CSCE 629-601 Homework 9
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1. Exercise 29.3-7, on Page 879

Solve for the following LP:

Minimize:

$$x_1 + x_2 + x_3 \tag{1}$$

Subject to

$$\begin{aligned} 2x_1 + 7.5x_2 + 3x_3 &\geq 10000 \\ 20x_1 + 5x_2 + 10x_3 &\geq 30000 \\ x_1, x_2, x_3, &\geq 0 \end{aligned}$$

The standard form of the above LP is
Maximize:

$$-x_1 - x_2 - x_3 \tag{2}$$

Subject to

$$\begin{aligned} -2x_1 - 7.5x_2 - 3x_3 &\leq -10000 \\ -20x_1 - 5x_2 - 10x_3 &\leq -30000 \\ x_1, x_2, x_3, &\geq 0 \end{aligned}$$

Converting to slack form we get,

$$z = -x_1 - x_2 - x_3 \tag{3}$$

$$\begin{aligned} x_4 &= -10000 + 2x_1 + 7.5x_2 + 3x_3 \\ x_5 &= -30000 + 20x_1 + 5x_2 + 10x_3 \end{aligned}$$

The above equations have negative constants term. To get rid of this, we can try to solve the dual LP.
Dual form:
Minimize:

$$-10000y_1 - 30000y_2 \tag{4}$$

Subject to

$$\begin{aligned} -2y_1 - 20y_2 &\geq -1 \\ -7.5y_1 - 5y_2 &\geq -1 \\ -3y_1 - 10y_2 &\geq -1 \\ y_1, y_2 &\geq 0 \end{aligned}$$

The standard form of the dual LP is
Maximize:

$$10000y_1 + 30000y_2 \tag{5}$$

Subject to

$$\begin{aligned}2y_1 + 20y_2 &\le 1\\ 7.5y_1 + 5y_2 &\le 1\\ 3y_1 + 10y_2 &\le 1\\ y_1, y_2 &\ge 0\end{aligned}$$

The slack form of the dual LP is given by

$$z = 10000y_1 + 30000y_2 \tag{6}$$

Subject to

$$\begin{aligned}y_3 &= 1 - 2y_1 - 20y_2\\ y_4 &= 1 - 7.5y_1 - 5y_2\\ y_5 &= 1 - 3y_1 - 10y_2\\ y_1, y_2 &\ge 0\end{aligned}$$

Now we can solve the above LP using Simplex method.
Let us pick y_2 and increase it. We get the following constraints for above equations.

$$\begin{aligned}y_2 &\le 1/20\\ y_2 &\le 1/5\\ y_2 &\le 1/10\end{aligned}$$

Choosing the tighter condition $y_2 \le 1/20$, and pivoting y_2 and y_3, we get

$$y_2 = 1/20 - y_1/10 - y_3/20$$

substituting y_2 in above equations, we get new slack form as

$$z = 1500 + 7000y_1 - 1500y_3 \tag{7}$$

Subject to

$$\begin{aligned}y_2 &= 1/20 - y_1/10 - y_3/20\\ y_4 &= 3/4 - 7y_1 + y_3/4\\ y_5 &= 1/2 - 2y_1 + y_3/2\\ y_1, y_2 &\ge 0\end{aligned}$$

Now picking y_1 and increasing it we ge the following conditions

$$\begin{aligned}y_1 &\le 1/2\\ y_1 &\le 3/28\\ y_1 &\le 1/4\end{aligned}$$

Choosing the tighter condition $y_1 \le 3/28$, and pivoting y_1 and y_4, we get

$$y_1 = 3/28 + y_3/28 - y_4/7$$

Substituting in the above equations, we get a new slack form given by

$$z = 2250 - 1250y_3 - 1000y_4 \tag{8}$$

Subject to

$$y_2 = 11/280 - 13/200y_3 + 1/70y_4$$
$$y_1 = 3/28 - 1/28y_3 - 1/7y_4$$
$$y_5 = 2/7 + 4/7y_3 + 2/7y_4$$

