08.02.2023

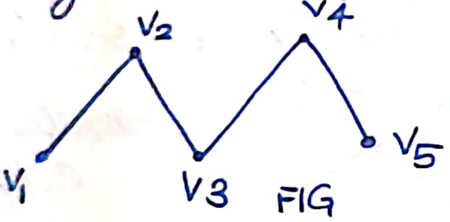
GRAPH THEORY

What is a graph?

A graph $G=(V,E)$ consists of a set of objects $V=\{v_1, v_2, v_3 \dots\}$ $E=\{e_1, e_2 \dots\}$ such that each edge e_i is identified with an unordered or ordered pair of vertices (v_i, v_j) $V=\{v_1 v_2 \dots$ called vertices OR nodes and $E=\{e_1 e_2 \dots\}$ is called edges

ADJACENT (VERTICES/NODES)

Any pair of vertices which are joined by an edge in a graph is known as adjacent vertices



FIG

(v_1, v_2), (v_2, v_3), (v_3, v_4) (v_4, v_5) adjacent

v_2 & v_4 not adjacent

ADJACENT EDGES

Two more edges in a graph is called adjacent edges if they have a common vertex

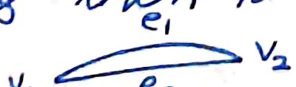
~~In the ab.~~

e_4, e_3 are adjacent

e_4 and e_5 not adjacent

PARALLEL EDGES

(3M) If two or more edges in a graph having same end vertices then the edges are called parallel edges



e_1 and e_2

SELF LOOP

An edge of a graph that joins a vertex to itself is called a loop

e_4 is a loop

DEGREE OF VERTEX

It is the number of edges incident on that

vertex.
Degree of a vertex v_i is usually denoted as $d(v_i)$
For a loop degree = 2

ORDER OF A GRAPH

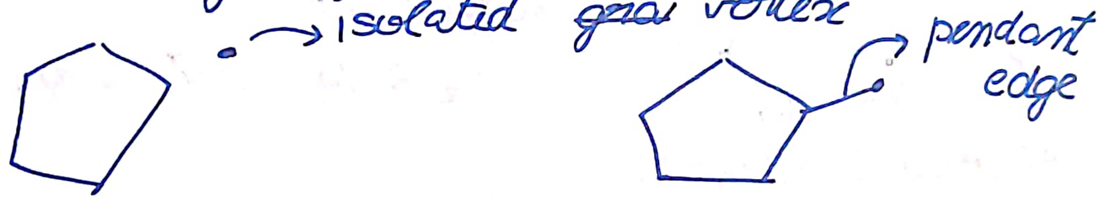
order of a graph is the no. of vertices in a graph

SIZE OF A GRAPH

Size of the graph is the no. of edges in that graph

ISOLATED VERTEX

A vertex having degree 0 is called isolated vertex.



PENDANT VERTEX

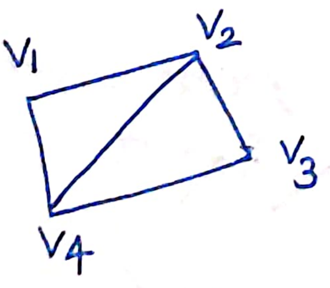
A vertex having degree 1 is called pendant vertex

PENDANT EDGE

Edge connecting a pendant vertex is called pendant edge

SIMPLE GRAPH

A graph in which no loops and parallel edg is called a simple graph

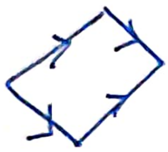


NULL GRAPH

A graph in which no edge is called null graph or all the vertices in a graph are isolated vertices

DIRECTED GRAPH

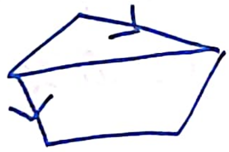
A graph in which every edge is directed is called directed graph / diagraph



graph ⇒ undirected

MIXED GRAPH

A graph G in which some edges are directed and some are undirected is called mixed graph



MULTI GRAPH

A graph in which there is some parallel edge but not loops

PSEUDO GRAPH

A graph in which loops and parallel edges are allowed is called a pseudo graph

WEIGHTED GRAPH

The graph in which the weights are assigned to each edge is called weighted graph

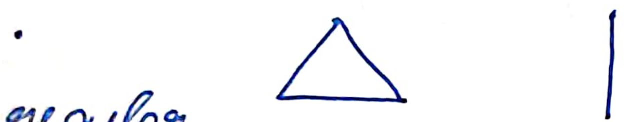
FINITE GRAPH

A graph in which finite no. of edges, finite no. of vertices is called finite graph

REGULAR GRAPH

A graph G is said to be regular if every vertex of a <u>simple graph</u> has same degree. If every vertex in a regular graph has degree r then the graph is called r-regular graph

eg:



0-regular graph　　2-regular graph　　1-regular graph

COMPLETE GRAPH:

IMP

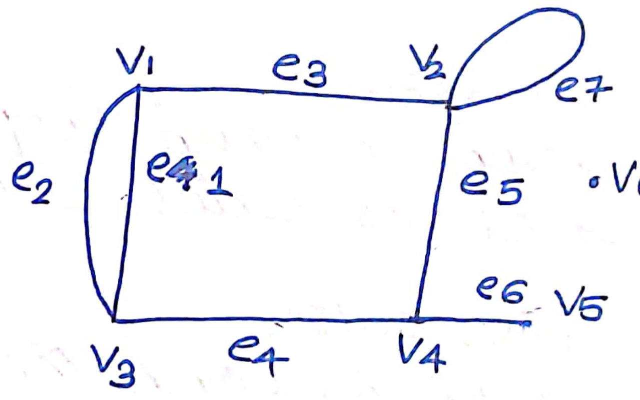
<u>A simple graph</u> in which there is exactly one edge between each pair of distinct vertices is called complete graph

OR

In a simple graph, every vertex connected to every other vertex

A complete graph of order n is represented by Kn where n is order

Degree of each vertex
Loop
Parallel edge
Pendant edge
pendant vertex



Degree of $V_1 = 3$, $V_2 = 4$, $V_3 = 3$, $V_4 = 3$ → sum = 1~~3~~4

$V_5 = 1$, $V_6 = 0$

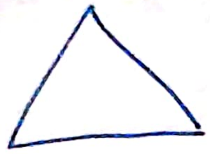
Order = 7

$\boxed{2 \times \text{order} = \text{Sum of degrees}}$

Parallel edges $e_2 \parallel$ e~~4~~1

Pendant edge e_6

Pendant vertex V_5



complete & regular graph

HANDSHAKING THEOREM

The sum of the degrees of the vertices of an undirected graph G is twice the no. of edges in G

Every edge contributes 2 degrees to each of the n vertices

PROOF

Let G be a graph having e edges, in a graph each edge is incident with exactly two vertices. Therefore every edge contributes two degrees to the n vertices. Therefore sum of the degrees of the vertices is equal to 2 x no. of edges

10.2.23

(3M) Prove that in a complete graph of order N the no. of edges is equal to $\frac{N(N-1)}{2}$

PROOF: In a complete graph of order N each vertex has degree N-1 ∴ sum of the degrees of the vertices is equal to N(N-1). By Hand shaking theorem

$$N(N-1) = 2 \times \text{no. of edges } (e)$$

$$\therefore e = \frac{N(N-1)}{2}$$

29 Prove that in an r regular graph of order N there are $\frac{Nr}{2}$ edges

Each vertex has degree r, there are n such vertices. So total no. i.e. sum of degrees is Nr, By hand shaking theorem

$Nr = 2e \quad \therefore \boxed{e = \frac{Nr}{2}}$

NOTE

For a simple graph of order N the maximum possible degree of a vertex is N-1

REASON: The max edges are possible only when simple graph is a complete graph

For a complete graph of order N the degree of a vertex is N-1

THEOREM: The no. of vertices of odd degree in an undirected graph is always even

PROOF: Let G be an undirected graph of order 'n' and let 'e' be the no. of edges in G

By Handshaking Theorem

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$\Rightarrow \sum_{j=1}^{m} d(v_j) + \sum_{k=1}^{p} d(v_k) = 2e \quad (1)$$ where m is no. of odd degree vertices, n → no. of even degree vertices

RHS of eq (1) is even and second term of LHS is even (∵ sum of even degree vertices)

∴ ~~sum~~ First term of eqn (1) must be even

ie First term is sum of odd nos

Sum of odd nos is even only when no. of

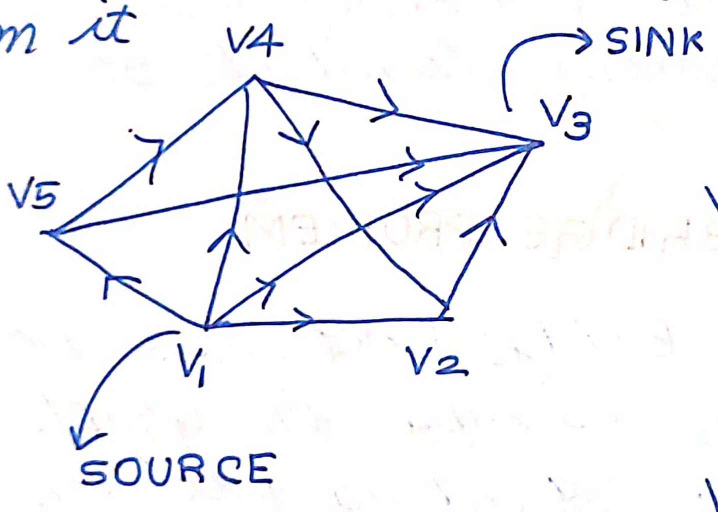
odd numbers is even.

Hence the number of odd degree vertices in an undirected graph is even.

INDEGREE AND OUTDEGREE

The indegree of a vertex v in a directed graph is the no. of edges ending in it

The outdegree of a vertex v in a directed graph is the no. of edges ~~ending~~ beginning from it



$V_1 \rightarrow$ indegree 0 (source
outdegree 4

$V_2 \rightarrow$ Indegree 2
Outdegree 1

$V_3 \rightarrow$ Indegree 4 (sink
~~V~~ outdegree 0

V4 Indegree 2
~~V~~ Outdegree 2

V5 Indegree 1
Outdegree 2

SOURCE

A vertex with indegree 0 is called source

SINK

A vertex with outdegree 0 is called sink

ASSIGNMENT

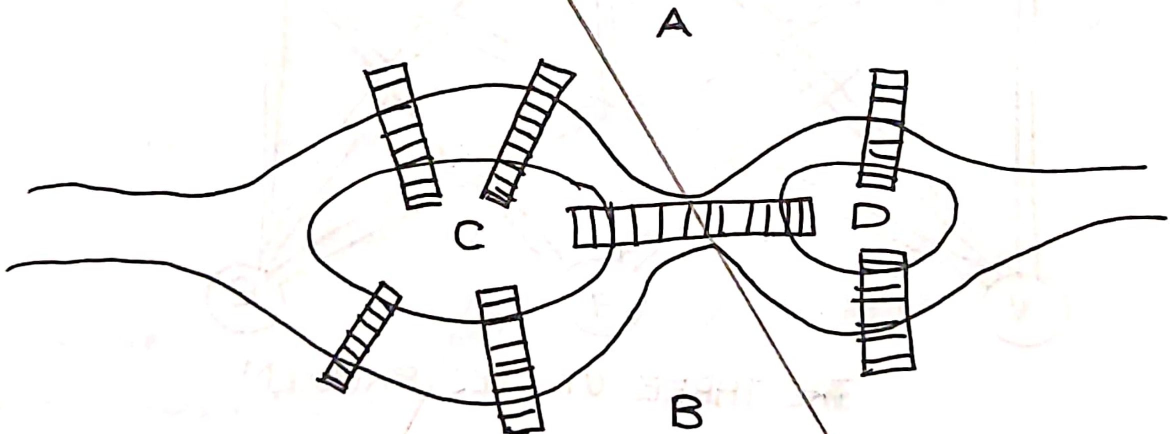
APPLICATION OF GRAPHS

Because of its inherent simplicity, graph theory has a very wide range of applications in engineering in physical and biological sciences in linguistics and in numerous other areas. A graph can be used to represent almost any physical situation involving discrete objects and a relationship among them. The following are four examples from among hundreds of such applications.

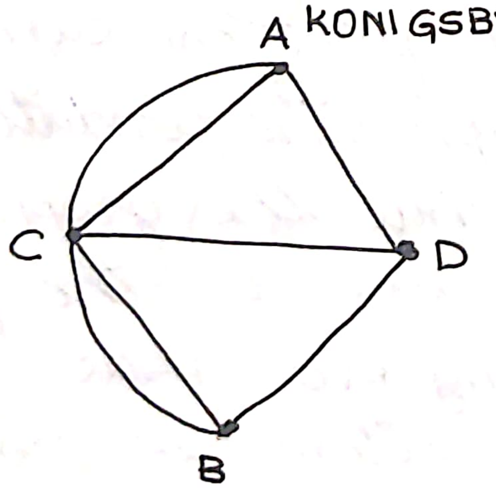
KONIGSBERG BRIDGE PROBLEM

The Konigsberg Bridge Problem is perhaps the best known example in graph theory. It was a long standing problem until solved by Leonhard Euler (1707-1783) in 1736 by means of a graph. Euler wrote the first paper ever in graph theory and thus became the originator of the theory of graphs as well as of the rest of topology. The problem is depicted in Fig 1-4. Two islands C and D formed by the Pregel river in Konigsberg (then the capital of East Prussia but now renamed Kaliningrad and in West Soviet Russia) were connected to each other and to the banks A and B with seven bridges as shown in Fig. The problem was to start at any of the four land areas of the city, A, B, C or D walk over each of the seven bridges exactly once and return to the starting point (without swimming across the river, of course).

Euler represented this situation by means of a graph as shown in Fig. 1-5. The vertices represent the land areas and the edges represent the bridges. As we shall see in chapter, Euler proved that a solution for this problem does not exist



KONIGSBERG BRIDGE PROBLEM



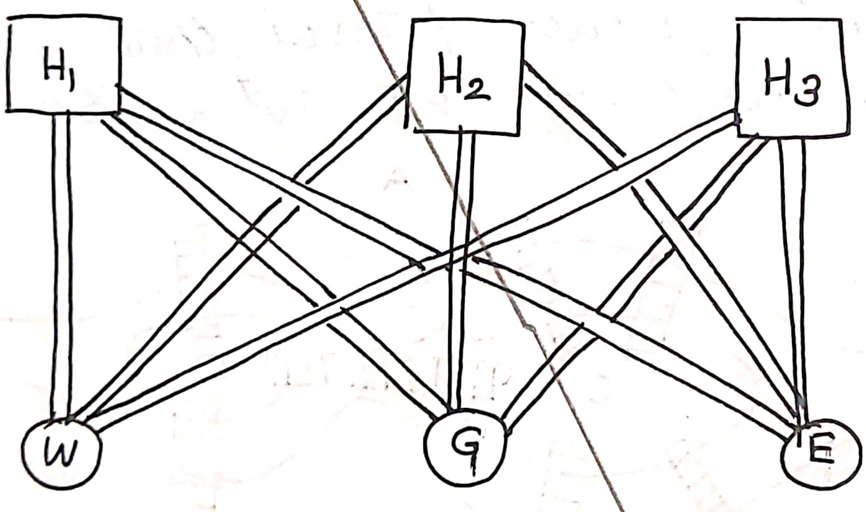
GRAPH OF KONIGSBERGS BRIDGE PROBLEM

The Konigsberg bridge problem is the same as the problem of drawing figures without lifting the pen from the paper

and without retracing a line (~~pro~~.
we have all been confronted with such problems at one time or another.

UTILITIES PROBLEM

There are three houses H_1, H_2 and H_3 each to be connected to each of the three utilities -water (W), gas (G) and electricity (E) -by means of conduits. Is it possible to make such connections without any cross overs of the conduits?



