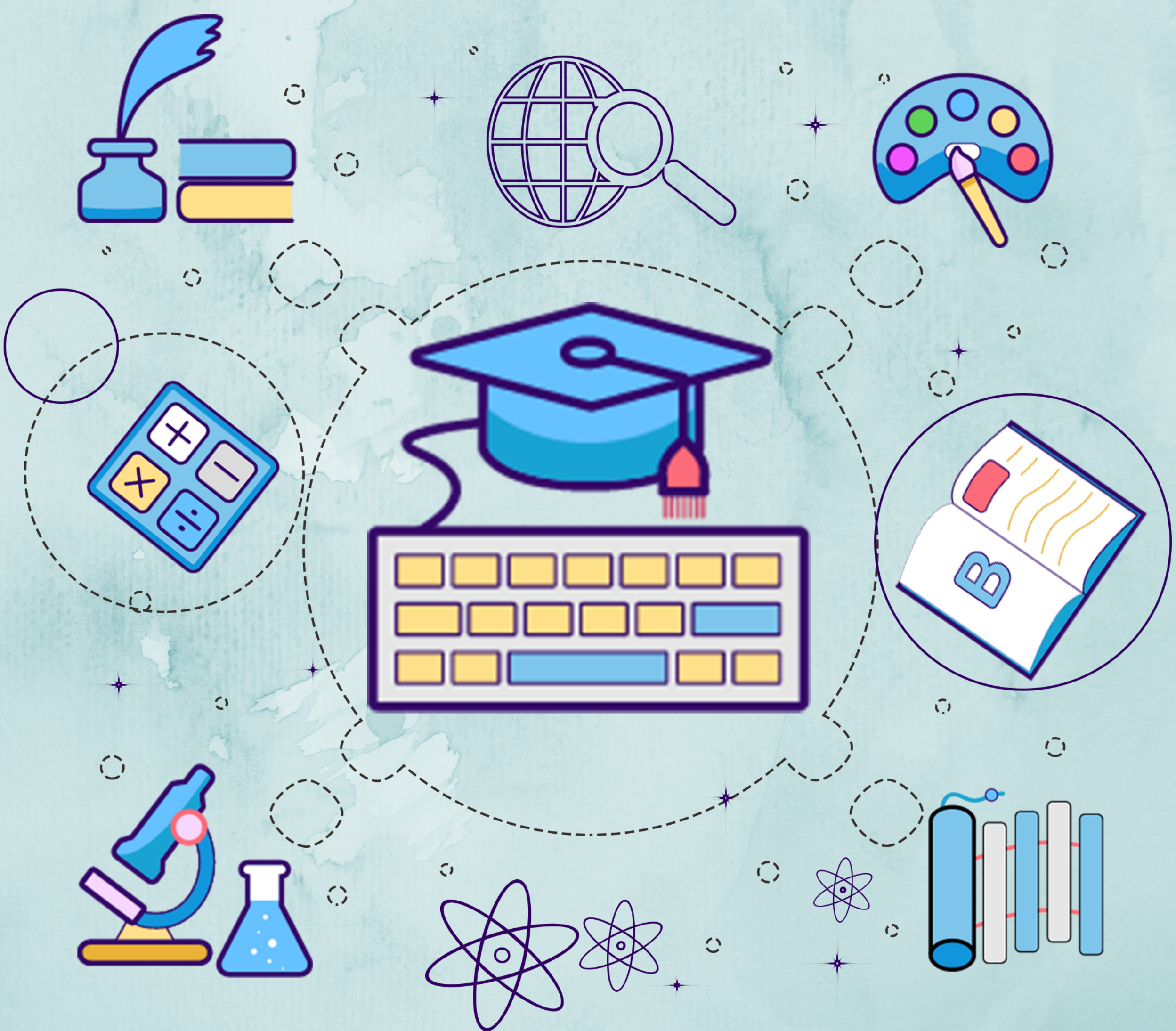


KTU S4 CSE NOTES

GRAPH THEORY (MAT206)

Module 4

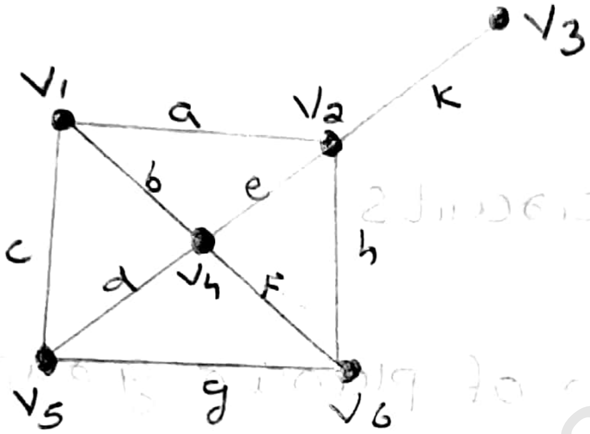
Module - 4

syllabus

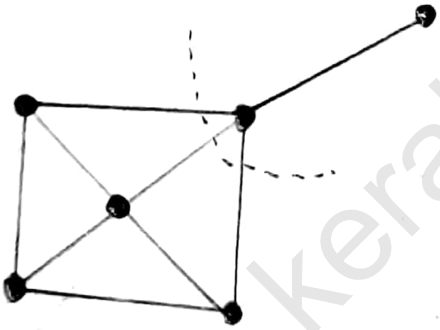
1. Vertex connectivity
2. Edge connectivity
3. Cut set
4. Cut vertices
5. Fundemental circuits
6. planar graph
7. Representation of planar graph
8. Euler's Theorem
9. Geometric dual
10. Combinational dual

Cut set

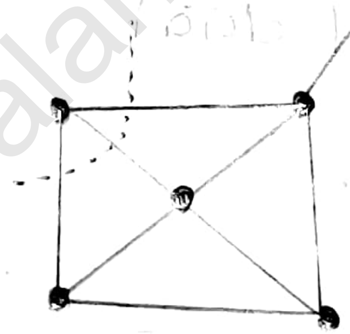
let G be a connected graph, a cutset is a set of edges whose removal from G leaves 'G' disconnected, provided removal of no proper subset of these edges disconnects 'G'



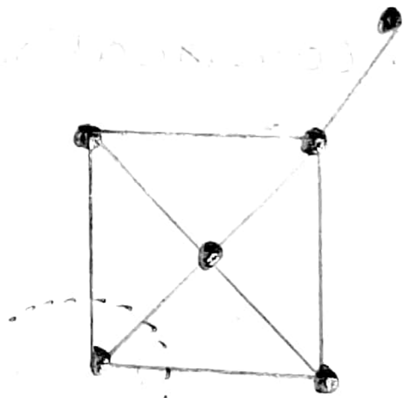
cutsets are:



