Analyzing a penalty shootout using Lyapunov theory and game theory

SE 762: Nonlinear Systems and Control

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Introduction:

This paper involves the analysis of a penalty shootout in a soccer using game theory and Lyapunov stability theory to analyze the optimal strategies for given payoffs depending on the direction the striker shoots the ball and the direction the goalkeeper jumps. The paper will introduce the analysis of this two-agent adversarial system using replicator dynamics and then dive into Nash equilibrium analysis using Lyapunov stability theory.

Game setup and background:

The game between the players is set out to be a zero-sum game between the players. This means that the total sum of benefits of the game is zero, meaning that if one player wins, the other player must lose. In the context of a penalty shootout, this assumption works since if the striker scores, the goalkeeper loses and vice-versa. The game is also simultaneous since both the goalkeeper and the players must decide simultaneously. [1]

Each striker and goalkeeper have three directions to shoot the ball or dive towards, being left right, and center and therefore the game is a mixed strategy game. Since the game is a mixed strategy game, replicator dynamics can be utilized to analyze the strategies of the agents as they evolve over time.

Nash equilibrium (NE) is defined as an outcome when achieved means that no player can increase payoff by changing their decisions unilaterally [1]. In the context of this penalty shootout, since the game is noncooperative, this implies that the game has at least one point that is NE for both the striker and the goalkeeper. [2]

Game dynamics:

$$A = \begin{bmatrix} 0 & 0.8 & 0.7 \\ 0.9 & 0 & 0.2 \\ 0.75 & 0.45 & 0 \end{bmatrix} \tag{1}$$

And

$$B = -A^T \text{ [3]} \tag{2}$$

A is the payoff matrix defined for the striker. To account for the stochastic nature of penalty kicks, all the payoff values are set to be in the interval [0,1] so that each payoff value represents the chance of a goal for a given shot direction and diving direction [4]. If the player shoots to the left and the goalkeeper stays in the center, the payoff for the striker is 0.8. This allows for misses or deflections in shots to be accounted for during the penalty, since each value in matrix A is the chance of scoring a goal given any mix of decisions made by the players. A key assumption in the game is that if the striker's shot direction and the goalkeeper's diving direction are the same, the payoff is zero so neither player wins nor loses.

The goalkeeper's playoff matrix B is defined by equation (2) since the game is asymmetric meaning that the roles cannot be switched as the striker's job is to kick the ball into the net and the goalkeeper's strategy involves diving to one side. Therefore, this penalty kick scenario is defined as a 'Non-symmetric Zero-sum' game.

To analyze the game dynamics and how the player strategies evolve over time, replicator dynamics were used to analyze this system. Replicator dynamics involve analyzing mixed strategy games that evolve over a continuous time and learning between agents over time. Since the game is zero-sum, the total payoff is zero [3], which means:

$$x^T Ay + y^T Bx = 0 \tag{3}$$

Since our game is a two-player game, the replicator dynamics can be defined as [3]:

$$\dot{x_i} = x_i(\{Ay\}_i - x^T Ay) \tag{4}$$

$$\dot{y_j} = y_j\left(\{Bx\}_j - y^T Bx\right) \tag{5}$$

Equation (4) covers the replicator dynamics for the striker and how their strategies evolve over time. $\{Ay\}_i$ is the payoff for the striker playing strategy i, given the goalkeeper plays a mixed strategy y and the replicator dynamics compares the payoffs of the pure strategy against the average payoff obtained using the mixed strategy x, which is denoted by $x^T Ay$. Equation (5) denotes the same thing but from the goalkeeper's perspective with $\{Bx\}_i$ denoting the payoff for the goalkeeper's current strategy against the striker's strategy mix of x and $y^T Bx$ being the average payoff for the goalkeeper.

Therefore, these dynamics allow strategies that outperform the average payoff to be more prevalent and strategies that underperform to be faded away. This allows the players to adopt their strategy to the other player as they aim to optimize their own payoffs.

