CALCULUS 3: EXAM 1 REVIEW

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1. Vectors And The Geometry Of Space

Problem 1. Suppose that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors in $\mathbb{R}^3$.

(a) For $t \in [0, 1]$, what sort of object does the equation $(1 - t)\mathbf{a} + t\mathbf{b}$ describe?

(b) For $t \in \mathbb{R}$, what sort of object does $\mathbf{a} + t\mathbf{b}$ describe? How (if at all) does this relate to your answer from part (a)?

(c) For $\mathbf{a}, \mathbf{b}$ nonparallel and $s, t \in \mathbb{R}$, what sort of object does $s\mathbf{a} + t\mathbf{b} + c$ describe?

Problem 2. For vector $\mathbf{v} = \langle x, y \rangle \in \mathbb{R}^2$, with $x, y \geq 0$, let θ be the angle $\mathbf{v}$ makes with the positive x-axis, and ϕ be the complementary angle (the angle with the positive y-axis).

(a) Show that $\sin\theta = \cos\phi$.

(b) Argue that $\mathbf{v} \cdot \mathbf{j} = |\mathbf{v} \times \mathbf{i}|$.

Problem 3. Suppose $\mathbf{a}$ and $\mathbf{b}$ are nonparallel vectors in $\mathbb{R}^3$, and that $\text{comp}_{\mathbf{b}}\mathbf{a} = c \in \mathbb{R}$.

(a) For $s, t > 0$, what is $\text{comp}_{s\mathbf{b}}(t\mathbf{a})$ equal to?

(b) Show that $\mathbf{a} - \text{proj}_{\mathbf{b}}(\mathbf{a})$ is orthogonal to $\mathbf{b}$.

(c) When do we have $\text{comp}_{\mathbf{b}}\mathbf{a} = \text{comp}_{\mathbf{a}}\mathbf{b}$?

Problem 4. Suppose A, B, C are the vertices of a triangle in $\mathbb{R}^2$. Compute $\vec{AB} + \vec{BC} + \vec{CA}$.

Problem 5. For $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r_0} = \langle 1, 1, 1 \rangle$, describe the set of points given by the equation $|\mathbf{r} - \mathbf{r_0}| = 2$.

Problem 6.

(a) Find the area of the triangle with vertices $A(2, -3, 4)$, $B(0, 1, 2)$, and $C(-1, 2, 0)$.

(b) Determine the volume of the parallelipiped with adjacent edges $\langle 3, -5, 1 \rangle$, $\langle 0, 2, -2 \rangle$, and $\langle 3, 1, 1 \rangle$.

(c) Determine whether the vectors $\langle 1, 5, -2 \rangle$, $\langle 3, -1, 0 \rangle$, $\langle 5, 9, -4 \rangle$ are coplanar.

(d) For $a, b, c \geq 0$, compute the triple scalar product of the vectors $a\mathbf{i}$, $b\mathbf{j}$, and $c\mathbf{k}$.

Problem 7. Given a plane equation in general form $ax + by + cz + d = 0$, extract the direction vector of the plane.

Problem 8. Consider the planes given by $x - 2y + z = 0$ and $2x + 3y - 2z = 0$.

(a) Find the angle between the two planes.

(b) Find a parametric equation of their line of intersection.

(c) Find the distance between $(1, 5, -4)$ and $3x - y + 2z - 6 = 0$.

(d) Find the distance between the (parallel) planes $3x-y+2z-6 = 0$ and $6x-2y+4z+4 = 0$. How do we know that the planes are parallel?

(e) Find the distance between point $Q(3, -1, 4)$ and the line $\langle -2 + 3t, -2t, 1 + 4t\rangle$.

(f) Find the distance between the skew lines $L_1(t) = \langle t, 2t, 3t\rangle$ and $L_2(t) = \langle 2t + 1, 3t + 2, 1\rangle$.

Problem 9. Verify that the lines

$$L_1 : \begin{cases} x = 2 - t \\ y = 3 + 2t \\ z = 4 + t \end{cases} \qquad \text{and} \qquad L_2 : \begin{cases} x = 3t \\ y = 1 - 6t \\ z = 4 - 3t \end{cases}$$

are parallel, and find the distance between them.

Problem 10. Suppose $\langle a, b, c\rangle$ is any vector in the plane given by $ex + fy + gz + d = 0$. What can you say about the quantity $ae + bf + cg$?

2. Vector Functions

Problem 11. Sketch the space curve represented by the intersection of the given surfaces, and represent the curve by a vector function using the given parameter.

a) $z = x^2 + y^2$ and $x + y = 0$ using $x = t$.

b) $z = x^2 + y^2$ and $z = 4$ using $x = 2\cos t$.

c) $4x^2 + 4y^2 + x^2 = 16$ and $x = z^2$ using $x = 2\sin t$.

d) $x^2 + y^2 + z^2 = 4$ and $x + z = 2$ using $x = 1 + \sin t$.

e) $x^2 + y^2 + z^2 = 10$ and $x + y = 4$ using $x = 2 + \sin t$.

f) $x^2 + z^2 = 4$ and $y^2 + z^2 = 4$ using $x = t$ (first octant).

g) $x^2 + y^2 + z^2 = 16$ and $xy = 4$ using $x = t$ (first octant).

