Module . IV

Cut-Sets and Cut Vertices.

Cut Vertex.

A cut vertex in a connected graph G is a vertex whose removal increase the number of components.

If v is a cut vertex of a connected graph G, then G − v is disconnected. A cut vertex is also called a cut point.

Cut Edge.

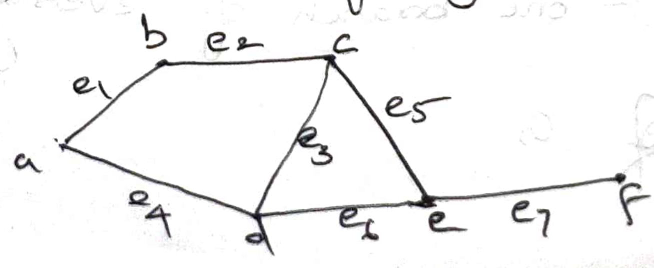
An edge whose removal increase the number of components. Cut edge is also called bridge.

Cutset of a graph G.

The set of all minimum number of edges of G whose removal disconnects a graph G is called a cutset of G.

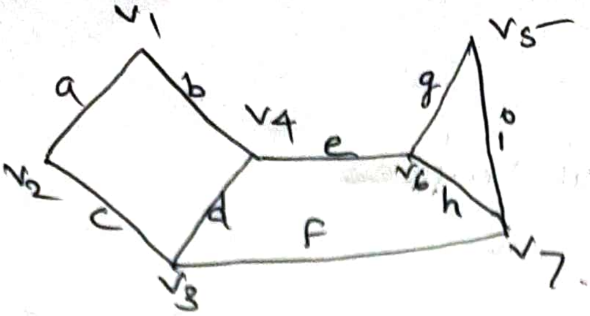
The cutset S of a graph G satisfy the following.

1) S is a subset of edge set E of G
2) No proper subsets of S disconnect the graph.
3) Removal of edges in S disconnect the graph.



'c' is a cut vertex
'd' is a cut vertex
but 'a' is not a cut vertex
$\{e_7\}$, $\{e_3, e_5, e_6\}$, $\{e_2, e_4\}$ are cutsets.

HW

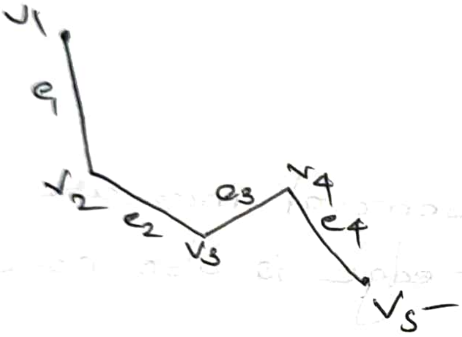


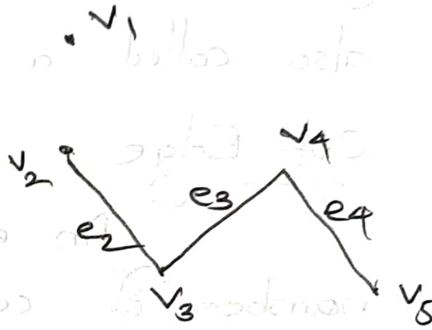
Write any 3 cut vertices.

Write any 3 cutsets.

Note

In a tree every edge is a cutset





∴ Removal of e_1 disconnects the tree

Removal of e_2 disconnect the tree.

Theorem

Every cutset in a connected graph must contain at least one branch of every spanning tree of G.

Proof.

Spanning tree is a tree that contain all vertices of G. If the cutset does not contain any branch of the spanning tree, the removal of S does not disconnects the graph. So S contains atleast one brach of the every spanning tree of G.

Converse of the above theorem.

Theorem

In a connected graph G, any minimal set of edges containing at least one branch of every spanning tree of G is a cutset.

