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Reg No.:

Name:

APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

Fourth Semester B.Tech Degree Examination June 2022 (2019 scheme)

Course Code: MAT206

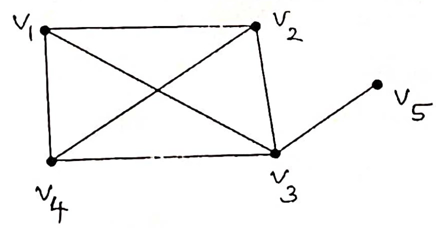
Course Name: GRAPH THEORY

Max. Marks: 100 Duration: 3 Hours

PART A

(Answer all questions; each question carries 3 marks) Marks

1	Prove that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.	3
2	Define walk, path and circuit with examples.	3
3	Draw a graph which is Eulerian but not Hamiltonian	3
4	Distinguish between strongly connected digraphs and weakly connected graphs with examples.	3
5	Prove that there is one and only one path between every pair of vertices in a tree.	3
6	Draw all unlabelled trees with 5 vertices.	3
7	Prove that the edge connectivity of a graph cannot exceed the degree of the vertex with the smallest degree in G.	3
8	Define planar graph and non-planar graph with examples.	3
9	Write the adjacency matrix for the following graph.	3



10	Prove that the chromatic polynomial of a complete graph with 4 vertices is $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$	3

PART B

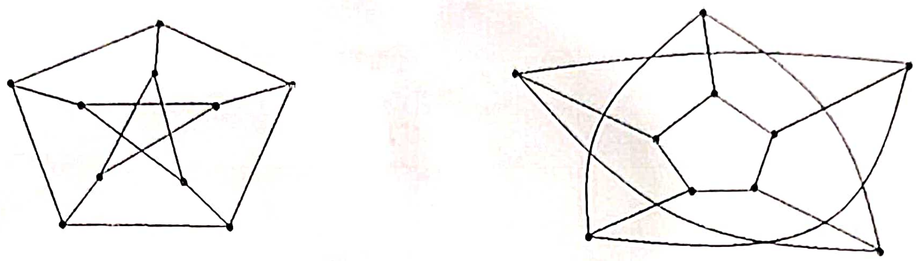
(Answer one full question from each module, each question carries 14 marks)

Module -1

11	a)	Prove that the number of vertices of odd degree in a graph is always even	7

b) If a connected graph G is decomposed into two subgraphs g_1 and g_2 then prove that there must be at least one vertex common between g_1 and g_2 7

12 a) Determine whether the following graphs are isomorphic or not. 7



b) If a graph has exactly two vertices of odd degree, then prove that there must be a path joining these two vertices. 7

Module -2

13 a) In a complete graph with n vertices, prove that there are $\frac{n-1}{2}$ edge-disjoint Hamiltonian circuits, if n is an odd number ≥ 3. 7

b) 1) For a binary relation "is greater than" on the set X= {3,4,7,5,8} 7

i) Draw the digraph representing the above relation

ii) Write its relation matrix

2) Define equivalence digraph with an example

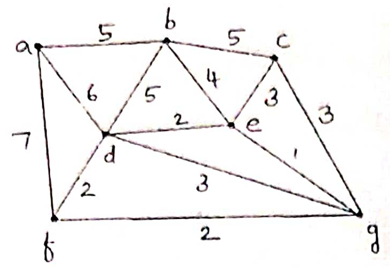
14 a) Prove that a connected graph G is an Euler graph if and only if all vertices of G are of even degree. 7

b) Define Hamiltonian circuit and Hamiltonian path. Give an example for each. Also draw a graph that has a Hamiltonian path but not a Hamiltonian circuit. 7