~~THE~~ THREE UTILITIES PROBLEM

Fig 2 shows how this problem can be represented by a graph -the conduits are shown as edges while the graph in Fig 2 cannot be drawn in the plane without edges crossing over. Thus the answer to the problem is no

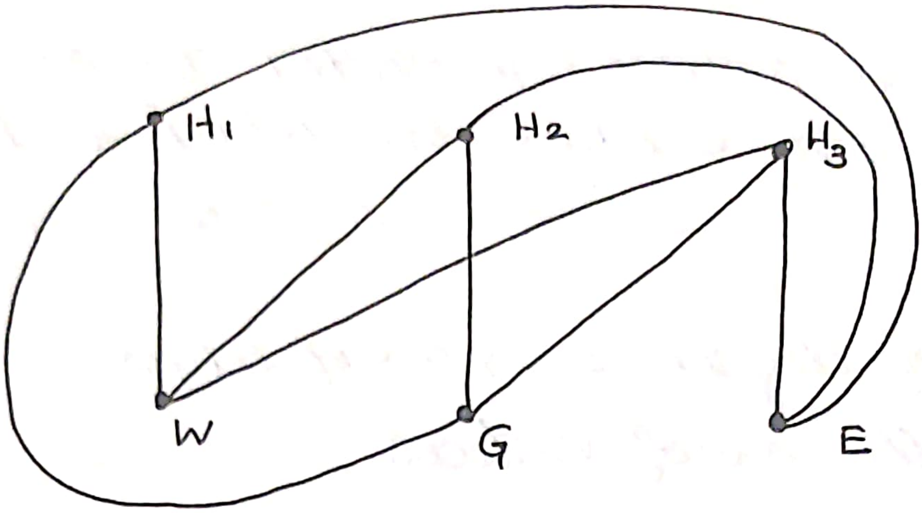


FIG: GRAPH OF THREE UTILITES PROBLEM

13.2.23

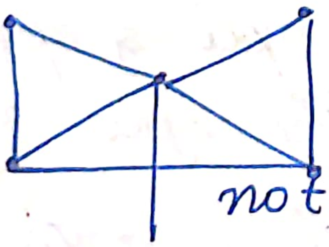
Determine the no. of edges in a graph with six vertices, two of degree 4 and 4 of degr 2. Draw two such graphs

$$\Sigma d(v_i) = 4\times 2 + 2\times 4 = 8+8 = 16$$

By handshaking theorem, sum of the degree of vertices = 2x no of edges

$\therefore$ No. of edges = 8

Drawing 2 graph with no. of edges = 8 and no of vertices = 6



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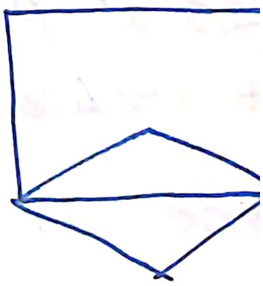
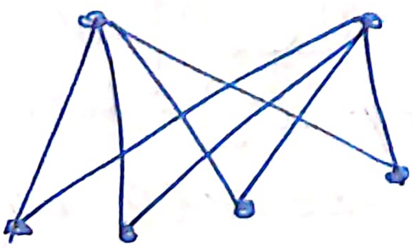


not possible



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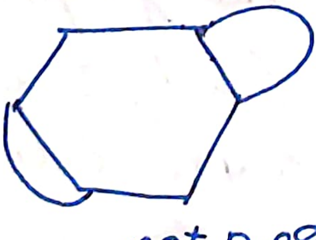


not possible

How many vertices are needed to construct a graph with 6 edges in which each of degree 2

sum of degree = 2 x no. of edges

Let p be the no of vertices

$$\Sigma d(v_i) = 2e$$

$$2 \times p = 2 \times 6 \quad \therefore p = 6$$

Is it possible to construct a graph with 12 vertices such that two of the vertices have degree 3 and the remaining vertices have degree four.

$$2 \times 3 + 10 \times 4 = 2e \Rightarrow e = \frac{46}{2} = 23$$

Here the sum of the degrees of vertices ie No. of edges is a whole number. Therefore we can draw such a graph

NOTE:- If sum is odd → no. of edges decimal

Is it possible to draw a simple graph with 4 vertices and 7 edges, justify your answer

Ans: In a simple graph with n vertices the maximum no. of edges will be $\frac{n(n-1)}{2}$ [when the simple graph becomes complete]. Here $n = 4$. Therefore the maximum possible edges is equal to 6, but here we have 7 edges

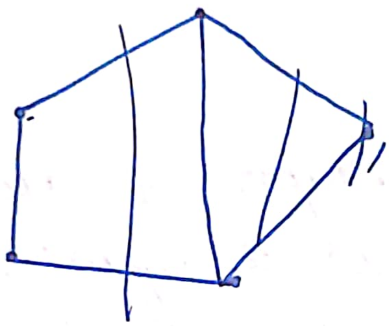
∴ we cannot draw a simple graph with 7 edges construct two graphs having the same degree sequence

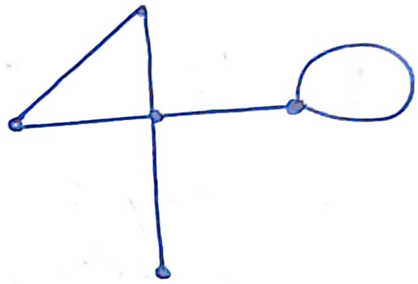
REM (no. of odd degree vertices should always be even)

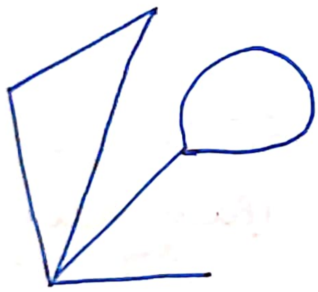
~~V_1 V_2 V_3 V_4~~

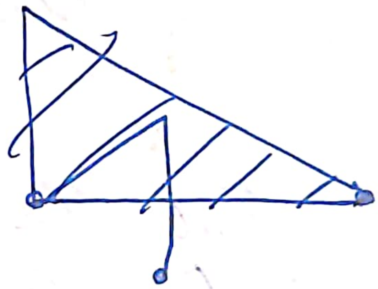
$\{2, 3, 4, 1, 2\}$ no. of edges $= 6$

5 vertices









Show that there exists no simple graph corresponding to the following degree sequen

1. $\{0, 2, 2, 3, 4\}$
2. $\{1, 1, 2, 3\}$
3. $\{2, 2, 3, 4, 5, 5\}$
4. $\{2, 2, 4, 6\}$

1. $\{0, 2, 2, 3, 4\}$

REASON: The no. of odd degree vertices is one 1, in a graph no. of odd degree vertices are always even.
Therefore there exists no ~~/~~ graph with given degree sequence

$\{\cancel{1}, \cancel{1}, \cancel{2}, \cancel{3}\}$ 2, 2, 4, 6 $\quad$ 7 edges $\quad$ 4 vertices, simple graph

No. of edges = $\frac{7}{}$

In a simple graph of order 4 the maximum possible edges is equal to $\frac{n(n-1)}{2} = \frac{4(4-1)}{2} = 6$ here no. of edges = 7

Show that the sequence 6, 3, 5, 2, 2 not represent a simple graph (Each vertex max deg $n-1 = 4$ here 6 ∴ selfloops)

Show that no simple graph has all degree of vertices distinct

eg $\{0, 1, 2, 3\}$ $\quad$ 1, 2, 3, 4 → not possible → max $n-1$

PROOF: Let the order of the graph be n the maximum possible degree of a vertex in a simple graph is $n-1$ ∴ the possible values of the degrees of the vertices are 0, 1, 2, 3, 4 up: $n-1$

Since the maximum degree
Since 1 vertex has degree $n-1$ it should b

adjacent to every other vertex of G. Therefore the minimum degree of the vertex is 1

$\therefore$ 0 deg vertex is not possible

$\therefore$ all other vertices ~~have~~ cannot have distinct degrees.

SUBGRAPH

A graph. $H=(V_1, e_1)$ is a subgraph of a graph $G=(V,E)$ if $V_1 \subset E$ and e_1 is a subset of E and each ~~b~~ edge of H has some end vertices as in G

PROPER SUBGRAPH

A subgraph $H(V_1, e_1)$ is a proper subgraph of $G(V,E)$ if either $V_1 \subset V$, $E_1 \subset E$ $V_1 \subset V$ or $E_1 \subset E$ or both

Every graph is its own subgraph, a single vertex in a graph G is a subgraph of G

A single edge in G together with its end vertices is also a subgraph of G

Null graph of a graph is also a subgraph

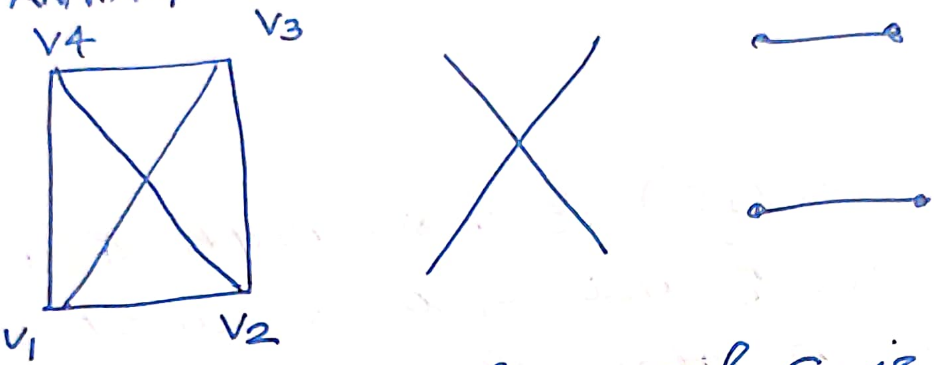
EDGE DISJOINT SUBGRAPHS

Two subgraphs g_1, g_2 of a graph G are said to be edge disjoint if g_1 and g_2 do not have any edges in common

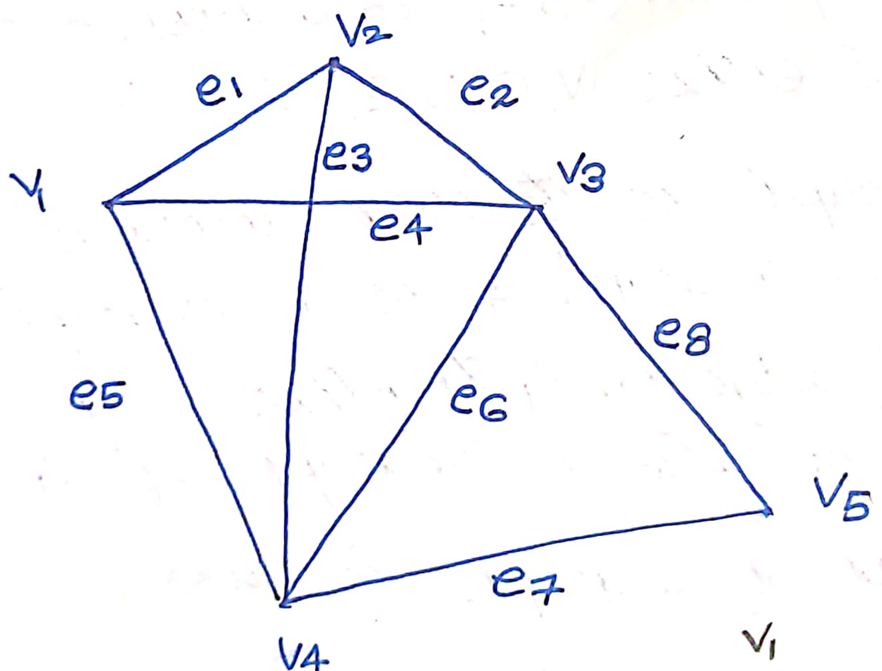
VERTEX DISJOINT SUBGRAPHS

Two subgraphs are vertex disjoint if they have no common vertex

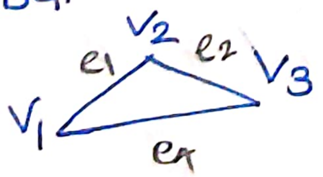
SPANNING

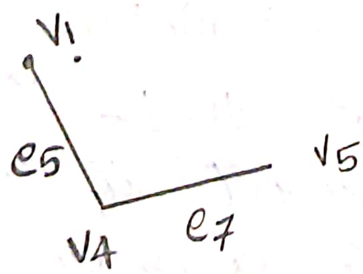


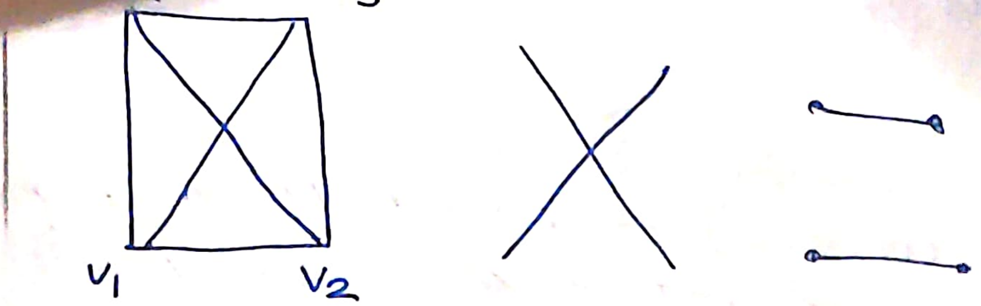
A subgraph h of a graph G is said to be a spanning subgraph if h contains all vertices of G



SUBGRAPH

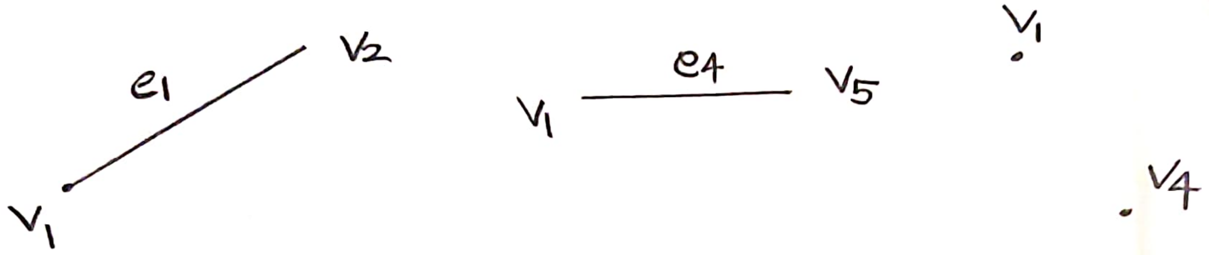




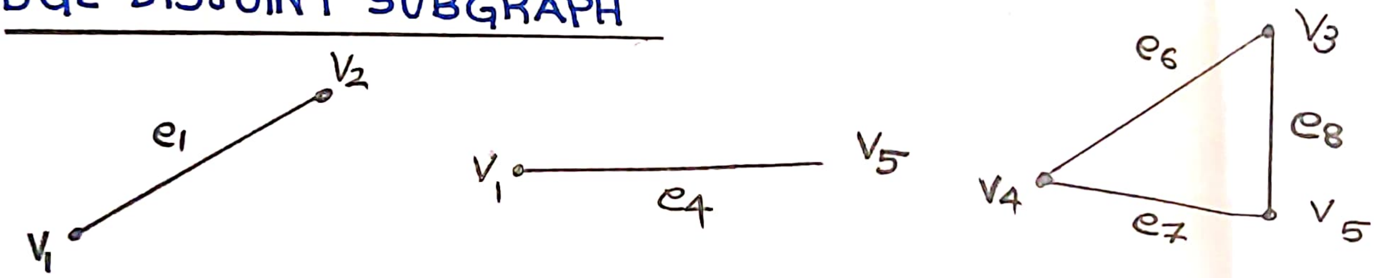


A sub graph h of a graph G is said t
a spanning subgraph if h contains
vertices of G

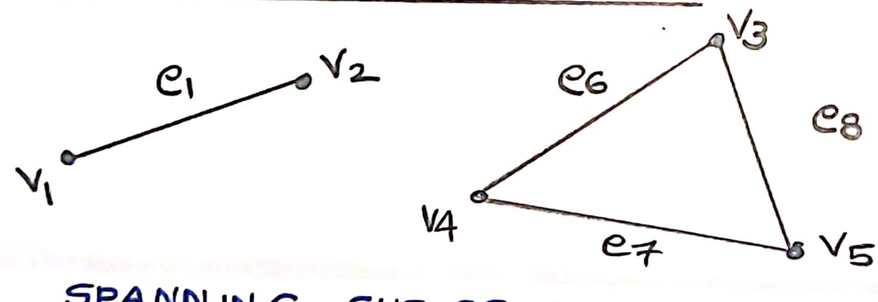
PROPER SUBGRAPH



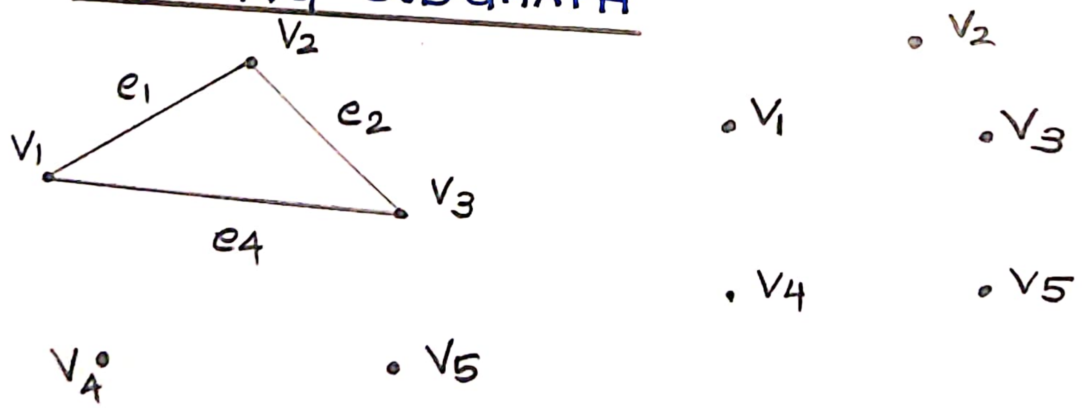
EDGE DISJOINT SUBGRAPH



VERTEX DISJOINT SUBGRAPH

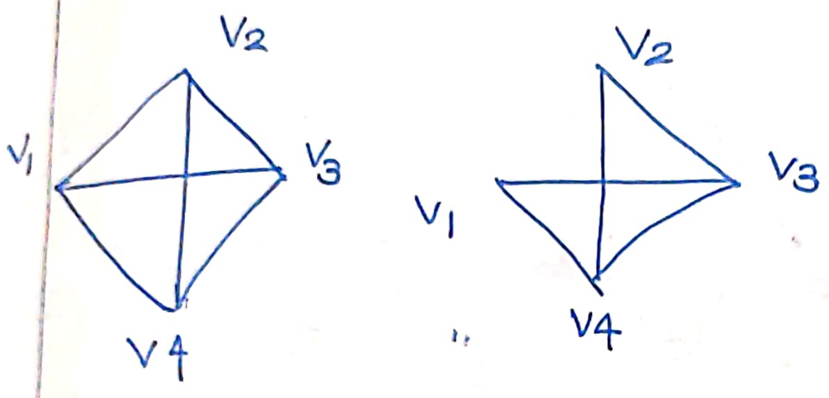


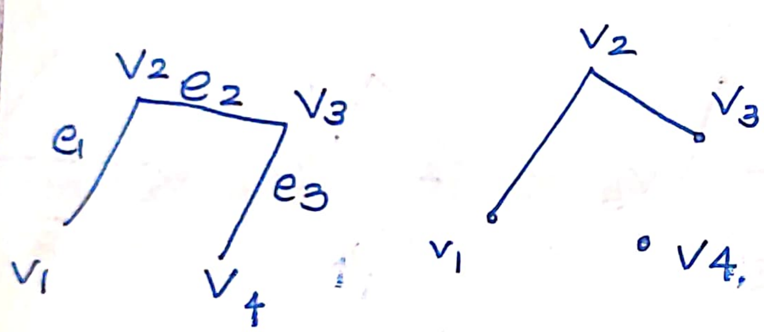
SPANNING SUBGRAPH



REMOVAL OF EDGES

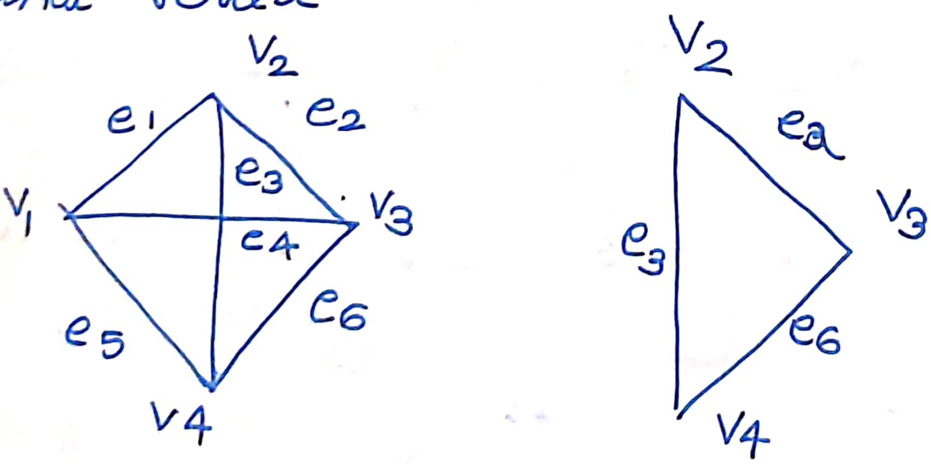
When we remove an edge from the graph we remove only the edge not the end vertices





