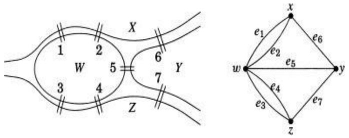
MOD 1

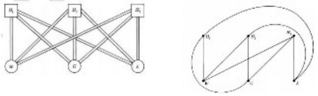
Applications of graphs

a)Konigsberg Bridge Problem: Two islands C and D were connected to each other and to the banks A and B with seven bridges as shown in figure. The problem was to start at any land areas A, B, C or D , walk over each of the seven bridges exactly once, and return to the starting point.

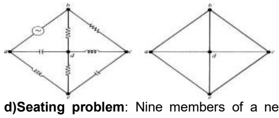


Euler represented this problem by means of a graph. Vertices represent the land areas and the edges represents the bridges. Euler proved that a solution

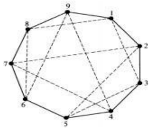
b)Utilities Problem: There are three houses H1 , H2 and H3, each to be connected to each of the three utilities water(W), gas(G) and electricity(E) by means of conduits. Is it possible to make such connections without any crossovers of conduits? This problem also can be represented by a graph where edges represented the conduits and vertices represented the houses and utility centres for this problem does not exists.



c)Electrical Network Problem: Topology of a electrical network is studied by means of graphs. Vertices represented the electrical network junctions and the edges represented the branches electrical network and it's graph.



d)Seating problem: Nine members of a new club meet each day for lunch at a round table. They decide to sit such that every member has different neighbours at ach lunch. How many days can this arrangement last? Solution: Seating arrangement of nine members in a round table can be represented by means of a graph. Each vertex represents a member and an edge joining two vertices represents the relationship



MOD 1

Walk, path, circuit

A **walk** is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the preceding and following it. No edge appeared more than once in a walk. A vertex may appear more than once. A walk is also referred to as an edge or chain. Vertices with which a walk begins and ends are called its terminal vertices.

A **path** is an open walk in which no vertices appears more than once. A path must not intersect with itself in between. The number of edges in a path is called the length of path. Degree of terminal vertices of a path is one and that of intermediate vertices is two.

A **circuit/ elementary cycle/ circular path** / polygon is a closed path in which no vertex appears more than once except the initial and final vertex. A circuit is a closed non intersecting walk. Every vertex of a circuit have degree two Circuit.

MOD 1

THE0REM 3

The maximum no. of edges in a simple graph with 'n' vertices is n(n-1)/2

Proof: Since it is a simple graph, no self loops or parallel edges

Consider a simple graph with n vertices.

Consider any vertex vi on it.

Since it is simple, each vertex is adjacent to atmost remaining n-1 vertices.

ie: degree of each vertex is atmost n-1

Sum of the degrees of all vertices is then atmost n (n-1).

∑_i=1^n▒d(v_i) = n(n-1)   2e = n(n-1)

max no.of edges, e = n(n-1)/2 Hence the proof

MOD 1

THEOREM 5

A complete graph with n vertices contains n(n-1)/2 edges"

Proof by the method of induction:

If n=1, max no. of edges, Emax = n(n−1)/2 = 1(1−1)/2 = 0

If n=2, Emax = n(n−1)/2 = 2(2−1)/2 = 1

The theorem is true for n=1 and n=2

Assume the theorem is true for n=k vertices.

Then, we have   Emax = k(k−1)/2

Then to prove that the theorem is true for n= k+1, add the (k+1)th vertex, to become a complete graph, need to connect it to the k original vertices requiring k additional edges.

Since for k vertices,   Emax = k(k−1)/2

Then for (k+1) vertices, Emax = k+k(k−1)/2

= 2k+k2−k/2   = k2+2k−k+1−1/2 = (k+1)2−(k+1)/2

= (k+1)[(k+1)−1]/2, true for n=k+1 Hence the proof

MOD 1

THEOREM-6

A graph G is disconnected if and only if its vertex set V can be partitioned into two non empty, disjoint subsets v1 and v2 such that there exists no edge in G whose one end vertex lies in subset v1 and the other in subset v2.

-PROOF:-Suppose that such a partition exists and vertex set V can be partitioned into two disjoint subsets V1 and V2Consider two arbitrary vertices a and b such that $a∈V1$ and $b∈V2$Then no path can exist between a and b because if such a path existed, then there has to be at least an edge whose one end vertex would lie in V1 and the other in V2. But there in no such edge.Hence if such a partition exists, then G is disconnected.Conversely, let G be a disconnected graph. Consider a vertex a in G. Let V1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V1 does not include all vertices of G. The remaining vertices will form a set V2 .No vertex in V1 is joined to any in V2 by an edge. Hence the partition.

MOD 1

THEOREM 8

A simple graph with n vertices and k components can have at most (n−k)(n−k+1)/2edges

The proof of the theorem depends on an Algebraic Inequality

∑_i=1^k▒ni2 ≤ n2 –(k-1)(2n-k)

Consider a Simple Graph G with n vertices and k components.

Let the no. of vertices in each of the k components of a Graph G be named as n1,n2,............nk.

Thus we have   ni≥1 ∑_i=1^k▒ni = n1]2 = (n-k)2

ie: [(n1-1)+ (n2-1)+ (n3-1)+ …….+ (nk-1)]2 = n2-2nk+k2

∑_i=1^k(ni-1) 2 +2 ∑_(i,j=1 i≠j)^k(ni -1)(nj – 1) = n2 -2nk+k2

∑_i=1^k(ni2-2ni+1) + 2 ∑_(i,j=1 i≠j)^k(ni -1)(nj – 1) = n2 -2nk+k2

∑_i=1^kni2 - 2 ∑_i=1^kni + ∑_i=1^k 1 + 2 ∑_(i,j=1i≠j)^k(ni -1)(nj – 1)   = n2 -2nk+k2

∑_i=1^k ni2 - 2n+k +2 ∑_(i,j=1 i≠j)^k(ni -1)(nj – 1) = n2 -2nk+k2

We know that ni≥1, then ni-1 ≥0

∑_i=1^kni2 ≤ n2 –(k-1)(2n-k)

The total no. of edges possible in the 1st component is n1(n1−1)/2

In the 2nd component is n2(n2−1)/2

Similarly, in the kth component is nk(nk−1)/2

Then the maximum no. of edges possible in G is the sum of maximum no. of edges possible in each component

emax = n1(n1−1)/2 + n2(n2−1)/2 +…+ nk(nk−1)/2

= 1/2 ∑_i=1^kni (ni-1)

emax = 1/2 [∑_i=1^k(ni2 −ni)]

= 1/2 [∑_i=1^kni2 − ∑_i=1^kni]

≤1/2[(n2−(k−1)(2n−k))−n]≤1/2[(n2−(2nk−k2−2n+k))−n] ≤ 1/2[n2−2nk+k2+2n−k−n]

≤1/2[(n−k)2+(n−k)]

≤(n−k)(n−k+1)/2

i.e, emax can be atmost (n−k)(n−k+1)/2.Hence the proof.

MOD 1

THEOREM-7

If a graph (connected or disconnected) has exactly 2 vertices of odd degree then there must be a path joining these two vertices

Proof:• Let G be a graph with all even-degree vertices except v1 and v2 which are of odd degree.According to theorem 2 " The number of odd degree vertices in a graph is always even". Therefore, every component of a disconnected graph, can be considered as a graph itself, no component can have odd no. of odd degree vertices. Hence v1 and v2 must belong to the same component for theorem 2 to holdIf they lie in the same component,then there has to be a path between them. Hence the proof.

MOD 1

Isomorphism:

Two graphs G and G' are said to be isomorphic if there is a one to one correspondence between their vertices and between their edges such that the incidence relationship is preserved.

The two isomorphic graph must have

a) same number of vertices

b) same number of edges

c) equal number of vertices with a given degree.

To check if two given graphs are isomorphic:

Isomorphic graph yields same adjacent matix (matrix with rows and columns indicating the vertices and if an edge existing between the pair of vertices that entry will be 1 else 0. Parallel edges cannot be represented) so interchanging rows and columns finally if giving the same adjacency matrix are isomorphic.

MOD 1

THEOREM

A graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty, disjoint subsets V1 and V2 such that there exists no edge in G whose one end vertex is in subset V1 and the other in subset V2.

Proof: Suppose that such a partitioning exists. Consider two arbitrary vertices a and b of G, such that a ∈ V1 and b ∈ V2. No path can exist between vertices a and b; otherwise, there would be at least one edge whose one end vertex would be in V1 and the other in V2. Hence, if a partition exists, G is not connected. Conversely, let G be a disconnected graph. Consider a vertex a in G.

Let V1 be the set of all vertices that are joined by paths to a. Since G is disconnected, V1 does not include all vertices of G. The remaining vertices will form a (nonempty) set V2. No vertex in V1 is joined to any in V2 by an edge. Hence the partition.

MOD 2

In a connected graph G with exactly 2k odd vertices, there exist k edge-disjoint subgraphs such that they together contain all edges of G and that each is a unicursal graph

Proof:

□ Let the odd vertices of the given graph G be named as v1,v2,...,vk;
w1,w2,...,wk in any arbitrary order. Add k edges to G between the vertex pairs (v1,w1),(v2,w2),...,(wk,wk) to form a new graph G'.

