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APJ ABDUL KALAM TECHNOLOGICAL UNIVERSITY

Fourth Semester B.Tech Degree Examination July 2021 (2019 Scheme)

Course Code: MAT206

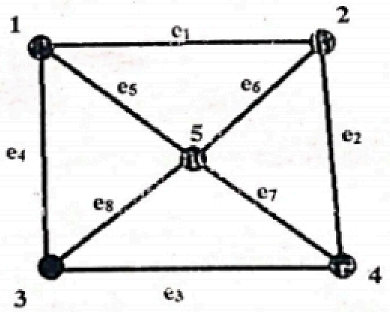
Course Name: GRAPH THEORY

Max. Marks: 100 Duration: 3 Hours

PART A

(Answer all questions; each question carries 3 marks) Marks

1	What is the maximum number of edges in a simple graph with n vertices? Justify your answer.	3
2	There are 25 telephones in Metropolis. Is it possible to connect them with wires so that each telephone is connected with exactly 7 others? Why?	3
3	Show that all vertices of an Euler graph G are of even degree	3
4	Explain strongly connected and weakly connected graphs with the help of examples.	3
5	Prove that a connected graph G with n vertices and n-1 edges is a tree.	3
6	How many labelled trees are there with n vertices? Draw all labelled trees with 3 vertices.	3
7	Define planar graphs. Is K_4, the complete graph with 4 vertices, a planar graph? Justify.	3
8	Define fundamental circuits and fundamental cut-sets.	3
9	Construct the adjacency matrix and incidence matrix of the graph .	3



10	Define chromatic number. What is the chromatic number of a tree with two or more vertices?	3

PART B

(Answer one full question from each module, each question carries 14 marks)

Module -1

11 a) Define complete graph and complete bipartite graph. Draw a graph which is a complete graph as well as a complete bipartite graph. 7

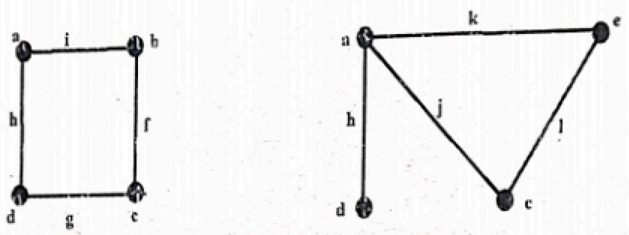
b) Explain walks, paths and circuits with the help of examples. 7

12 a) Define isolated vertex, pendant vertex, even vertex and odd vertex. Draw a graph that contains all the above. 7

b) Prove that simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges. 7

Module -2

13 a) 9



Find the union, intersection and ring sum of the above graphs.

b) State travelling salesman problem. How it is related to Hamiltonian circuits? 5

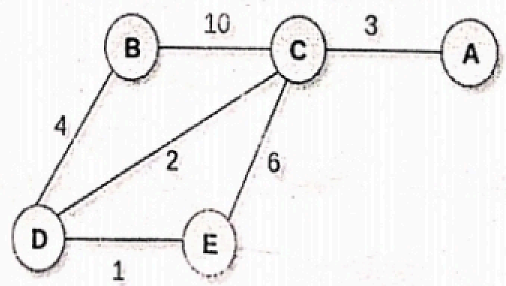
14 a) Prove that in a complete graph with n vertices there are $(n-1)/2$ edge disjoint Hamiltonian circuits, if n is an odd number and $n \geq 3$. 7

b) For which values of m, n is the complete graph $K_{m,n}$ an Euler graph ? Justify your answer. 7

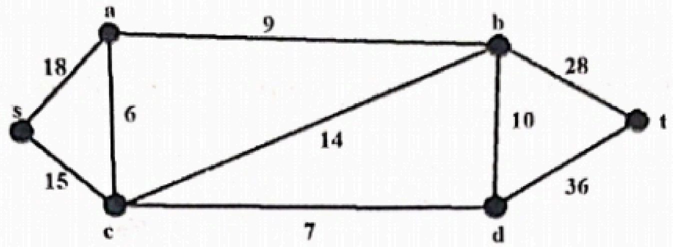
Module -3

15 a) Prove that a binary tree with n vertices has $(n+1)/2$ pendant vertices. 7

b) Using Prims algorithm, find a minimal spanning tree for the following graph. 7



16 a) Write down Dijkstra's algorithm and use it to find the shortest path from s to t. 9

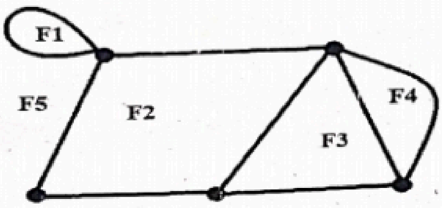


b) Prove that every tree has either one or two centers. 5

Module -4

17 a) Define cut-set. Prove that every circuit in G has an even number of edges in common with any cut-set. 8

b) Construct the geometric dual of the graph below 6



18 a) Prove that a connected planar graph with n vertices and e edges has e-n+2 regions. 9

b) Let G be a connected graph and e an edge of G. Show that e is a cut-edge if and only if e belongs to every spanning tree. 5

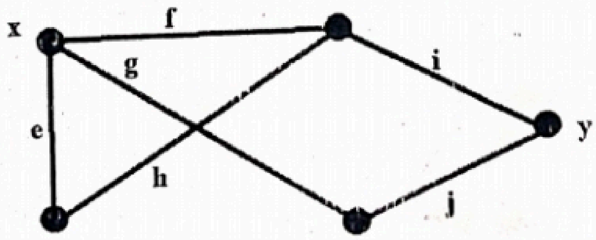
Module -5

19 a) Explain *four colour problem* using the concept of chromatic number. 5

b) Let B and A be the circuit matrix and the incidence matrix of a graph G which is free from loops, whose columns are arranged using the same order of edges. Show that $AB^T = BA^T = 0 \pmod 2$. 9

20 a) Show that chromatic polynomial of a tree with n vertices is $P_n(\lambda) = \lambda(\lambda-1)^{n-1}$ 7

b) Define path matrix of a graph. Find the path matrix P(x, y) for the graph below. 7



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(1) Maximum no. of edges in a simple graph with n vertices $= \frac{n(n-1)}{2}$

Max. Degree of a vertex $= (n-1)$

Sum of degrees of n vertices $= (n-1) + (n-1) e \cdots + n$ times

$= n(n-1)$

we know

$$2e = \sum d(v_i)$$
$$= n(n-1)$$
$$\therefore e = \frac{n(n-1)}{2}$$

(2) Consider a graph with telephones are vertices and connections are edges. Then the graph will be such that it has 25 vertices connected to to exactly 7 other vertices. This is not possible since in a graph it is not possible to have odd number of vertices with odd degree. i.e, 25 vertices with 7 degree each is not possible

3) Suppose that G is an Euler graph i, G contains a closed walk which contain every edge of G exactly once. (Euler line)

In tracing this walk we observe that every time the walk meets a vertex 'v' it goes through two new edges incident on v - with one we 'entered' v and the other 'exited'.

