Gaussian Radial Basis Function Applied with a Hermite Collocation Scheme for a Coupled Nonlinear PDE

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The main purposes of this study are firstly, to

1. Introduction

2. The Collocation Method

The collocation scheme starts with considering the following elliptical partial differential equation defined on a bounded and connected domain Ω ;

a $$\Phi\left[u(\mathbf{x})\right] = f(x) \quad \text{for} \quad \mathbf{x} \in \Omega \subset \mathbb{R}^n \tag{6.1}$$

b $$B_1 u(\mathbf{x}) = g(x) \quad \text{for} \quad \mathbf{x} \in \Gamma_1 \tag{6.2}$$

c $$B_1 u(\mathbf{x}) = h(x) \quad \text{for} \quad \mathbf{x} \in \Gamma_1 \tag{6.3}$$

Where $\partial\Omega$ is the domain boundary containing two non-overlap sections; Γ_1 and Γ_2, with $\Gamma_1 \cap \Gamma_2 = \phi$. These differential operators ; Φ, and B_1, B_2 are applied on the domain, and the two boundary sections respectively. Three known functions $f(\mathbf{x}), g(\mathbf{x}), h(\mathbf{x})$ can well be dependent of space and/or time. Let $X^c = \{\mathbf{x}_j\}_{j=1}^{N}$ be a set of randomly selected points, known as 'centers', on the domain where $\{\mathbf{x}_j\}_{j=1}^{N_i}$ are those contained within, where $\{\mathbf{x}_j\}_{j=N_i+1}^{N_i+N_1}$ and $\{\mathbf{x}_j\}_{j=N_i+N_1+1}^{N}$ are those on the boundary Γ_1 and Γ_2 respectively. The collocation scheme writes the approximate solution, $\tilde{u}(\mathbf{x})$, as the linear combination of the basis function $\{\varphi(:-)\}_j^N$, shown in the following form;

d $$u(\mathbf{x}) \simeq \tilde{u}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \varphi\left(\left\|\mathbf{x} - \mathbf{x}_j\right\|_2\right) \tag{6.4}$$

Where α_j are coefficients and $\|\,.\,\|_2$ being the Euclidean norm. The basis function used now is the radial type as defined previously. Applying Φ, and B_1, B_2 to on both domain and boundary sections, satisfying the governing system of equations, allows the system to arrive at;

e $$\mathbf{A\alpha} = \mathbf{b} \tag{6.5}$$

Where $\boldsymbol{\alpha} = \left[\alpha_1 \quad \alpha_2 \quad \ldots \quad \alpha_N\right]$, the known $\mathbf{b}$ vector is as follows;

$$\mathbf{b} = \left[\, f(\mathbf{x}_1) \quad f(\mathbf{x}_2) \quad \ldots \quad f\left(\mathbf{x}_{N_i}\right) \quad g\left(\mathbf{x}_{N_i+1}\right) \quad \ldots \quad g\left(\mathbf{x}_{N_i+N_1}\right) \quad h\left(\mathbf{x}_{N_i+N_1+1}\right) \quad \ldots \quad h(\mathbf{x}_N) \,\right]^T$$

and by setting $\boldsymbol{\varphi}$ to be a matrix with entries $\varphi_{ij} = \varphi\left(\left\|\mathbf{x}_i - \mathbf{x}_j\right\|_2\right)$ for $i, j = 1, 2, \ldots, N$ we have;

f $$\mathbf{A} = \begin{bmatrix} \Phi[\boldsymbol{\varphi}] \\ B_1[\boldsymbol{\varphi}] \\ B_2[\boldsymbol{\varphi}] \end{bmatrix} \tag{6.6}$$

Once α_j are obtained by e , the approximate solution are straightforward yielded. This method is known as Kansa method and is known to suffer the problem of unsymmetric interpolation matrix, $\mathbf{A}$, and very often produces low quality results particularly in boundary-adjacent region.

In order to alleviate this difficulty , Fasshauer , proposed a new way of interpolating by applying the self-adjoint operators Φ, and B_1, B_2 to the governing system of equations and rewrite the approximate solution as g

$$u(\mathbf{x}) \simeq \tilde{u}(\mathbf{x}) = \sum_{j=1}^{N_i} \alpha_j \Phi^* \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right) + \sum_{j=N_i+1}^{N_i+N_1} \alpha_j B_1^* \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right) + \sum_{j=N_i+N_1+1}^{N} \alpha_j B_2^* \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right) \tag{6.7}$$

This leads to a new interpolation matrix $\mathbf{A}$, shown as follows;

h

$$\mathbf{A} = \begin{bmatrix} \Phi\Phi^*[\boldsymbol{\varphi}] & \Phi B_1^*[\boldsymbol{\varphi}] & \Phi B_2^*[\boldsymbol{\varphi}] \\ B_1\Phi^*[\boldsymbol{\varphi}] & B_1 B_1^*[\boldsymbol{\varphi}] & B_1 B_2^*[\boldsymbol{\varphi}] \\ B_2\Phi^*[\boldsymbol{\varphi}] & B_2 B_1^*[\boldsymbol{\varphi}] & B_2 B_2^*[\boldsymbol{\varphi}] \end{bmatrix} \tag{6.8}$$

3. Discretization of 2D Burgers' equations

3.1) Space Discretization using Hermite Scheme

From the governing equations previously described, with Hermite interpolation technique, it begins with writing the approximation of solution for $\tilde{u}(\mathbf{x})$ and $\tilde{v}(\mathbf{x})$, respectively, with the same radial basis function $\varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right)$, as follows;

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$$u(\mathbf{x},t) \simeq \tilde{u}(\mathbf{x},t) = \sum_{j=1}^{N_i} \alpha_j^t \Phi^* \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right) + \sum_{j=N_i+1}^{N} \alpha_j^t B^* \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right) \tag{6.9}$$

and

j

$$v(\mathbf{x},t) \simeq \tilde{v}(\mathbf{x},t) = \sum_{j=1}^{N_i} \beta_j^t \Phi^* \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right) + \sum_{j=N_i+1}^{N} \beta_j^t B^* \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right) \tag{6.10}$$

At $n^{th}-$ derivative, this leads to;

i

$$\frac{\partial^{(n)} u(\mathbf{x},t)}{\partial x^{(n)}} \simeq \sum_{j=1}^{N_i} \alpha_j^t \frac{\partial^{(n)}\left[\Phi^* \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right)\right]}{\partial x^{(n)}} + \sum_{j=N_i+1}^{N} \alpha_j^t \frac{\partial^{(n)}\left[B^* \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right)\right]}{\partial x^{(n)}} \tag{6.11}$$

and

i

$$\frac{\partial^{(n)} v(\mathbf{x},t)}{\partial y^{(n)}} \simeq \sum_{j=1}^{N_i} \beta_j^t \frac{\partial^{(n)}\left[\Phi^* \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right)\right]}{\partial y^{(n)}} + \sum_{j=N_i+1}^{N} \beta_j^t \frac{\partial^{(n)}\left[B^* \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|_2\right)\right]}{\partial y^{(n)}} \tag{6.12}$$

3.2) Time Discretization by 4^{th}-Runge-Kutta method

4. Numerical Results

4.1) The effect of the shape parameter

4.2) The effect of the number of centers

4.3) At moderate to high Reynolds number

5. General Discussion

6. Conclusion

References