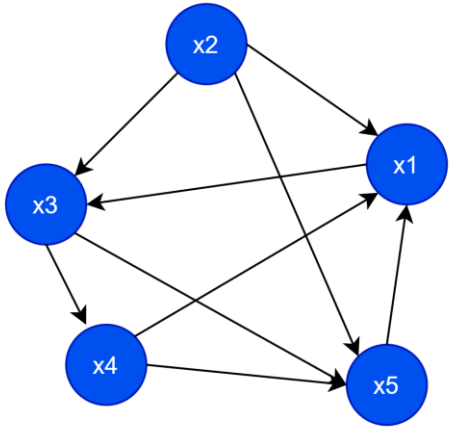
Prestige (Eigenvector centrality)

$$\boldsymbol{p = A^T p}$$

Let's do an example



Let's start with a guess: $p_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$$A^T = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad p = \begin{pmatrix} \mathrm{p}(\mathrm{x}_1) \\ \mathrm{p}(\mathrm{x}_2) \\ \mathrm{p}(\mathrm{x}_3) \\ \mathrm{p}(\mathrm{x}_4) \\ \mathrm{p}(\mathrm{x}_5) \end{pmatrix}$$

$$p_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

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We need to do a slight change in each iteration...

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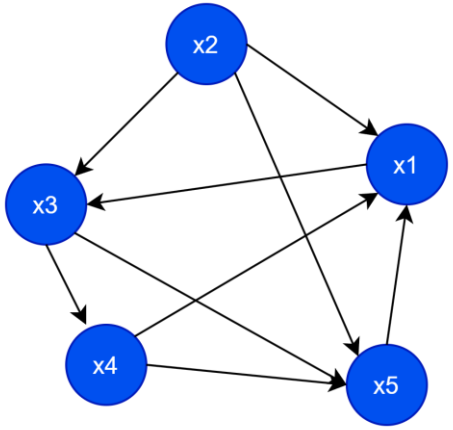

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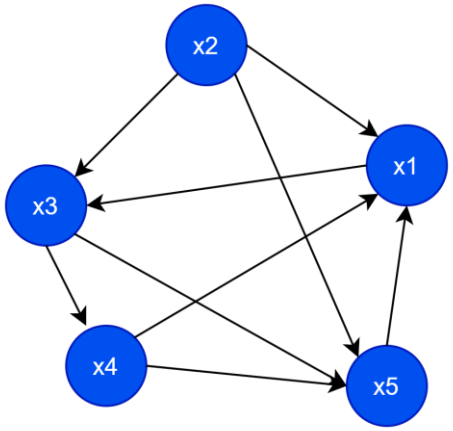

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After each iteration we need to divide the resultant vector by its largest component.

The ratio of the maximum entry in iteration k yields an estimate for the eigenvalue.

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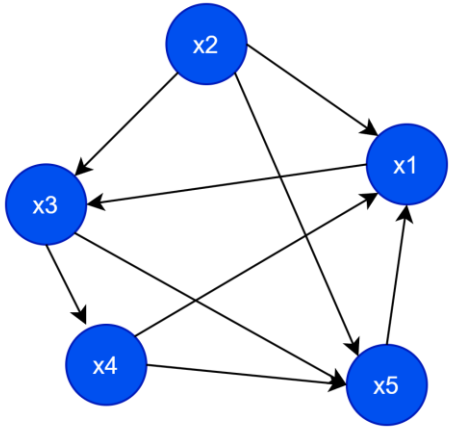
Prestige (Eigenvector centrality)



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$$\begin{pmatrix} 5 \\ 0 \\ 4 \\ 3 \\ 5 \end{pmatrix} \rightarrow \frac{1}{5} \begin{pmatrix} 5 \\ 0 \\ 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.8 \\ 0.6 \\ 1 \end{pmatrix}$$

$$p_4 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 4 \\ 3 \\ 5 \end{pmatrix} =$$

$$\begin{pmatrix} 8 \\ 0 \\ 5 \\ 4 \\ 7 \end{pmatrix} \rightarrow \frac{1}{8} \begin{pmatrix} 8 \\ 0 \\ 5 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.625 \\ 0.5 \\ 0.875 \end{pmatrix}$$

After each iteration we need to divide the resultant vector by its largest component.