This is true not only for all intermediate vertices of the walk but also of the terminal vertex, because we 'exited' and 'entered' the same vertex at the beginning and end of the walk respectively.

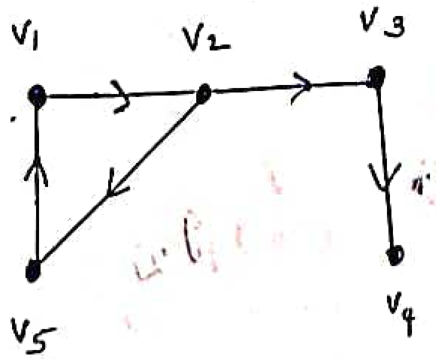
4) **Strongly connected digraphs**

If there is at least one directed path from every vertex to every other vertex then the digraph G is said to be strongly connected.

Weakly connected digraphs

A digraph is said to be weakly connected if its corresponding undirected graph is connected.

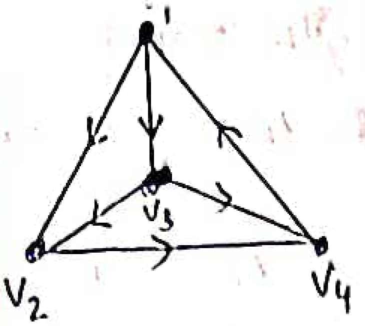
Eg. i)



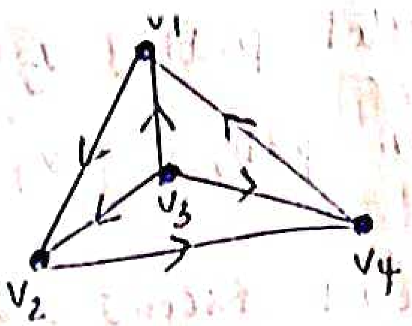
Not strongly connected

But weakly connected

ii)



strongly connected graph



weekly connected graph

5) Any connected graph with n vertices and n-1 edges is a tree.

Proof:

Let us assume the contradiction. Suppose G be a connected graph with n vertices and n-1 edges but not a tree. This means G contains at least one circuit.

Now remove an edge e from the circuit. Then G-e is again a circuitless connected graph with n-2 edges. That is G-e is a tree with n vertices & n-2 edges which contradicts the theorem that a graph with ~~n-1~~ tree with n vertices has n-1 edges. Hence our assumption was wrong and G is a tree.

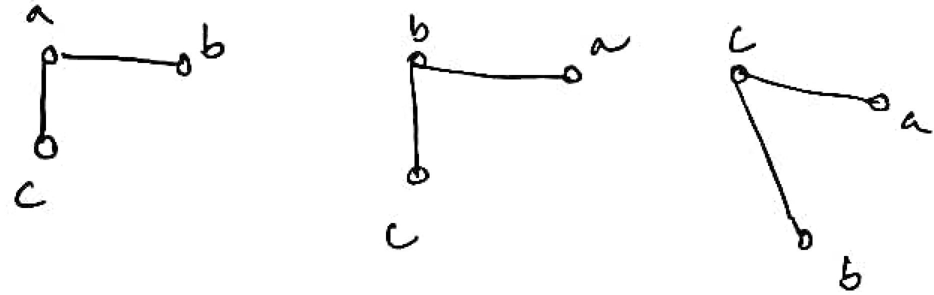
1 6)

Number of labelled trees with n vertices

is n^{n-2}

$\therefore$ with 3 vertices $3^{3-2} = \underline{\underline{3}}$

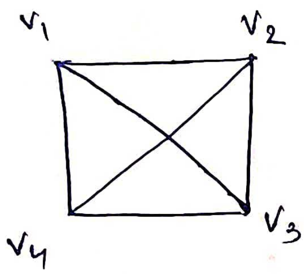
They are,

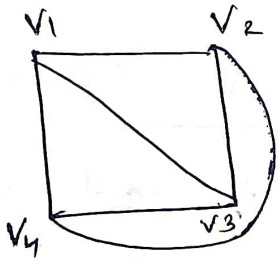


27) Define planar graph and non planar graph with example.

Planar graph

A graph is called planar if it can be drawn in the plane without any edge crossing such a drawing is called a planar representation of graph





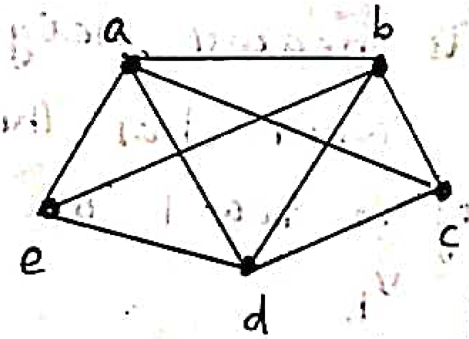
planar representation

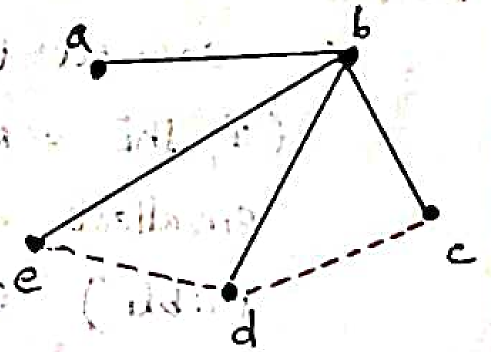
Non planar

8) Fundamental Circuits

Consider a spanning tree T in a connected graph G. Adding any one chord (an edge of G that is not in a given spanning tree) to T will create exactly one circuit. Such a circuit formed by adding a chord to a spanning tree is called a fundamental circuit.

Example: G: T:





Adding the chord (c,d) to the spanning tree T will form a fundamental circuit bcdb

Adding the chord (d,e) to the spanning tree T will creat a fundamental circuit bdeb

But adding both the chords will create a circuit bcdea also which is not a fundamental circuit.

Fundamental cut-sets.

Defn: A cut-set containing exactly one branch of a tree is called a fundamental cut-sets, w.r.t that tree. It is also called a basic cut-sets.

