Equally Likely Outcomes

For many experiements it is reasonable to assume that all possible outcomes are equally likely. For example:

- Flip a coin. $S = \{H, T\}$. We intuitively know that heads and tails are equally likely, so $P(H) = 0.5$ and $P(T) = 0.5$.
- Flip a coin 2 times and observe the sequence of H and T that results. $S = \{HH, HT, TH, TT\}$, and each sequence is equally likely to appear with probability $1/4 = 0.25$.
- Roll two dice and observe the ordered sequence of numbers. There are $6 \cdot 6 = 36$ possible outcomes, and each is equally likely to appear with probability $1/36$.

Equally Likely Outcomes

In general, for any sample space S with N equally likely outcomes, the probability of each outcome is $\frac{1}{n(S)} = \frac{1}{N}$.

Example: Flip a 4-sided die. The sample space has 4 equally likely outcomes: $S = \{1, 2, 3, 4\}$. Assign probabilities to these outcomes.

$$P(1) = P(2) = P(3) = P(4) = \frac{1}{4} = 0.25$$

Example: Flip a fair coin twice and record the sequence of H and T. Each of the four outcomes $\{HH, HT, TH, TT\}$ have the same probability. What is the probability of getting a single H? The event that we get a single H is $E = \{HT, TH\}$, so $P(E) = 1/4 + 1/4 = 1/2$.

Equally Likely Outcomes

If $E \subseteq S$ is an **event** in a sample space containing equally likely outcomes, the probability that E occurs is the sum of the probabilities of the outcomes in E:

$$\boxed{P(E) = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in S}} = \frac{n(E)}{n(S)}}$$

Notice that this formula displays the probability as the quotient of the answers to two counting problems.

Equally Likely Outcomes

Example: A pair of six sided dice, one red and one green, are rolled and the pair of numbers on the uppermost face is observed. We record red first and then green.

Equally Likely Outcomes

The sample space for the experiment is shown below:

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

(a) Let E be the event that the sum of the numbers is 7. What is the probability of E?

Equally Likely Outcomes

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(3,1) & (3,2) & (3,3) & \boxed{(3,4)} & (3,5) & (3,6) \\
(4,1) & (4,2) & \boxed{(4,3)} & (4,4) & (4,5) & (4,6) \\
(5,1) & \boxed{(5,2)} & (5,3) & (5,4) & (5,5) & (5,6) \\
\boxed{(6,1)} & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

(a) Let E be the event that *the sum of the numbers is 7*. What is the probability of E?

Since $n(S) = 36$ and $n(E) = 6$, we have $P(E) = \dfrac{6}{36} = \dfrac{1}{6}$.

Equally Likely Outcomes

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
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(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

(b) Let F be the event that *the sum of the numbers is 11.* List the elements of the set F and calculate $P(F)$.

$F = \{(5,6),(6,5)\}$, so $P(F) = \dfrac{2}{36}$.

(c) Let G be the event that *the numbers on both dice are the same.* What is $P(G)$?

$G = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$, so $P(G) = \dfrac{6}{36}$.

Equally Likely Outcomes

Example: A 6 sided die and a 4 sided die are rolled and the pair of numbers on the up face is observed. The sample space shown below has equally likely outcomes. Calculate the probability of the event that *the numbers add to 4*.

$$\begin{array}{cccc}
(1,1) & (1,2) & (1,3) & (1,4) \\
(2,1) & (2,2) & (2,3) & (2,4) \\
(3,1) & (3,2) & (3,3) & (3,4) \\
(4,1) & (4,2) & (4,3) & (4,4) \\
(5,1) & (5,2) & (5,3) & (5,4) \\
(6,1) & (6,2) & (6,3) & (6,4)
\end{array}$$

$E = \{(1,3),(2,2),(3,1)\}$. Since $n(E) = 3$ and $n(S) = 24$, the probability that E occurs is $P(E) = \dfrac{3}{24}$.

Probability and Counting

Suppose we take a sample of size k from n available objects. The probability that the sample satisfies a certain property is

$$\boxed{\frac{\#\text{ samples with desired property}}{\#\text{ of possible samples}} = \frac{\#\text{ desired samples}}{\binom{n}{k}}}$$

Example: A bag of marbles contains 4 red and 3 white. Randomly pick 2 marbles from the bag. What is the probability of getting 2 white marbles?

$$\frac{\#\text{ ways to get 2 white}}{\#\text{ ways to get any 2 marbles}} = \frac{\binom{3}{2}}{\binom{7}{2}} = \frac{3}{21} \approx 0.143$$

Probability and Counting

Example: A bag contains 8 red marbles and 4 white marbles. Take a random sample of 2 marbles.

