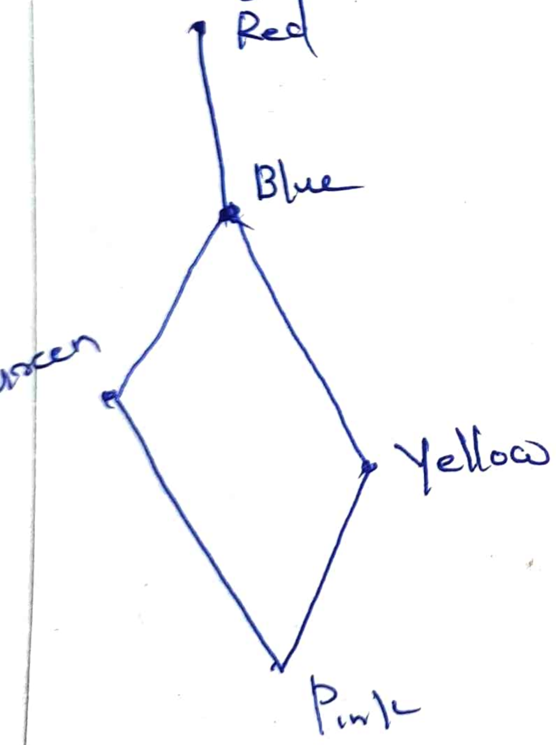
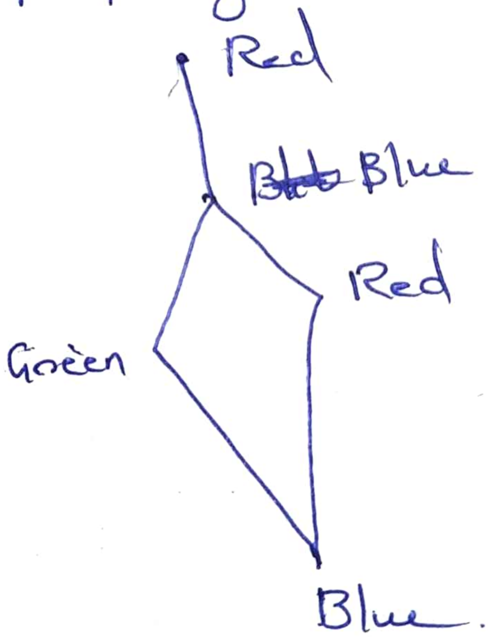
Chromatic Number:

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called proper coloring of a graph.

A graph in which every vertex has been assigned a color according to a proper coloring is called a properly colored graph.





A graph G that requires k different colours for its proper coloring and no less, is called k-chromatic graph and the number 'k' is called the chromatic number of G.

Note: For coloring problems we need to consider only simple connected graphs.

1. A graph consisting of only isolated vertices is 1-chromatic.
2. A graph with one or more edges is ~~atleas~~ atleast 2 chromatic
3. A completegraph of n vertices is n-chromatic, as all its vertices are adjacent.

   A graph containing a complete graph of 'r' vertices is at least r-chromatic.
4. A graph consisting of simply one circuit with $n \geq 3$ vertices is 2 chromatic if n is even and 3 chromatic is n is odd.

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Theorem-

Every tree with two or more vertices is 2 chromatic.

Proof.

Select any vertex v in the given tree T. Consider T as a rooted tree at vertex v. Paint v with color 1. Paint all the vertices adjacent to v with color 2. Next paint vertices adjacent to these using color 1. Continue this process till every vertex in T ~~be~~ has been painted. Now in T we find that all vertices at odd distances from v have color 2 and vertices at even distances from v have color 1.

Along any path in T the vertices are of alternating colors. Since there is one and only one path between any two vertices in a tree, no two adjacent vertices have the same color. Thus T has been properly

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colored with two colors.

Theorem

A graph with atleast one edge is 2 chromatic iff it has no circuits of odd length.

Proof:

Let G be a connected graph with circuits of only even lengths. Consider a spanning tree T in G. Since every tree is 2 chromatic, spanning tree T properly colored with 2 colors. Now add the chords to T one by one. Since had no circuits of odd length, the end vertices of every chord being replaced are differently colored in T. Thus G is properly colored with two colors

$\therefore$ G is 2 chromatic.

Conversely if G has a circuit of odd length we would need at least 3 colors just for that circuit.

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Theorem.

If d_{max} is the maximum degree of the vertices in a graph G.

chromatic number of G $\leq 1 + d_{max}$.

If G has no complete graph of $d_{max} + 1$ vertices then chromatic number of G $\leq d_{max}$.

Bipartite graph.

A graph G is called bipartite if its vertex set V can be decomposed into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2.

Every tree is a bipartite graph.

Every 2 chromatic graph is bipartite because coloring partitions the vertex set into two subsets V_1 and V_2 such that no vertex in V_1 (or V_2) are adjacent.

A graph G is called p-partite if its vertex set can be decomposed into p disjoint subsets $V_1, V_2, \ldots V_p$, such that no edge in G ~~joint~~ joins the vertices in the same subset. A k-chromatic graph is p partite iff $k \leq p$.