9)

<u>Adjacency Matrix</u>

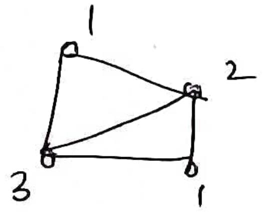
$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

<u>Incidence Matrix</u>

$$\begin{array}{cc} & \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

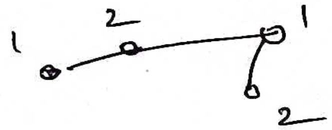
10) The chromatic number of a graph is the minimum number of colors needed to color the vertices in such a way that no two adjacent vertices has the same color.

eg



A Tree is 2-chromatic. Needs two colors only

eg



11)a 8.a) Define complete graph & complete bipartite graph. Draw a graph which is a comple graph ass well as complete bipartite graph.

b) (Already solved)

Answer a) Complete Graph :- A complete graph is a simple graph in which each pair of distinct vertices are joined by an edge

Complete bipartiate graph : The vertices are divided into 2 disjoint sets. Lets say, sets A & B. Each vertex in A is connected to every vertex in B resulting in a complete bipartite graph.

eg:

$A = \{V_1\}$

$B = \{V_2\}$

G is a complete Graph as well as a complete bipartite graph.

11)b Define Walk, path & circuit with example.

Ans: Walk :

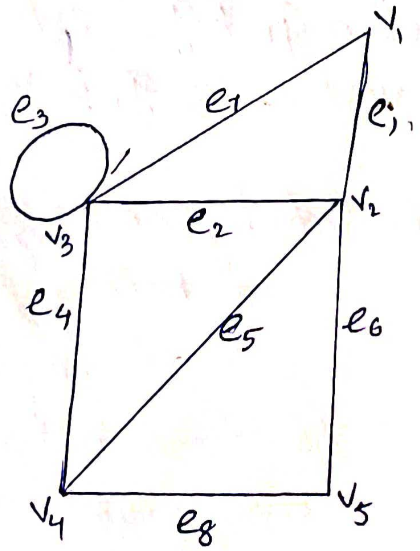
A walk in a graph is a finite alternating sequence of vertices & edges beginning & ending with vertices such that each edge is incident with the vertices preceding & following it.

Path

An open ~~op~~ walk in which no vertex appear more than once is called a path

Circuit : A closed path is called a circuit.

eg:



Walk : $v_1 e_1 v_2 e_2 v_3 e_3 v_3 e_4 e_5 v_2 e_6 v_5$

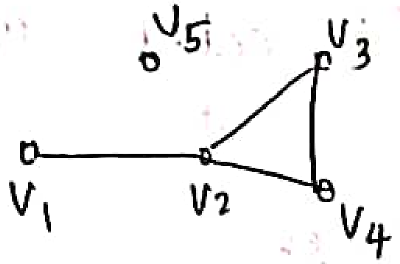
Path : $v_1 e_1 v_2 e_2 v_3 e_4 v_4$

Circuit : $v_2 e_2 v_3 e_4 v_4 e_5 v_2$

12)a

Isolated Vertex:

A vertex is said to be isolated if it is not an end vertex of any edge in the graph.



V_5 is an isolated vertex

Pendant Vertex (End vertex)

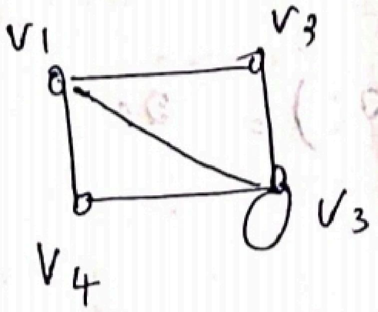
A vertex of degree 1 is called a pendant vertex or end vertex

Note:

For an isolated vertex v, $d(v) = 0$

Even and odd vertices

A vertex with even degree is known as an even vertex. and a vertex with odd degree is known as an odd vertex.



$d(v_1) = 3$

$d(v_2) = 2$

$d(v_3) = 5$

$d(v_4) = 2$

v_1 & v_3 are odd degree vertices.

v_2 & v_4 are even degree vertices.

12)b A simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof

Let G be a graph with k components.

Let $H_1, H_2, H_3, \ldots H_k$ be the k components of G with ~~vertices~~ no. of vertices $n_1, n_2, \ldots n_k$ respectively with $n_i \geq 1$ for $i = 1, 2, 3 \ldots k$.

We know that,

$$H_1 \cup H_2 \cup \ldots \cup H_k = G$$

i.e. $n_1 + n_2 + \cdots + n_k = n$

$$\sum_{i=1}^{k} n_i = n \quad \text{(1)}$$

Now, $\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$

$= n - k$

i.e. $\sum_{i=1}^{k} (n_i - 1) = n - k$

squaring both sides.

$$\left[\sum_{i=1}^{k}(n_i-1)\right]^2 = (n-k)^2$$

i, $\sum (n_i-1)^2 +$ some of +ve terms $= (n-k)^2$

i, $$\sum_{i=1}^{k}(n_i-1)^2 \le (n-k)^2$$

$$\sum_{i=1}^{k}(n_i^2 - 2n_i + 1) \le (n-k)^2$$

$$\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \le (n-k)^2$$

$$\sum_{i=1}^{k} n_i^2 - 2n + k \le (n-k)^2$$

i, $$\sum_{i=1}^{k} n_i^2 \le (n-k)^2 + (2n-k) \longrightarrow (1)$$

Now the maximum number of edges of the i^{th} component H_i with n_i vertices is, $\frac{n_i(n_i-1)}{2}$

∴ Maximum no. of edges in G

$$= \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \cdots + \frac{n_k(n_k-1)}{2}$$

$$= \frac{1}{2} \sum_{i=1}^{k} (n_i(n_i-1))$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

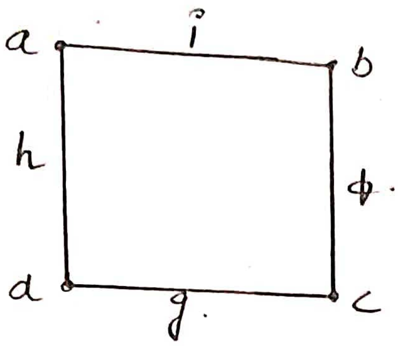
$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - n\right]$$