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- **Instead of feeding the vector calculated at each step to the next iteration directly, We divide the vector by the largest component at each step and feed it to the next iteration.**
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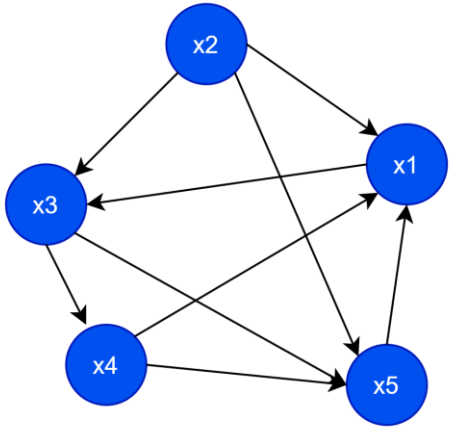

$$\boldsymbol{p} = \boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{p}$$

$$p_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Algorithm 4.1: Power Iteration Method: Dominant Eigenvector

PowerIteration (A, ϵ):

$k \leftarrow 0$ // iteration
$\mathrm{p}_0 \leftarrow 1 \in \mathbb{R}^n$ // initial vector
repeat
  $k \leftarrow k + 1$
  $\mathrm{p}_k \leftarrow \mathrm{A}^T \mathrm{p}_{k-1}$ // eigenvector estimate
  $i \leftarrow \arg\max_j \{\mathrm{p}_k[j]\}$ // maximum value index
  $\lambda \leftarrow \mathrm{p}_k[i] / \mathrm{p}_{k-1}[i]$ // eigenvalue estimate
  $\mathrm{p}_k \leftarrow \frac{1}{\mathrm{p}_k[i]} \mathrm{p}_k$ // scale vector
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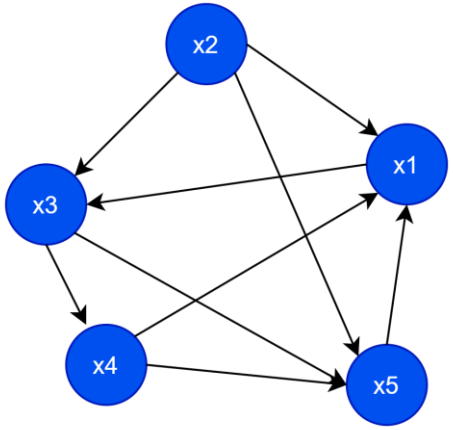

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return $\mathbf{p}, \lambda$

$$\boldsymbol{p} = \boldsymbol{A}^{\boldsymbol{T}} \boldsymbol{p}$$

$$p_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad p_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

$$max = 3 \rightarrow p_1 = \frac{1}{3} \begin{pmatrix} 3 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.6 \\ 0.3 \\ 1 \end{pmatrix}$$

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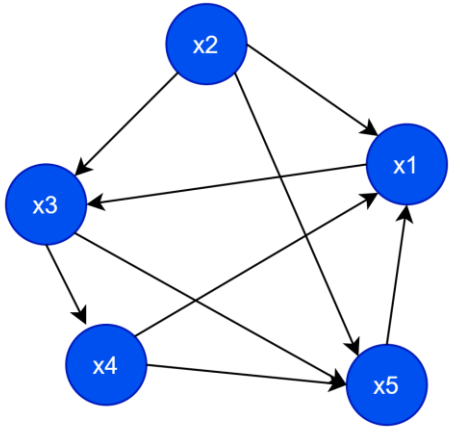

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return p, λ

$$\boldsymbol{p} = \boldsymbol{A}^{\boldsymbol{T}} \boldsymbol{p}$$

$$p_1 = \begin{pmatrix} 1 \\ 0 \\ 0.6 \\ 0.3 \\ 1 \end{pmatrix} \quad p_2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.6 \\ 0.3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.3 \\ 0 \\ 1 \\ 0.6 \\ 1 \end{pmatrix}$$

$$max = 1.3 \rightarrow p_2 = \frac{1}{1.3} \begin{pmatrix} 1.3 \\ 0 \\ 1 \\ 0.6 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.75 \\ 0.5 \\ 0.75 \end{pmatrix}$$

