9 members of a club meet everyday for a dinner, they sit in a round table for dinner, but not two members who sat together will sit together in future, how long is this possible

SEATING PROBLEM

9 Members meet for lunch every day in a round table

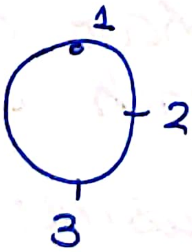
Every day → new neighbour

How many ways can they be seated?

consider only 2 members

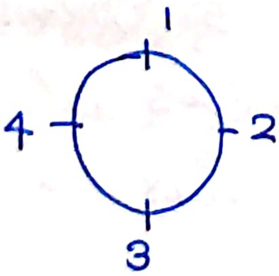
Day 1: 1 2
Day 2: — } 1 way

3 members



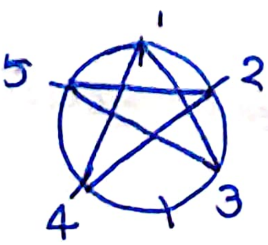
Day ~~2~~ 1: 1 2 3 1
Day 2: – } 1 way

4 members



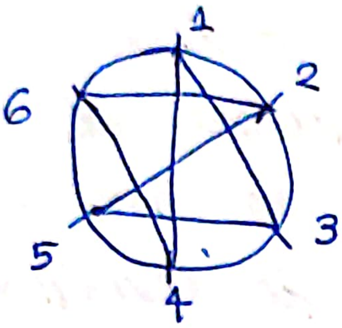
Day 1: 1 2 3 4 1
Day 2: 1 3 × Not possible } 1 way

5 members



Day 1: 1 2 3 4 5 1
Day 2: 1 3 5 2 4 1 } 2
Day 3: Not possible

6 members

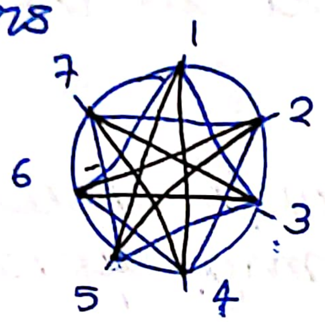


Day 1 : 1 2 3 4 5 6 1

Day 2 : 1 3 5 2 6 4 1

Day 3 : - 2 ways

7 members



Day 1: 1 2 3 4 5 6 7 1

Day 2: 1 3 5 7 2 4 6 1

Day 3: 1 4 7 3 6 2 5 1

Day 4 : -
3

Initially given a graph K_9 ie nine vertices as nine members

Similarly any cycle can be shown from from K_9.

No. of possible arrangements are the edge disjoint Hamiltonian cycles of K_9, the no. of edge disjoint hamiltonian circuits are $\frac{n-1}{2}$ if n is an odd no > 3

$$\therefore \frac{9-1}{2} = 4 \rightarrow \text{hamiltonian circuits}$$

Since they meet every day & each day a hamiltonian circuit is obtained they can meet for 4 days.

DIRECTED PATH & CONNECTEDNESS

A directed path from a vertex V_i and V_j is an alternating sequence of vertices and edges, beginning with V_i and ending with V_j such that each edge is oriented from the vertex preceeding it to the vertex following it

No edge in directed walk appears more than once but a vertex may appear more than once

SEMIWALK

semiwalk in a directed graph is a walk in the corresponding undirected graph

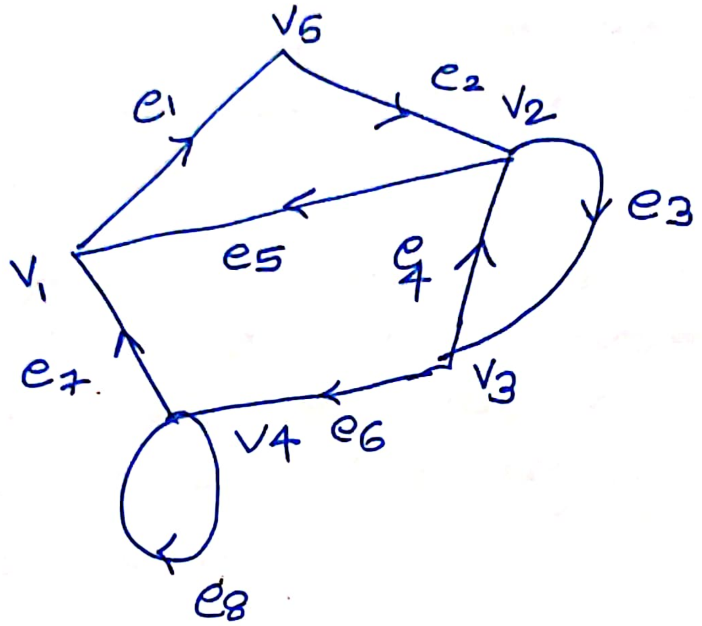
PATH, DIRECTED PATH, SEMIPATH

I

CONNECTED DIAGRAPH

A diagraph G is said to be strongly connected if there is at least one directed path from every vertex to every other vertex

A diagraph G is said to be weakly connected if its corresponding undirected graph is connected



Path
A ~~ed~~ open walk such that no ~~edges~~ vertices are repeated in a directed graph (moving along same dir)

Semi Path
~~walk~~ is path in undirected graph

Directed walk : $V_4 e_7 V_1 e_1 V_5 e_2 V_2$
Semi walk : ~~e_4~~ $V_4\ e_6 V_3\ e_4\ V_2$
Directed Path : $V_1\ e_1 V_5\ e_2\ V_2$
Semi Path : $V_1\ e_5\ V_2\ e_3\ V_3$
circuit : $V_1\ e_1\ V_5\ e_2\ V_2\ e_5\ V_1$
Semi circuit $V_1\ e_5\ e_4\ V_3\ V_4\ e_7\ V_1$

THEOREM

A connected graph G is an Euler graph iff only it can be decomposed into circuits

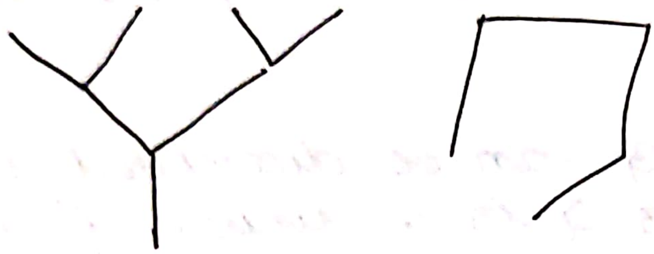
Textbook (28,29)

suppose graph G can be decomposed into circuits; that is G is a union of edge disjoint circuits. Since degree of every vertex in a circuit is two the degree of every vertex in G is even

conversely, let G be an Euler graph. Consider a vertex v_1. There are at least two edges incident at v_1. Let one of these edges be b/w v_1 and v_2, ∴ v_2 is also of even degree it can also be traversed and finally forming a circuit. T. Let us remove T from G. The remaining vertices must also be of even degree. From this circuit remove another circuit in exactly same way as we removed T from G. Continue until no edges are left. Thus we proved the graph can be decomposed into circuits

MODULE 3

TREES



tree is a connected graph without any circuit, tree is a simple graph

THEOREM 1

There is one and only one path b/w every pair of vertices in a tree T

since T is a connected graph there must exist at least one path b/w every pair of vertices in T. Suppose that b/w two vertices a and b of T there are two distinct paths, the union of these two paths will contain a circuit, then T cannot be a tree, hence there is only one path b/w every pair of vertices in a tree T

THEOREM 2

If in a graph G there is one and only one path b/w every pair of vertices then G is a tree

Existence of a path b/w every pair of vertices assures that G is connected then to prove G is a tree it is enough to prove that G has no circuit.

A circuit in a graph G implies that there is at least one pair of vertices (a,b) such that there are two distinct paths b/w a&b

Since G has one and only one path b/w every pair of vertices, G has no circuit Therefore G is a tree.

A tree with n vertices has n-1 edges

THEOREM

A graph is a tree iff and only if it is minimally connected

A connected graph is said to be minimally connected if the removal of any one edge from it disconnects the graph

A minimally connected graph can't have a circuit otherwise removal of one edge from a circuit does not disconnect the graph

Thus the minimally connected graph is a tree

Conversely if a connected graph G is not minimally connected there must exist an e_i in G such that $G-e_i$ is connected.

Therefore e_i is in some circuit which implies that G is not a tree. Therefore ~~a minimally conne~~ the graph G is minimally connected.

THEOREM

A graph G with n vertices n-1 edges and no circuits is connected

PROOF

A graph G with n vertices

suppose that there exists a circuitless with n vertices and n-1 edges which is disconnected, in that case G will consist of two or more circuitless components. Without loss of generality let G consist of two components g_1 and g_2, add an edge b/w v_1 in g_1 and v_2 in g_2. Since there ~~is~~ was no path b/w v_1 and v_2 in G adding ~~e~~ e did not create a circuit. Therefore G∪E is a circuitless connected graph with n vertices and n edges. This is not possible. Therefore our assumption is wrong. Hence ~~any~~ G is connected

3M Write any 3 properties of trees

A graph G with n vertices is called a tree if G is connected and circuitless

G is connected and has n-1 edges

G is minimally connected.

There exists ~~one~~ exactly one path b/w every pair of vertices in G.

PENDANT VERTICES IN A TREE

In any tree with two or more vertices there are atleast two pendant vertices

For n vertices, no. of edges is $n-1$

sum of degrees $= 2 \times (n-1)$

$= 2n - 2$

$d(v_1) + d(v_2) + \dots + d(v_n) = 2n-2$

$\Sigma d(v_i) = 2(n-1)$

↳ give 2 to each but $2n$ → decrease by 2

Trees with n vertices have $n-1$ edges moreover it is connected therefore no vertex can be of degree 0

sum of degrees of vertices in a graph is equal to 2x no. of edges

$$\Sigma d(v_i) = 2(n-1)$$

Therefore we must have at least 2 vertices of degree 1

DISTANCE B/W ANY TWO VERTICES

It is the no. of edges in the shortest path connecting these two vertices

A function $f(x,y)$ is called a metric if it satisfies the following conditions

1. Non Negativity

$f(x,y) = 0$ only if $x = y$

symmetry

$f(x,y) = f(y,x)$

Triangle inequality

$\cancel{f(x,z)}$ $f(x,y) \leq f(x,z) + f(z,y) \ \forall z$

Let $f(x,y\cancel{z})$ be defined in R^2 as distance b/w two points in the plane

Distance b/w Two vertices in a graph

In a connected graph G the distance b/w vertices V_i and V_j is denoted as $d(v_i, v_j)$ and is defined as the length of the shortest path connecting V_i and v_j

The distance b/w the vertices of a connected graph G is a metric

$d(v_i, v_j) \geq 0$

Since distance b/w two vertices is the length of the shortest path connecting these vertices, distance b/w $v_i, v_j = 0 \Rightarrow v_i = v_j$