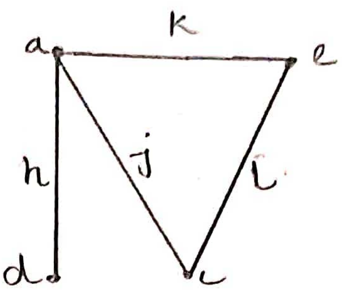
$$\leq \frac{1}{2}\left[(n-k)^2 + (2n-k) - n\right] \text{ from } 1$$

$$\leq \frac{1}{2}\left[(n-k)^2 + (n-k)\right]$$

$$\leq \frac{1}{2}(n-k)(n-k+1)$$

13)a





Find the union intersection and the ring sum of the above graphs

b) State travelling salesman problem; How it is related to Hamiltonian circuit

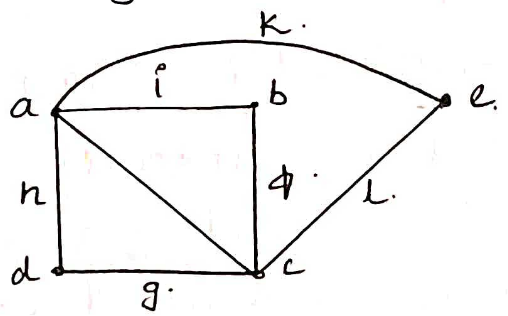
Union.

Union of two graphs $G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$ is another graph $G_3 = (V_3, E_3)$ where $V_3 = V_1 \cup V_2$

$E_3 = E_1 \cup E_2$.

Here $V_1 = \{a, b, c, d\}$, $E_1 = \{i, h, f, g\}$

$V_2 = \{a, d, e, c\}$ $E_2 = \{k, h, j, l\}$

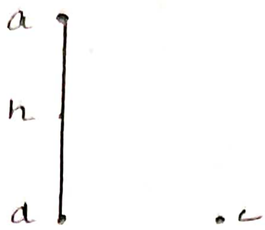


Intersection.

Intersection of 2 graph $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ is another graph $G_3 = (V_3, E_3)$ where $V_3 = V_1 \cap V_2$

$E = E \cap E_2$

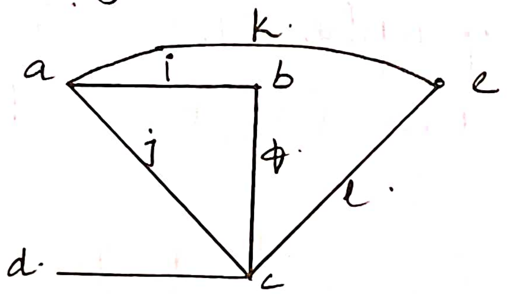
$V_1 = \{a, b, c, d\}$ $E_1 = \{i, f, h, g\}$
$V_2 = \{a, d, e, c\}$ $E_2 = \{k, h, j, l\}$
$V_3 = \{a, d, c\}$ $E_3 = \{h\}$



Ring Sum.

The ring sum of G_1 & G_2 is denoted by $G_1 \oplus G_2$. It is a graph consisting of vertex set $V_1 \cup V_2$ and edges that are either in G_1 or G_2 or not in both.

$V_1 = \{a, b, c, d\}$, $E_1 = \{i, f, h, g\}$
$V_2 = \{a, d, e, c\}$ $E_2 = \{k, h, j, l\}$
$V_3 = \{a, b, c, d, e\}$ $E_3 = \{i, f, g, k, j, l\}$



13)b

Travelling Salesman Problem (TSP)

Problem:

A salesman is required to visit a number of cities during a trip. Given the distance between the cities. In what order should he travel so as to visit every city precisely once and return home, with the minimum mileage travelled.

Before giving the solution for TSP we will see the following definition and result.

Solution to TSP

Represent the cities by vertices and the roads between them by edges. In this graph with every edge e_i there is associated a real number $w(e_i)$, the distance in miles.

Here in this graph we have $\frac{(n-1)!}{2}$ different Hamiltonian circuit and we will pick up the one that has the smallest sum of distances.

14)a In a complete graph with n vertices there are $(n-1)/2$ edge disjoint Hamiltonian circuits if n is an odd number $(n \geq 3)$

Proof

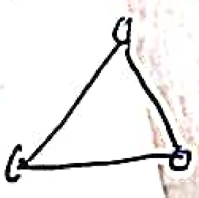
A complete graph with n vertices has exactly $\frac{n(n-1)}{2}$ edges.

A Hamiltonial cycle in a graph with n vertices contains n edges.

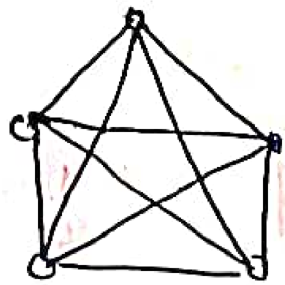
Hence the number of edge disjoint cycles in a Hamiltonian cycle does not exceed $\frac{n-1}{2}$

Now we illustrate that if n is an odd number $(n \geq 3)$ there are $\frac{n-1}{2}$ edge disjoint Hamiltonian Circuit

K_3



K_5



1^{st} Hamiltonian cycle - 1 2 3 9 4 8 5 7 6 1

rotation: 2^{nd} Hamiltonian circuit - 1 9 2 8 3 7 4 6 5 1

" - 3^{rd} " - 1 8 9 7 2 6 3 5 4 1

" - 4^{th} " - 1 7 8 6 9 5 2 4 3 1

Now the 4^{th} rotation gives the same circuit 1 6 7 5 8 4 9 3 2 1 as first one.