□ Since every vertex of G' is of even degree, G' consists of anEuler line .

□ Now if we remove ρ the k edges we just added, ρ will split intok walks, each of which is a unicursal line

□ The first removal will leave a single unicursal line; the second removal will split that into two unicursal line; and each successive removal will split a unicursal line into two, until there are k of them. Thus the theorem.

MOD 2

Theorem 2.1: A given connected graph G is an Euler graph if and only if all vertices of G are of even degree

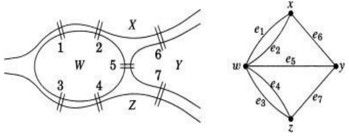
Proof:

Suppose that G is and Euler graph. Which contains a closed walk called Euler line. In tracing this walk, observe that every time the walk meets a vertex v it goes through two "new" edges incident on v –with one we entered v and with the other "exited". This is true not only of all vertices of the walk but also of the terminal vertex, because we "exited" and entered the same vertex at the beginning and end of the walk, respectively. Thus if G is an Euler graph, the degree of every vertex is even Conversely, assume that each vertex of G has even degree. We need to show that G is Eulerian. Let us start with a vertex v0 ∈ (G). Assume G is connected, there exists a vertex v1 ∈ (G) that is adjacent to v0. Since G is a simple graph and $d(vi)$ ≥ 2, for each vertex vi ∈ $V(G)$, there exists a vertex v2 ∈ $V(G)$, that is adjacent to v1 with $v2 \neq v0$.Similarly, there exists a vertex v3 ∈ $V(G)$, that is adjacent to v2 with $v3 \neq v1$. If C contains every edge of G, then C gives rise to a closed Eulerian trail and we are done. Let us assume that $E\ C$ is a proper subset of . Now consider the graph $G1$ that is obtained by removing all the edges in C from G. Then, $G1$ may be a disconnected graph but each vertex of $G1$ still has even degree. Hence, we can do the same process explained above to 1 also to get a closed Eulerian trail, say $C1$. If $\boldsymbol{C1}$ contains all edges of $\boldsymbol{G1}$, then $\boldsymbol{C} \cup \boldsymbol{C1}$ is a closed Euler trail in G. If not, let $G2$ be the graph obtained by removing the edges of $C1$ from $G1$. Since G is a finite graph, we can proceed to find out a finite number of cycles only. Le the process of finding cycles, as x plained above, ends after a finite number of steps say, r. Then the reduced graph $Gr = Gr{-}1 - E\ Cr{-}1 = G - (C \cup C1 \cup Cr{-}1)$ will be an empty graph(null graph). Then $C \cup C1 \cup C2 \ldots \cup Cr{-}1$ is a closed Euler trail in G. Therefore, G is Eulerian, This completes the proof.

MOD 2

Konigsberg Bridge Problem:

Two islands C and D were connected to each other and to the banks A and B with seven bridges as shown in figure. The problem was to start at any land areas A, B, C or D , walk over each of the seven bridges exactly once, and return to the starting point



Euler represented this problem by means of a graph. Vertices represent the land areas and the edges represents the bridges. Euler proved that a solution for this problem does not exists. The problem can be solved by checking whether the following graph is an Euler or not. If it is Euler graph then we can solve the problem. Otherwise not. From the figure we find that not all its vertices are of even degree . Hence it is not an Euler graph. Thus it is not possible to walk over each of the seven bridges exactly once and return to the starting point.

MOD 2

A connected graph G is an Euler graph if and only if it can be decomposed into circuits

Proof:

 Suppose graph G can be decomposed into circuits; that is , G is a union of edge disjoint circuits.

 Since the degree of every vertex in a circuit is two , the degree of every vertex in G is even. Hence G is an Euler graph

Conversely. Let G be euler graph(all edges must be of even degree).

 Consider a vertex v1. There are atleast two edges incident withv1. Let one of these edges are formed between v1 and v2. Here v2 also of even degree so definitely v2 will be neighbor of another vertex v3.

 Proceeding like this eventually we will arrive at a vertex that is being already traversed, that forms a circuit Γ.

 Remove circuit Γ from g. all vertices remaining will be of even degree.

 In the remaining graph remove another circuit just like what we did. Continue till no edges left.

 Hence theorem proved.

MOD 2

Travelling Salesman problem:

Problem: A salesman is required to visit a number of cities during a trip. Given the distance between the cities, in what order should he travel so as to visit every city precisely once and return home, with the minimum mileage travelled?

We can represent the cities by vertices and the roads between them by edges in a graph. In this graph, with every edge ei there is associated a real number (ei). Such a graph is called a weighted graph; $w\ ei$ being the weight of edge ei.

In this problem, if each of the cities has a road to every other city, we have a completed weighted graph . This graph has numerous Hamiltonian circuits and we are to pick the one that has the smallest sum of distances. Theoretically , the problem of travelling salesman can always be solved by enumerating all $(n-1)$!/2 Hamiltonian circuits, calculating the distance traveled in each and then picking the shortest one.

Complete graph:

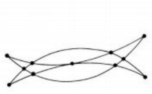
A simple graph G is said to be a Complete graph if every vertex in G is connected to all other vertices. So G contains exactly one edge between each pair of vertices. Since every vertex is joined with every other vertex through one edge , the degree of every vertex is n-1 in a completed graph G of n vertices A complete graph is denoted by Kn and Kn has exactly n(n-1)/2 edges.

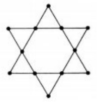
Kn with n = 1,2,3,4,5,6,7,8 is given below

MOD 2

EULER GRAPHS:

A closed walk in a graph containing all the edges of the graph, is called an Euler Line and a graph that contain Euler line is called Euler graph. Euler graph is always connected.





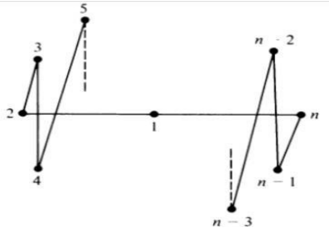
MOD 2

In a complete graph with n vertices there are (n - 1)/2 edge- disjoint Hamiltonian circuits, if n is an odd number > 3.

Proof:

A complete graph G of n vertices has n(n-1)/2 edges, and a Hamiltonian circuit in G consists of n edges. Therefore, the number of edge-disjoint Hamiltonian circuits in G cannot exceed (n - 1) / 2.

That there are (n - 1) / 2 edge-disjoint Hamiltonian circuits, when n is odd, can be shown as follows:



The subgraph (of a complete graph of n vertices) is a Hamiltonian circuit. Keeping the vertices fixed on a circle, rotate the polygonal pattern clockwise by 360/(n - 1), 2.360/(n - 1), 3. 360/(n - 1), . . . , (n - 3)/2. 360/(n - 1) degrees.

Observe that each rotation produces a Hamiltonian circuit that has no edge in common with any of the previous ones.

Thus we have (n - 3)/2 new Hamiltonian circuits, all edge disjoint from the one in Figure above and also edge disjoint among themselves. Hence the theorem

MOD 2

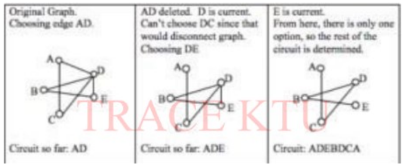
FLEURY'S ALGORITHM

1. Start at any vertex if finding an Euler circuit. If finding an Euler path, start at one of the two vertices with odd degree.

2. Choose any edge leaving your current vertex, provided deleting that edge will not separate the graph into two disconnected sets of edges.

3. Add that edge to your circuit, and delete it from the graph.

4. Continue until you're done.



Original Graph.
Choosing edge AD.

Circuit so far: AD

AD deleted. D is current.
Can't choose DC since that would disconnect graph.
Choosing DE

Circuit so far: ADE

E is current.
From here, there is only one option, so the rest of the circuit is determined.

Circuit: ADEBDCA

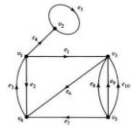
MOD 2

DIRECTED GRAPH

A directed graph or digraph G consists of a set of vertices $V = v1, v2, \dots,$ a set of edges $E = e1, e2, \dots$ and a mapping φ that maps every edge onto some ordered pair of vertices (vi, vj)

In digraph, vertices are represented by points and edges by a line segment between vi and vj with an arrow directed from vi to vj The vertex vi, which edge ek is incident out of is called the initial vertex of ek The vertex vj, which edge ek is incident into is called the terminal vertex of ek A digraph is also referred to as oriented graph. The number of edges incident out of a vertex vi is called the out-degree (outvalence or outward demidegree) of vi and is written $d+(vi)$

The number of edges incident into vertex vi is called the in-degree (in valence or inward demidegree) of vi and is written $d-(vi)$



v_5 is the initial vertex and v_4 is the terminal vertex of edge e_7

$$d^+(v_1) = 3, \quad d^-(v_1) = 1,$$
$$d^+(v_2) = 1, \quad d^-(v_2) = 2,$$
$$d^+(v_5) = 4, \quad d^-(v_5) = 0.$$

Fig. Directed graph with 5 vertices and 10 edges.

