A Variable Inverse-Multiquadric Shape Parameter Applied with a Meshless Method for Nonlinear PDEs

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Abstract. It is known that all RBF-based meshfree methods suffer from a lack of reliable judgment on the choice of shape parameter, appearing in most of the RBFs. Many attempts on providing promisingly useful information on how to define this parameter have been made during the part years. Nevertheless, it is known that it is practically impossible to have one form of parameter that yields reasonable results for all kinds of problems. In this work, we aim to complete three important tasks. Firstly, we apply the methodology of RBF-collocation method, one of the classical form of meshfree/meshless schemes, to PDEs with nonlinear nature. Secondly, several forms of variable Multiquadric shape parameters proposed and available in literature are gathered and applied to the same type of PDEs. Lastly, we propose in this work a new form of shape parameter under the form of inverse-multiquadric type of RBF. In order to justify the quality of our proposed variable, all forms gathered from literature are applied to the same PDEs, and all the results are compared against one another. The information gathered and presented in this work shall be useful for the future users in making decision as well as will provide useful guide to further invent another potential shape adaptive approaches.

INTRODUCTION

Considered as another comparably-new research area, finding numerical solutions to a given partial differential equations (PDEs) by means of meshfree methods is now receiving more and more attention. The main feature is its simplicity and the dependence of mesh/grid connection and topology making it much simpler to implement. The whole class of meshfree/meshless method can be categorized into three classes; weak forms, strong forms, and mixed, all nicely documented in [1]. Each of these has its own advantages/disadvantages depending on several factors involved including domain geometry, governing equations, boundary/initial conditions, computer arithmetic, etc. Amongst those being proposed and developed nowadays, one of the well-known meshfree method is that called 'RBF-collocation' or sometimes called 'Kansa's method' [2], where it uses a set of global approximation function to approximate the field variables on both the domain and the boundary when solving PDEs. This is of the following expression;

$$u(\mathbf{x}) = \sum_{j=1}^{N} c_j \varphi_j(r) = \sum_{j=1}^{N} c_j \varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|\right) \tag{1}$$

Over some given scattered data nodes $X = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N\} \subset \Omega$, Ω is the problem domain. The Radial Basis Functions (RBF), φ, are commonly found as multivariate functions whose values are dependent only on the distance from the origin and commonly assumed to be strictly positive definite. This means that $\varphi(\|\mathbf{x}\|) = \varphi(r) \in \mathbb{R}$

with $\mathbf{x} \in \mathbb{R}^n$ and $r \in \mathbb{R}$; or, in other words, on the distance from a point of a given set $\{\mathbf{x}_j\}$, and $\varphi(\|\mathbf{x}-\mathbf{x}_j\|)=\varphi(r_j) \in \mathbb{R}$ where can normally defined as follows;

$$r=\|\mathbf{x}-\mathbf{x}^{\Theta}\|_2=\sqrt{(x_1-x_1^{\Theta})^2+...+(x_n-x_n^{\Theta})^2} \tag{2}$$

For some fixed points $\mathbf{x}\in\mathbb{R}^n$. Nevertheless, in this work, $r_j=\|\mathbf{x}-\mathbf{x}_j\|_2$ is the Euclidean distance and the radial basis function (RBF), φ. Some mostly-used and investigated RBFs are listed in Table 1.

TABLE 1. Radial Basis Functions (RBF) widely-used in literature.

RBF types	**Mathematical Formula**
Linear (LR)	$1+r$
Gaussian (GU)	$\exp(-r^2)$
Cubic (CU)	r^3
Polyharmonic (PY)	r^{2n-1}, $n\in\mathbb{N}$
Multiquadric (MQ)	$\sqrt{r^2+\varepsilon^2}$
Inverse Multiquadric (IMQ)	$1/\sqrt{r^2+\varepsilon^2}$
Thin-plate Spline (TPS)	$r^2\ln(r)$
Matern/Sobolev (MS)	$^*\left[K_V(r)r^V\right]$

$^*K_V$ is an order modified Bessel function

In this work, nevertheless, we the whole investigation is on the Inverse Multiquadric type, defined in general form as;

$$\varphi(r,\varepsilon)=\left(\varepsilon^2+r^2\right)^{\beta} \tag{3}$$

where $\beta=...,-3/2,-1/2,1/2,3/2,...$ and ε is the so-called 'shape parameter', known to play a crucial role in determining the quality of the final results and have always been an open topic for decades. Hardy [3] suggests that under the context of multiquadric type, by fixing the shape at $\varepsilon=1/(0.815d)$, where $d=(1/N)\sum_{i=1}^{N}d_i$, and d_i is the distance from the node to its nearest neighbor, good results should be anticipated. Also, in the work of Franke [4] where the choice of a fixed shape of the form $\varepsilon=0.8\sqrt{N}/D$ where D is the diameter of the smallest circle containing all data nodes, can also be a good alternative.