For this game, x and y are 3x1 column vectors which represent the probability distribution for the kicker's strategies and goalkeeper strategies respectively, which are defined as:

$$i\epsilon\{kick\ left, kick\ center, kick\ right\} = \{1,2,3\}$$

$$j\epsilon\{dive\ left, dive\ center, dive\ right\} = \{1,2,3\}$$

Since the game is non-cooperative, there exists at least one Nash equilibrium which is true for our system. Nash equilibrium for a system is defined as a point where unilaterally changing the strategy does not benefit. [2] For replicator dynamics, a pair of mixed strategies x^* and y^* are Nash equilibrium points if the agents interpret their respective payoffs to be optimized. For zero-sum games, Nash strategies are also referred to as max-min strategies. The following expressions describe the criteria for the Nash equilibrium [3]:

$$\forall i\epsilon[n]: \{Ay^*\}_i = {x^*}^T Ay^* \tag{6}$$

$$\forall j\epsilon[m]: \{Bx^*\}_j = {y^*}^T Bx^* \tag{7}$$

By setting equations (4) and (5) for all the possible strategies and setting them to 0 such that $\frac{dx_i}{dt} = 0\ and \frac{dy_j}{dt} = 0$, the equilibrium strategy mixes can be calculated and comparing them with equations (6) and (7), it can determined whether or not the strategies obtained are Nash or not. Pure strategies can also be fixed points in equations (6) and (7), however those will be exploited by the other player easily and therefore are not considered to be Nash equilibrium for the game.

Lyapunov Stability:

Since our penalty kick game is dynamic and nonlinear, a suitable Lyapunov candidate function can be used to analyze our two-agent system. From existing literature, the Kullback-Leibler (K-L) divergence can be used as a Lyapunov function as defined below [5]:

$$V(x,y) = \sum_{i=1}^{3} x_i^* \ln\left(\frac{x_i^*}{x_i}\right) + \sum_{j=1}^{3} y_j^* \ln\left(\frac{y_j^*}{y_j}\right) \quad (8)$$

The stability criteria using Lyapunov function:

- $\frac{dV}{dt} \leq 0 : system\ is\ stable$
- $\frac{dV}{dt} < 0: system\ is\ asymptotically\ stable$
- $\frac{dV}{dt} > 0: system\ is\ unstable$

The KL divergence measured how different the current strategy mix is different than the Nash equilibrium. It meets the following conditions:

- $V(x^*, y^*) = 0$ and this is the only zero point of the function
- $V(x,y) > 0$ for all $x \neq x^*, y \neq y^*$

The time derivative equation (8) ends up being 0 at the Nash equilibrium, meaning that the Nash equilibrium is neutrally stable such that if they started at the equilibrium, the system stays there but starting at a local neighborhood of the equilibrium, the trajectories for the strategy will oscillate about the equilibrium point. The proof for this is shown below:

Proof 1: Neutral stability

$$\dot{V} = -\sum_{i=1}^{3} \frac{x_i^*(\dot{x}_i)}{x_i} - \sum_{j=1}^{3} \frac{y_j^*(\dot{y}_j)}{y_j}$$

$$\dot{x}_i = x_i(\{Ay\}_i - x^T Ay)\ and\ \dot{y}_j = y_j(\{Bx\}_j - y^T Bx)$$

$$\dot{V} = -(\sum_{i=1}^{3} x_i^* \frac{(x_i(\{Ay\}_i - x^T Ay))}{x_i} + \sum_{j=1}^{3} y_j^* \frac{(y_j(\{Bx\}_j - y^T Bx))}{y_j})$$

$\dot{V} = -(\sum_{i=1}^{3} x_i^*(\{Ay\}_i - x^T Ay) + \sum_{j=1}^{3} y_j^*(\{Bx\}_j - y^T Bx)) \rightarrow x^T Ay + y^T Bx = 0$

$\dot{V} = -(\sum_{i=1}^{3} x_i^* (\{Ay\}_i) + \sum_{j=1}^{3} y_j^*(\{Bx\}_j)) \rightarrow B = -A^T$

$\dot{V} = -(x^* - x)^T A(y - y^*) \rightarrow$ For Nash equilibrium $(x,y) = (x^*, y^*)$, $\dot{V} = 0 \rightarrow$ neutrally stable

Near equilibrium: $\dot{V} = \delta x^T A \delta y \rightarrow\ \delta x \approx 0$ and $\delta y \approx 0$, then $\dot{V} \approx 0$

This means if the system starts at the equilibrium, it stays there, otherwise it oscillates around the equilibrium without ever converging to it if the initial conditions are close enough to the equilibrium. Since the payoff matrix A is not symmetric (i.e. $A \neq A^T$), then the matrix has to be averaged to make it symmetric as shown below [6]:

$$A_s = \frac{1}{2}(A + A^T)$$

This results in the following matrix:

$$A = \begin{bmatrix} 0 & 0.85 & 0.725 \\ 0.85 & 0 & 0.325 \\ 0.725 & 0.325 & 0 \end{bmatrix}$$

This matrix is symmetric so by finding the eigenvalues, the matrix definiteness can be determined. The eigenvalues are $\lambda_1 = 1.29, \lambda_2 = -0.970, \lambda_3 = -0.320$. Since the eigenvalues are both positive and negative, the matrix is determined to be indefinite and does not allow for inference of the behavior of the system for trajectories near the equilibrium. If the initial trajectories stay close to equilibrium, meaning $\delta x \approx 0$ and $\delta y \approx 0$, then $\dot{V} \approx 0$, resulting in the system being neutrally stable.