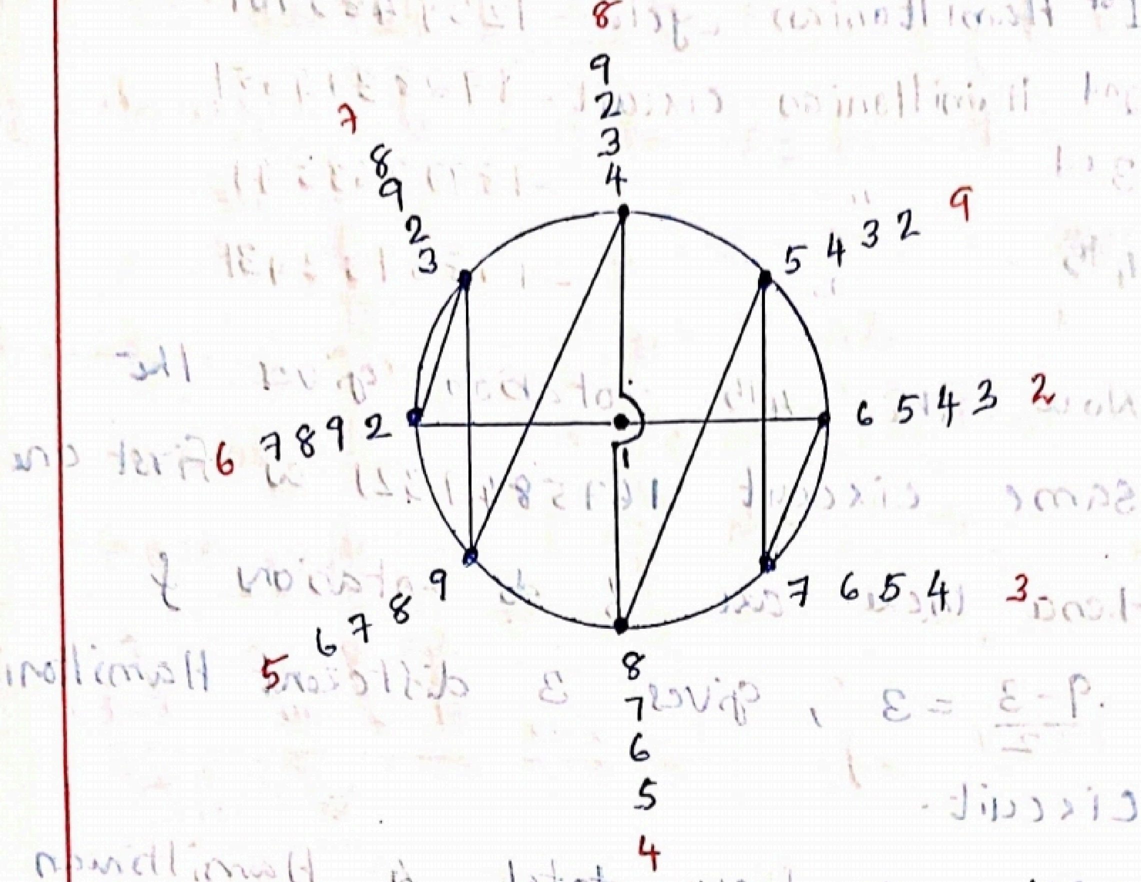
Hence there are 3 rotations & $\frac{9-3}{2} = 3$, gives 3 different Hamiltonian circuit.

Hence we have total 4 Hamiltonian circuits i.e. $\frac{n-1}{2} = \frac{9-1}{2} = 4$

Hence in general $\frac{(n-3)}{2} \times \frac{2\pi}{(n-1)}$ rotations produces $\frac{n-3}{2}$ Hamiltonian circuits. Hence total $\frac{n-3}{2} + 1 = \frac{n-1}{2}$ Edge disjoint Hamiltonian circuits are there.

(For K_5, $1 \cdot \frac{2\pi}{4}$, $2 \cdot \frac{2\pi}{4}$, rotations gives 2 different Hamiltonian circuits

Hence $\frac{5-1}{2} = 2$, edge disjoint circuits are present.) [Verify]



Consider the complete graph with 9 vertices & K_9. $n=9$

Consider a Hamiltonian circuit with 9 vertices keeping the vertices fixed (fix one vertex at the centre) on a circle.

Rotate the polygon pattern clockwise by $1 \cdot \frac{2\pi}{8}$, $2 \cdot \frac{2\pi}{8}$, $3 \cdot \frac{2\pi}{8}$

Each rotation produces a Hamiltonian cycle that has no edge in common with any of the previous one.

4)b A graph is Euler iff degree of every vertex is even.

Since $K_{m,n}$ is complete bipartite, the vertex set V is partitioned into two non-empty subset X & Y such that every vertex in X is ~~mapped~~ joined to every other vertex of Y by an edge.

Since X contains m vertices and Y contains n vertices, the degree of all vertices will be even only if both m & n are even.

15)a Let p be the number of pendant vertices in a Binary tree. T. Then $n-p-1$ is the number of vertices of degree 3. Therefore number of edges in T equals. For (tree, $e = n-1$)

We have, $2e = \sum d(v_i)$

i.e., $2(n-1) = p \times 1 + (n-p-1) \times 3 + 2$

$2n - 2 = p + 3n - 3p - 3 + 2 = -2p + 3n - 1$

i.e., $2p = 3n - 1 - 2n + 2 = n + 1$

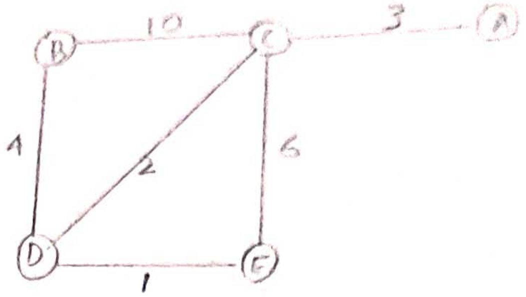
i.e., $p = \frac{n+1}{2}$

$\therefore$ No. of pendant vertices in $T = \frac{n+1}{2}$ //

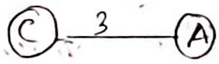
15)b Step 1: Tabulating a 5x5 table with weights of the the given graph.

• weights of non-existent edges are set to ∞

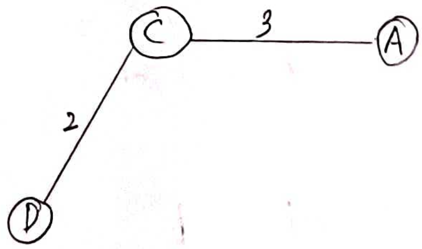
$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & - & \infty & 3 & \infty & \infty \\ B & \infty & - & 10 & 4 & \infty \\ C & 3 & 10 & - & 2 & 6 \\ D & \infty & 4 & 2 & - & 1 \\ E & \infty & \infty & 6 & 1 & - \end{array}$$



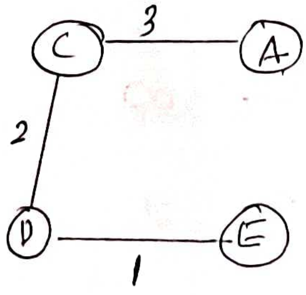
Step 2: Starting from vertex A and connecting to its nearest neighbour with smallest weight. say C.



