Coloring of maps-

The regions of a planar graph are said tobe properly colored if no two adjacent regions have the same color. Two regions are said tobe adjacent if they have a common edge between them. Note that one or more vertices in common does not make two regions adjacent. The proper coloring of regions is also called map coloring.

The four color problem

Every map (a planar graph) can be properly colored ~~by~~ with four colors-
(no proof)

A graph has a dual if and only if it is planar. Therefore, coloring the regions of a planar graph G is equivalent to coloring vertices of its dual G^* and vice versa

Five color theorem.

The vertices of ~~a~~ every plannar graph can be properly colored with five colours.

Proof:-

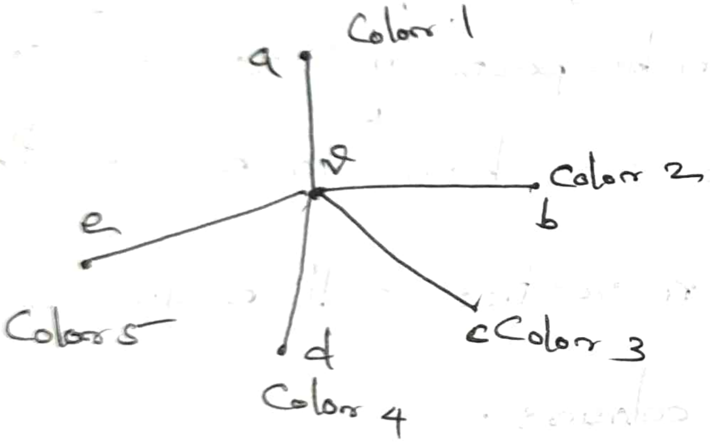
The theorem will be proved by induction. ~~Since~~ The vertices of all graphs with 1, 2, 3, 4 or 5 vertices can be properly colored with five colors. Assume that vertices of every planar graph with n-1 vertices can be properly coloured with five colours. Then we have to prove that planar graph with n vertices will require no more than five colours.

Consider the plannar G with n vertices. Since G is plannar, it must have at least one vertex with degree five or less. Let ~~be~~ this vertex be v. Delete v from G. Let G' be the remaining graph after deleting v.

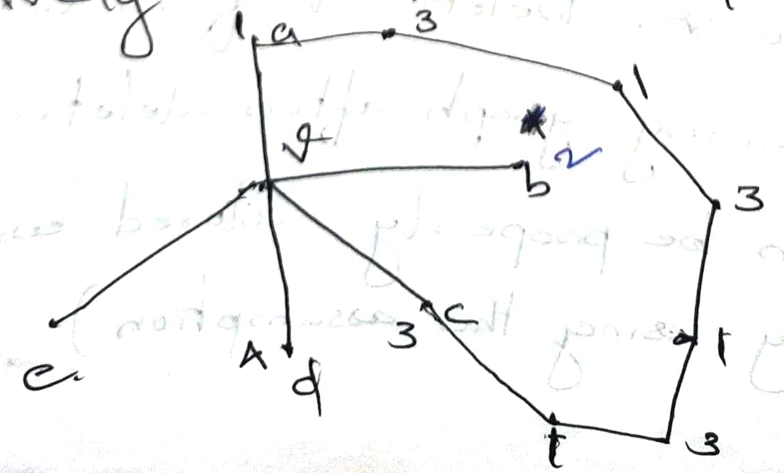
∴ G' can be properly colored with 5 colors. (~~Since~~ By using the assumption)

Suppose that the vertices in G' have been properly colored. Now we add v to G' and all edges incident ~~of~~ on v. If the degree of v is 1, 2, 3 or 4, we have no difficulty in assigning a proper color to v.

Consider the case when degree of v is 5 and all the five colors ~~are~~ are used for coloring the adjacent vertices of v.



Suppose that there is a path in G' between the vertices a and c colored alternatively with colors 1 and 3.



Then a similar path between b and d,
colored alternatively with colors 2 and 4 can
not exists. Otherwise these two paths will intersect
and G become non planar.