The basic solution for the dual problem is $(y_1, y_2, y_3, y_4, y_5) = (3/28, 11/280, 0, 0, 2/7)$.
Since there are no more positive coefficients in objective function, $z = 2250$ is the optimal solution. The optimal solution of the primal and dual problem are same.
The solution set for the primal LP is

$$x_i = \begin{cases} -C'_{n+i} & if(n+i) \in N \\ 0 & otherwise \end{cases}$$

Using the above formula with $n = 2$, $m = 3$, $N = \{3, 4\}$ and $B = \{1, 2, 5\}$, we get

$$(x_1, x_2, x_3) = (1250, 1000, 0)$$

And the value of our Objective Funtion is Z= 1250 + 1000 + 0 = 2250 Which is minimized

Problem 2. Exercise 29.4-5, on Page 885

Prove that the Dual LP of the Dual LP is Primal LP.

Let us take a generalized LP of the following form
Maximize:

$$\sum_{j=1}^{n} c_j x_j \tag{9}$$

Subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, 2, 3, \ldots, m$$
$$\text{and } x_j \geq 0 \text{ for } j = 1, 2, 3, \ldots, n$$

By the principle of the duality, we will have the following dual LP for our primal LP
Minimize:

$$\sum_{i=1}^{m} b_i y_i \tag{10}$$

Subject to:

$$\sum_{i=1}^{m} a_{ij} y_j \geq c_j \text{ for } i = 1, 2, 3, \ldots, n$$
$$\text{and } y_i \geq 0 \text{ for } i = 1, 2, 3, \ldots, m$$

Considering the above Linear Program in standard LP form we have
Maximize:

$$\sum_{i=1}^{m} -b_i y_i \tag{11}$$

Subject to:

$$\sum_{i=1}^{m} -a_{ij} y_j \leq -c_j \text{ for } i = 1, 2, 3, \ldots, n$$
$$\text{and } y_i \geq 0 \text{ for } i = 1, 2, 3, \ldots, m$$

Now since we have the above linear program in the standard form. Lets change the negative coefficients to be substituted by the new coefficients as follows:

$$
\begin{aligned}
b'_i &= -b_i \\
a'_{ij} &= -a_{ij} \\
c'_j &= -c_j \\
\text{for } i &= 1,2,3,\ldots,n \\
\text{for } i &= 1,2,3,\ldots,m
\end{aligned}
$$

So the equation becomes:
Maximize:

$$\sum_{i=1}^{m} b'_i y_i \tag{12}$$

Subject to:

$$
\begin{aligned}
\sum_{i=1}^{m} a'_{ij} y_j \leq c'_j \text{ for } i = 1,2,3,\ldots,n \\
\text{and } y_i \geq 0 \text{ for } i = 1,2,3,\ldots,m
\end{aligned}
$$

Now we have the LP in standard form, lets get the dual of this LP, which means dual of the dual of original LP, a we have:
Minimize:

$$\sum_{j=1}^{n} c'_j z_j \tag{13}$$

Subject to:

$$
\begin{aligned}
\sum_{j=1}^{n} a'_{ij} z_j \geq b'_i \text{ for } i = 1,2,3,\ldots,m \\
\text{and } z_j \geq 0 \text{ for } j = 1,2,3,\ldots,n
\end{aligned}
$$

Now lets change it back to standard form and we have :
Maximize:

$$\sum_{j=1}^{n} -c'_j z_j \tag{14}$$

Subject to:

$$
\begin{aligned}
\sum_{j=1}^{n} -a'_{ij} z_j \leq -b'_i \text{ for } i = 1,2,3,\ldots,m \\
\text{and } z_j \geq 0 \text{ for } j = 1,2,3,\ldots,n
\end{aligned}
$$

Now lets get back the values we substituted earlier: That is:

$$
\begin{aligned}
b'_i &= -b_i \\
a'_{ij} &= -a_{ij} \\
c'_j &= -c_j \\
\text{for } i &= 1,2,3,\ldots,n \\
\text{for } i &= 1,2,3,\ldots,m
\end{aligned}
$$