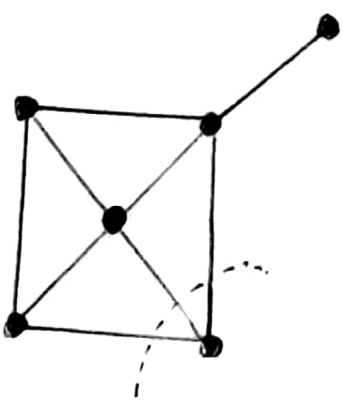
$A = \{a, e, h\}$



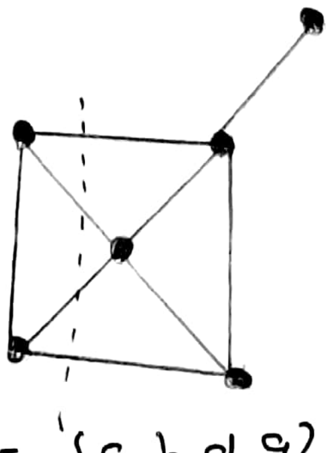
$B = \{a, b, c\}$



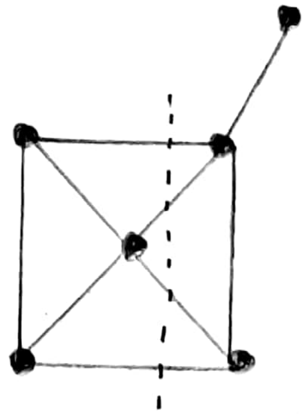
$C = \{c, d, g\}$



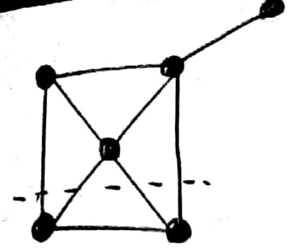
$D = \{g, f, h\}$



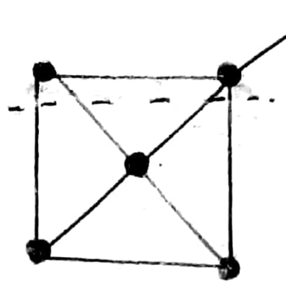
$E = \{a, b, d, g\}$



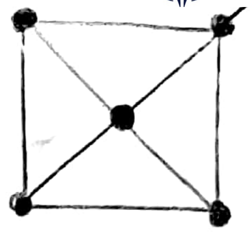
$F = \{a, e, f, g\}$



G = {c, d, f, h}



H = {c, b, e, h}

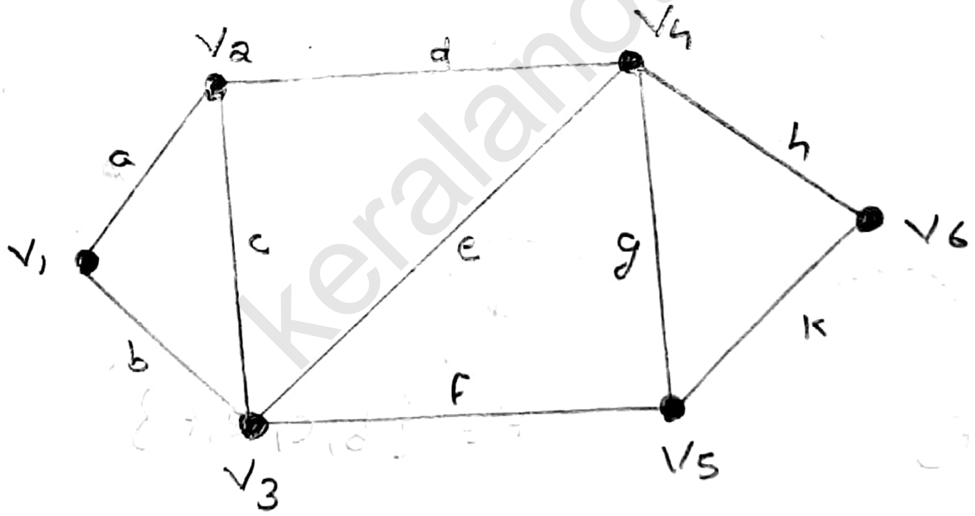


I = {k}

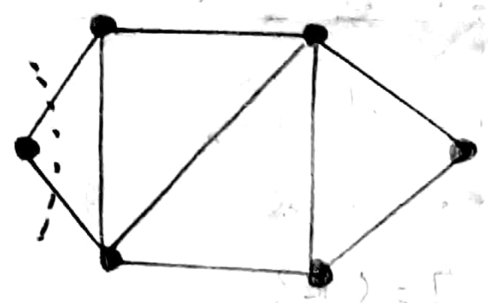
Eg: Here {a, e, h, f} is not a cut set because {a, e, h} is a cut set. It is a subset of {a, e, h, f}

Assignment

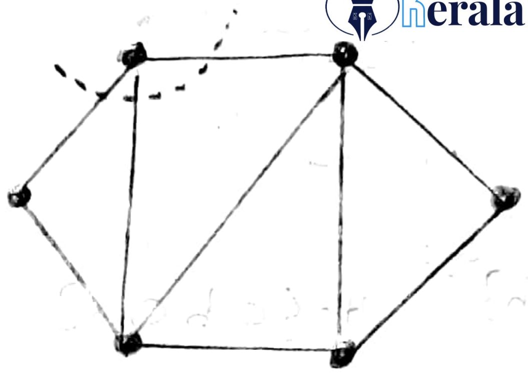
1. Find all cutsets of



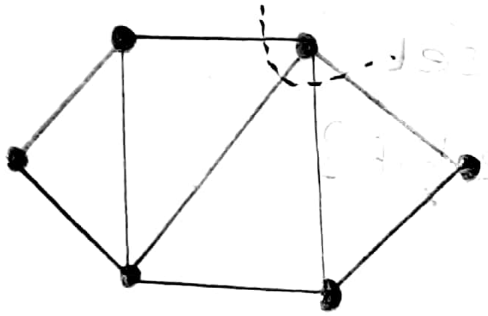
cutsets are:



