Compound Events

Set theory allows us to use unions, intersections, and complements to break complex events into simpler pieces which will make probability calculations easier.

An event that can be described in terms of the union/intersection/complement of simper events is called a **compound event**.

Let E and F be events in a sample space.

- $E \cup F$ is the event that *E or F occurred*.
- $E \cap F$ is the event that *both E and F occurred*.
- E' is the event that *E failed to occur*.

Compound Events

Example: Roll two of six-sided dice, one red and one green, and observe the pair of numbers that come up (red first, then green). The sample space for this experiment is shown below.

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

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(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

a) Let E be the event: *the sum of numbers is 7.* List the elements of E. $E = \{(6,1),(5,2),(4,3),(3,4),(2,5),(1,6)\}$

b) Let F be the event: *at least one of the numbers is a 6.* How many outcomes are in the event F? The outcomes we want are in the bottom row, or in the last column. There are 11 total.

c) Give a verbal description of the event $E \cap F$, and list its outcomes.

We want all the outcomes where *6 appears at least once, and the numbers add up to 7.* $E \cap F = \{(6,1),(1,6)\}$.

Compound Events

d) How many outcomes are in the event $E \cup F$?

This is the event that *one of the numbers is a 6, or that the numbers add up to 7.* The easiest way to count the desired outcomes is to use Inclusion-Exclusion!

$$n(E \cup F) = n(E) + n(F) - n(E \cap F) = 6 + 11 - 2 = 15$$

e) How many outcomes are in the event E'?

This time we use the Complement Principle:

$$n(E') = n(S) - n(E) = 36 - 6 = 30$$

Compound Events

Example: An experiment with outcomes $\{a, b, c, d\}$ is described by the following probability table:

Outcome	Probability
a	0.20
b	0.10
c	0.15
d	0.55

Consider the events $E = \{a, b, c\}$ and $F = \{b, c, d\}$. What is $P(E \cap F)$? We have $E \cap F = \{b, c\}$, so

$$P(E \cap F) = P(b) + P(c) = 0.10 + 0.15 = 0.25$$

IE and CP for Probability

Let E, F be events in a sample space S. We learned that the **Complement Principle** for probability is

$$\boxed{P(E) = 1 - P(E')}$$

We have a similar formula for **Inclusion-Exclusion** for probability:

$$\boxed{P(E \cup F) = P(E) + P(F) - P(E \cap F)}$$

Note: both formulas work for ANY sample space (not just for equally likely outcomes which we studied last time).

Deriving IE For Probability

We start with our familiar formula from sets

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

and for equally likely outcomes, if we divide by the size of the sample space $n(S)$ we obtain

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E) + n(F) - n(E \cap F)}{n(S)}$$

Split the fraction to get

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$

but this is the same as

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Using IE and CP

Example: Let E and F be events in a sample space S. If $P(E) = 0.30$, $P(F) = 0.80$ and $P(E \cap F) = 0.20$, what is $P(E \cup F)$?

$$\begin{aligned} P(E \cup F) =& P(E) + P(F) - P(E \cap F) \\ =& 0.30 + 0.80 - 0.20 = 0.90 \end{aligned}$$

Example: Let A and B be two events. If $P(A \cup B) = 0.70$, $P(A) = 0.30$, and $P(B) = 0.40$, what is the probability that both A and B occur?

We want $P(A \cap B)$. Using IE, $0.7 = 0.3 + 0.4 - P(A \cap B)$, which means $P(A \cap B) = 0$. It is impossible for both A and B to occur (they are disjoint events).

Using IE and CP

Example: Let E and F be events in a sample space S. If $P(E) = 0.5$, $P(F') = 0.4$, and $P(E \cup F) = 0.9$, find $P(E \cap F)$.

We first find $P(F)$ by using the complement principle: $P(F) = 1 - 0.4 = 0.6$. Now apply Inclusion-Exclusion:

$$0.9 = 0.5 + 0.6 - P(E \cap F)$$

so $P(E \cap F) = 0.2$.

More Examples

Example: At the Bad Donkey Stables, 50% of the donkeys bite, 40% kick, and 20% do both. You are assigned a donkey at random to groom. What is the probability that the donkey you get will either bite or kick (or both)?

Let B be the event that the donkey will bite, and K be the event that the donkey will kick.

We are given $P(B) = 0.5$, $P(K) = 0.4$ and $P(B \cap K) = 0.2$. We need to find $P(B \cup K)$, so we apply IE:

$$P(B \cup K) = P(B) + P(K) - P(B \cap K) = 0.5 + 0.4 - 0.2 = 0.7$$

More Examples

Example: Out of 10,000 undergraduates at the University of Notthe Same, 2,000 are enrolled in a math class, 2,500 are enrolled in an English class, and 500 enrolled in both. Randomly pick a student.

a) Find the probability that they are enrolled a class.

Let M be the event *enrolled in math*, and E be the event *enrolled in English*. We want $P(M \cup E)$:

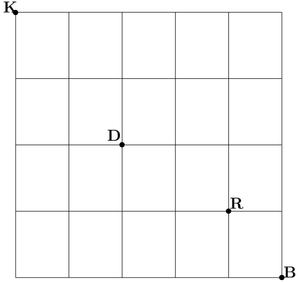
$$\begin{aligned} P(M \cup E) &= P(M) + P(E) - P(M \cap E) \\ &= 0.20 + 0.25 - 0.05 = 0.40 \end{aligned}$$

b) Find the probability that the student is not enrolled in either of the two classes.

$$P(M \cup E)' = 1 - P(M \cup E) = 1 - 0.40 = 0.60$$

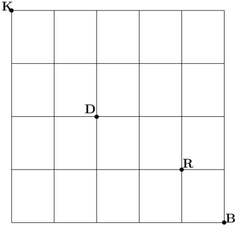
More Examples

Example: Kristina randomly chooses a route from K to B (see map below) with no backtracking for her morning run.