We have :
Maximize:

$$\sum_{j=1}^{n} c_j z_j \tag{15}$$

Subject to:

$$
\begin{aligned}
\sum_{j=1}^{n} a_{ij} z_j \leq b_i \text{ for } i = 1,2,3,\ldots,m \\
\text{and } z_j \geq 0 \text{ for } j = 1,2,3,\ldots,n
\end{aligned}
$$

Now since the z_j is just the name of the variable, lets change the name of the variable from z to x, then we have :
Maximize:

$$\sum_{j=1}^{n} c_j x_j \tag{16}$$

Subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \text{ for } i = 1, 2, 3, \ldots, m$$
$$\text{and } x_j \geq 0 \text{ for } j = 1, 2, 3, \ldots, n$$

Which is exactly the same as we have our Primal LP. Hence we have proved that the Dual of the Dual of the primal LP is back to the Primal LP.
Hence Proved.

Problem 2. Exercise 29.5-5, on Page 893

Solution: In the problem we have been given:
maximize $x_1 + 3x_2$
subject to:
$x_1 - x_2 \leq 8$
-$x_1 - x_2 \leq$ -3
-$x_1 + 4x_2 \leq 2$
$x_1 \geq 0$, $x_2 \geq 0$
Writing the above equations in the Slack LP form:
z = $x_1 + 3x_2$
$x_3 = 8 - x_1 + x_2$
$x_4 = -3 + x_1 + x_2$
$x_5 = 2 + x_1 - 4x_2$
Basic Solution is (0,0,8,-3,2). Since we have x_4 value as -3 , Hence the initial basic solution is not feasible.
Lets make an auxiliary LP:
maximize $z = -x_0$ (1)
$x_3 = 8 - x_1 + x_2 + x_0$ (2)
$x_4 = -3 + x_1 + x_2 + x_0$ (3)
$x_5 = 2 + x_1 - 4x_2 + x_0$ (4)
Since there is only one term with negative constant value, so we will pivot around the most negative coefficient.
Pivoting around the equation (3), we have:
$x_0 = 3 - x_1 - x_2 + x_4$ (5) Changing the value in each equation
$x_3 = 11 - 2x_1 + x_4$ (6)
$x_5 = 5 - 5x_2 + x_4$ (7)
and
$z = -3 + x_1 + x_2 - x_4$ (8)
We select x_1 to increase the value of z.
In each equation, constraints for x_1 are:
$x_1 \leq 11/2$, $x_1 \leq 3$, $x_1 \leq \infty$
We choose equation (5) for pivoting:
$x_1 = 3 - x_0 - x_2 + x_4$ (5) Putting the value of x_1 in each equation
$x_3 = 5 + 2x_0 + 2x_2 - x_4$ (6)
$x_5 = 5 - 5x_2 + x_4$ (7)
$z = -x_0$
Since we have reached the end of solution for auxilliary LP, as objective function can not be maximized further
Basic solution is (x_0,x_1,x_2,x_3,x_4,x_5) = (0,3,0,5,0,5)
and objective value is Z=0.
Since we have got optimal value for z as 0, so same slack form can be used for our originial LP, but with x_0 as 0.
We have Original LP as
z = $x_1 + 3x_2$
Using the value of x_1 from above LP,
$z = 3 - x_2 + x_4 + 3x_2$
$z = 3 + 2x_2 + x_4$

$x_1 = 3 - x_2 + x_4$
$x_3 = 5 + 2x_2 - x_4$
$x_5 = 5 - 5x_2 + x_4$
Basic solution for the above LP is (x_1,x_2,x_3,x_4,x_5) = (3,0,5,0,5)
and the objective value is 3.
Since now we have a slack form which is feasible, we will start applying simplex algorithm.
We wil use x_2 to increase the value of z
We have constraints as