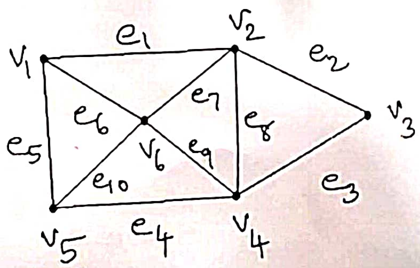
Module -3

15 a) Prove that every tree has either one or two centers 7

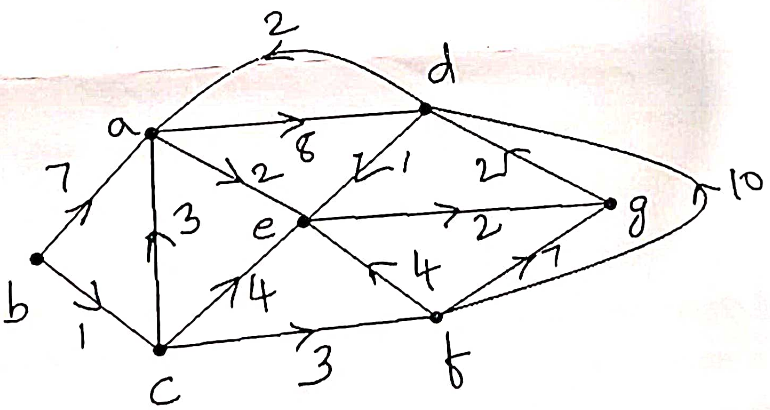
b) Apply Kruskal's algorithm to find the minimal spanning tree for the following weighted graph. 7



16 a) For any spanning tree of a connected graph with n vertices and e edges, prove that there are n-1 tree branches and e-n+1 chords. For the following graph find two spanning trees and hence show that an edge that is a branch of one spanning tree can be a chord with respect to another spanning tree of same graph. 7

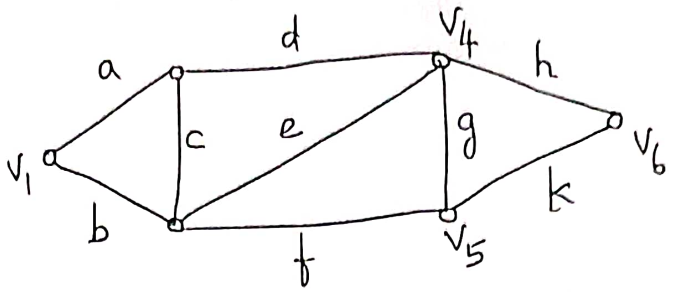


b) Use Dijkstra's algorithm to find the shortest path for the following weighted digraph and find the shortest distance from vertex a to other vertices. 7



Module -4

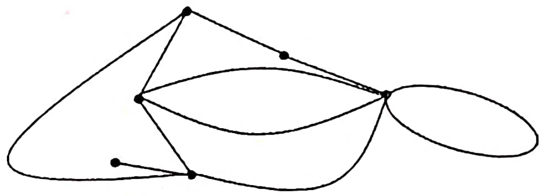
17 a) Illustrate the statement: "The ring sum of any two cut-sets in a graph is either a third cut-set or an edge disjoint union of cut-sets", in the following graph. 7



b) Define edge connectivity, vertex connectivity separable and non-separable graph. Give an example for each. 7

18 a) Prove that the complete graph on 5 vertices is non-planar 7

b) Draw the geometric dual of the following graph 7



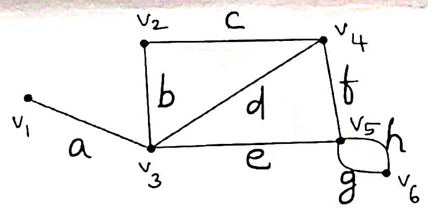
Module -5

19 a) For the following graph find the 7

i. Incidence matrix

ii. Path matrix between v_2 and v_5

iii. Circuit matrix



b) Draw a connected graph and show that the rank of its incidence matrix is one less than the number of vertices. 7

20 a) Prove that every tree with two or more vertices is 2-chromatic 7

b) Prove that a covering g of a graph is minimal if and only if g contains no path of length three or more. 7

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Total Pages: 10

APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

Scheme for Valuation/Answer Key

Scheme of evaluation (marks in brackets) and answers of problems key

FOURTH SEMESTER B.TECH DEGREE(R,S) EXAMINATION JUNE 2022(2019 Scheme)

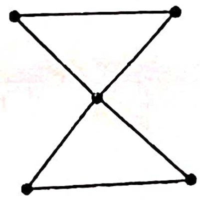
Course Code: MAT206

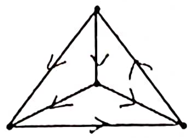
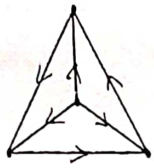
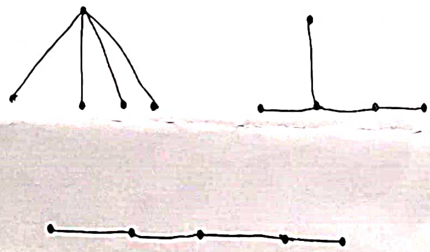
Course Name: GRAPH THEORY