In any digraph, sum of all in degree is equal to the sum of all out-degree, each sum being equal to the number of edges in G.

$$\sum_{i=1}^{n} d^+(v_i) = \sum_{i=1}^{n} d^-(v_i) = e$$

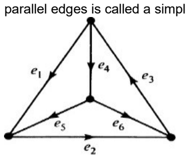
An isolated vertex is a vertex in which the in-degree and the out-degree are both equal to zero. A vertex in a digraph is called pendant if its is of degree one.

$$d^+(v_i) + d^-(v_i) = 1$$

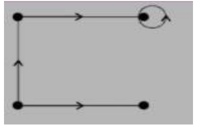
MOD 2

Types of Digraphs:

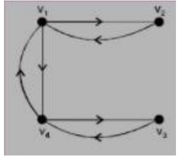
Simple Digraphs: A digraph that has no self loop or parallel edges is called a simple digraph



Asymmetric Digraphs: Digraphs that have atmost one directed edge between a pair of vertices, but contains self loops are called asymmetric or antisymmetric



Symmetric Digraphs: Digraphs in which for every edge(a,b), there is also an edge(b,a)



A digraph that is both simple and symmetric is called a simple symmetric digraph. Similarly, a digraph that is both simple and asymmetric is called a simple asymmetric digraph.

MOD 2

HAMILTONIAN PATHS AND CIRCUITS

A circuit in a connected graph G is said to be Hamiltonian circuit if it includes every vertex of G. a, starting at vertex v, if one traverse along the edges shown in heavy lines- passing through each vertex exactly once- one gets a Hamiltonian circuit.Hamiltonian circuit in a graph of n vertices consists of exactly n edges.

Since Hamiltonian circuit traverses every vertex exactly once, it cannot include a self loop or a set of parallel edges.

Hamiltonian Path:

If we remove any one edge from a Hamiltonian circuit , we are left with a path. This path is called a Hamiltonian Path. The length of a Hamiltonian path in a connected graph of n vertices is n-1.

MOD 3

DIJKSTRA'S ALGORITHM

At each stage in the algorithm some vertices have permanent labels and others temporary labels. The algorithm begins by assigning a permanent label 0 to the starting vertex s, and a temporary label ∞ to the remaining n — 1 vertices. From then on, in each iteration another vertex gets a permanent label, according to the following rules :

1. Every vertex j that is not yet permanently labeled gets a new temporary label whose value is given by where i is the latest vertex permanently labeled, in the previous iteration, and dij is the direct distance between vertices i and j. If i and j are not joined by an edge, then dij = ∞.

2. The smallest value among all the temporary labels is found, and this becomes the permanent label of the corresponding vertex. In case of a tie, select any one of the candidates for permanent labeling.

Steps 1 and 2 are repeated alternately until the destination vertex t gets a permanent label.

The first vertex to be permanently labeled is at a distance of zero from s. The second vertex to get a permanent label (out of the remaining n − 1 vertices) is the vertex closest to s. From the remaining n − 2 vertices, the next one to be permanently labeled is the second closest vertex to s. And so on. The permanent label of each vertex is the shortest distance of that vertex from s

MOD 3

PRIM'S ALGORITHM

 draw n isolated vertices and label them v1, v2, . . . , vn.

 Tabulate the given weights of the edges of G in an n by n table. (Note that the entries in the table are symmetric with respect to the diagonal, and the diagonal is empty.)

 Set the weights of nonexistent edges (corresponding to those pairs of cities between which no direct road can be built) as infinity.

 Start from vertex v1 and connect it to its nearest neighbor (i.e., to the vertex which has the smallest entry in row 1 of the table), say vk.

 Consider v1 and vk as one subgraph, and connect this subgraph to its closest neighbor (i.e., to a vertex other than v1and vk that has the smallest entry among all entries in rows 1 and k). Let this new vertex be vi.

 Next regard the tree with vertices v1, vk, and vi as one subgraph,

 Continue the process until all n vertices have been connected by n − 1 edges.

MOD 3

Kruskal's algorithm

Step-01: Sort all the edges from low weight to high weight.

Step-02: Take the edge with the lowest weight and use it to connect the vertices of the graph. If adding an edge creates a cycle, then reject that edge and go for the next least weight edge.

Step-03: Keep adding edges until all the vertices are connected and a Minimum Spanning Tree (MST) is obtained.

MOD 3

ROOTED TREES

A tree in which one vertex (called the root) is distinguished from all the others is called a rooted tree. All rooted trees with four vertices are shown in figure where the roots are enclosed in a small triangle.

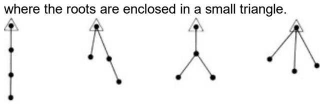


Fig. Rooted trees with four vertices.

APPLICATION: For instance, in Fig given below vertex N, from where all the mail goes out, is distinguished from the rest of the vertices. Hence N can be considered the root of the tree, and so the tree is rooted.

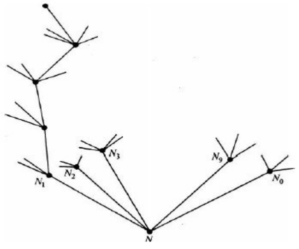


Fig. 3-3 Decision tree.

MOD 3

BINARY TREES:

A binary tree is defined as a tree in which there is exactly one vertex (root) is of degree two, and each of the remaining vertices is of degree one or three. Every binary tree is a rooted tree since the vertex of degree two is distinct from all other vertices, this vertex serves as a root. Most straightforward application of binary tree is in search procedures, binary identification problems, and variable-length binary codes. Two **properties of binary trees**

The number of vertices n in a binary tree is always odd. This is because there is exactly one vertex of even degree, and the remaining n − 1 vertices are of odd degrees. Since from Theorem 1-1: the number of vertices of odd degrees is even, n − 1 is even. Hence n is odd. Let p be the number of pendant vertices in a binary tree T. Then n − p − 1 is the number of vertices of degree three. Therefore, the number of edges in T equals

$$\frac{1}{2}[p+3(n-p-1)+2]=n-1;$$

MOD 3

Theorem

Any connected graph with n vertices and n − 1 edges is a tree.

Proof : the minimum number of edges required to make a graph of n vertices connected is (n-1) edges. We can observe that removal of one edge from the graph G will make it disconnected. Thus a connected graph of n vertices and (n-1) edges cannot have a circuit. Hence a graph G is a tree. Vertices of a tree are connected together with the minimum number of edges. A connected graph is said to be minimally connected if removal of any one edge from it disconnects the graph. A minimally connected graph cannot have a circuit; otherwise, we could remove one ofthe edges in the circuit and still leave the graph connected. Thus a minimally connected graph is a tree. Conversely, if a connected graph G is not minimally connected, there must exist an edge ei in G such that G − ei is connected. Therefore, ei is in some circuit, which implies that G is not a tree.

MOD 3

Theorem 3

A tree with n vertices has n − 1 edges.

Proof: The theorem will be proved by induction on the number of vertices.

Proof:

Let n be the number of vertices in a tree (T).

If n=1, then the number of edges=0.

Graphs with n=1, 2,3 :

If n=2 then the number of edges=1.

If n=3 then the number of edges=2.

Hence, the statement (or result) is true for n=1, 2, 3.

Let the statement be true for n=m. Now we want to prove that it is true for n=m+1.

Let e be the edge connecting vertices say Vi and Vj. Since G is a tree, then there exists only one path between vertices Vi and Vj. Hence if we delete edge e it will be disconnected graph into two components G1 and G2 say. These components have less than m+1 vertices and there is no circuit and hence each component G1 and G2 have m1 and m2 vertices.

Now, the total no. of edges = (m1-1) + (m2-1) +1
= (m1+m2)-1
= m+1-1
= m.

Hence for n=m+1 vertices there are m edges in a tree (T). By the mathematical induction the graph exactly has n-1 edges.

Mod 3

CAYLEY'S THEOREM

The number of labelled trees with 'n' vertices (n>=2) is nn-2

Proof of Theorem: Algorithm for generating Prufer sequence (encoding) • Suppose we have a treeT with n uniquely labeled vertices

Suppose we have a treeT with n uniquely labeled vertices • Label all vertices with numbers from 1 to n

Suppose we have a treeT with n uniquely labeled vertices • Label all vertices with numbers from 1 to n • Find the pendant vertex with the smallest label(least number) • Let it be a1 . • Let b1 be the vertex adjacent to a1 • Delete vertex a1 along with the incident edge (a1, b1) from the tree • Start a prufer sequence; add b1 to the prufer sequence (b1)

From the remaining tree with (n-1) vertices, find the pendant vertex with the least number. • Let it be a2 . • Let b2 be the vertex adjacent to a2 • Delete vertex a2 along with the incident edge (a2,b2) from the tree • Add b2 to the prufer sequence (b1 , b2) • Continue the process until only two vertices are left in the tree • Now the tree defines the prufer sequence (b1 , b2 , …, bn-2) uniquely

Conversely, Algorithm for constructing tree using prufer sequence (Decoding) • Given a sequence (b1 , b2 , …, bn-2) of n-2 labels, an n vertex tree can be constructed uniquely as follows: • Determine the first number in the sequence 1,2,3,…….,n that does not appear in sequence (b1 , b2 , …, bn-2). • This number clearly is a1 • And thus the edge (a1, b1) is defined • Remove b1 from sequence (b1 , b2 , …, bn-2) and a1 from 1,2,3,…….,n

• In the remaining sequence of 1,2,3,…….,n, find the first number that does not appear in the remainder of (b1 , b2 , …, bn-2) • This would be a2, and thus the edge (a2,b2) is defined • The construction is continued till the sequence (b1 , b2 , …, bn-2) has no element left. • Finally the last two vertices remaining in 1,2,3,…….,n are joined.

