LINEAR ALGBRA

ASSIGNMENT # 02

TOPIC :- DETERMINANTS.

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QUESTIONS #

What is matrix determinant?
What are the properties of a determinant?
Explain each property with an example.

DETERMINANTS OF A MATRIX.

The determinant is a scalar value.
It is function of the elements of a square matrix.

It is also an element that indentifies or determines nature of a matrix.

e.g. A matrix is invertible if its determinat is non-zero.

We can determine a determinant by following method:-

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot ad - bc \Rightarrow \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad -(1)$$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - eg)$$
$$= aei - ahf - bdi - bgf + cdh - ceg \quad -(2)$$

For 2X2 matrix $-(1)$
for 3X3 matrix $-(2)$

A general definition on how to determine a matrix determinant with order $n \times n$ is:-

$$\det A = a_{11} \det A_{11} - a_{12} + \dots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{ij} (\det A_{ij}) \quad -(3)$$

where $(-1)^{i+j} A_{ij}$ is known as cofactor of an element.

$\det A_{ij}$ can also be written as A_{ij}.

Cofactor of matrix A.

$$C_{ij} = (-1)^{i+j} \det A_{ij} \quad -(4)$$

matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & \dots & \dots & a_{3n} \\ a_{41} & a_{42} & \dots & \dots & \dots & a_{4n} \\ \vdots & \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \dots & \dots & \dots & a_{nn} \end{bmatrix}$

Area of parallelogram

The determinant of a matrix can also determine the area of parallelogram.

$$\text{Area of parallelogram} = |\det(A)| \quad -(15)$$

Volume of parallelpiped.

The determinant of a matrix determines the volume of parallelpiped.

$$\text{Volume of parallelpiped} = |\det A| \quad -(6)$$

Examples for calculating square matrix determinants.

-(1)

let $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} = 5\times4 - 2\times3 = 20 - 6 = 14$

-(2), -(3), -(4)

let $A = \begin{bmatrix} 1 & 3 & 6 \\ 5 & 2 & 9 \\ 4 & 7 & 8 \end{bmatrix} = 1 \begin{bmatrix} 2 & 9 \\ 7 & 8 \end{bmatrix} + (-1)^{1+2}(3) \begin{bmatrix} 5 & 9 \\ 4 & 8 \end{bmatrix}$

$+ (-1)^{1+3}(6) \begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}$

$= 1(2\times8 - 7\times9) + (-1)^3(3)(5\times8 - 4\times9) + (-1)^4(6)(5\times7 - 4\times2)$

