Network Formation and Social Norms Under Complentarities: Statics and Dynamics

1 Static Model

1.1 Set Up

- Consider a set of N individuals. Each individual has a type $\theta_i = (\theta_1, ..., \theta_K) \in \Theta$. Individuals know the game.
- Define the *undirected network* $g \in G$ as an $N \times N$ symmetric matrix where $g_{i,j} = g_{j,i} = 1$ there is a link between i and j and $g_{i,j} = g_{j,i} = 0$ otherwise. The space of undirected networks G is the set of all symmetric $N \times N$ matrices with elements in $\{0, 1\}$.
- Define i's *neighbors* $N_i(g) = \{j \neq i : g_{i,j} = 1\}$ as the set of individuals with whom i shares a link. i's *degree* $d_i(g) = |N_i(g)|$.
- First, players form links consensually (starting from the empty network). Then, players choose actions from a set of actions X.
- A strategy of player $i \in N$ is a vector $s_i = (\gamma_i, x_i)$.
  - $\gamma_i = (\gamma_{i,1}, ..., \gamma_{i,i-1}, 1, \gamma_{i,i+1}, ..., \gamma_{i,N})$ with $\gamma_{i,j} \in 0, 1$
  - $x_i = (x_{i,1}, ..., x_{i,K}) \in X \subseteq \Re^K_+$ with $K < \infty$.
- The network $g \in G$ is then created where $g_{ij} = g_{ji} = 1$ if $\gamma_{ij} = \gamma_{ji} = 1$ and $g_{ij} = g_{ji} = 0$ otherwise.
- Utilities take the following form:

$$u_i(x, g) = \sum_{k=1}^{K} \left\{ \underbrace{\theta_{i,k} x_{i,k}}_{\text{Private benefit}} + \underbrace{\left[a x_{i,k} \sum_{j \in N_i(g)} x_{j,k} \right]}_{\text{Complementarities}} \right\} - \underbrace{c(x_i, d_i(g))}_{\text{Cost}}$$

  where

  - a is a K-vector of complentarities
  - $c(x_i)$ is a cost function $c_X : X \times G \to \Re_+$

1.2 Equilibrium Concept(s)

- We are interested in two equilibrium concepts.

  1. **Pairwise Stability:** No two nodes i and j with $g_{i,j} = 0$ are better off forming a link and no two nodes i and j with $g_{i,j} = 1$ are better of breaking their link.
  2. **Efficiency:** $U(x, g) = \Sigma_i u_i(x, g) \geq \Sigma_i u_i(x', g') = U(x', g')\ \forall x' \in X,\ \forall g' \in G$

1.3 Motivating "Simple" Example

There are N individuals, n of type θ_a and N-n of type θ_b. Individuals of type a have $\theta_{a,1} > \theta_{a,2}$ and individuals of type b have $\theta_{b,1} < \theta_{b,2}$. Let the action space $X = \{0, 1\}^2$ consist of two possible actions: $x_{i,1}$ is going to the bar on Friday and $x_{i,2}$ is going to the bar on Saturday. The cost of taking each action is the same; $c_1 = c_2 (> \theta_1, \theta_2)$. The complementarities for each action are the same; $a_1 = a_2$. The cost of keeping a link is $c_d \in \Re$.

We are interested in the set of networks for which there exists an $x = (x_1, ...x_{10})$ which is a Nash EQ that makes the network pairwise stable. Then, we are interested in the efficiency of said network.

One result that this formulation predicts is that i will not do an action unless at least one of their neighbors is doing this action, and i will not share a link with someone that isn't doing at least one of the actions i is doing.