Step 3: choose the smallest weighted neighbour of C ie choose (C,D)

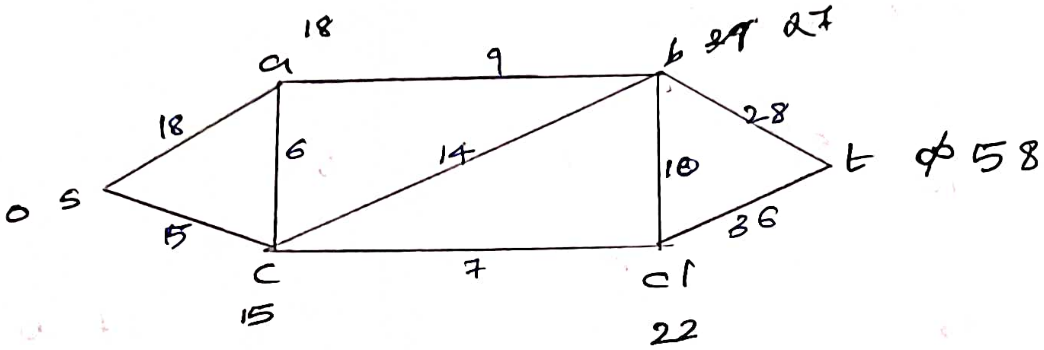


Step 4: select vertex D and choose (D,E)



Step 5: Choosing next vertex as E. The smal weighted edge of has been already selected

steps. permanantly mark d and assign value to its connected vertices b and t.



Step 6

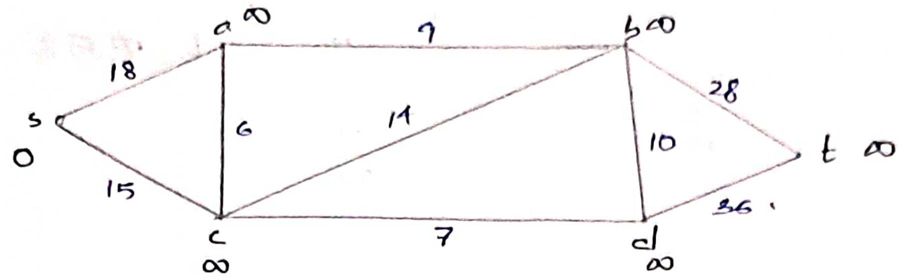
permanantly mark b.

Assign value to t min (27+28, 58) = 55

and permanantly mark t as last vertex.

Hence the length of the shortest path from s to t is 55 first step 1.

Assign a permanent label '0' to the starting vertex s and a temporary label ∞ to the remaining (n-1) vertices.



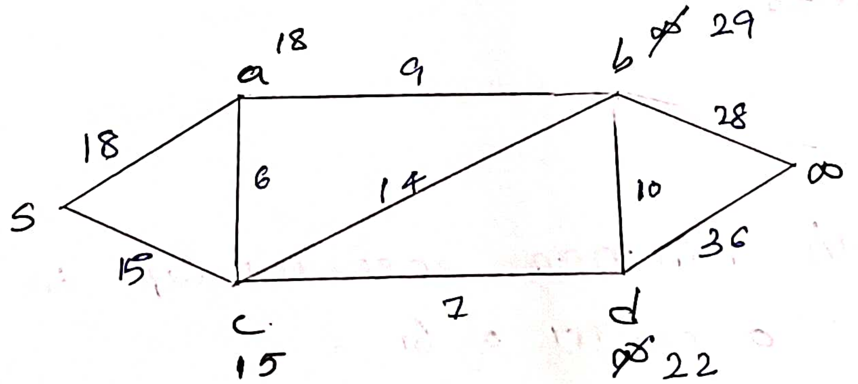
Step 2.

Assign min (0+18, ∞) and min(0+15, ∞) to connected vertex of s i.e a and c.

permanantly mark c as it has the smallest value in the assigned row.

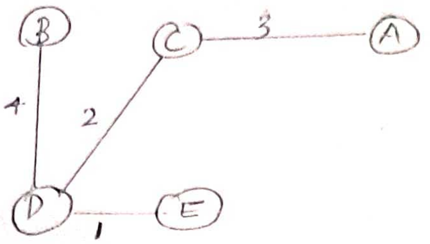
Step 3.

Assign values to the connected vertices of c i.e s, a, b, d. and choose the next smallest vertex.



Step 4. permanantly mark a with the smallest weight, and assign its connected vertex b.

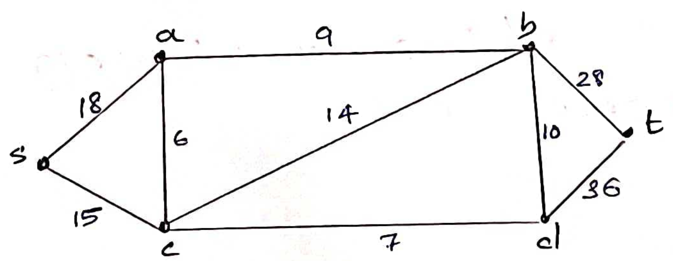
choosing (E,C) will form a circuit. Hence choose (D,B)



All the vertices of the graph has been included without a circuit.
Hence the shortest spanning tree has a total weight of $1 + 4 + 2 + 3 = \underline{10}$.

16)a

a). Write down Dykstra's algorithm and use it to find the shortest path from s to t.



s	a	b	c	d	t
0	∞	∞	∞	∞	∞
	18	∞	15	∞	∞
	18	29		22	∞
		27		22	∞
		27			58
					55
0	18	27	15	22	55

16)b

Proof

The maximum distance, $\max d(v, v_i)$ from a given vertex v to any other vertex v_i occurs only when v_i is a pendant vertex.

Let T be a tree having more than two vertices. Then T must have two or more pendant vertices.

Delete all the pendant vertices from T. Then the resulting graph T' is still a tree.

The removal of pendant vertices from T will uniformly reduce the eccentricities of the remaining vertices by one.

Therefore all vertices that T had as centers will still remain centers in T'.

From T' we can again remove all pendant vertices and get another tree T''.

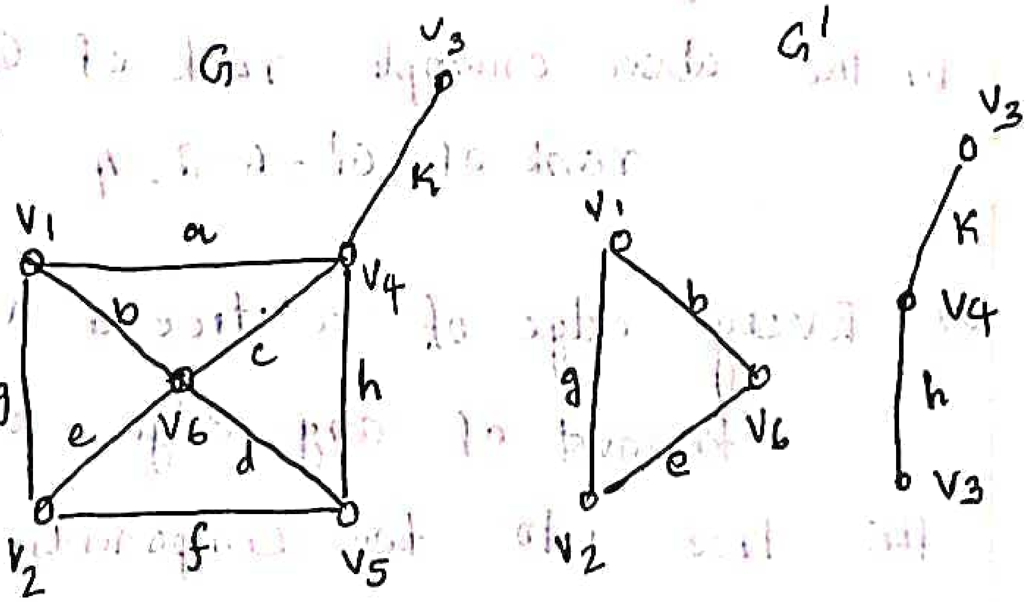
We continue this process until there is left either a vertex (which is the center of T) or an edge (whose end vertices are the two centers of T).