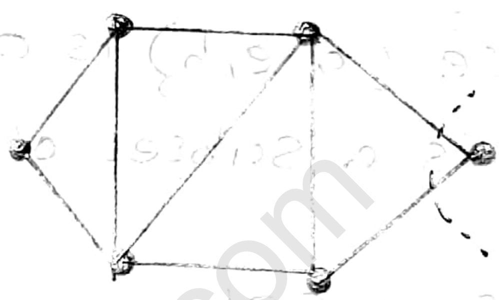
A = {a, b}



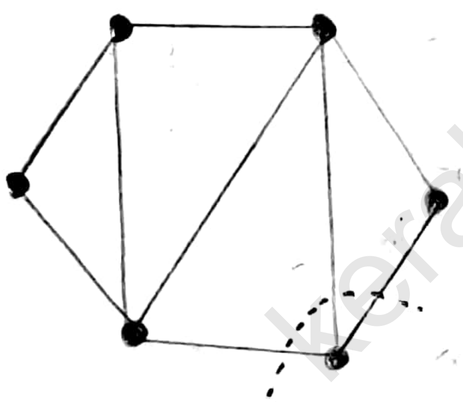
B = {a, c, d}



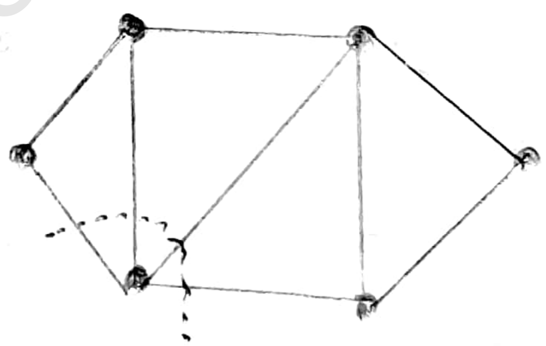
C = {d, e, g, h}



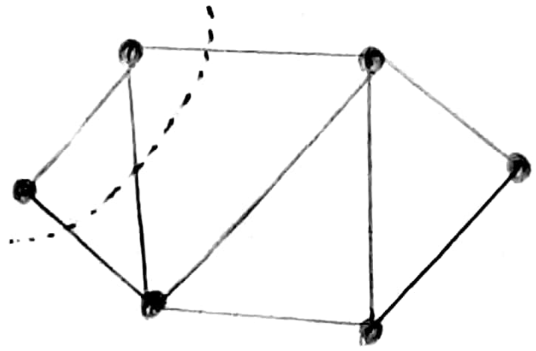
D = {h, k}



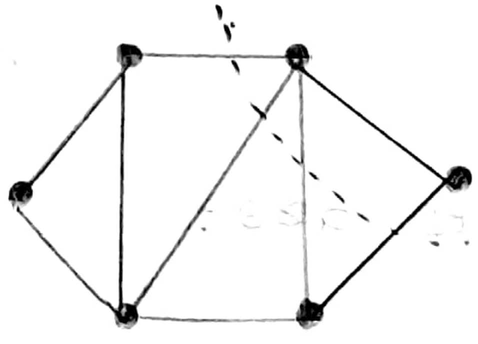
E = {k, g, F}



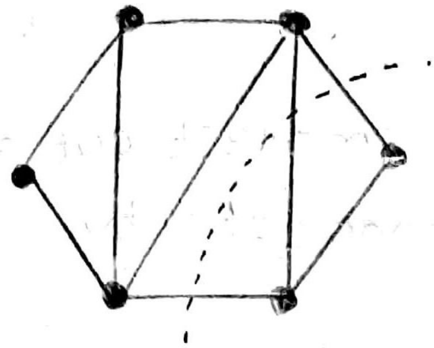
F = {b, c, e, F}



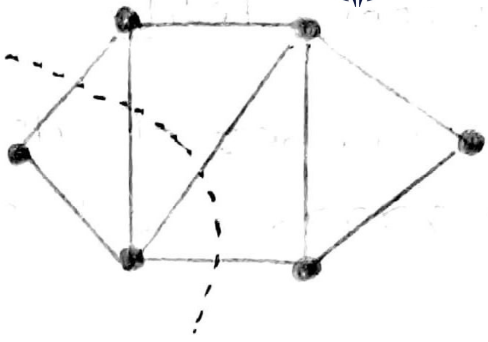
G = {b, c, d}



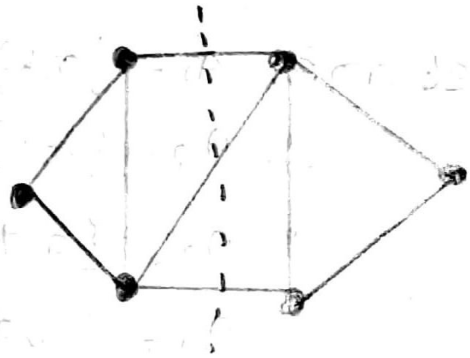
H = {d, e, g, k}



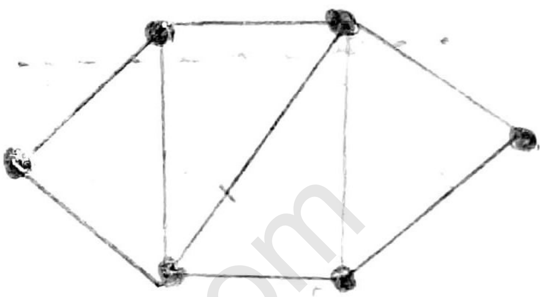
I = {h, g, F}



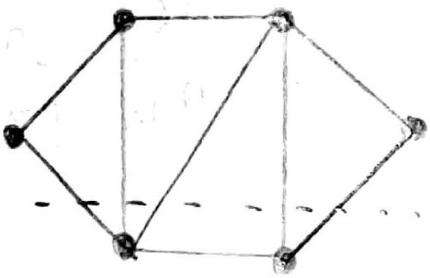
J = {a, c, e, F}



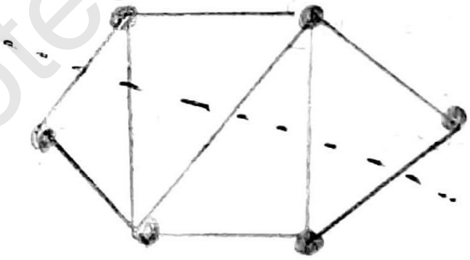
K = {d, e, F}



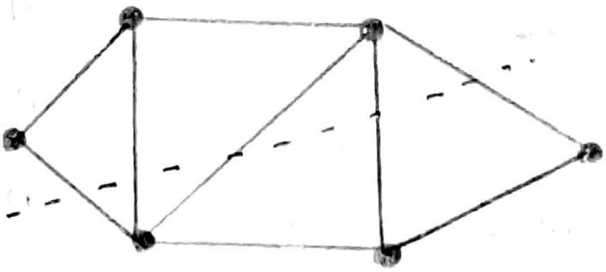
L = {a, c, e, g, h}



M = {b, c, e, g, k}



N = {a, c, e, g, k}



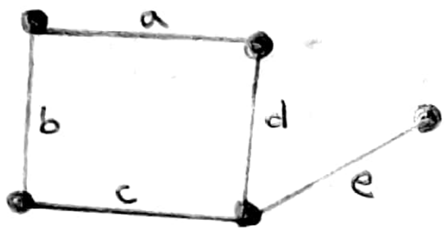
O = {b, c, e, g, h}

Edge connectivity

The number of edges in the smallest cut set of a graph is called edge connectivity

Eg: Find edge connectivity of the given graphs

1.



cut set are: $A_1 = \{a,b\}$

$A_2 = \{b,c\}$

$A_3 = \{a,d\}$

$A_4 = \{c,d\}$

$A_5 = \{b,d\}$

$A_6 = \{a,c\}$

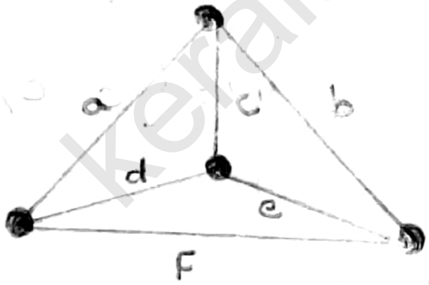
$A_7 = \{e\}$

So smallest cut set is

$A_7 = \{e\}$

$\Rightarrow$ Edge connectivity = 1

2.



cut set are: $A_1 = \{a,c,b\}$

$A_2 = \{F,e,b\}$

$A_3 = \{a,d,F\}$

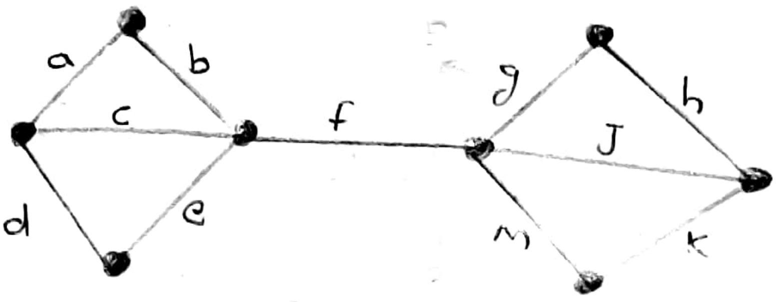
$A_4 = \{d,c,e\}$

So all cutset contains 3 element

$\Rightarrow$ Edge connectivity = 3

Assignment

3.



cut set are:

