CALCULUS 3: EXAM 3 REVIEW

ADRIAN PĂCURAR

Contents

1. Polar Coordinates

Problem 1. Set up an integral expressing the volume of the solid bounded by the cylinders $x^2+y^2=R^2$ and $y^2+z^2=R^2$, both in cartesian and polar coordinates. Use Wolfram Alpha to evaluate this integral.

Problem 2. Evaluate the integral.

(a) $\iint_D xy\ dA$ over the disk D centered at the origin and radius 3.

(b) $\iint_R \sqrt{4-x^2-y^2}\ dA$ over $R=\{(x,y): x^2+y^2\leq 4, x\geq 0\}$.

(c) $\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\ dy\ dx$.

(d) $\int_0^a \int_{-\sqrt{a^2-y^2}}^0 x^2y\ dx\ dy$.

Problem 3. Find the volume of the given solid.

(a) Under the cone $z=\sqrt{x^2+y^2}$ and above the disk $x^2+y^2\leq 4$.

(b) Enclosed by the hyperboloid $-x^2-y^2+z^2=1$ and the plane $z=2$.

(c) Above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$.

(d) Bounded by the paraboloids $y=3x^2+3z^2$ and $y=4-x^2-z^2$.

(e) Inside both the cylinder $x^2+y^2=4$ and the ellipsoid $4x^2+4y^2+z^2=64$.

Problem 4. We computed in tutorial the value of the **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$$

The following variation plays a very important role in probability theory, and it is intimately connected to the **normal distribution** with mean zero and standard deviation σ:

$$\int_{-\infty}^{\infty} e^{-x^2/(2\sigma^2)} dx = \sigma\sqrt{2\pi}$$

Prove this result.

Problem 5. Use polar coordinates to set up and evaluate the double itnegral $\iint_R f(x, y)\, dA$.

(a) $f = \arctan(y/x)$ where R is defined by $1 \leq x^2 + y^2 \leq 4$ and $0 \leq y \leq x$.

(b) $f = 9 - x^2 - y^2$, where R is defined by $x^2 + y^2 \leq 9$ and $x, y \geq 0$.

Problem 6. The area of a surface S given by $z = f(x, y)$ on a closed region R can be computed using

$$\text{Surface Area} = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$$

Use this to find the surface area of the following:

(a) The paraboloid $z = 1 - x^2 - y^2$ that lies above the unit circle.

(b) The sphere $x^2 + y^2 + z^2 = 5^2$. (Hint: compute it for the region $0 \leq x^2 + y^2 \leq a < 5$, then take the limit as $a \to 5$)

2. Triple Integrals

Problem 7. Evaluate the triple integral.

$$\text{(a)} \quad \iiint_{[1,e]\times[1,e]\times[1,e]} \frac{1}{xyz} dV \qquad \text{(b)} \quad \int_0^1 \int_{1+y}^{2y} \int_z^{y+z} z\, dx\, dz\, dy$$

Problem 8. Convert the given integral to both cylindrical coordinates and spherical coordinates, then evaluate the simplest of the two forms.

$$\text{(a)} \quad \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} x\, dz\, dy\, dx \qquad \text{(b)} \quad \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\, dz\, dy\, dx$$

$$\text{(c)} \quad \int_{-a}^{a} \int_{-\sqrt{a-x^2}}^{\sqrt{a-x^2}} \int_{a}^{a+\sqrt{a^2-x^2-y^2}} x\, dz\, dy\, dx$$

Problem 9. Find the volume of the specified solid.

(a) Inside $x^2 + y^2 + z^2 = 9$, outside $z = \sqrt{x^2 + y^2}$, and above the xy-plane.

(b) Below $x^2 + y^2 + z^2 = z$ and above $z = \sqrt{x^2 + y^2}$.

(c) Between the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$ (where $b > a$), and inside the cone $z^2 = x^2 + y^2$.

3. Change Of Variables

Problem 10. By using an appropriate change of variable, show that the curve $y = 1/x$ is a hyperbola (i.e. show that it satisfies an equation of the form $u^2 - v^2 = c^2$).

Problem 11. Compute the Jacobian of the transformation.