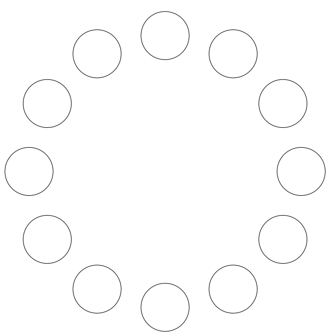

Figure 1: Empty

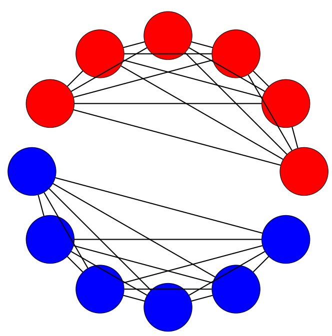

Figure 2: Perfectly Split

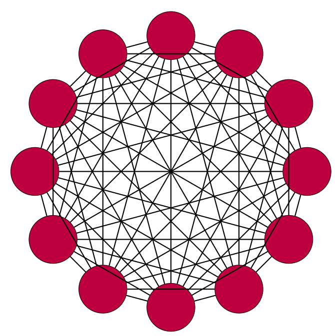

Figure 3: Complete

Note: This figure displays several examples of networks of possible networks structures and the resulting *social norms*: the set of actions. White refers to $x_{i,1} = 0$ and $x_{i,2} = 0$ (not going to the bar at all). Red refers to $x_{i,1} = 1$ and $x_{i,2} = 0$ (going to the bar on Friday but not Saturday). Blue refers to $x_{i,1} = 0$ and $x_{i,2} = 1$ (going to the bar on Saturday but not Friday). Purple refers to $x_{i,1} = 1$ and $x_{i,2} = 1$ (going to the bar on Friday *and* Saturday).

For example, the *empty network* is pairwise stable for $x = (0, ..., 0)$. The sum of utilities is 0.

Consider the *perfectly-split network* where individuals of type a are linked with all other individuals of type a, and individuals of type b are linked with all other individuals of type b. In this network, individuals of type a choose $x_{i,1} = 1$ and $x_{i,2} = 0$, while individuals of type b choose $x_{i,1} = 0$ and $x_{i,2} = 1$. This network is pairwise stable as long as:

- $\theta_{a,1} + na - c_x - nc_d > 0 \implies n(a - c_d) > c_x - \theta_{a,1} \implies$ the benefit-cost of forming the 5 links > the cost-benefit of doing the action! With this, people will want to form the links with their type
- $\theta_{a,2} + a - c_x - c_d < 0 \implies a - c_d < c_x - \theta_{a,2} \implies$ the benefit-cost of forming a new link < the cost-benefit of doing the action the new link is doing.
- also need the same for b!

The sum of utilities is then positive!

Consider the *complete network* where everybody is linked with everybody and doing both actions. This network is pairwise stable as long as:

- $\theta_{a,1} + \theta_{a,2} + 2Na - 2c_x - Nc_d > 0 \implies N(2a - c_d) > 2c_x - (\theta_{a,1} + \theta_{a,2}) \implies$ the benefit-cost of forming the N links > the cost-benefit of doing the 2 actions.
- Need the same for b!

The sum of utilities is positive, and is more efficient than the above network if $n(\theta_{a,1} + \theta_{a,2} + 2Na - 2c_x - Nc_d) + (N - n)(\theta_{b,1} + \theta_{b,2} + 2Na - 2c_x - Nc_d) > n(\theta_{a,1} + na - c_x - nc_d) + (N - n)(\theta_{b,1} + na - c_x - nc_d)$.

Each network, thus, defines a social norm. In the empty network, the social norm is that no one goes to the bar ever. The perfect split networks has that type a's go on Friday, and b's go on Saturday. The complete network has that both types go on both days.