Some recent attempts to pinpoint the optimal value of ε involve the work of Zhang et al. [5] where they demonstrated and concluded that the optimal shape parameter is problem dependent. In 2002, Wang and Lui [6] pointed out that by analyzing the condition number of the collocation matrix, a suitable range of derivable values of ε can be found. Later in 2003, Lee et. al. [7] suggested that the final numerical solutions obtained are found to be less affected by the method when the approximation in equation (1) is applied locally rather than globally.

Regarding the inverse form of RBF expressed in equation (3), however, up to the author's knowledge, there has not been much of investigation on searching for optimal shape parameter. This prompts one of the main objectives of the work.

RADIAL BASIS FUNCTION COLLOCATION MESHFREE METHOD FOR PDES

For the methodology of RBF-collocation meshless method for numerically solving PDEs, it begins with considering a linear elliptic partial differential equation with boundary conditions, where $g(\mathbf{x})$ and $f(\mathbf{x})$ are known. We seek $u(\mathbf{x})$ from;

$$Lu(\mathbf{x})=f(\mathbf{x}),\ \mathbf{x}\ \text{in}\ \Omega \tag{4}$$

$$Mu(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \text{ on } \partial\Omega \tag{5}$$

where $\Omega \in \mathbb{R}^d$, $\partial\Omega$ denotes the boundary of domain , L and M are the linear elliptic partial differential operators and operating on the domain Ω and boundary domain $\partial\Omega$, respectively. For Kansa's method, it represents the approximate solution $u(\mathbf{x})$ by the interpolation, using an RBF interpolation as expressed in equation (1). We can see that linear dependent equations are required for solving N unknowns of c_j Substituting $u(\mathbf{x})$ into equation (4)and equation (5), we obtain the system of equations as follows;

$$L\left(\sum_{j=1}^{N_I} c_j \varphi\left\|\mathbf{x}-\mathbf{x}_j\right\|\right) = \sum_{j=1}^{N_I} c_j L\varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|\right) = f(\mathbf{x}_i), \tag{6}$$

$$M\left(\sum_{j=1}^{N_I} c_j \varphi\left\|\mathbf{x}-\mathbf{x}_j\right\|\right) = \sum_{j=1}^{N_I} c_j M\varphi\left(\left\|\mathbf{x}-\mathbf{x}_j\right\|\right) = g(\mathbf{x}_i), \tag{7}$$

Above equations, we choose N collocation points on both domain Ω and boundary domain , and divide it into N_I interior points and N_B boundary points $(N = N_I + N_B)$. Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ denotes the set of collocation points, $I = \{I_1, \ldots, I_{N_I}\}$ denotes the set of interior points and $B = \{B_1, \ldots, B_{N_B}\}$ the set of boundary points. The centers $\mathbf{x}_j$ used in equation (6) and equation (7), are chosen as collocation points. The previous substituting yields a system of linear algebraic equations which can be solved for seeking coefficient by rewriting equation (6) and equation (7) equation in matrix form as;

$$\mathbf{Ac} = \mathbf{F} \tag{8}$$

where $\mathbf{A} = [\mathbf{A}_L, \mathbf{A}_M]^T$, $(\mathbf{A}_L)_{ij} = L\varphi\left(\left\|\mathbf{x}_i - \mathbf{x}_j\right\|\right)$, $\mathbf{x}_j \in X$ $i = 1, 2, \ldots, N_I$, $j = 1, 2, \ldots, N$, $(\mathbf{A}_M)_{ij} = M\varphi\left(\left\|\mathbf{x}_i - \mathbf{x}_j\right\|\right)$, $\mathbf{x}_i \in B$, $\mathbf{x}_j \in X$, $i = N_I + 1, \ldots, N$, $j = 1, 2, \ldots, N$, and $\mathbf{F} = [f(\mathbf{x}_i), g(\mathbf{x}_i)]$, $f(\mathbf{x}_i); \mathbf{x}_i \in I$, $i = 1, 2, \ldots, N_I$, $g(\mathbf{x}_i); \mathbf{x}_i \in B$, $i = N_I + 1, \ldots, N$.