What is the probability that the route she chooses will take her to either the doberman at D or the rottweiler at R (or both)?

More Examples



What is the probability that the route she chooses will take her to either the doberman at D or the rottweiler at R (or both)?

All routes: $n(S) = \binom{9}{5} = 126.$

Routes through D: $n(D) = \binom{4}{2} \cdot \binom{5}{3} = 6 \cdot 10 = 60.$

Routes through R: $n(R) = \binom{7}{3} \cdot \binom{2}{1} = 35 \cdot 12 = 70.$

Routes through both: $n(D \cap R) = \binom{4}{2} \cdot \binom{3}{2} \cdot \binom{2}{1} = 6 \cdot 3 \cdot 2 = 36.$

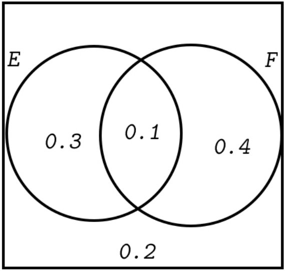
We are interested in $P(D \cup R)$:

$$\frac{n(D \cup R)}{n(S)} = \frac{60 + 70 - 36}{126} = \frac{94}{126} \approx 0.746$$

Venn Diagrams in Probability

We can use Venn diagrams to represent probabilities by recording the appropriate probability in each region. The sum of all probabilities in the sample space must be 1.

Example: Suppose E and F are events in a sample space, with $P(E) = 0.40$, $P(F) = 0.50$ and $P(E \cap F') = 0.30$. Use a Venn diagram to find $P(E \cap F)$ and $P(E \cup F)$



$P(E \cap F) = 0.10$

$P(E \cup F) = 0.80$

Mutually Exclusive Events

Two events E and F are **mutually exclusive** whenever they have no outcomes in common. This is equivalent to the following conditions:

- as subsets, E and F are disjoint, so $E \cap F = \emptyset$
- $P(E \cap F) = 0$, so the two events cannot occur at the same time (e.g. rolling an even or an odd number at the same time)

For mutually exclusive events, the probability $P(E \cup F)$ is additive:

$$\boxed{P(E \cup F) = P(E) + P(F)}$$

Mutually Exclusive Events

Example: Two six sided dice are rolled, and the numbers that come up are recorded. Consider the following events:

E: both numbers are odd

F: the sum is is odd

G: at least one of the numbers is a 5

$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(1,6)$
$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$
$(3,1)$	$(3,2)$	$(3,3)$	$(3,4)$	$(3,5)$	$(3,6)$
$(4,1)$	$(4,2)$	$(4,3)$	$(4,4)$	$(4,5)$	$(4,6)$
$(5,1)$	$(5,2)$	$(5,3)$	$(5,4)$	$(5,5)$	$(5,6)$
$(6,1)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$	$(6,6)$

Which of the following statements is true?

a) E and G are mutually exclusive. False. Consider $(5,3)$.

b) E and F are mutually exclusive. True.

(c) F and G are mutually exclusive False. Consider $(5,2)$.

Mutually Exclusive Events

Example: Draw 5 cards from a standard deck at random. What is the probability that the hand of cards we draw will have either 4 Aces (event A) or 4 Kings (event K)?

The sample space: $n(S) = C(52, 5) = 2,598,960$.
Four Aces: $n(A) = C(4, 4) \cdot C(48, 1) = 48$.
Four Kings $n(K) = C(4, 4) \cdot C(48, 1) = 48$.

Since we only draw 5 cards, it is impossible to get 4 aces and 4 kings in the same hand. Hence A and K are mutually exclusive, and

$$P(A \cup K) = \frac{n(A)}{n(S)} + \frac{n(K)}{n(S)} = \frac{96}{2,598,960} \approx 0.000037$$

Mutually Exclusive Events

Example: Flip a coin 20 times and observe the resulting ordered sequence of H and T.

a) What is the probability that we get either *exactly 0 Heads or 1 Head or 2 Heads* in the resulting sequence?

Let H_k be the event that we get exactly k Heads total, so $P(H_k) = \frac{C(20,k)}{2^{20}}$.

We want $P(H_0 \cup H_1 \cup H_2)$ and these events are mutually exclusive, so

$$P(H_0 \cup H_1 \cup H_2) = P(H_0) + P(H_1) + P(H_2) = \frac{\binom{20}{0} + \binom{20}{1} + \binom{20}{2}}{2^{20}}$$

$$P(H_0 \cup H_1 \cup H_2) = \frac{1 + 20 + 190}{1048576} = \frac{211}{1048576} \approx 0.0002$$

Mutually Exclusive Events

Example: Flip a coin 20 times and observe the resulting ordered sequence of H and T.

b) what is the probability that we get at least three heads?

We use the complement principle, so we are looking at

$$1 - P(H_0 \cup H_1 \cup H_2) = 1 - \frac{\binom{20}{0} + \binom{20}{1} + \binom{20}{2}}{2^{20}} \approx 0.9998$$