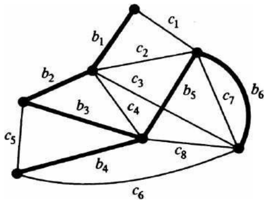
For each of the n-2 elements in sequence (b1 , b2 , …, bn-2) we can choose any one of n numbers , thus forming n n-2 tuples, each defining a distinct labelled tree of n vertices. • And since each tree defines one of these sequences uniquely, there is one-to-one correspondence between the trees and the n n-2 sequences. • Hence the theorem.

MOD 3

SPANNING TREES

A graph with e edges has 2e subgraphs combinations. A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices of G.

A spanning tree is sometimes referred to as a skeleton or scaffolding of G. Since spanning trees are the largest (with maximum number of edges) trees among all trees in G, it is also quite appropriate to call a spanning tree a maximal tree subgraph or maximal tree of G. A disconnected graph with k components has a spanning forest consisting of k spanning trees.



Finding a spanning tree of a connected graph G :

If G has no circuit, it is its own spanning tree. If G has a circuit, delete an edge from the circuit.. If there are more circuits, repeat the operation till an edge from the last circuit is deleted—leaving a connected, circuit-free graph that contains all the vertices of G.

MOD 3

A spanning tree t (of a given weighted connected graph g) is a shortest spanning tree (of g) if and only if there exists no other spanning tree (of g) at a distance of one from t whose weight is smaller than that of t.

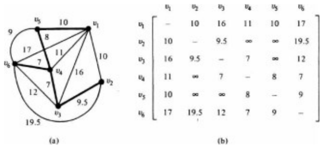
Proof:

Let T1 be a spanning tree in G satisfying the theorem: there is no spanning tree at a distance of one from T1 which is shorter than T1. Let T2 be a shortest spanning tree in G. Clearly, T2 must also satisfy the theorem. Consider an edge e in T2 which is not in T1. Adding e to T1 forms a fundamental circuit with branches in T1. Some of the branches in T1 that form the fundamental circuit with e may also be in T2; each of these branches in T1 has a weight smaller than or equal to that of e, because of the assumption on T1. Amongst all those edges in this circuit which are not in T2 at least one, say bj, must form some fundamental circuit (with respect to T2) containing e. Because of the minimality assumption on T2 weight of bj cannot be less than that of e. Therefore bj must have the same weight as e. Hence thespanning tree = (T1 ∪e − bj), obtained from T1 through one cycle exchange, has the same weight as T1. But T1 has one edge more in common with T2, and it satisfies the condition. This argument can be repeated, producing a series of trees of equal weight, T1 , T1 , T1, . . . , each a unit distance closer to T2, until we get T2 itself.

This proves that if none of the spanning trees at a unit distance from T is shorter than T, no spanning tree shorter than T exists in the graph.

MOD 3

Shortest spanning tree in a weighted graph.



	v_1	v_2	v_3	v_4	v_5	v_6
v_1	–	10	16	11	10	17
v_2	10	–	9.5	∞	∞	19.5
v_3	16	9.5	–	7	∞	12
v_4	11	∞	7	–	8	7
v_5	10	∞	∞	8	–	9
v_6	17	19.5	12	7	9	–

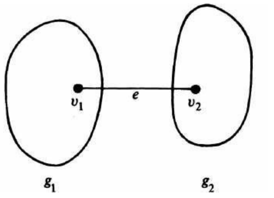
A connected weighted graph with 6 vertices and 12 edges is shown in Fig. 3- 20(a). The weight of its edges is tabulated in Fig. 3-20(b). We start with v1 and pick the smallest entry in row 1, which is either (v1, v2) or (v1, v5). Let us pick (v1, v5). [Had we picked (v1, v2) we would have obtained a different shortest tree with the same weight.] The closest neighbor of subgraph (v1, v5) is v4 , as can be seen by examining all the entries in rows 1 and 5. The three remaining edges selected following the above procedure turn out to be (v4, v6), (v4, v3), and (v3, v2) in that sequence. The resulting tree—a shortest spanning tree—is shown in Fig. 3-20(a) in heavy lines. The weight of this tree is 41.5 units.

MOD 3

A graph G with n vertices, n − 1 edges, and no circuits is connected.

Proof: Suppose there exists a circuitless graph G with n vertices and n – 1 edges which is disconnected. In that case G will consist of two or more circuitless components. Without loss of generality, let G consist of two components, g1 and g2. Add an edge e between a vertex v1 in g1 and v2 in g2

Since there was no path between v1 and v2 in G, adding e did not create a circuit. Thus G ⋃ e is a circuitless, connected graph (i.e., a tree) of n vertices and n edges, which is not possible, because of theorem - A tree with n vertices has n − 1 edges.



MOD 3

PENDANT VERTEX

A pendant vertex was defined as a vertex of degree one. In a tree no vertex can be of zero degree, we must have at least two pendant vertices. Concept of pendant vertex can be used in computer programmes as data tree.

Theorem 3.6

In any tree (with two or more vertices), there are atleast two pendant vertices.

Proof: Pendant vertices are vertex of degree one. For a tree of n vertices we have n-1edges and hence 2(n − 1) degrees to be divided among n vertices. Since no vertex can be of zero degree, we must have atleast two vertices of degree one in a tree. Hence the theorem

MOD 3

Theorem 3

Every tree has either one or two centers

Proof: The maximum distance,$max\ dvi,\ v$from a given vertex v to any other vertexvi occurs only when $vi$is a pendant vertex. With this observation, let us start with a tree T having more than one two vertices. Tree T must have two or more pendant vertices. Delete all the pendant vertices from T. The resulting graph T' is still a tree.. Removal of all pendant vertices from T uniformly reduced the eccentricity of the remaining vertices (i.e., vertices of T')by one. Therefore all vertices that T had as centers will still remain centers in T' .From T'we can again remove all pendant vertices and get another tree T''. We continue this process until there is left either a vertex (which isthe centre of T) or an edge (whose end vertices are the two centers of T). Thus the theorem.

MOD 3

Applications of trees:

The genealogy of a family is often represented by means of a tree.

A river with its tributaries and subtributaries can be represented by a tree.

The sorting of mail according to zip code and the sorting of punched cards are done according to a tree (called decision tree or sorting tree).

PROPERTIES OF TREES:

A graph with n vertices is called a tree if

□ G is connected and is circuit-less

□ G is connected and has n-1edges

□ G is circuit-less and has n-1edges

□ There is exactly one path between every pair of vertices inG

□ G is a minimally connectedgraph

MOD 4

Theorem

A connected planar graph with n vertices and e edges has e − n + 2 regions

Proof:

□ prove the theorem for a simple graph, because adding a self-loop or a parallel edge simply adds one region to the graph and simultaneously increases the value of e by one. We can also disregard all edges that do not form boundaries of any region.

□ Since any simple planar graph can have a plane representation such that each edge is a straight line, any planar graph can be drawn such that each region is a polygon(a polygonalnet)

□ Let the polygonal net representing the given graph consist of f regions or faces, and let kpbe the number of p-sided regions.

□ Since each edge is on the boundary of exactly tworegions,3. $k3$ + 4. $k4$ + 5. $k5$ + … + $r.\ kr$ =2.e………………………..Eqn (1)

where $kr$is the number of polygons with maximum edges.