SYMMETRY

Dist b/w v_i, v_j is equal to v_j, v_i

$d(v_i, v_j) = d(v_j, v_i)$ which is equal to length of the shortest path connecting v_i, v_j

Let v_k be any of the other vertices in G then distance b/w v_i v_j is less than or equal to $d(v_i, v_j) \leq d(v_i, v_k) + d(v_k, v_j)$

Therefore distance b/w any two points in a connected graph is a metric

ECCENTRICITY OF A GRAPH

The eccentricity of a graph E(V) of a vertex V in G is the distance from V to the vertex farthest from V in G.

eccentricity of V is equal to maximum of

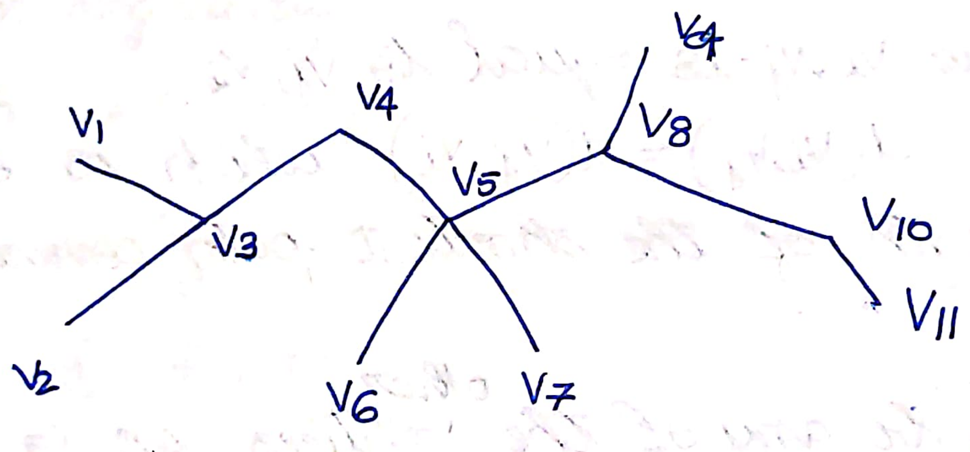
$E(v) = \max(d(v, v_i))$ $v_i \in G$

CENTRE OF A GRAPH

A vertex with minimum eccentricity in graph G is called centre of G

RADIUS OF A TREE

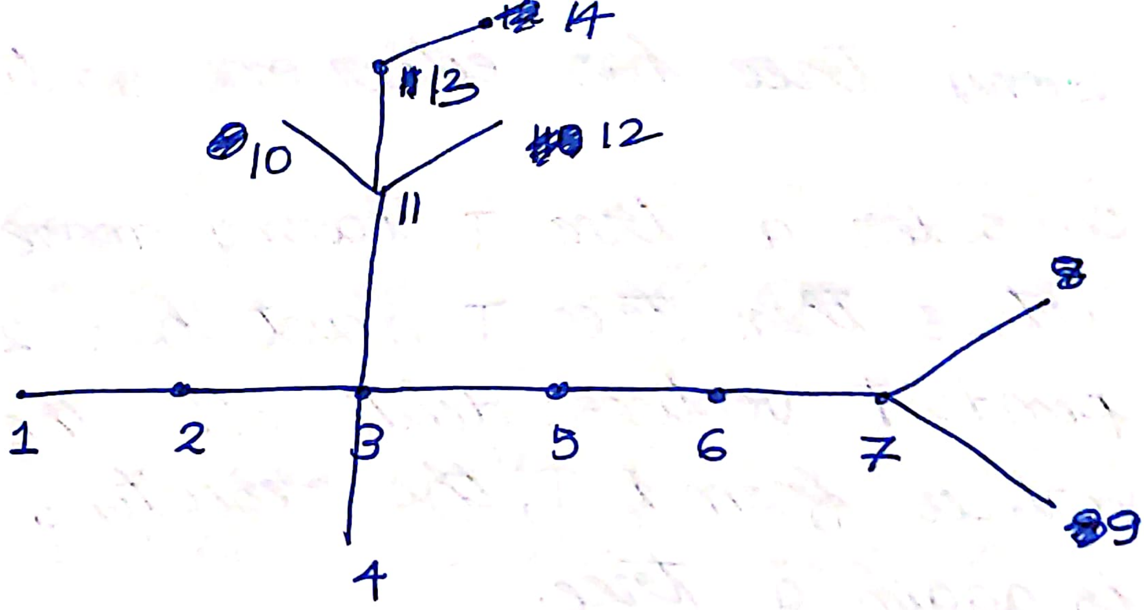
The eccentricity of the centre in a tree is called radius of a tree



DIAMETER OF A TREE

It is defined as the longest path in G

$e(V_1)=6$
$e(V_2)=6$
$e(V_3)=5$
$e(V_4)=4$
$e(V_5)=3$
$e(V_6)=4$
$e(V_7)=4$
$e(V_8)=4$
$e(V_9)=5$
$e(V_{11})=6$



E(1) = 6

E(2) = 5

E(3) = 4 → centre

E(4) = 5

E(5) = 4 → centre

E(6) = 5

E(7) = 6

E(8) = 7

E(9) = 7

E(10) = 5

E(11) = 5

E(12) = 6

E(13) = 6

E(14) = 7

Diameter max 7

Every tree has either one or two centre

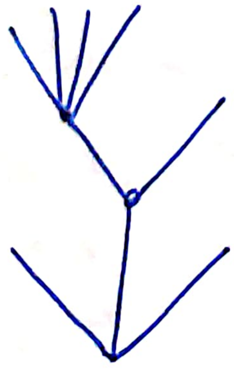
Consider a tree T having more than 2 vertices then tree T must have 2 or more pendant vertices, delete all the pendant vertices from T, the resulting graph T' is again a tree.

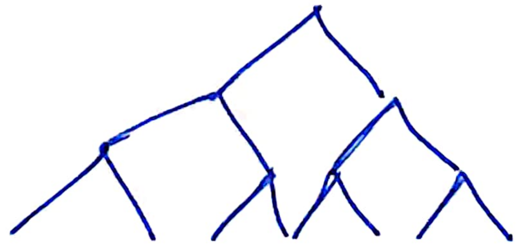
From T' we again remove all pendant vertices and obtain another tree T'', we continue this process until there is left an edge or vertex, if we get a vertex only, tree has only one centre. If we get an edge, tree has 2 centres.

ROOTED & BINARY TREE

A tree in which one vertex called (root) is distinguished from all other vertices is called a rooted tree

BINARY TREE





BINARY TREE

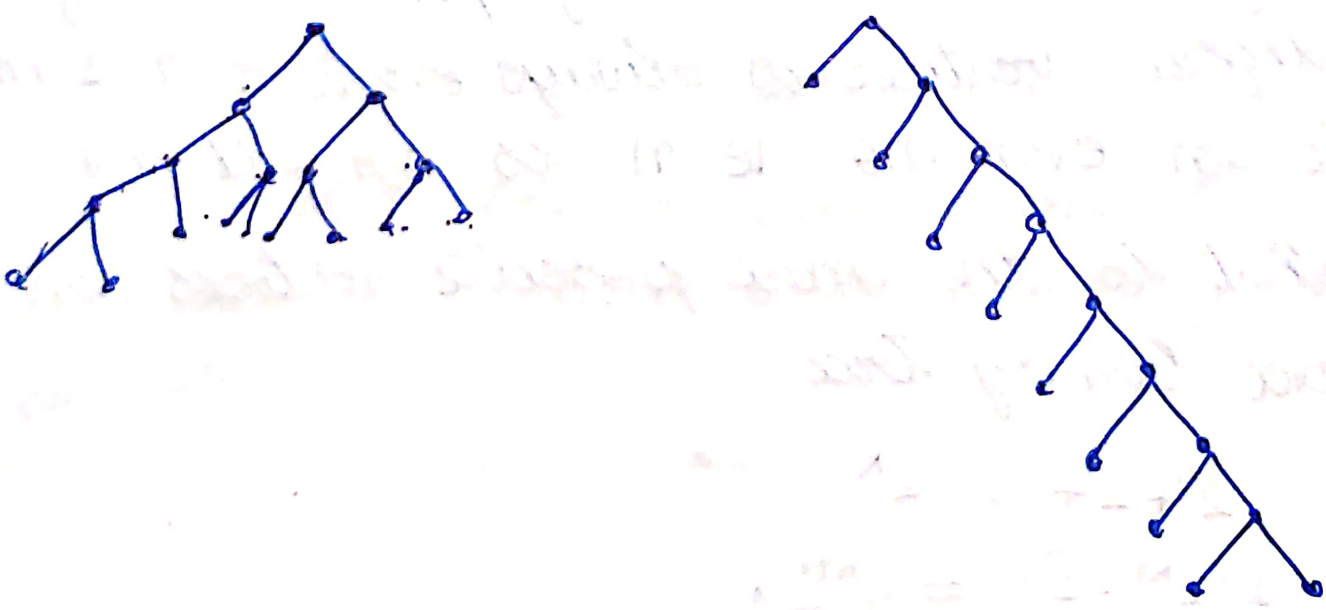
A binary tree is defined as a rooted tree in which there is exactly one vertex of degree 2 and each of the remaining vertices of degree 1 or 3, a non pendant vertex in a tree is called internal vertex, ~~level of a vertex~~

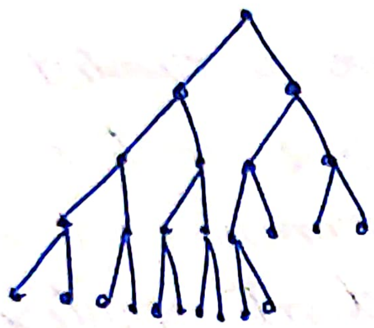
LEVEL OF A VERTEX

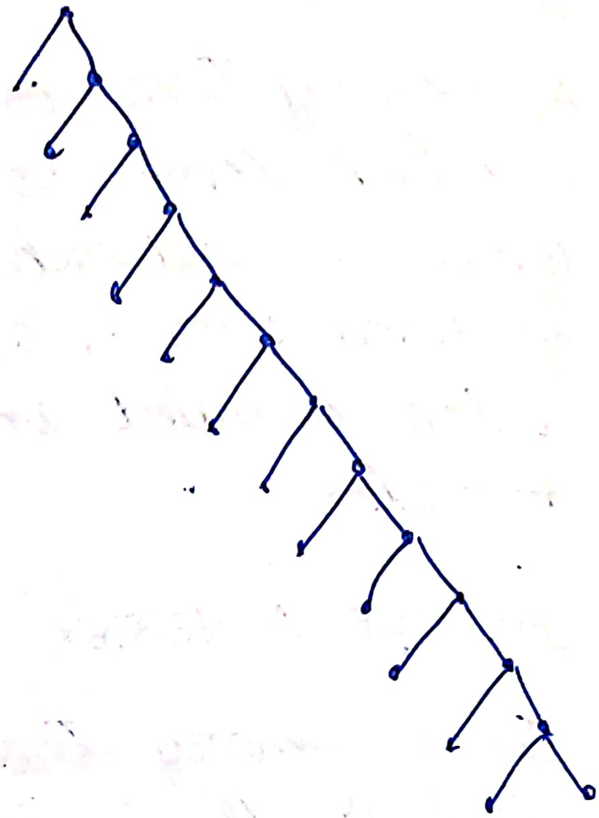
In a binary tree a vertex v_i is said to be level v_i if v_i is at a distance e_i from the root

The root is at the level zero

The number of vertices in a tree is 17 Draw the binary tree which has the maximum height and minimum height







PROVE that the number of vertices in a binary tree is always odd

Ans: One vertex has degree 2 all other vertices have degree 1 or 3 ie $n-1$ vertices are odd degree vertices. In a graph the no. of odd degree vertices is always even, ie $n-1$ must be an even no ie n is an odd no

What is the max possible vertices in k level binary tree

$$2^0 + 2^1 + \dots 2^k$$

$$\frac{1(2^{k+1}-1)}{2-1} = 2^{k+1} - 1$$

The no. of pendant vertices in a binary tree having n vertices is $\frac{n+1}{2}$.