This means that if the system starts at equilibrium, the strategy mixes do not converge to the Nash equilibrium, rather the game evolves into a cycle of outwitting the opponent and the opponent catching up, resulting in cyclical dynamics for the system.

<u>Simulation and results:</u>

Using the background and game setup, this problem was simulated in Python (code attached with this project report). The Nash equilibria for the system were as follows:

$$x^* = [0.503, 0.422, 0.075], y^* = [0.416, 0.276, 0.308]$$

This shows that the striker prefers to shoot left and center more often than to the right while the goalkeeper adopts a more balanced approach as to which direction to dive towards with the left being preferable since the striker tends to shoot that way more often. Examining this equilibrium point using equations (6) and (7), proves that the system is a Nash equilibrium. Tables 1 and 2 show the results of this analysis for the striker and goalkeeper, respectively.

Table 1: Striker Payoff Analysis

Strategy	$\{Ay\}_i$	Optimal payoff for the striker-${x^*}^T A y^*$	$\Delta = \{Ay\}_i - {x^*}^T A y^*$
Kick left (i=1)	0.436	0.436	0
Kick center (i=2)	0.436	0.436	0
Kick right (i=3)	0.436	0.436	0

Table 2: Goalkeeper payoff analysis

Strategy	$\{Bx\}_j$	Optimal payoff for the striker-${y^*}^T B x^*$	$\Delta = \{Bx\}_j - {y^*}^T B x^*$

Dive left (j=1)	-0.436	-0.436	0
Dive center (j=2)	-0.436	-0.436	0
Dive right (j=3)	-0.436	-0.436	0

Starting the trajectories exactly at the equilibrium point, they stay there as shown in figure 1:

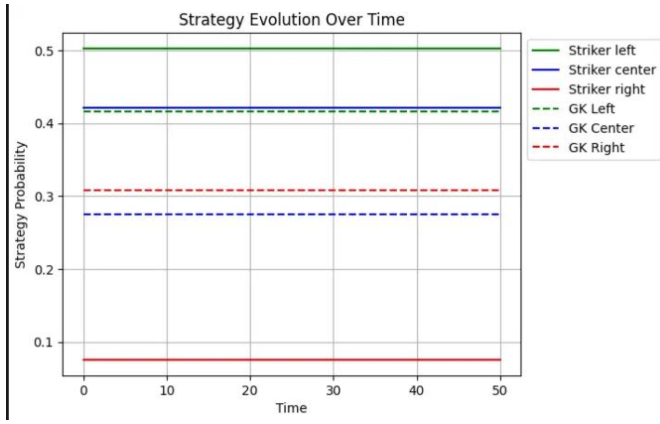

Figure 1: Initial conditions are the same as the Nash equilibrium, allowing the trajectories to stay where they started

Lyapunov stability theory indicated that the system was neutrally stable meaning that if we started at a neighborhood of the equilibrium point, the trajectories oscillate about the equilibrium points. Figure 2 shows this phenomenon happening for a randomly chosen initial probability distribution for the striker and goalkeeper.

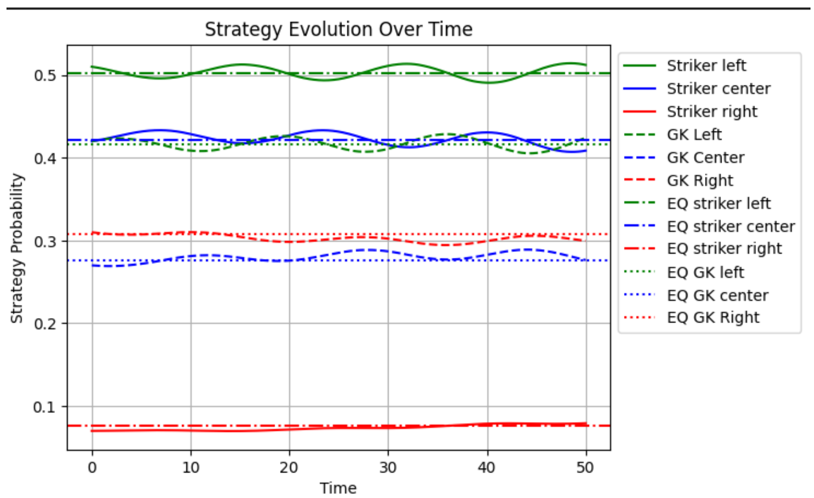

Figure 2: Strategy evolution over time for the initial probability distributions, x= [0.51,0.42,0.07] and y= [0.42,0.27,0.31]

Further examining the system under varying perturbations, randomly generated Gaussian noise with a mean of 0 and varying standard distributions was added to the known equilibrium point, allowing for randomly generated initial conditions at neighborhood away from the equilibrium. Figures 3-4 show the generated trajectories with σ=0.02 and σ=0.2, respectively.