□ Also, $k3$ + $k4$ + $k5$ + ⋯ + kr=f…………Eqn(2)

Sum of angles subtended at each vertex in the polygonal net = 2πn………..Eqn(3)

Sum of all interior angles of a p-sided polygon = πp − 2

Sum of exterior angles of a p-sided polygon = πp + 2

□ $Eqn3becomes$

Sum of angles subtended at each vertex in the polygonal net = Sum of all interior angles+ Sum of exterior angles of a p-sided polygon = $\pi 3$ − 2. $k3$ + $\pi 4$ − 2. $k4$ + $\pi 5$ − 2. $k5$ + … + πr − 2. kr + 4π= $\pi 2e$ − 2f+4π Eqn(4)

□ $From\ Eqn3and\ Eqn4$

$\pi 2e$ − 2f+ 4π = 2$\pi n2\pi e$ − f + 2= 2πn

MOD 4

EULER'S THEOREM:

Theorem

A given connected graph G is an Euler graph if and only if all vertices of G are of even degree

Proof:

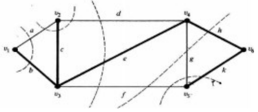
Suppose that G is and Euler graph. Which contains a closed walk called Euler line. In tracing this walk, observe that every time the walk meets a vertex v it goes through two —new‖ edges incident on v –with one we entered v and with the other —exited‖. This is true not only of all vertices of the walk but also of the terminal vertex, because we —exited‖ and entered the same vertex at the beginning and end of the walk, respectively. Thus if G is an Euler graph, the degree of every vertex is even Conversely, assume that each vertex of G has even degree. We need to show that G is Eulerian. Let us start with a vertex v0 ∈ (G). Assume G is connected, there exists a vertex v1 ∈ (G) that is adjacent to v0. Since G is a simple graph and $(vi) \geq 2$, for each vertex vi ∈ $V(G)$, there exists a vertex v2 ∈ $V(G)$, that is adjacent to v1 with $v2 \neq v0$.Similarly, there exists a vertex v3 ∈ $V(G)$, that is adjacent to v2 with $v3 \neq v1$. If C contains every edge of G, then C gives rise to a closed Eulerian trail and we are done. Let us assume that $E\ C$ is a proper subset of

. Now consider the graph $G1$ that is obtained by removing all the edges in C from G. Then, $G1$ may be a disconnected graph but each vertex of $G1$ still has even degree. Hence, we can do the same process explained above to 1 also to get a closed Eulerian trail, say $C1$. If $\boldsymbol{C1}$ contains all edges of $\boldsymbol{G1}$, then $\boldsymbol{C} \cup \boldsymbol{C1}$ is a closed Euler trail in G. If not, let $G2$ be the graph obtained by removing the edges of $C1$ from $G1$. Since G is a finite graph, we can proceed to find out a finite number of cycles only. Le the process of finding cycles, as xplained above, ends after a finite number of steps say, r. Then the reduced graph $Gr = Gr-1 - E\ Cr-1 = G - (C \cup C1 \cup Cr-1)$ will be an empty graph(null graph). Then $C \cup C1 \cup C2 \ldots \cup Cr-1$ is a closed Euler trail in G. Therefore, G is Eulerian, This completes the proof.

MOD 4

Fundamental CutSet/Basic Cutset:

Consider a spanning tree T of a connected graph G. Take any branch b in T. Since {b} is a cut set in T, {b} partitions all vertices of T into two disjoint sets—one at each end of b. Consider the same partition of vertices in G, and the cut set S in G that corresponds to this partition. Cutset S will contain only one branch b of T, and the rest (if any) of the edges in S are chords with respect to T. Such a cut-set S containing exactly one branch of a tree T is called a fundamental cut-set with respect to T. a spanning tree T (in heavy lines) and all five of the fundamental cutsets with respect to T are shown (broken lines —cutting‖ through each cut-set).

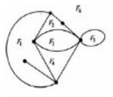


Every chord of a spanning tree defines a unique fundamental circuit, every branch of a spanning tree defines a unique fundamental cut-set. The fundamental cut-set has meaning only with respect to a given spanning tree.

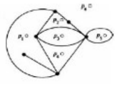
MOD 4

GEOMETRIC DUAL

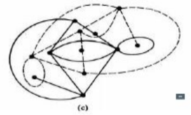
Consider the Graph with six regions or faces F1, F2, F3, F4, F5, and F6



Let us place six points p1, p2, . . . , p6, one in each of the regions,



Join these six points according to the following procedure:If two regions Fi and Fj are adjacent (i.e., have a common edge), draw a line joining points pi and pj that intersects the common edge between Fi and Fj exactly once. If there is more than one edge common between Fi and Fj, draw one line between points pi and pj for each of the common edges. For an edge e lying entirely in one region, say Fk, draw a self-loop at point pk intersecting e exactly once



By this procedure we obtain a new graph G* consisting of six vertices, p1, p2, . . . , p6 and of edges joining these vertices. Such a graph G* is called a dual or a geometric dual of G. There is a one-to-one correspondence between the edges of graph G and its dual G*−one edge of G* intersecting one edge of G.

Relationship between G and its dual $\boldsymbol{G}$

1. An edge forming a self-loop in G yields a pendant edge in G*.
2. A pendant edge in G yields a self-loop in G*.
3. Edges that are in series in G produce parallel edges in G*.
4. Parallel edges in G produce edges in series in G*.
5. the number of edges constituting the boundary of a region Fi in G is equal to the degree of the corresponding vertex pi in G*, and vice versa.
6. Graph G* is also embedded in the plane and is therefore planar.
7. Considering the process of drawing a dual G* from G, it is evident that G is a dual of G* . so we can say that G and G* are dual graphs.
8. If n, e, f, r, and μ denote as usual the numbers of vertices, edges, regions, rank, and nullity of a connected planar graph G, and if n*, e*, f*, r*, and μ* are the corresponding numbers in dual graph G*, then

n* = f,
e* = e,
f* = n.
r* = μ,
μ* = r.

MOD 4

Cut-Sets

• In a connected graph G, a cut-set is a set of edges removal of which leaves the graph disconnected, provided removal of no proper subset of the set disconnects G • Also defined as a minimal set of edges in a connected graph, removal of which reduces the rank of the graph by one • A cut-set cuts the graph into two, such that the vertex set of the graph is partitioned into two subsets with no elements in common; Hence no path exists between the two subgraphs.

Cut-Vertex

In a connected graph G, a cut-vertex is a set of vertices removal of which leaves the graph disconnected, provided removal of no proper subset of the set disconnects G • Also defined as a minimal set of vertices in a connected graph, removal of which reduces the rank of the graph by one • A cut-vertex cuts the graph into two, such that the vertex set of the graph is partitioned into two subsets with no elements in common; Hence no path exists between the two subgraphs.

Edge Connectivity (Ec)

• The number of edges in the smallest cut set (i.e., cut-set with fewest number of edges) is defined as the edge connectivity of G • Edge connectivity of a connected graph can be defined as the minimum number of edges whose removal (i.e., deletion) reduces the rank of the graph by one.

Edge Connectivity of a Tree

• Since a tree can be broken by the removal of a single edge, edge connectivity of a tree is always 1

Vertex Connectivity (Vc) • Vertex connectivity of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph dis connected and reduces the rank of the graph by one. • It is given by the size of the smallest cut-vertex

Vertex Connectivity of a Tree

• Since a tree can be broken by the removal of a single non-pendant vertex, vertex connectivity of a tree is always 1

Vertex Connectivity (Vc)

• Vertex connectivity of a connected graph G is defined as the minimum number of vertices whose removal from G leaves the remaining graph dis connected and reduces the rank of the graph by one.

• It is given by the size of the smallest cut-vertex

Fundamental Cut-Set

• With respect to a given Spanning Tree, a cut-set of the graph is said to be fundamental, if it contains exactly one branch of the Spanning Tree along with some/all of the chords • Since each branch can generate one cut-set, the no. of fundamental cut-sets possible for a graph is given by the no. of branches in the graph • Just as every chord of a spanning tree defines a unique fundamental circuit, every branch of a spanning tree defines unique fundamental cut-set.

• The term fundamental cut-set has meaning only with respect to a given spanning tree

Properties of Cut-Set

Every cut-set in a connected graph G must contain at least one branch of every spanning tree of G

In a connected graph G, any minimal set of edges containing at least one branch of every Spanning tree of G is a cut-set

Every circuit has an even no. of edges in common with any cut-set

The ring sum of any two cut-sets in a graph is either a third cutset or an edge disjoint union of cut-sets

MOD 4

PLANAR GRAPHS

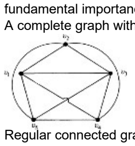
A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersect. A graph that cannot be drawn on a plane without a crossover between its edges is called nonplanar. A drawing of a geometric representation of a graph on any surface such that no edges intersect is called embedding. Thus, to declare that a graph G is nonplanar, we have to show that of all possible geometric representations of G none can be embedded in a plane. Equivalently, a geometric graph G is planar if there exists a graph isomorphic to G that is embedded in a plane. Otherwise, G is nonplanar. An embedding of a planar graph G on a plane is called a plane representation of G.

MOD 4

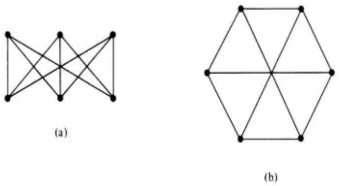
KURATOWSKI'S TWO GRAPHS

Two specific nonplanar graphs which are of fundamental importance. Kuratowski's Graphsare

A complete graph with five vertices



Regular connected graph with six vertices and nine



Two common geometric representation of Kuratowski's second graph is shown in the above figure.

Properties common to the two graphs of Kuratowski are:

1. Both are regular graphs.
2. Both are nonplanar.
3. Removal of one edge or a vertex makes each a planar graph.
4. Kuratowski's first graph is the nonplanar graph with the smallest number of vertices, and Kuratowski's second graph is the nonplanar graph with the smallest number of edges. Thus both are the simplest nonplanar graphs. Kuratowski's first graph is usually denoted by K5 and the second graph by K3.3

MOD 4

1-ISOMORPHISM

A separable graph consists of two or more nonseparable subgraphs. Each of the largest nonseparable subgraphs is called a block / component. The graph has two blocks.