$x_2 \leq 3$, $x_2 \leq \infty$, $x_2 \leq 1$

Solving the last equation using pivoting
$x_2 = 1 + \frac{x_4}{5} - \frac{x_5}{5}$
$x_1 = 2 + \frac{4x_4}{5} + \frac{x_5}{5}$
$x_3 = 7 - \frac{3x_4}{5} - \frac{2x_5}{5}$
$z = 5 + \frac{7x_4}{5} - \frac{2x_5}{5}$
Basic solution for the above slack form = (x_1,x_2,x_3,x_4,x_5) = (2,1,7,0,0)
and the objective value is 5.
We will increase z now by increasing x_4
We have constraints as

$x_4 \leq \infty$, $x_4 \leq \infty$, $x_4 \leq \frac{35}{3}$

So we will pivot around x_3
$x_4 = \frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3}$
$z = \frac{64}{3} - \frac{7x_3}{3} - \frac{4x_5}{3}$ (Using value of x_4 in z)
$x_1 = \frac{34}{3} - \frac{4x_3}{3} - \frac{x_5}{3}$
$x_2 = \frac{10}{3} - \frac{x_3}{5} - \frac{x_5}{5}$
$x_4 = \frac{35}{3} - \frac{5x_3}{3} - \frac{2x_5}{3}$
Since now we don't have any variable with positive coefficient in z, we have got the optimal solution.
Solution is (x_1,x_2,x_3,x_4,x_5) = ($\frac{34}{3}$,$\frac{10}{3}$,0,$\frac{35}{3}$,0)
and optimal objective value is z = $\frac{64}{3}$
Lets check in the original LP. Since (x_1,x_2) = ($\frac{34}{3}$,$\frac{10}{3}$)
So z = $\frac{34}{3}$ + 3*$\frac{10}{3}$ = $\frac{64}{3}$
Also the optimal values for x_1 and x_2 are satisfying every constraint in the original LP in the question
So final solution is (x_1,x_2) = ($\frac{34}{3}$,$\frac{10}{3}$)
and Objective value z is z= $\frac{64}{3}$

Problem 4. Exercise 29.5-7, on Page 893 Solve the Problem Using Simplex Algorithm:

Maximize:

$$x_1 + 3x_2 \tag{17}$$

Subject to

$$\begin{aligned} -x_1 + x_2 &\leq -1 \\ -x_1 - x_2 &\leq -3 \\ -x_1 + 4x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Converting the given Equations in to Slack Form:
The Slack form is:

$$Z = x_1 + 3x_2 \tag{18}$$

Subject to

$$\begin{aligned} x_3 &= -1 + x_1 - x_2 \\ x_4 &= -3 + x_1 + x_2 \\ x_5 &= 2 + x_1 - 4x_2 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

So basic solution is : 0,0,-1,-3,2 , which is not feasible.

So we would first convert it to Auxiliary LP, to verify the feasibility of the Linear Equations given:
The Auxiliary LP is:

$$Z = -x_0 \tag{19}$$

Subject to

$$\begin{aligned} x_3 &= -1 + x_1 - x_2 + x_0 \\ x_4 &= -3 + x_1 + x_2 + x_0 \\ x_5 &= 2 + x_1 - 4x_2 + x_0 \\ x_0, &x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Since Constrain 2 has the most negative coefficient we choose that equation to do the pivoting, Which gives us the equation:

$$Z = -3 + x_1 + x_2 - x_4 \tag{20}$$

Subject to:

$$\begin{aligned} x_0 &= 3 - x_1 - x_2 + x_4 \\ x_3 &= 2 - 2x_2 + x_4 \\ x_5 &= 5 - 5x_2 + x_4 \end{aligned}$$

Now Lets increase the value of x_1 variable in Z.
Where the constraints are:

$$\begin{aligned} x_1 &\leq 3 \\ x_1 &\leq \infty \\ x_1 &\leq \infty \end{aligned}$$

So we would pivot equation containing x_0.
We will get the following LP form:

$$Z = -x_0 \tag{21}$$

Subject to:

$$\begin{aligned} x_1 &= 3 - x_0 - x_2 + x_4 \\ x_3 &= 2 - 2x_2 + x_4 \\ x_5 &= 5 - 5x_2 + x_4 \end{aligned}$$