Chromatic Polynomial.

A given graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed by means of a polynomial. This polynomial is called the chromatic polynomial.

The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the no. of ways of properly coloring the graph using λ or fewer colors.

Let c_i be the different ways of properly coloring G using exactly i different colors. ∴ Different ways of selecting i colors out of λ colors is λC_i

∴ Different ways of properly coloring G using exactly i colors out of λ colors is

$c_i \times \lambda C_i$

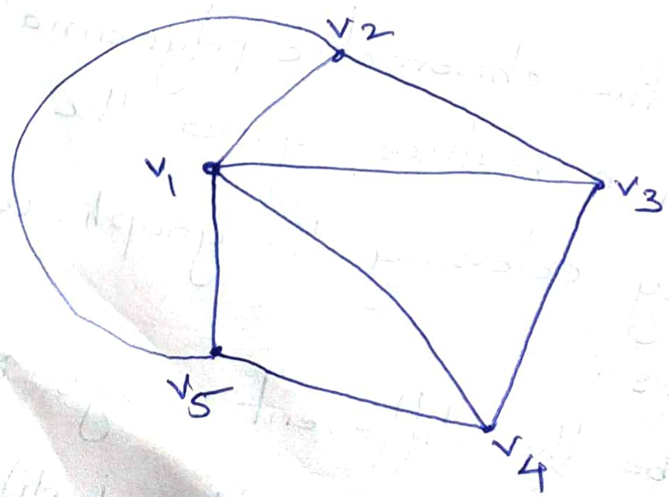
Since i can be any positive integer from 1 to n

$$P_n(\lambda) = \sum_{i=1}^{n} c_i \lambda C_i$$

$$= c_1 \lambda + c_2 \frac{\lambda(\lambda-1)}{2!} + c_3 \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots$$

$$+ c_n \frac{\lambda(\lambda-1)\cdots(\lambda-(n-1))}{n!}.$$

<u>Example</u>



$$P_5(\lambda) = c_1\lambda + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!}$$

$$+ c_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{\cancel{4}\,5!}$$

Since the graph contains triangle, it requires at least 3 colors

$\therefore c_1 = c_2 = 0$ and $c_5 = 5!$

To evaluate c_3, suppose we have 3 colors. These 3 colors can be assigned properly to the vertices v_1, v_2 and v_3 in $3! = 6$ ways. v_5 must have the same color as v_3 and v_4 must have the same color as v_2

$\therefore c_3 = 6$

To evaluate c_4, with the four colors, v_1, v_2 and v_3 can be properly colored in $4 \cdot 6 = 24$ ways. The forth color can be assigned to v_4 or v_5, thus providing 2 choices.

$\therefore c_4 = 24 \times 2 = 48$

$$\therefore P_5(\lambda) = 3! \times \frac{\lambda(\lambda-1)(\lambda-2)}{3!} + 48\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{24}$$

$$+ 5! \times \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{5!}$$

$$= \lambda(\lambda-1)(\lambda-2)[\lambda^2-5\lambda+7]$$

Theorem.

A graph of n vertices is a complete graph iff its chromatic polynomial is

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)$$

Proof.

With λ colors, the first ~~a~~ vertex of a graph can be colored in λ different ways. Second vertex can be colored properly in exactly $\lambda-1$ ways, the third in $\lambda-2$ ways, ... and nth vertex in $\lambda-(n-1)$ ways. iff every vertex is adjacent to every other. That is iff the graph is complete.

Theorem.

An n-vertex graph is a tree iff its chromatic polynomial

$$P_n(\lambda) = \lambda(\lambda-1)^{n-1}.$$

Proof.

Let G be a tree on n vertices.
We prove the result by induction on n
If $n=1$, then G contains only one vertex which can be colored in λ distinct ways only.
Hence the result holds in this case.
If $n=2$, then G contains one edge, so that exactly two colors are required for the proper coloring of the graph. Hence $c_1 = 0$ and two colors can be assigned in two different ways for the vertices of the graph. $\therefore c_2 = 2$.

$$\therefore P_n(\lambda) = 0 + \frac{\lambda(\lambda-1)}{2!} \times 2 = \lambda(\lambda-1)$$

Hence the result holds with $n=2$.
Now assume the result is true for $n-1$ vertices.

ie $\cancel{P_n(\lambda) = \lambda(\lambda-1)^{n-2}}$

$$P_{n-1}(\lambda) = \lambda(\lambda-1)^{n-2}.$$

Since the graph is a tree, it contains a pendant vertex. Let v be the pendant vertex of the graph. Let G' be the graph obtained by deleting the vertex v. Then by inductive hypothesis the chromatic polynomial of G' is $\lambda(\lambda-1)^{n-2}$.

Now for each proper coloring of G' the given graph can be properly colored by painting the vertex v with the color other than vertex adjacent to the vertex v.

Thus we can choose $(\lambda-1)$ colors to v

Hence total $\lambda(\lambda-1)^{n-1}(\lambda-1) = \lambda(\lambda-1)^{n-1}$ ways we can properly color the given tree.

Thus the result hold by induction