The graph has five blocks (and three cut-vertices a, b, and c)

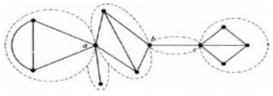


Fig. 4-8 Separable graph with three cut-vertices and five blocks.

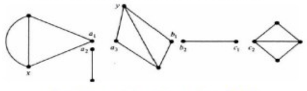


Fig. 4-9 Disconnected graph 1-isomorphic to Fig. 4-8.

The blocks of the graph are isomorphic to the components of the graph in Such graphs are said to be 1-isomorphic. Two graphs G1 and G2 are said to be 1-isomorphic if they become isomorphic to each other under repeated application of the following operation.

Operation 1: —Split‖ a cut-vertex into two vertices to produce two disjoint subgraphs. From this definition it is apparent that two nonseparable graphs are 1-isomorphic if and only if they are isomorphic.

If G1 and G2 are two 1-isomorphic graphs, the rank of G1 equals the rank of G2 and the nullity of G1 equals the nullity of G2 If we join two components of by gluing‖ together two vertices(say vertex x to y)-

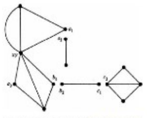


Fig. 4-10 Graph 1-isomorphic to Figs. 4-8 and 4-9.

2-connected graphs - graphs with vertex connectivity of two

2-ISOMORPHISM

In a 2-connected graph G let vertices x and y be a pair of vertices whose removal from G will leave the remaining graph disconnected. Suppose that we perform the following operation 2 onG:

Operation 1: —Split‖ a cut-vertex into two vertices to produce two disjoint subgraphs.

Operation 2: "Split‖ the vertex x into x1 and x2 and the vertex y into y1 and y2 such that G split into g1 and g2 . Let vertices x1 and y1go with g1 and x2 and y2with g2. Now rejoin the graphs g1 and g2 by merging x1 with y2 and x2 with y1.

Two graphs are said to be 2-ismorphic if they become isomorphic after undergoing operation 1 or operation 2 or both operation any number of times

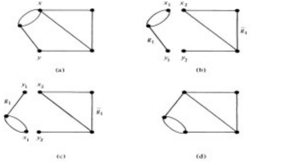


Fig. 2-isomorphic graphs (a) and (d).

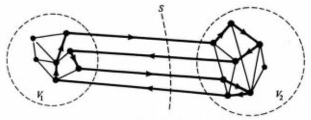
2-isomorphic graphs are equal in rank and equal in nullity

MOD 4
Theorem
With respect to a given spanning tree T, a chord ci that determines a fundamental circuit occurs in every fundamental cut-set associated with the branches in and in no other.

Proof:
Because of there must be an even number of edges common to Γ and S1. Edge b1 is in both Γ and S1, and there is only one other edge in Γ (which is ci) that can possibly also be in S1. Therefore, we must have two edges b1 and ci common to S1 and Γ. Thus the chord ci
is one of the chords c1, c2, . . . cq.
Exactly the same argument holds for fundamental cutsets associated with b2, b3, . . . , and bk.
Therefore, the chord ci is contained in every fundamental cut-set associated with branches in Γ.
Is it not possible for the chord ci to be in any other fundamental cut-set S′ (with
respect to T, of course) besides those associated with b1, b2, . . . and bk.
Otherwise (since none of the branches in Γ are in S′), there would be only one edge ci common
to S′ and Γ, a contradiction to Theorem 4-3.hence the theorem proved.
As an example, consider the spanning tree {b, c, e, h, k}, shown in heavy
lines, in Fig. 4-3. The fundamental circuit made by chord f is {f, e, h, k}.
The three fundamental cutsets determined by the three branches e, h, and k are
determined by branch e: {d, e, f},
determined by branch h: {f, g, h},
determined by branch k: {f, g, k}.
Chord f occurs in each of these three fundamental cutsets, and there is no other fundamental cut-set that contains f. The converse of Theorem 4-5 is also true.

MOD 4
Theorem
Every circuit has an even number of edges in common with any cut-set.

Proof:
Consider a cut-set S in graph G . Let the removal of S partition the vertices of G into two (mutually exclusive or disjoint) subsets V1and V2.



Circuit Γ shown in heavy lines, and is traversed along the direction of the arrows

Consider a circuit in G. If all the vertices in are entirely within vertex set V1 (or V2), the number of edges common to S and Γ is zero; that is, N(S ∩ Γ) = 0, an even number. Some vertices in Γ are in V1and some in V2, we traverse back and forth between the sets V1 and V2 as we traverse the circuit . Because of the closed nature of a circuit, the number of edges we traverse between V1 and V2 must be even. And since very edge in S has one end in V1 and the other in V2, and no other edge in G has this property (of separating sets V1and V2), the number of edges common to S and is even.

MOD 4
Theorem
With respect to a given spanning tree T, a branch bi that determines a fundamental cut-set S is contained in every fundamental circuit associated with the chords in S, and in no others.

Proof: The proof consists of arguments similar to those that led to Theorem 4- 5. Let the fundamental cut-set S determined by a branch bi be S = {bi, c1, c2, . . . , cp},
and let Γ1 be the fundamental circuit determined by chord c1: T1 = {c1, b1, b2, . . . , bq}.
Since the number of edges common to S and Γ1 must be even, bi must be in Γ1. The same is true for the fundamental circuits made by chords c2, c3, . . . , cp.
On the other hand, suppose that bi occurs in a fundamental circuit Γp+1 made by a chord other than c1, c2, . . . , cp. Since none of the chords c1, c2, . . . , cp is in Γp+1, there is only one edge bi common to a circuit Γp+1 and the cut-set S, which is not possible. Hence the theorem.
Example:
Consider the graph Fig. 4-3, consider branch e of spanning tree {b, c, e, h, k}. The fundamental cut-set determined by e is {e, d, f}.
The two fundamental circuits determined by chords d and f are determined by chord d: {d, c, e}, determined by chord f: {f, e, h, k}.
Branch e is contained in both these fundamental circuits, and none of the remaining three fundamental circuits contains branch e.

MOD 5
GREEDY COLOURING ALGORITHM
1. Color first vertex with first colour.
2. Do following for remaining V-1 vertices
Consider the currently picked vertex
Colour it with the lowest numbered colour that has not been used on any previously colored vertices adjacent to it
If all previously used colors appear on vertices adjacent to v, assign a new color to it .

MOD 5
Five-Color Theorem: every planar map can be properly colored with five colors
The vertices of every planar graph can be properly colored with five colors.

Proof: The theorem will be proved by induction. Since the vertices of all graphs (self-loop free, of course†) with 1, 2,3,4, or 5 vertices can be properly colored with five colors, let us assume that vertices of every planar graph with n – 1 vertices can be properly colored with five colors. Then, if we prove that any planar graph G with n vertices will require no more than five colors, we shall have proved the theorem. Consider the planar graph G with n vertices. Since G is planar, it must have at least one vertex with degree five or less . Let this vertex be v. Let G′ be a graph (of n – 1 vertices) obtained from G by deleting vertex v (i.e., v and all edges incident on v). Graph G′ requires no more than five colors, according to the induction hypothesis. Suppose that the vertices in G′ have been properly colored, and now we add to it v and all edges incident on v. If the degree of v is 1, 2, 3, or 4, we have no difficulty in assigning a proper color to v. This leaves only the case in which the degree of v is five, and all the five colors have been used in coloring the vertices adjacent to v, is part of a planar representation of graph G′.

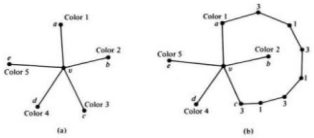


Fig. 8-15 Reassigning of colors.

Suppose that there is a path in G′ between vertices a and c colored alternately with colors 1
and 3. Then a similar path between b and d, colored alternately with colors 2 and 4, cannot exist; otherwise, these two paths will intersect and cause G to be nonplanar. (This is a consequence of the Jordan curve theorem, used in Section 5-3, also.) If
there is no path between b and d colored alternately with colors 2 and 4, starting from vertex b we can interchange colors 2 and 4 of all vertices connected to b through vertices of alternating colors 2 and 4. This interchange will paint vertex b with color 4 and yet keep G′ properly colored. Since vertex d is still with color 4, we have color 2 left over with which to paint vertex v. Had we assumed that there was no path between a and c of vertices painted alternately with colors 1 and 3, we would have released color 1 or 3 instead of color 2. And thus the theorem.

MOD 5
FOUR COLOR PROBLEM
The regions of a planar graph are said to be properly colored if no two contiguous or adjacent regions have the same color. (Two regions are said to be adjacent if they have a common edge between them. Note that one or more vertices in common does not make two regions adjacent.) The proper coloring of regions is also called map coloring, referring to the fact that in an atlas different countries are colored such that countries with common boundaries are shown in different color.
Every map (i.e., a planar graph) can be properly colored with four colors. The four-color conjecture.
Vertex Coloring Versus Region Coloring: we know that a graph has a dual if and only if it is planar.
Therefore, coloring the regions of a planar graph G is equivalent to coloring the vertices of its dual G*, and vice versa. Thus the four-color conjecture can be restated as follows: Every planar graph has a chromatic number of four or less.

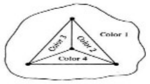


Fig. 8-14 Necessity of four colors.

