CALCULUS 2
EXAM 4 PRACTICE PROBLEMS

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Contents

1. Curves Defined by Parametric Equations

Problem 1. Sketch the following curves

a) $\begin{cases} x(t) = t^2 + t \\ y(t) = t^2 - t \end{cases} \quad (-2 \le t \le 2)$ b) $\begin{cases} x(t) = \cos^2 t \\ y(t) = 1 - \sin t \end{cases} \quad (0 \le t \le \pi/2)$

c) $\begin{cases} x(t) = e^{-t} + t \\ y(t) = e^{-t} \end{cases} \quad (0 \le t \le \infty)$

Problem 2. Sketch the curves and eliminate the parameter to find a Cartesian equation:

a) $\begin{cases} x(t) = 3 - 4t \\ y(t) = 2 - 3t \end{cases}$ b) $\begin{cases} x(t) = 1 - t^2 \\ y(t) = t - 1 \end{cases}$ c) $\begin{cases} x(t) = t - 1 \\ y(t) = t^3 + 1 \end{cases}$

d) $\begin{cases} x(t) = \sqrt{t} \\ y(t) = 1 - t \end{cases}$ e) $\begin{cases} x(t) = t^2 \\ y(t) = t^3 - t^4 \end{cases}$

Problem 3. Eliminate the parameter and sketch the curve, indicating the direction in which the curve is traced as the parameter increases.

a) $\begin{cases} x(t) = \sin(\theta/2) \\ y(t) = \cos(\theta/2) \\ -\pi \le \theta \le \pi \end{cases}$ b) $\begin{cases} x(t) = 0.5\sin(\theta) \\ y(t) = 2\cos(\theta) \\ 0 \le \theta \le \pi \end{cases}$ c) $\begin{cases} x(t) = e^t - 1 \\ y(t) = e^{2t} \end{cases}$

d) $\begin{cases} x(t) = e^{2t} \\ y(t) = t - 1 \end{cases}$ e) $\begin{cases} x(t) = \tan^2\theta \\ y(t) = \sec\theta \\ -\pi/2 \le \theta \le \pi/2 \end{cases}$ f) $\begin{cases} x(t) = 2\sin t \\ y(t) = 4 + \cos t \end{cases}$

2. Calculus With Parametric Curves

The derivative for a parametric curve is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

as long as dx/dt is not zero. If $dx/dt = 0$, and $dy/dt \neq 0$, we have a vertical tangent.

If we wanted to calculate areas, the old way was to look at $\int_a^b y dx$. This time, both x and y are functions of t given by $x(t)$ and $y(t)$, so we have

$$A = \int_a^b y \, dx = \int_{t_1}^{t_2} y(t)x'(t) \, dt \quad \text{OR} \quad \int_a^b x \, dy = \int_{t_1}^{t_2} x(t)y'(t) \, dt$$

where the second formula will work if we are integrating along the y axis.

One last thing we can talk about is arc length, and recall we used to have

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

but we derived this formula from $\int \sqrt{dx^2 + dy^2}$, so in parametric equation form we have

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Problem 4. Find the equation of the tangent line to the parametric curve at the specified value of the parameter.

$$\text{a)} \begin{cases} x(t) = t\cos t \\ y(t) = t\sin t \\ \text{at } t = \pi \end{cases} \qquad \text{b)} \begin{cases} x(t) = t - t^{-1} \\ y(t) = 1 + t^2 \\ \text{at } t = 1 \end{cases} \qquad \text{c)} \begin{cases} x(t) = \sin^3\theta \\ y(t) = \cos^3\theta \\ \text{at } \theta = \pi/6 \end{cases}$$

Problem 5. Find the equation of the tangent line to the parametric curve at the specified point. Notice that for these problems, you have the option of eliminating the parameter first, and finding the tangent line the old way.

$$\text{a)} \begin{cases} x(t) = 1 + \ln t \\ y(t) = t^2 + 2 \\ \text{at } (1, 3) \end{cases} \qquad \text{b)} \begin{cases} x(t) = 1 + \sqrt{t} \\ y(t) = e^{t^2} \\ \text{at } (2, e) \end{cases}$$