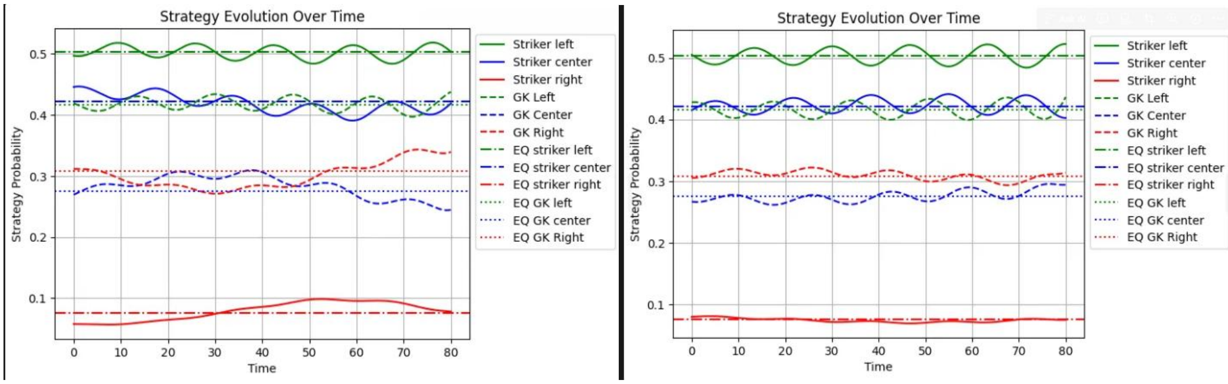

Figure 3 (a) (b): Strategy evolution over time for the initial probability distributions, $\sigma = 0.02$

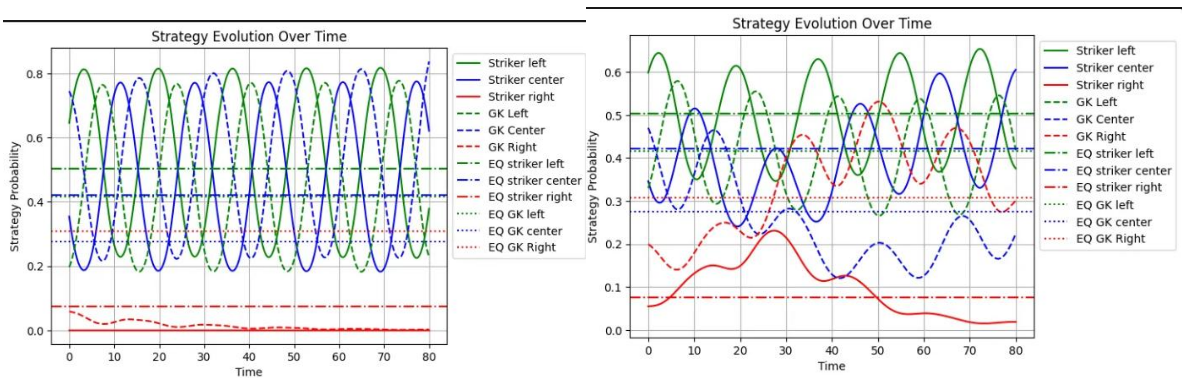

Figure 4 (a) (b): Strategy evolution over time for the initial probability distributions, $\sigma = 0.2$

It was observed that for these initial conditions, the trajectories kept oscillating within a neighborhood of the equilibrium, meaning that neither agent learned the optimal strategy for this penalty shootout, leaving more to be desired to achieve a strategic balance between the striker and the goalkeeper. Observing from figures 3 and 4, increasing the width of the distribution of the noise, which increases the distance between the initial conditions and the equilibrium point of the system. This results in an increase in oscillations around the equilibrium point and reflects the neutral stability of the system as highlighted by the Lyapunov theory. To examine this further, the payoffs for each player over time are analyzed. Modifying equations (6) and (7) allow us to examine how the player strategies are paying off for the striker and goalkeeper with respect to the average payoff for each player:

$$\{Ay\}_i - x^T Ay \neq 0 \; and \; \{Bx\}_j - y^T Bx \neq 0$$

Comparing the payoffs against the average for both players through plotting these differences for figures 3 (a) and 4 (a), we obtain the following plots as shown in figures 5-6:

