6/4/22

ASSIGNMENT II

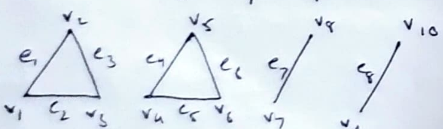
Q1. Consider a graph G with 4 vertices v1, v2, v3 and v4 and the degrees of vertices are 3, 5, 2 and 1 respectively. Is it possible to construct such a graph G? If not, why?

A1. Acc. to theorem –
" The no. of odd degree vertices in any graph is even "

- In the question, the no. of odd degree vertices are 3 (3, 5, 1) which is an odd no. Therefore, since it contradicts the above theorem, the given graph can't be constructed.

Q2. Draw a disconnected simple graph G1 with 10 vertices and 4 components and also calculate max. no. of edges possible in G1.

A2.



G1

- For a simple graph w/ n vertices and k components it can have at most –

$$\frac{(n-k)(n-k+1)}{2} \text{ edges.}$$

- Here, $n = 10$, $k = 4$.

Therefore, max. possible edges $= \frac{(10-4)(10-4+1)}{2}$

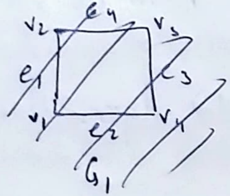
$= \frac{6 \times 7}{2} = 21$ edges

Q3. State Dirac's theorem for ~~harm~~ hamiltonicity and why it is not a necessary condition for a simple graph to have a Hamiltonian circuit?

A3. Dirac theorem –

Let G be a connected graph with p vertices and degree of each vertex is at least $p/2$. Then G has a Hamiltonian circuit i.e., G is Hamiltonian.

- ~~Consider the following graph G1 with 4 vertices and 4 edges~~

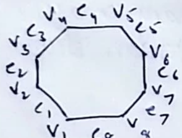


• Here, we can easily (find) a ~~Hamiltonian circuit~~

$v_1 e_1 v_2 e_4 v_3 e_3 v_4 e_2 v_1$

Better

- consider an octagon -



Here, $p = 8$; $p/2 = 4$.

Here we can definitely find a Hamiltonian circuit

$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_6 e_6 v_7 e_7 v_8 e_8$.

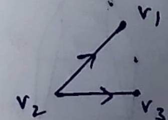
but $d(v_i) = 2 < \frac{p}{2}$

Therefore, Dirac's theorem is not a necessary condition.

Q4. Differentiate b/w symmetric and asymmetric digraphs with examples. and draw a complete symmetric digraph with 4 vertices.

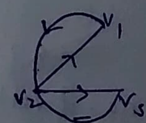
A4. Asymmetric digraph

• A digraph that have atmost one directed edge b/w a pair of vertices.

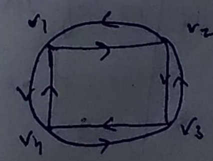


Symmetric digraph

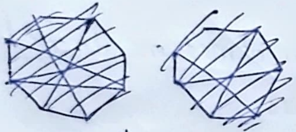
- A digraph in which if there is an edge from v_i to v_j there exists edge b/w v_j to v_i.

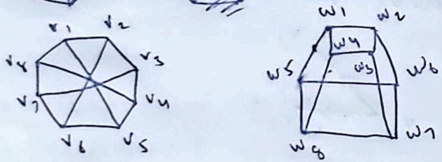


complete symmetric digraph with 4 vertices



... satisfied for two graphs to be isomorphic? Are the two graphs below isomorphic? Explain with valid reasons.





Ans a) For 2 graphs to be isomorphic –

- no. of vertices must be equal.
- no. of edges must be equal.
- equal no. of vertices w/ given degree.

