RIC for Delaunay

- build Δtion & point location DS
- Given n-points in the plane $P = \{p_1, \dots, p_n\}$

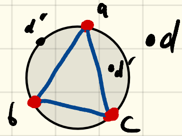
How to build up:
- insert in a random order
- show rand. → $O(n \lg n)$ expected time algo

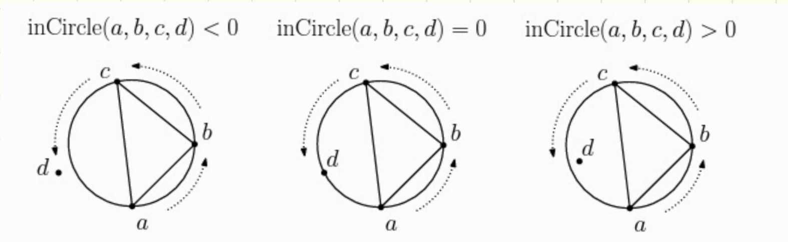
  <u>key</u> $O(1)$ expected structural change w/ each insertion

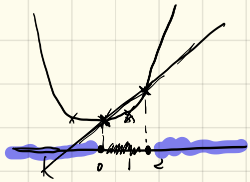
InCircle:

Δabc in Delaunay ⟺ circumcircle of Δabc has no sites of P in its interior

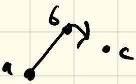
$$\text{InCircle}(a,b,c,d) = \det \begin{bmatrix} a_x & a_y & a_x^2+a_y^2 & 1 \\ b_x & b_y & b_x^2+b_y^2 & 1 \\ c_x & c_y & c_x^2+c_y^2 & 1 \\ d_x & d_y & d_x^2+d_y^2 & 1 \end{bmatrix} \substack{<\\=\\>} 0$$







Where does this come from



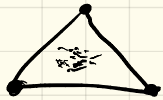
$$\det \begin{pmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{pmatrix}$$

w/ plane

$$\det \begin{pmatrix} a_x & a_y & a_z & 1 \\ b_x & b_y & b_z & 1 \\ c_x & c_y & c_z & 1 \\ q_x & q_y & q_z & 1 \end{pmatrix}$$

Inc upate

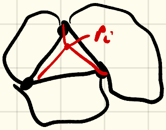
assume all points are in a very big Δ



Assume we have the Del of $p_1, \dots, p_{i-1}$

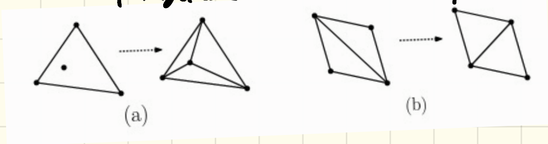
insert p_i

1. find Δ w/ p_i
2. Make up a star
3. flip to Del



triangulate

flip



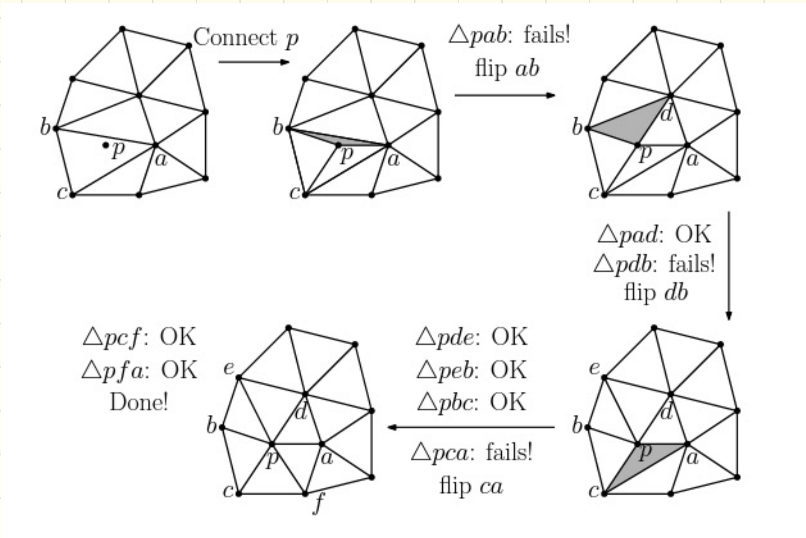


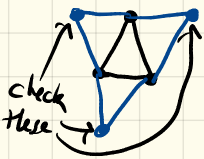
Fig. 65: Point insertion.

