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Worksheet 3, Math 10560

Times indicate the amount of time that you would be expected to spend on the problem in an exam. All problems except the last have appeared on old exams for Calculus 2.

1. (4 mins.) Evaluate $\displaystyle\int \frac{2^x}{\sqrt{1-4^x}}\,dx$.

Solution: To solve this integral we must first note that the form of this integral resembles one that we know; namely, $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C$. We make the substitution $u = 2^x$, $du = \ln 2 \cdot 2^x dx$ and our integral becomes

$$\begin{aligned}\int \frac{2^x}{\sqrt{1-4^x}}\,dx &= \int \frac{1}{\ln 2\sqrt{1-u^2}}\,du \\ &= \frac{1}{\ln 2}\arcsin(u) + C \\ &= \frac{\arcsin(2^x)}{\ln 2} + C.\end{aligned}$$

2. (4 mins.) Evaluate $\displaystyle\lim_{x\to\infty} \frac{x^2}{2^x}$.

Solution:

This limit is an indeterminate form of type $\frac{\infty}{\infty}$ since both the numerator and denominator approach ∞ as $x \to \infty$. This is one of the forms for which l'Hôpital's rule applies.

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2}{2^x} &= \lim_{x\to\infty} \frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(2^x)} \\ &= \lim_{x\to\infty} \frac{2x}{\ln 2 \cdot 2^x}\end{aligned}$$

This is also of indeterminate form $\frac{\infty}{\infty}$ so we apply l'Hôpital's rule one more to obtain

$$\lim_{x\to\infty} \frac{\frac{d}{dx}(2x)}{\frac{d}{dx}(\ln 2 \cdot 2^x)} = \lim_{x\to\infty} \frac{2}{(\ln 2)^2 \cdot 2^x} = 0.$$

Thus $\displaystyle\lim_{x\to\infty} \frac{x^2}{2^x} = 0$.

3. (2-3 mins) Evaluate the expression $\arcsin\left(\sin\left(\dfrac{3\pi}{4}\right)\right)$.

   (Note: $\arcsin(x) = \sin^{-1}(x)$.)

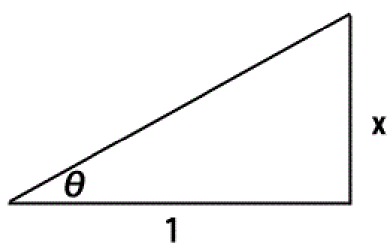
Solution: We first evaluate the inner expression, i.e. $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) = \arcsin\left(\frac{\sqrt{2}}{2}\right)$. Note that the range of $\arcsin(x)$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and so $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right) \neq \frac{3\pi}{4}$ even though $\sin(x)$ and $\arcsin(x)$ are inverses. The angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ which makes $\sin(\theta) = \frac{\sqrt{2}}{2}$ is $\theta = \frac{\pi}{4}$, so $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$. Thus, $\arcsin\left(\sin\left(\dfrac{3\pi}{4}\right)\right) = \dfrac{\pi}{4}$.

4. (2-3 mins) Use a triangle to express

$$\cos(\tan^{-1}(x))$$

   as a function of x with a formula which does not use trigonometric functions or their inverses.

Solution: First suppose $x \geq 0$. Let $\theta = tan^{-1}(x)$ be the base angle of the triangle shown below.



So $\tan(\theta) = x$, and since $\tan(\theta) = \frac{opp}{adj}$ we can label the opposite and adjacent sides of the triangle x and 1, respectively. Now, by the Pythagorean theorem we know that the length of the hypotenuse must be $\sqrt{1+x^2}$.

Hence $\cos(\tan^{-1}(x)) = \cos\theta = \frac{1}{\sqrt{1+x^2}}$

For $x < 0$, $\cos(\tan^{-1}(x)) = \cos(-\tan^{-1}(x)) = \cos(\tan^{-1}(-x)) = \frac{1}{\sqrt{1+x^2}}$ by the previous case.

5. (7 mins) Evaluate the limit

$$\lim_{x\to 0}(\cos x)^{1/x^2}.$$

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Solution: Substituting in 0 to the expression, we see that this is indeterminate of form 1^∞. Let $L = \lim_{x\to 0}(\cos x)^{1/x^2}$ and use l'Hôpital to find $\ln(L)$. We have,

$$\begin{aligned}
\ln(L) &= \ln(\lim_{x\to 0}(\cos x)^{1/x^2}) \\
&= \lim_{x\to 0} \ln(\cos x)^{1/x^2} \\
&= \lim_{x\to 0}(1/x^2)\ln(\cos x) \\
&= \lim_{x\to 0} \frac{\ln(\cos x)}{x^2}
\end{aligned}$$

If we substitute in 0 we see that this is indeterminate of form $\frac{0}{0}$. Applying, l'Hôpital's rule we see

$$\begin{aligned}
\ln(L) &= \lim_{x\to 0} \frac{\ln(\cos x)}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{1}{\cos x}\cdot(-\sin x)}{2x} \\
&= \lim_{x\to 0} \frac{-\tan x}{2x} \\
&= \lim_{x\to 0} \frac{-\sec^2 x}{2} \\
&= \lim_{x\to 0} \frac{-1}{2\cos^2 x} \\
&= -\frac{1}{2}
\end{aligned}$$

So, $\ln(L) = -\frac{1}{2}$, and it follows $L = e^{\ln(L)} = e^{-\frac{1}{2}}$.

