LP RIC

Randomized Incremental d-Dimensional Linear Programming

Input: A set $H = \{h_1, \ldots, h_n\}$ of $(d-1)$-dimensional halfspaces, such that the first d define an initial feasible vertex v_d, and the objective vector c.

Output: The optimum vertex v or an error status indicating that the LP is infeasible.

(1) If the dimension is 1, solve the LP by brute force in $O(n)$ time.

(2) Let v_d be the intersection point of the hyperplanes bounding $h_1, \ldots, h_d$, which we assume define an initial feasible vertex. Randomly permute the remaining halfspaces, and let $\langle h_{d+1}, \ldots, h_n \rangle$ denote the resulting sequence.

(3) For $i = d+1$ to n do:

  (a) If $(v_{i-1} \in h_i)$ then $v_i \leftarrow v_{i-1}$.

  (b) Otherwise, intersect $\{h_1, \ldots, h_{i-1}\}$ with the $(d-1)$-dimensional hyperplane ℓ_i that bounds h_i and project onto $\mathbb{R}^{d-1}$. Let c' be the projection of c onto ℓ_i and then onto $\mathbb{R}^{d-1}$. Solve the resulting $(d-1)$-dimensional LP recursively.

    (i) If the $(d-1)$-dimensional LP is infeasible, terminate and report that the LP is infeasible.

    (ii) Otherwise, let v_i be the solution to the $(d-1)$-dimensional LP.

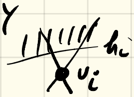
(4) Return v_n as the final solution.

Analysis for randomized LP:

Let $T_d(n)$ be the expected run time of RIC LP alg
we show by induction on n
that $T_d(n) \leq \gamma_d\, d!\, n$

w/ γ_d is dependent on the dim
then we remove dim dependency

for $i \in \{d+1, \ldots, n\}$
let p_i be the prob that h_i changed the opt vertex



Case 1: prob $(1-p_i)$ that there is no change
$\Rightarrow O(d)$ time to determine

Case 2: prob p_i that there is a change in opt vertex
1) project c onto ℓ_i - $O(d)$ time
2) intersect $i-1$ halfspaces w/ ℓ_i - $O(d(i-1))$ time
$\Big]$ takes $O(di)$ time
3) recurse to solve $(d-1)$-dim LP on $i-1$ halfspaces
- by induction hyp it takes $T_{d-1}(i-1)$ time

cobine (and omit constant factors)

$$T_d(n) \leq \sum_{i=d+1}^{n} \left((1-p_i)d + p_i \left(di + T_{d-1}(i) \right) \right)$$

$$\leq \sum_{i=d+1}^{n} \left(d + p_i \left(di + T_{d-1}(i) \right) \right)$$

What is p_i? ... Backwards analysis

Let S_i be an arbitray set of i hyperplanes from the input
how many permutation $\rightarrow$ $i!$

Want to count: Nuber of permutions in which the opt vert changes on the i^{th} step?

Let v_i be the opt vertex of i- halfspaces
... note v_i depends only on S_i (not the order)

Observations:
- there are d halfspaces that define v_i
- if none of them are $h_i \Rightarrow v_{i-1} = v_i$
- if any of them are $h_i \Rightarrow$ opt vertex changed

$\Rightarrow$ opt vert changed $\Leftrightarrow$ definer of v_i is h_i

Since any halfspace could be last
and there are d halfspaces that can define v_i
$\Rightarrow p_i = \frac{d}{i}$

plug our p_i into our bound for $T_d(n)$

$$T_d(n) \le \sum_{i=d+}^{n} \left(d + p_i\left(di + T_{d-1}(i)\right)\right)$$

assume $T_{d-1}(i) = \gamma_{d-1}(d-1)!\,i$ (by induction hyp) side note

$$\le \sum_{i=d+}^{n} \left(d + \frac{d}{i}\left(di + \gamma_{d-1}(d-1)!\,i\right)\right)$$

$$\le \sum_{i=d+1}^{n} \left(d + d^2 + \gamma_{d-1}\, d!\right) \le \left(d + d^2 + \gamma_{d-1}\, d!\right) n$$

Find a const γ_d so that

$\gamma_d\, d!\, n \ge d + d^2 + \gamma_{d-1}\, d!$

choose $\gamma_d = \frac{d+d^2}{d!} + \gamma_{d-1}$

$$\Rightarrow T_d(n) \le \left(d + d^2 + \gamma_{d-1} d!\right) n = \left(\frac{d+d^2}{d!} + \gamma_{d-1}\right) d!\, n \le \gamma_d\, d!\, n$$

What about our const

for large enough d

$$\frac{d+d^2}{d!} \le \frac{1}{2^d} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{2^i} = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \ldots$$

is geometric series

$\sum_{i=0}^{\infty} (ar)^i$ 1 $\frac{1}{2}$

w/ $r < 1$ (e.g. $\frac{1}{2}$)

$\Rightarrow$ geometric series converges to a const!!

$\Rightarrow T_d(n) = O(d!\, n)$

since d is const $d!$ is a const

$\Rightarrow T_d(n) = O(n)$

Seidel: the prob of time as 6-times the expected running time is $O\left(\left(\frac{1}{c}\right)^{6d!}\right)$ for any fixed const c

e.g. in 2d 10x expected running time is at most $6.5e^{-12}$