PROOF : $1+p+(n-1-p)=n$

$$2\times 1+p\times 1+3(n-1-p)=\cancel{2(n)}\ 2(n-1)$$

$$2+p+3n-3-3p=2n-2$$

$$3n-2n+1=2p$$

$$n+1=2p$$

$$p=\frac{n+1}{2}$$

Prove that the no. of internal vertices in a binary tree is one less than the no. of pendant vertices.

PROOF: Let p denote the no. of pendant vertices in a tree. Therefore no. of internal vertices is equal to $n-p$ ie $p=\frac{n+1}{2}$ $\therefore n-p=n-\frac{(n+1)}{2}$

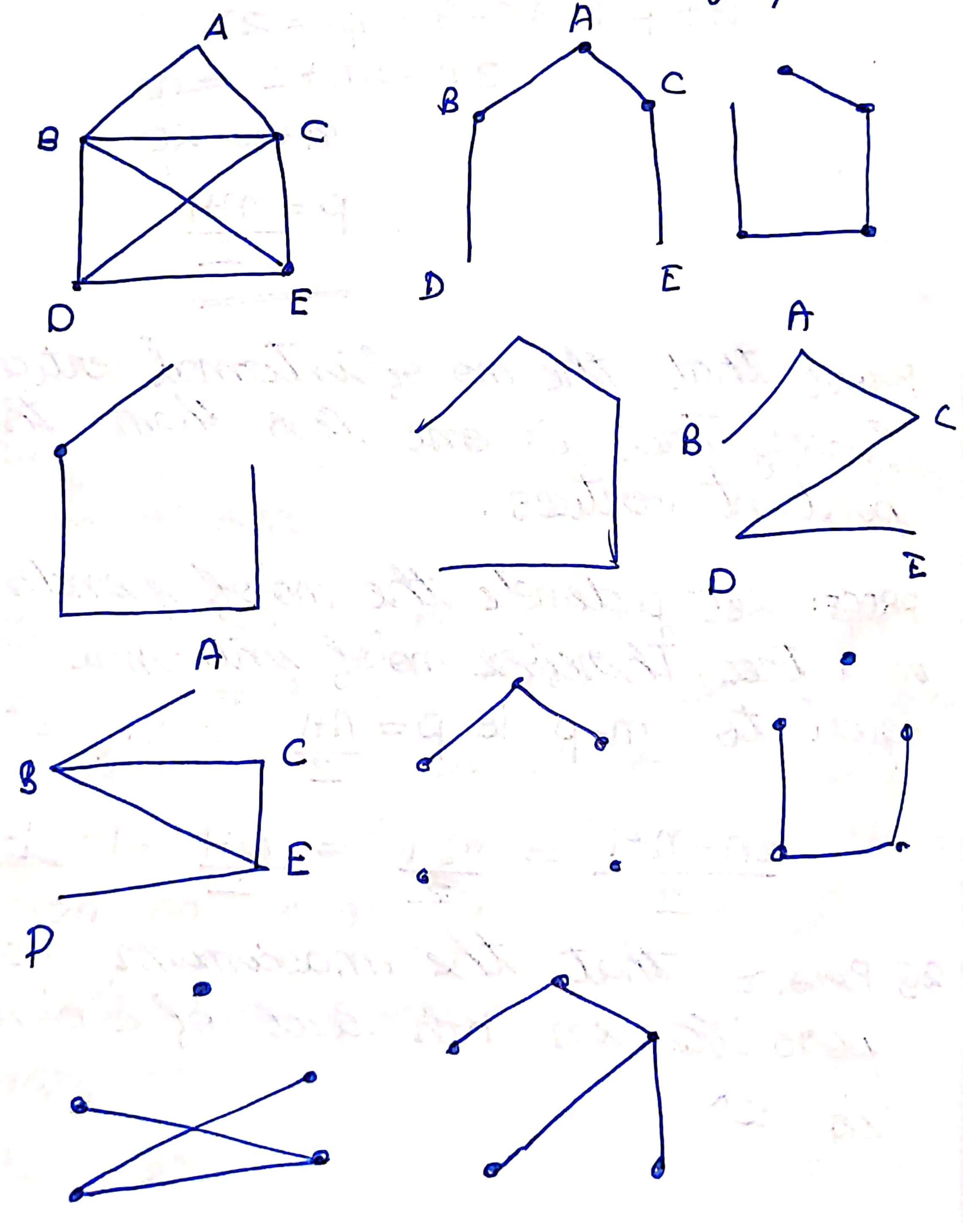
$$\frac{2n-n+1}{2}=\frac{n-1}{2}=\frac{n+1}{2}-1=p-1$$

29 Prove that the maximum no. of vertices possible in kth level of a binary tree is 2^k

SPANNING TREE

A tree T is said to be a spanning tree of a connected graph G, if T is a subgraph of G and T contains all ~~subgraphs~~ vertices of G

Find all spanning trees of the graph



Since spanning trees are the largest tree in all trees of G. it is called maximal tree of G

Each component of a disconnected graph have a spanning tree, thus a disconnected graph with k components has a spanning forest consisting of K spanning trees

A collection of trees is called a forest.

Every connected graph has at least one spanning tree

If G has no circuit it is its own spanning tree, if G has a circuit, delete an edge from the circuit, this will still leave the graph connected if there are more circuits repeat the operation till an edge from the last circuit is deleted.

Then we get a circuitry graph that contains all the vertices of G. Therefore every connected graph has at least one spanning tree.

THEOREM

With respect to any of its spanning tree a connected graph with n vertices and e edges has n-1 branches & e-n+1 chords

If a connected graph has n vertices then its spanning tree also has n vertices.

the edges in the spanning tree are called branches

If a tree has n vertices it has $n-1$ edges

$\therefore$ no. of branches $= n-1$

The remaining edges not included in the spanning tree are called ~~pots~~ chords.

$\therefore$ no. of chords is $e-(n-1) = e-n+1$

RANK AND NULLILITY

Let G be a graph, e be the no. of edges, K be the no. of components in G then rank is equal to $r = n-K$

Nullility $(\mu) = e-n+K$

The rank of a connected graph G is $n-1$ $(K=1)$

Nullility of a connected graph is $e-n+1$

Rank of G = No. of branches in ~~a~~ any spanning tree of G

Nullity of G = No. of chords in G

Nullity of a graph is also referred to as ~~cyclomapic or~~ cyclomatic no or first Betti no.

FUNDAMENTAL CIRCUIT

consider a spanning tree T in a connected graph G, add anyone chord to T it will create exactly one circuit such a circuit is called fundamental circuit.

Such

If the circuit in G contains only one chord & others are branches in G →

DISTANCE B/W TWO SPANNING TREES

$d(T_1, T_2)$

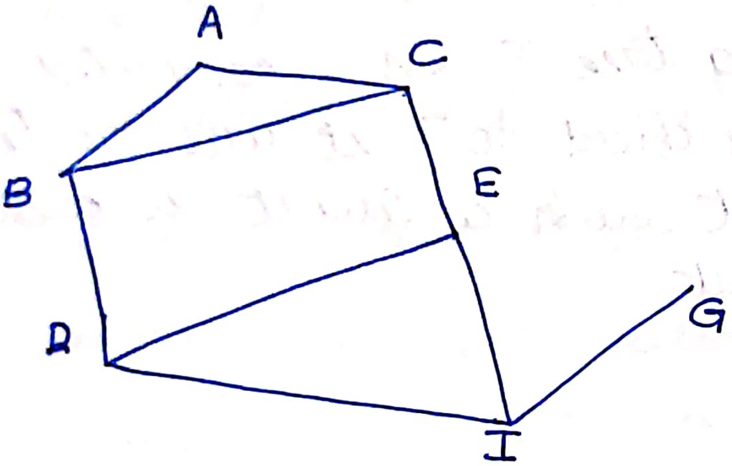
Distance b/w two spanning trees T_i and T_j of a connected graph G is defined as the no of edges of G present in tree but not in another, it is denoted as $d(T_i, T_j)$

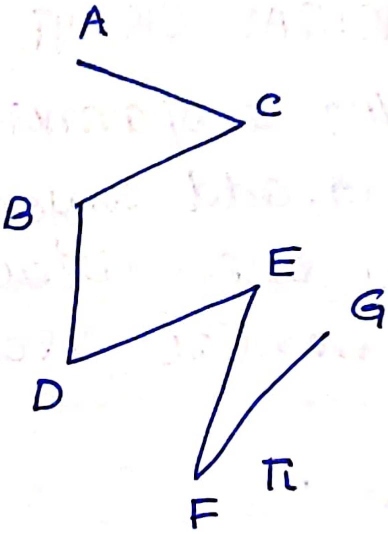
RING SUM OF TWO SPANNING TREES (T_i, T_j)

Ring sum of two trees T_i and T_j of a connected graph G is defined as a subgraph of G consisting of all edges of G that are either in T_i or T_j but not in both.