Equation (8), the coefficient $\mathbf{c}'s$ are computed from the following system; $u(\mathbf{x})$

$$\mathbf{c} = \mathbf{A}^{-1}\mathbf{F} \tag{9}$$

Therefore, the matrix $\mathbf{c}$ is substituted into equation (1) and the approximate solution of can be determined by;

$$u(\mathbf{x}) = \sum_{j=1}^{N} c_j \varphi\left(\left\|\mathbf{x} - \mathbf{x}_j\right\|\right) \tag{10}$$

Being one of the most popular choices of meshless method, its existence, uniqueness, and convergence have been nicely documented in [8,9]. The system is known to provide solution if and only if the matrix $\mathbf{A}$ is non-singular, its inverse exists. This aspect is related directly to its condition number, as can be defined as;

$$\kappa_\delta(\mathbf{A}) = \left\|\mathbf{A}\right\|_\delta \left\|\mathbf{A}^{-1}\right\|_\delta, \ \delta = 1, 2, \infty \tag{11}$$

In the RBF-meshless method, it is well-known that this condition number is strongly affected by the magnitude of the shape parameter and the number of nodes involved, N.

LINEARIZATION AND COMPUTATION ALGORITHM

As this work focusses on applying the RBF-collocation method to non-linear type of PDEs, there are several aspects needed to be taken into consideration. The computation process starts with considering the nonlinear PDEs of the form

$$Lu = f \tag{12}$$

with the following steps;

(1) Create the collocation point sets $X \subset \Omega$ and start with an initial guess u_0

(2) For $k = 1, 2, \ldots, K$

(a) The linearized problem

$$\Lambda_{u_{k-1}} v = f - L u_{k-1} \quad \text{on} \quad X$$

(b) Perform the Newton update

$$\tilde{v} = v$$

(c) Update the previous approximation

$$u_k = u_{k-1} + \tilde{v}$$

In this algorithm $\Lambda_{u_{k-1}}$ is the linearization of the nonlinear differential operator L at u_{k-1}. Here we provide an example in order to elaborate the algorithm described above.
Considering the nonlinear PDE of the form;

$$-\mathrm{E}^2 \nabla^2 u - u + u^3 = f \quad \text{in } \Omega = (0,1) \times (0,1) \tag{13}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{14}$$

On the right hand side, f is chosen so that equation (9) has an analytic solution of the form

$$u(x,y) = \psi(x)\psi(y) \tag{15}$$

with $\psi(t) = 1 + e^{-1/\mathrm{E}} - e^{t/\mathrm{E}} - e^{(t-1)/\mathrm{E}}$, here (x,y) denotes the Cartesian coordinates of $\mathbf{x} \in \mathbb{R}^2$, and the parameter ε determines the size of the boundary layers near the edges of the domain Ω. We use a value of $\mathrm{E} = 0.1$. For this model problem the linearization $\Lambda_{u_{k-1}}$ of L is given by

$$\Lambda_{u_{k-1}} v = -\mathrm{E}^2 \nabla^2 v + \left(3u_{k-1}^2 - 1\right) v \tag{16}$$

and therefore the equation to be solved in step 2a) of the algorithm is of the form;

$$-\mathrm{E}^2 \nabla^2 v + \left(3u_{k-1}^2 - 1\right) v = f + \mathrm{E}^2 \nabla^2 u_{k-1} + u_{k-1} - u_{k-1}^3 \tag{17}$$

when perform step (2a), it is necessary to solve the following linear system, arising from the nonsymmetic collocation procedure(1), (2), (3),(6), (7) and (9):

$$\sum_{j=1}^{n^{(k)}} c_j^{(k)} \Lambda_{u_{k-1}} \left[\varphi\left(\left\| \mathbf{x} - \mathbf{x}_j^{(k)} \right\| \right) \right]_{x = x_j^{(k)}} = f + \Lambda u_{k-1}\left(\mathbf{x}_i^{(k)}\right),\ i = 1,\ldots,n_I^{(k)} \tag{18}$$

$$\sum_{j=1}^{n} c_j^{(k)} \left[\varphi\left(\left\| \mathbf{x}_i^{(k)} - \mathbf{x}_j^{(k)} \right\| \right) \right] = 0\ ,\ i = n_I^{(k)} + 1,\ldots,n^{(k)} \tag{19}$$