Proof

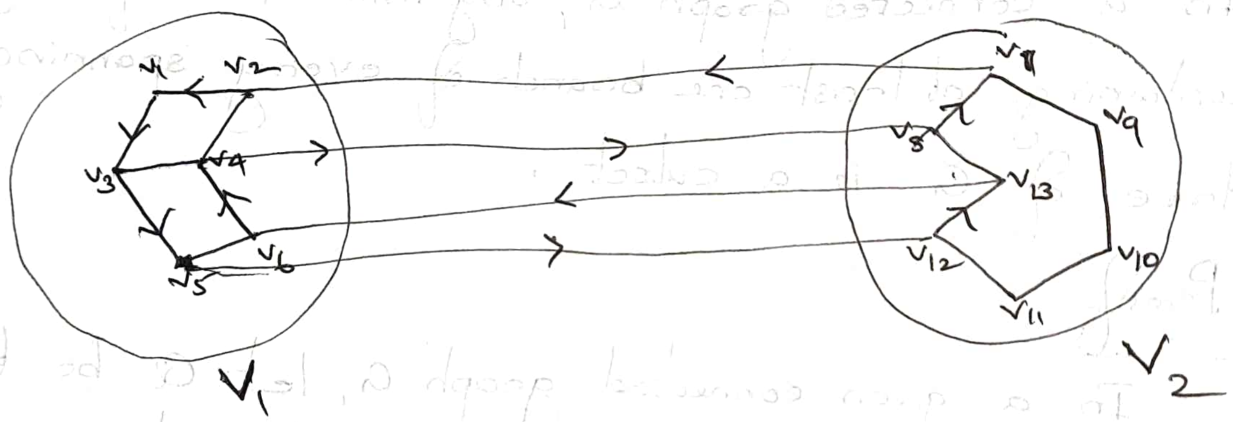
In a given connected graph G, let Q be ~~the~~ a minimal set of edges containing at least one branch of every spanning tree of G.

Consider G-Q, the subgraph that remains after removing the edges in Q from G. Since the subgraph G-Q contains no spanning tree of G, G-Q is ~~disc~~ disconnected. Also since Q is a minimal set of edges with this property, an edge e from Q returned to G-Q will create at least one spanning tree. Thus the subgraph G-Q+e will be a connected graph. $\therefore$ Q is a minimal set of edges whose removal from G disconnects G. $\therefore$ Q is a cutset. Hence the theorem.

Theorem

Every circuit has an even number of edges in common with any cut-set.

Proof



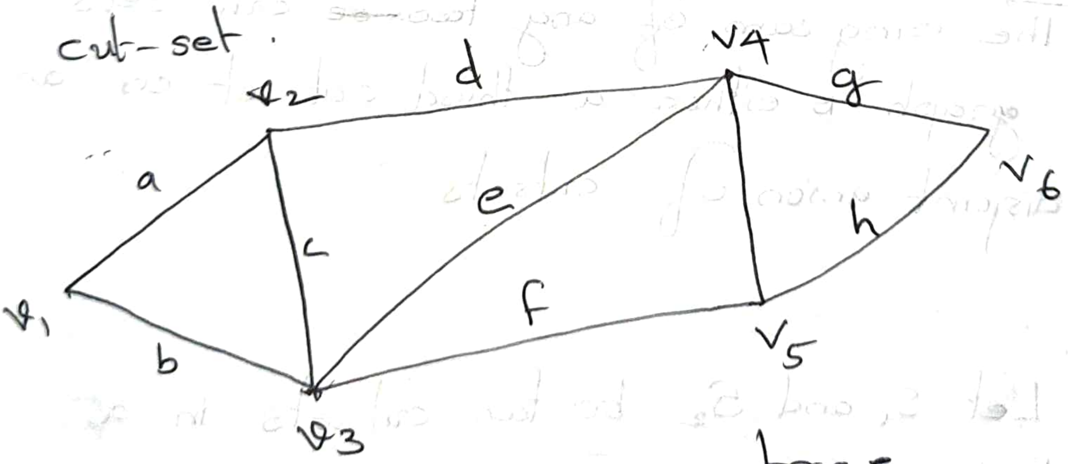
Consider a cutset S in a graph G. Let the removal of S partition the vertices of G into two ~~mutually~~ disjoint subsets V_1 and V_2. Consider a circuit Γ in G. If all the vertices in Γ are entirely within one of the vertex set V_1 or V_2, the number of edges common to S and Γ is zero, which is even.

If on the otherhand, some vertrices of Γ are in V_1 and some in V_2. Because of the closed nature of a circuit, the number of edges we traverse between V_1 and V_2 must be even. And since every edge in S has one end in V_1 and the other in V_2, and no other edge in G has this property, the number of edges common to S and Γ is even.