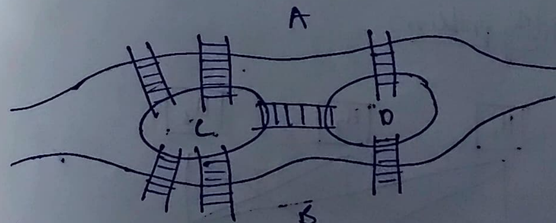
G1	G2.
- no. of vertices = 8.	no. of vertices = 8.
- no. of edges = 12	no. of edges = 12
- no. of vertices with degree 3 = 8	no. of vertices with degree 3 = 8.

since both graphs ... regular graph, every vertex in G1 has a mapping in G2, making them ISOMORPHIC.

b) Write 2 applications of graphs with examples. –

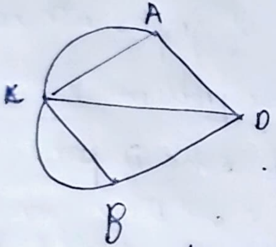
Ans. b) i) Konigsberg bridge problem

There are 2 islands C and D formed by Pregel river. If C and D are connected to each other and to banks A and B by 7 bridges as follows –



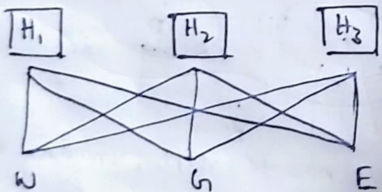
The problem was to start from any one of the 4 land areas A, B, C or D and walk over each bridge exactly once and to return back to starting point.

- Euler represented this situation by means of graph shown below with ABCD as vertices and bridges as edges



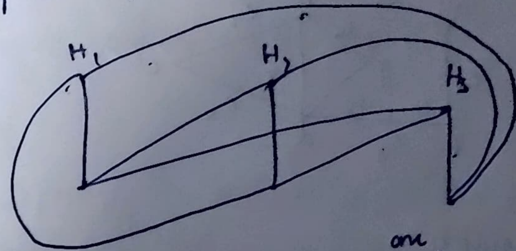
Euler proved that there is no solution available for this problem.

ii) Utility problem.



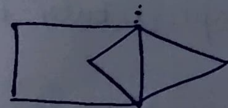
we have to provide 3 utilities - water, gas and electricity to each of 3 houses H_1, H_2 and H_3 by means of conduits without crossover.

graphical representation —



This graph can't be drawn in on plane without edges crossing over.

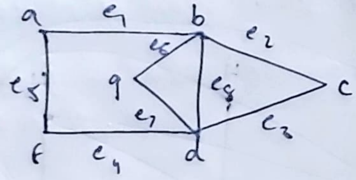
Q6. a) Consider the graph G given below -



Define Euler graph. Is G a Euler? If yes, write a Euler line from G.

b) what is necessary and sufficient condition for a graph to be Euler? And also prove it.

A6. a)



Euler graph -

A connected graph G is called a Euler graph, if there is a closed trail which includes every edge of the graph G also known as a Euler circuit.

- Yes, the given graph is Euler by theorem given below -

'If degree of every vertex is even in a simple connected graph, it is a euler graph'.

In the given graph -

$d(a) = 2$, $d(b) = 4$, $d(c) = 2$, $d(d) = 4$

$d(g) = 2$, $d(f) = 2$.

- Euler line in G -

$a\, e_1\, b\, e_2\, c\, e_3\, d\, e_8\, b\, e_6\, g\, e_7\, d\, e_4\, f\, e_5\, a$.

b) A ~~g~~ connected graph is euler iff. all vertices of G are of even degree.

Proof -

- suppose G is a Euler graph and also connected. Then G contains a Euler line which is also a closed walk containing all edges of G exactly once.

Then, as we trace the walk, we find that the walk meets a vertex v along one edge and leaves that vertex along another edge.

This is true for all vertices of the walk, including the terminal vertex.

Hence, we can say that every vertex has to be of even degree.

- Conversely, suppose that all vertices of G are of even degree and G is connected.

Then, starting a walk from an arbitrary vertex v and going through the edges of G, one by one, without repetition and finally coming back to the vertex v.

It is possible as every vertex is of even degree and since G is connected, we can trace through all edges of G.

- Then G is a Euler graph.
- Hence, proved!