By using (13),(14) and(15), the above then becomes;

$$\sum_{j=1}^{n} c_j \left[-\mathrm{E}^2 \left[\nabla^2 \varphi\right]_{ij} + \left(3u_i^2 - 1\right)\varphi_{ij} \right] = f + \mathrm{E}^2 \left[\tilde{\nabla}^2 u\right]_i + u_i - u_i^3\ \ , i = 1,\ldots,n_I \tag{20}$$

$$\sum_{j=1}^{n} c_j \varphi_{ij} = 0,\ , i = n_I + 1,\ldots,n_I + n_B = n \tag{21}$$

where, for transparency, the index k's are being omitted on the quantities n, n_I, n_B, and c_j. The Newton update $\tilde{v} = v$ used in step (2c) of the algorithm is then given by;

$$v(\mathbf{x}) = \sum_{j=1}^{n^{(k)}} c_j^{(k)} \varphi\left(\left\| \mathbf{x} - \mathbf{x}_j^{(k)} \right\| \right) \tag{22}$$

The next section, the whole process of numerical computing is implemented for solving nonlinear partial differential equations and for this, four problems are selected and tested. All solutions obtained from this investigation are validated against with their corresponding exact solutions and also those obtained from literature when possible.

VARIABLE PARAMETER SCHEMES AND THE PROPOSED FORM

It is known that the effectiveness of the RBF-collocation method can well be influenced by the choice of the shape parameter. Many researchers have agreed that using variable shape parameters provides superior in solution quality (See [10] and references herein). The result of allowing the shape parameter to vary locally is that each column of the interpolation matrix, matrix $\mathbf{A}$ in (8), no longer contains constant entries leading to lower condition number. Several strategies have been proposed for providing reliable numerical solution accuracy and they are revisited as shown in Table 2.

The variable abbreviated as **Var-1** is clearly in an exponential manner and was used in Kansa [2] before its further modified version was later invented in Kansa and Carlson [11], noted as **Var-2**. In their work, it was demonstrated that if $\varepsilon_{\min}^2$ and $\varepsilon_{\max}^2$ varied by several orders of magnitude, then a very satisfactory result quality can well be expected. Later, a linear form of variable shape parameter was proposed and applied to both interpolation and some benchmark partial differential equations by Sarra [12] and Sarra and Sturgill [10], as noted in Table 2 by **Var-3** and **Var-4** respectively. Here, the command 'rand' is the MATLAB function that returns uniformly distributed pseudo-random numbers on the unit interval. It was proven in their work that the variable shape outperformed the fixed value of parameter especially when the scheme includes the information about the minimum distance of a center to its nearest neighbor, h_n, with also a user input value μ. In terms or the condition number produced by equation (11), it was also found to be considerable smaller over most of the average shape range.

TABLE 2. Variable shaper parameter schemes used in literature.

Reference {Abbreviation used in this work}	**Formulation of for j^{th}-element**
Kansa [2] { Var-1}	$\varepsilon_j = \left[\varepsilon_{\min}^2\left(\frac{\varepsilon_{\max}^2}{\varepsilon_{\min}^2}\right)^{\frac{j-1}{N-1}}\right]^{\frac{1}{2}}, \ j=1,2,...,N$
Kansa and Carlson [11] { Var-2}	$\varepsilon_j = \varepsilon_{\min} + \left(\frac{\varepsilon_{\max}-\varepsilon_{\min}}{N-1}\right)j, \ j=0,1,2,...,N-1$
Sarra [12] { Var-3}	$\varepsilon_j = \varepsilon_{\min} + \left(\varepsilon_{\max}-\varepsilon_{\min}\right)\times rand(1,N)$
Sarra and Sturgill [10] { Var-4}	$\varepsilon_j = \frac{\mu}{h_n}\left[\varepsilon_{\min} + \left(\varepsilon_{\max}-\varepsilon_{\min}\right)\times rand(1,N)\right]$

In this work, we proposed a new form of variable shape parameter where both linear and exponential manners are taken into consideration, expressed as in equation (22)

$$\varepsilon_j = (1-\zeta)\left[\varepsilon_{\min}^2\left(\frac{\varepsilon_{\max}^2}{\varepsilon_{\min}^2}\right)^{\zeta}\right]^{\frac{1}{2}} + \zeta\left[\varepsilon_{\min} + \left(\varepsilon_{\max}-\varepsilon_{\min}\right)\zeta\right] \tag{23}$$

where $\zeta = \frac{j-1}{N-1}$ and $j=1,2,...,N.$ The automatically self-adjusted parameter ζ is introduced and act as a weight function leading ε_j to equal to the exponential manner when $j=1.$ This weight then sets ε_j to become its linear mode when $j=N$ This proposed variable shape is referred to as **Var-5** throughout the work