$A_1 = \{a, c, d\}$

$A_2 = \{a, b\}$

$A_3 = \{b, c, e\}$

$A_4 = \{d, e\}$

~~$A_5 = \{a, b, c, d, e\}$~~

$A_6 = \{g, j, m\}$

$A_7 = \{g, h\}$

$A_8 = \{j, h, k\}$

$A_9 = \{m, k\}$

~~$A_{10} = \{g, h, j, k, m\}$~~

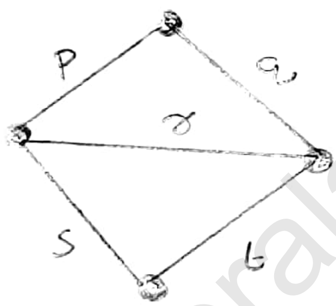
$A_{11} = \{F\}$

So smallest cutset is ~~$A_2 = \{a, b\}$~~, $A_{11} = \{F\}$

~~$A_9 = \{m, k\}$~~

=> Edge connectivity = ~~2~~ 1

4.



cut set are

$A_1 = \{p, r, s\}$

$A_2 = \{p, q\}$

$A_3 = \{q, r, t\}$

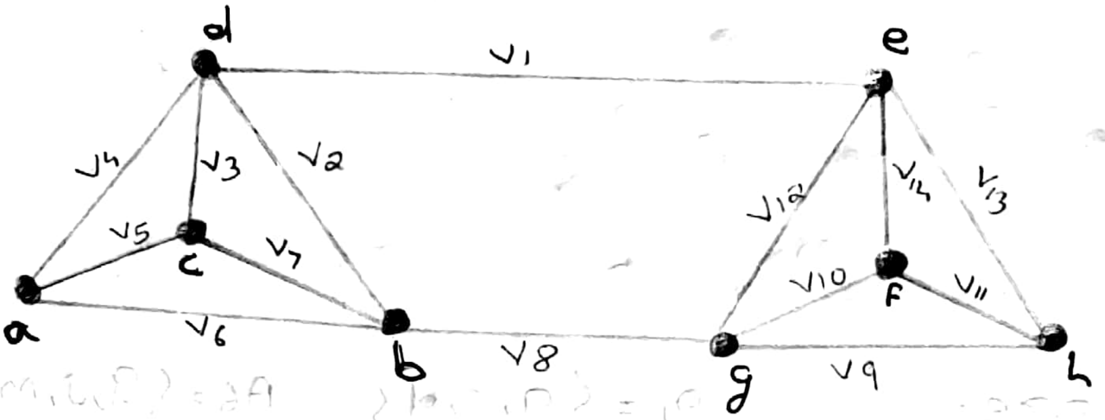
$A_4 = \{s, t\}$

So smallest cut set is $A_2 = \{p, q\}$ / $A_4 = \{s, t\}$

=> Edge connectivity = 2

Assignment

1.



cutsets are: $A_1 = \{V_1, V_5, V_6\}$

$A_2 = \{$ ~~V4, V3, V2~~ V6, V7, V8, V1 $\}$

$A_3 = \{V_4\ V_3, V_2\}$

$A_4 = \{V_1, V_8\}$

$A_5 = \{V_{12}, V_{14}, V_{13}$

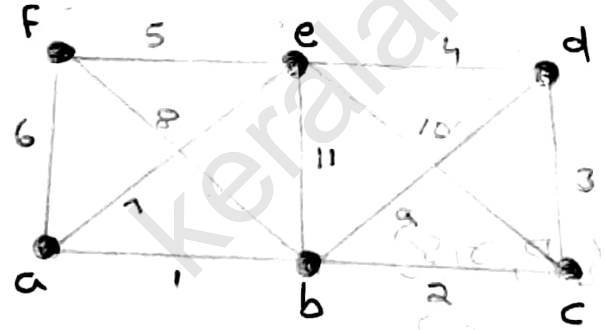
$A_6 = \{V_{12}, V_{10}, V_9$

$A_7 = \{V_9, V_{11}, V_{13}$

so smallest cutset is $A_4 = \{V_1, V_8\}$

$\Rightarrow$ Edge connectivity = 2

2.



cutsets are $A_1 = \{6, 7, 1\}$ $A_2 = \{6, 8, 5\}$

$A_3 = \{4, 10, 3\}$ $A_4 = \{2, 10, 3\}$

$A_5 = \{5, 8, 11, 10, 4\}$ $A_6 = \{1, 8, 11, 9, 2\}$

so smallest cutset is $A_1 = \{6, 7, 1\}$ $A_2 = \{6, 8, 5\}$

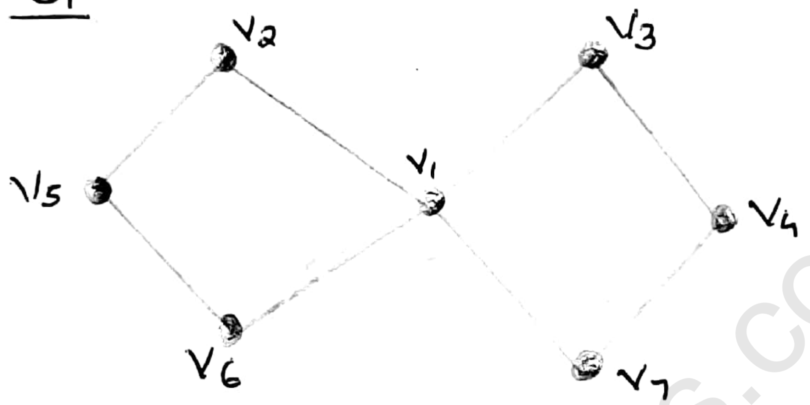
$A_3 = \{4, 10, 3\}$ $A_4 = \{2, 10, 3\}$

$\Rightarrow$ Edge connectivity = 3

Vertex connectivity

The vertex connectivity of a graph G is defined as the minimum number of vertices whose removal from G becomes the graph disconnected