A $T_i \oplus T_j$ $\quad$ $N(T_i \oplus T_j)$ denote no. of edges in $T_i \oplus T_j$

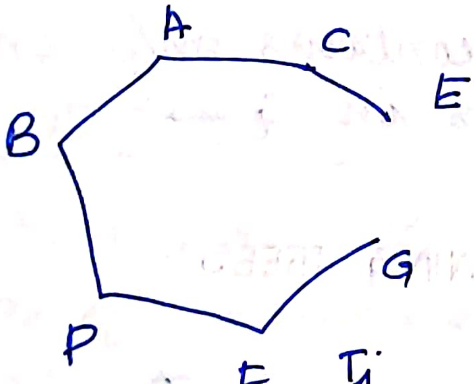
$$N(T_i \oplus T_j) = 2 \times d(T_i T_j)$$
$$= 2d(T_j, T_i)$$





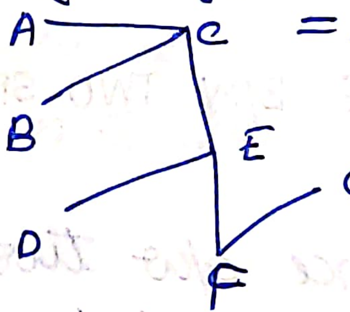
T_i

~~T_j~~ T_i



T_j

$$N(T_i \oplus T_j) = 3+3 = 6$$
$$= 2d(T_i T_j)$$



DISTANCE BETWEEN THE SPANNING TREES OF A CONNECTED GRAPH G is a metric

Non negativity
Let G be a connected graph, Ti and Tj be the spanning trees of G, then distance b/w Ti and Tj. $d(T_i, T_j) \geq 0$ since it is no. of edges in Ti but not in Tj

$d(T_i, T_j) = 0$ when $T_i = T_j$

SYMMETRY

$d(T_i\ T_j) = d(T_j, T_i) = \underbrace{n-1}_{\text{(no. of edges in a first tree)}} - (\text{No of common edges})$

TRANSITIVITY

Let $T_i\ T_j\ T_K$ be any three spanning trees of a connected graph G

$d(T_i T_j) \leq d(T_i, T_K) + d(T_K, T_j)$ for any T_K in G

$\therefore$ distance b/w two spanning trees is a metric

If a graph G is a weighted graph then the weight of a spanning tree Tg is defined as the sum of the weights of all branches in T

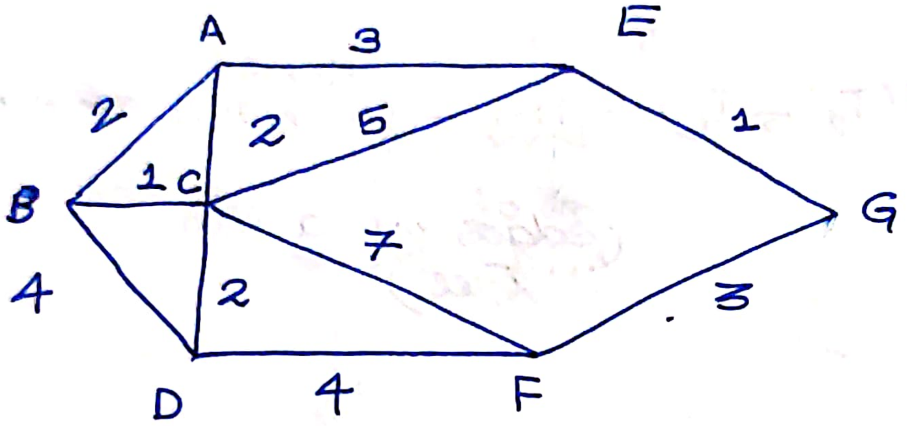
A spanning tree with smallest weight in a weighted graph is called shortest/minimal spanning tree

ALGORITHMS FOR SHORTEST SPANNING TREE

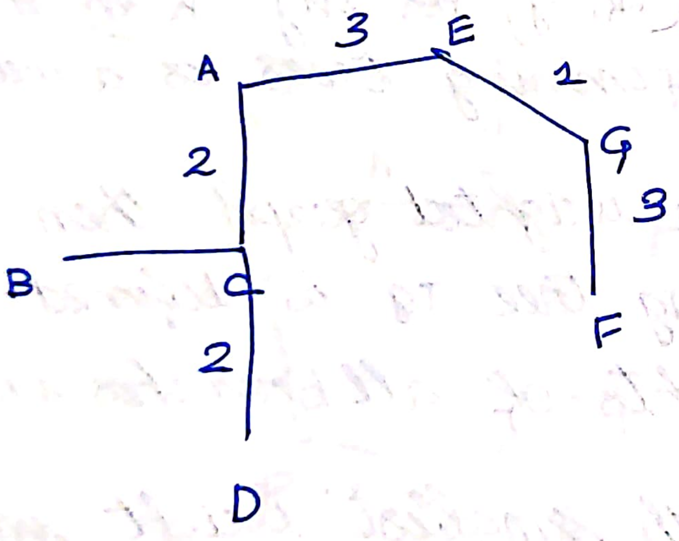
KRUSKAL'S ALGORITHM

PRIM'S ALGORITHM

KRUSKAL'S ALGORITHM pg 62, 63

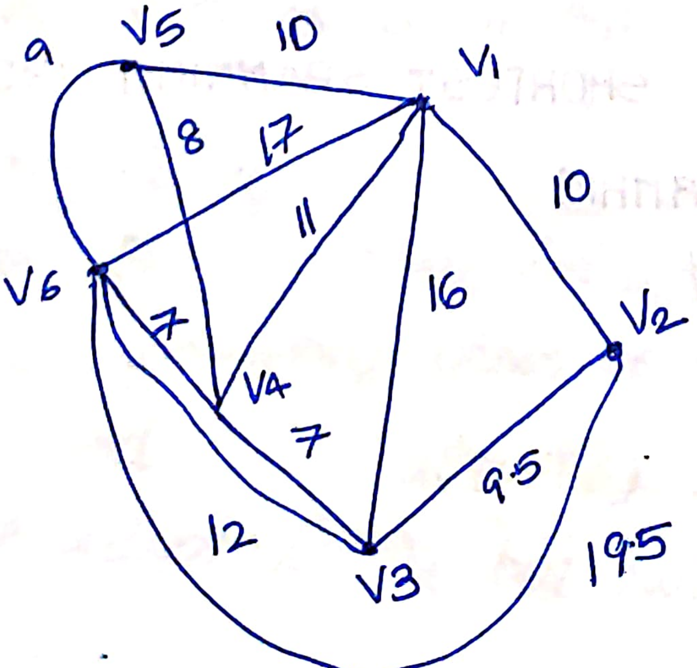


BC – 1 ✓
EG – 1 ✓
AC – 2 ✓
CD – 2 ✓
AB – 2 ✗
AE – 3 ✓
FG – 3 ✓
DF – 4 ✗
BD – 4 ✗
CE – 5 ✗
CF – 7 ✗

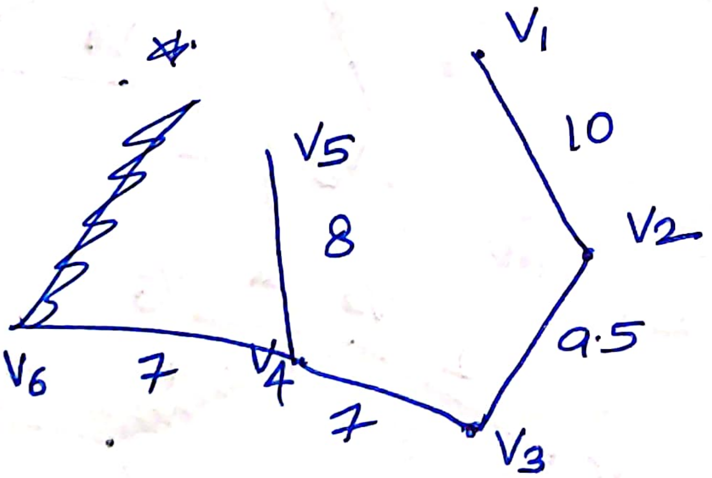


Minimal weight of spanning tree = 12

PRIM'S ALGORITHM

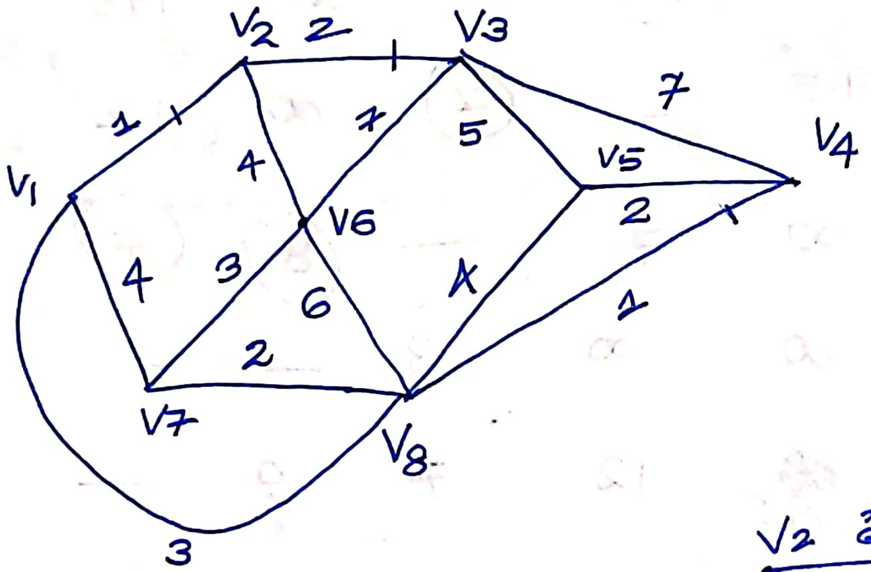


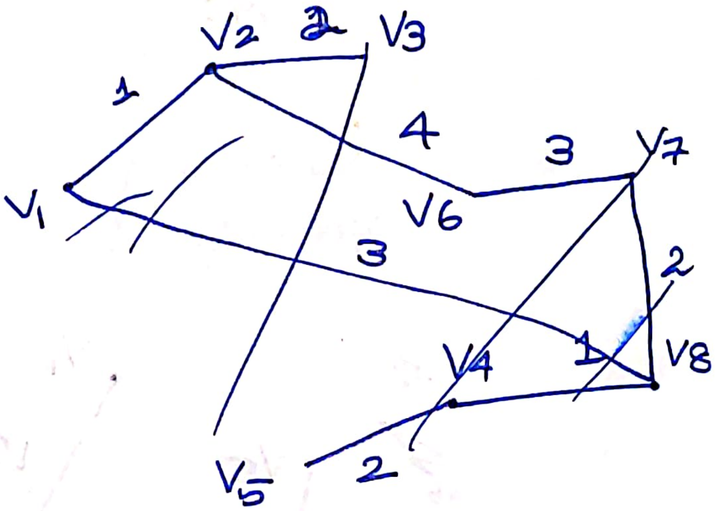
	V_1	V_2	V_3	V_4	V_5	V_6
V_1	—	10	16	11	10	17
V_2	10	—	9.5	∞	∞	19.5
V_3	16	9.5	—	7	∞	12
V_4	11	∞	7	—	8	7
V_5	10	∞	∞	8	—	9
V_6	17	~~∞~~ 19.5	12	7	9	—