2 Dynamic Model

2.1 Set Up

- Consider a set of N *infinitely-lived individuals*. Each individual has a *type* $\theta_i = (\theta_1, ..., \theta_K) \in \Theta$. At $t = 0$, individuals do not know the players.
- Define the *undirected network* $g(t) \in G$ as an $N \times N$ symmetric matrix where $g_{i,j}(t) = g_{j,i}(t) = 1$ there is a link between i and j and $g_{i,j}(t) = g_{j,i}(t) = 0$ otherwise. The space of undirected networks G is the set of all symmetric $N \times N$ matrices with elements in $\{0, 1\}$.
- Define i's *neighbors* $N_i(g(t)) = \{j \neq i : g_{i,j}(t) = 1\}$ as the set of individuals with whom i shares a link. i's *degree* $d_i(g(t)) = |N_i(g(t))|$. Individuals observe the actions $x_j(t)$ of their neighbors.

- When individuals i and j meet in period t, they learn the action choices $x_i(t-1)$ and $x_j(t-1)$.

- Let $p(g_{ij}(t-1), x_i(t-1), x_j(t-1))$ represent the *probability that individuals* i *and* j *meet in period* t, which is a function of their action choices $x_i(t-1)$ and $x_j(t-1)$ from the previous period. If $g_{ij}(t-1) = 1$, $p(\cdot) = 0$

- In each period t, individuals first "meet" a random subset $n_i(t) \subseteq N$ where $i \in n_j(t) \iff j \in n_i(t)$. They can choose to make links with the people they meet, and break any existing links. Then, players choose from a set of actions X.

- A *strategy* of player $i \in N$ in period t is a vector $s_i(t) = (\gamma_i(t), x_i(t))$.
  - $\gamma_i(t) = (\gamma_{i,1}(t), ..., \gamma_{i,i-1}(t), 1, \gamma_{i,i+1}(t), ..., \gamma_{i,N}(t))$ with $\gamma_{i,j}(t) \in \{0,1\}$ and $\gamma_{i,j}(t) - \gamma_{i,j}(t-1) > 0 \implies j \in n_i(t)$
    * Individuals can only form links with people they "meet" in a period t, but can break links at any time.
  - $x_i(t) = (x_{i,1}(t), ..., x_{i,K}(t)) \in X \subseteq \Re_+^K$ with $K < \infty$.

- The network $g(t) \in G$ is then created where $g_{ij}(t) = g_{ji}(t) = 1$ if $\gamma_{ij}(t) = \gamma_{ji}(t) = 1$ and $g_{ij}(t) = g_{ji}(t) = 0$ otherwise.

- Utilities in each period take the same form, but are indexed by time:

$$U_i(x(t), g(t), t) = \sum_{k=1}^{K} \left\{ \underbrace{\theta_{i,k} x_{i,k}(t)}_{\text{Private benefit}} + \underbrace{\left[a x_{i,k}(t) \sum_{j \in N_i(g(t))} x_{j,k}(t) \right]}_{\text{Complementarities}} \right\} - \underbrace{c(x_i(t), d_i(g(t)))}_{\text{Cost}}$$

  where
  - a is a K-vector of complentarities
  - $c(x_i(t), d_i(g(t)))$ is a cost function $c_d : X \times G \to \Re_+$.

2.2 Equilibrium Concept(s)

- We are interested in two(three) equilibrium concepts.
  1. **Pairwise Stability:** No two individuals i and j with $g_{i,j}(t) = 0$ are better off forming a link and no two nodes i and j with $g_{i,j}(t) = 1$ are better of breaking their link.

$\vdots$

Period t: Meeting set $n_i(t)$ is revealed

$\searrow$

Individuals choose $\gamma_i(t)$

$\searrow$

Network $g(t)$ is formed

$\searrow$

Individuals choose action $x_i(t)$

$\searrow$

Utility $U_i(t)(x(t), g(t))$ realized

Period t+1:

$\vdots$

Figure 4: Game Tree for One Period

  - **(Feasible) Pairwise Stability** is a weakening of this equilibrium concept where we require pairwise stability only for individuals who have a non-zero probability of meeting.

2. **Efficiency:** $U(x(t), g(t), t) = \Sigma_i u_i(x(t), g(t), t) \geq \Sigma_i u_i(x', g', t) = U(x', g', t)$ $\forall x' \in X$, $\forall g' \in G$