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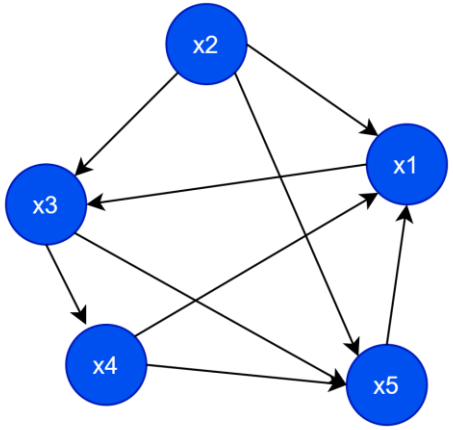

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return p, λ

$$\boldsymbol{p} = \boldsymbol{A}^{\boldsymbol{T}} \boldsymbol{p}$$

$$p_2 = \begin{pmatrix} 1 \\ 0 \\ 0.75 \\ 0.5 \\ 0.75 \end{pmatrix} \quad p_3 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.75 \\ 0.5 \\ 0.75 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 0 \\ 1 \\ 0.75 \\ 1.25 \end{pmatrix}$$

$$max = 1.25 \rightarrow p_3 = \frac{1}{1.25} \begin{pmatrix} 1.25 \\ 0 \\ 1 \\ 0.75 \\ 1.25 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.8 \\ 0.6 \\ 1 \end{pmatrix}$$

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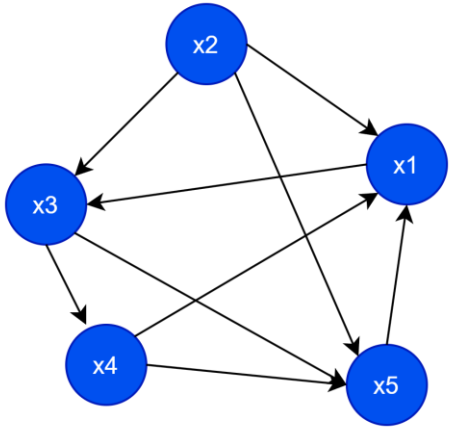

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$$\boldsymbol{p} = \boldsymbol{A}^{\boldsymbol{T}} \boldsymbol{p}$$

$$p_3 = \begin{pmatrix} 1 \\ 0 \\ 0.8 \\ 0.6 \\ 1 \end{pmatrix} \quad p_4 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.8 \\ 0.6 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.6 \\ 0 \\ 1 \\ 0.8 \\ 1.4 \end{pmatrix}$$

$$max = 1.6 \rightarrow p_4 = \frac{1}{1.6} \begin{pmatrix} 1.6 \\ 0 \\ 1 \\ 0.8 \\ 1.4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.625 \\ 0.5 \\ 0.875 \end{pmatrix}$$

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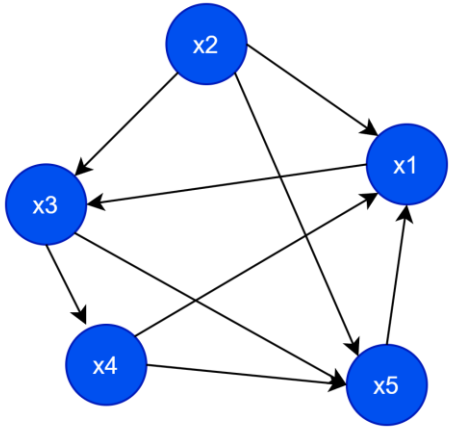