NUMERICAL EXPERIMENTS

For error analysis, the two error indicators, over the domain; relative error and Root Mean Square (RMS) error, are adopted in this work and are given respectively by;

$$\text{Relative Error} = \left|\frac{u_i^{exact} - u_i^{num}}{u_i^{exact}}\right| \tag{24}$$

$$\text{RMS} = \sqrt{\frac{\sum_{i=1}^{N}\left(u_i^{exact} - u_i^{num}\right)^2}{N}} \tag{25}$$

All solutions presented in this section were obtained by using 289 uniform collocation nodes within the domain and 64 nodes on its boundary.

Example 1. The nonlinear PDE as given in GE. Fasshauer [13] on a square domain $0 < x < 1, 0 < y < 1$ is taken a look at. The governing equation is as follows;

$$-\varepsilon^2\nabla^2 u - u + u^3 = f \tag{26}$$

with the boundary condition $u = 0$ on $\partial\Omega$ and the right hand side of the equation is chosen from the analytical solution provided as follows;

$$u(x, y) = \psi(x)\psi(y) \tag{27}$$

with

$$\psi(t) = 1 + e^{-1/\varepsilon} - e^{-t/\varepsilon} - e^{(t-1)/\varepsilon} \tag{28}$$

and (x, y) denotes the Cartesian coordinate in $\mathbb{R}^2$.

TABLE 3. Solution quality comparison at different values of $\varepsilon_{\min}$ and $\varepsilon_{\max}$.

Shape Type	$(\varepsilon_{\min}, \varepsilon_{\max}) = (1,10)$			$(\varepsilon_{\min}, \varepsilon_{\max}) = (1,20)$			$(\varepsilon_{\min}, \varepsilon_{\max}) = (0.1,20)$		
	Opt Value	**Relative Error**	**RMS**	**Opt Value**	**Relative Error**	**RMS**	**Opt Value**	**Relative Error**	**RMS**
Var-1	1.26E+00	1.25E-04	9.03E-08	1.31E+00	1.36E-04	9.78E-08	1.24E+00	1.38E-04	9.95E-08
Var-2	1.31E+00	1.48E-04	1.07E-07	1.13E+00	1.90E-04	1.37E-07	1.21E+00	2.72E-04	1.96E-07
Var-3	1.37E+00	1.80E-04	1.30E-07	1.08E+00	1.90E-04	1.37E-07	1.18E+00	2.21E-04	1.59E-07
Var-4	1.64E+00	7.80E-04	5.63E-07	1.64E+00	7.59E-04	5.47E-07	1.22E+00	4.75E-04	3.42E-07
Var-5	1.13E+00	1.50E-04	1.08E-07	1.33E+00	1.49E-04	1.08E-07	1.33E+00	1.49E-04	1.07E-07

Example 2. The following nonlinear equation as given in Linesawat [14, 15] is studied. The governing equation is as follows;

$$\Delta^2 u = 2u^3 \tag{29}$$

defined on a $(1,5)\times(1,5)$ domain with the Dirichlet boundary condition $u = -1/x$ on $\partial\Omega$. The analytical solution of the above problem is as follows;

$$u = -1/x \tag{30}$$

TABLE 4. Solution quality comparison at different values of $\varepsilon_{\min}$ and $\varepsilon_{\max}$.

Shape Type	$(\varepsilon_{\min}, \varepsilon_{\max}) = (1,10)$			$(\varepsilon_{\min}, \varepsilon_{\max}) = (1,20)$			$(\varepsilon_{\min}, \varepsilon_{\max}) = (0.1,20)$		
	Opt Value	**Relative Error**	**RMS**	**Opt Value**	**Relative Error**	**RMS**	**Opt Value**	**Relative Error**	**RMS**
Var-1	1.15E+00	1.75E-03	3.07E-03	1.16E+00	1.73E-03	3.05E-03	1.64E-01	1.17E-06	2.05E-06
Var-2	1.16E+00	1.73E-03	3.05E-03	1.13E+00	3.86E-03	6.79E-03	1.69E-01	1.20E-06	2.11E-06
Var-3	1.17E+00	2.61E-03	4.60E-03	1.17E+00	2.83E-03	4.99E-03	1.03E-01	2.40E-05	4.22E-05
Var-4	2.78E-01	1.69E-04	2.97E-04	2.71E-01	1.36E-04	2.39E-04	1.60E-01	1.22E-06	2.15E-06
Var-5	1.16E+00	1.73E-03	3.05E-03	1.16E+00	1.83E-03	3.22E-03	1.58E-01	2.89E-06	5.08E-06