REMOVAL OF VERTICES

When we remove a vertex from a graph we remove the vertex and all edges incident on that vertex

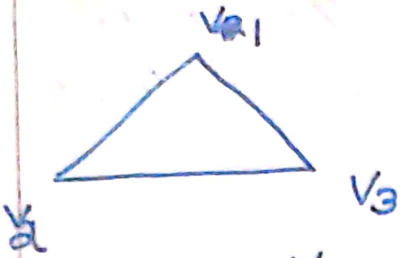


RESULT

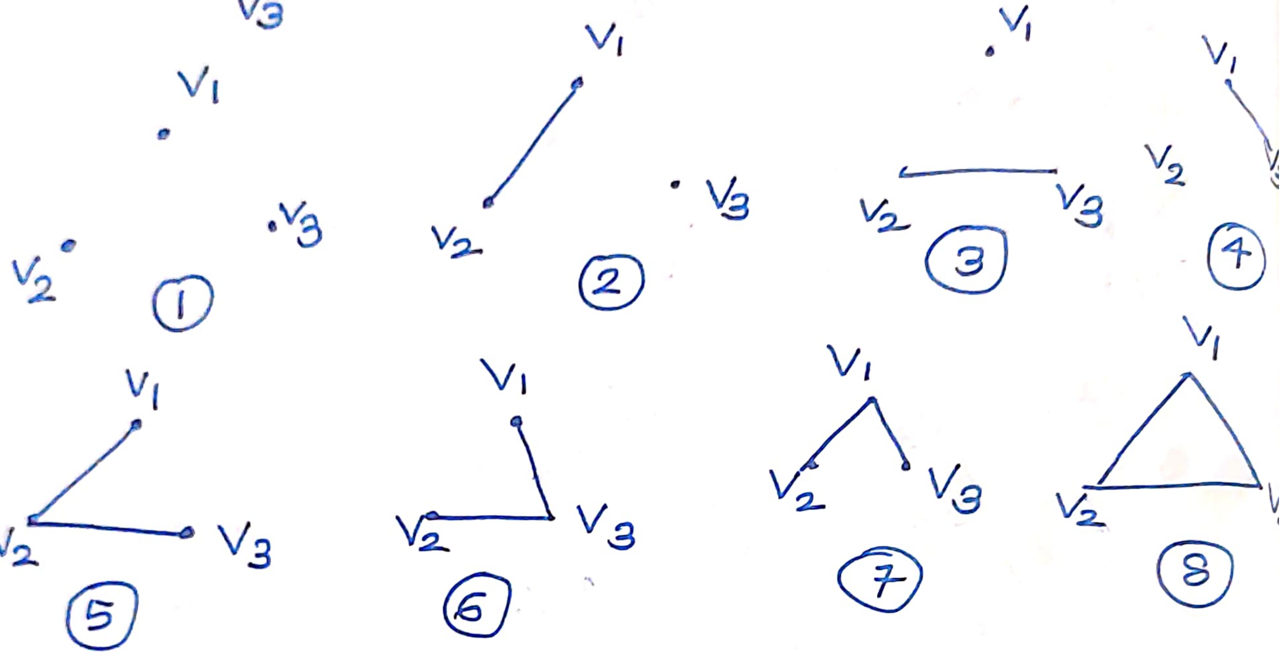
The total no. of subgraphs of a graph having m vertices and n edges is equal to $(2^m - 1)2^n$

The number of spanning subgraphs is equal to 2^n

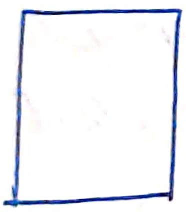
Find all spanning subgraphs of the graph

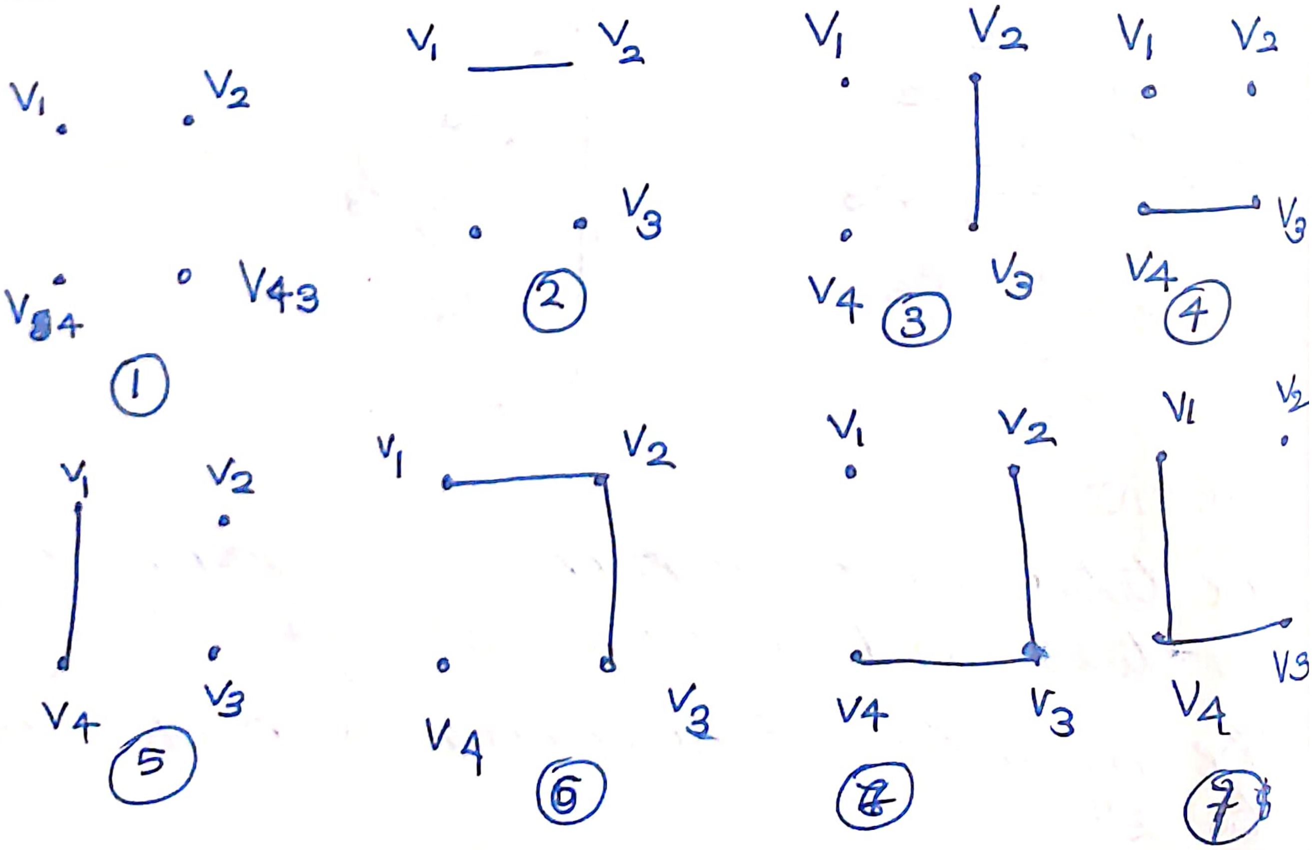


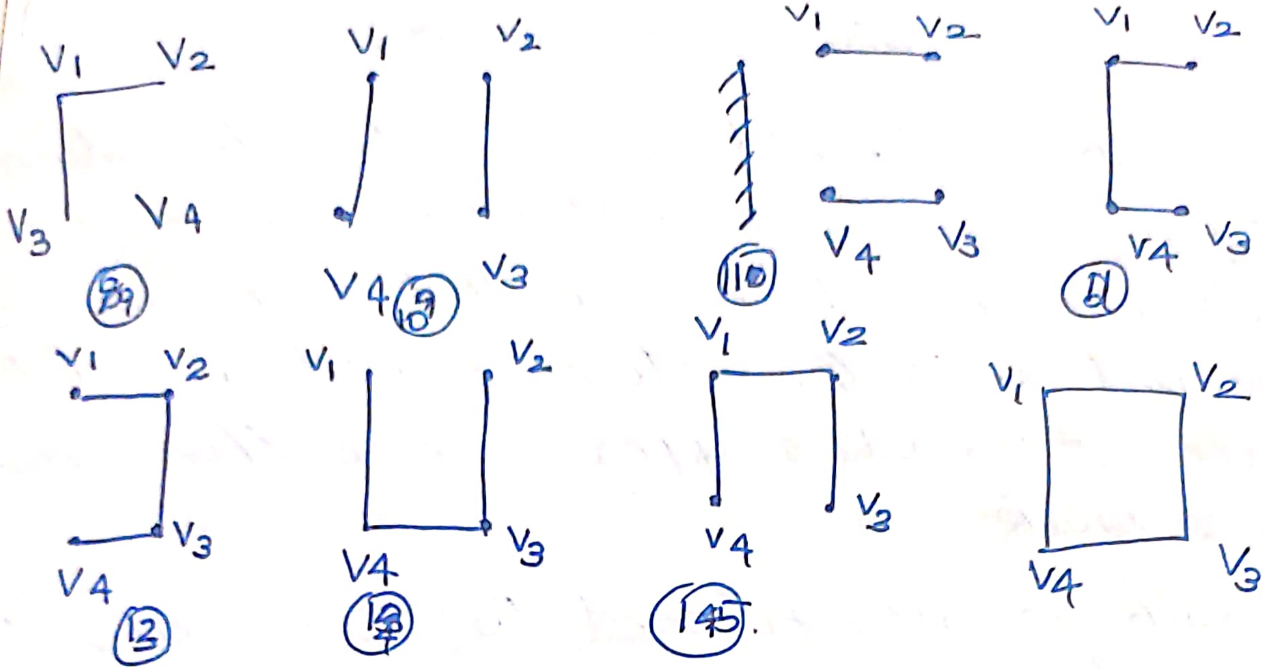
$2^3 = 8$ spanning subgraphs



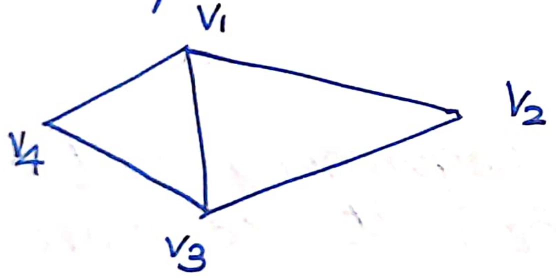
Draw all spanning subgraph







How many sub graphs and spanning subgraphs are possible for the graph below



$m = 4$

$n\ 5$

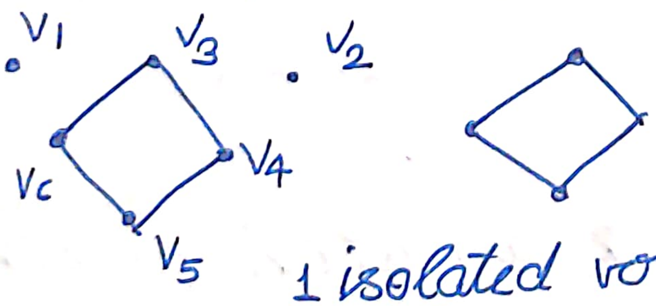
$(2^4 - 1) \times 2^5$

$= 32 \times 15$

$sp \rightarrow 2^5$

Does there exist a proper subgraph of G whose size is equal to size of the graph

Yes it is possible, when a graph has isolated vertex



1 isolated vertex

WALKS PATHS CIRCUITS

A walk is defined as a finite alternating sequences of vertices and edges, beginning and ending with vertices such that each edge is incident with the vertices preceding and following it. No edges appears more than once in a walk.

A walk is also referred to as an edge train or a chain

Vertices with which a walk begins and ends are called its terminal vertices.

OPEN WALK

In a walk where the beginning and end vertices are different then it is called open walk

CLOSED WALK

In a walk where the beginning and end vertices are the same it is called closed walk

PATH

Path is an open walk in which no edges and vertices are repeating

CIRCUIT

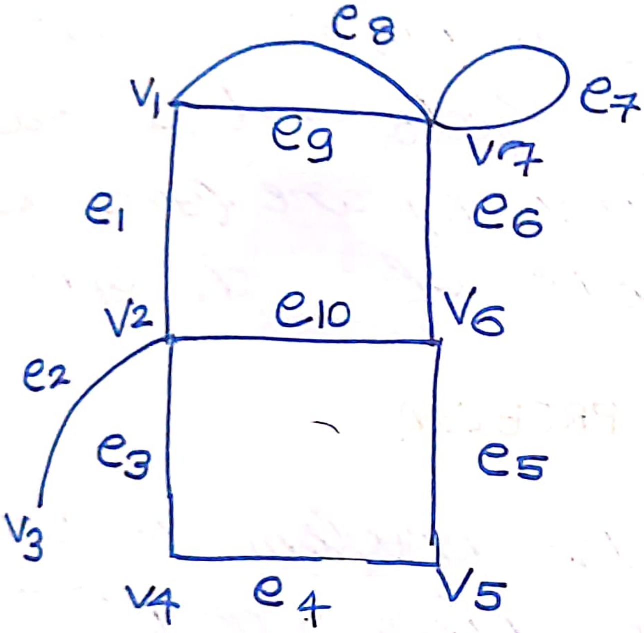
A circuit is a closed walk in which no edges and vertices are repeating

The ... length of a path

An edge which is not a self loop is a path of length 1

The terminal vertices of a path are of degree 1 and the rest of the vertices are of degree 2

A circuit is also called cycle



2 open walk

$V_1 e_9 V_7 e_6 V_6 e_5 V_5$

$V_1 e_1 V_2 e_{10} V_6 e_6 V_7 e_7$

2 closed walk

$V_1 e_9 V_7 e_8 V_1$

$V_6 e_6 V_7 e_7 V_7 e_9 V_1 e_1 V_2 e_{10} V_6$

path

$V_3 e_2 V_2 e_{10} V_6$

$V_6 e_5 V_5 e_4 V_4$

circuit $V_1 e_9 V_7 e_6 V_6 e_{10} V_2 e_1 V_1$

27.02.2023
MONDAY

ASSIGNMENT - I

APPLICATIONS OF GRAPHS

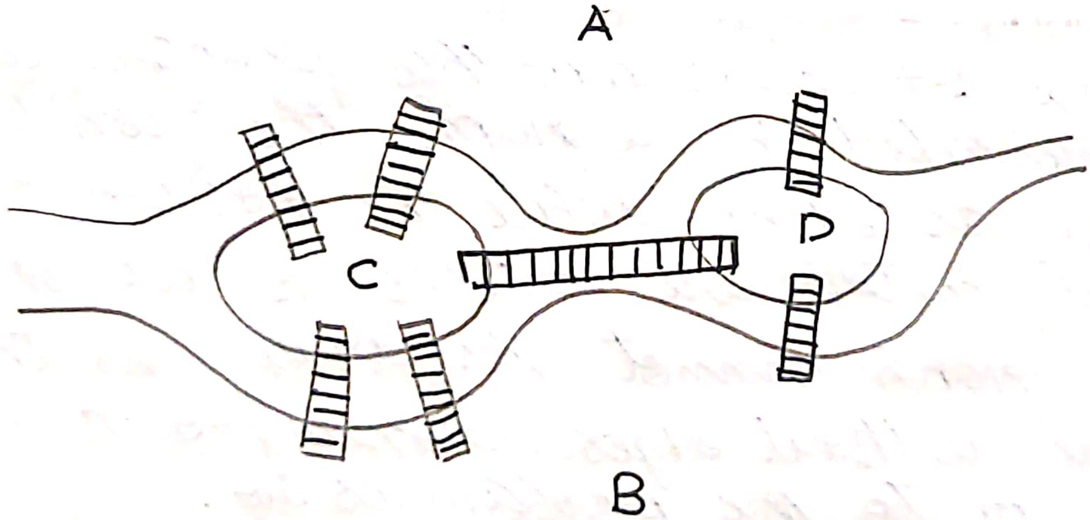
Because of its inherent simplicity graph theory has a very wide range of applications in engineering in physical, social and biological sciences in linguistics and in numerous other areas. A graph can be used to represent almost any physical situation involving discrete objects and a relationship among them. The following are four examples from among hundreds of such applications.