Q7. a) Define Hamiltonian circuits and paths with examples. Find out the number of edge-disjoint Hamiltonian circuits possible in a complete graph with five vertices.

b) State Travelling-salesman Problem and how TSP solution is related with Hamiltonian circuits?

A7. a) Hamiltonian path -

simple path with all vertices appearing exactly once.

Hamiltonian circuit

circuit in G that contains all vertices exactly once except the end vertices.

- By theorem, maximum possible edge disjoint Hami-circuits possible in a complete graph with n vertices is $\frac{(n-1)}{2}$ given n is odd and $n \geq 3$.

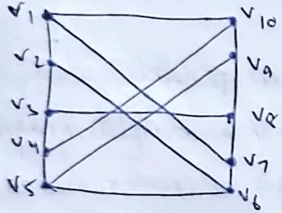
Here, $n=5$ $\therefore$ max. edge disjoint Hami circuits $= \frac{5-1}{2} = 2$ circuits

b) Travelling salesman Problem

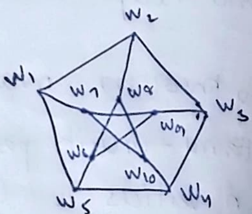
It is a problem where salesman has to travel to many cities. To solve a TSP, we need to find cheapest way for the travelling salesman to visit every city and to reach back where he started.

- This is done by finding the Hami-circuit in a complete graph that has the lowest overall weight.

Q8. Are the two graphs isomorphic to each other? Justify your answer.



G_1



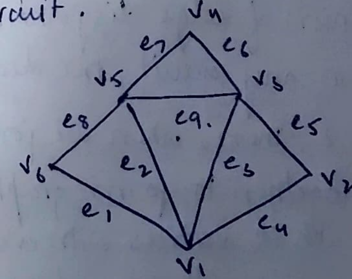
G_2

A8.

	G_1	G_2
(i)	No. of vertices = 8	No. vertices = 8
(ii)	No. of edges = 12	No. of edges = 12
(iii)	No. of vertices with degree 3 = 8.	No. of vertices with degree 3 = 8.

since both given graphs are 3 regular graphs every vertex in G_1 has mapping ~~in~~ in G_2 making them ISOMORPHIC.

Q9. For the following graph, find the shortest path b/w v1 and v4. Also find a Euler circuit.



A9.
- shortest path from v_1 to v_4 -

$v_1\ e_3\ v_3\ e_6\ v_4$

or

$v_1\ e_2\ v_5\ e_7\ v_4$

- A Euler circuit -

$v_1\ e_1\ v_6\ e_8\ v_5\ e_7\ v_4\ e_6\ v_3\ e_5\ v_2\ e_4\ v_1\ e_2\ v_5\ e_9\ v_3\ e_3\ v_1$

Q10. For a Eulerian graph G, prove that G is an edge disjoint union of cycles.

A10. Proof -

suppose G can be decomposed into circuits. Then G is a union of edge disjoint circuits. We know that in any circuit, the degree of each vertex is 2. Then, when we join all these circuits together to form graph G, G will have all its vertices with even degree. Since G has all even degree vertices G is a Euler graph.

- conversely, suppose G is a Euler graph. Then G has all its vertices with even degree. Consider a vertex v_1. Degree of v_1 is at least 2 i.e., 2 edges are incident on v_1. Let one of these edges be connecting v_1 and v_2. Since v_2 is also of even degree, it must have atleast 2 edges incident on it. Let one of them be connecting v_2 and v_3.

Proceeding in this fashion we eventually arrive at a vertex that has been previously traversed, thereby forming a circuit.

~~Q11. a) Prove that a simple graph with n vertices must be connected, if it has more than $\frac{(n-1)(n-2)}{2}$ edges.~~

~~b) 19 students in a nursery school play a game each day, where they hold hands to form a circle. For how many days can they do this, with no students holding hands with the same playmates more than once? Substantiate your answer with graph theoretic concepts.~~

~~A11. a) To prove - A simple graph is connected graph if it has more than $\frac{(n-1)(n-2)}{2}$ edges or $\frac{n(n-1)}{2} - (n-1)$ edges.~~

~~Proof - We know,~~

Q11. 19 students in a nursery school play a game each day, where they hold hands to form a circle. For how many days can they do this, with no students holding hands with the same playmates more than once? Substantiate your answer with graph theoretic concepts.