Algorithm 4.1: Power Iteration Method: Dominant Eigenvector

```
PowerIteration (A, ϵ):
k ← 0 // iteration
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    p_k ← (1 / p_k[i]) p_k // scale vector
until ‖p_k − p_{k−1}‖ ≤ ϵ
p ← (1 / ‖p_k‖) p_k // normalize eigenvector
return p, λ
```

$$\boldsymbol{p} = \boldsymbol{A^T p}$$

$$p_4 = \begin{pmatrix} 1 \\ 0 \\ 0.625 \\ 0.5 \\ 0.875 \end{pmatrix} \quad p_5 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.625 \\ 0.5 \\ 0.875 \end{pmatrix} = \begin{pmatrix} 1.375 \\ 0 \\ 1 \\ 0.625 \\ 1.125 \end{pmatrix}$$

$$max = 1.375 \rightarrow p_5 = \frac{1}{1.375} \begin{pmatrix} 1.375 \\ 0 \\ 1 \\ 0.625 \\ 1.125 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.7272 \\ 0.45 \\ 0.8181 \end{pmatrix}$$

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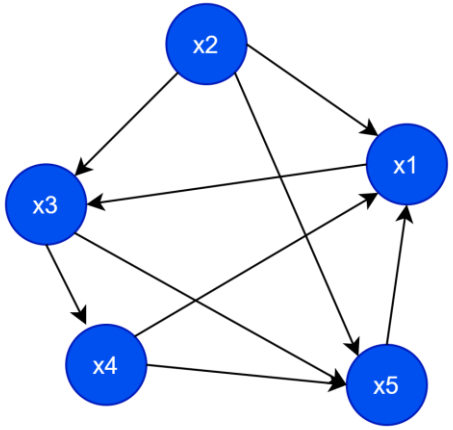
Prestige (Eigenvector centrality)