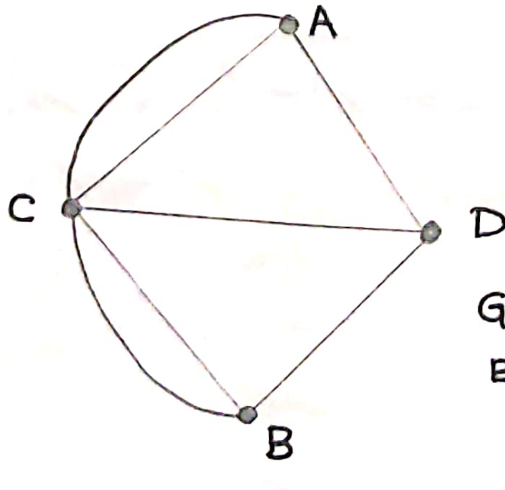
KONIGSBERG BRIDGE PROBLEM

The königsberg bridge problem is perhaps the best-known example in graph theory. It was a long-standing problem until solved by Leonhard Euler (1707 - 1783) in 1736 by means of a graph. Euler wrote the first paper ever in graph theory and thus became the originator of the theory of graphs as well as of the rest of topology. The problem is depicted in Fig 1-4.

Two islands c and D formed by the Pregel River in Konigsberg (then the capital of East Prussia but now renamed Kaliningrad and in West Soviet Russia) were connected to each other and to the banks A and B with seven bridges as shown in figure. The problem was to start at any of the four land areas of the city A, B, C, or D walk over each of the seven bridges exactly once and return to the starting point (without swimming across the river of course).

Euler represented this situation by means of a graph as shown in Fig. The vertices represent the land areas and the edges represent the bridges. As we shall see in chapter Euler proved that a solution for this problem does not exist.





GRAPH OF KONIGSBERG BRIDGE PROBLEM

The Konigsberg bridge problem is the same as the problem of drawing figures without lifting the pen from the paper and without retracing a line. We all have been confronted with such problems at one time or another

UTILITIES PROBLEM

There are three houses H_1, H_2 and H_3 each to be connected to each of the three utilities water (W), gas (G) and electricity (E) by means of conduits. Is it possible to make such connections without any crossovers by the conduits?

Figure 1-7 shows how this problem can be represented by a graph the conduits are shown as edges while the houses and utility supply centers are vertices. As we shall see the graph cannot be drawn in the plane without edges crossing over. Thus the answer to the problem is no.

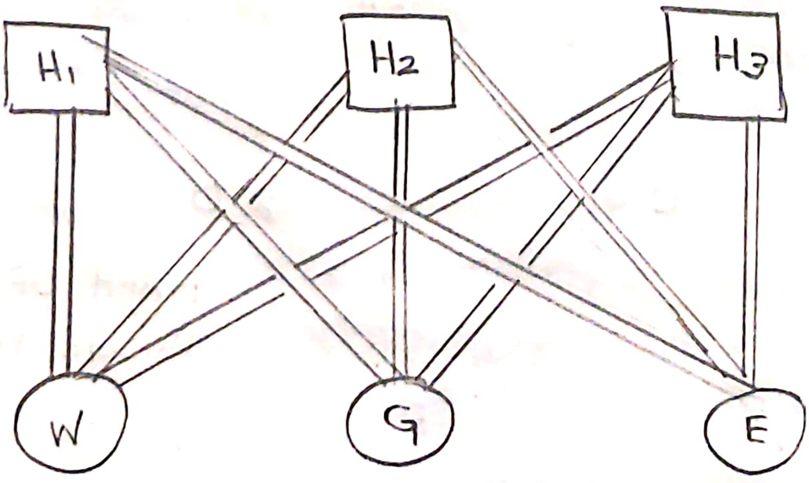


FIG: THREE UTILITIES PROBLEM

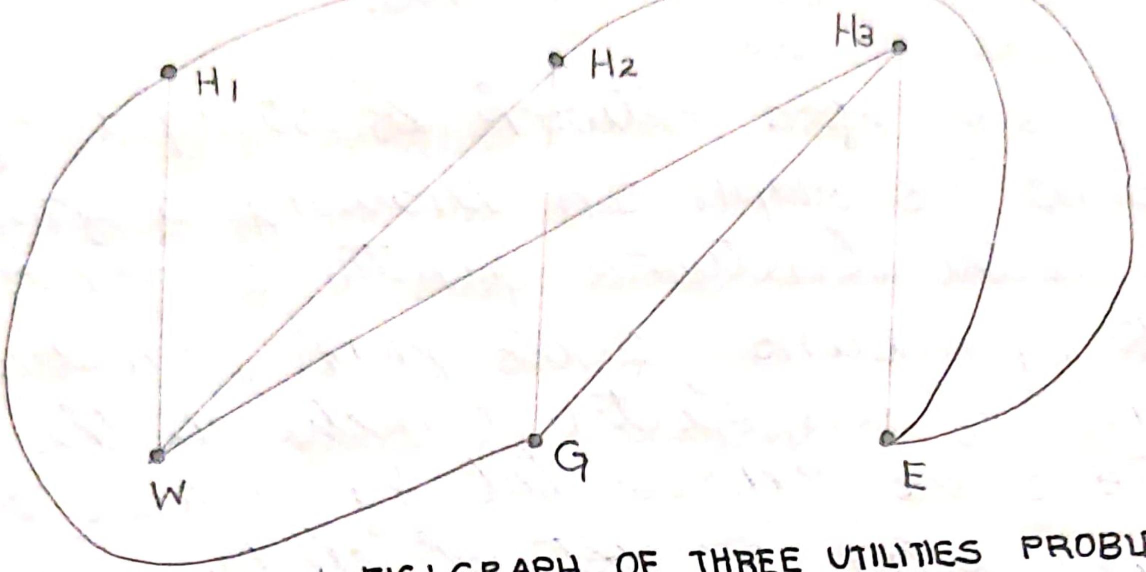


FIG: GRAPH OF THREE UTILITIES PROBLEM

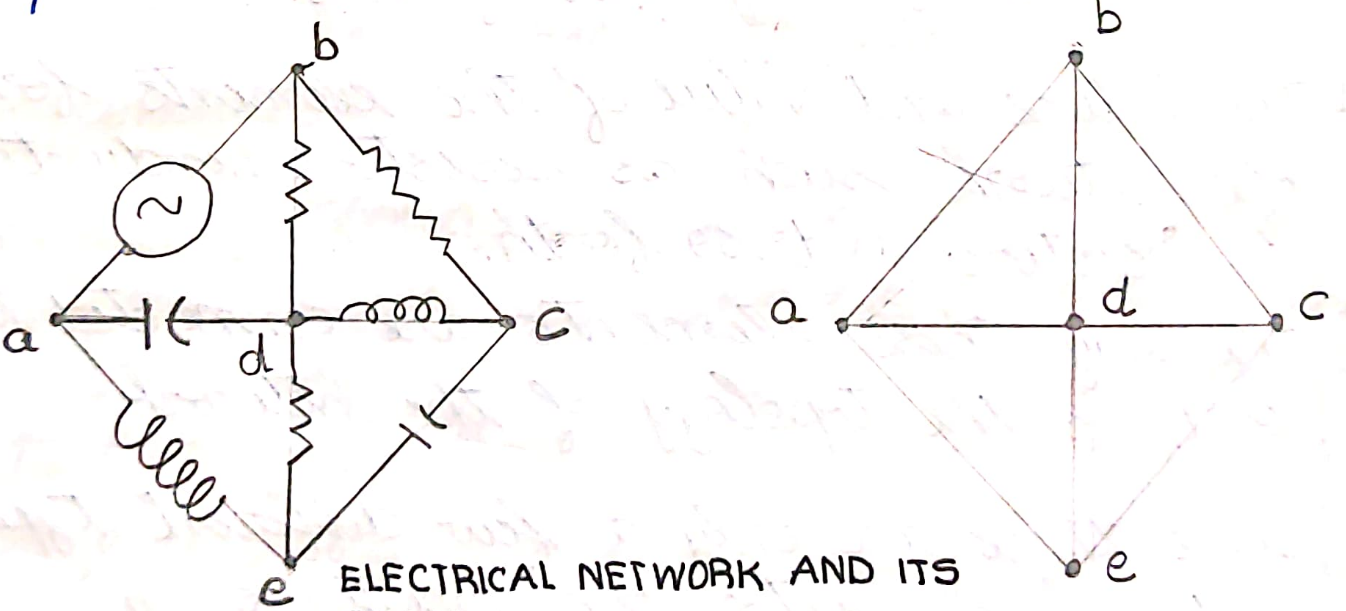
III ELECTRICAL NETWORK PROBLEMS

Properties (such as transfer function and input impedance) of an electrical network are functions of only two factors:

1. The nature and value of the elements forming the network such as resistors, inductors, transistors and so forth.
2. The way these elements are connected together that is the topology of the network.

Since there are only a few different types of electrical elements the variations in network are chiefly due to the variations in topology. Thus electrical network analysis and synthesis are mainly the study of network topology. In the topological study of electrical networks, factor 2 is separated from 1 and is studied independently. The advantage of this approach will be clearer in in chapter , a chapter devoted to solely to applying graph theory to electrical networks.

The topology of a network is studied by means of its graph. In drawing a graph of an electrical network junctions are represented by means of its graph. In drawing a graph of an electrical network the junctions are represented by vertices and branches which consist of electrical elements) are represented by edges, regardless of the nature and size of the electrical elements. An electrical network and its graph are shown in Figure.



ELECTRICAL NETWORK AND ITS GRAPH

IV SEATING PROBLEM

Nine members of a new club meet each day for lunch at a round table. They decide to sit such that every member has different neighbours at each lunch. How many days can this arrangement

last?
This situation can be represented by a graph with nine vertices such that each vertex represents a member and an edge joining two vertices represents the relationship of sitting next to each other. Figure shows two possible seating arrangements these are 1 2 3 4 5 6 7 8 9 (solid lines) and 1 3 5 2 7 4 9 6 8 (dashed lines)

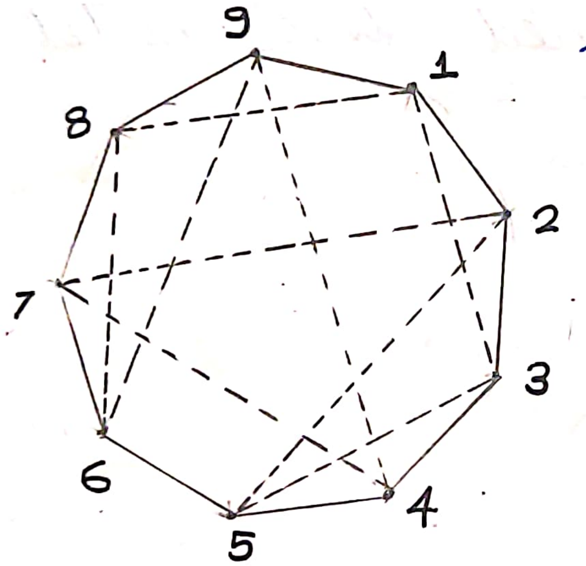


FIG: ARRANGEMENTS IN A DINNER TABLE

It can be shown by graph theoretic considerations that there are only two more arrangements possible They are 1 5 7 3 9 2 8 4 6 1 and 1 7 9 5 8 3 6 2 4 1. In general it can be shown that for n people the number of such possible arrangements is

$$\frac{n-1}{2} \text{ if } n \text{ is odd}$$

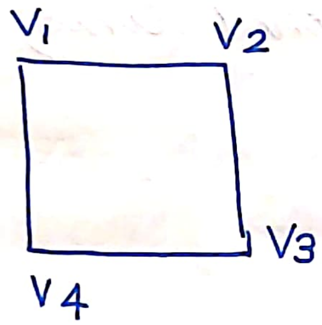
and $$\frac{n-2}{2} \text{ if } n \text{ is even}$$

It is noticed that three of the four examples are puzzles and not engineering problems. This was done to avoid introducing at this stage background material not pertinent to graph theory. More substantive applications will follow particularly in the last four chapters.

1.3.23

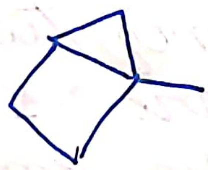
CONNECTED GRAPH

A graph is connected if there exists a path b/w ~~any~~ every pair of vertices in graph G otherwise it is called disconnected graph.



COMPONENT

The connected subgraph of a disconnected graph is called component of graph G.



STATEMENT 1

If a graph (connected/disconnected has exactly 2 vertices of odd degree there must be a path joining these 2 vertices

Pf: Let G be a connected graph

If the graph is connected there must exist a path ~~conn~~ joining every pair of vertices in G. Hence there must be a path joining odd degree vertices.

suppose the graph G is disconnected with exactly two odd degree vertices.

Then by previous theorem the odd degree vertices must lie in the same component of G, because each component is a subgraph. Therefore no. of odd degree vertices must be even.

Since the odd degree vertices lie in the same component there must be a path connecting these 2 vertices.

TH2

A graph G is disconnected iff and only if its vertex at V can be partitioned into 2 non empty disjoint subsets V1 and V2 such that there exists no edge whose one end vertex is in subset V_1 and the other in V_2

Suppose that the vertex at V can be partitioned into 2 non empty disjoint subsets V1 and V2 such that there exists no edge V whose one end vertex is in subset V1 and V2.

consider two arbitrary vertices a and b of G such that $a \in V_1$ and $b \in V_2$. No path can exist b/w vertices a and b otherwise there would be at least one edge whose one end end at V_1 and other at V_2. Hence, if a partition exists G is disconnected.

conversely let G be a dis connected graph. consider a vertex a in G. let V_1 be the set of all vertices that are joined by paths to a $\because$ G is disconnected V_1 does not include all the vertices of G. The remaining vertices will form a set V_2. No vertex in V_1 is joined to any in V_2 by an edge. Hence a partition exists.

THEOREM 3.

IMP

A simple graph with n vertices and k components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges

Suppose that the graph G is a simple connected graph with n vertices. since the graph is connected it has only one component ie $k=1$, a simple graph has maximum edges when it is complete. Therefore max possible edges $\frac{(n-1)n}{2}$ when we put $k=1$ $\frac{(n-k)(n-k+1)}{2}$ we get $\frac{n(n-1)}{2}$

Let the graph G be simple, disconnected and has k components and n vertices

Let n_i denote the number of vertices in the ith component $\therefore n_1+n_2+\ldots+n_k=n$

A component with n_i vertices will have the maximum possible no. of edges when it is complete. Hence the maximum possible edges in the ith component is $\frac{n_i(n_i-1)}{2}$

Therefore maximum no of edges in G =

$$\sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \sum_{i=1}^{k} \frac{n_i^2 - n_i}{2} = \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i$$

We use the algebraic inequality

$$\boxed{\sum_{i=1}^{k} n_i^2 \leq n^2 - (k-1)(2n-k)}$$

$$\leq \frac{1}{2}\left(n^2 - (k-1)(2n-k) - n\right)$$

$$\leq \frac{1}{2}\left(n^2 - 2nk + k^2 + 2n - k - n\right)$$

$$\leq \frac{1}{2}\left((n-k)^2 + n - k\right)$$

$$\leq \frac{1}{2}\left((n-k)(n-k+1)\right)$$

PROOF: $$\sum_{i=1}^{k} (n_i - 1) = (n-k)^2$$

$$\sum_{i=1}^{k} n_i^2 - 2n_i + k = n^2 - 2nk + k^2$$

+ some +ve terms

$$\sum_{i=1}^{k} n_i^2 - 2n + k \leq n^2 - 2nk + k^2$$

$$\sum_{i=1}^{k} n_i^2 \le n^2 - 2nk + k^2 + 2n - k$$

$$= n^2 - (k-1)(2n-k)$$

PROVE

Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Consider a simple graph of n vertices v_1, v_2 ... ~~chose~~ v_{n-1} ...

choose $n-1$ vertices $v_1, v_2 \ldots v_{n-1}$ maximum no. of edges can be drawn b/w these vertices

$$(n-1)C_2 = \frac{(n-1)(n-2)}{2}$$

since the graph is connected. There must exist at least one edge b/w nth vertex v_n and any one of the vertices $v_1 v_2 \ldots v_{n-1}$

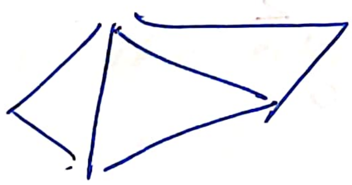
Therefore there must exist more than $\frac{(n-1)(n-2)}{2}$ edges.

PROVE

Prove that a connected graph G remains connected after removing an edge E from G ~~if E~~ iff only if ~~then~~ E lies in some circuits of G.

If an edge e lies in a circuit of G then b/w the end vertices of e there exists a path, hence after the removal of such an edge e from the graph, the graph remains connected

conversely if e does not lie in any circuit of G, then the removal of e disconnect the end vertices of e, $\therefore$ ~~G~~ G becomes disconnected



THEOREM

Let G be a disconnected graph with n vertices where n is even, if G has two components each of which is complete prove that G has a minimum $\frac{n(n-2)}{4}$ edges.

Let G be a disconnected graph with n vertices n is even.