So from here we cannot increase the value of Objective function more and the maximum value of the objective function would 0 for our auxiliary LP.
Now coming to our Original LP using the slack form LP of the Auxillary LP and using the value of $x_0 = 0$, we write the equations:

$$Z = 3 + x_4 + 2x_2 \tag{22}$$

Subject to:

$$\begin{aligned} x_1 &= 3 - x_2 + x_4 \\ x_3 &= 2 - 2x_2 + x_4 \\ x_5 &= 5 - 5x_2 + x_4 \end{aligned}$$

Where Basic Solution set is: 3,0,2,0,5 and the value of the objective function is 3.
Now Increasing the value of x_2, we get the following constrain:

$$x_2 \leq 3$$
$$x_2 \leq 1$$
$$x_2 \leq 1$$

Choosing the pivot equation with least Index and doing the pivoting we get.

$$Z = 5 - x_3 + 2x_4 \tag{23}$$

Subject to:

$$x_1 = 2 + x_3/2 + x_4/2$$
$$x_2 = 1 - x_3/2 + x_4/2$$
$$x_5 = 5x_3/2 - 3x_4/2$$

Basic Solution here is: 2,1,0,0,0
Increasing the value of x_4, we get the following constrain:

$$x_4 \leq \infty$$
$$x_4 \leq \infty$$
$$x_4 \leq 0$$

So, we will pivot around x_5, which would give us the equation:

$$Z = 5 + 7x_3/3 - 4x_5/3 \tag{24}$$

Subject to:

$$x_1 = 2 + 4x_3/3 - x_5/5$$
$$x_2 = 1 + x_3/3 - x_5/5$$
$$x_4 = 5x_3/3 - 2x_5/3$$

Basic solution is : 2,1,0,0,0 , where the objective function is Z= 5.
So we will increase the value of x_3, where the constrain is given by:

$$x_3 \leq \infty$$
$$x_3 \leq \infty$$
$$x_3 \leq \infty$$

Since, we have reached a point, where x_3 is not bounded,as it can take any value, So the maximum value of the objective function is not bounded., That means our LP here has a feasible solution, but does not contain any upper bound.
Showing it graphically since we have two variable in out LP, so we can present the LP on the cartesian coordinate system in the given figure:
So as we can see the we got two values when we have done the pivoting but we couldn't find the next vertex of the feasible solution as it is not bounded.

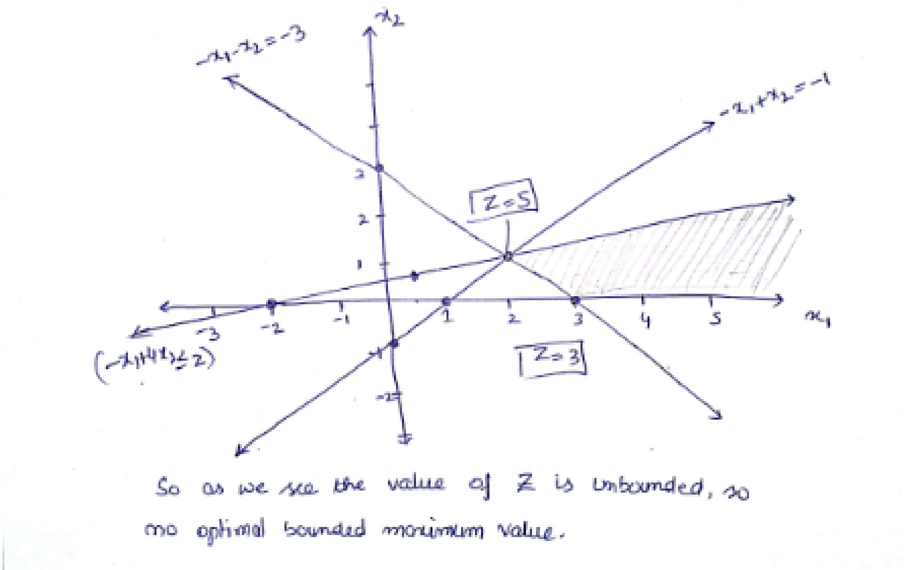

Figure 1: Figure showing the graph using the equations in our LP