$= 16 - 63 + (-1)(3)(40 - 36) + (+1)(6)(35 - 8)$

$= -47 + (-3)(4) + 6(27)$

$= 162 - 47 - 12$

$= 103.$

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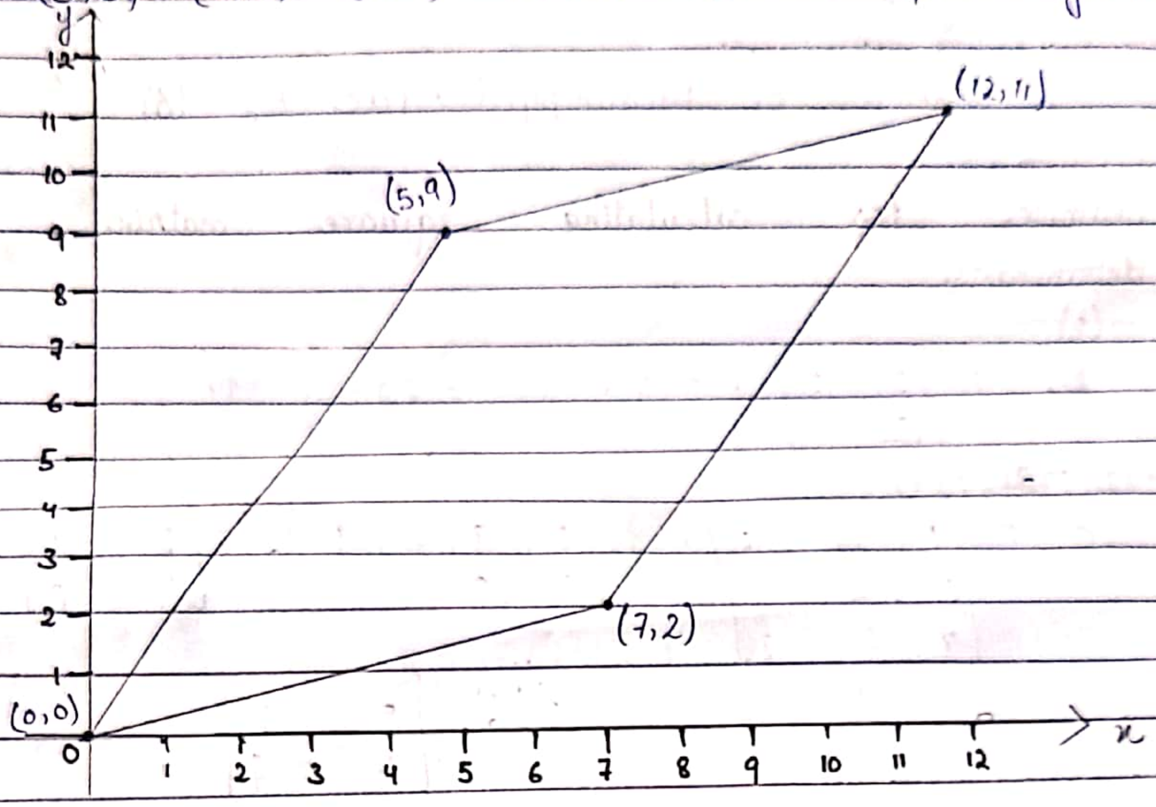
Example for calculating area of parallelogram using determinants.

let (0,0) (7,2) (5,9) (12,11) be a parallelogram.



The area can be determined by vectors.

$V_1 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$ & $V_2 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 5 & 7 \\ 9 & 2 \end{bmatrix} \Rightarrow$ area of parallelogram $= [\det A]$

$\Rightarrow$ Area of parallelogram $= \left[\begin{bmatrix} 5 & 7 \\ 9 & 2 \end{bmatrix}\right] = [5 \times 2 - 9 \times 7]$

$= [10 - 63]$

$= [-53] = 53$

Example of calculating volume of parallelpiped using determinants.

let $(0,0,0)$, $(2,2,-1)$, $(1,3,0)$, $(-1,1,4)$ makes a parallelpiped

Then volume of parallelpiped will be absol value of determinant of matrix.

$$\begin{vmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{vmatrix}$$

volume of parallelpiped $= \left| \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{bmatrix} \right|$

$$= \left| 2\begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix} - 1\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \right|$$

$$= \left[2(3\times4-0) - 2(1\times4-0) - 1(1\times1-(-1)(3)) \right]$$

$$= \left[2(12) - 2(4) - 1(4) \right]$$

$$= \left[24 - 8 - 4 \right]$$

$$= \left[12 \right]$$

$$= 12.$$

PROPERTIES OF DETERMINANTS.

(1) The matrix is only invertible when determinant is non-zero.
ie matrix A is only invertible i-e A^{-1} when $\det A \neq 0$
because if it is 0 then $A^{-1} = \frac{Adj\, A}{\det A}$ will be undefined.

$$A^{-1} = \frac{Adj\ A}{\det A}, \quad \det A \neq 0.$$

(2) The matrix's determinant has same value as the determinant of its transpose.
i-e, $\det A = \det A^t$.
→ if A is a matrix then the determinant of its transpose will be same as its determinant.

Proof

e.g, let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

$\det A = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22} - a_{21}a_{12}$

$= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}$.

Example.

if $A = \begin{bmatrix} 6 & 8 \\ 7 & 11 \end{bmatrix}$ ⇒ $\det A = 11 \times 6 - 7 \times 8 = 66 - 56$
$= 10.$

then $A^t = \begin{bmatrix} 6 & 7 \\ 8 & 11 \end{bmatrix}$ ⇒ $\det A^t = 6 \times - 8 \times 7 = 66 - 56$
$= 10.$

(3) When rows or columns of matrix are interchanged the determinant is negative.

i-e if we interchange rows or columns of matrix A then its determinant is $-\det A$.