Max. Marks: 100		Duration: 3 Hours

PART A

(Answer all questions; each question carries 3 marks)

		Marks
1	The maximum degree of any vertex in a simple graph with n vertices is n−1.	1
	The total maximum degree of all n vertices is n(n − 1) That is, $\sum_{i=1}^{n} d(v_i)= 2e \le n(n-1) \Rightarrow e \le \frac{n(n-1)}{2}$	2
	or	OR
	The maximum number of edges incident to a vertex is n − 1.	1
	Thus, maximum number of edges incident to all n vertices is n (n − 1). But each edge contributes a degree 2 to each vertex. ie. each edge is counted twice, so n (n − 1) divided by 2. So the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$	2
2	A walk is defined as a finite alternating sequence of vertices and edges, beginning, and ending with vertices, such that each edge is incident with the vertices preceding and following it. An open walk in which no vertex appears more than once is called a path. A closed walk in which no vertex (except the initial and the final vertex) appears more than once is called a circuit.	2
	Draw a graph and mention these three.	1
3	 (Any other correct answer)	3
4	A digraph G is said to be strongly connected if there is at least one directed path from every vertex to every other vertex.	1.5

		 (any example) A digraph G is said to be weakly connected if its corresponding undirected graph is connected but G is not strongly connected.  (any example)	1.5
5		Since T is a connected graph, there must exist at least one path between every pair of vertices in T. Now suppose that between two vertices a and b of T there are two distinct paths. The union of these two paths will contain a circuit and T cannot be a tree.	1 2
6			3x1=3
7		Let the vertex v_i be the vertex with the smallest degree in G. Let $d(v_i)$ be the degree of v_i. Vertex v_i can be separated from G by removing the $d(v_i)$ edges incident on v_i.	3
8		A graph is planar if there exists some geometric representation of graph which can be drawn on a plane such that no two of its edges intersect. A graph that cannot be drawn on a plane without cross over between its edges is called non-planar Example for each	1 1 1
9		$\begin{array}{c c} & \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$	3

10		There are 4 vertices and edges between every pair of vertices. So first vertex is coloured with λ colors, second with λ-1 colors, third with λ-2 colors and fourth with λ-3 colors. Hence the chromatic polynomial of a complete graph with 4 vertices is $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$	3

PART B

(Answer one full question from each module, each question carries 14 marks)

Module -1

11	a)	Consider the vertices with odd degree and even degree separately. Let the number of vertices be n namely $v_1, v_2, \ldots v_n$ and e edges $\sum_{i=1}^{n} d(v_i) = \sum_{even} d(v_j) + \sum_{odd} d(v_k)$ → (1) where . $\sum_{even} d(v_j)$the sum of even degree vertices and $\sum_{odd} d(v_k)$ the sum of odd degree vertices. Since$\sum_{i=1}^{n} d(v_i)$= 2e, an even number and , $\sum_{even} d(v_j)$the sum of even degree vertices is an even number.	3
		from (1). $\sum_{odd} d(v_k) = \sum_{i=1}^{n} d(v_i) - \sum_{even} d(v_j)$ an even number Each $d(v_k)$is an odd number and for the sum to be even, the total number of odd vertices should be an even number.	2
			2
	b)	Let V be the vertex set of G such that V be partitioned into V_1 the vertex set of g_1 and V_2 the vertex set of g_2.	2
		To prove $V_1 \cap V_2 \neq \varphi$.	
		Suppose that $V1 \cap V2 = \varphi$.	2
		That is there is no edge from g_1 to g_2. Thus g_1 and g_2 are two components of G . This shows that G is disconnected. which is a contradiction.	
		Hence $V_1 \cap V_2 \neq \varphi$.	3
		That is there must be at least one vertex common between g_1 and g_2.	
12	a)	It is isomorphic.	2
		Justify by one-to-one correspondence	5
	b)	Let G be a graph with all even vertices except vertices v_1 and v_2, which are odd. For a connected graph, there is a path between every pair of vertices.	2
		By theorem, the number of vertices of odd degree in a graph (true for component also) is always even.	2
		Therefore, for every component of a disconnected graph, no graph can have an odd number of odd vertices. Therefore, in graph G, v_1 and v_2 must belong to the same component, and hence must have a path between them.	3