- **More iterations you do algorithm will converge into dominant eigenvector**



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until ||p_k − p_{k−1}|| ≤ ε
p ← (1 / ||p_k||) p_k // normalize eigenvector
return p, λ
```

$$\boldsymbol{p = A^T p}$$

$$p_5 = \begin{pmatrix} 1 \\ 0 \\ 0.7272 \\ 0.45 \\ 0.8181 \end{pmatrix} \quad p_6 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0.7272 \\ 0.45 \\ 0.8181 \end{pmatrix} = \begin{pmatrix} 1.272 \\ 0 \\ 1 \\ 0.7272 \\ 1.8181 \end{pmatrix}$$

$$max = 1.8181 \rightarrow p_6 = \frac{1}{1.8181} \begin{pmatrix} 1.272 \\ 0 \\ 1 \\ 0.7272 \\ 1.8181 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0.7857 \\ 0.57 \\ 0.9285 \end{pmatrix}$$

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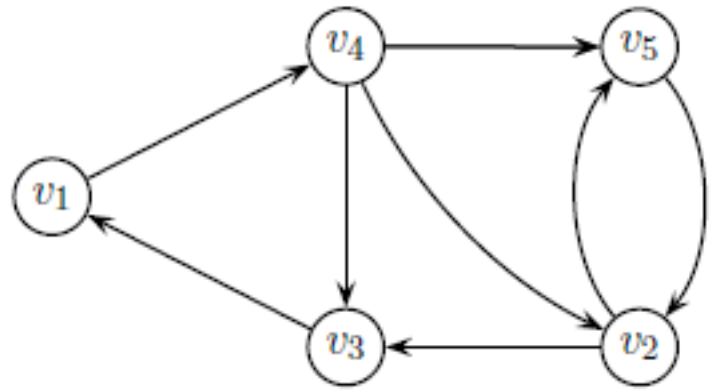

(a)

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

(b)

$$\mathbf{A}^T = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

(c)

$\mathbf{p}_0$	$\mathbf{p}_1$	$\mathbf{p}_2$	$\mathbf{p}_3$
$\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\2\\2\\1\\2\end{pmatrix} \to \begin{pmatrix}0.5\\1\\1\\0.5\\1\end{pmatrix}$	$\begin{pmatrix}1\\1.5\\1.5\\0.5\\1.5\end{pmatrix} \to \begin{pmatrix}0.67\\1\\1\\0.33\\1\end{pmatrix}$	$\begin{pmatrix}1\\1.33\\1.33\\0.67\\1.33\end{pmatrix} \to \begin{pmatrix}0.75\\1\\1\\0.5\\1\end{pmatrix}$
λ	2	1.5	$1.3\overline{3}$

$\mathbf{p}_4$	$\mathbf{p}_5$	$\mathbf{p}_6$	$\mathbf{p}_7$
$\begin{pmatrix}1\\1.5\\1.5\\0.75\\1.5\end{pmatrix} \to \begin{pmatrix}0.67\\1\\1\\0.5\\1\end{pmatrix}$	$\begin{pmatrix}1\\1.5\\1.5\\0.67\\1.5\end{pmatrix} \to \begin{pmatrix}0.67\\1\\1\\0.44\\1\end{pmatrix}$	$\begin{pmatrix}1\\1.44\\1.44\\0.67\\1.44\end{pmatrix} \to \begin{pmatrix}0.69\\1\\1\\0.46\\1\end{pmatrix}$	$\begin{pmatrix}1\\1.46\\1.46\\0.69\\1.46\end{pmatrix} \to \begin{pmatrix}0.68\\1\\1\\0.47\\1\end{pmatrix}$
1.5	1.5	1.444	1.462

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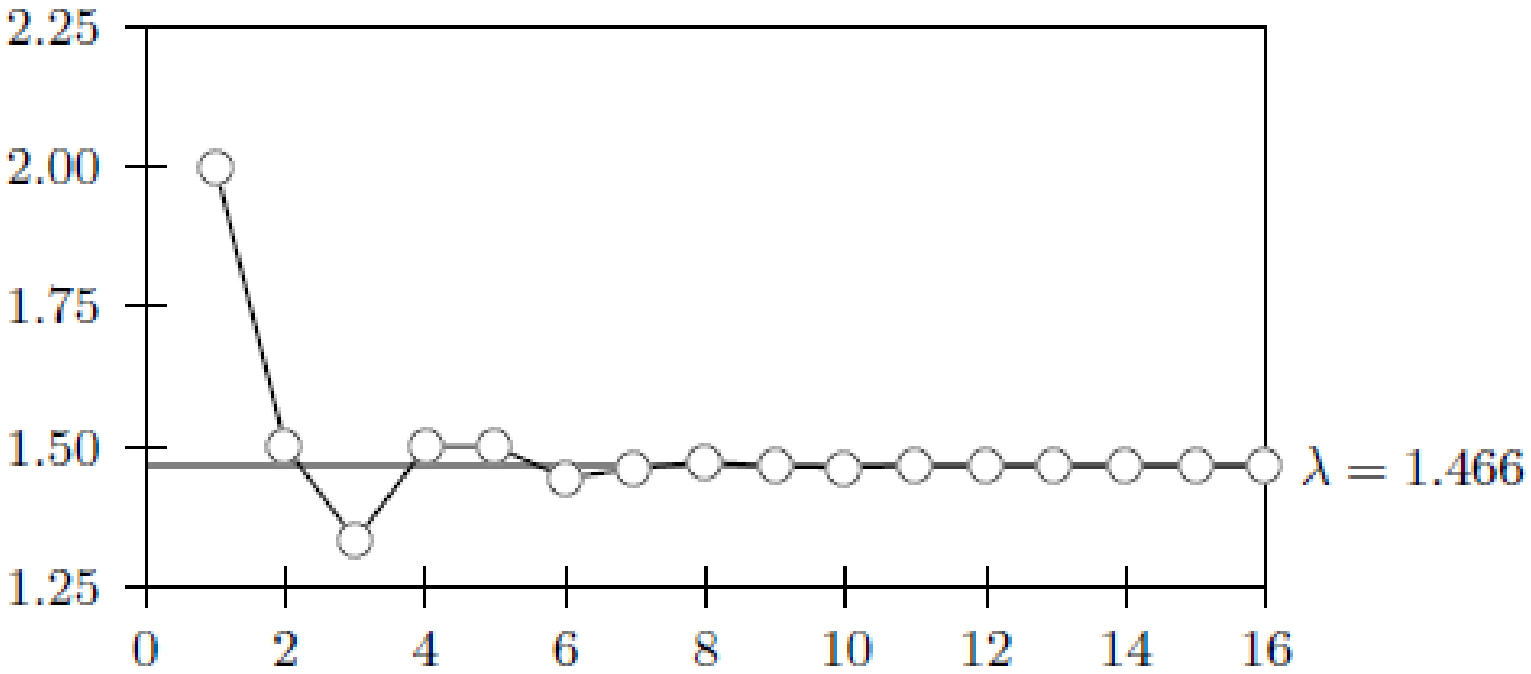

Figure 4.7: Convergence of the Ratio to Dominant Eigenvalue

PageRank centrality (or just PageRank)

- PageRank is a method for computing the prestige or centrality of nodes in the context of web search.
- The web graph consists of pages (the nodes) connected by hyperlinks (the edges).
- The method uses the so-called random surfing assumption that a person surfing the web randomly chooses one of the outgoing links from the current page, or with some very small probability randomly jumps to any of the other pages in the web graph.
- The PageRank of a web page is defined to be the probability of a random web surfer landing at that page.