(a) What is the probability of getting two red marbles?

$$\frac{\#\text{ ways to get 2 red}}{\#\text{ ways to get any 2 marbles}} = \frac{\binom{8}{2}}{\binom{12}{2}} = \frac{28}{66} \approx 0.424$$

(b) What the probability of getting one red and one white marble in the sample?

$$\frac{\#\text{ ways for 1 red and 1 white}}{\#\text{ ways to get any 2 marbles}} = \frac{\binom{8}{1}\cdot\binom{4}{1}}{\binom{12}{2}} = \frac{32}{66} \approx 0.485$$

Probability and Counting

Example: When you buy a Powerball ticket, you choose 5 different white numbers from $\{1, 2, \ldots, 59\}$, and 1 red number from $\{1, 2, \ldots, 35\}$. What is the probability of winning the lottery?

The total number of samples is

$$\binom{59}{5} \cdot \binom{35}{1} = 175,223,510$$

There is only one winning combination of numbers, so your probability of winning the lottery is

$$\frac{1}{175,223,510} \approx 0.0000006\%$$

Probability and Counting

Example: A poker hand is dealt randomly (if the deck is well shuffled). What is the probability of getting a Full House (three cards from one rank and two from another rank)?

There are 13 different ranks (A, 2, 3, . . . , 10, J, Q, K). We choose 4 cards from the first rank, and 2 cards from the second (different) rank:

$$\frac{\binom{13}{1}\binom{4}{3} \times \binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = \frac{13 \cdot 4 \times 12 \cdot 6}{2,598,960} = \frac{3,744}{2,598,960} \approx 0.00144$$

This is about 0.144 %, a very small chance!

Probability and Counting

Example: A box ready for shipment contains 100 light bulbs, 10 of which are defective. The quality control test is to take a random sample of 5 light bulbs from the box. If one (or more) of the 5 is defective, the box will not be shipped. What is the probability that the box is shipped?

We want to know when the box gets shipped, so we need to count the number of ways in which the 5 selected bulbs are all good. We have 90 good bulbs.

$$\frac{\binom{90}{5}}{\binom{100}{5}} = \frac{43,949,268}{75,287,520} \approx 0.584$$

There is a 58.4% chance that the box gets shipped, so if all the boxes were like this one, more than half the customers get 10 bad bulbs on their order!

Probability and Counting

Example: A coin is flipped 4 times and the sequence of heads and tails is recorded. All of these sequences are equally likely.

(a) How many elements are there in this sample space?

$2^4 = 16$

(b) How many outcomes have exactly 3 heads?

$\binom{4}{3} = 4$

(c) Let E be the event "we get exactly 3 heads." Find $P(E)$?

$$P(E) = \frac{4}{16} = 0.25$$

The Complement Principle

If E is an event in a sample space S, we know from set theory that $n(E) = n(S) - n(E')$. Therefore

$$\frac{n(E)}{n(S)} = \frac{n(S)}{n(S)} - \frac{n(E')}{n(S)}$$

This gives **the complement principle:**

$$\boxed{P(E) = 1 - P(E')}$$

Note: If we define "success" to be the occurrance of E and "failure" to be occurrence of E' (or failure of E to occur), we can think of the complement principle as

$$\boxed{P(\text{success}) = 1 - P(\text{failure})}$$

The Complement Principle

Example: Flip a coin 10 times and observe the sequence of heads and tails.

(a) How many outcomes are in this sample space?

$2^{10} = 1,024.$

(b) What is the probability that you get exactly 5 heads?

$$\frac{\binom{10}{5}}{2^{10}} = \frac{252}{1024} \approx 0.246$$

(c) What is the probability that you get at least one tail?

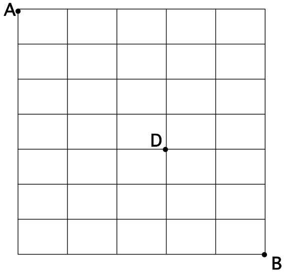
$$1 - P(\text{no tails}) = 1 - \frac{\binom{10}{0}}{1024} = 1 - \frac{1}{1024} \approx 0.99902$$

(d) What is the probability that we will observe at least two heads?

$$1 - P(0 \text{ or } 1 \text{ heads}) = 1 - \frac{\binom{10}{0} + \binom{10}{1}}{1024} = 1 - \frac{11}{1024} \approx 0.9892$$

The Complement Principle

Example: Kristina, on her morning run, wants to get from point A to point B. Being an efficient runner, she always travels South or East.



(a) How many routes can she take?

$\binom{7+5}{7} = \binom{12}{7} = 792$ (you could do $\binom{12}{5}$ instead)

(b) If she chooses a route at random, find the probability that she will avoid the doberman at D? It's easier to find the routes through D: $\binom{7}{3}\binom{5}{2} = 35 \cdot 10 = 350.$

$$1 - \frac{350}{792} = \frac{442}{350} \approx 0.558$$

The Complement Principle

Example A box ready for shipment contains 100 light bulbs, 10 of which are defective. The quality control test is to take a random sample of 5 light bulbs from the box. If one (or more) of the 5 is defective, the box will not be shipped. What is the probability that the box will not be shipped.