Eg: G



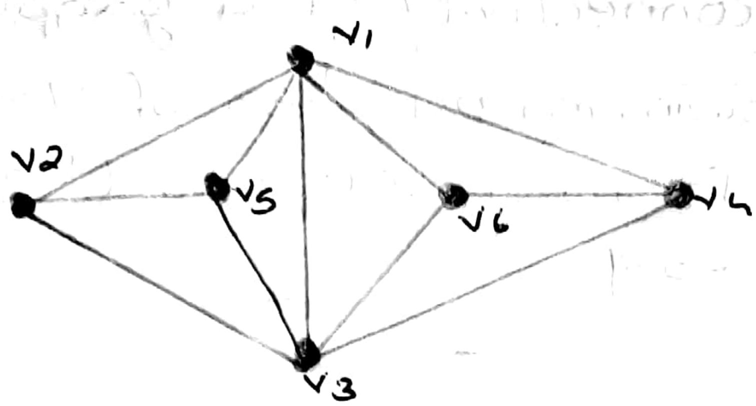
Here minus $G - \{v_1\}$

By removal of v_1, graph became disconnected. so

vertex connectivity = 1

Assignment

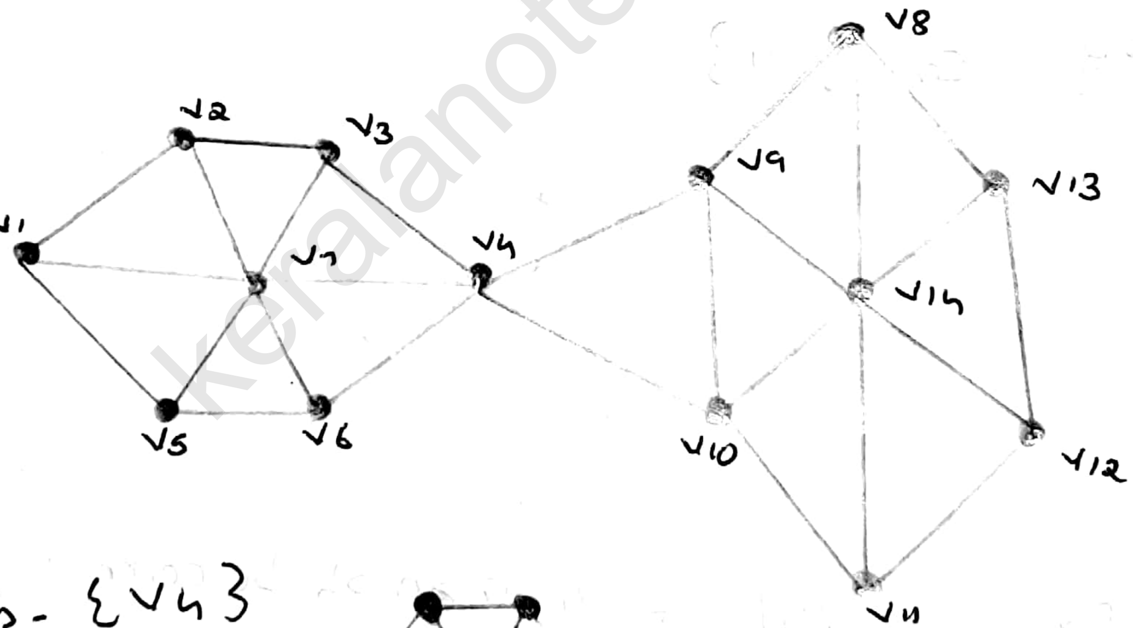
1.



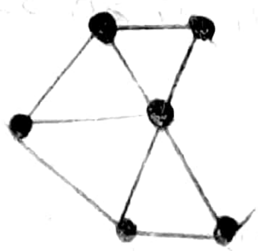
$G - \{v_1, v_3\}$

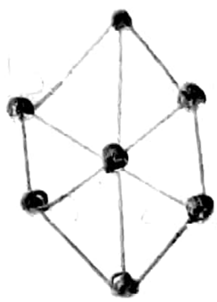
vertex connectivity = 2

2.



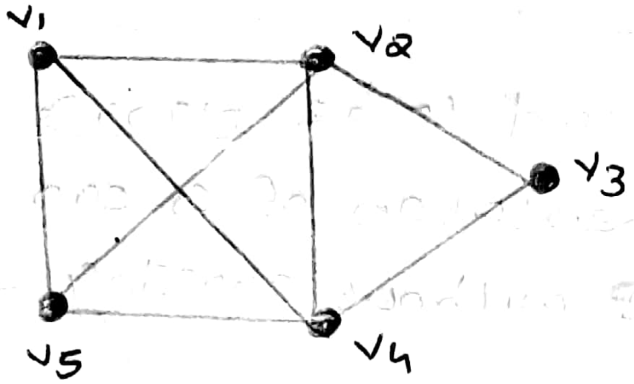
$G - \{v_4\}$



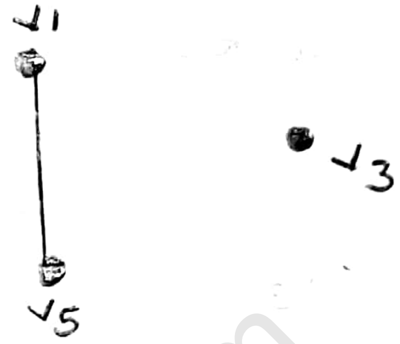


vertex connectivity = 1

3.



G - {v2, v4}



Separable graph

A connected graph is said to be separable if its vertex connectivity is one

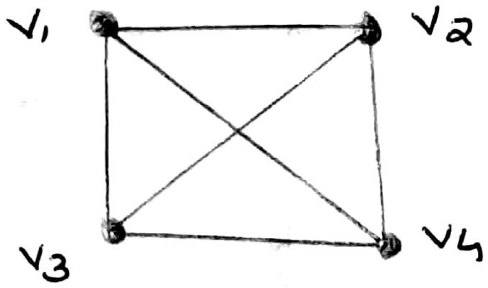
Eg: ①

② Tree (every tree has vertex connectivity one). so every tree is separable

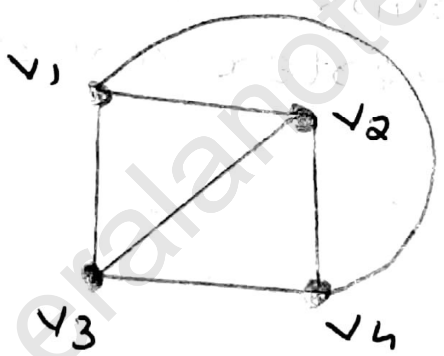
Planar graph

A graph G is said to be planar if the geometric representation of G can be drawn on plane without crossing edges.

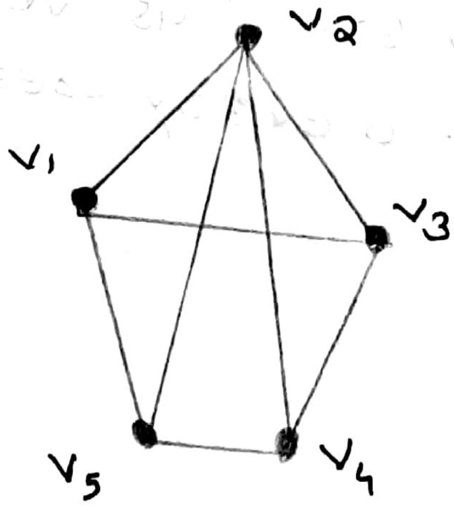
Eg: ①



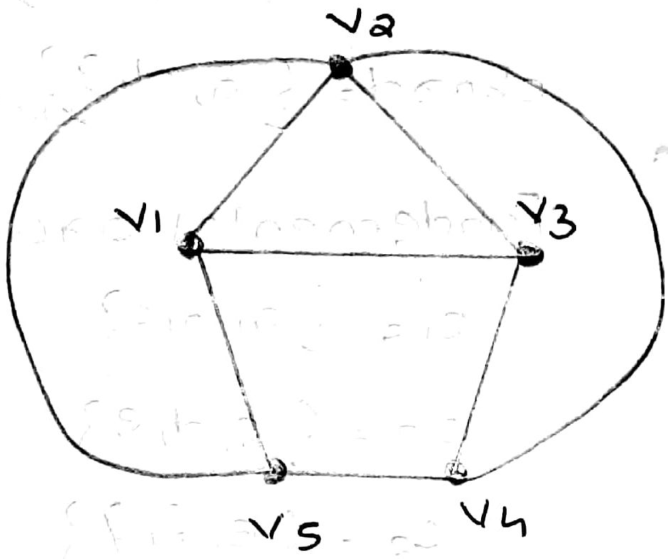
The above graph is planar because it can be drawn without crossing edge



②



The above graph is planar because it can be drawn without crossing edges



Fundemental Circuit

Branch:

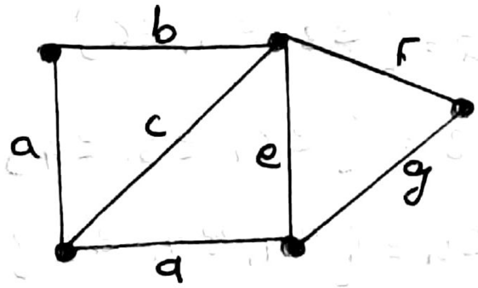
Edges in a spanning tree

chord : Edges in graph which are not in spanning tree

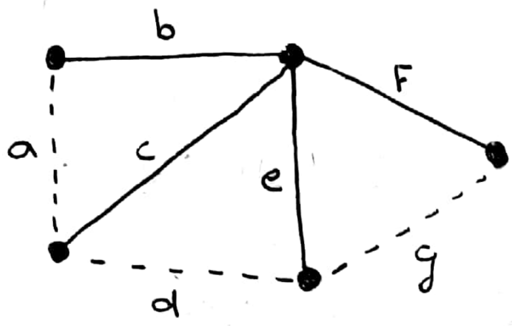
Fundemental circuit : circuit formed by adding one edge from chord to the spanning tree

Q Find Fundemental circuits of the given graph

1.