- Like prestige, the PageRank of a node v, recursively depends on the PageRank of other nodes that point to it.

Normalized prestige

Let $od(u) = \sum_v A(u, v)$ denote the outdegree of node u.

Since a random surfer can choose among any of its outgoing links, if there is a link from u to v, then the probability of visiting v from u is $\frac{1}{od(u)}$.

Starting from an initial probability or PageRank $p_0(u)$ for each node:

$$\sum_u p_0(u) = 1$$

We can update the normalized prestige vector for v as follows

$$p(v) = \sum_u \frac{A(u, v)}{od(u)} \cdot p(u)$$

$$p(v) = \sum_u N(u, v) \cdot p(u) = p(v) = \sum_v N^T(v, u) \cdot p(u)$$

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Normalized prestige

Normalized adjacency matrix of a graph is given by:

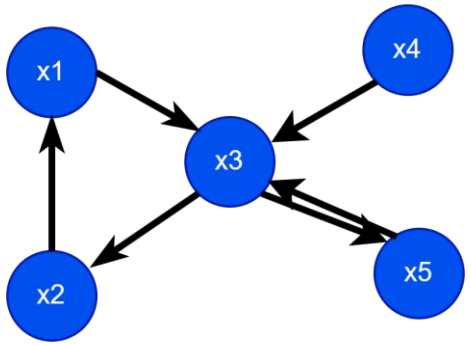
$$N(u,v) = \begin{cases} \dfrac{1}{od(u,v)} & if\ (u,v) \in E \\ 0 & if\ (u,v) \notin E \end{cases}$$

Across all nodes, we can express normalized prestige vector as follows:

$$p = N^T p$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \rightarrow N^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$



$$p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

$$\begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

It's similar to prestige calculation. We can use the power iteration and find the normalized prestige values.

The difference here is we use the **normalized adjacency matrix** instead of the normal adjacency matrix.

Random Jumps

In the random surfing approach, there is a small probability of jumping from one node to any of the other nodes in the graph, even if they do not have a link between them.

Therefore, we can view the graph as a fully connected directed graph

$$A_r = 1_{n\times n} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

For this random surfer matrix probability of jumping from u to v is $\frac{1}{od(u)} = \frac{1}{n}$.

Hence, if we only allow random jumps, we can define the random jump PageRank vector as….

Random Jumps

$$p(v) = \sum_u \frac{A_r(u,v)}{od(u)} \cdot p(u)$$

$$p(v) = \sum_u N_r(u,v) \cdot p(u)$$

$$p(v) = \sum_u N_r^T(v,u) \cdot p(u)$$

N_r is the normalized adjacency matrix.

Then the random PageRank vector can be defined analogously.

$$p = N_r^T p$$

We do not use the random PageRank alone, rather combine it with the normalized prestige. Otherwise, this vector would be all 1s.

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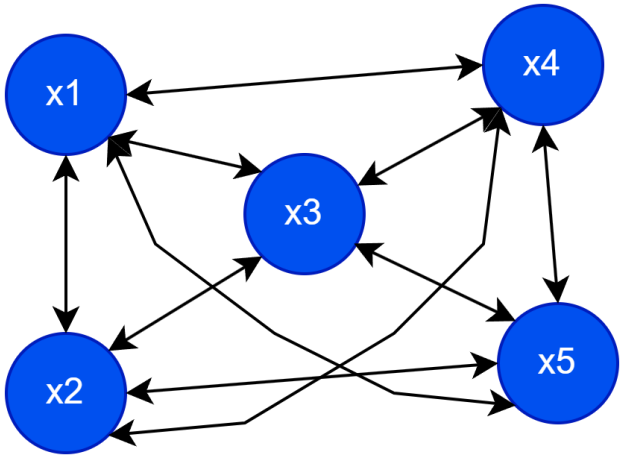
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \rightarrow A^T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$N_r = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} \rightarrow N_r^T = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$