We already calculated the probability that the box will get shipped (it was 0.583).

The probability that it will not get shipped in $1 - 0.583 = 0.416$.

More Examples

Example: An experiment consists of drawing 3 marbles at random from a bag containing 2 red and 4 white. What is the probability of getting at least 2 white balls?

$$\underline{a})\ \frac{4}{5} \qquad b)\ \frac{2}{3} \qquad c)\ \frac{1}{5} \qquad d)\ \frac{2}{5} \qquad e)\ \frac{3}{5}$$

$$\frac{\binom{4}{2}\cdot\binom{2}{1}+\binom{4}{3}\cdot\binom{2}{0}}{\binom{6}{3}} = \frac{16}{20} = \frac{4}{5}$$

More Examples

Example: Three out of 25 new cars are selected at random to check for steering defects. Suppose that 7 of the 25 cars have such defects. What is the probability that all 3 of the selected cars are defective?

$$\frac{C(7,3)}{C(25,3)}$$

Example: A fair coin is tossed 10 times. What is the probability of observing exactly 3 heads?

$$\frac{C(10,3)}{2^{10}}$$

The Birthday Problem

Q: What is the probability that at least two people in a group will share a birthday (month and day)? Assume we only have 365 days in a year.

The table shows the approximate probability, according to group size:

Group Size	Probability
10	0.1169
15	**0.2529**
20	0.4114
30	0.7063
40	0.8912
50	0.9704
60	0.9941
70	0.9991
80	0.9999

The Birthday Problem

We'll compute the probability for a group of 20. Let E be the event that "at least two people share a birthday".

It is much easier to calculate $P(E')$ (all bdays different), and use the complement principle: $P(E) = 1 - P(E')$.

Size of sample space: $365 \cdot 365 \cdot 365 \ldots 365 = 365^{20}$

Size of E': $365 \cdot 364 \cdot 363 \cdot 362 \cdots 345 = P(365, 20)$

Probability of E': $P(E') = P(365, 20)/365^{20} \approx 0.588$
Finally, we are able to compute the desired probability:

$$P(E) = 1 - P(E') \approx 0.4114$$

Test your intuition!

Each of the 15 people in the room today selects a whole number between 1 and 100, at random (no collaboration!).

How likely is it that at least two people pick the same number?

A: less than 10%

B: close to 60%

C: more than 95%

It turns out that the probability of at least 2 people picking the same number is about 66.87%.

Test your intuition!

Example: If 10 people each choose a number (randomly) from $\{1, 2, 3, \ldots, 50\}$, what are the chances that at least two people choose the same number?

$a)\ < 20\%$ $\qquad b)\ \approx 40\%$ $\qquad c)\ \approx 60\%$ $\qquad d)\ > 80\%$

$$1 - \frac{P(50, 10)}{50^{10}} \approx 0.618$$

Note: The answer isn't much different from 66.87% (from the previous slide). Why?

The # of people decreased (which would make the previous probability of 66.87% go down).

At the same time, they have less numbers to choose from (which would increase the previous probability of 66.87%).

Coincidences

The following series of coincidences between the life events of Abraham Lincoln and John F. Kennedy often strike people as unusual or even spooky:

Lincoln was elected to Congress in 1846; Kennedy in 1946.

Lincoln was elected president in 1860; Kennedy in 1960.

Lincoln's secretary was named Kennedy; Kennedy's was named Lincoln.

Andrew Johnson, who succeeded Lincoln, was born in 1808; Lyndon Johnson, who succeeded Kennedy, was born in 1908.

John Wilkes Booth, who assassinated Lincoln, was born in 1839; Lee Harvey Oswald, who assassinated Kennedy, was born in 1939.

Coincidences

Given the amount of information we have about these two men, is it really surprising that we might find 5 such coincidences in their parallel lives?

Experiment: Create 2 fictitious characters, A and B and I give each a profile by randomly choosing years of occurrence of 1000 life events, out of 100 years total.

- year born
- year died
- year they got their first dog
- year they got married
- year they visited Ireland
- ...

Coincidences

a) What is the probability that I assign the same year to both person A and person B for a **single** life event?

Probability that I assign different years: $\frac{100 \cdot 99}{100 \cdot 100} = 0.99$.
Probability of coincidence: $1 - 0.99 = 0.01$ or 1%.

b) How many coincidences would you expect among the 1000 life events?

$\approx 1000 \cdot 0.01 = 10$ coincidences

c) How likely is it to have at least 5 coincidences?

It turns out the probability is about 0.97, which is very high! We will learn how to calculate this later when we study the Binomimal Distribution.