A11. Let each child represent a vertex in a graph, and two children holding hands be represented by an edge connecting the vertices.

Therefore, the formation each day would be an edge disjoint ~~Hami-gra~~ Hmi-circuit.

According to theorem, a connected graph with n vertices can have atmost $\frac{(n-1)}{2}$ Hami-circuits given that n is odd and $n \geq 3$.

Hence, no. of ~~days~~ days $= \frac{19-1}{2} = \frac{18}{2}$

$= \underline{\underline{9 \text{ days}}}$

Q12. Show that in a simple graph with n vertices, the maximum no. of edges is $n(n-1)/2$ and the maximum degree of any vertex is $(n-1)$.

A12. Proof – Let G be a simple graph.

Since it is a simple graph, G will have no self loops / parallel edges.

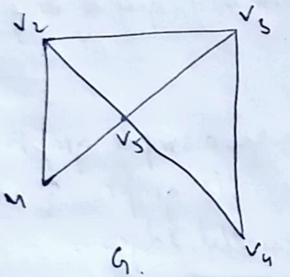
- Since there are total n vertices in G, each vertex can have a max. degree of $(n-1)$.

- Sum of the degree of all ~~vert~~ vertices is therefore – $n \times (n-1)$.

- Since, each edge contributes '2' degree to the graph, max. no. of edges could be $\frac{n(n-1)}{2}$

- Hence, proved!

Q13. Check applicability of Dirac's Theorem in the following graph, G.



A13. Total no. of vertices in G = p = 5.

According to Dirac's theorem, a sufficient condition for a simple graph G with p vertices to have a Hamiltonian circuit is that, the degree of every vertex in G be be atleast $p/2$ (for $p \geq 3$).

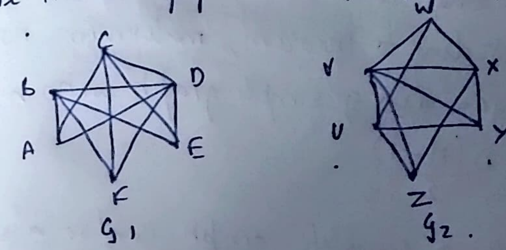
- Here $p/2 = 2.5 \simeq 3$.

In the given graph G, Dirac's theorem is not satisfied as $d(v_1) = 2$ and $d(v_4) = 2$

Therefore, Dirac's theorem is not applicable for graph i.e, G may or may not be Hamiltonian.

Q14. Are the two graphs below isomorphic? Justify.

a)



b) Consider a complete graph G with 11 vertices.

1. Find the max. no. of edges possible in G.
2. Find the number of edge disjoint Hamiltonian circuits in G.

A14. a).

	G_1	G_2.
(i)	no. of vertices = 6	no. of vertices = 6
(ii)	no. of edges = 11	no. of edges = 9.

- Since, both G_1 and G_2 have diff. no. of edges, every vertex in G_1 won't have a mapping in G_2. Therefore, G_1 and G_2 are NOT ISOMORPHIC

b) ~~According~~

1. According to theorem, maximum no. of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Since, a complete graph is also a simple graph, max. edges in complete graph w/ 11 vertices $= \frac{11(11-1)}{2} = \underline{55 \text{ edges}}$

2. No. of edge disjoint ~~vertices~~ Hami-circuits in a ~~connected~~ complete graph with 11 vertices $= \frac{11-1}{2} = \frac{10}{2} = \underline{5 \text{ circuits}}$

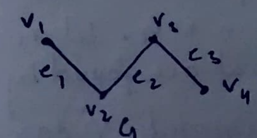
Q15. There are 37 telephones in the city of Istanbul, Turkey. Is it possible to connect them with wires so that each telephone is connected with exactly 7 others? substantiate your answer with graph concepts.