(a) $x = -\frac{1}{2}(u - v)$, $y = \frac{1}{2}(u + v)$

(b) $x = au + bv$, $y = cu + dv$.

(c) $x = u - v^2$, $y = u + v$.

(d) $x = u + a$, $y = v + a$.

(e) $x = e^u \sin v$, $y = e^u \cos v$.

(f) $x = u/v$, $y = u + v$.

Problem 12. Sketch the region $T(R)$ in the uv-plane of the specified region R (in the xy-plane) under the specified transformation T. What would be the new limits of integration?

(a) R : triangle with vertices $(0, 0)$, $(3, 0)$, $(2, 3)$, and $T : x = 3u + 2v$, $y = 3v$.

(b) R : parallelogram with vertices $(0, 0)$, $(2, 2)$, $(6, 3)$, $(4, 1)$, and $T : x = \frac{1}{3}(4u - v)$, $y = \frac{1}{3}(u - v)$.

(c) R : polygon with vertices $(1/2, 1/2)$, $(0, 1)$, $(1, 2)$, $(3/2, 3/2)$, and $T : x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$.

Problem 13. Use a change of variable to find the volume of the solid lying below the surface $z = f(x, y)$, and above the plane region R.

(a) $f(x, y) = 48xy$ and R : square with vertices $(1, 0)$, $(0, 1)$, $(1, 2)$, $(2, 1)$.

(b) $f(x, y) = (3x + 2y)^2\sqrt{2y - x}$ and R : parallelogram with vertices $(0, 0)$, $(-2, 3)$, $(2, 5)$, $(4, 2)$.

(c) $f(x, y) = (x + y)e^{x-y}$ and R : square with vertices $(4, 0)$, $(6, 2)$, $(4, 4)$, $(2, 2)$.

(d) $f(x, y) = (x + y)^2 \sin^2(x - y)$ and R : square with vertices $(\pi, 0)$, $(3\pi/2, \pi/2)$, (π, π), $(\pi/2, \pi, 2)$.

(e) $f(x, y) = \sqrt{(x - y)(x + 4y)}$ and R : parallelogram with vertices $(0, 0)$, $(1, 1)$, $(5, 0)$, $(4, -1)$.

(f) $f(x, y) = \frac{xy}{1+x^2y^2}$ and R : region bounded by the graphs $xy = 1$, $xy = 4$, $x = 1$, $x = 4$.

Problem 14. Derive the Jacobian of the polar and cylindrical change of variable.

4. Vector Fields And Line Integrals

Problem 15. Determine which of the following vector fields in $\mathbb{R}^2$ are conservative. If so, find a potential function.

(a) $\vec{F} = \langle 2x, 4y\rangle$ (b) $\vec{F} = \left\langle xy^2, x^2y\right\rangle$ (c) $\vec{F} = \dfrac{1}{x^2}(y\mathbf{i} - x\mathbf{j})$

(d) $\vec{F} = \langle \sin y, x\cos y\rangle$ (e) $\vec{F} = \left\langle 5y^3, 15xy^2\right\rangle$ (f) $\vec{F} = \dfrac{1}{xy}(y\mathbf{i} - x\mathbf{j})$

(g) $\vec{F} = e^x(\cos y\mathbf{i} - \sin y\mathbf{j})$ (h) $\vec{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{x^2+y^2}$ (i) $\vec{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$

Problem 16. Determine which of the following vector fields in $\mathbb{R}^3$ are conservative. If so, find a potential function.

(a) $\vec{F}(x,y,z) = \langle xy^2z^2, x^2yz^2, x^2y^2z\rangle$

(b) $\vec{F}(x,y,z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$

(c) $\vec{F}(x,y,z) = \langle \sin z, \sin x, \sin y\rangle$

(d) $\vec{F}(x,y,z) = \langle ye^z, ze^x, xe^y\rangle$

(e) $\vec{F}(x,y,z) = \dfrac{z}{y}\mathbf{i} - \dfrac{xz}{y^2}\mathbf{j} + \dfrac{x}{y}\mathbf{k}$

(f) $\vec{F}(x,y,z) = \dfrac{x}{x^2+y^2}\mathbf{i} + \dfrac{y}{x^2+y^2}\mathbf{j} + \mathbf{k}$