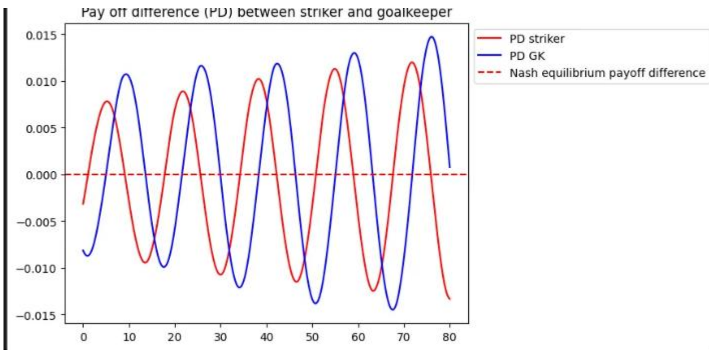

Figure 5: Payoff difference for 3(a)

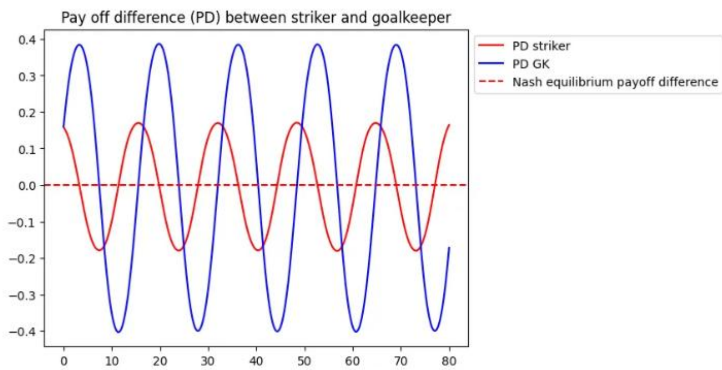

Figure 6: Payoff difference for 4(a)

The payoff differences for both the striker and goalkeeper oscillate about 0, meaning that neither player was able to reach the Nash equilibrium, and the players were constantly adapting to each other's strategy. The increased width of the Gaussian distribution allowed for the differences to be more pronounced meaning that the trajectory for the payoffs was deviating further away from the equilibrium. A positive difference indicates that the current strategy is generating a higher payoff than the average for a player and vice versa meaning the strategies that perform better are being chosen while underperforming ones are diminishing. Both the striker and goalkeeper's payoff differences also move in the same direction, indicating that they are adapting to their opponent's strategy.

This results in the game evolving into a cycling of adapting to each other's strategies and since the strategies do not converge to the Nash equilibrium, it leaves players with incentives to be able to benefit from unilaterally changing their strategies. Since the strategic balance is never achieved by the replicator dynamics themselves, more work had to be done through feedback linearization to achieve the strategic balance, allowing for more predictability in the game.

<u>Feedback Linearization:</u>

Since replicator dynamics by themselves proved to be ineffective to teach the agents to optimal strategy, full-state feedback linearization was introduced to the system. Feedback linearization allows the system to be global asymptotically stable, allowing convergence to the Nash equilibrium regardless of the initial starting condition. This allows both agents to adapt the optimized strategy for both agents so that they maximize their own playoffs relative to the other player. Equations (9) and (10) show the updated replicator dynamics for feedback linearization being added to the system.

$$\frac{dx_i}{dt} = x_i(\{Ay\}_i - x^T Ay) + u_i, i \,\epsilon\{1,2,3\} \tag{9}$$

$$\frac{dy_j}{dt} = y_j\big(\{Bx\}_j - y^T Bx\big) + v_j, j \,\epsilon\{1,2,3\} \tag{10}$$

Since each of the systems has a degree of 3, due to three state equations, there must be 3 control outputs for each agent, meaning a total of 6 control outputs to the system. The full state linearization can be defined by the following equation:

$$\begin{aligned} \frac{dx}{dt} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{11}$$

For the updated dynamics as shown in equations (9) and (10), equation (11) parameters for the striker and goalkeeper can be defined as:

$$\frac{dx}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix}, f(x) = \begin{bmatrix} x_1\{Ay\}_1 - x^T Ay \\ x_2\{Ay\}_2 - x^T Ay \\ x_3\{Ay\}_3 - x^T Ay \end{bmatrix}, g(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{12}$$

$$\frac{dy}{dt} = \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{bmatrix}, f(y) = \begin{bmatrix} y_1\{Bx\}_1 - y^T Bx \\ y_2\{Bx\}_2 - y^T Bx \\ y_3\{Bx\}_3 - y^T Bx \end{bmatrix}, g(y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{13}$$