Module -2

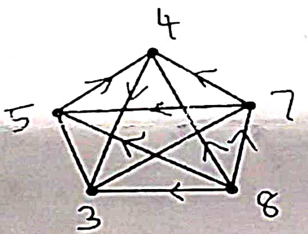
13 a) A complete graph G of n vertices has $\frac{n(n-1)}{2}$ edges, and a Hamiltonian circuit in G consists of n edges. 2

Therefore, the number of edge-disjoint Hamiltonian circuits in G cannot exceed $\frac{n(n-1)}{2}$. 1

That there are $\frac{(n-1)}{2}$ edge disjoint Hamiltonian circuits, when n is odd. Keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise by $\frac{360}{n-1}$, $\frac{2.360}{n-1}$, . . . $\frac{n-3}{2}\frac{360}{n-1}$degrees. 2

Each rotation produces a Hamiltonian circuit that has no edge in common with any of the previous ones. Thus we have $\frac{n-3}{2}$ new Hamiltonian circuits, all edge disjoint and also edge disjoint among themselves. 2

b) 1. (i) 2

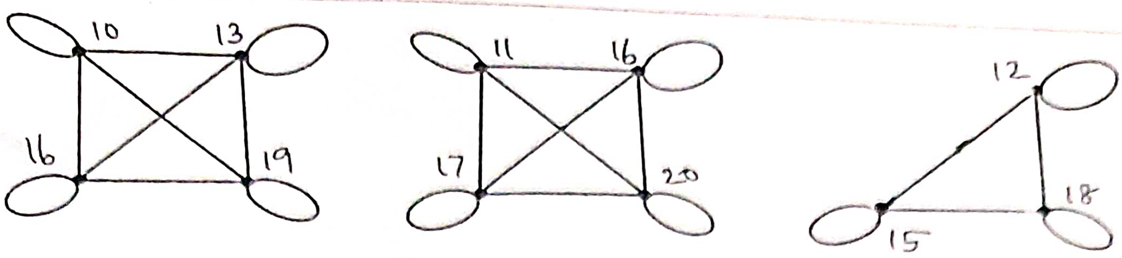


(ii) 2

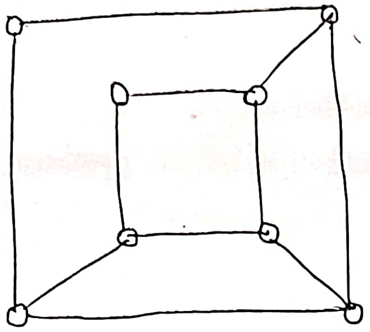
$$\begin{array}{c|ccccc} & 3 & 4 & 7 & 5 & 8 \\ \hline 3 & & & & & \\ 4 & & & & & \\ 7 & & & & & \\ 5 & & & & & \\ 8 & & & & & \end{array}$$

$$\begin{array}{cc} & \begin{matrix} 3 & 4 & 7 & 5 & 8 \end{matrix} \\ \begin{matrix} 3 \\ 4 \\ 7 \\ 5 \\ 8 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

2) The graph representing an equivalence relation is called an equivalence graph. For example, consider the binary relation " is congruent to modulo 3" defined on the set {10, 11, 12, ... 20} 3

			
14	a)	Suppose that G is an Euler graph. That is it contains an Euler line. Since Euler line is a closed walk, while tracing the walk, it is observed that every time the walk entered v and with the other exited. This is true not only for intermediate vertices, but also for terminal vertex, because we exited and entered the same vertex at the beginning and end of the walk respectively. This shows that the degree of all vertices are even.	3
		Conversely assume that all vertices of G are of even degree. Now we construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than once. We continue tracing as far as possible. Since every vertex is of even degree, we can exit from every vertex we enter; the tracing cannot stop at any vertex but v. And since v is also of even degree, we shall eventually reach v when the tracing comes to an end. If this closed walk h we just traced includes all the edges of G, G is an Euler graph. If not, we remove from G all the edges in h and obtain a subgraph h_1 of G formed by the remaining edges. Since both G and h have all their vertices of even degree, the degrees of the vertex of h_1 are also even. Moreover, h_1 must touch h at least at one vertex a, because G is connected. Starting from a, we can again construct a new walk in graph h_1. Since all the vertices of h_1 are of even degree, this walk in h_1 must terminate at vertex a; but this walk in h_1 can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h. This process can be repeated until we obtain a closed walk that traverses all the edges of G. Thus G is an Euler graph	4
	b)	A Hamiltonian Circuit in a connected graph is defined as a closed walk that traverses every vertex exactly once, except the terminal vertices. Example	2
		If an edge is removed from a Hamiltonian circuit, then a path is obtained. This path is called Hamiltonian path Example(any graph)	2