(g) $\vec{F}(x,y,z) = y\ln z\mathbf{i} - x\ln z\mathbf{j} + \dfrac{xy}{z}\mathbf{k}$

(h) $F(x,y,z) = \sin yz\mathbf{i} + xz\cos yz\mathbf{j} + xy\sin yz\mathbf{k}$

Problem 17. Consider two potential functions $f(x,y,z)$ and $g(x,y,z)$, with their respective gradient fields $\vec{F}(x,y,z)$ and $\vec{G}(x,y,z)$, which we know to be conservative vector fields. Show that the vector field $\vec{F} + \vec{G}$ is also conservative. Find a counterexample to show that, in general, $\vec{F} \times \vec{G}$ is not conservative.

Problem 18. Compute $\int_C f\ ds$, $\int_C f\ dx$, and $\int_C f\ dy$ for the following:

(a) $f = xy$, $\vec{r}(t) = 4t\mathbf{i} + 3t\mathbf{j}$, $t \in [0,1]$.

(b) $f = x^2 + y^2 + z^2$, $\vec{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$, $t \in [0, \pi/2]$.

(c) $f = 2xyz$, $\vec{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 2t\mathbf{k}$, $t \in [0,1]$.

(d) $f = z$, $\vec{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$, $t \in [1,3]$.

Problem 19. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following:

(a) $\mathbf{F} = \langle x, y\rangle$, $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, $t \in [0,1]$.

(b) $\mathbf{F} = \langle xy, y\rangle$, $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$, $t \in [0, \pi/2]$.

(c) $\mathbf{F} = \langle xy, xz, yz\rangle$, $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$, $t \in [0, 1]$.

(d) $\mathbf{F} = \langle y, x, 2z\rangle$, $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, \sin(t/2)\rangle$, $t \in [0, 2\pi]$.

Problem 20. Check if the given vector field $\mathbf{F}$ is conservative. If so, use the Fundamental Theorem Of Line Integrals to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ over the specified path, otherwise evaluate the integral the usual way.

(a) $\mathbf{F} = \langle 2xy, x^2\rangle$, $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}$, $t \in [0, 1]$.

(b) $\mathbf{F} = \langle ye^{xy}, xe^{xy}\rangle$, $\mathbf{r}(t) = t\mathbf{i} - (t-3)\mathbf{j}$, $t \in [0, 3]$.

(c) $\mathbf{F} = \langle ye^{xy}, xe^{xy}\rangle$, along the closed path consisting of line segments $(0,3) \to (0,0) \to (3,0) \to (0,3)$.

(d) $\mathbf{F} = \langle yz, xz, xy\rangle$, $\mathbf{r}(t) = \langle t^2, t, t^3\rangle$, $t \in [0, 2]$.

(e) $\mathbf{F} = \langle y, -x\rangle$, along $\mathbf{r}_1(t) = \langle t, t\rangle$, $\mathbf{r}_2(t) = \langle t, t^2\rangle$, $\mathbf{r}_3(t) = \langle t, t^3\rangle$ (evaluate along each curve individually and compare), for $t \in [0, 1]$.

Problem 21. We saw on the review sheet for Exam 2 that a function $f(x, y)$ which satisfies **Laplace's equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

is called a **harmonic function**. Show that for any harmonic function, we have

$$\int_C \left(\frac{\partial f}{\partial y}dx - \frac{\partial f}{\partial x}dy\right) = 0$$

where C is a smooth closed curve in the plane.

Problem 22. Consider the vector field

$$\mathbf{F}(x, y) = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}$$

Show that $\nabla\left(\arctan\frac{x}{y}\right) = \mathbf{F}$.

Problem 23. Let $\mathbf{F}$ be the gravitational force field of a mass M on a particle of mass m:

$$\mathbf{F}(x, y, z) = -\frac{GMm}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

Given that G, M, and m are all constants, show that the work done by $\mathbf{F}$ as the particle moves from $\mathbf{v}_0 = \langle x_0, y_0, z_0\rangle$ to $\mathbf{v}_1 = \langle x_1, y_1, z_1\rangle$ depends only on $|\mathbf{v}_0|$ and $|\mathbf{v}_1|$.