Since the dynamics of the system are well-known in this instance, it is convenient to cancel out the non-linearities in the system and add a linear term that involves the convergence of the probability for each strategy to reach the Nash equilibrium. The control inputs u and v are the following:

$$u = \begin{bmatrix} -x_1\{Ay\}_1 + x^T Ay - k_1(x_1 - x_1^*) \\ -x_2\{Ay\}_2 + x^T Ay - k_2(x_2 - x_2^*) \\ -x_3\{Ay\}_3 + x^T Ay - k_3(x_3 - x_3^*) \end{bmatrix} v = \begin{bmatrix} -y_1\{Bx\}_1 + y^T Bx - c_1(y_1 - y_1^*) \\ -y_2\{Bx\}_2 + y^T Bx - c_2(y_2 - y_2^*) \\ -y_3\{Bx\}_3 + y^T Bx - c_3(y_3 - y_3^*) \end{bmatrix} \tag{14}$$

Inserting the control inputs u and v displayed in equation (14) into equations (12) and (13), the nonlinearities of the replicator dynamics are cancelled out, which results in the following dynamic systems equations:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix} = \begin{bmatrix} -k_1(x_1 - x_1^*) \\ -k_2(x_2 - x_2^*) \\ -k_3(x_3 - x_3^*) \end{bmatrix}, \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{bmatrix} = \begin{bmatrix} -c_1(y_1 - y_1^*) \\ -c_2(y_2 - y_2^*) \\ -c_3(y_3 - y_3^*) \end{bmatrix}, k_i \ and \ c_i \ are \ constant \ values \ more \ than \ 0 \quad (15)$$

Using equation (15), asymptotic stability for the system can be achieved for any value to be greater than zero to achieve global asymptotic stability. To prove this notion, $\frac{dV}{dt}$ was calculated again and it was less than zero, proving the asymptotic stability as shown Proof 2. The value of these constants determines the rate of convergence to the Nash equilibrium with a higher value allowing for faster convergence and vice versa and can be chosen arbitrarily if they are both greater than 0. Figures 7-8 show how the trajectories for the probability distribution converge to x for different perturbations being applied to the system.

Proof 2: Asymptotic stability

$$\dot{V} = -\sum_{i=1}^{3} \frac{x_i^*(\dot{x}_i)}{x_i} - \sum_{j=1}^{3} \frac{y_j^*(\dot{y}_j)}{y_j}$$

$$\dot{x}_i = -k_i(x_i - x_i^*), \ \dot{y}_j = -c_j(y_j - y_j^*)$$

$$\dot{V} = -(\sum_{i=1}^{3} \frac{x_i^*(-k_i(x_i - x_i^*))}{x_i} + \sum_{j=1}^{3} \frac{y_j^*(-c_j(y_j - y_j^*))}{y_j})$$

$$\dot{V} = -(\sum_{i=1}^{3} x_i^*(k_i\left(-\frac{x_i}{x_i} + \frac{x_i^*}{x_i}\right)) + \sum_{j=1}^{3} y_j^*(c_j\left(-\frac{y_j}{y_j} + \frac{y_j^*}{y_j}\right)))$$

$\dot{V} = \sum_{i=1}^{3} x_i^*(k_i\left(1 - \frac{x_i^*}{x_i}\right)) + \sum_{j=1}^{3} y_j^*(c_j\left(1 - \frac{y_j^*}{y_j}\right))$, for $\frac{y_j^*}{y_j}, \frac{x_i^*}{x_i} > 1$→ $\dot{V} < 0$, therefore it is asymptotically stable, $\forall k, c > 0$

This however fails to prove global asymptotic stability proposed by the feedback linearization theorem. Therefore, a different Lyapunov function must be used to prove global asymptotic stability. Consider:

$$V(x, y) = \sum_{i=1}^{3} \frac{1}{2}(x_i - x_i^*)^2 + \sum_{j=1}^{3} \frac{1}{2}\left(y_j - y_j^*\right)^2$$

This candidate function also meets the following conditions:

- $V(x^*, y^*) = 0$ and this is the only zero point of the function
- $V(x, y) > 0$ for all $x \neq x^*, y \neq y^*$