MOD 5
Theorem: Proving rank of incident matrix of a connected graph with n vertices is n-1
Two graphs G1 and G2 are isomorphic if and only if their incidence matrices A(G1) and A(G2) differ only by permutations of rows and columns.

Proof:
Rank of the Incidence Matrix: Each row in a incidence matrix A(G) may be regarded as a
vector over GF(2) in the vector space of graph G. Let the vector in the first row be called A1, in the second row A2, and so on. Thus There are exactly two 1's in every column of A, the sum of all these vectors is 0. Vectors A1, A2, . . . , An are not linearly independent therefore rank A ≤ n − 1.
Consider the sum of any m of these n vectors (m ≤ n − 1). If the graph is connected, A(G) cannot be partitioned, A(g1) is with m rows and A(g2) with n − m rows. Since there are only two constants 0 and 1 in this field, the additions of all vectors taken m at a time for m = 1, 2, . . . , n− 1 exhausts all possible linear combinations of n − 1 row vectors.
Thus we have just shown that no linear combination of m row vectors of A (for m ≤ n − 1)
can be equal to zero. Therefore, the rank of A(G) must be at least n − 1. Since the rank of A(G) is no more than n − 1 and is no less than n − 1, it must be exactly equal to n − 1. Hence theorem.

MOD 5
Theorem
Let G be a directed graph with adjacency matrix A (with respect to the ordering v1, v2, . . . ,
Vn of the vertices). For every positive integer k, the (i, j) entry of Ak is the number of walks of length k from vi to vj in G
Let G be a directed graph with adjacency matrix A (with respect to the ordering v1, v2, . . . , vn of the vertices). For every positive integer k, the (i, j) entry of Ak is the number of walks of length k from vi to vj in G.
Proof.
By induction on k. The case k = 1 is just the definition of the adjacency matrix. So assume that the theorem is true for k = m − 1 and consider k = m. Then

$$(\mathbf{A}^m)_{ij} = \sum_{l} \mathbf{A}_{il}\mathbf{A}^{m-1}_{lj}.$$

Every path of length m from vi to vj must start with an edge from vi to some vl and follow that
with a path of length m − 1 from vl to vj.
For each l, the entry Ail is 1 if (vi, vl) is an edge and 0 otherwise. By the induction hypothesis Am−1lj is the number of directed walks of length m – 1 from vl to vj in G. Thus for each vertex l, the integer AilAm−1lj is the number of walks of length m from vi to vj that have vl as their second vertex. The sum over l of these is the total number of walks of length m from vi to vj in G. This completes the induction proof

MOD 5
Theorem: Proving rank of incident matrix of a connected graph with n vertices is n-1 Two graphs G1 and G2 are isomorphic if and only if their incidence matrices A(G1) and A(G2) differ only by permutations of rows and columns.

Proof:
Rank of the Incidence Matrix: Each row in an incidence matrix A(G) may be regarded as a
vector over GF(2) in the vector space of graph G. Let the vector in the first row be called A1, in the second row A2, and so on. Thus

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_n \end{bmatrix},$$

There are exactly two 1's in every column of A, the sum of all these vectors is 0. Vectors A1, A2, . . . , An are not linearly independent therefore rank A ≤ n − 1.
Consider the sum of any m of these n vectors (m ≤ n − 1). If the graph is connected, A(G) cannot be partitioned, A(g1) is with m rows and A(g2) with n − m rows.
Since there are only two constants 0 and 1 in this field, the additions of all vectors taken m at a time for m = 1, 2, . . . , n− 1 exhausts all possible linear combinations of n − 1 row vectors. Thus we have just shown that no linear combination of m row vectors of A (for m ≤ n − 1) can be equal to zero. Therefore, the rank of A(G) must be at least n − 1.
Since the rank of A(G) is no more than n − 1 and is no less than n − 1, it must be exactly equal to n − 1. Hence theorem.

MOD 5
Theorem:
If A(G) is an incidence matrix of a connected graph G with n vertices, the rank of A(G) is n −1
Proof:
The rank of A(G) is n − k, if G is a disconnected graph with n vertices and k components.
If we remove any one row from the incidence matrix of a connected graph, the
remaining (n − 1) by e submatrix is of rank n − 1.
The remaining n − 1 row vectors are linearly independent. So only n −1 rows of an incidence matrix to specify the corresponding graph completely, for n − 1 rows contain the same
amount of information as the entire matrix.
Such an (n − 1) by e submatrix Af of A is called a reduced incidence matrix.
The vertex corresponding to the deleted row in Af is called the reference vertex.
Clearly, any vertex of a connected graph can be made the reference vertex.
Since a tree is a connected graph with n vertices and n − 1 edges, its reduced incidence matrix
is a square matrix of order and rank n − 1.
The reduced incidence matrix of a tree is nonsingular.
The reduced incidence matrix of a graph is nonsingular if and only if the graph is a tree..

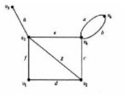
MOD 5

PATH MATRIX

A path matrix is defined for a specific pair of vertices in a graph, say (x, y), and is written as P(x,y). The rows in P(x, y) correspond to different paths between vertices x and y, and the columns correspond to the edges in G. That is, the path matrix for (x, y) vertices is P(x, y)= [pij], where pij = 1, if jth edge lies in ith path, and = 0, otherwise.

Consider all paths between vertices v3 and v4. There are three different paths; {h, e}, {h, g, c}, and {h, f, d, c}. Let us numberthem 1, 2, and 3, respectively.

Then we get the 3 by 8 path matrix P(v3, v4)



$$\mathbf{B}(G)=\begin{array}{c} \\ 1\\ 2\\ 3\\ 4\end{array}\begin{array}{c}\begin{array}{cccccccc} a & b & c & d & e & f & g & h\end{array}\\ \begin{bmatrix}1&1&0&0&0&0&0&0\\0&0&1&0&1&0&1&0\\0&0&0&1&0&1&1&0\\0&0&1&1&1&1&0&0\end{bmatrix}\end{array}.$$

Observations/Properties of a path matrix P(x, y) of a graph G are

1. A column of all 0's corresponds to an edge that does not lie in any path between x and y.
2. A column of all 1's corresponds to an edge that lies in every path between x and y.
3. There is no row with all 0's.
4. The ring sum of any two rows in P(x, y) corresponds to a circuit or an edge-disjoint union of circuits

MOD 5

COVERING

In a graph G, a set g of edges is said to cover G if every vertex in G is incident on at least one edge in g. A set of edges that covers a graph G is said to be an edge covering, a covering subgraph, or simply a covering of G. A graph G is trivially its own covering. A spanning tree in a connected graph (or a spanning forest in an unconnected graph) is another covering. A Hamiltonian circuit (if it exists) in a graph is also a covering. The minimal covering–a covering from which no edge can be removed without destroying its ability to cover the graph.

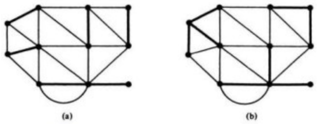


Fig. 8-9 Graph and two of its minimal coverings.

1. A covering exists for a graph if and only if the graph has no isolated vertex.
2. A covering of an n-vertex graph will have at least ⌈n/2⌉ edges. (⌈n⌉ denotes the smallest integer not less than x.)
3. Every pendant edge in a graph is included in every covering of the graph.
4. Every covering contains a minimal covering.
5. If we denote the remaining edges of a graph by (G – g), the set of edges g is a covering if and only if, for every vertex V, the degree of vertex in (G – g) ≤ (degree of vertex v in G) – 1.
6. No minimal covering can contain a circuit, for we can always remove an edge from a circuit without leaving any of the vertices in the circuit uncovered. Therefore, a minimal covering of an n-vertex graph can contain no more than n – 1 edges.
7. A graph, in general, has many minimal coverings, and they may be of different sizes (i.e., consisting of different numbers of edges). The number of edges in a minimal covering of the smallest size is called the covering number of the graph.

MOD 5

THEOREM

Every tree with two or more vertices is 2-chromatic.

Proof:

Select any vertex v in the given tree T. Consider T as a rooted tree at vertex v. Paint v with color 1. Paint all vertices adjacent to v with color 2. Next, paint the vertices adjacent to these (those that just have been colored with 2) using color 1. Continue this process till every vertex in T has been painted.

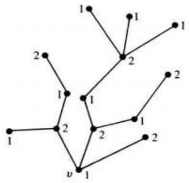


Fig. 8-2 Proper coloring of a tree.

Now in T we find that all vertices at odd distances from v have color 2, while v and vertices at even distances from v have color 1. Now along any path in T the vertices are of alternating colors. Since there is one and only one path between any two vertices in a tree, no two adjacent vertices have the same color. Thus T has been properly colored with two colors. One color would not have been enough (observation 2 in this section). Though a tree is 2- chromatic, not every 2-chromatic graph is a tree

MOD 5

THEOREM 8-2

A graph with at least one edge is 2-chromatic if and only if it has no circuits of odd length.