Spanning tree:



Branch = { b, c, e, f }

chord = { a, d, g }

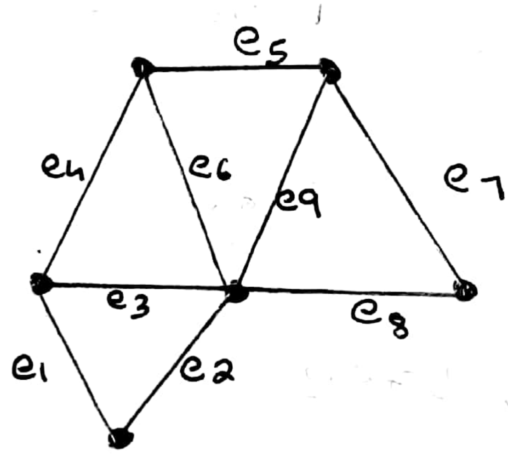
Fundemental circuits:

$c_1 = \{a, b, c\}$

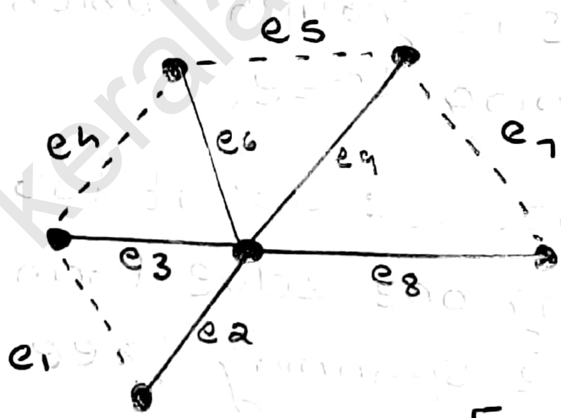
$c_2 = \{c, d, e\}$

$c_3 = \{e, f, g\}$

2.



No. of Fundemental element = no. of chord



Branch = $\{e_2, e_3, e_6, e_9, e_8\}$

chord = $\{e_1, e_4, e_5, e$

Fundemental circuits are:

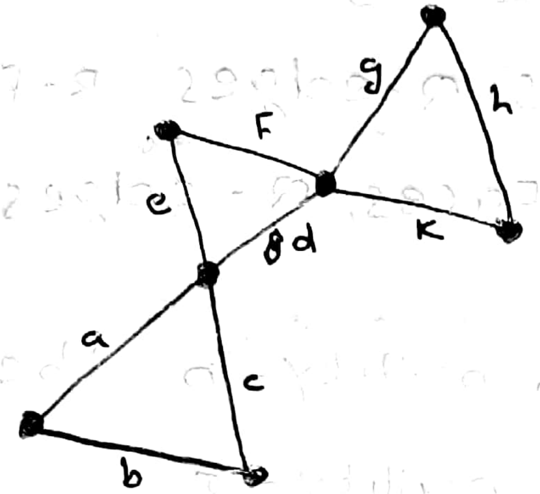
$c_1 = \{e_1, e_2, e_3\}$

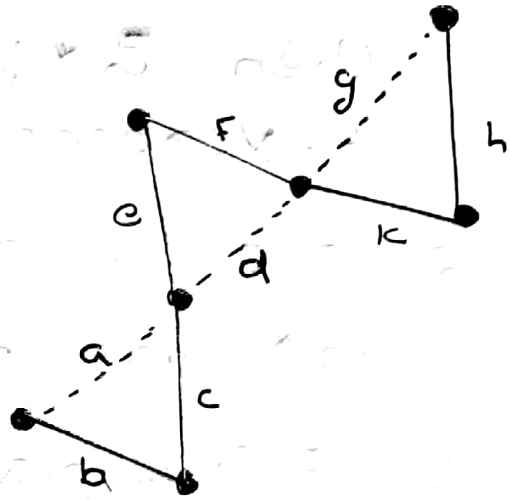
$c_2 = \{e_3, e_4, e_6\}$

$c_3 = \{e_6, e_5, e_9\}$

$c_4 = \{e_9, e_7, e_8\}$

3.





Branch = { b, c, e, F, k, h }

chord = { a, d, g }

Fundemental circuit are = {

c_1 = { a, b, c }

c_2 = { e, F, d }

c_3 = { g, h, k }

Dual of a graph

Let G be a graph its dual is G* is a graph that has vertex for each Face of G The dual graph has an edge when two ~~regio~~ ~~phases~~ Faces of G are seperated From each other by an edge dual has a loop when the same ~~phases~~ Faces appears ~~on both~~ For u side of an edge.

properties of dual (G^*)

1. If G has p-vertices Q-edges R-faces then G^* has p-faces, Q-edges and R-vertices

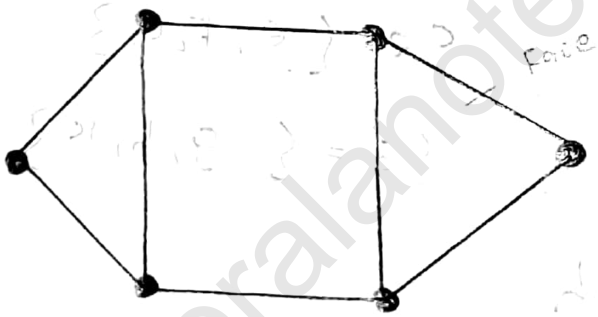
2. If G has rank = r, nullity = n then G^* has rank = n, nullity = r

3. If G is planar then G^* also planar

4. A pendent edge in G is a loop in G^*

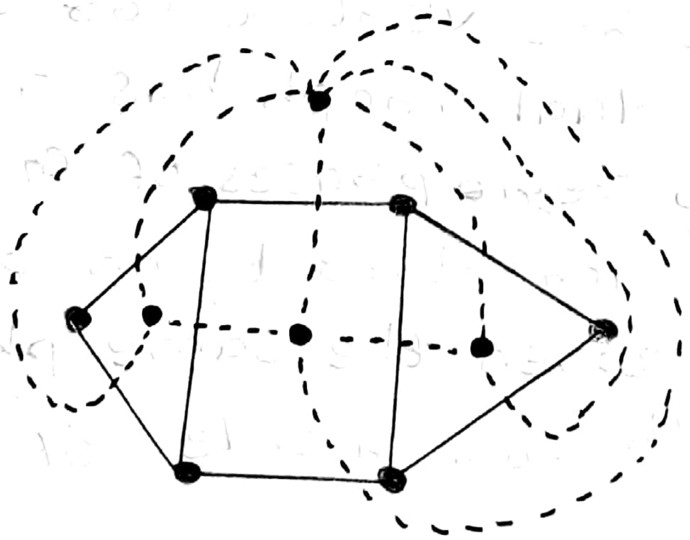
Q Draw dual of the given graphs

i.