Proof 2 modified:

$$\dot{V}(x,y) = \sum_{i=1}^{3} \dot{x}_i(x_i - x_i^*) + \sum_{j=1}^{3} \dot{y}_j(y_j - y_j^*)$$

$$\dot{x}_i = -k_i(x_i - x_i^*),\ \dot{y}_j = -c_j(y_j - y_j^*)$$

$$\dot{V}(x,y) = \sum_{i=1}^{3} -k_i(x_i - x_i^*)(x_i - x_i^*) + \sum_{j=1}^{3} c_j(y_j - y_j^*)(y_j - y_j^*)$$

$$\dot{V}(x,y) = \sum_{i=1}^{3} -k_i(x_i - x_i^*)^2 + \sum_{j=1}^{3} -c_j\left(y_j - y_j^*\right)^2,\quad \forall k, c > 0 \rightarrow \dot{V} < 0 \longrightarrow asymptotic\ stability$$

This second candidate Lyapunov function was able to prove that the dynamics are globally asymptotically stable. This means that regardless of the starting point, the dynamics will guarantee convergence to the Nash equilibrium, resulting in both players learning the optimal strategy for the penalty shootout.

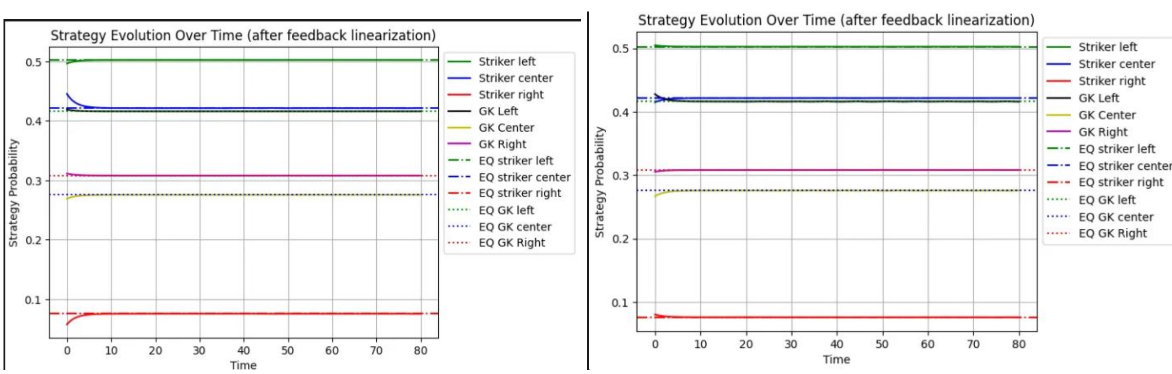

Figure 7 (a) (b): Strategy evolution over time after feedback linearization, $\sigma = 0.02$, k=c=0.5

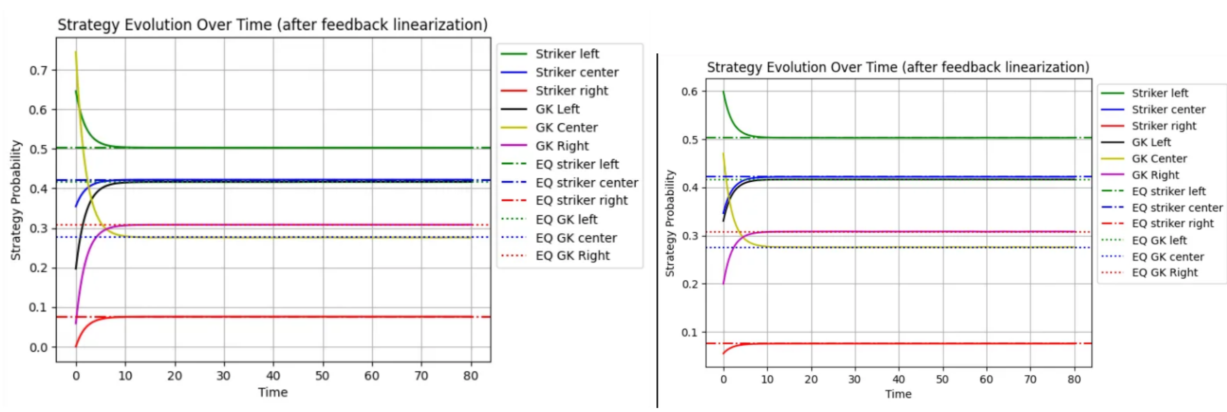

Figure 8 (a) (b): Strategy evolution over time after feedback linearization, $\sigma = 0.2$, k=c=0.5,

Figures 7 and 8 display that regardless of the magnitude of the perturbation added to the initial conditions, the feedback linearization controls displayed in equation (15) allow the player strategies to converge to the desired Nash equilibrium of the system, making the system asymptotically stable.