Proof:

Let G be a connected graph with circuits of only even lengths. Consider a spanning tree T in G. Using the coloring procedure and the result of Theorem 8-1, let us properly color T with two colors. Now add the chords to T one by one. Since G had no circuits of odd length, the end vertices of every chord being replaced are differently colored in T. Thus G is colored with two colors, with no adjacent vertices having the same color. That is, G is 2- chromatic. Conversely, if G has a circuit of odd length, we would need at least three colors just for that circuit (observation 4 in this section). Thus the theorem.

MOD 5

A covering g of a graph is minimal if and only if g contains no paths of length three or more. Star graphs of one, two, three, and four edges.

Proof: Suppose that a covering g contains a path of length three, and it is v1e1v2e2v3e3v4 .

Edge e2 can be removed without leaving its end vertices v2 and v3 uncovered. Therefore, g is not a minimal covering. Conversely, if a covering g contains no path of length three or more, all its components must be star graphs (i.e., graphs in the shape of stars; see Fig. 8- 10). From a star graph no edge can be removed without leaving a vertex uncovered. That is, g must be a minimal covering.

MOD 5

MATCHINGS

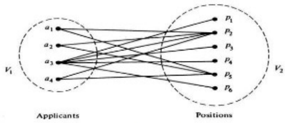


Fig. 8-6 Bipartite graph.

Suppose that four applicants a1 , a2 , a3 , and a4 are available to fill six vacant positions p1 , p2 , p3 , p4 , p5 , and p6 . Applicant a1 is qualified to fill position p2 or p5 . Applicant a2 can fill p2 or p5 . Applicant a3 is qualified for p1 , p2 , p3 , p4 , or p6 . Applicant a4 can fill jobs p2 or p5 . This situation is represented by the graph in The vacant positions and applicants are represented by vertices. The edges represent the qualifications of each applicant for filling different positions. The graph clearly is bipartite, the vertices falling into two sets V1 = {a1 , a2 , a3 , a4} and V2 = {p1 , p2 , p3 , p4 , p5 , p6}.

Is it possible to hire all the applicants and assign each a position for which he is suitable? If the answer is no, what is the maximum number of positions that can be filled from the given set of applicants?

This is a problem of matching (or assignment) of one set of vertices into another. More formally, a matching in a graph is a subset of edges in which no two edges are adjacent. A single edge in a graph is obviously a matching. A maximal matching is a matching to which no edge in the graph can be added. For example, in a complete graph of three vertices (i.e., a triangle) any single edge is a maximal matching. The edges shown by heavy lines in Fig. 8-7 are two maximal matchings. Clearly, a graph may have many different maximal matchings, and of different sizes. Among these, the maximal matchings with the largest number of edges are called the largest maximal matchings.), a largest maximal matching is shown in heavy lines. The number of edges in a largest maximal matching is called the matching number of the graph

MOD 5

CHROMATIC POLYNOMIAL

A given graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the chromatic polynomial of G and is defined as follows:

The value of the chromatic polynomial Pn (λ) of a graph with n vertices gives the number of ways of properly coloring the graph, using λ or fewer colors.

Let ci be the different ways of properly coloring G using exactly i different colors. Since I colors can be chosen out of λ colors in

$$\binom{\lambda}{i} \qquad \textbf{different ways,}$$

there are different ways of properly coloring G using exactly i colors out of λ colors. Since I can be any positive integer from 1 to n, the chromatic polynomial is a sum of these terms; that is,

$$\begin{aligned}P_n(\lambda) &= \sum_{i=1}^{n} c_i\binom{\lambda}{i}\\ &= c_1\frac{\lambda}{1!} + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \cdots\\ &\quad + c_n\frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{n!}.\end{aligned}$$

Any graph with even one edge requires at least two colors for proper coloring, and therefore

C1 = 0.

A graph with n vertices and using n different colors can be properly colored in n! ways; that

is, Cn = n!.

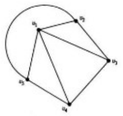
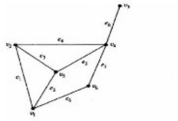


Fig. 8-4 A 3-chromatic graph.

MOD 5

ADJACENCY MATRIX/CONNECTION MATRIX

The adjacencymatrix of a graph G with n vertices and no parallel edges is an n by n symmetric binary matrix X = [xij] defined over the ring of integers such that xij = 1, if there is an edge between I th and j th vertices, and = 0, if there is no edge between them



Observations/Properties of the adjacency matrix X of a graph G are :

1. The entries along the principal diagonal of X are all 0'= the graph has no parallel edges. A self-loop at the ith vertex corresponds to xii = 1.
2. the adjacency matrix X was defined for graphs without parallel edges.
3. If the graph has no self-loops (and no parallel edges, of course), the degree of a vertex equals the number of 1′s in the corresponding row or column of X.
4. two graphs G1, and G2 with no parallel edges are isomorphic if and only if their adjacency matrices X(Gt) and X(G2) are related: X(G2) = R−1 X(G1)·R, where R is a permutation matrix.
5. A graph G is disconnected and is in two components g1 and g2 if and only if its adjacency matrix X(G) can be partitioned as

$$\mathbf{X}(G)=\left[\begin{array}{c:c}\mathbf{X}(g_1) & 0\\ \hdashline 0 & \mathbf{X}(g_2)\end{array}\right],$$

X(g1) is the adjacency matrix of the component g1 and X(g2) is that of the component g2. This partitioning clearly implies that there exists no edge joining any vertex in subgraph g1 to any vertex in subgraph g2.

6.Given any square, symmetric, binary matrix Q of order n, one can always construct a graph G of n vertices (and no parallel edges) such that Q is the adjacency matrix of G.

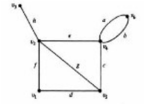
MOD 5

INCIDENCE MATRIX/VERTEX-EDGE INCIDENCE MATRIX

also written as A(G).

Let G be a graph with n vertices, e edges, and no self-loops. Define an n by e matrix A =[aij], whose n rows correspond to the n vertices and the e columns correspond to the e edges, as follows :

The matrix element Aij = 1, if jth edge ej is incident on ith vertex vi,and = 0, otherwise



Observations/Properties of the incidence matrix A(G) of a graph G are :

1.Every edge is incident on exactly two vertices, so each column of A has exactly two I's.

2.The number of I's in each row = the degree of the corresponding vertex. 3. A row with all 0's, represents an isolated vertex.

4.Parallel edges in a graph produce identical columns in its incidence matrix,eg:columns 1 and 2

5.If a graph G is disconnected and consists of two components g1 and g2, theincidencematrix A(G) of graph G can be written in a block-diagonal form

$$\mathbf{A}(G)=\left[\begin{array}{c:c}\mathbf{A}(g_1) & 0\\ \hdashline 0 & \mathbf{A}(g_2)\end{array}\right],$$

where A(g1) and A(g2) are the incidence matrices of components g1 and g2.

7. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph

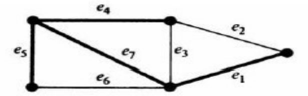
MOD 5

FUNDAMENTAL CIRCUIT MATRIX

A submatrix (of a circuit matrix) in which all rows correspond to a set of fundamental circuit is called a fundamental circuit matrix Bf. A graph and its fundamental circuit matrix with respect to a spanning tree If n is the number of vertices and e the number of edges in a connected graph, then Bf is an (e− n + 1) by e matrix, because the number of fundamental circuits is e − n + 1, each fundamental circuit being produced by one chord.

Arrange the columns in Bf such that all the e − n + 1 chords correspond to the first e − n + 1

columns. Rearrange the rows such that the first row corresponds to the fundamental circuit made by the chord in the first column, the second row to the fundamental circuit made by the second, and so on.



MOD 5

VERTEX COLOURING

Given a graph G with n vertices and are asked to paint its vertices such that no two adjacent vertices have the same color. What is the minimum number of colors that you would require? This constitutes a coloring problem. Having painted the vertices, you can group them into different sets— one set consisting of all red vertices, another of blue, and so forth. This is a partitioning problem. The coloring and partitioning can, of course, be performed on edges or vertices of a graph.

MOD 5

CHROMATIC NUMBER

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the proper coloring (or sometimes simply coloring) of a graph. A graph in which every vertex has been assigned a color according to a proper coloring is called a properly colored graph. Usually a given graph can be properly colored in many different ways

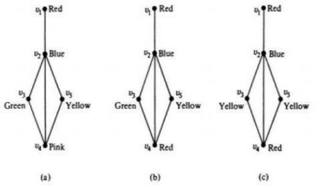


Fig. 8-1 Proper colorings of a graph.

The proper coloring which is of interest to us is one that requires the minimum number of colors. A graph G that requires κ different colors for its proper coloring, and no less, is called a κ-chromatic graph, and the number κ is called the chromatic number of G. You can verify that the graph is 3- chromatic. In coloring graphs there is no point in considering disconnected graphs.

Properties / Obsevations of chromatic numbers

1.A graph consisting of only isolated vertices is 1-chromatic.

2.A graph with one or more edges (not a self-loop, of course) is at least 2- chromatic (also 3. A complete graph of n vertices is n -chromatic, as all its vertices are adjacent. Hence a called bichromatic).

graph containing a complete graph of r vertices is at least r-chromatic. For instance, every graph having a triangle is at least 3- chromatic.