Pseudo-code

```
Insert(p) {
    Find the triangle △abc containing p;
    Insert edges pa, pb, and pc into triangulation;
    SwapTest(ab);                          // check/fix the surrounding edges
    SwapTest(bc);
    SwapTest(ca);
}

SwapTest(ab) {
    if (ab is an edge on the exterior face) return;
    Let d be the vertex to the right of edge ab;
    if (inCircle(p, a, b, d) {             // d violates the incircle test
        Flip edge ab for pd;
        SwaptTest(ad);                     // check/fix the new suspect edges
        SwaptTest(db);
    }
}
```

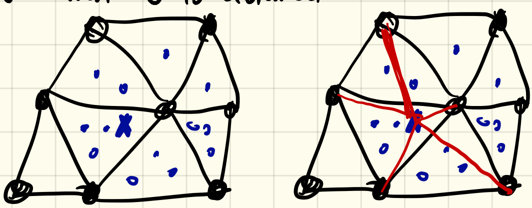
Correctness:
- only check Del prop for Δs w/ p and Δs on oppisite edge



Locally Delanay
for each Δ
verts on oppistie side
satisfy delaun prop

Delamay Theorem
all Δs are locally Del ⇒ Δtion is globally Del

Point location
- maintain uninserted sites in 'buckets'
- update bucket when Δ is disturbed



Bound?:

1) # of structural changes on avg when a new site is inserted
2) # of rebucketings

worst case avered over all insertion orderings

Structural changes:

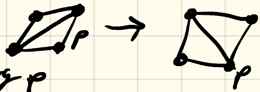
Claim expected # of structural changes is $O(1)$

Backwards analysis

+ Delaunay is ind of insertion order
  ⇒ any site is equally likely to be the last inserted
  suppose that we just added p

What is the expected work of inserting p?
What causes work?

-edge flipping



w/ edges containing p

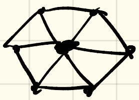
⇒ total # of edge flips is proportional to the deg of p after insertion

Consider after inserting i^{th} site

- Dtion struct is ind of insertion order
  any site could be last site

⇒ expected time to insert p_i proportional to the average deg of a vert in Δ of i sites

Use Euler ($v-e+f=2$) ⇒ expected deg is 6 ⇒ expected deg is $O(1)$ ⇒ expected # of edge flips is $O(1)$



Rebucketing

Claim* total time spent rebucketing is expected $O(n \lg n)$

if claim * $\Rightarrow$ claim * $\Rightarrow$ expected $O(n \lg n)$
dominates
running
time

Goal: Show that the expected # of
times that a site is rebucketed is $O(\lg n)$ times

Fix $q \in P$
and consider just after inserting p_i

case 1 q already inserted $\Rightarrow$ no rebucketing

case 2: q is not yet inserted
(as before any site can be p_i)

Claim is: $\Pr(q \text{ is rebucked}) = \frac{3}{i}$

Pf (sketch)

let t be Δ containing q after i^{th} insertion
$\Rightarrow$ t came into existence because of p_i
$\Rightarrow$ p_i is a vertex of t
t has 3 verts defining it
$\Rightarrow \Pr(t \text{ in existence}) = \frac{3}{i}$

At i^{th} stage of alg $(n-i)$ pts that can be rebuckted

$\Rightarrow E[\# \text{ of rebucketed points at step } i] \leq (n-i)\frac{3}{i}$

lin of expectation

$$\Rightarrow E[\text{total \# of rebucketings}] = \sum_{i=1}^{n} E[\# \text{ of rebucketings at } i] \leq \sum_{i=1}^{n} (n-i)\frac{3}{i} \leq \sum_{i=1}^{n} \frac{3n}{i} = 3n \sum_{i=1}^{n} \frac{1}{i} = O(n \lg n)$$