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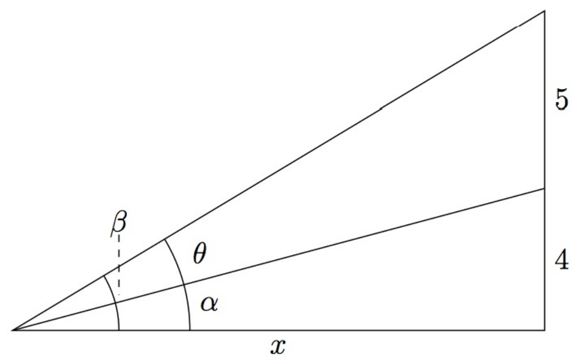
6. (Extra Problem) A painting in an art gallery has height 5 ft. and is hung so that its lower edge is a distance 4 ft. above the eye of an observer (see the figure) standing distance x from the wall. θ is the angle subtended at the observer's eye by the painting.



(a) Rewrite θ in terms of α and β.

Solution: It is clear from the picture that, $\theta = \beta - \alpha$.

(b) Use inverse trigonometric functions to write α and β in terms of x.

Solution: Looking at the smaller triangle at the bottom of the diagram we notice that $\tan(\alpha) = \frac{4}{x}$, thus $\alpha = \arctan(\frac{4}{x})$. Similarly by looking at the whole triangle we see that $\tan(\beta) = \frac{9}{x}$, thus $\beta = \arctan(\frac{9}{x})$.

(c) Combining (a) and (b), write θ as a function of x.

Solution: $\theta = \beta - \alpha = \arctan(\frac{9}{x}) - \arctan(\frac{4}{x})$.

(d) How far should the observer stand from the wall to get the best view? In other words, where should the observer stand so as to *maximize* the angle θ subtended at his eye by the painting (see diagram above). [Recall from Calculus 1 the method of finding the maximum or minimum of a function on a closed interval].

Solution: Differentiating with respect to x we obtain

$$\begin{aligned}\theta'(x) &= \frac{d}{dx}[\theta] = \frac{d}{dx}(\beta - \alpha) = \frac{d}{dx}\left[\arctan\left(\frac{9}{x}\right) - \arctan\left(\frac{4}{x}\right)\right] \\ &= \frac{1}{1 + (\frac{9}{x})^2} \cdot \left(-\frac{9}{x^2}\right) - \frac{1}{1 + (\frac{4}{x})^2} \cdot \left(-\frac{4}{x^2}\right)\end{aligned}$$

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To find the maximum of this function we need to first find its critical numbers. These occur when $\theta'(x) = 0$ or $\theta'(x)$ is undefined for x in the domain of $\theta(x)$. The only value of x for which $\theta'(x)$ is undefined is when $x = 0$, but this is not in the domain of $\theta(x)$. Furthermore, we only need to look at values when $0 < x \leq w$, where w denotes the width of the room. Now, if $x \neq 0$,

$$\begin{aligned}\theta'(x) = \frac{1}{1+(\frac{9}{x})^2} \cdot \left(-\frac{9}{x^2}\right) - \frac{1}{1+(\frac{4}{x})^2} \cdot \left(-\frac{4}{x^2}\right) &= \frac{-9}{x^2+81} + \frac{4}{x^2+16} \\ &= \frac{-9(x^2+16)+4(x^2+81)}{(x^2+81)(x^2+16)} \\ &= \frac{180-5x^2}{(x^2+81)(x^2+16)}.\end{aligned}$$

So $\theta'(x) = 0$ provided $180 - 5x^2 = 0$, or when $x = 6$ ($x = -6$ is also a critical value but in our example we only want to consider positive values of x. Since $\theta'(x) > 0$ when $0 \leq x < 6$, and $\theta'(x) < 0$ when $6 < x < w$, $x = 6$ is an absolute maximum on this interval. Thus, the observer should stand 6 ft away from the wall in order to get the best view.

Note: It is safe to assume that the room is greater than 6 ft across, but if we had $w < 6$ we would have the maximum view occurring when the observer is standing at position $x = w$, i.e. at the opposite side of the room.