G has two components and the components are complete, let x be the no. of vertices in one component then the other component has $n-x$ vertices, then the total no. of vertices in graph G

$\frac{n(n-1)}{2}$ each component is complete

$$m = \frac{x(x-1)}{2} + \frac{(n-x)(n-x-1)}{2}$$

$$= \frac{1}{2}\left[x^2 - \cancel{x} + n^2 - nx - n - nx + x^2 + \cancel{x}\right]$$

$$= \frac{1}{2}\left[2x^2 - 2nx + n^2 - n\right]$$

$$= x^2 - nx + (n^2 - n)/2$$

To find min value of m

$$m' = 2x - n$$

$$m' = 0 \Rightarrow x = \frac{n}{2}$$

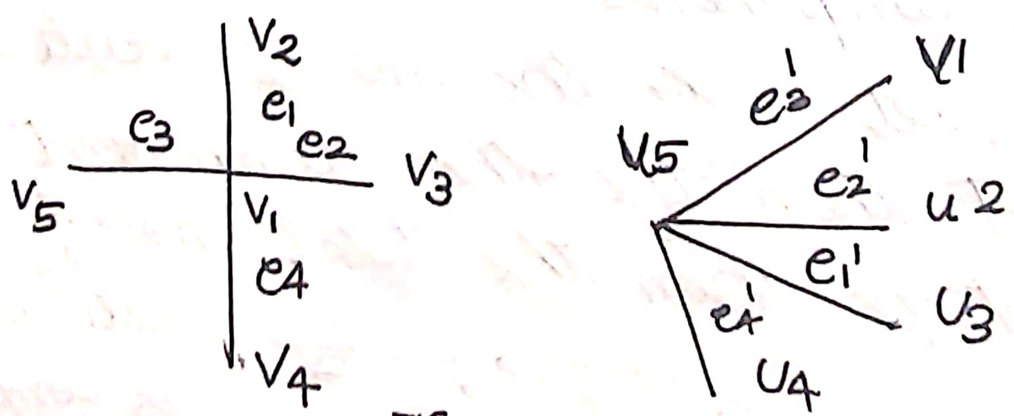
$$m(n/2) = 2 > 0$$

$\therefore$ $x = \frac{n}{2}$ is the minimum

$$\therefore m = \frac{\frac{n}{2}\left(\frac{n}{2} - 1\right)}{2} + \frac{\left(n - \frac{n}{2}\right)\left(n - \frac{n}{2} - 1\right)}{2}$$

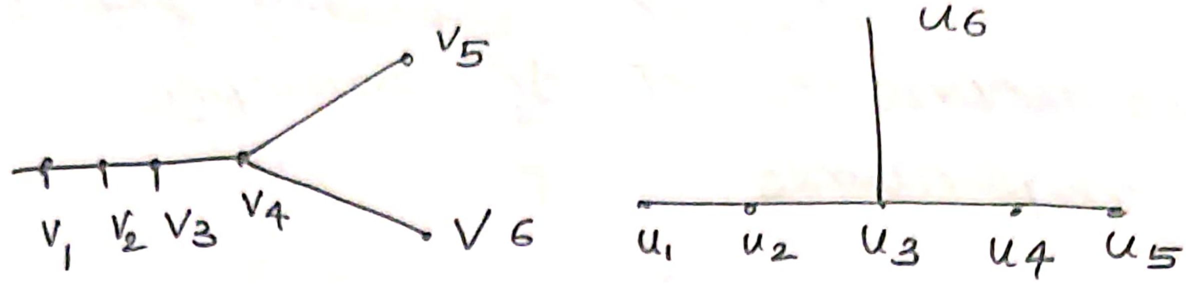
$$= \cancel{\frac{n(n-2)}{}} \quad \frac{n^2}{4} - \frac{n}{2} = \frac{n^2 - 2n}{4} = \frac{n(n-2)}{4}$$

IP GRAPH ISOMOR~~S~~PHI~~R~~SM



$V_1 \to u_5$
$V_2 \to u_3$
$V_3 \to u_2$
$V_4 \to u_4$
$V_5 \to u_1$

FIG: 1
ISOMORPHIC



Two graphs G_1 G_2 are said to be isomorphic if there is a one-one correspondence b/w vertices and edges such that incidence relationship is preserved

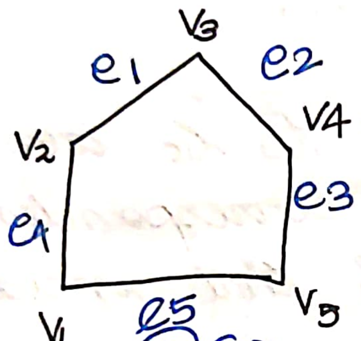
suppose that the edge e is incident on the vertices v_1 and v_2 in G_1, then the corresponding edge e' in G_2 must be incident on the vertices v_1' and v_2' that correspond to v_1 and v_2 respectively.

Two graphs will be isomorphic if they have the same no. of vertices, same no. of edges, equal no. of vertices with a given degree

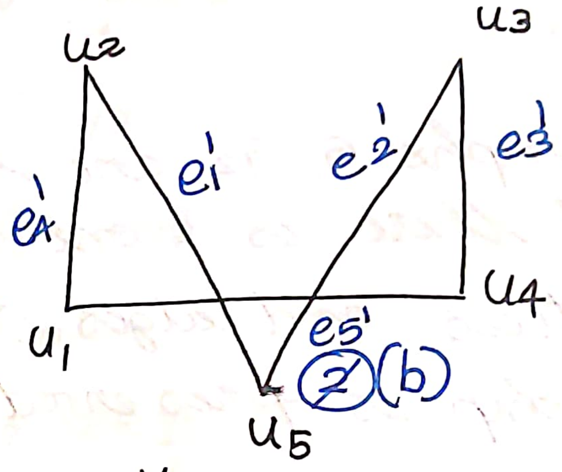
But this condition is not sufficient but just the necessary part

In fig 4.2
Corresponding vertex of V4 is u3 but V4 is connected to two pendant vertices but u4 is connected to only one pendant vertex. Therefore it is not isomorphic.

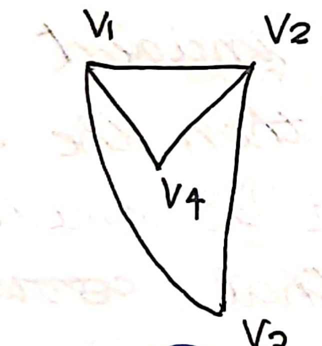
Check whether the following graphs are isomorphic or not. If isomorphic write the correspondance



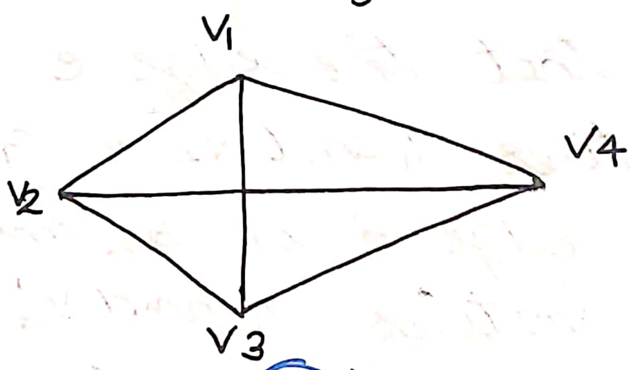
1 (a)



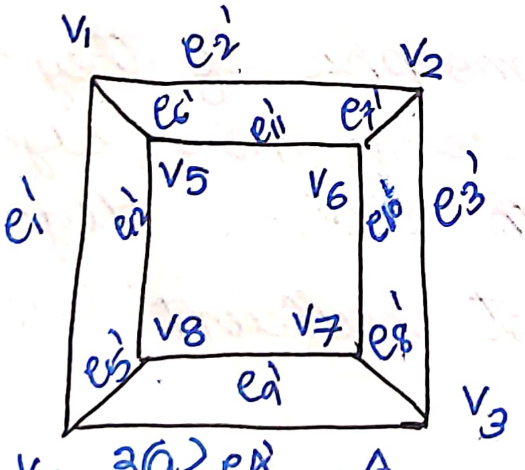
(b)



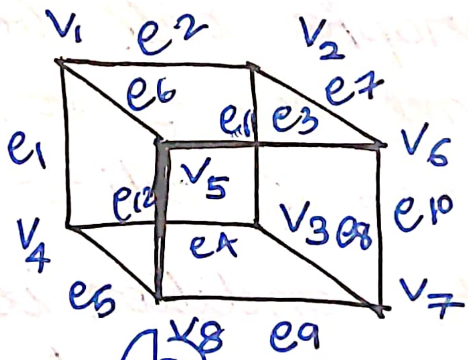
2 (a)



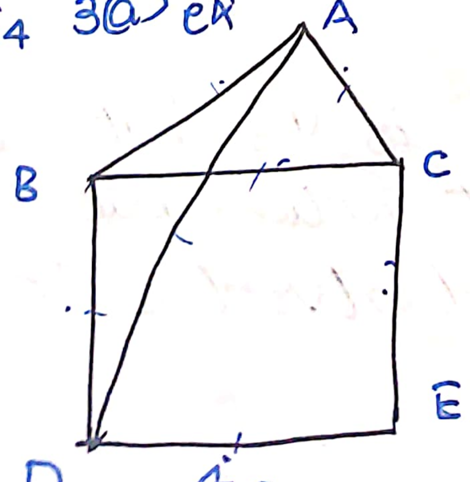
2 b



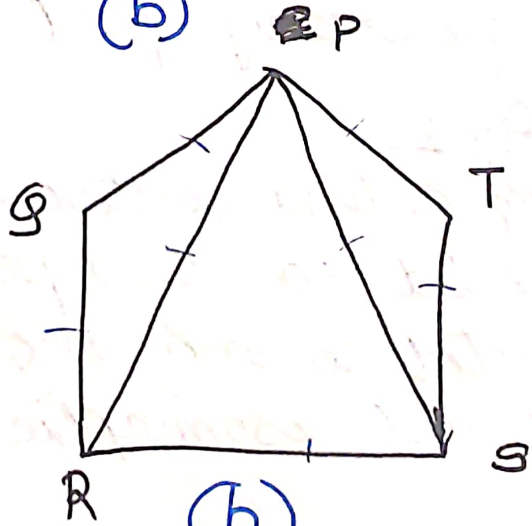
3(a)



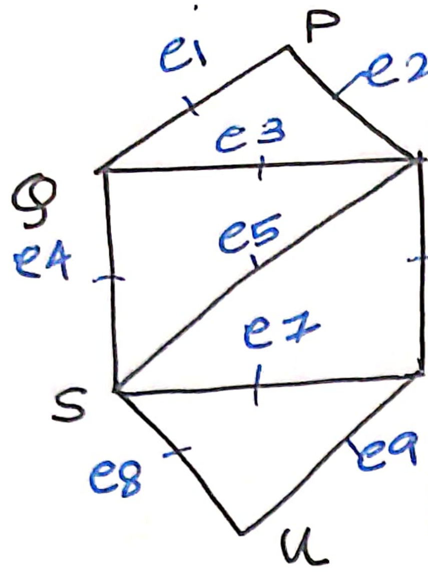
(b)



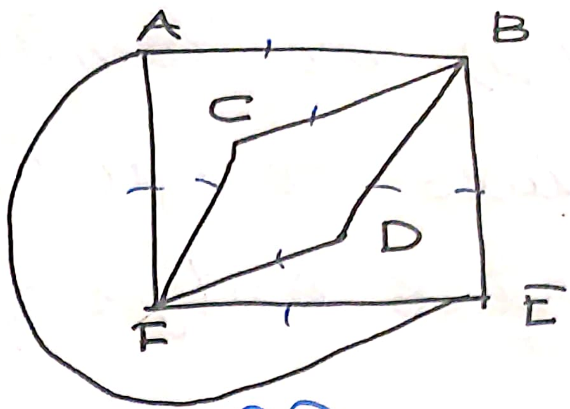
4 (a)



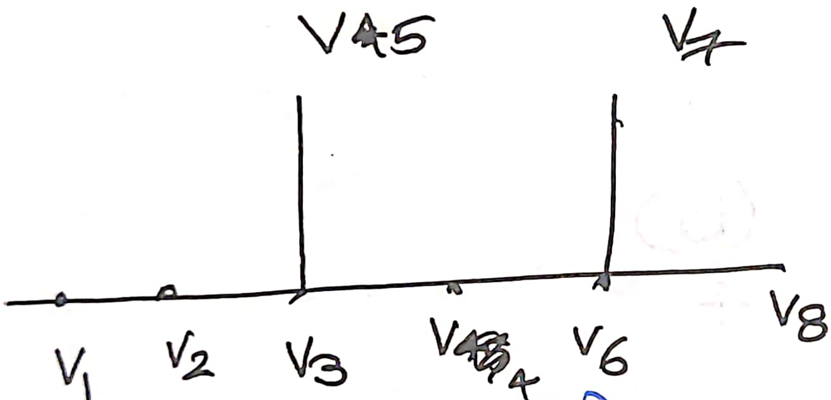
(b)


P
Q
R
S
T
U
e1
e2
e3
e4
e5
e6
e7
e8
e9

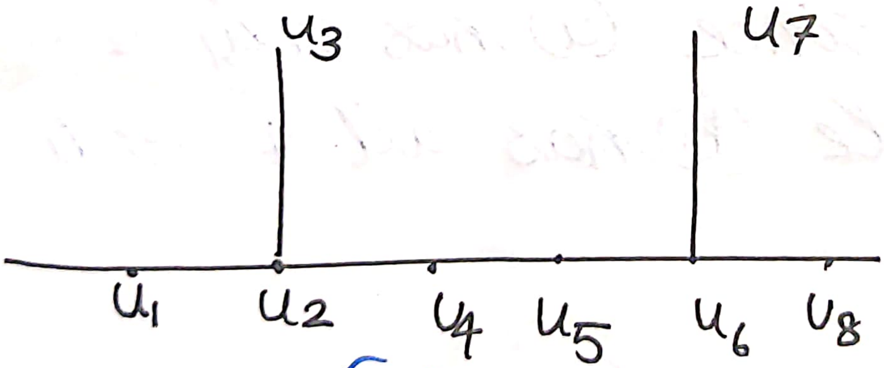
5(a)


A
B
C
D
E
F

(b)


V45
V7
V1
V2
V3
V4
V6
V8

6(a)


u3
u7
u1
u2
u4
u5
u6
u8

(b)

1(a) and (b) are isomorphic

No. of vertices are equal = 5

No. of ~~vertices~~ edges = 5

$V_3 \leftrightarrow U_5$
$V_2 \leftrightarrow U_2$
$V_4 \leftrightarrow U_3$
$V_1 \leftrightarrow U_1$
$V_5 \leftrightarrow U_4$

In 2	(a)	(b)
No. of vertices	= 4	4
No. of edges	= 5	5

Not isomorphic since (a) has only 2 vertices of degree 3 while (b) has all 4 vertices of degree 3

3)	(a)	(b)
No. of vertices	8	8
No. of edges	~~8~~ 12	~~8~~ 12
No. of 3 degree vertices	8	8

Isomorphic

$V_1 \leftrightarrow V_1$
$V_2 \leftrightarrow V_2$
$V_3 \leftrightarrow V_3$
$V_4 \leftrightarrow V_4$
$V_5 \leftrightarrow V_5$
$V_6 \leftrightarrow V_6$
$V_7 \leftrightarrow V_7$
$V_8 \leftrightarrow V_8$

	(a)	(b)
NO of vertices	5	5
No. of edges	7	7
No. of 3 degree vertices	2	2
No. of 4 degree vertices	0	1

Not isomorphic

	(a)	(b)
5. No. of vertices	6	6
No. of edges	9	9
No. of 2 degree vertices	2	2
No. of 3 degree vertices	2	2
No. of 4 degree vertices	2	2

~~Isomorphic~~

~~Q ⇔ A~~
~~T ⇔ E~~
~~R ⇔ B~~
~~S ⇔ F~~
~~P ⇔ C~~
~~U ⇔ D~~

Not isomophic since in (a) a 2 degree vertice is conn to 3 degree vertice but not in b

	(a)	(b)
6. No. of vertices	8	8
No. of edges	8	8

No. of pendant vertices in

(a)	(b)
4	4

Not isomorphic since 2 vertices of deg 2 connected to each other in b but not

MODULE 2

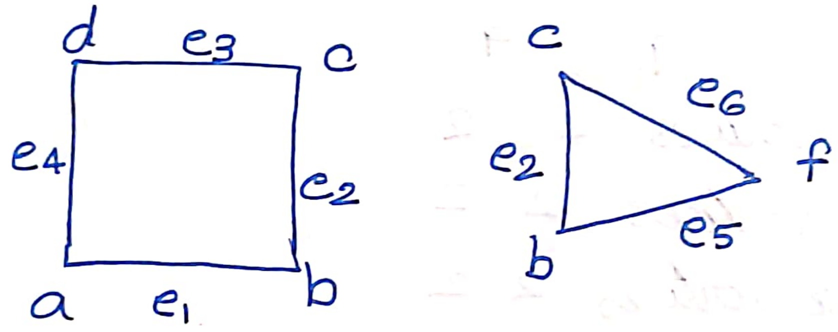
OPERATIONS ON GRAPHS

UNION OF TWO GRAPHS

given two graphs G_1 and G_2 their union will be the graph

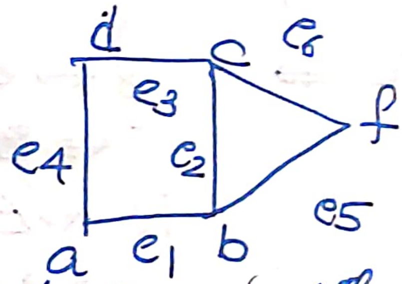
$V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$

$E(G_1 \cup G_2) = E(G_1) + E(G_2)$



$V(G_1 \cup G_2) = \{a, b, c, d, f\}$

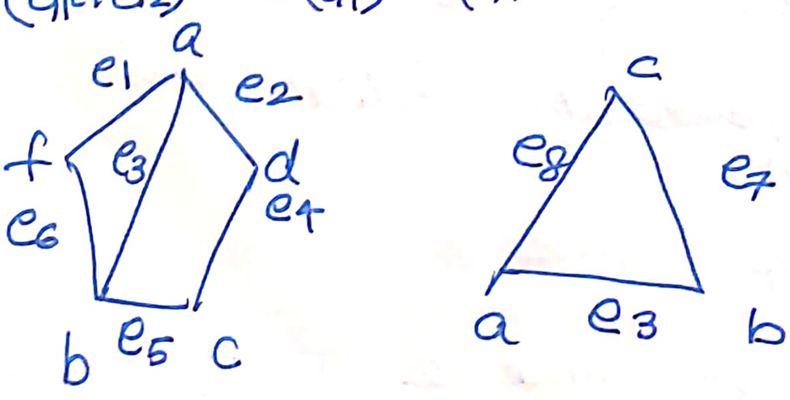
$E(G_1 \cup G_2) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$



given two graphs G_1 and G_2 with at least one vertex in common then their intersection will be a graph such that