Hence the theorem.

17)a cut-sets

In a connected graph G, a cut-set is a set of edges (minimal set) whose removal from G leaves G disconnected provided removal of no proper subset of these edges disconnects G. A cut-set always cuts a graph into two.

Eg:



{a, c, d, f} is a cut-set which disconnects the graph into two.

There are many cut-sets in a connected graph G.

{a, b, g}, {a, b, e, f}, {d, h, f}

{K} are some cut sets of G

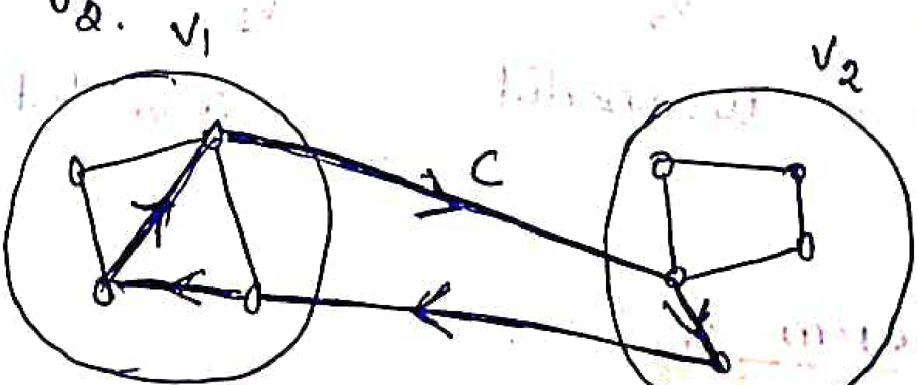
Proof

Let S be a cut-set in G. Then by definition of cut-set, the removal of S partition the vertices of G into two subsets V_1 & V_2.

Consider a circuit C in G.

If all vertices of C are entirely within set V_1 (or V_2) the number of edges common to S & C are zero (clearly even)

If on the other hand some vertices in C are in V_1 and some in V_2.

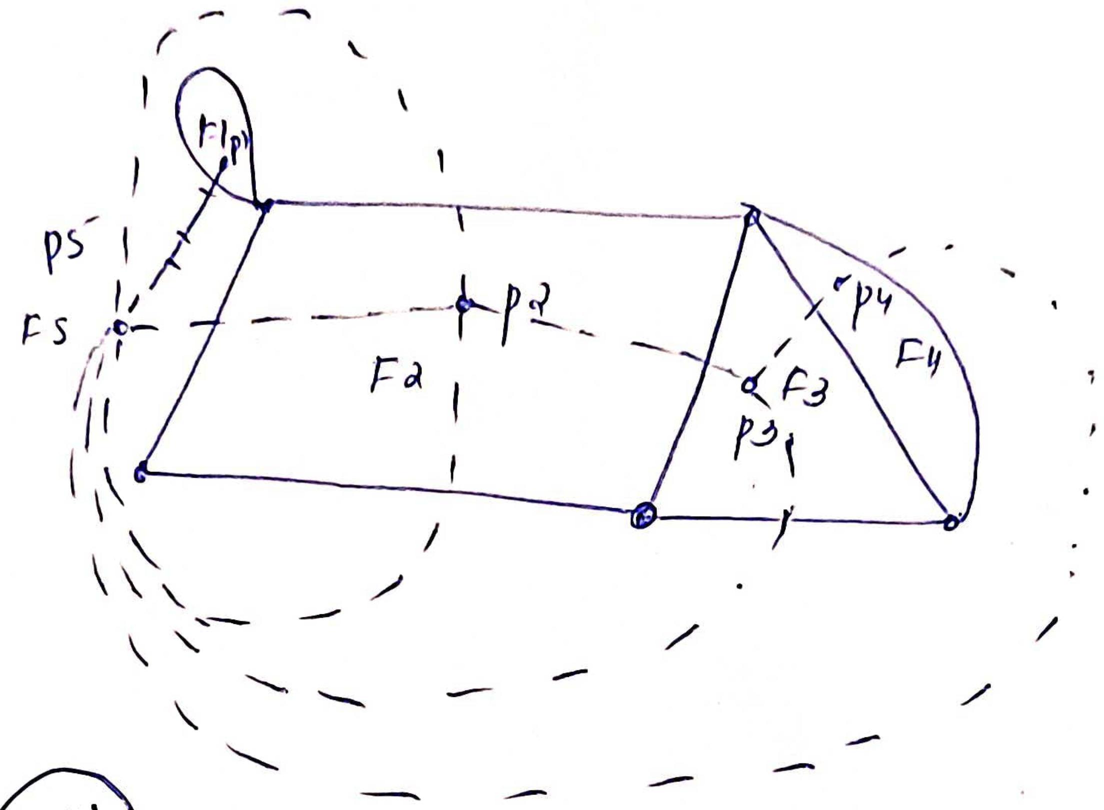


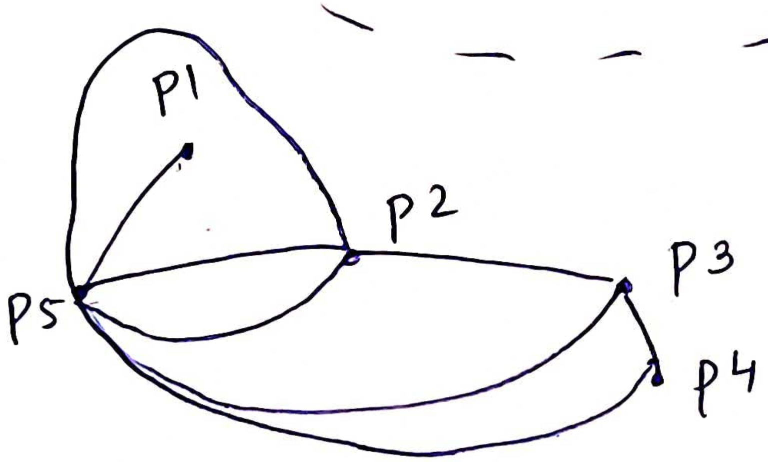
Circuit C shown in heavy lines, and is traversed along the direction of arrows we travel back & forth between two sets V_1 & V_2 to get a circuit. Because of the closed nature of a circuit, the number of edges we traverses b/w V_1 & V_2 must be even.