Example 3. The nonlinear PDE of the form shown below is studied; [16]

$$-\Delta^2 u = u(1-u) + q(x, y) \text{ in } (0,1)\times(0,1) \tag{31}$$

with boundary condition $u = 0$ on $\partial\Omega$ and with $q(x, y)$ given by;

$$q(x, y) = \left(2\pi^2 - 1 + \sin(\pi x)\sin(\pi y)\right)\sin(\pi x)\sin(\pi y) \tag{32}$$

The analytical solution of this particular problem is as follows;

$$u = \sin(\pi x)\sin(\pi y) \tag{33}$$

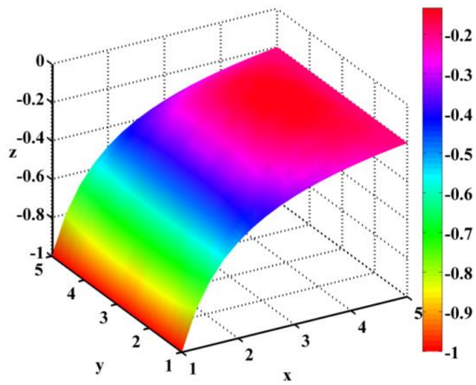

FIGURE 1. Solution profile obtained from the proposed variable for Example 2.

TABLE 5. Solution quality comparison at different values of $\varepsilon_{\min}$ and $\varepsilon_{\max}$.

Shape Type	$(\varepsilon_{\min}, \varepsilon_{\max}) = (1,10)$			$(\varepsilon_{\min}, \varepsilon_{\max}) = (1,20)$			$(\varepsilon_{\min}, \varepsilon_{\max}) = (0.1,20)$		
	Opt Value	**Relative Error**	**RMS**	**Opt Value**	**Relative Error**	**RMS**	**Opt Value**	**Relative Error**	**RMS**
Var-1	1.00E+00	2.50E-01	1.74E-04	1.00E+00	2.50E-01	1.74E-04	2.71E-01	8.96E-04	6.22E-07
Var-2	1.02E+00	2.69E-01	1.87E-04	1.05E+00	2.90E-01	2.02E-04	2.99E-01	2.08E-03	1.45E-06
Var-3	1.04E+00	2.81E-01	1.95E-04	1.05E+00	2.91E-01	2.02E-04	2.89E-01	1.14E-03	7.90E-07
Var-4	1.96E+00	1.11E+00	7.73E-04	2.02E+00	1.17E+00	8.12E-04	2.77E-01	1.41E-03	9.79E-07
Var-5	1.00E+00	2.50E-01	1.74E-04	1.00E+00	2.50E-01	1.74E-04	2.67E-01	1.26E-03	8.79E-07

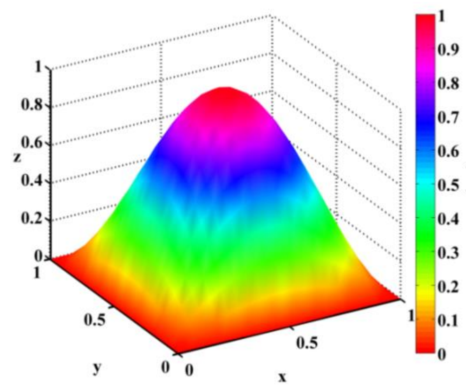

FIGURE 2. Solution profile obtained from the proposed variable for Example 3.

CONCLUSION

There are three main objectives of this numerical study. Firstly, we have successfully applied a type of meshless known as 'Kansa's or RBF-collocation' method to the convective-diffusive type of problem. Secondly, different types/forms of variable multiquadric shape parameters proposed and widely used in literature were gathered and tested out with the same type of PDEs. Lastly, a new form of variable shape parameter containing both linear and exponential manners is proposed and implemented to the problem. Several useful findings obtained from this work are as follows. It has been found that our form of parameter provide more convenient ways to specify the possibly-optimal choice of Inverse-multiquadric shape parameter. Further investigation would be on more complicated problems.

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