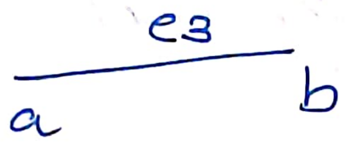
$V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$

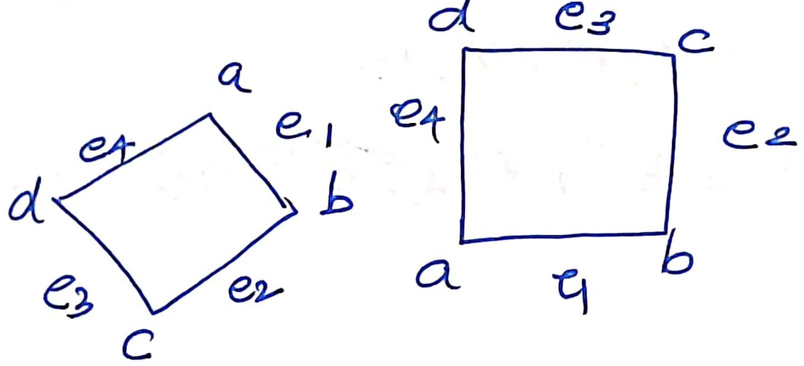
$E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$ (common edges)



$V(G_1 \cap G_2) = \{a, b\}$

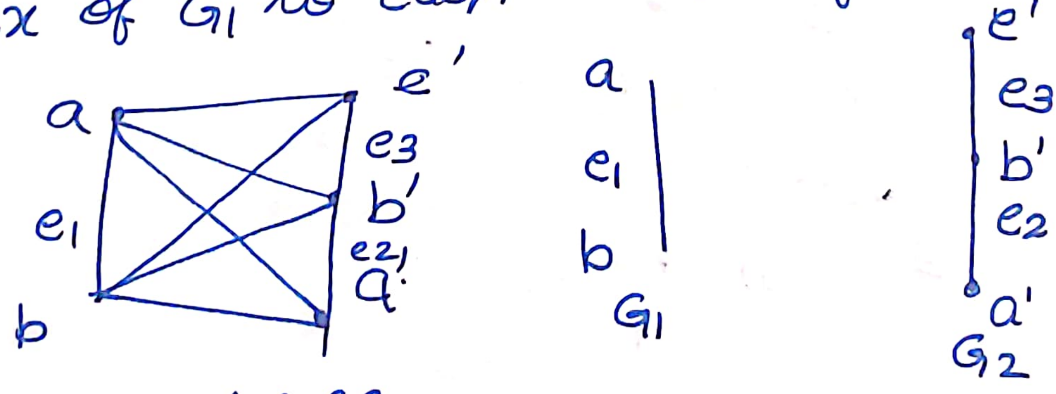
$E(G_1 \cap G_2) = \{e_3\}$





SUM OF GRAPHS.

If the graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \phi$. Then the sum of G_1 and G_2 is denoted as $G_1 + G_2$ and is defined as the graphs whose vertex set is vertices $V(G_1) + V(G_2)$ and the edge set is consisting of those edges which are in G_1 and in G_2 and the edges obtained by joining each vertex of G_1 to each vertex of G_2



sum of G_1 & G_2

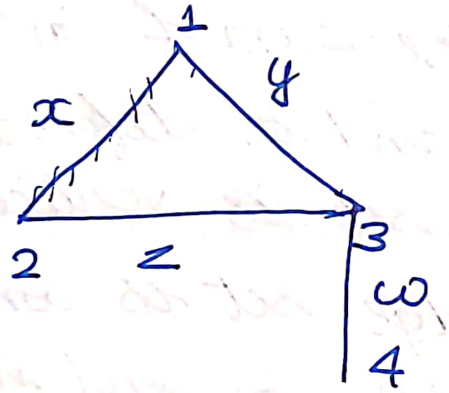
RING SUM OF TWO GRAPHS

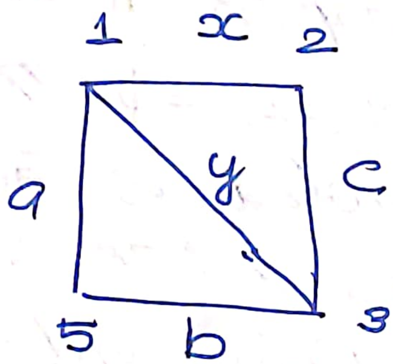
Let $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ be two graphs then the ring sum of G_1 and G_2 is denoted by $G_1 \oplus G_2$ is defined as the graph G such that vertices

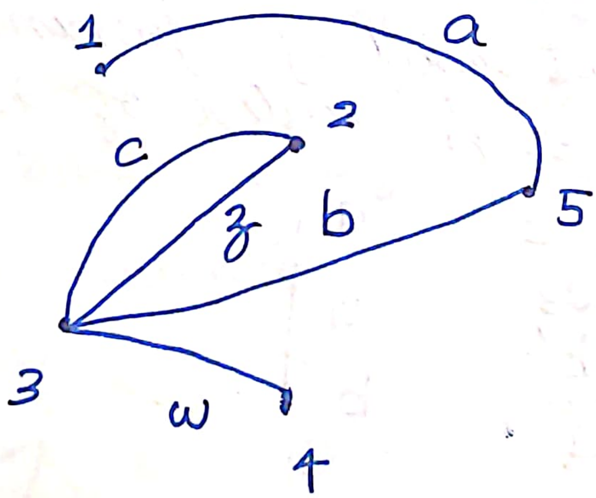
$$V(G_1 \oplus G_2) = V(G_1) \cup V(G_2)$$

$$E(G_1 \oplus G_2) = E(G_1) \cup \cancel{E(} E(G_2) - E(G_1 \cap G_2)$$

ie edges that are either in G_1, or in G_2 but not in both

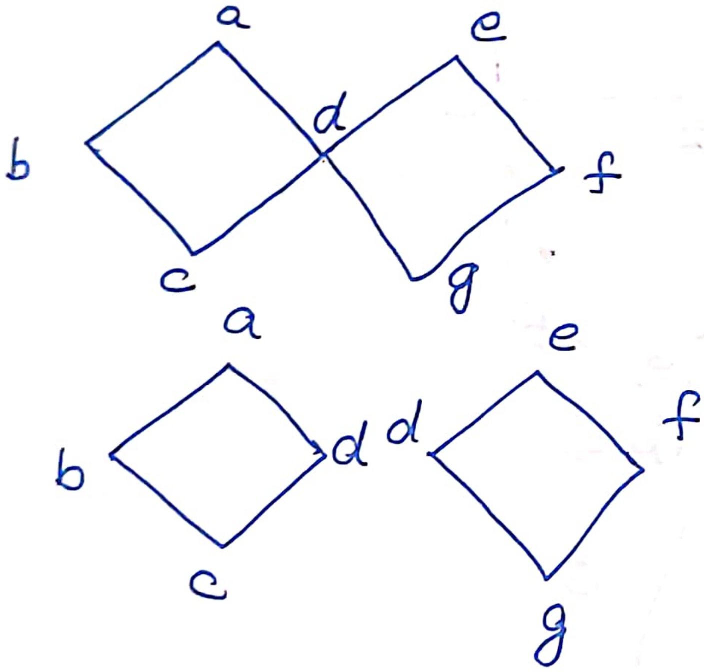






DECOMPOSITION

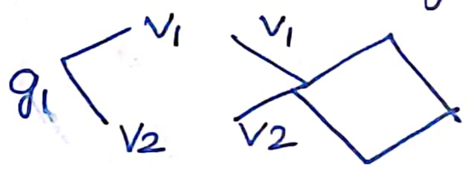
A graph G is said to have been decomposed into two subgraphs G_1 and G_2 if $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \phi$



$g_1 \cup g_2 = G$

$g_1 \cap g_2 = \phi$

If only 1 common vertice null graph

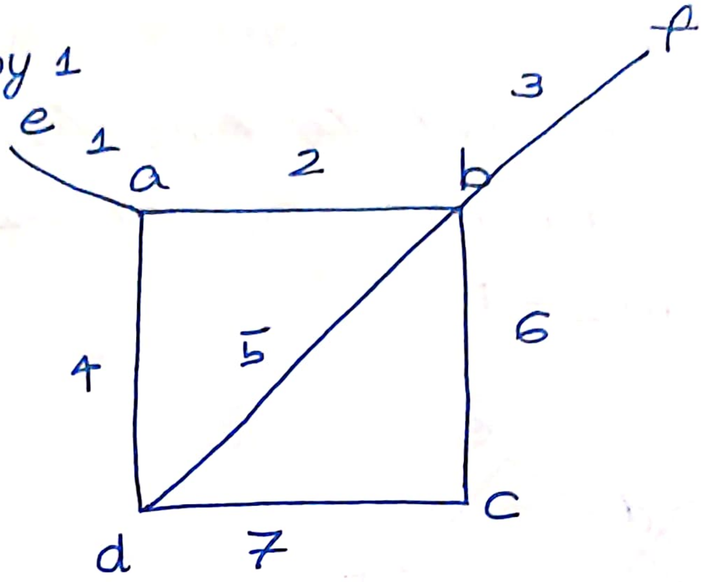


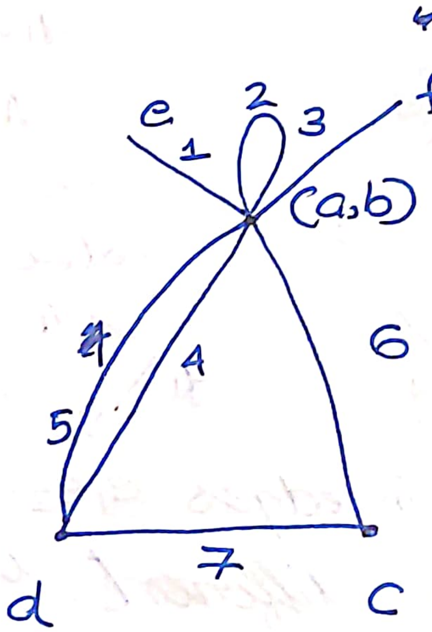
A graph containing m edges $e_1 e_2 \dots e_m$ can be decomposed in $2^{m-1} - 1$ different ways into pairs of subgraphs G_1, G_2

FUSION

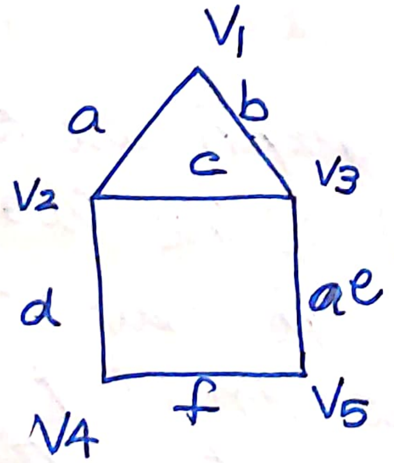
A pair of vertices (a,b) in a graph are said to be fused (merged/identified). If the two vertices are replaced by a single new vertex such that every edge that was incident either on a, or b or both is incident on the new vertex.

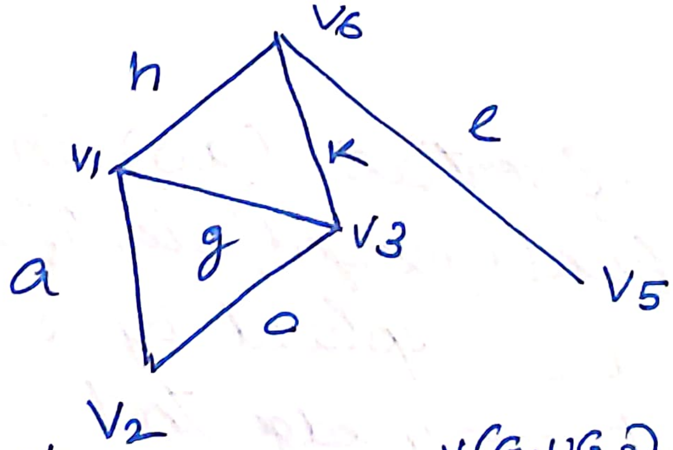
Thus fusion of two vertices does not alter the no. of edges but it reduces the no. of vertices by 1

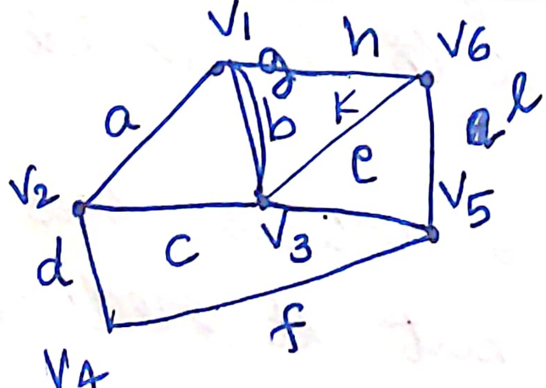




(V1,V6) → fuse





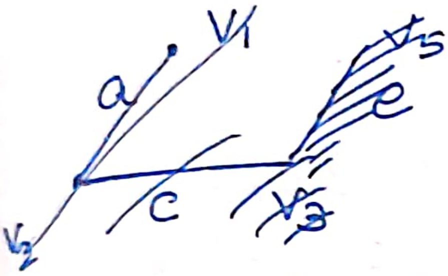


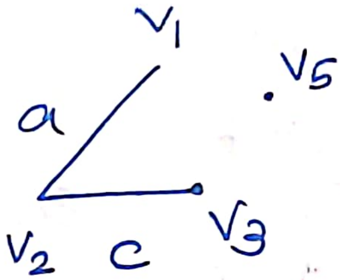
$V(G_1 \cup G_2) = \{V_1, V_2, V_3,$
$V_4, V_5, V_6\}$

$E(G_1 \cup G_2)$

$= \{a, b, c, d, e, f,$
h, K, l

Intersection

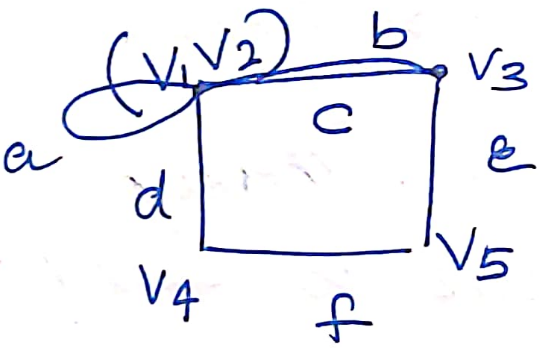


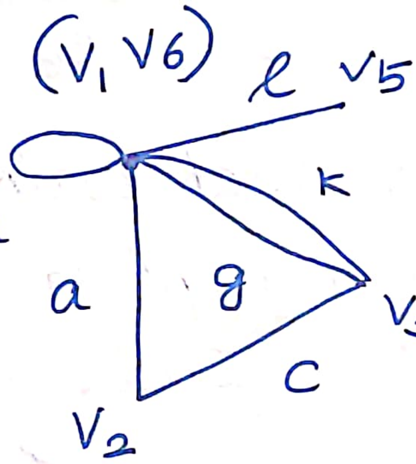


$V(G_1 \cap G_2) = \{V_1, V_2, V_5\}$

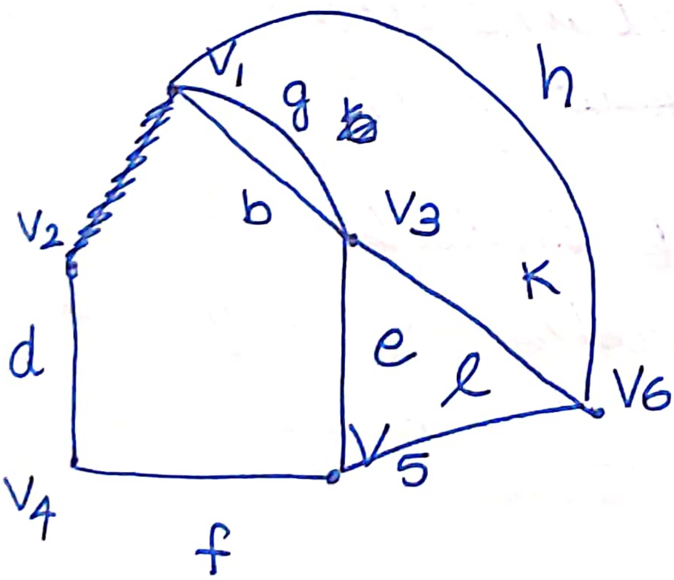
$V(E_1 \cap E_2) = \{a, c\}$

Fusion (V_1, V_2)





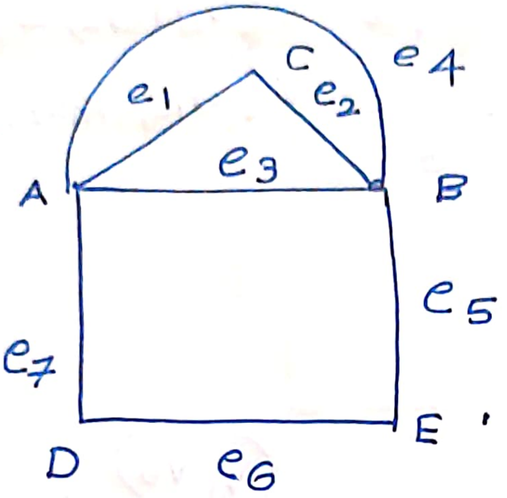
RING SUM



$V(G_1 \oplus G_2) = \{V_1, V_2, V_3, V_4, V_5, V_6\}$

$E(G_1 \oplus G_2) = \{b, d, e, f, g, h, k, l\}$

EULERIAN GRAPH



Eulerian graph

$Ae_3Be_2Ce_1Ae_4Be_5E$
e_6De_7A

Eulerian circuit

A closed walk running through every edge of a graph G exactly once is called Euler line (Euler circuit)

A graph which contains Euler line is called Eulerian ~~circuit~~ graph

An open walk that includes all edges of a graph without retracing any edge is called Unicursal line or open Euler line.

