Northwestern University
EECS 336
Fall 2015
A Quick Guide to Mathematical Induction

1 Induction Proofs

You should format every proof by induction you write in the following way:

Four parts:

1. State what you are proving. State what you are inducting on.
2. State your base case(s).
3. State your inductive hypothesis (IH).
4. State your inductive case. Make clear where IH is used.

Example:

Claim 1. *The vector $A^k z$ contains the kth and $k+1$st Fibonacci numbers, where*

- $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
- $z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Proof. (Induction on k).

Base Case: $k = 0$.

- $F_0 = 0 = z_1$
- $F_1 = 0 = z_2$

Inductive Hypothesis: Assume $A^k z$ contains the kth and $k+1$st Fibonacci numbers.

Inductive Case: Consider $A^{k+1} z$.

$$\begin{aligned} A^{k+1} z &= A A^k z \\ &= A \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} && \text{(IH)} \\ &= \begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix} && \text{(Linear Algebra)} \end{aligned}$$

□

1.1 Loop Invariants

Idea: Use to prove that algorithm is doing what you want at each step. Often, this means proving that internal variables satisfy some invariant properties. At the end, invariant properties mean algorithm is correct.

Example: Iterative Fib

Property: After the kth iteration of the loop, last[0] is the kth Fibonacci number, and last[1] is the $k+1$st. (Sanity check: at the end, this shows our algorithm to be correct.)

Proof. *(Loop Invariant) (Alternatively: Induction on the iteration k of the loop.)*

Base Case: Before the 1st iteration, property holds.

Inductive Hypothesis: Assume property holds after step k.

Inductive Case:

- After the $k + 1$st step, we have set last[0] = old last[1] and last[1] to sum of two old values.
- By IH, last contained the kth and $k + 1$st Fibonacci numbers.
- By definition of Fibnoacci numbers, Property still holds after step $k + 1$.

$\square$