Total wt is same

use Prims and Kruskal algorithm find minimum spanning tree





$V_1 - V_2 = 1$ ✓

$V_4 - V_8 = 1$ ✓

$V_2 - V_3 = 2$ ✓

$V_4 - V_5 = 2$ ✓

$V_7 - V_8 = 2$ ✓

$V_6 - V_7 = 3$ ✓

$V_1 - V_8 = 3$ ✓

$V_2 - V_6 = 4$

$V_5 - V_8 = 4$

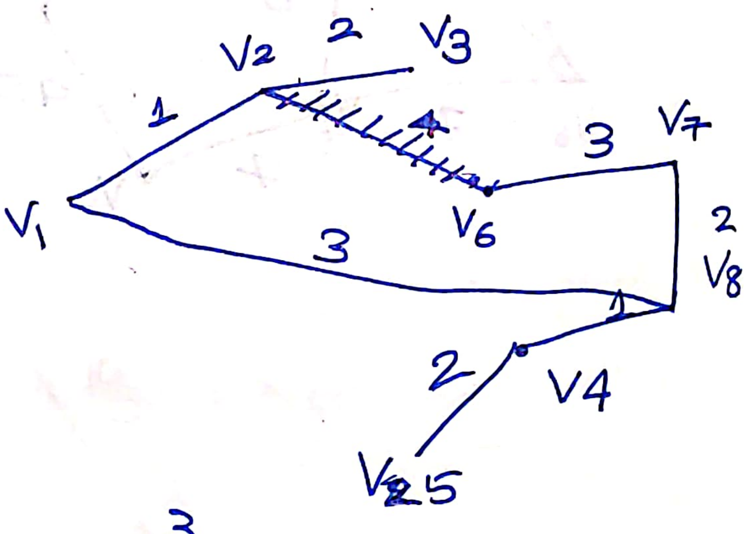
$V_1 - V_7 = 4$

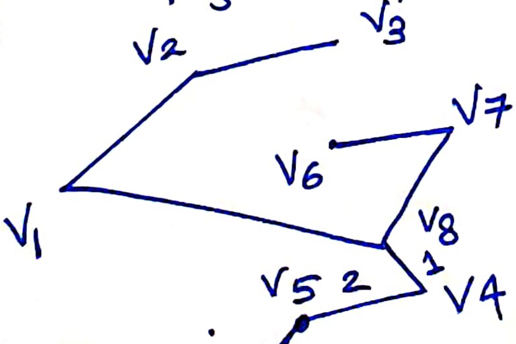
$V_3 - V_5 = 5$

$V_6 - V_8 = 6$

$V_3 - V_4 = 7$

$V_3 - V_6 = 7$



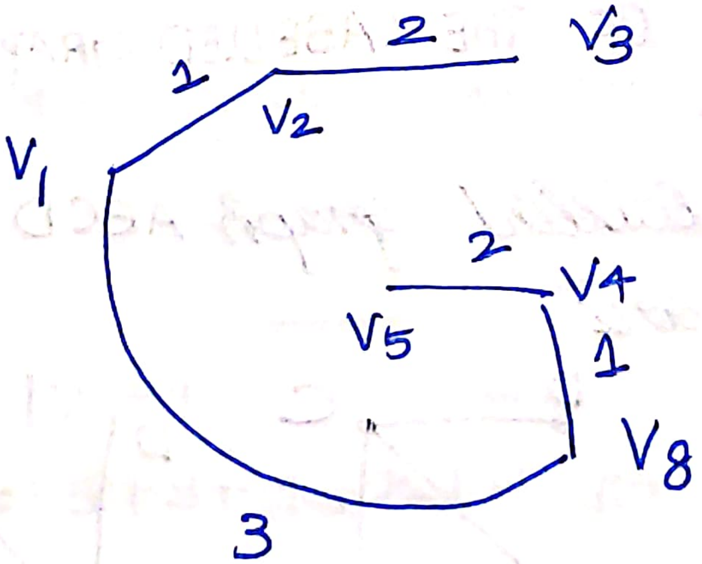


$1+2+\cancel{4}\,3+3+2+1+2$

$= \cancel{12}\,14$

~~Kruska~~ Prims

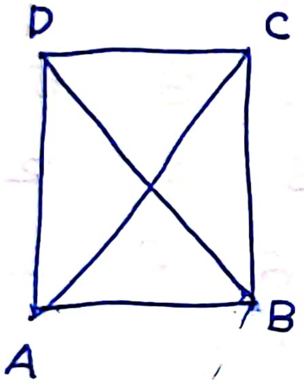
	V_1	V_2	V_3	V_4	V_5	V_6	V_7	
V_1	—	1	∞	∞	∞	∞	4	3
V_2	1	—	2	∞	∞	4	∞	∞
V_3	∞	2	—	7	5	7	∞	∞
V_4	∞	∞	7	—	2	∞	∞	1
V_5	∞	∞	5	2	—	∞	∞	4
V_6	∞	4	7	∞	∞	—	3	6
V_7	4	∞	∞	∞	∞	3	—	2
V_8	3	∞	∞	1	4	6	2	—



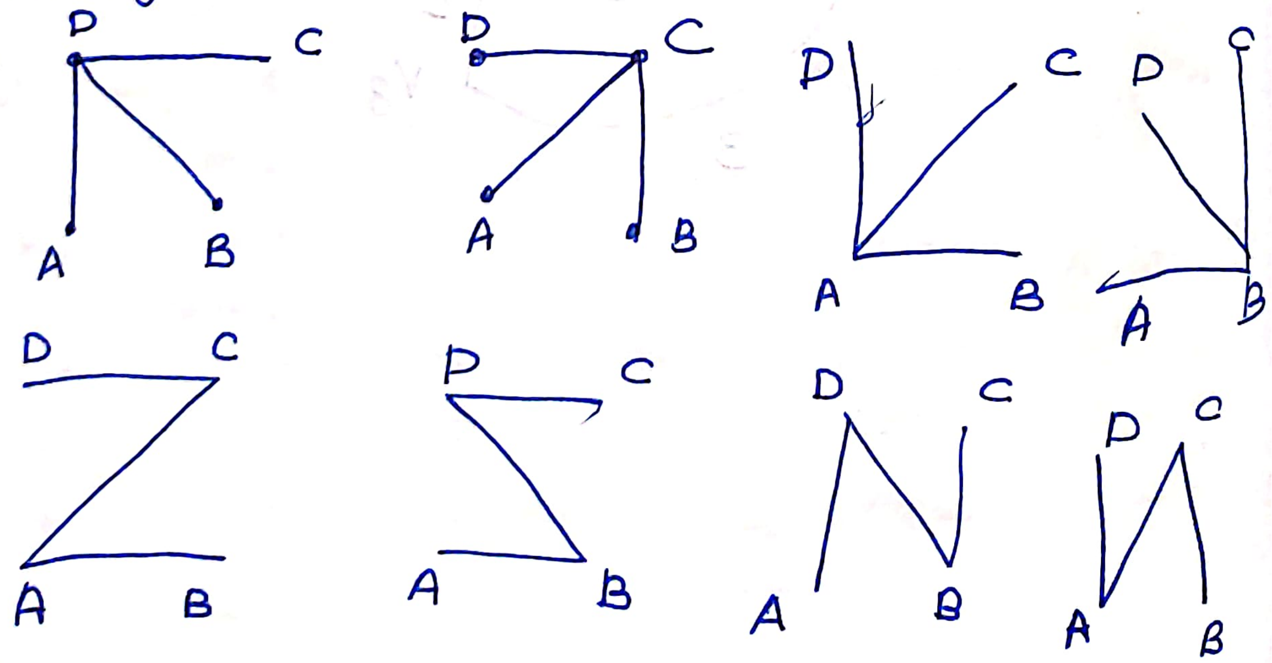
COUNTING TREES

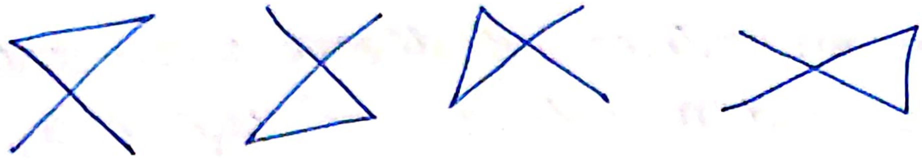
A graph in which every vertex is assigned a unique name or label, no two vertices are assigned the same label is called a labelled graph

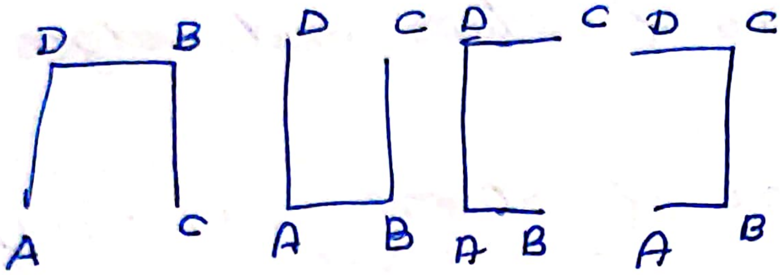
The no. of labelled trees with n vertices $n \geq 2$ is n^{n-2}



SPANNING TREES OF THE LABELLED GRAPH is called counting trees
consider the labelled graph ABCD The possible counting trees are







Q A tree has 5 vertices of degree 2, 3 vertices of degree 3 4 vertices of degree 4
How many vertices of degree 1 does it have

Let x denote the no. of pendant vertices

$$5+3+4+x = 2\times(n-1)$$
$$\times 2 \quad \times 3 \quad \times 4 \quad \times 1$$

$$10+9+16+x = 2(n-1)$$

$$35+x = 2(n-1)$$

$$e = |V|-1$$
$$= 5+3+4+x-1 = 11+x$$

$$35+x = 2(11+x)$$
$$35+x = 22+2x$$
$$\underline{x = 13}$$

A tree has 2n vertices of degree 1, 3n vertices of degree 2 and n vertices of degree 3, determine the no. of vertices and edges in a tree

sum of degrees of vertices = 2 x no. of edges

$$2n + 3n \times 2 + 3n = 2 \times \text{no of edges}$$

$$11n = 2e \quad \text{—} (1)$$

$$e = 2n + 3n + n - 1$$

$$e = 6n - 1 \quad \text{—} (2)$$

$$11n = 2(6n-1)$$

$$11 = 12n - 2$$

$$\underline{\underline{n = 2}}$$

$$\text{Edges} = 6n - 1 = \underline{\underline{11}}$$

Show that a complete graph K_n $n \geq 2$ is not a tree

When $n > 2$ there exists a circuits

Show that if a tree has exactly two pendant vertices the degree of every other vertex is 2

Let the vertices be $V_1, V_2 \cdots, V_n$

$d(v_1)=1$
$d(v_2)=1$

$$1+1+d(v_3)+d(v_4)+\cdots+d(v_n)=2(n-1)$$

$$\underbrace{d(v_3)+d(v_4)+\cdots+d(v_n)}_{n-2}=2n-2-2$$
$$=2(n-2)$$

Sum of degrees of vertices

$$\Sigma d(v_i)=2(n-1)$$

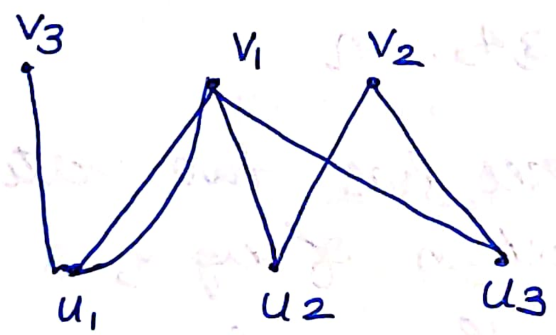
$$\Rightarrow d(v_i)=2 \quad i=3,4,\ldots n$$