2



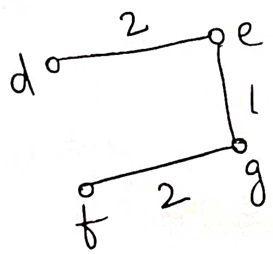
Hamiltonian path no Hamiltonian circuit- justification 1

Module -3

15 a) The maximum distance max $d(v, v_i)$ from a given vertex v to any other vertex vi occurs only when v_i is a pendant vertex. Let T be tree with more than two vertices. T must have at least two pendant vertices. Delete all the pendant vertices from T. The resulting graph say T_1 is still a tree. Removal of pendant vertices from T reduces the eccentricity of all vertices uniformly by one. Therefore all vertices that T had as centers will still remain centers in T_1 . 2 3

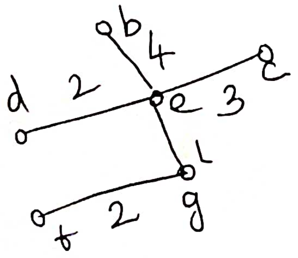
From T_1 again remove all pendant vertices and get another tree T_2. Continue this process until there is left either a vertex(center of T) or an edge (whose end vertices are the centers of T). Hence the theorem 2

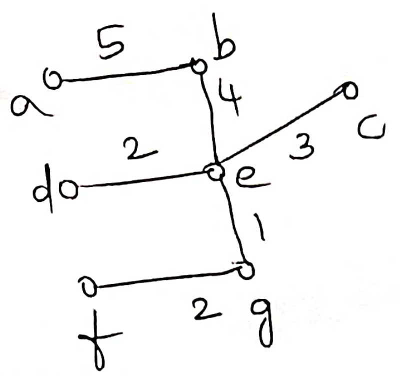
b) Writing the algorithm 2



1

1

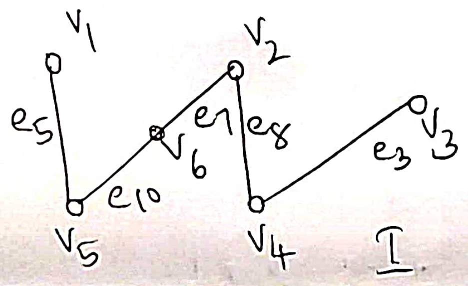


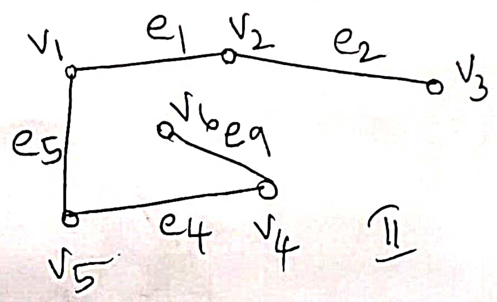


3

(Give full marks for drawing the spanning tree directly)

16 a) The graph has n vertices and e edges. Any spanning tree of the graph will contain n vertices and n - 1 edges ⇒ n − 1 branches. The remaining edges of the graph are the chords. i.e e − (n − 1) = e − n + 1 3





2

For spanning tree I. branch=$\{e_5, e_7, e_8, e_3, e_{10}\}$, chord= $\{e_1, e_2, e_4, e_6, e_9\}$

For spanning tree II. branch= $\{e_1, e_2, e_5, e_4, e_9\}$, chord=$\{e_3, e_6, e_7, e_8, e_{10}\}$ 2