A.15. ~~The~~ Let each telephone be represented by a vertex G which would represent the network of telephones. Therefore, the total no. of vertices in G = 37.

- Now, since each telephone is to be connected with exactly 7 other telephones, the degree of each vertex in G would be equal to 7 (connection b/w two telephones will be represented by an edge in G).
- Since, in any graph the total no. of odd degree vertices should be equal, the given network is <u>not possible</u>.

Q16. Draw a graph that has a Hamiltonian path but not a Hamiltonian circuit.

A16.



Hamiltonian path –

$v_1 e_1 v_2 e_2 v_3 e_3 v_4$

Q17. Prove that if a connected graph G is decomposed into two subgraphs g1 and g2, there must be atleast one vertex common between g1 and g2.

A17. Proof -

Suppose a graph G can be partitioned into g1 and g2 with vertex sets V_1 and V_2

- consider two arbitrary vertices a and b of g1 and g2 such that $a \in V_1$ and $b \in V_2$.

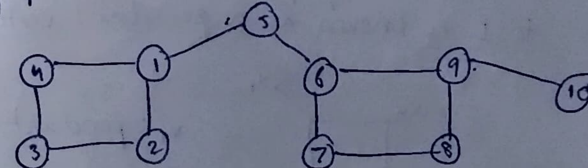
No path can exist b/w vertices a and b otherwise, there would be atleast one edge whose one end vertex would be in V_1 and the other is in V_2.

Hence if a partition exists, G is not connected.

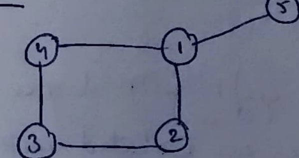
- conversely, let G be a disconnected graph. consider a vertex a in G. Let V_1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V_1 does not include all vertices of G. The remaining vertices will form V_2

- No vertex in V_1 is joined to any ~~edge~~ in V_2 by an edge. Hence the partition.

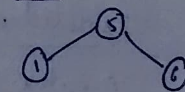
Q18. Print a walk, trail, path and cycle on the graph below.



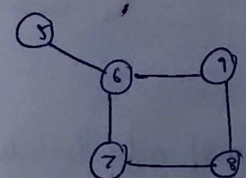
walk.



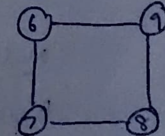
Path



~~Pe~~ Trail



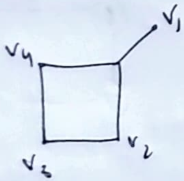
Cycle



Q19. Define pendant vertex, ~~si~~ isolated vertex and null graph with an example each.

A19. <u>Pendant vertex</u>

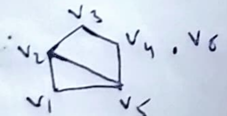
A vertex in a graph with degree equal to 1 is known as a pendant vertex.



v_1 : pendant ~~at~~ vertex

<u>Isolated vertex</u>

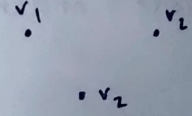
A vertex in a graph with degree equal to 0 is known as an isolated vertex



v_6 : isolated vertex

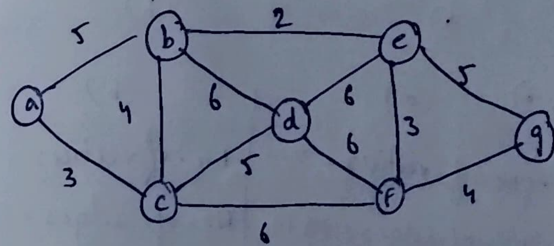
<u>Null graph</u>

A graph comprising of only isolated vertices is referred to as a null graph.

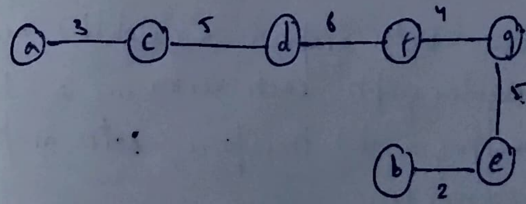


A null graph

Q20. Print a travelling salesman's tour in the graph below.