Problem 6. Find the second derivative. For which values of t is the curve concave upward?

$$\text{a)} \begin{cases} x(t) = t^2 + 1 \\ y(t) = t + t^2 \end{cases} \qquad \text{b)} \begin{cases} x(t) = e^t \\ y(t) = e^{-t} \end{cases} \qquad \text{c)} \begin{cases} x(t) = 2\sin t \\ y(t) = 3\cos t \\ 0 < t < 2\pi \end{cases}$$

$$\text{c)} \begin{cases} x(t) = t^3 + 1 \\ y(t) = t^2 - t \end{cases} \qquad \text{d)} \begin{cases} x(t) = t^2 + 1 \\ y(t) = e^t - 1 \end{cases} \qquad \text{e)} \begin{cases} x(t) = \cos 2t \\ y(t) = \cos t \\ 0 < t < \pi \end{cases}$$

Problem 7. Find points with horizontal or vertical tangents:

$$\text{a)} \begin{cases} x(t) = t^3 - 3t \\ y(t) = t^2 - 3 \end{cases} \qquad \text{b)} \begin{cases} x(t) = t^3 - 3t \\ y(t) = t^3 - 3t^2 \end{cases} \qquad \text{c)} \begin{cases} x(t) = \cos t \\ y(t) = \cos 3t \end{cases}$$

Problem 8. Derive the area enclosed by the circle of radius r and the ellipse with intercepts $a, b > 0$.

$$\text{a)} \begin{cases} x(t) = r\cos\theta \\ y(t) = r\sin\theta \end{cases} \qquad \text{b)} \begin{cases} x(t) = a\cos\theta \\ y(t) = b\sin\theta \end{cases}$$

Problem 9. Find the area enclosed by the y-axis and the curve

$$\begin{cases} x(t) = t^2 - 2t \\ y(t) = \sqrt{t} \end{cases}$$

Problem 10. Find the area enclosed by the x-axis and the curve

$$\begin{cases} x(t) = 1 + e^t \\ y(t) = t - t^2 \end{cases}$$

Problem 11. Set up an integral that represents the length of the curve.

a) $\begin{cases} x(t) = r\cos\theta \\ y(t) = r\sin\theta \end{cases}$ b) $\begin{cases} x(t) = a\cos\theta \\ y(t) = b\sin\theta \end{cases}$ c) $\begin{cases} x(t) = t + e^{-t} \\ y(t) = t - e^{-t} \\ 0 \le t \le 2 \end{cases}$

d) $\begin{cases} x(t) = t^2 - t \\ y(t) = t^4 \\ 1 \le t \le 4 \end{cases}$ e) $\begin{cases} x(t) = t - 2\sin t \\ y(t) = 1 - 2\cos t \\ 0 \le t \le 4\pi \end{cases}$

Problem 12. Find the exact length of the curve.

a) $\begin{cases} x(t) = r\cos\theta \\ y(t) = r\sin\theta \end{cases}$ b) $\begin{cases} x(t) = 1 + 3t^2 \\ y(t) = 4 + 2t^3 \\ 0 \le t \le 1 \end{cases}$ c) $\begin{cases} x(t) = e^t + e^{-t} \\ y(t) = 5 - 2t \\ 0 \le t \le 3 \end{cases}$

d) $\begin{cases} x(t) = t\sin t \\ y(t) = t\cos t \\ 0 \le t \le 1 \end{cases}$ e) $\begin{cases} x(t) = e^t\cos t \\ y(t) = e^t\sin t \\ 0 \le t \le \pi \end{cases}$

3. Polar Coordinates

The derivative can be computed from the parametric method, using $x = r\cos\theta$ and $y = r\sin\theta$ (and keeping in mind that r is a function of θ, so product rule applies):

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Problem 13. Plot the following points given in polar coordinates (r, θ), and find the corresponding Cartesian coordinates.

a) $(1, \pi/4)$ b) $(-1, \pi/4)$ c) $(-1, 5\pi/4)$ d) $(-1, -3\pi/4)$ e) $(3, \pi/3)$

f) $(2, 3\pi/2)$ g) $(\sqrt{2}, \pi/4)$ h) $(-1, -\pi/6)$ i) $(-3, -\pi/3)$ j) $(4, 4\pi/3)$