Proof

let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det A = a_{11}a_{22} - a_{12}a_{21}$

if $A = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$ → Rows interchanged.

$\det A = a_{21}a_{12} - a_{22}a_{11} = a_{12}a_{21} - a_{11}a_{22}$

$= -(a_{11}a_{22} - a_{12}a_{21}) = -\det A.$

Example.

$A = \begin{bmatrix} 11 & 10 \\ 2 & 3 \end{bmatrix} \Rightarrow \det A = 33 - 20 = 13$

$A = \begin{bmatrix} 10 & 11 \\ 3 & 2 \end{bmatrix} \Rightarrow \det A = 20 - 33 = -13.$

$\Rightarrow -\det A.$

(4) The determinant of a matrix is 0.

$\det A = 0$ when.

→ i) A matrix A has two identical rows or two identical columns.

→ ii) A matrix A has all elements zero.

i-e when all the entries of a matrix are zero.

(i) $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det A = a_{11}a_{21} - a_{11}a_{21} = 0.$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow \det A = 2-2 = 0.$$

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 8 \\ 3 & 3 & 9 \end{bmatrix} \Rightarrow \det A = \begin{vmatrix} 1 & 1 & 4 \\ 1 & 1 & 8 \\ 3 & 3 & 9 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 8 \\ 3 & 9 \end{vmatrix} - 1\begin{vmatrix} 1 & 8 \\ 3 & 9 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}$$

$$= 1(9-24) - 1(9-24) + 0 = -15 + 15 = 0.$$

(ii)

It is obvious that if all the elements of a matrix are 0 then determinant will be zero.

Which concludes that

A null matrix A of order nxn has det A = 0 (always.)

(5)

Determinant of the identity matrix is always.

i.

i-e $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \det(I) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

$$= 1 \times 1 - 0 \times 0 = 1$$

Same goes for oder $n \times m$.

(6)
If each entry of row or a column consists of two teams as sum than its determinant can be expressed as.

Example.

$$A = \begin{bmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow |A| = \begin{vmatrix} a_{11}+b_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22} & a_{23} \\ a_{31}+b_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} b_{11} & a_{12} & a_{13} \\ b_{21} & a_{22} & a_{23} \\ b_{31} & a_{32} & a_{33} \end{vmatrix}$$

(7)
When to each entry of a column or row of a matrix is added a non-zero multiple of the corepondPng entries of another ~~zero~~ row or column then determinant of matrix remains same.

Example.

let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightarrow |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$\rightarrow$ if $|B| = \begin{vmatrix} a_{11} & a_{12} + ka_{11} \\ a_{21} & a_{22} + ka_{21} \end{vmatrix}$

then $|B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & ka_{11} \\ a_{21} & ka_{21} \end{vmatrix}$

$\Rightarrow |B| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + k \begin{bmatrix} a_{11} & a_{11} \\ a_{21} & a_{21} \end{bmatrix}$

$\Rightarrow |B| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + 0$

$\Rightarrow |B| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = |A|$

(8) If the matrix is triangular then product of the entries in its diagnol gives the determinant of that matrix.

Example.

e.g let $A = \begin{bmatrix} 21 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$|A| = \begin{bmatrix} 21 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 21 \times 6 \times 2$
$= 252.$

let $B = \begin{bmatrix} 3 & 2 & 6 \\ 0 & 7 & 1 \\ 0 & 0 & 6 \end{bmatrix}$

$|B| = \begin{bmatrix} 3 & 2 & 6 \\ 0 & 7 & 1 \\ 0 & 0 & 6 \end{bmatrix} = 3 \times 7 \times 6$
$= 126$

Other way round.

$|B| = 3 \begin{bmatrix} 7 & 1 \\ 0 & 6 \end{bmatrix} - 2 \begin{bmatrix} 0 & 1 \\ 0 & 6 \end{bmatrix} + 6 \begin{bmatrix} 0 & 7 \\ 0 & 0 \end{bmatrix}$

$= 3(7 \times 6 - 0) - 2(0) + 6(0)$
$= 3(42)$
$= 126$

Hence proved.