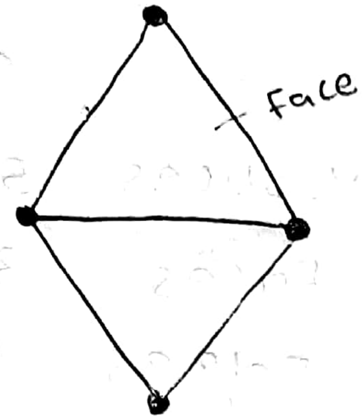


	G	G^*
vertices	6	4
Faces	4	6
Edges	8	8

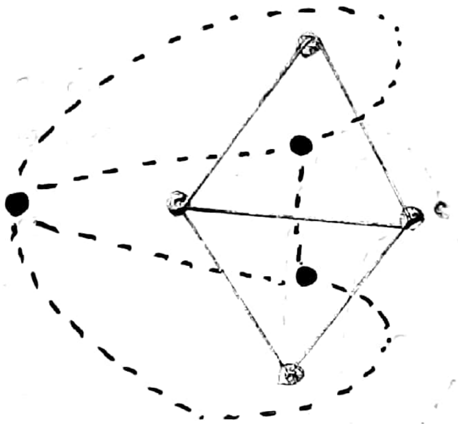
Interchanging



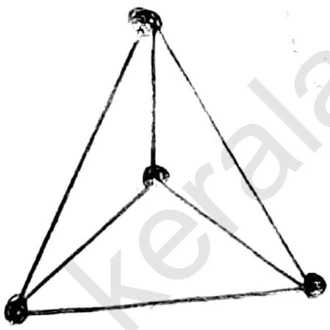
2.



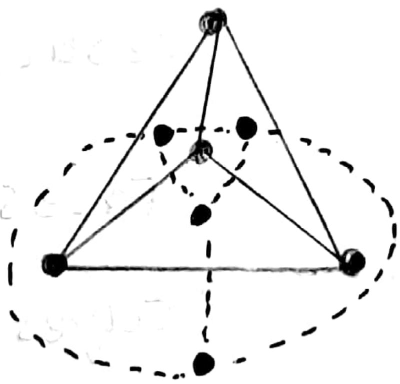
	G	G*
Vertices	4	3
Faces	3	4
Edges	5	5



3.



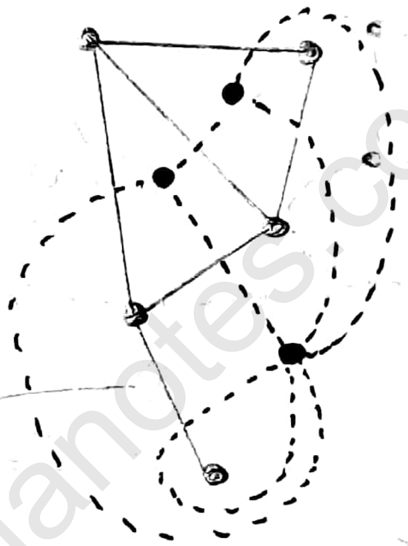
	G	G*
Vertices	4	4
Faces	4	4
Edges	6	6



4.

	G	G*
Vertices	5	3
Faces	3	5
Edges	6	6

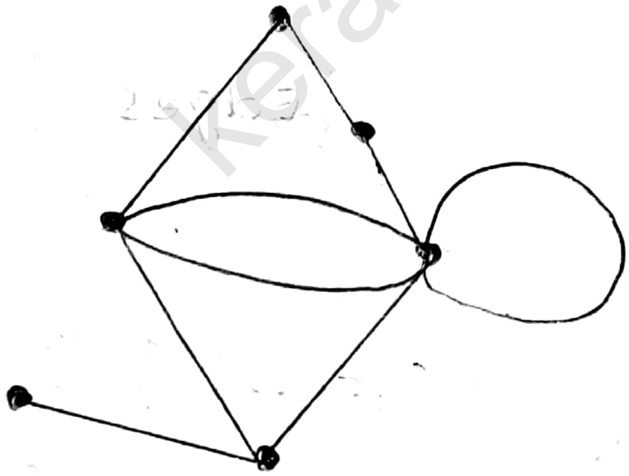
when pendent vertex comes it should be covered as a loop



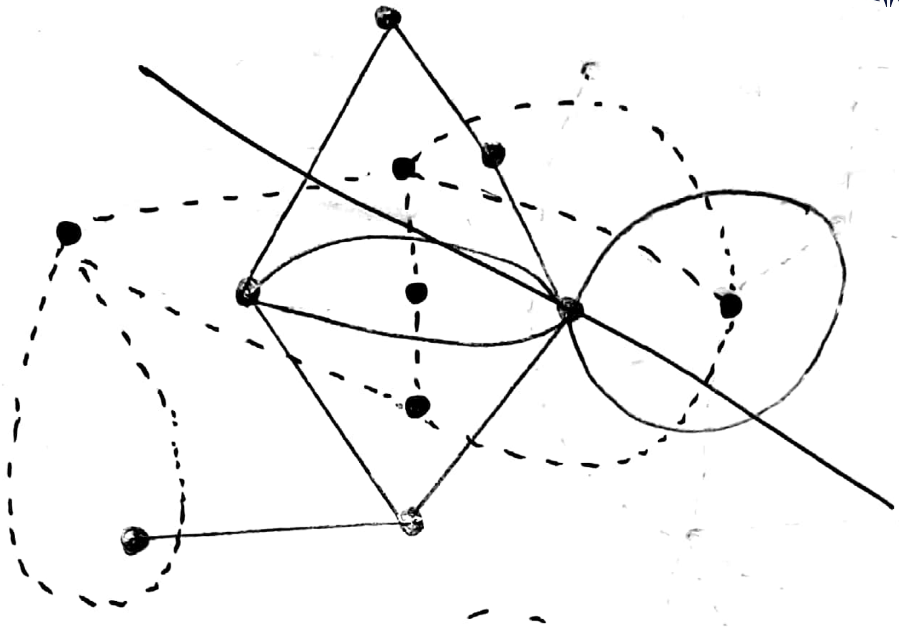
pendent vertex
[with degree 1]

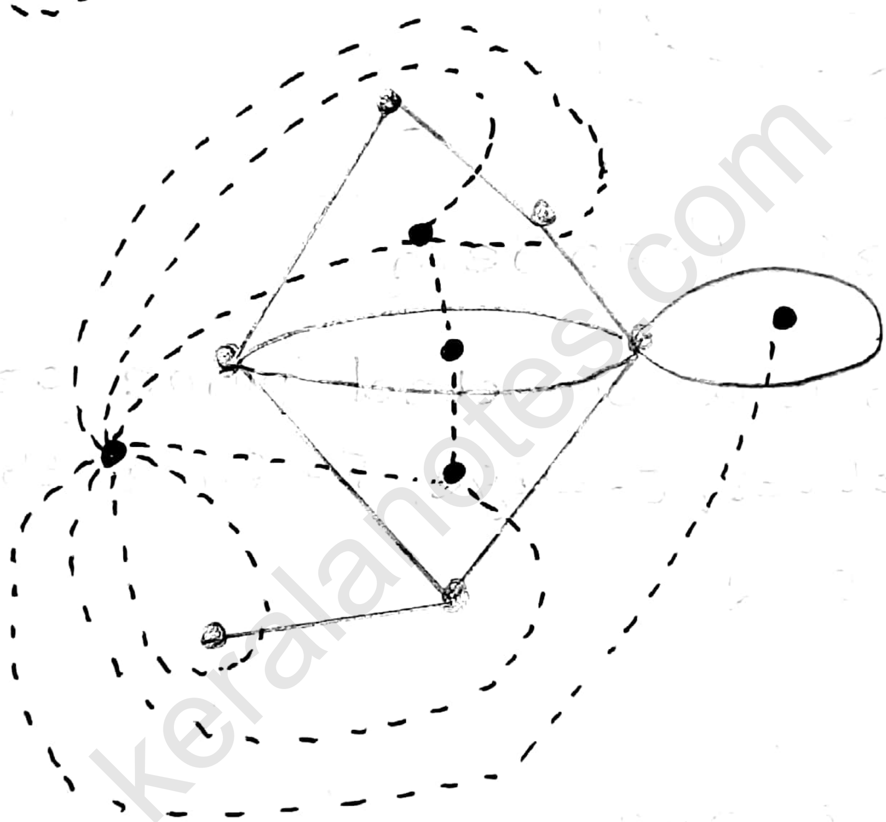
imp

5.