Suppose that a tree has two vertices of degree 2, 4 vertices of deg 3, 3 vertices of deg 4 find the no. of pendant vertices

BIPARTITE GRAPH

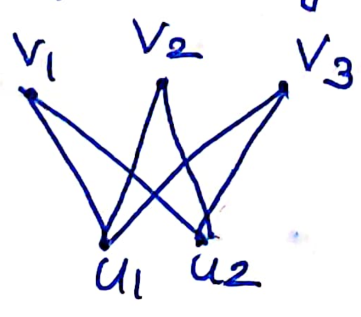
A graph $G=(V,E)$ is bipartite if the vertex set V can be ~~parlion~~ partitioned into 2 subsets V_1 and V_2 such that every edge in E connects a vertex in V_1 and a vertex in V_2. ~~connects~~ the ordered pair (V_1, V_2) is called bipartition of G and G is called a bipartite graph

No loops are allowed in bipartite graph but parallel edges may be possible



COMPLETE BIPARTITE GRAPH

A complete bipartite graph of m and n vertices $K_{m,n}$ is the graph whose vertex set is partitioned into sets ~~V~~ V_1 with m vertices and V_2 with n vertices in which there is an edge b/w each pair of vertices v_i and v_j where v_i is in V_1 and v_j is in V_2

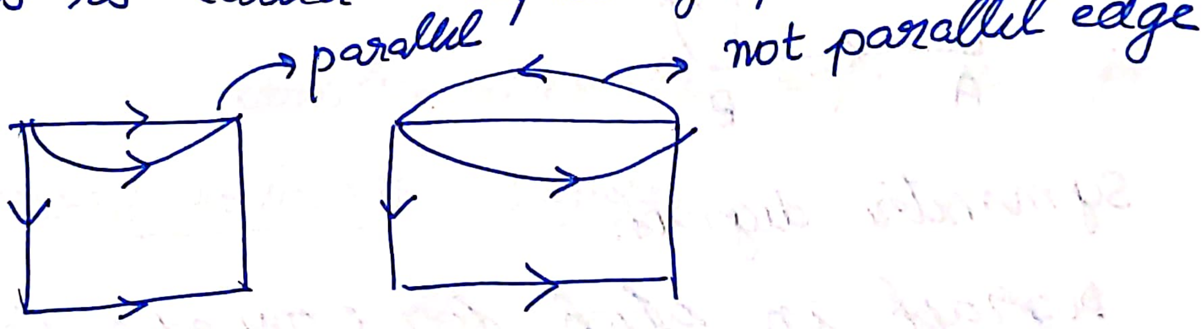


ASYMMETRIC AND SYMMETRIC DIGRAPH

Digraphs that have at most one directed edge b/w a pair of vertices but are allowed to have self loops are called asymmetric or antisymmetric

SIMPLE DIGRAPH

A digraph that has no self loops or parallel edges is called simple graph



Parallel edges in a digraph

Two edges e and e' of a digraph are said to be parallel if e and e' have the same initial vertex and same terminal vertex

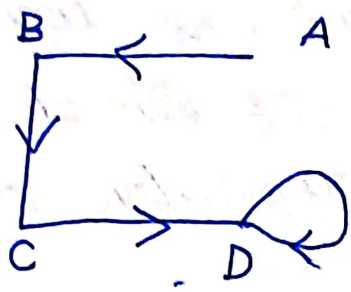
<u>Pendant vertex</u>

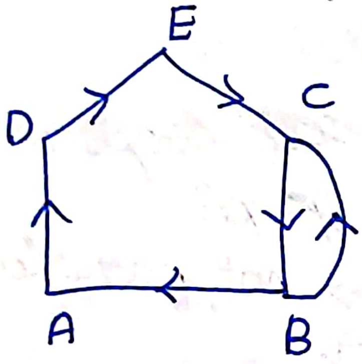
If v is a vertex of a digraph D then v is called a pendant vertex of D if $d^+(v) + d^-(v) = 1$

d^+v (out degree)

d^-v (indegree)

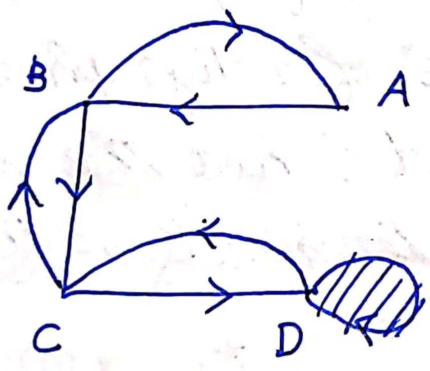
eg ASYMMETRIC DIGRAPH





Not asymmetric
But simple

Symmetric digraph
Digraph in which for every edge (a,b), there is also an edge (b,a)

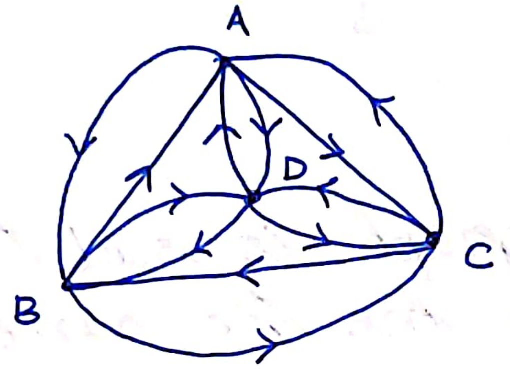


A digraph that is both simple and ~~simp~~ symmetric is called simple symmetric digraph

A digraph that is both simple and asymmetric is called simple asymmetric digraph

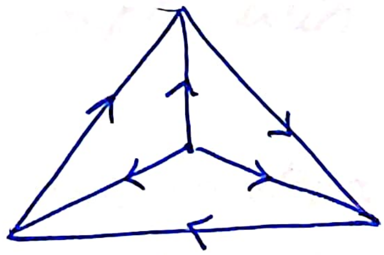
COMPLETE SYMMETRIC DIGRAPH

~~For dio~~ A complete symmetric digraph is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex



complete symmetric, simple digraph

COMPLETE ASYMMETRIC DIGRAPH

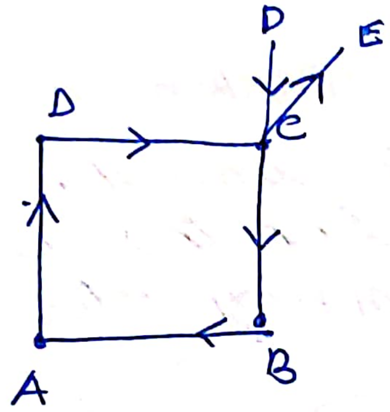


A complete asymmetric digraph is an asymmetric digraph in which there is exactly one edge b/w every pair of vertices.

A complete asymmetric digraph of n vertices contains $\frac{n(n-1)}{2}$ edges but a symmetric digraph of n vertices contain $n(n-1)$ edges.

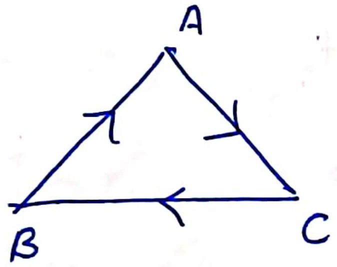
BALANCED DIGRAPH

A digraph D is said to be a balanced digraph / isograph, indegree = outdegree for every vertex



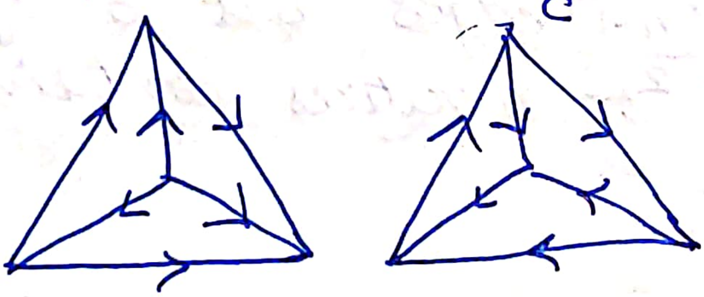
REGULAR DIGRAPH

A balanced digraph is said to be regular if every vertex has the same indegree as outdegree as every other vertex



ISOMORPHIC DIGRAPHS

Two digraphs are isomorphic not only must their corresponding undirected graphs be isomorphic but also the directions of the corresponding edges must also agree



Not isomorphic

MODULE 5

INCIDENCE MATRIX

Matrix representation of graphs, ~~inci~~.

(i) Incidence matrix

(ii)

	a	b	c	e	d
V_1					
V_2					
V_3					
V_4					
V_5					
V_6					

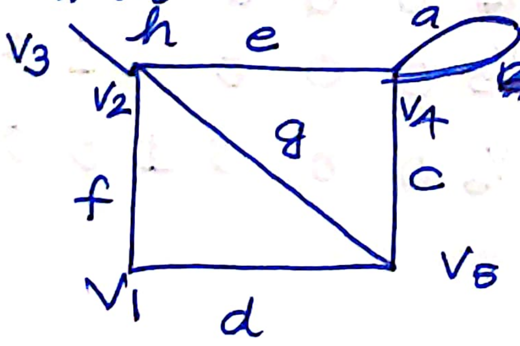
INCIDENCE MATRIX

Let G be a graph with n vertices, e edges and no self loops

Define an $n \times e$ matrix $A=[a_{ij}]$ whose n rows corresponding to n vertices and e columns correspond to edges as follows