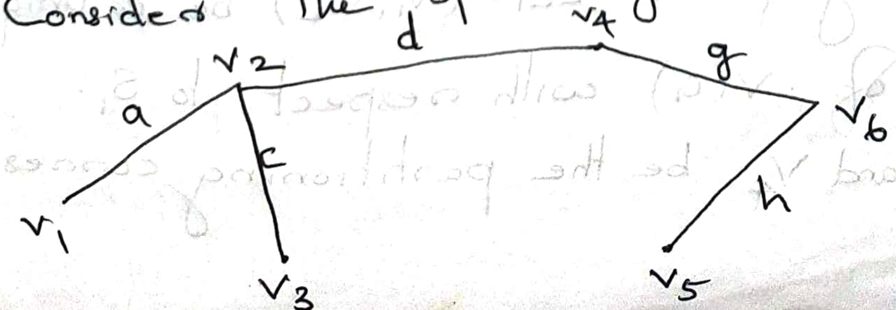
Fundamental Cut-Sets.

Consider a spanning tree T of a connected graph G. Cutsets S of G that contain only one branch of the spanning tree are called fundamental cutset of G with respect to T.

Fundamental cut-set is also called a basic cut-set.



Consider the spanning tree



$\{d, e, f\}$ is a fundamental cut-set w.r.t. T
$\{c, d, f\}$ is another fundamental cut-set.

Ring Sum.

Consider the graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$. The ring sum of two graphs G_1 and G_2 is a ~~sub~~graph consisting of the vertex set $V_1 \cup V_2$ and of the edges that are either in G_1 or G_2 but not in both.

If g is a subgraph of G, then $G \oplus g$ is a subgraph of G which remain after all the edges in g have been removed from G.

Theorem-

The ring sum of any ~~two~~ two cut-sets in a graph is either a third cut-set or an edge disjoint union of cutsets.

Proof.

Let S_1 and S_2 be two cutsets in a connected graph G. Let (V_1, V_2) be a unique partition of $V(G)$ with respect to S_1. Let V_3 and V_4 be the partitioning corresponding to the cut-set S_2. Thus we have

$$V_1 \cup V_2 = V \qquad V_1 \cap V_2 = \phi$$

$$V_3 \cup V_4 = V \qquad V_3 \cap V_4 = \phi$$

Now we have

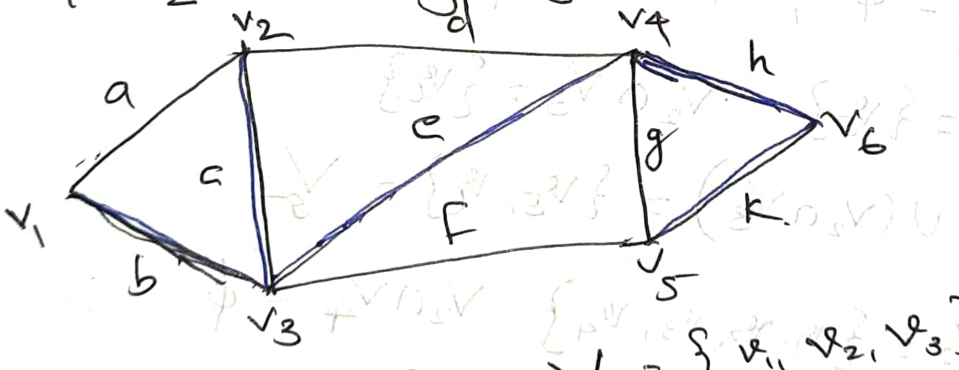
$$(V_1 \cap V_4) \cup (V_2 \cap V_3) = V_1 \oplus V_3 = V_5$$

$$(V_1 \cap V_3) \cup (V_2 \cap V_4) = V_2 \oplus V_4 = V_6$$

The ring sum $S_1 \oplus S_2$ consists of the edges that join vertices in V_5 to those in V_6. Thus the ~~st~~ set of edges $S_1 \oplus S_2$ partitions $V(G)$ into two sets V_5 and V_6 such that

$$V_5 \cup V_6 = V \qquad V_5 \cap V_6 = \phi .$$

Hence $S_1 \oplus S_2$ is a cutset if the subgraphs containing V_5 and V_6 each remain connected after $S_1 \oplus S_2$ is removed from G. Otherwise $S_1 \oplus S_2$ is an edge disjoint union of cutsets.