To further visualize the asymptotic stability of the system, the payoff deviation from the average can be analyzed since the convergence to the Nash condition suggests that the payoff difference (PD) should converge to 0, signaling the convergence to the Nash equilibrium. Analyzing this, for figures 7a and 8a, are shown below.

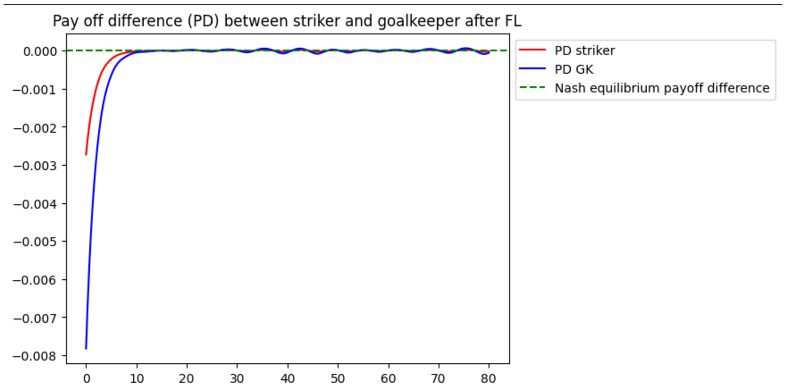

Figure 9: PD after FL was implemented 7(a)

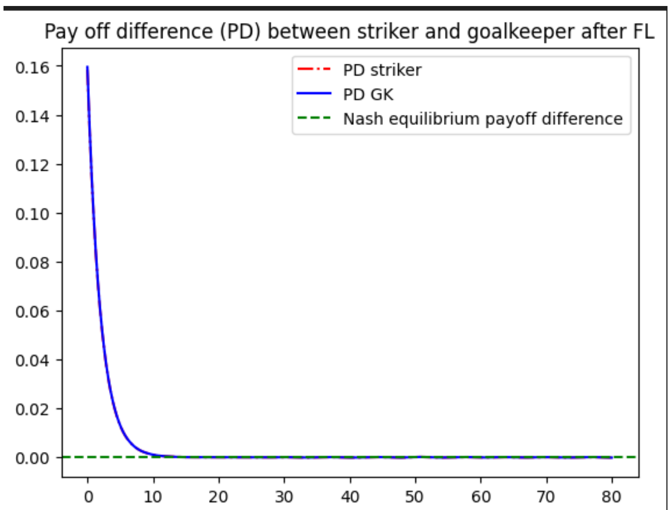

Figure 10: PD after FL was implemented 8(a)

Compared to the neutral stability of the replicator dynamics by themselves, the feedback linearization guarantees convergence to the Nash equilibrium, meaning that players choose a strategy that does as well as the optimal mixed strategy. Therefore, by converging to the Nash equilibrium, this feedback linearization allows both players to learn their optimal strategies and achieve balance in the game to prevent it from cycling forever and minimizes the risk of exploitation or bugs in the game, creating a strategic balance between the player and the goalkeeper. In real-life this can be imagined as coaching based on scouting reports and data from previous penalty shootouts which allow players to optimize their respective strategies in a penalty shootout. The system stabilizes- not because everyone is winning but because neither player wins by unilaterally changing their strategy.

Further work:

To extend this work further, uncertainties could be added to the system dynamics, which would involve using sliding mode control to ensure convergence to the Nash equilibrium under model uncertainty by approximating the system dynamics and bounding the error between the actual system and the estimated system. In addition, this paper assumes a fixed payoff matrix for the players which do not consider the characteristics of the players as some players develop fixed kicking and diving strategies and this theory can be used to create curated strategies for players. This also does not consider match conditions and how the ball was kicked, leaving room for improvement in the implementation of game theory and feedback linearization with a more detailed dynamic systems model. A potential extension for the project will be using policy gradient methods in reinforcement learning to allow the system to converge to the Nash equilibrium.

Conclusion:

This paper broke down the dynamics of penalty kicks, a high pressure and random moment in the soccer and explained how the system dynamics evolve over time using game theory. Using Lyapunov stability theory and KL divergence, it was shown that the system is neutrally stable and leaves room for work to achieve the strategic balance, which was achieved using feedback linearization to drive the system to the Nash equilibrium to achieve the strategic balance. In a game dictated by mental attributes, math can be used to analyze an important moment quantitatively and qualitatively in any soccer game.

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Appendix:

GitHub repo for this project:

https://github.com/adilfaisal01/SE762--Game-theory-and-Lyapunov-calculations

Feel free to leave comments and questions and I will be happy to answer them.