e_3 is in branch of spanning tree I and is in chord of spanning tree II

b) Writing the algorithm 2

Label a as 0 and other vertices as ∞

Unvisited neighbours of a are d and e

d(d) = d(a) + wt(a, d) = 0 + 8 = 8 < ∞

d(e) = d(a) + wt(a, e) = 0 + 2 = 2 < ∞

Unvisited neighbour of e is g

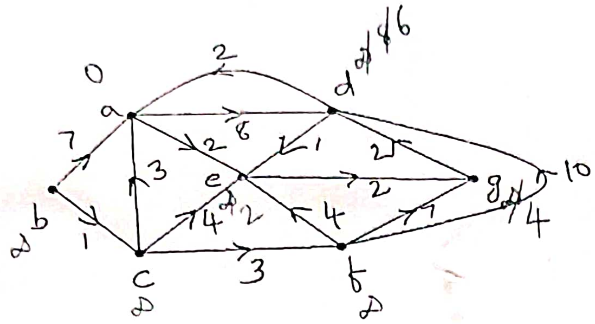
d(g) = d(e) + wt(e, g) = 2 + 2 = 4 < ∞

Unvisited neighbour of g is d

$d(d) = d(g) + wt(g, d) = 4 + 2 = 6 < 8$ 2

shortest path from a is $a \to e \to g \to d$.

$d(a, e) = 2$, $d(a, g) = 4$, $d(a,d) = 6$ 3



Module -4

17 a) Consider cut-sets $S_1 = \{d, e, f\}$ and $S_2 = \{f, g, h\}$ $S_1 \oplus S_2 = (S_1 \cup S_2) - (S_1 \cap S_2) = \{d, e, g, h\}$, a cut-set 2 2

Also Consider cut-sets $S_1 = \{a, b\}$ and $S_2 = \{b, c, e, f\}$ $S_1 \oplus S_2 = (S_1 \cup S_2) - (S_1 \cap S_2) = \{a, c, e, f\}$, another cut-set 3

Also Consider cut-sets $S_1 = \{d, e, g, h\}$ and $S_2 = \{f, g, k\}$ $S_1 \oplus S_2 = (S_1 \cup S_2) - (S_1 \cap S_2) = \{d, e, f\} \cup \{h, k\}$, an edge-disjoint union of cut-sets

b) The number of edges in the smallest cut set or minimum number of edges whose removal reduces the rank by one is called edge connectivity of the graph.

Example 2

The minimum number of vertices whose removal leaves the graph disconnected is called connectivity of the graph.

Example 2

A connected graph is separable if its vertex connectivity is one. Otherwise, non-separable

Example 3

18 a) By theorem, If G is triangle free and planar then $e \le 3n - 6$ 2

In complete graph of five vertices $e = 10$, $n = 5$ 2

Substituting in equation, we get $10 \le 9$ 2

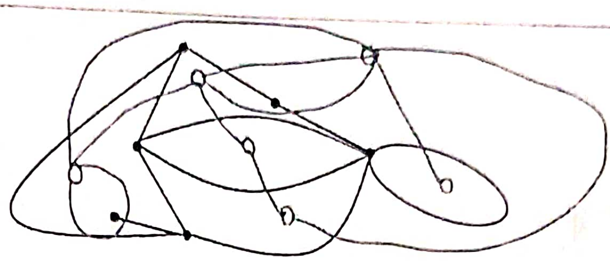
Which is a contradiction

Hence K_5 is non planar 1

310
115
155

OR

Using Jordan Curve theorem

b)  7

Module -5

19 a) i)

$$\begin{array}{c|cccccccc} & a & b & c & d & e & f & g & h \\ \hline v_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ v_3 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ v_6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

2

ii) Paths between v_2 and v_5 are{c, f}, {b, e}, {b, d, f}. Name the paths as 1,2,3 respectively

$$\begin{array}{c|cccccccc} & a & b & c & d & e & f & g & h \\ \hline 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{array}$$

2

		iii) Circuits are {h, g},{c, b, d}, {d, e, f},{c, f, e, b}. Name them as 1,2, 3,4 respectively $$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$	3
	b)	Draw a graph	2
		Write its incidence matrix	3
		Find its rank	2
20	a)	Select a vertex v in tree T. Consider T as rooted tree at v. Paint v with color 1. Paint all adjacent vertices of v with color 2. Then paint adjacent to these with color 1 and continue this. Thus, all vertices with odd distances have color 2 and that of even with 1.	4
		Since there is only one path between every pair of vertices with alternate colors T is properly colored with 2 colors. So, 2-chromatic	3
	b)	Suppose a covering g of a graph is minimal. Assume covering g contains no path of length three. Deleting an edge make its end vertices uncovered. That is g is not minimal.	3
		Conversely a covering g contains no path of length three or more. All its components are star graphs. Here no edge can be removed without a vertex uncovered. So, g is minimal	4

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