$a_{ij}=1$ if jth edge is incident on ith vertex
else $a_{ij}=0$

Write the incident matrix of the given graph

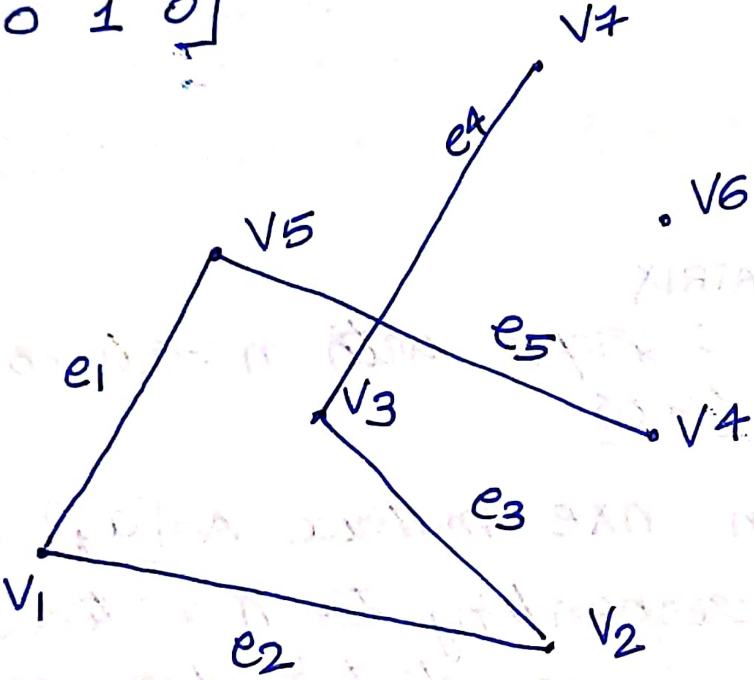


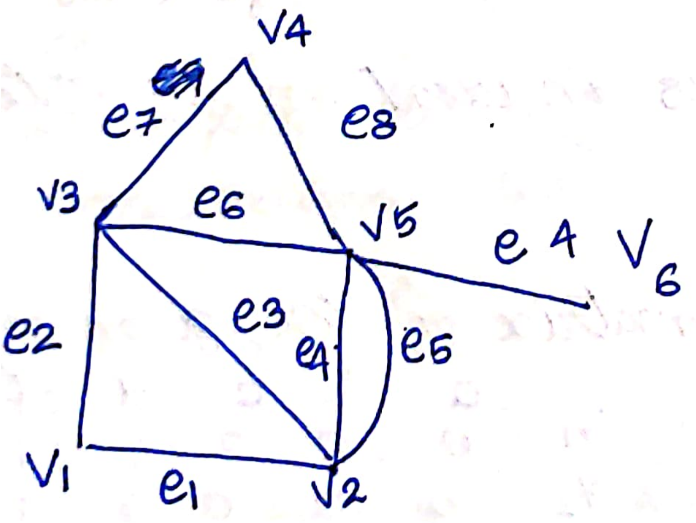
	a	b	c	d	e	f	g	h
V_1	0	0	0	1	0	1	0	0
V_2	0	0	0	0	1	1	1	1
V_3	0	0	0	0	0	0	0	1
V_4	1	1	1	0	1	0	0	[illegible]
V_5	0	0	1	1	0	0	1	[illegible]
V_6	1	1	0	0	0	0	0	[illegible]

Find a graphs whose incident matrix is

pg 139

$$
\begin{array}{c|c}
 & \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \\
\begin{matrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \end{matrix} &
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$





	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
V_1	1	1	0	0	0	0	0	0	0
V_2	1	0	1	1	1	0	0	0	0
V_3	0	1	1	0	0	1	1	0	0
V_4	0	0	0	0	0	0	1	1	0
V_5	0	0	0	1	1	1	0	1	1
V_6	0	0	0	1	0	0	0	[illegible]	1

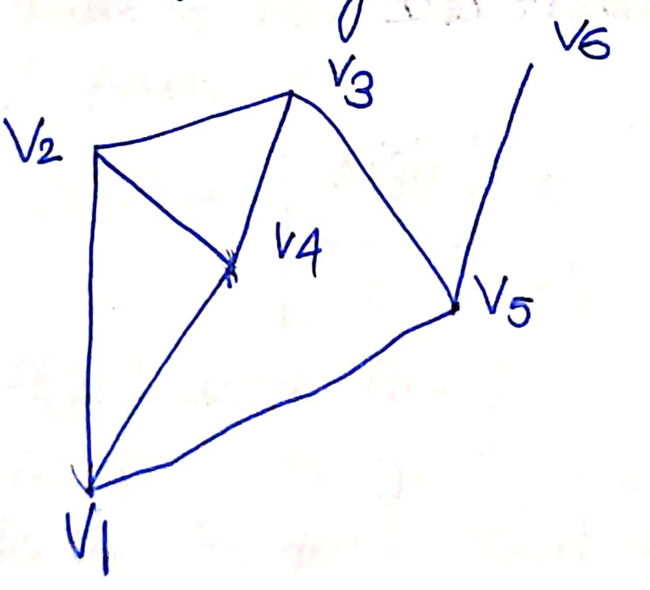
$$\begin{bmatrix} A(g_1) & 0 \\ 0 & A(g_1) \end{bmatrix}$$

ADJACENCY MATRIX

The adjacency matrix of a graph G with n vertices and no parallel edges is an $n \times n$ symmetric matrix $X = [X_{ij}]$ which is defin as $X_{ij} = 1$ if ith vertex is incident to jth vertex

$X_{ij} = 0$ otherwise

Q. Write the adjacency matrix of the graph



e9
0
0
0
0
1
1

	V_1	V_2	V_3	V_4	V_5	V_6
V_1	~~1~~0	1	0	1	1	0
V_2	1	0	1	1	0	0
V_3	0	1	0	1	1	0
V_4	1	1	1	0	0	0
V_5	0	0	1	0	0	1
V_6	0	0	0	0	1	0

V_6 - pendant

All 0's $\rightarrow$ isolated vertex

All diagonal entries non zero $\rightarrow$ there is a loop

The degree of a vertex

$$\begin{matrix} X(G_1) & 0 \\ 0 & X(G_2) \end{matrix}$$

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PROPERTIES

Since every edge is incident on exactly 2 vertices, each column of A has exactly two 1s

2. the number of 1s in each row equals the degree of the corresponding vertex

3. A row with all 0's therefore represents an isolated vertex

4. Parallel edges in a graph produce identical columns in its incidence matrix for eg column 1 and 2 in Fig 7.

5 If a graph G is disconnected and consists of two components g_1 and g_2 the incidence matrix $A(G)$ of graph G can be written in a block diagonal form as

$$A(G) = \left[\begin{array}{c|c} A(g_1) & 0 \\ \hline 0 & A(g_2) \end{array}\right]$$

where $A(g_1)$ and $A(g_2)$ are the incidence matrices of components g_1 and g_2. This observation results from the fact that no edge in g_1 is incident on vertices of g_2 and vice versa. Obviously this remark is also true for a disconnected graph with any no. of components.

6. Permutation of any two rows or columns in an incidence matrix simply corresponds to relabeling the vertices and edges of the same graph.

A simple graph and its adjacency matrix are shown in figure

1. The entries ~~...~~ along the principal diagonal are all 0s if and only if the graph has no self loops. A self loop at the ith vertex corresponds to $x_{ii} = 1$

2. The definition of adjacency matrix makes no provision for parallel edges. This is why the adjacency matrix X was defined for graphs without parallel edges.

3. If the graph has no self loops and no parallel edges of course the degree of a vertex equals the ~~..~~ number of 1's in the corresponding row or column of X

4. Permutations of rows & of the corresponding columns imply reordering the vertices. It must be noted. however that the rows & columns must be arrange in the same order. Thus if two rows are interchanged in X, the corresponding cols must be

interchanged. Hence 2 graphs G_1 and G_2 with no parallel edges are isomorphic if and only if their adjacency matrix $X(G_1)$ and $X(G_2)$ are related

$$X(G_2) = R^{-1} \cdot X(G_1) \cdot R$$ where R is a permutation matrix

2. A graph is disconnected and is in two components g_1 and g_2 if and only its adjacency matrix $X(G)$ can be partitioned as

$$X(G) = \left[\begin{array}{c|c} X(g_1) & 0 \\ \hline 0 & X(g_2) \end{array}\right]$$

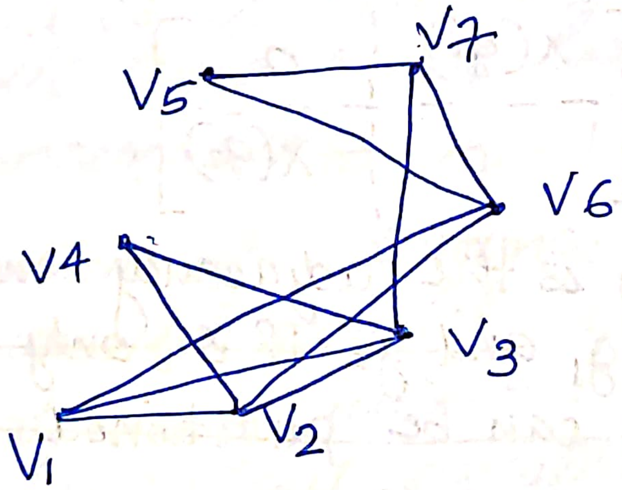
where $X(g_1)$ is the adjacency matrix of the components g_1 ~~and g_2 if & only if its adjacency matrix $X(G)$ can be partitioned as~~

$X(g_2)$ is that of the component g_2.

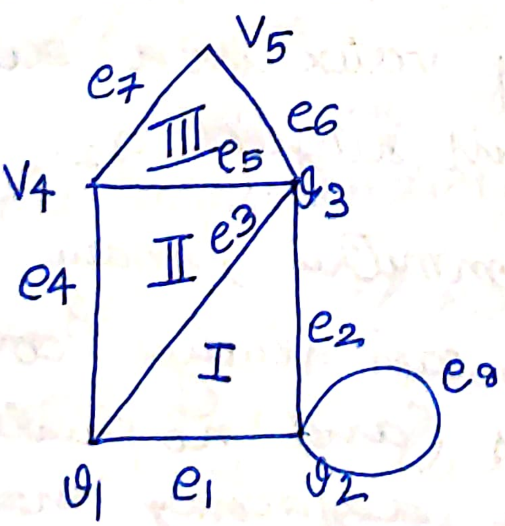
this partitioning implies that there exists no edge joining any vertex in a subgraph g_1 to any vertex in subgraph g_2.

3. Given any square, symmetric, binary matrix Q of order n, one can always construct a graph G of n vertices (and no parallel edges) such that Q is the adjacency matrix of G.

$$
\begin{array}{c|ccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\hline
v_1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
v_2 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
v_3 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
v_4 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
v_6 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
v_7 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{array}
$$



CIRCUIT MATRIX



I = $\{e_1, e_2 e_3\}$

II - $\{e_3, e_4, e_5\}$

III - $\{e_5, e_6, e_7\}$

IV - $\{e_3, e_6, e_7, e_4\}$

V - $\{e_1, e_2, e_5, e_4\}$

VI - $\{e_1, e_2, e_6, e_7, e_4\}$

No. of circuits = no. of rows

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
I	1	1	1	0	0	0	0	0
II	0	0	1	1	1	0	0	0
III	0	0	0	0	1	1	1	0
IV	0	0	1	1	0	1	1	0
V	1	1	0	1	1	0	0	0
VI	0	0	0	0	0	0	0	1
VII	1	1	0	1	0	1	1	0

consider a graph G with e edges, let Q be the different circuits in G then then the circuit matrix

is a $Q \times e$ matrix, defined as follows

$b_{ij} = \{1\}$ if ith circuit includes jth edge otherwise it is zero.

If all entries are 0's in a particular col then the edge is not included in any circ

If there is only '1' in a row → it is a loop.