$V_1 = \{v_1, v_2, v_3\}$ $\qquad V_2 = \{v_4, v_5, v_6\}$

$S_1 = \{d, e, f\} \rightarrow$ $\qquad V_4 = \{v_5, v_6\}$

$S_2 = \{f, g, h\}$ $\qquad V_3 = \{v_1, v_2, v_3, v_4\}$

$V_1 \cap V_4 = \phi$ $\qquad \cancel{V_2 \cap V_3 = \{v_4\}}$ $V4$

$(V_1 \cap V_4) \cup (V_2 \cap V_3) = \{v_4\} = v_1 \oplus v_3 = V_5$

$\cancel{(V_1 \cap V_3)}$

$V_1 \cap V_4 = \phi \ , \ V_2 \cap V_3 = \{v_4\}$

$(V_1 \cap V_4) \cup (V_2 \cap V_3) = \{v_4\} = V_5$

$V_1 \cap V_3 = \{v_1, v_2, v_3\} \quad V_2 \cap V_4 = \{v_5, v_6\}$

$(V_1 \cap V_3) \cup (V_2 \cap V_4) = \{v_1, v_2, v_3, v_5, v_6\} = V_6$

$V_5 \cup V_6 = V \ , \ V_5 \cap V_6 = \phi$

$S_1 \oplus S_2 = \{d, e, g, h\}$ it is a cut-set

Again.

$S_1 = \{d, e, f, h\} \qquad S_2 = \{F, g, k\}$

$V_3 = \{v_1, v_2, v_3, v_4, v_6\} \quad V_4 = \{v_5\}$

$V_3 \cup V_4 = V, \quad V_3 \cap V_4 = \phi$

$V_1 = \{v_1, v_2, v_3, v_4, v_5\} \ V_2 = \{v_6\}$

$V_1 \cap V_2 = \phi \ , \ V_1 \cup V_2 = \phi$

$V_1 \cap V_4 = \{v_5\} \quad V_2 \cap V_3 = \{v_6\}$

$(V_1 \cap V_4) \cup (V_2 \cap V_3) = \{v_5, v_6\} = V_5$

$V_1 \cap V_3 = \{v_1, v_2, v_3, v_4\} \quad V_2 \cap V_4 = \phi$

$(V_1 \cap V_3) \cup (V_2 \cap V_4) = \{v_1, v_2, v_3, v_4\} = V_6$

Then $V_5 \cup V_6 = V, \ V_5 \cap V_6 = \phi$.

$S_1 \oplus S_2 = \{d, e, F, h, k\}$

$= \{d, e, f\} \cup \{h, k\}$

$\{d, e, f\}$ and $\{h, k\}$ are cut-sets of the given graph.

Theorem.

With respect to a given spanning tree T, a chord c_i that determines a fundamental circuit Γ occurs in every fundamental cutset associated with ~~the~~ the branches in Γ and in no other.

Proof

Consider a spanning tree T in a given connected graph G. Let c_i be a chord with respect to T. Let the fundamental circuit made by c_i be called Γ, consisting of k branches $b_1, b_2, \ldots b_k$ in addition to the chord c_i

$$\Gamma = \{c_i, b_1, b_2, \ldots b_k\}$$

Every branch of any spanning tree has fundamental cut-set associated with it. Let S_1 be the fundamental cut-set associated associated with b_1 consisting of q chords $S_1 = \{b_1, c_1, c_2, \ldots c_q\}$. It consisting of q chords in addition to the branch b_1

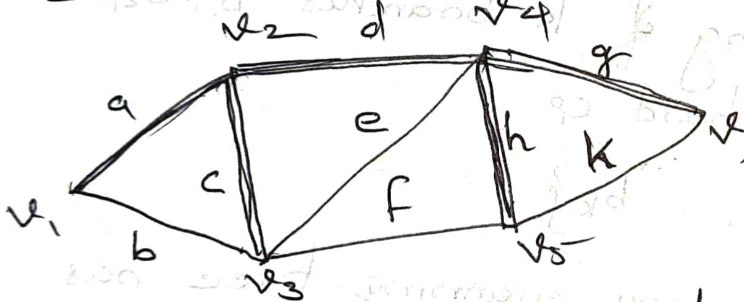
Since there must be an even number of edges common to Γ and S_1, c_i is one of the chord $c_1, c_2, \ldots c_q$.

Exactly same arguement holds for fundamental cut-sets associated with $b_2, b_3 \ldots b_k$.

Therefore, the chord c_i contained in every fundamental cut-sets associated with Γ.

If possible suppose that S' is a fundamental cut-set contain the chord c_i. Then ~~no~~ S' does not contain $b_1, b_2, \ldots b_k$. Thus the only edge common to S' and Γ is c_i. This is a contradiction. Therefore chord c_i contained in every fundamental cut-set associated with branches in Γ.

Example.



Consider the spanning tree $\{a, c, d, h, g\}$

Consider the chord f. The fundamental circuit Γ associated with f is

$\Gamma: \{d, c, f, h\}$

Fundamental cutset determined by d, c, h are respectively

$\{d, e, f\}$, $\{b, c, e, f\}$ and $\{f, h, k\}$

f belong to all cut-sets.

