a. Let OPT be total value of the toys of the least happy child under the optimal solution. Our bound will be:

$$\text{OPT} \leq \frac{1}{n}\sum_{j=1}^{m} v_j.$$

This follows from the fact that the righthand size above is the solution to the fractional relaxation of the toy distribution problem.

b. Our algorithm is unsorted greedy: considering the items sequentially, assign each item to the child who currently has the lowest value for their bundle of toys among all children.

c. The above algorithm is a $\frac{5}{4}$-approximation. To see this, we will relate the performance of the algorithm, denoted ALG, to the bound we found in part a. In particular, we show that:

$$\text{ALG} \geq \frac{4}{5n}\sum_{j=1}^{m} v_j.$$

We prove this by contradiction. Assume ALG $< \frac{4}{5n}\sum_{j=1}^{m} v_j$. For each child i, let $j^*(i)$ be the index of the final toy given to child i by the greedy algorithm. Let S_i be the set of items given to the child by the greedy algorithm. By our assumption and the way the greedy algorithm works, it must be that for each child i,

$$\sum_{j \in S_i \setminus \{j^*(i)\}} v_j \leq \frac{4}{5n}\sum_{j=1}^{m} v_j,$$

with the inequality being strict for the least happy child.

To derive our contradiction we now compute a bound on the total value of items by summing the value of each child's toys. We have

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j \in S_i} v_j &= \sum_{i=1}^{n}\sum_{j \in S_i \setminus \{j^*(i)\}} v_j + \sum_{i=1}^{n} v_{j^*(i)} \\
&< \frac{4}{5}\sum_{j=1}^{m} v_j + \sum_{i=1}^{n} v_{j^*(i)} && \text{By above discussion.} \\
&\leq \frac{4}{5}\sum_{j=1}^{m} v_j + \frac{2n}{m}\sum_{j=1}^{m} v_j && \text{By assumption that toys are similar.} \\
&\leq \frac{4}{5}\sum_{j=1}^{m} v_j + \frac{2n}{10n}\sum_{j=1}^{m} v_j && \text{By assumption that } m \geq 10n. \\
&= \sum_{j=1}^{m} v_j.
\end{aligned}$$

This is a contradiction. Putting together our bound on OPT and the preceding proof, we get

$$\text{OPT} \leq \frac{1}{n}\sum_{j=1}^{m} v_j \leq \frac{5}{4}\text{ALG}.$$