A connected graph that has a Unicursal line is called unicursal graph

DCBADEFGH

IMP: (6-8M) STATE NECESSARY & SUFFICIENT CONDITION TO BE EULER

THEOREM A given connected graph G is an Euler graph if and only if all the vertices of G are of even degree

PROOF: (NECESSARY CONDITION)

Let G be an Eulerian graph then G has an Eulerian circuit which begins and ends at an arbitrary vertex U

If we travel along the trial then each time when we visit a vertex by using two edges one in and one out. ~~because~~ This is also ~~This~~ True for the starting vertex because V also ends there

Since an Eulerian circuit uses every vertex ~~4~~, once. each occurrence of V represents a contribution of ~~two~~ 2 to its degree. ~~Their degree~~ Thus degree of each vertex is even.

CONVERSE

(SUFFICIENT CONDITION)

To prove the sufficient condition assume that all vertices of G are of even degree and G is a connected graph ie Euler

Now we construct a walk starting at an arbitrary vertex V and going through the edges of G such that no edge is traced more than once.

Continue tracing as far as possible. Since every vertex is of even degree, we can exit from every vertex we enter and the tracing can stop at V

Let the resultant circuit be H. If the closed walk H includes all edges of G then G is Eulerian

If not we remove all the edges of H from G and obtain a subgraph H'(G), formed by the remaining edges. Since G and H has all the vertices of even degree, the vertices of H' is also in even degree. Moreover since the graph G is connected, H' must touch H, at least one vertex, a then ~~from~~ form a new walk starting from a and ending at a such that no edges are repeated then on combining these two walks we get a circuit and if which cover all the edges of G the graph becomes Eulerian Otherwise we repeat the process until we obtain a closed walk covering all edges of G, the graph becomes Eulerian

THEOREM 2

In a connected graph with exactly 2K odd vertices there exist K edge disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph.

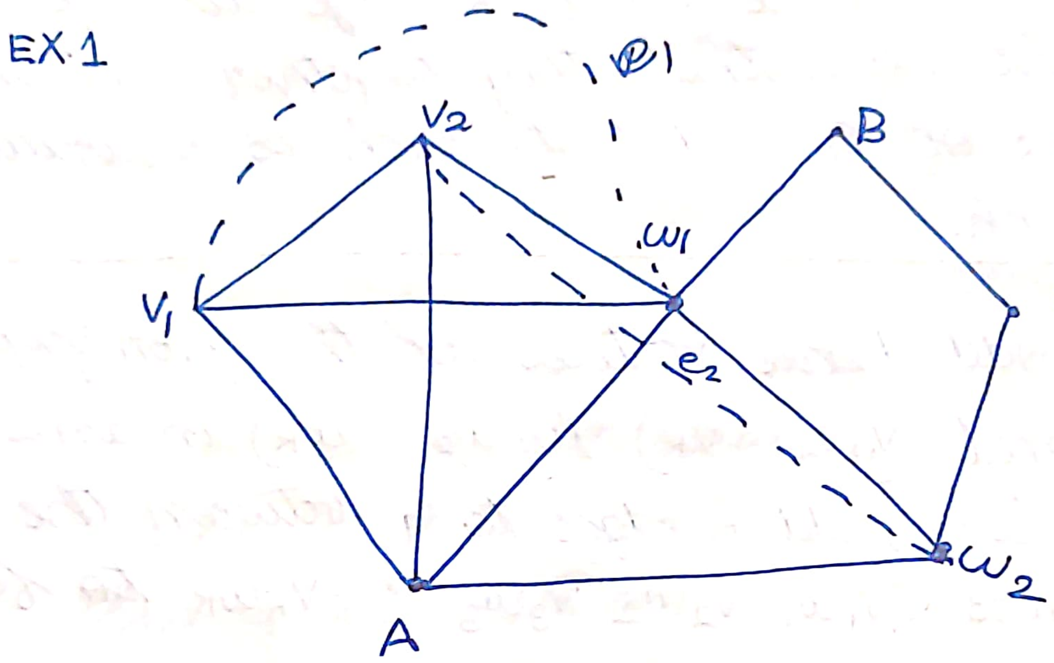
PROOF: Let the odd degree vertices of the given graph G be named $(V_1, V_2, \ldots V_K)$, $(W_1 W_2 \ldots W_K)$ in an arbitrary order, add K edges to G between the vertex pairs V_1W_1 V_2W_2 $V_3W_3 \ldots$ V_KW_K to form a new graph G'

Since every vertex of G' is of even degree G' consists of Eulerian line ~~k~~ l Now if we remove from l, the k edges that we just added l will split into k walks, each of which is a unicursal line.

The first removal will leave a single unicursal line.

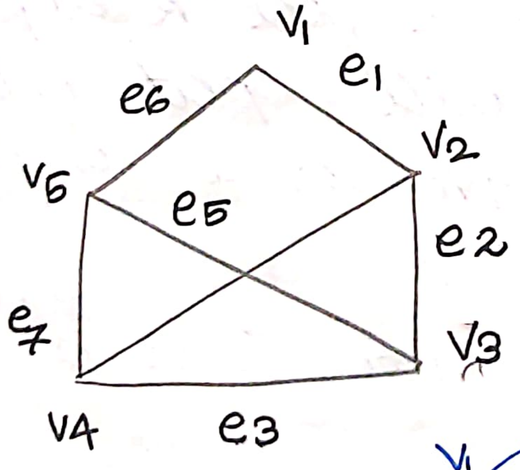
The second removal will split that into two unicursal lines and each successive removal will split a unicursal line into two unicursal lines. Since there are K edges removal of all K edges splits l i k unicursal lines and they are edge disjoint

EX.1



$V_1\, e_1\, w_1\, V_1\, V_2\, A\, w_1\, B\, C\, w_2\, w_1\, A\, w_2\, e_2\, V_2\, V_1$

HAMILTONIAN GRAPH

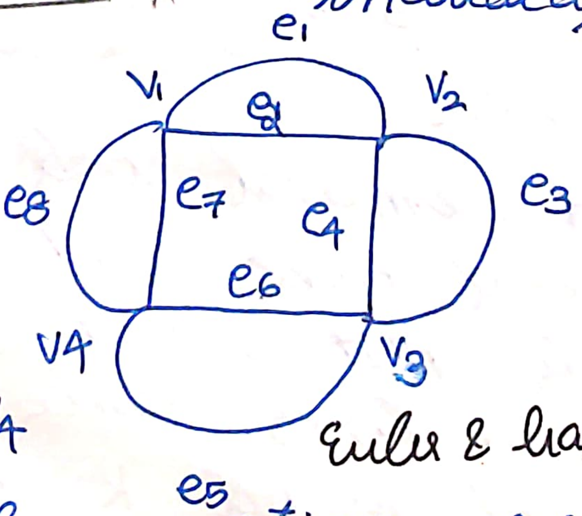


All vertices present but not edges

(path) → edges & vertices not repeated

$V_1\, e_1\, V_2\, e_2\, V_3\, e_3\, V_4\, e_7\, V_5\, e_6\, V_1$

(include all vertices but all edges may not be included)



Euler circuit

$V_1 e_1\, V_2 e_2\, V_1\, e_8 V_4\, e_5$
$V_3 e_3\, V_2\, e_4 V_3\, e_6\, V_4$
$e_7 V_1$ } → here vertices are repeated

Euler & hamiltonian graph

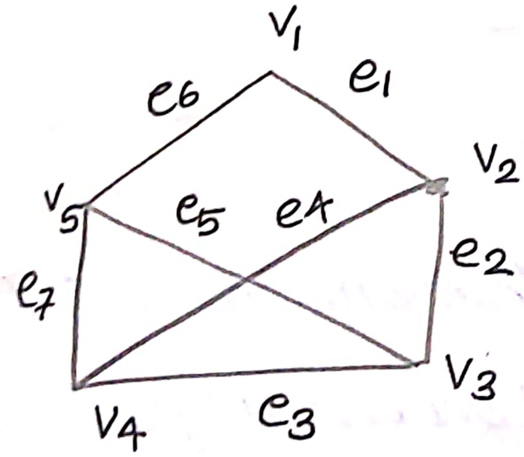
HAMILTONIAN PATH is a simple path that contains all vertices of Graph G exactly once

HAMILTONIAN CIRCUIT

A circuit in a graph G that contains each vertex in G exactly once, except for the starting and ending vertices that appears twice

HAMILTONIAN GRAPH

A graph G is called a Hamiltonian graph if it contains a Hamiltonian circuit

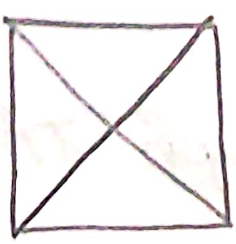


$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_7 v_5 e_6 v_1$

DIRAC THEOREM

Let G be a graph of order $p \geq 3$ and if degree of $v \geq p/2$ for any vertex v in G then G is a hamiltonian graph

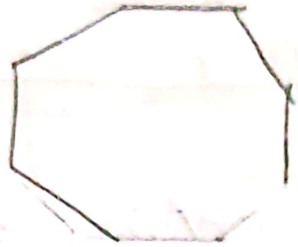
Converse not true

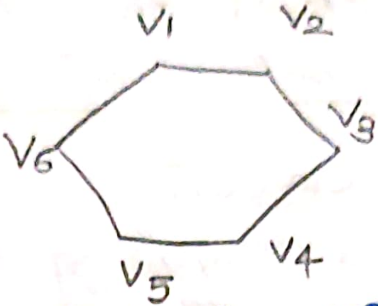


$p = 4$

$d(v) = 3 \geq p/2 = 2$

$\therefore$ Hamiltonian graph





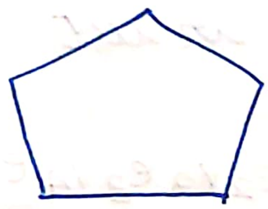
here $d(V) < 3$

But hamiltonian

$n \geq 5 \rightarrow$ regular polygon

Justify converse not true

Draw vertices $\geq 5 \rightarrow$ polygon (any regular polygon with



$d(V) = 2$

$\frac{P}{2} = 2.5$

$\therefore$ Converse not true

The converse of Dirac Theorem is not true

If the graph is hamiltonian degree of each vertex $\geq P/2$

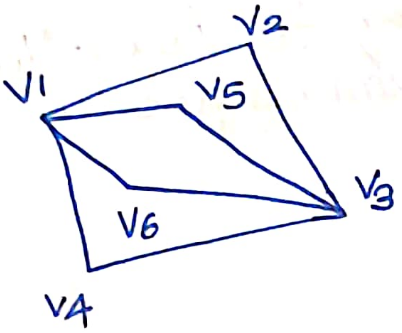
For eg: Any regular polygon having vertices greater than or equal to 5

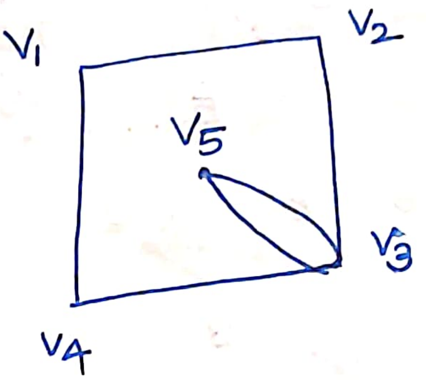
EXAMPLE OF GRAPH (EULER & HAMILTONIAN)

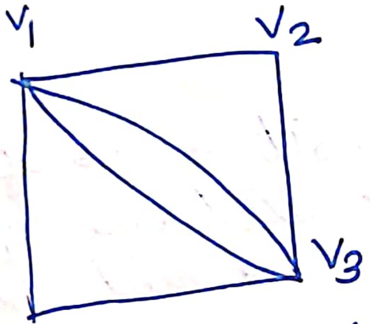
A polygon having vertices / edges greater than or equal to 3

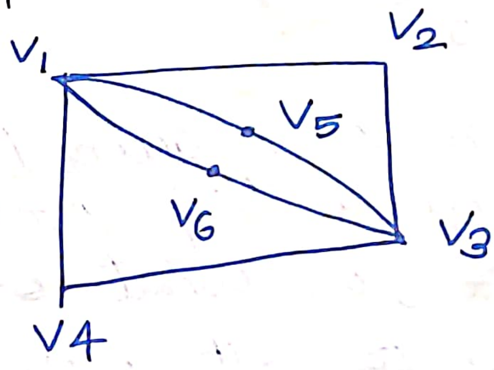
give an example of Euler graph that is not Hamiltonian

Draw a polygon like a square and inside draw another a parallel edges starting and ending at the same vertice with two more vertice

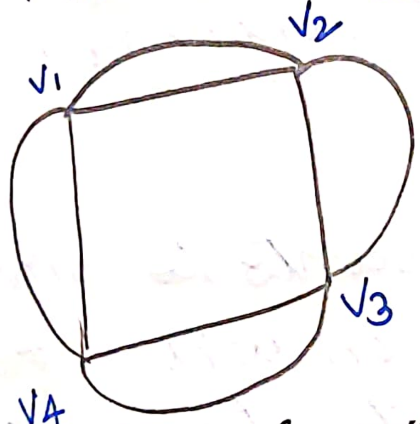








Hamiltonian & euler

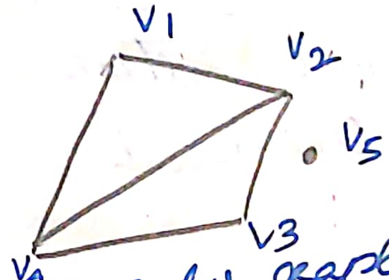


$p=4$

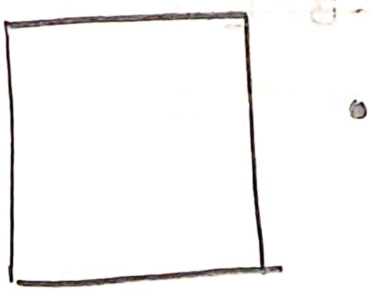
$d(v)=4$

$d(v) \geq p/2$

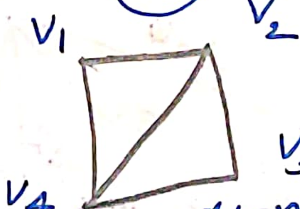
Euler & hamiltonian



Not euler graph
Not hamitonian

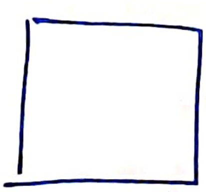
① 

euler but not hamiltonian

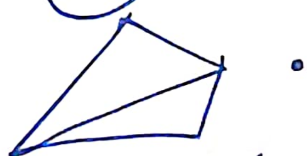
② 

hamiltonian but not euler

4 casses euler & hamiltonian

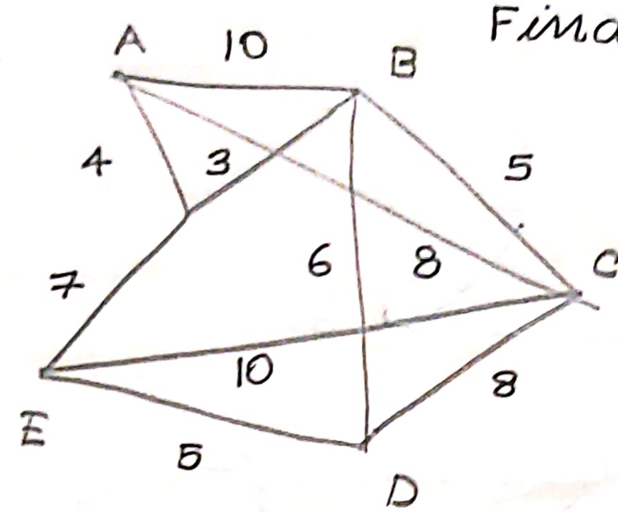
③ 

euler & hamiltonian

④ 

Not euler & not hamiltonian

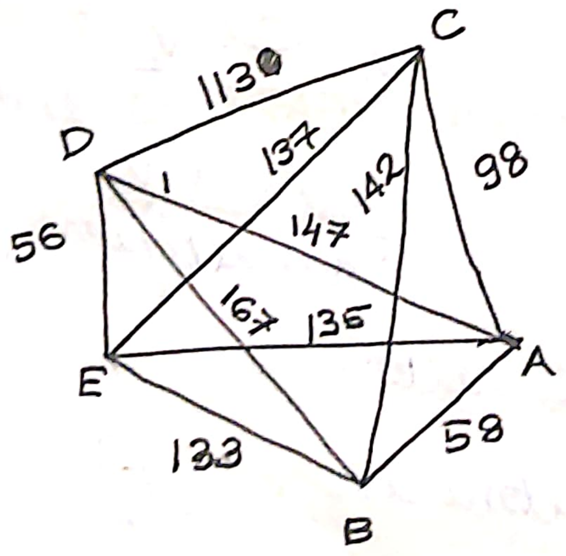
TRAVELLING SALESMAN PROBLEM



Find shortest path
not eq to travelling salesman prob

A salesman is required to visit a no. of cities during a trip, given the distance b/w the cities ~~the~~, in what order should he travel so as to visit every city precisely once and return home with the minimum mileage travelled

suppose that a salesman wants to visit five cities namely A, B, C, D, E, in which order should he visit these cities to travel the minimum total visits