Problem 12. Show that the vector-valued function $\mathbf{r}(t) = t\mathbf{i} + 2t\cos t\mathbf{j} + 2t\sin t\mathbf{k}$ lies on the cone $4x^2 = y^2 + z^2$.

Problem 13. Find the limits, if they exist:

$$\lim_{t\to 2}\left(3t\mathbf{i} + \frac{2}{t^2-1}\mathbf{j} + \frac{1}{t}\mathbf{k}\right), \qquad \lim_{t\to 1}\left(\sqrt{t}\mathbf{i} + \frac{\ln t}{t^2-1}\mathbf{j} + \frac{1}{t-1}\mathbf{k}\right)$$

Problem 14. Suppose that f, g, and h are first degree polynomials. What sort of object is the curve $\langle f(t), g(t), h(t)\rangle$?

Problem 15. Suppose that the trajectories of two particles are given by the vector functions

$$\mathbf{r_1}(t) = \left\langle t^2, 7t - 12, t^2 \right\rangle \text{ and } \mathbf{r_2}(t) = \left\langle 4t - 3, t^2, 5t - 6 \right\rangle$$

Do the particles ever collide?

Problem 16. Find the point and angle of intersection of the curves:

(a) $\langle t, t^2, t^3\rangle$ and $\langle \sin t, \sin 2t, t\rangle$.

(b) $\langle t, 1 - t, 3 + t^2\rangle$ and $\langle 3 - s, s - 2, s^2\rangle$.

Problem 17. For the vector function $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$, find the following at the point $(1, 1, 1)$:

(a) an equation of the tangent line;

(b) an equation of the normal plane;

(c) the vectors $\mathbf{T}$, $\mathbf{B}$, and $\mathbf{N}$;

(d) an equation of the osculating plane;

(e) the normal and tangential components of acceleration.

Problem 18. For $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t, e^t\rangle$, find $\mathbf{T}$, $\mathbf{N}$, a_T and a_N at $t = 0$. (Note: it will be easier to find $\mathbf{N}$ last using $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$).

Problem 19. Suppose a particle moves according to position vector $\mathbf{r}(t) = \langle a \cos \omega t, b \sin \omega t\rangle$.

(a) Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, a_t and a_n.

(b) Determine the directions of $\mathbf{T}$ and $\mathbf{N}$ relative to the position vector $\mathbf{r}$.

(c) Determine the speed of the object at any time t and explain its value relative to the value of a_T.

(d) When the angular velocity ω is halved, by what factor is a_N changed?

Problem 20. Show that $\dfrac{d}{dt}[\mathbf{r} \times \mathbf{r}'] = \mathbf{r} \times \mathbf{r}''$.

Problem 21. For a smooth curve C given by $\mathbf{r}(t)$, the curvature κ of C is defined as the change in $\mathbf{T}$ with respect to arc length, but it can be more easily computed using

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$

Find the curvature of a circle with radius a.

Problem 22. For a curve given by a function in rectangular coordinates $y = f(x)$, the curvature at point (x, y) can be computed using

$$\kappa = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

Find the maximum curvature of the parabola $y = ax^2$. Your answer should depend on a. What is the radius of the *circle of curvature* (I showed what this looks like in tutorial Thursday) at the point of maximum curvature? What is the center of this circle?

Problem 23. For a curve given by $x = x(t)$ and $y = y(t)$, the curvature is given by

$$\kappa = \frac{|x'y'' + x''y'|}{[(x')^2 + (y')^2]^{3/2}}$$

Consider the ellipse $(x/a)^2 + (y/b)^2 = 1$. Find the minimum and maximum curvatures, and the radii of the corresponding circles of curvature. How are these connected to a and b?

Problem 24. Show that the magnitude of the acceleration of a particle moving in uniform circular motion (i.e. constant speed) is v^2/r, where v is the speed and r is the radius.

3. Functions Of Several Variables

Problem 25. Find the domain and range of the functions:

(a) $f(x, y) = e^{xy}$.

(b) $z = \dfrac{x + y}{xy}$.

(c) $z = \dfrac{xy}{x - y}$.

(d) $f(x, y) = \ln(xy - 6)$.

Problem 26. Describe the level curves (for 2 variable) or level surfaces (for 3 variables) of the functions:

(a) $z = x + y$.

(b) $f(x, y) = xy$.

(c) $f(x, y) = ax^2 + by^2$.

(d) $f(x, y, z) = x^2 + y^2 + z^2$.

Problem 27. The electric potential V at any point (x, y) is given by

$$V(x, y) = \frac{5}{\sqrt{25 + x^2 + y^2}}$$

Describe the shape of the equipotential curves $V = c$ for $c > 0$.

Problem 28. A Pythagorean triple is a point (x, y, z) that lies on the cone $x^2 + y^2 = z^2$ where all the coordinates are integers (usually taken to be positive, though the cone does have two branches, one "positive" and one "negative"). Show that the parametric function of two variables r, s given by

$$\left\langle r^2 - s^2, 2rs, r^2 + s^2 \right\rangle$$

satisfies the cone equation. Fun fact: for $r > s$ positive integers, this gives rise to ALL possible Pythagorean triples (there are infinitely many of them)!

Problem 29. The Cobb-Douglass production function is given by

$$z = Cx^a y^{1-a}$$

Show that this can be rewritten as

$$\ln \frac{z}{y} = \ln C + a \ln \frac{x}{y}$$