A20. The tour with the least weight possible is given below -

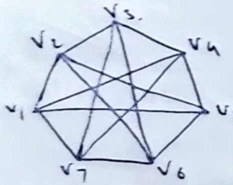


overall weight = 3+5+6+4+2

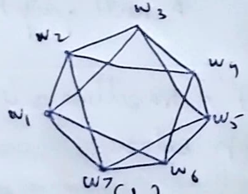
= <u>20</u>.

Q21. Show that the graphs (a) and (b) are isomorphic.

a)



(a)



(b)

A21. a)

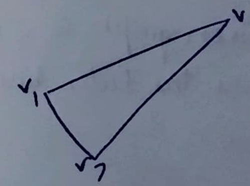
(a)	(b)
- No. of vertices = 7	No. of vertices = 7
- No. of edges = ~~14~~ 14	No. of edges = 14.
- No. of vertices w/ degree 4 = 7	No. of vertices w/ degree 4 = 7.

- Since both graph's (a) and (b) are a 4 regular graph, each vertex in (a) has a mapping in (b) therefore, both graphs are <u>ISOMORPHIC</u>

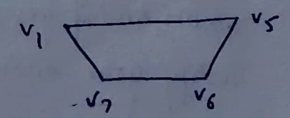
b) Define subgraph. Give 2 subgraphs of graph (a)

- A graph g is said to be a subgraph of a graph G, ~~iff~~ if all the vertices and all the edges of g are in G, and each edge of g has the same end vertices in g as in G.

- 2 ~~sub~~ subgraphs of (a) are —



g_1



Q22. Draw a simple disconnected graph with 10 vertices, 4 components and maximum number of edges.

A22. max. edges $= \frac{(n-k)(n-k+1)}{2}$ edges

where – n = no. of vertices

k = no. of components.

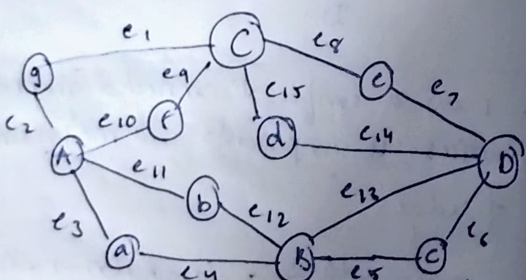
$= \frac{(10-4)(10-4+1)}{2} = \underline{21 \text{ edges}}.$

G.

- graph G has 21 edges and 4 components.

Q23. check whether the given graph is a Euler graph and if yes, give the Euler line. Justify your answer.

A23:-



A23. A Euler line is a closed walk that contains all edges of the graph exactly once.

- In the given graph a q Euler line can be drawn as -

$C\ e_1\ g\ e_2\ A\ e_3\ a\ e_4\ B\ e_5\ c\ e_6\ D\ e_7\ e\ e_8\ C\ e_9$ -

- $f\ e_{10}\ A\ e_{11}\ b\ e_{12}\ B\ e_{13}\ D\ e_{14}\ d\ e_{15}\ g$.

- A Euler graph is a graph that contains a Euler line.

- Therefore, given graph is Euler.

Q24. In a graph G let p1 and p2 be 2 different paths between two given vertices. Prove that ringsum of p1 and p2 is a circuit or a set of circuits.

A24. Assume, that the ring sum is not a circuit. Let us traverse from p1 to p2 in any one of the sub-graphs and back from p2 to p1 from the other sub-graph.

Now, if p1-p2-p1 is NOT a circuit as we've assumed, then, either of paths (p1-p2 or p2-p1) is broken.

But, we're given that there exists 2 different paths p1 and p2. This is a contradiction and therefore ring sum of p2 and p1 is a circuit.

- Similarly if the paths p1 and p2 have common vertices their ring sum would lead to a set of circuits.

Q25. What is the number of vertices in an undirected connected graph ~~of~~ with 27 edges, 6 vertices of degree 2, 3 vertices of degree 4 and remaining of degree 3?