Since S is a cut set every edge in S has one end in V_1 and the other end in V_2, and no other edge in G has this property.

∴ The number of edges common to S & C are even.

17)b





18)a Let, f_k - numbers of edges

e_k - number of edges

n_k - numbers of vertices of the planar graph G

Now apply induction on number of edges.

Let the number of edges $k=1$

ie, 

then, no of region $= 1$

no. of edges $= 1$

no of vertices $= 2$

$f=1, e=1, n=2$

$f_1 = e_1 - n_1 + 2 = 1 - 2 + 2 = 1$

The planar graph has only 1 region

∴ The result is true for $k=1$

Now assume that the result is true for k number of edges

ie, $f_k = e_k - n_k + 2$ (assumption)

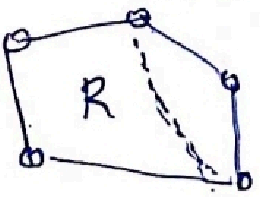
Now we will prove that the result is true for $k = k+1$

(ie, $f_{k+1} = e_{k+1} - n_{k+1} + 2$)

Let $\{a_{k+1}, b_{k+1}\}$ be the edge that is to added to G_k to obtain G_{k+1}. Now there are two possibilities to consider.

First case: Vertex a_{k+1} & b_{k+1} are already in G_k.

The addition of this new edge splits R into two regions (faces) Consequently,



$$f_{k+1} = f_k + 1$$
$$e_{k+1} = e_k + 1$$
$$V_{k+1} = V_k$$

Now consider,

$$f_{k+1} = e_{k+1} - n_{k+1} + 2$$

i, $f_k + 1 = e_k + 1 - n_k + 2$

i, $f_k = e_k - n_k + 2$

the formula is still true.

Second case: let one of the ~~two~~ ~~edge~~ two vertices of the new edge is not already in G_K

Suppose that a_{k+1} is in G_K and b_{k+1} is not in G_K. Adding this new edge doesnot produce any new region because,

b_{k+1} must be in a region that has a_{k+1} on its boundary

Consequently,

$$f_{k+1} = f_k$$

$$e_{k+1} = e_{k+1}$$

$$v_{k+1} = v_k + 1$$

$$\therefore f_{k+1} = e_{k+1} - n_{k+1} + 2$$

$$f_k = e_k + 1 - (n_k + 1) + 2$$

$$f_k = e_k - n_k + 2$$

Hence the formulas is still true.

Hence $f = e - n + 2$

or

$r = e - v + 2$ //

18)b

Let G be a connected graph. e is a cut edge. Suppose that e does not belong to every spanning tree. Then the removal of e does not disconnect the graph since, there will exists a spanning tree for the graph which does not contain 'e' and also hence the graph is connected. This is contradiction to the fact that e is a cut edge. Hence e belongs to every spanning tree.

Conversely suppose that every spanning tree of G contains 'e'. Then G-e will be surely disconnected since it does not have a spanning tree. That means e is a cut edge.

19)a Four color Problem (Four color conjecture)

Consider the proper coloring of regions in a planar graph. Just as in coloring of vertices and edges, the regions of a planar graph are said to be properly colored if no two contiguous or adjacent regions have the same color. (Two regions are said to be adjacent if they have a common edge between them)

The proper coloring of regions is called map coloring.

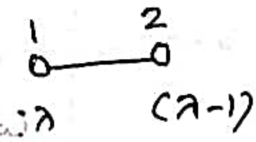
The Four colour conjecture is that every map (ie, a planar graph) can be properly colored with 4 colors.

No one has yet been able to either prove the theorem or come up with a map that requires more than four colors.

20)a Proof is by induction on number of vertices. Let $n=1$, i.e., an isolated vertex. Then clearly the vertex can be coloured in λ ways with λ colors.

$$\therefore P_n(\lambda) = \lambda = \lambda(\lambda-1)^{1-1} = \lambda$$

when $n=2$,



$$P_2(\lambda) = \lambda.(\lambda-1)$$
$$= \lambda(\lambda-1)^{2-1}$$

Theorem holds for $n=1$ & 2.

Now assume the theorem holds for $n=k$ vertices.

i.e., $P_n(\lambda) = \lambda(\lambda-1)^{k}$.

We will prove that the theorem is true for $k+1$ vertices.

Consider a tree with $k+1$ vertices.

Every tree with $n \geq 2$ has minimum 2 pendant vertices.

Remove one of the pendant vertex then we left with a tree with k vertices. Then by hypothesis

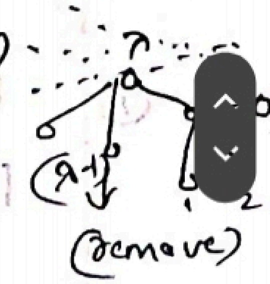
$$P_k(\lambda) = \lambda.(\lambda-1)^{k-1}$$

After coloring all vertices in $\lambda(\lambda-1)^{k-1}$ ways attach the pendant vertex

Then the pendant vertex can be coloured ~~with~~ in $(\lambda-1)$ ways

Hence all vertices can be coloured in $\lambda(\lambda-1)^{k-1}(\lambda-1)$ ways in tree with k+1 vertices.

$$P_{k+1}(\lambda) = \lambda(\lambda-1)^{k-1}(\lambda-1)$$
$$= \lambda(\lambda-1)^{k}$$
$$= \lambda(\lambda-1)^{(k+1)-1}$$



Hence the theorem holds for a tree with k+1 vertices also.

So by induction theorem is proved.

20)b

Path Matrix

A path matrix is defined for a specific pair of vertices in a graph say (x, y), and is written as $P(x, y)$. i.e. the path matrix for (x, y) vertices is

$P(x, y) = [P_{ij}]$ where,

$P_{ij} = 1$ if j^{th} edge lies in i^{th} path and

$= 0$, otherwise

Consider all paths