$$p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

$$\begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

Again, it's similar to prestige calculation. We can use the power iteration and find the random jump PageRank values.

The difference here is we use the **normalized adjacency matrix of the fully connected web graph** instead of the normal adjacency matrix.

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PageRank

Now, we combine this idea of normalized prestige and random jump PageRank into PageRank.

We assume that there is some small probability, α, a random web surfer jumps from the current node u to any other random node v.
Then with $1 - \alpha$ probability user follows the existing links from u to v.

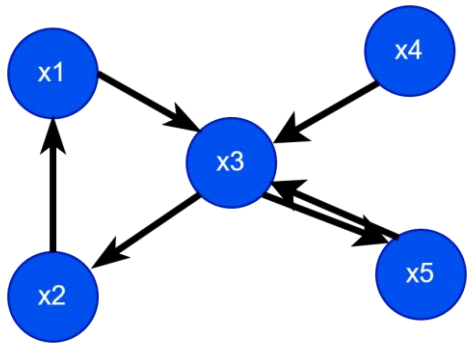
Final PageRank vector:

$$p' = (1 - \alpha)N^T p + \alpha N_r^T p$$
$$p' = \left((1 - \alpha)N^T + \alpha N_r^T\right)p$$
$$p' = M^T p$$

$$M = (1 - \alpha)N + \alpha N_r$$

M is the combined normalized adjacency matrix.

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



$$N^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$N_r^T = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

$$p = \begin{pmatrix} \mathrm{p}(\mathrm{x}_1) \\ \mathrm{p}(\mathrm{x}_2) \\ \mathrm{p}(\mathrm{x}_3) \\ \mathrm{p}(\mathrm{x}_4) \\ \mathrm{p}(\mathrm{x}_5) \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{p}(\mathrm{x}_1) \\ \mathrm{p}(\mathrm{x}_2) \\ \mathrm{p}(\mathrm{x}_3) \\ \mathrm{p}(\mathrm{x}_4) \\ \mathrm{p}(\mathrm{x}_5) \end{pmatrix} = (1-\alpha) \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{p}(\mathrm{x}_1) \\ \mathrm{p}(\mathrm{x}_2) \\ \mathrm{p}(\mathrm{x}_3) \\ \mathrm{p}(\mathrm{x}_4) \\ \mathrm{p}(\mathrm{x}_5) \end{pmatrix} + \alpha \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} \mathrm{p}(\mathrm{x}_1) \\ \mathrm{p}(\mathrm{x}_2) \\ \mathrm{p}(\mathrm{x}_3) \\ \mathrm{p}(\mathrm{x}_4) \\ \mathrm{p}(\mathrm{x}_5) \end{pmatrix}$$

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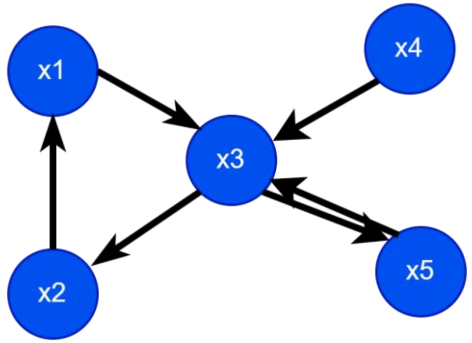
PageRank Centrality : Recap



$$p = (1-\alpha)N^T p + \alpha N_r^T p$$

$$p = M^T p$$

$$p = \begin{pmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \\ p(x_4) \\ p(x_5) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad A^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \quad N^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

$$N_r = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} \quad N_r^T = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

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