Rows → Row Vector

Columns → column Vector

Observations

→ A column of all zeroes correspond to a non circuit edge

→ Each row of B(G) is a circuit vector, unlike incidence matrix. a circuit matrix is capable of representing a self loop, the corresponding row ~~col~~ will have

→ ~~The co~~ a single 1.

the number of 1s in a row is equal to no. of edges in the corresponding circuit

If graph G is separable and consists of two components g_1 and g_2 the circuit can be written in a block diagonal form

$$\begin{bmatrix} B(g_1) & 0 \\ 0 & B(g_2) \end{bmatrix}$$

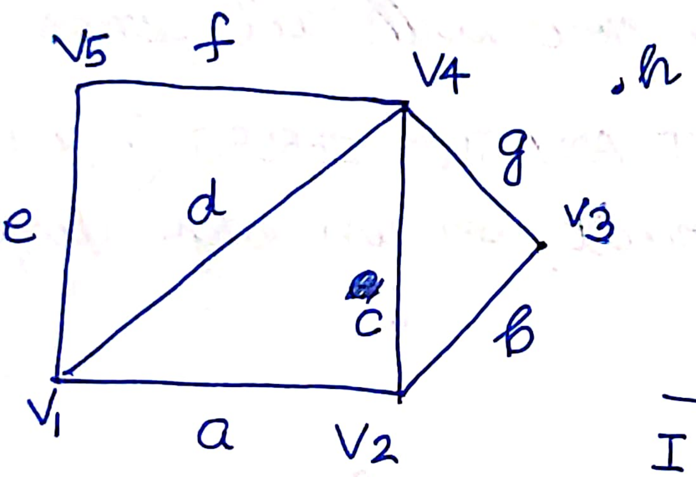
Permutation of any two rows ~~in a~~ / columns in a circuit matrix, simply corresponds to relabelling the circuits & edges

PATH MATRIX

Path matrix is defined for a specific pair of vertices in a graph say (x, y) and is written as $P(x, y)$ path matrix connecting x and y.

The rows in path matrix correspond to diff paths b/w x and y. ~~Row~~ Columns correspond to the edges in G.

$P_{ij} = 1$ if jth edge lies in the ith path
equal to 0 otherwise.



$P(V_1, V_3)$ I $\{a, b\}$

II $\{d, g\}$

III $\{a, c, g\}$

IV $\{e, f, g\}$

V $\{e, f, \cancel{d}\, c, b\}$

VI $\{d, c, b\}$

	a	b	c	d	e	f	g
I	1	1	0	0	0	0	0
II	0	0	0	1	0	0	1
III	1	0	1	0	0	0	1
IV	0	0	0	0	1	1	1
V	0	1	1	0	1	1	0
VI	0	1	1	1	0	0	0

All 0's in a column → that edge is not included in any of the paths

In case it is 1 → all the column's 1s

A column of all zero corresponds to an edge that lies b/w any path b/w x and y

A column of all 1s correspond to an edge that lies in every path b/w x and y.

There is no row with all zeroes.

RING SUM OF ANY TWO ZEROES (P(x,y)) corresponds to a circuit. or an edge disjoint union of circuit

Incidence matrix for the graph shown earlier

	a	b	c	d	e	f	g	h
V_1	1	0	0	1	1	0	0	1
V_2	1	1	1	0	0	0	0	0
V_3	0	1	0	0	0	0	1	0
V_4	0	0	1	1	0	1	1	0
V_5	0	0	0	0	1	1	0	1

B^T.

B^T	V_1	V_2	V_3	V_4	V_5
a	1	1	0	0	0
b	0	1	1	0	0
c	0	1	0	1	0
d	1	0	0	1	0
e	1	0	0	0	1
f	0	0	0	1	1
g	0	0	1	1	0

$\cancel{A \cdot B^T =}$

~~V_1~~ ~~V_2~~ ~~V_3~~

	a	b	c	d	e	f	g
V_1							
V_2							
V_3							
V_4							
V_5							

	V_1	V_2	V_3	V_4	V_5
V_1	3	0	0	0	0
V_2					
V_3					
V_4					
V_5					

$$
B^T = \begin{array}{c}
 \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} & \mathrm{V} & \mathrm{VI} \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

$$
A \cdot B^T = \begin{array}{c}
 \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ \
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} & \mathrm{VI} & \mathrm{VI} \end{array} \\
\begin{bmatrix}
2 & \cancel{4}2 & 0 & 2 & 2 & 2 \\
0 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 2 & 0 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 \\
2 & 0 & 0 & 2 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
\equiv 0 \bmod 2
$$

Let G be a

Let B and A be respectively the circuit and incidence matrix of a self loop-free graph whose columns are arranged as the same order of edges

Then every row of B is orthogonal to every row of A ie $A \cdot B^T = B \cdot A^T \cong 0 \bmod m$

Let G be a graph without self loops, and let A and B respectively be the incidence and cycle matrix of G, note that for any matrix v_i and any circuit c_j in G either

$v_i \in c_j$ or $v_i \notin c_j$

In the former case there are exactly two edges of c_j which are incident on v_i and in later case there is no edge of c_j which is incident on v_i

Now consider ith row of a and jth row of b, which is the jth column of B^T

Since the edges are arranged in the same order, the rth entries in these two rows are both non zero iff only if the edge e_r is incident on the ith vertex v_i and also in the jth circuit c_j

$$= \cancel{\Sigma[A]_{ir}[}$$

$$(A \cdot B^T)_{ij} = \cancel{\Sigma A_{ir} B^T_{rj}}$$

$$= \cancel{\Sigma a_{ir} b_{jr}}$$

$$= \Sigma [A]_{ir} [B^T]_{rj}$$

$$= \Sigma [A]_{ir} [B]_{jr}$$

$$= \Sigma a_{ir} b_{jr}$$

For each edge e_r of G we have one of the following cases, e_r is incident on v_i and e_r does not belong to C_j

Here $a_{ir} = 1$, $b_{ir} = 0$ $a_{ir} b_{jr} = 0$

e_r is not incident on v_i and e_r is ~~not~~ an element of C_j

$a_{ir} = 0$, $b_{jr} = 1$ $a_{ir} b_{jr} = 0$

e_r is ~~in~~ not incident on V_i e_r is not an element in C_j

e_r is incident on V_i. e_r is an element of C_j then $a_{ir} = 1$, $b_{jr} = 1$

∴ sum

Here we have, ~~d_{ii}~~ exactly 2 edges, say e_r and e_t incident on v_i such that

$a_{ir}=1$

$a_{it}=1$

$b_{jr}=1$ $\quad\quad AB^T = 1+1 = 0 \bmod 2$

$b_{jt}=1$

Thus we can say that $AB^T \equiv 0 \bmod 2$

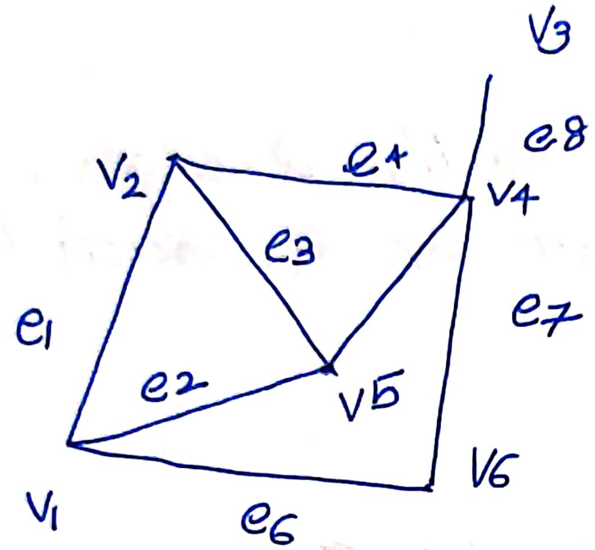
1P properties of Adjacency Matrix

POWERS OF ADJACENCY MATRIX(X)

the value of an offdiagonal entry in X^2 is equal to

That is ijth entry i not equal j in X^2 is equal to the no. of 1s in the dot product of ith row and jth column which is equal to the no. of positions in which both ith and jth rows of x have 1s which is equal to no. of vertices that are adjacent to both ith and jth vertices which is equal to the no. of different paths of lengths b/w ith & jth vertices

$$X^r = X^{r-1} \cdot X$$



$$
X = \begin{array}{c|cccccc}
 & V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \\
\hline
V_1 & 0 & 1 & 0 & 0 & 1 & 1 \\
V_2 & 1 & 0 & 0 & 1 & 1 & 0 \\
V_3 & 0 & 0 & 0 & 1 & 0 & 0 \\
V_4 & 0 & 1 & 1 & 0 & 1 & 1 \\
V_5 & 1 & 1 & 0 & 1 & 0 & 0 \\
V_6 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

$$
X^2 = \begin{bmatrix}
3 & 1 & 0 & 3 & 1 & 0 \\
1 & 3 & 1 & 1 & 2 & 2 \\
0 & 1 & 1 & 0 & 1 & 1 \\
3 & 1 & 0 & 4 & 1 & 0 \\
1 & 2 & 1 & 1 & 2 & 2 \\
0 & 2 & 1 & 0 & 2 & 2
\end{bmatrix}
$$

$$X^3 = \begin{bmatrix} 2 & 7 & 3 & 2 & 7 & 6 \\ 7 & 4 & 1 & 8 & 5 & 2 \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 2 & 8 & 4 & 2 & 8 & 7 \\ 7 & 5 & 1 & 8 & 4 & 2 \\ 6 & 2 & 0 & 7 & 2 & 0 \end{bmatrix}$$

IMP If a_g is an incident matrix of a connecta graph G with n vertices then rank of $A(G) = n-1$

$$A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$$

No of singular matrix

Rank of $A(G) \leq n-1$

I If a matrix is non singular rank is its order

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THEOREM

Two graphs G_1 and G_2 are isomorphic if and only if their incidence matrix $A(G_1)$ and $A(G_2)$ differ only by permutations of rows and columns.

Rank of the incidence matrix Each row in an incidence matrix $A(G)$ may be regarded as a vector over GF(2) in the vector space of graph G. Let the vector in the first row be called A_1 in the second row A_2 and so on. Thus $A(G) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$.

Since there are exactly two 1s in every column of A the sum of all these vectors is 0 (this being a modulo 2 sum of the corresponding entries). Thus ~~A is less than~~ vectors $A_1, A_2 \dots A_n$ are not linearly independent. Therefore rank of $A \leq n-1$

Now consider the sum of any m of these n vectors $(m \leq n-1)$. If the graph is connected A(G) cannot be partitioned as in eg such that $A(g_1)$ is with m rows and $A(g_2)$ with $n-m$ rows. In other words no m by n submatrix of A(G) can be found for $m \leq n-1$

such that modulo 2 sum of those m rows is equal to 0.

since there are only two constants 0 & 1 in this field, the additions of all vectors taken m at a time for $m = 1, 2 \ldots n-1$ exhausts all possible linear combinations of $n-1$ row vectors. thus we have just shown that no linear combination of m row vectors of A for $m \leq n-1$ can be equal to 0. Therefore rank of $A(G)$ must be ~~less~~ least $n-1$. Since the rank of $A(G)$ is no more than $n-1$ and is no less than $n-1$, it must exactly be equal to $n-1$. Hence proved.