If there is no path between b and d
colored alternately with colors 2 and 4,
starting from vertex b we can interchange
colors 2 and 4 of all vertices connected
to b through vertices of alternating color
2 and 4. This interchange will paint vertex
b with color 4 and yet keep G' properly
colored. Since vertex d is still with color 4,
we have color 2 left over with which to
paint vertex v.

Coverings:-

In a graph G, a set g of edges is
said to cover G, if every vertex in G is
incident on atleast one edge in g.

A set of edges that covers a graph
G is said tobe an edge covering.

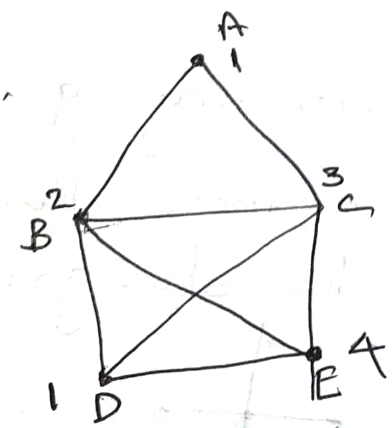
For a star graphs no edge can be removed without leaving a vertex ~~or~~ uncovered.

∴ g must be a minimal covering.

Greedy Coloring Algorithm

The algorithm processes the vertices in the given ordering, assigning a color to each one as it is processed. The colors may be represented by the numbers 1, 2, 3 ... and each vertex is given the color with the smallest number that is not already used by one of its neighbours. To find the smallest available ~~color~~ color, one may use an array to count the number of neighbours of each color.

Eg.



1 Blue

2 Red

3 Green

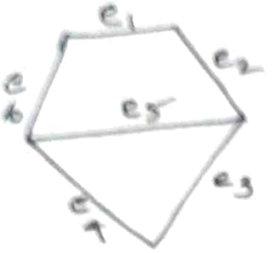
4 Black

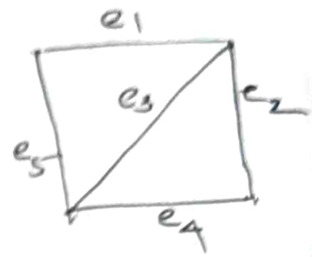
No. of colors required – 4.

Matchings

A matching in a graph is a subset of edges in which no two edges are adjacent.
A single edge in a graph is a matching.

Eg





$\{e_1, e_4\}$ is a matching

$\{e_1, e_5\}$ is a matching

$\{e_2\}$ is a matching.

$\{e_3\}$ is a maximal matching.

$\{e_1, e_4\}$ is a maximal matching.

Maximal matching.

A maximal matching is a matching to which no edge in the graph can be added.

Eg: In the above graph, $\{e_2, e_6\}$ is a maximal matching.

In K_3, any single edge is a maximal matching.

Largest Maximal Matching.

Maximal matchings with the largest number of edges are called the largest maximal matchings.

Matching number.

The number of edges in a largest maximal matching is called the matching number of the graph.

In a bipartite graph having a vertex partition V_1 and V_2, a complete matching of vertices in set V_1 into those in V_2 is a matching in which there is one edge incident with every vertex in V_1. In otherwords, every vertex in V_1 is matched against some vertex in V_2.

Clearly a complete matching is a largest ~~matching~~ maximal matching, where as the converse is not necessarily true.

Theorem-

In a bipartite graph a complete matching of V_1 into V_2 exists if there is a positive integer m for which the following condition is satisfied

degree of every vertex in $V_1 \geq m \geq$ degree of every vertex in V_2.

Proof:-

Consider a subset of r vertices in V_1. These 'r' vertices have at least $m \cdot r$ edges incident on them. Each $m \cdot r$ edge is incident to some vertex in V_2. Since the degree of every vertex in set V_2 is no greater than m, these $m \cdot r$ edges are incident on at least $\frac{m \cdot r}{m} = r$ vertices in V_2.

Thus any subset of r vertices in ~~V_2~~ V_1 is collectively adjacent to r or more vertrices in V_2. Therefore there exists a complete matching of V_1 into V_2

Theorem.

A complete matching of V_1 into V_2 in a bipartite graph exists if and only if every subset of 'r' vertices in V_1 is collectively adjace to r or more vertices in V_2 for all values of r.