~~A-B-E~~

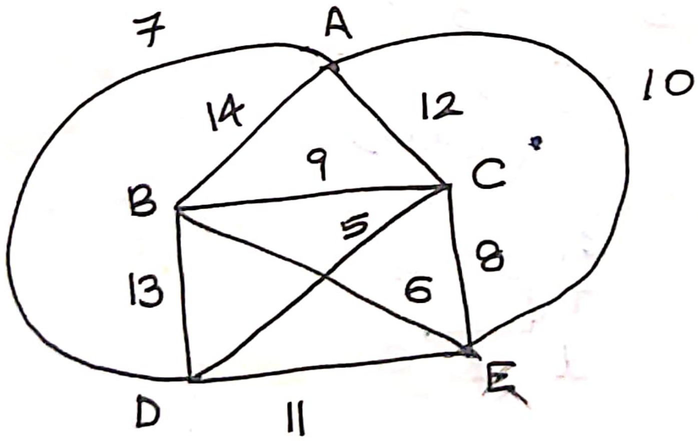
HAMILTONIAN CIRCUIT	DIST	TOTAL
A-B-E-D-C-A	58+133+56+113 +98	458
C-D-E-B A-C-D	113+56+58+98	458
A-D-B-C-E-A		
⋮	⋮	

the shortest path is 458

1a hamiltonian circuit

No. of hamiltonian circuit $\frac{(n-1)!}{2} = \frac{4!}{2} = 1$

HW



Starting from D

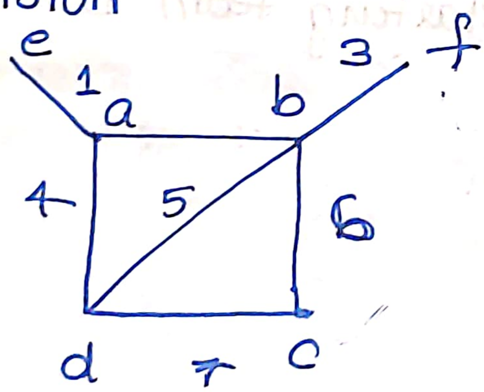
HAMILTONIAN CIRCUIT	DIST	TOTAL
DABCED	7+14+9+8+11	49
DEACBD	~~11+10~~ 11+10+12+9+13	55
DCAEBD	5+12+10+6+13	46
DBAECD	13+14+10+8+5	50
DBCAED	13+9+12+10+11	55
DCBAED	5+9+14+10+11	49

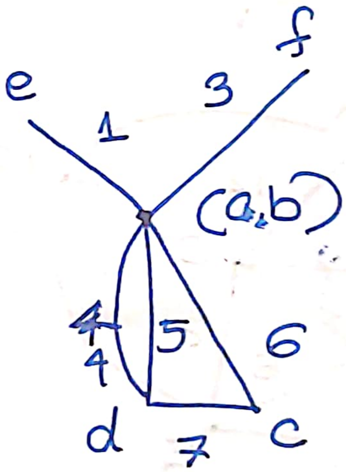
DEBCAD	11+6+9+12+7	45
DECABD	11+8+12+14+13	58
DBACED	13+14+12+8+11	58
DECBAD	11+8+9+14+7	49
~~DABCED~~	~~7+14+9+8+11~~	
DACBED	7+12+9+6+11	45
DAECBD	7+10+8+9+13	47
DAEBCD	7+10+6+9+5	37

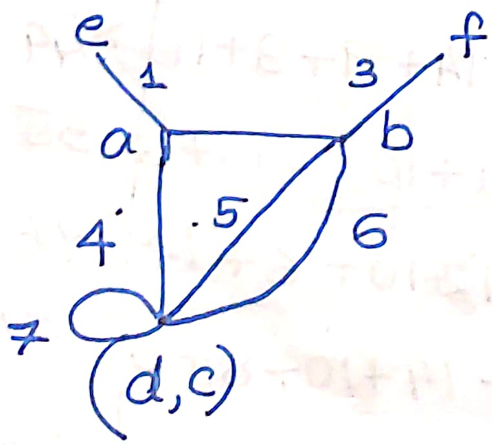
~~DAE~~

Shortest : DAEBCD = 37

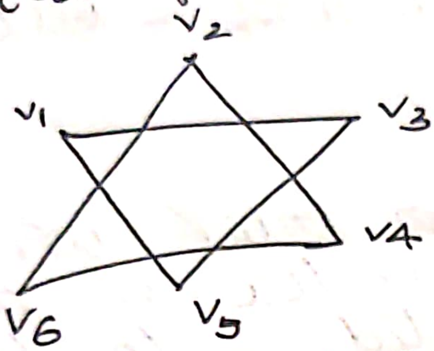
REVISION



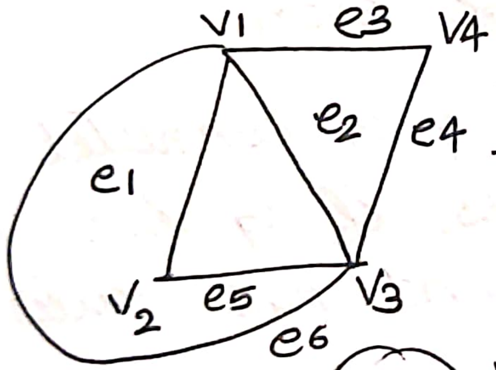




Why only a connected graph with all vertices of even degree — Euler



This is not a connected graph, hence even if all vertices are of even degree the graph is not Eulerian



Has an eulerian ~~trail~~ trail but not euler circuit

$\underbrace{V_1 e_3 V_4 e_4 V_3}$ e_6 $\underbrace{V_1 e_1 V_2 e_5 V_3\ e_2 V_1}$

$V_1\ e_3\ V_4\ e_4\ V_3\ e_2 V_1\ e_1\ V_2\ e_5\ V_3 \boxed{e_6} V_1$

d of given last first only then we get a trail with continu

A connected graph G has an eulerian trail if and only if it has at most 2 odd degree vertices

suppose G has an euler trail which is not closed, since each vertex in the middle of the trail is associated with two edges and since only one edge is associated with each end vertex of the trail these end vertices must be odd and other vertices must be even.

conversely suppose that G is connected with at most 2 degree vertices, if G has no odd vertices then G euler and has euler trail

~~If~~

suppose that G has two odd degree vertices. then join the two odd degree vertices by an edge, now all vertices of even degree. Therefore graph contains an Euler circuit, remove edge e from circuit we get euler trial

Find all +ve integers n such that the complete graph Kn is eulerian

For a complete graph Kn each vertex has degree n-1, Kn is eulorian, Kn is Euler if all vertices are of even degree ie n-1 is an even no implies n is a odd no

Show that any K regular simple graph with $2K-1$ vertices is Hamiltonian (State Dirac's theorem)

Here $P = 2K-1$

$2K-1$ is an odd number

$$\frac{P}{2} = \frac{2K-1}{2} = K - \frac{1}{2}$$

$\deg(v) = K$ (K regular graph)

ie $\deg(v) > K - \frac{1}{2} = \frac{P}{2}$

$\therefore$ By dirac theorem the graph is hamiltonian

Prove that a complete graph K_n, n greater than or equal to 3 is Hamiltonian

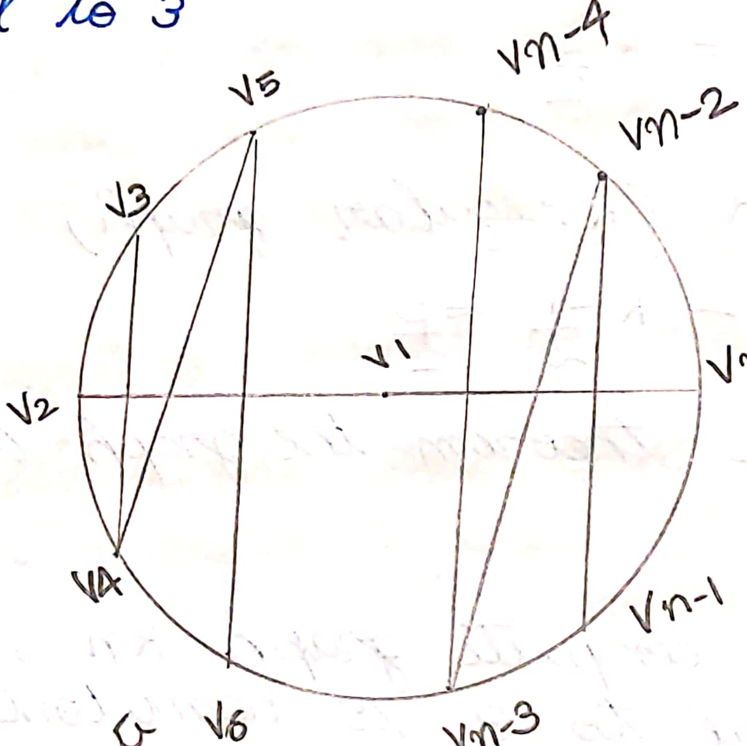
Here $p = n$, $n \geq 3$, degree of each vertex in K_n is $n-1$

$\frac{P}{2} = \frac{n}{2} < n-1 \quad \forall\, n \geq 3$

$\therefore$ By dirac theorem there exists a Hamiltonian circuit

IMP

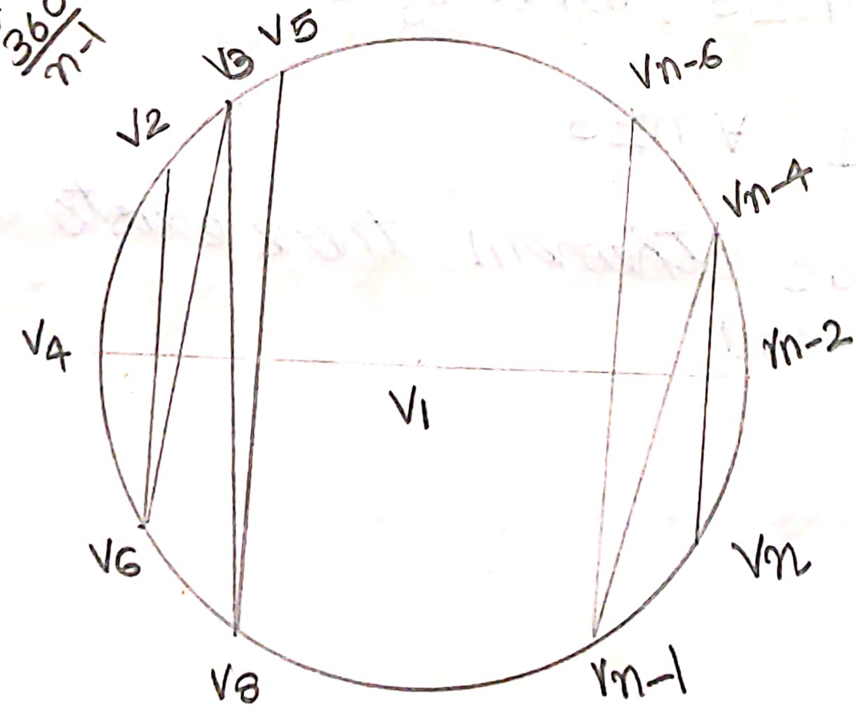
In a complete graph with n vertices there are $\frac{(n-1)}{2}$ edge disjoint Hamiltonian circuits if n is an odd no greater than or equal to 3



$\frac{360}{n-1}$

we can rotate $2 \times \frac{360}{n-1} \ldots n \times \frac{(360)}{n-1}$

$\frac{360}{n-1}$ rotate



↳ we get $\frac{n-3}{2}$ diff Ham circuits

$\frac{n-3}{2}+1 = \frac{n-1}{2}$

PROOF:

A complete graph G of n vertices has $\frac{n(n-1)}{2}$ edges and a Hamiltonian circuit in G consists of n edges. Therefore the no. of edge disjoint Hamiltonian circuits in G, cannot exceed $\frac{(n-1)}{2}$

Now assume that $n \geq 3$ and is odd, construct a subgraph g of K_n as explained below

The vertex V_1 is placed at the centre of the circle and the remaining n-1 vertices are placed on a circle at equal distances along the circle such that the angle made at the centre by two points is $\left(\frac{360}{n-1}\right)^{\circ}$, ~~the verti~~

The vertices with odd suffices are placed along upper half of circle, vertices with even suffices are placed along the lower half of the circle. Draw edges $V_i V_{i+1}$ where $1 \leq i \leq n$ with $V_{i+1} = V_i$ (treat V_{n+1} as V_1)

clearly the reduced graph g is a cycle covering all vertices of K_n clearly a hamiltonian circuit

If we rotate the vertices along the circle by $\frac{360}{n-1}$ degrees in clockwise dirn we get another Hamiltonian circuit similarly rotate the polygonal pattern clockwise by $\frac{2 \times 360}{n-1}$, $3 \times \frac{360}{n-1}$ upto $\frac{n-3}{2} \frac{360}{n-1}$, in degrees

each rotation produces a Hamiltonian circuit that has no edge in common with any of the previous ones.

Thus we have Total $\frac{n-3}{2} + 1 = \frac{n-3+2}{2} = \frac{n-1}{2}$ edge disjoint Hamiltonian circuit

FLEURY'S ALGORITHM

Let $G = (V, E)$ be a connected graph with each vertex of even degree.

STEP 1: select an edge e_1 that is not a bridge in G

Let its vertices be V_1, V_2

Let π be specified by V_π, V_1, V_2 and E_π ; e_1

Remove e_1 from E and V_1 and V_2 from V to create G_1

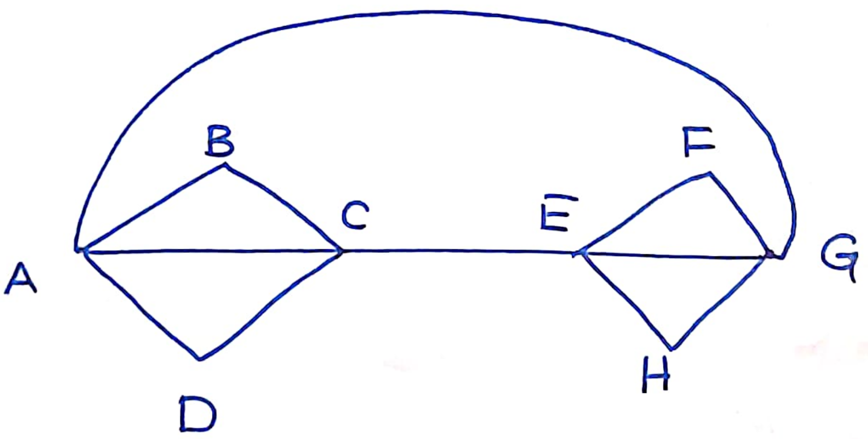
Step 2: suppose that $V_\pi : V_1, V_2 \ldots\ldots V_K$ and $E_\pi : e_1, e_2 \ldots\ldots e_{K-1}$ have been constructed so far and that all of these edges and vertices have been removed from V and E to form G_{K-1}

since v_K has even degree and e_{K-1} ends there, there must be an edge e_K in G_{K-1} that also has v_K as a vertex

If there is more than one such edge, select one that is not a bridge for G_{K-1}. Denote the vertex of e_K other than v_K by v_{K+1} and extend V_π and E_π to $V_\pi : v_1, v_2, v_K, v_{K+1}$ and $E_\pi : e_1, e_2, \ldots e_{K-1}, e_K$.

step 3: Repeat the step 2 until no edges remain in E

End



π: A

π AB

π: A B C

π A B C E

A B C E F

A B C E F G

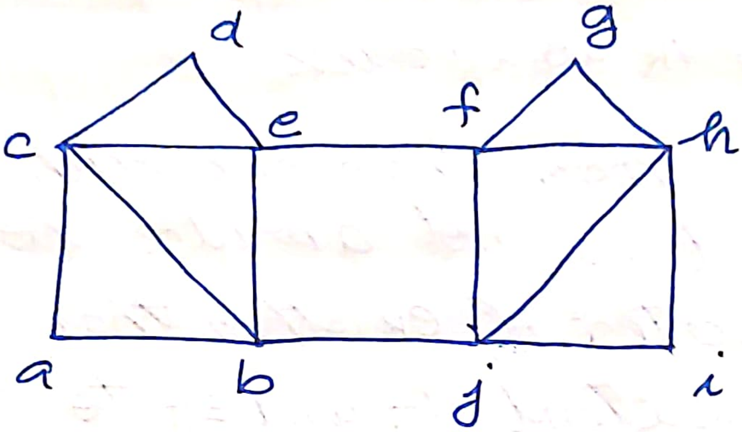
A B C E F G H E

A B C E F G H E G

A B C E F G H E G A

ABCEFGHEGBA

A BC E F G H E G A CD



a c b e d c e f g h f j h i j b a

Π : A

Π : AC

Π : ACB

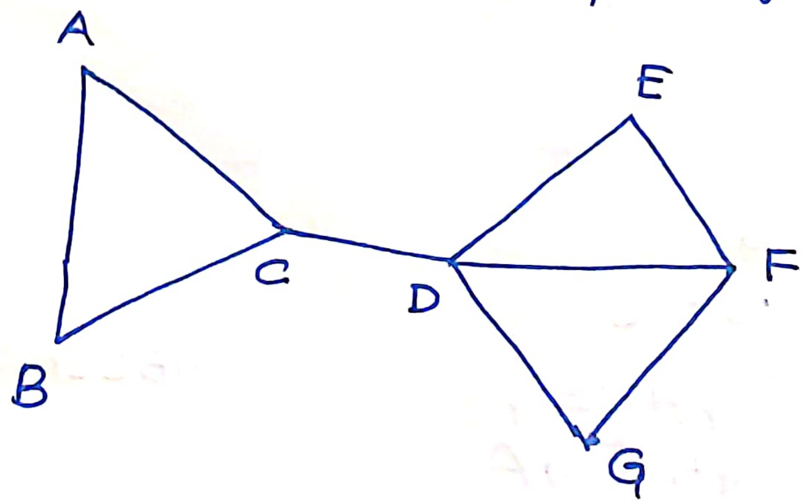
Π : ACBE

Π : ACBED

Π : ACBEDC

Π : ACBEDCE

Determine whether a Hamiltonian path/~~has~~ circuit exist in the graph of fig



Hamiltonian path : B A C D E F G

circuit is not possible since we moved through a bridge and vertices cannot be repeated

At a committee meeting of 10 people every member of the commitee has previously sat next to at most 4 other members. show that the members may be seated round a circular table in such a way that no one is next to someone they have previously sat beside

Ans: Consider a graph with 10 vertices representing 10 members, let 2 vertices be joined by an edge if the corresponding members have not previously sat next to ~~too~~ each other

since any member has not previously sat next to at least 5 members, the degree of every vertex is at least 5. therefore the graph has a hamiltonian circuit.

This circuit provides a seating arrangement of desired type

9 members of a club meet everyday for a dinner, they sit in a round table for dinner, but not two members who sat together will sit together in future, how long is this possible