Theorem.

With respect to a given spanning tree T, a branch b_i that determines a fundament cut set S is contained in every fundamental circuit associated with the chords in S, and in no others.

Proof:

Let the fundamental ~~circuit~~ cut set S determined by a branch b_i be

$S = \{b_i, c_1, c_2 \ldots c_k\}$

and let Γ_1 be the fundamental circuit determined by chord c_1

$\Gamma_1 = \{c_1, b_1, b_2, \ldots b_p\}$

Since the number of edges common to S and Γ_1 must be even, b_i must be in Γ_1.

Exactly same arguements holds for the fundamental circuits made by chords $c_2, c_3 \ldots c_k$.

If possible suppose that Γ_{k+1} is a fundamental circuit ~~in which b_i occurs and~~ made by chords other than $c_1, c_2 \ldots c_k$ in which b_i occurs. ~~Since $c_1, c_2 \ldots c_k$ do not~~. Since none of the chords $c_1, c_2 \ldots c_k$ is in Γ_{k+1}, there is only one edge b_i common to Γ_{k+1} and S.

This is not possible. Hence the theorem.

Example.

$S = \{d, e, f\}$ is a fundament cut set determined by the branch d.

The fundamental circuits determine by the chords e and F are

$\Gamma_e = \{d, c, e\}$

$\Gamma_F = \{d, c, h, f\}$

d included in the fundamental circuits determined by e and F.

None of the remaining fundamental circuits contain branch 'd'.

Connectivity.

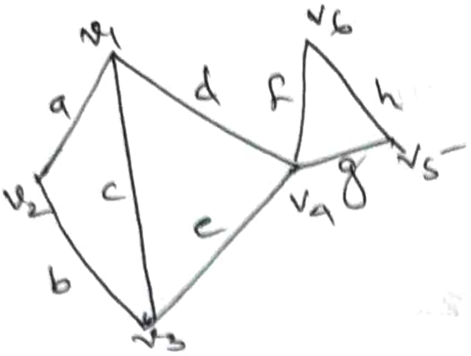
Edge Connectivity -

Let G be a graph having k components. The minimum number of edges whose deletion from G increases the number of components of G is called the edge connectivity of G.

The number of edges in the smallest cut-set of a graph is its edge connectivity.

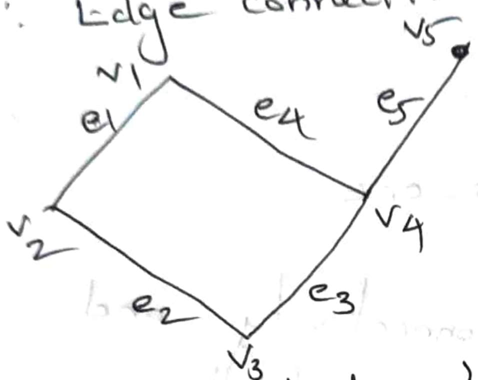
The edge connectivity of a tree is 1.

Eg:

1) 

Smallest cutset - {f, h} or {g, h} or {a, b}

∴ Edge connectivity 2.

2) 

Smallest cutset - {e5}

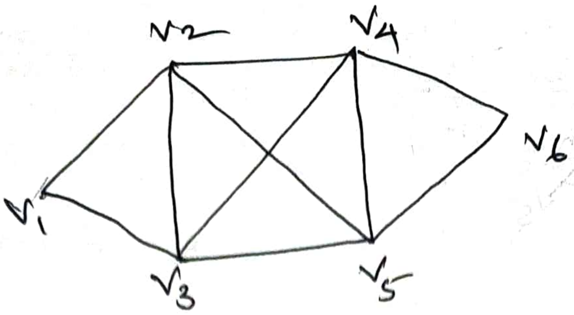
∴ Edge connectivity is 1

Vertex Connectivity.

Let G be a graph. The minimum number of vertices whose deletion from G increases the number of components of G is called the vertex connectivity of G

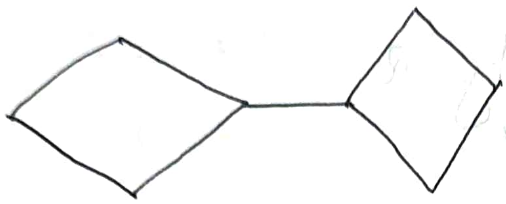
The vertex connectivity of a connected graph G is defined as the minimum number of vertices whose removal makes the graph disconnected. Vertex connectivity of a tree is 1

1)



Vertex Connectivity is 2.