Problem 14. For the given Cartesian coordinates, give at least two polar representations.

a) $(-4, -4)$ b) $(3, 3\sqrt{3})$ c) $(-6, 0)$ d) $(\sqrt{3}, -1)$

Problem 15. Sketch the set of points in the plane whose polar coordinates satisfies the given condition. Draw boundaries which are not included with dotted lines, and boundaries which are included with solid lines.

a) $r < 1$ b) $1 \leq r \leq 2$ c) $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$ d) $2 \leq r \leq 4, \theta > 0$ e) $r^2 = 1$

Problem 16. Find the Cartesian equation for the given curve.

a) $r^2 = 5$ b) $r = 5\cos\theta$ c) $\theta = \pi/3$ d) $r^2 \sin(2\theta) = 1$ e) $r^2 \cos(2\theta) = 1$

Problem 17. Find the polar equation for the given curve.

a) $y = 2$ b) $y = 1 + 3x$ c) $x^2 + 4y^2 = -2cx$ d) $x^2 - y^2 = 4$ e) $x = y^2$

Problem 18. Sketch the graph of the curve $r = \cos(n\theta)$ for various values of n. Do you notice a pattern when n is even vs n odd?

Problem 19. Practice the various graphing problems on page 707 in the book (29-46, 47, 48, 54).

Problem 20. Find the equation of the tangent line at the specified point.

a) $r = 2\cos\theta$ at $\theta = \pi/3$.

b) $r = 1/\theta$ at $\theta = \pi$.

c) $r = \cos(2\theta)$ at $\theta = \pi/4$.

Problem 21. Determine where the given curve has a horizontal or vertical tangent.

a) $r = 2\cos\theta$ b) $r = 1 - \sin\theta$ c) $r = e^\theta$

Problem 22. Argue that the equation $r = a\sin\theta + b\cos\theta$ represents a circle

Problem 23. Recall that the sine and cosine graphs are translates (shifts) of each other:

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right)$$

What can you say about the graphs of $r_1(\theta) = \sin(n\theta)$ and $r_2(\theta) = \cos(n\theta)$, where n is some positive integer? How are they related?

4. Areas and Lengths in Polar Coordinates

To find the area of a region enclosed by a function $r(\theta)$ (in polar coordinates) between θ_1 and θ_2, we use

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$$

Similarly, to find the arc length, we have

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Problem 24. Find the area of the region specified.

a) $r = e^{-\theta}$ for $\pi/2 \leq \theta \leq \pi$.

b) $r = 2\cos\theta$.

c) $r = 1/\theta$ for $\pi/2 \leq \theta \leq \pi$.

d) $r = \sin(4\theta)$ (region enclosed by one loop).

e) $r = \cos(3\theta)$ (region enclosed by the entire curve).

f) inside $r = 4\sin\theta$ and outside $r = 2$.

g) inside $1 + \cos\theta$ and outside $r = 2 - \cos\theta$.

h) inside $3\cos\theta$ and outside $1 + \cos\theta$.

i) inside both $r = 3\sin\theta$ and $r = 3\cos\theta$.

j) inside both $r = a\sin\theta$ and $r = b\cos\theta$, where $a, b > 0$.

k) inside both $r = 1$ and $r = 2\sin\theta$.

Problem 25. Find the points of intersection of the given curves

a) $r = \sin\theta$ and $r = 1 - \sin\theta$.

b) $r = 2\sin(2\theta)$ and $r = 1$.

c) $r = \sin\theta$ and $r = \sin(2\theta)$.

Problem 26. Compute the length of the polar curve

a) $r = 2\cos\theta$ for $0 \leq \theta \leq \pi$.

b) $r = e^{\theta}$ for $0 \leq \theta \leq 2\pi$.

c) $r = \theta^2$ for $0 \leq \theta \leq 4\pi$.

d) $r = \cos^2 2\theta$.

Good luck on the final, and I wish you all a great summer!