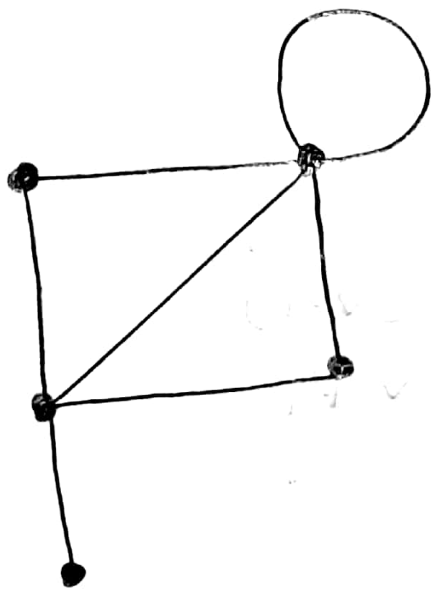


	G	G*
Vertices	6	5
Faces	5	6
Edges	9	9

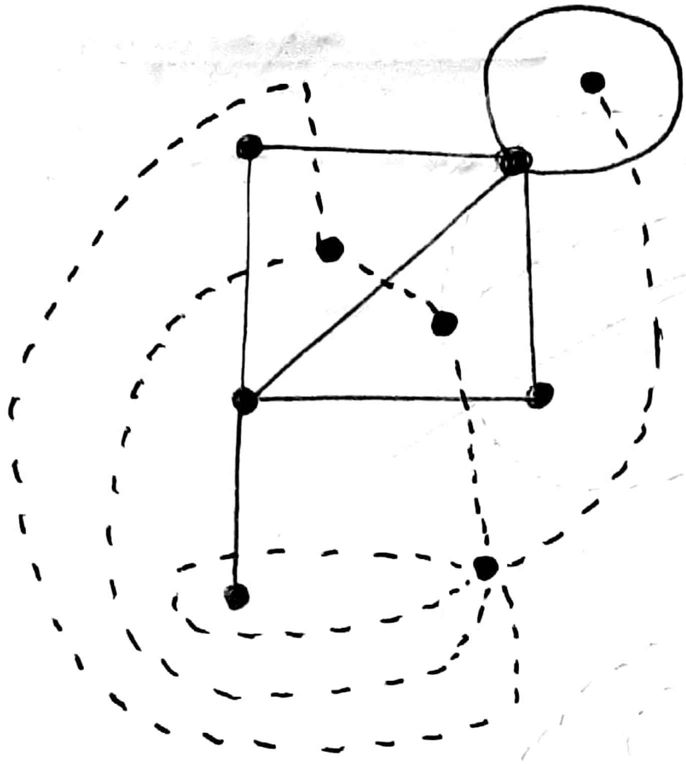




6.



	G	G*
vertices	5	4
faces	4	5
Edges	7	7



Planar: edges should not cross

Euler's Theorem

If G is a connected plane graph with V - vertices, E - edge, F - face then

$V + F - E = 2$

proof

If G is a tree

$F = 1$

$E = V - 1$

Then $V + F - E = V + 1 - (V - 1)$

$= V + 1 - V + 1$

$= 2$

It is true

IF G is not a tree

By mathematical induction. Assume it is true for V - vertices E - edges, F - Faces

ie, $V + F - E = 2$

Then we have to prove, it is true for E-1 edges

Remove an edge ~~spanning~~ seperating two faces.

Then no. of edges = $E - 1$

no. of Faces = $F - 1$

no. of vertices = V

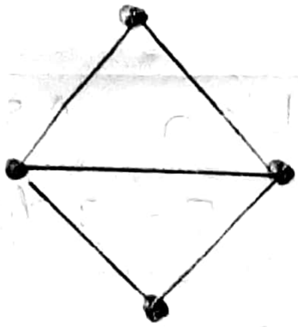
Then $V + (F-1) - (E-1)$

$= V + F - 1 - E + 1$

$= V + F - E = 2$

Hence proved

Q verify Euler's Theorem in the given graph

1. 

$V = 4$

$F = 3$

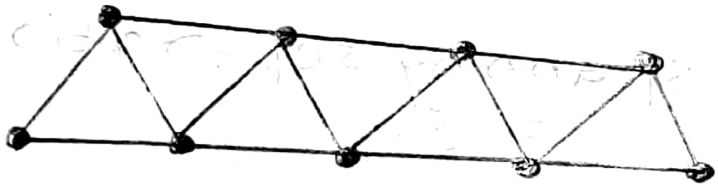
$E = 5$

$V + F - E = 3 + 4 - 5$

$= 7 - 5$

$= 2$

Hence Euler's theorem verified

2. 

$V = 9$

$F = \cancel{5}\ 8$

$E = \cancel{12}\ 15$

$V + F - E = 9 + 8 - \cancel{12}\ 15$

$= 2$

Hence Euler's Theorem verified.

Theorem - 1

(i) If G is a simple graph with n greater than or equal to 3 then

$E \leq 3V - 6$

(ii) If G is triangle free then $E \leq 2V - 4$

PROOF

(i) If G is simple

$2E \geq 3F$

$\Rightarrow \frac{2E}{3} = F$

By Euler's theorem

$V + F - E = 2$

$\Rightarrow V + \frac{2E}{3} - E \geq 2$

$\Rightarrow 3V + 2E - 3E \geq 6$

$\Rightarrow 3V - E \geq 6 \quad = \underline{\underline{3V - 6 \geq E}}$

(ii) If G is triangle free

$2E \geq 4F$

$\Rightarrow \frac{2E}{4} \geq F$

By Euler's theorem

$V + F - E = 2$

$\Rightarrow V + \frac{2E}{4} - E \geq 2$

$\Rightarrow 4V + 2E - 4E \geq 8$

$\Rightarrow 4V - 2E \geq 8 \quad \Rightarrow 4V - 8 \geq 2E \quad \div 2$

$\Rightarrow \underline{\underline{2V - 4 \geq E}}$

Theorem - 2

Rank of circuit matrix is $e-n+1$

[E = No. of edges, n = no of vertices]

Proof

lets A = Incidence matrix of G [rank = n-1]

B = circuit matrix of G

we have $A \cdot B^T \overset{\text{congruent}}{\equiv} 0 \pmod 2$

By sylvesters theorem

$\text{rank}(A) + \text{rank}(B) = e$

$\Rightarrow \text{rank}(B) \leq e - \text{rank}(A)$

$\Rightarrow \text{rank}(B) \leq e-(n-1)$

$\Rightarrow \text{rank}(B) \leq e-n+1$ ——— (1)

For any connected graph

$\text{rank}(B) \geq e-n+1$ ——— (2)

$a \geq b$
$a \leq b$
$\therefore a = b$

From (1) and (2)

$\text{rank}(B) = e-n+1$

Kuratowski's Theorem

A Graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or K_3