2)

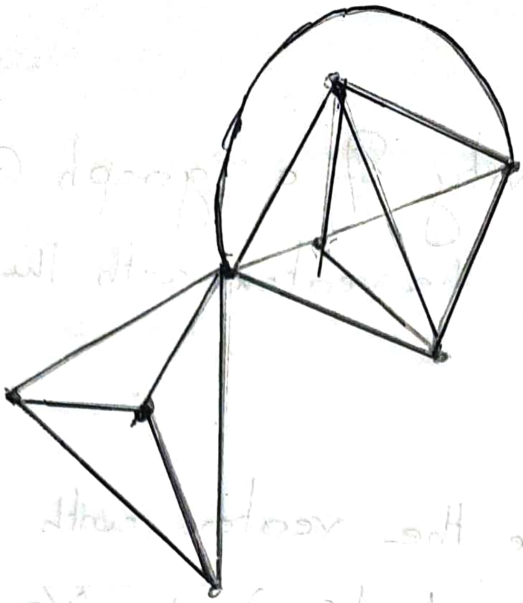


Vertex connectivity is one.

Application of Vertex Connectivity and Edge Connectivity.

Suppose we are given n stations that are to be connected by means of e lines (telephone lines, bridges, rail roads etc). where $e \geq n-1$. Construct a graph with n vertices and 'e' edges that has maximum possible edge connectivity and vertex connectivity

For example take $n=8$, $e=16$



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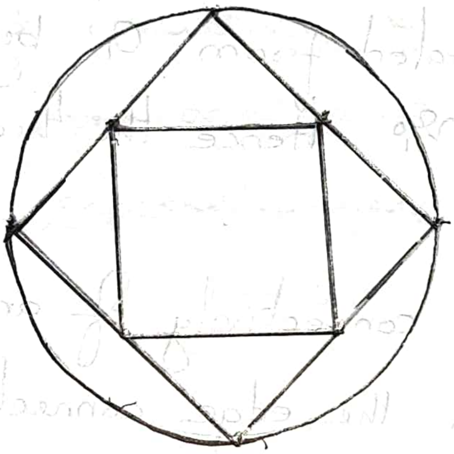


II

Edge connectivity as well as vertex connectivity of II graph is 4. But for frist graph. vertex connectivity is 1 and edge connectivity is 3.

In the second graph even after any 3 stations are bombed or any 3 lines are destroyed, the remaining stations can still continue to communicate with each other.

Theorem.

The edge connectivity of a graph G cannot exceed the degree of the vertex with the smallest degree in G.

Proof:-

Let vertex v_i be the vertex with the smallest degree in G. Let $d(v_i) = k$. Vertex v_i can be separated from G by removing k edges incident on v_i. Hence the theorem.

Theorem

The vertex connectivity of any graph G can never exceed the edge connectivity of G.

Proof.

Let α denote the edge connectivity of G. Therefore $\exists$ a cut-set S in G with α edges. S partition the vertices of G into two subsets V_1 and V_2. By removing at most α vertices from V_1 (or V_2) on which the edges in S are incident, we can effect the removal of S together with all other edges incident on these vertices from G. Hence vertex connectivity of any graph G never exceed the edge connectivity of G.

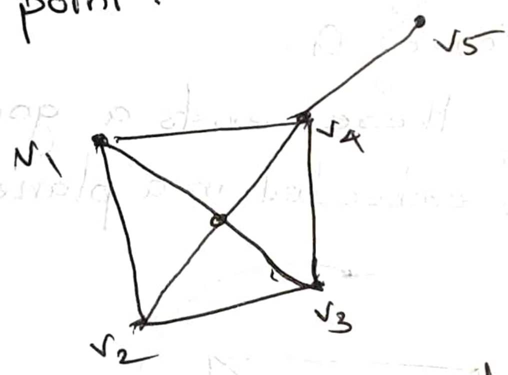
Theorem

The maximum vertex connectivity one can archieve with a graph G on n vertices and e edges $(e \geq n-1)$ in the integral part of $\frac{2e}{n}$ $\left(\left[\frac{2e}{n}\right]\right)$

Separable graph

A connected graph is said tobe separable if its vertex connectivity is one. All other connected graphs are called non-separable.

In a separable graph a vertex whose removal disconnects the graph is called a cut-vertex, a cut-node or an articulation point.



v_4 is an articulation point