A25. From theorem -
Sum of degrees of all vertices in a connected graph is equal to twice the number of edges,

$\therefore 2e = \sum d(v_i)$ (e = total edges)

$\therefore 2 \times 27 = \sum d(v_i) = 54$

Let no. of vertices with degree 3 be x.

$6 \times 2 + 3 \times 4 + x \times 3 = 54$

$x \times 3 = 30$

$x = 10$

$\therefore$ The graph has $6 + 3 + 10 = 19$ vertices.

Q26. There are 20 people in a room. Suppose some pairs of the people shake hands and some don't. As the people leave the room you ask each person whether they shook hands an odd number of times or an even number of times. Prove that the number of people who answered "odd" is an even number.

A26. Let each person be represented as a vertex of a graph G.
Let a handshake b/w two persons be represented by an edge. Therefore, depending on which ~~the~~ person shook/didn't shook their hand with another person, G may be connected or disconnected.

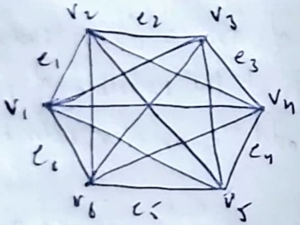
- But according to the theorem, "No. of vertices with odd degree is always even", the number.

of people who answered odd will be even.

Hence, proved!

Q27. Label the vertices and edges in the graph given below, find and separately draw any 4 Hami-circuits contained in the given graph

A27.



Remaining edges —

v1 —e7— v3

v3 —e8— v5

v5 —e9— v1

v1 —e10— v4

v4 —e11— v2

~~[illegible]~~

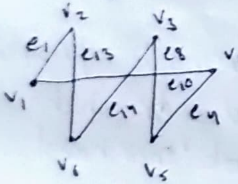
v5 —e12— v2

v2 —e13— v6

v6 —e14— v3

v4 —e15— v6

— 4 possible Hami-circuits —



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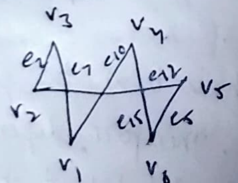
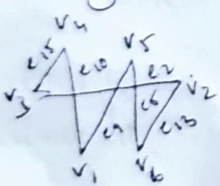
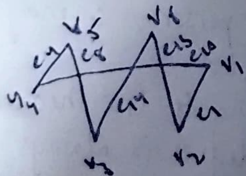


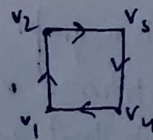




Q28. Explain strongly connected and weakly connected graphs with the help of examples.

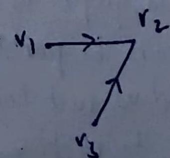
A28. A graph is said to be strongly connected if every pair of vertices (u, v) in the graph has a path connecting them.

Eg —



— A ~~gra~~ directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

Eg —



Q29. Define complete graph and complete bipartite graph. Draw a graph which is a complete graph. Draw a graph as well as a complete bipartite graph.

A29. complete graph

A simple graph with n vertices is said to be complete if there is an edge between every pair of vertices.

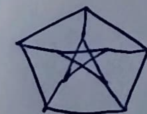
bipartite graph.

A bipartite graph is a graph in which the vertices can be partitioned into two disjoint sets V and W such that each edge is an edge b/w a vertex in V and a vertex in W.

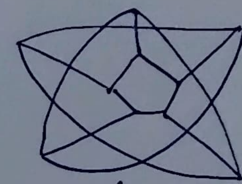
complete bipartite graph

A complete bipartite graph is a simple graph in which the vertices can be partitioned into two disjoint sets V and W such that each vertex in V is adjacent to each vertex in W.

Q30. Determine whether the following graphs are isomorphic or not.



G_1



G_2

G_1	G_2
- no. of vertices = 10	- no. of vertices = 10
- no. of edges = 15	- no. of edges = 15
- no. of vertices w/ degree 3 = 10	- no. of vertices w/ degree 3 = 10

- since both graphs are 3 regular graphs, there exists a one-one correspondence b/w every vertices in G_1 and G_2. Therefore both graphs are ISOMORPHIC