Names ______________________________ Date ____________

Math 10560 Worksheet 5
Show all your work to receive credit.

1. Consider the integral

$$\int_1^3 x^5\,dx.$$

(a) Estimate the integral using Simpson's Rule and n = 4. You do not need to simplify your answer.

Solution: Since we are using 4 steps, $\Delta x = \frac{1}{2}$. Let $f(x) = x^5$. Thus we have

$$\frac{1}{2\cdot 3}\left[f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + f(3)\right] =$$
$$\frac{1}{6}\left[1 + 4\left(\frac{3}{2}\right)^5 + 2(2)^5 + 4\left(\frac{5}{2}\right)^5 + 3^5\right] = 121.5.$$

(b) Estimate the error using the error bound for Simpson's Rule:

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}, \quad K \geq |f^{(4)}(x)|.$$

Solution: First we find K. We note that

$$f^{(1)}(x) = 5x^4$$
$$f^{(2)}(x) = 20x^3$$
$$f^{(3)}(x) = 60x^2$$
$$f^{(4)}(x) = 120x$$

On the interval of $[1,3]$ we have that $f^{(4)}(x) = 120x \leq 120\cdot 3 = 360$. Thus we can take $K = 360$. Furthermore we have $a = 1, b = 3$ and $n = 4$. This give us

$$|E_S| \leq \frac{360(3-1)^5}{180\cdot 4^4} = \frac{360\cdot 2^5}{180\cdot 4^4} = \frac{1}{4}.$$

2. Compute the integral $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$ by completing the following steps:

(a) Write $\dfrac{10}{(x-1)(x^2+9)}$ in partial fraction decomposition form (leaving A, B, C, etc. in the numerators).

Solution: $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$

(b) Solve for the partial fraction coefficients A, B, C, etc. above.

Solution: Multiplying both sides of the equation above by $(x-1)(x^2+9)$ we get:

$$A(x^2+9) + (Bx+C)(x-1) = 10.$$

Equating coefficients, we get:
$Ax^2 + Bx^2 = 0 \cdot x^2 \Rightarrow A + B = 0$
$-Bx + Cx = 0 \cdot x \Rightarrow C - B = 0$
$9A - C = 10$
Using $C = B$ in the last equation, we get $9A - B = 10$. Adding the first equation to this one: $10A = 10 \Rightarrow A = 1 \Rightarrow B = C = -1$.

(c) Evaluate the integral $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$.

Solution: Using the partial fraction decomposition from above, we get:

$$\begin{aligned}\int \frac{10}{(x-1)(x^2+9)}\,dx &= \int \left(\frac{1}{x-1} + \frac{-x-1}{x^2+9}\right) dx \\ &= \int \left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\right) dx \\ &= \ln|x-1| - \frac{1}{2}\ln(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.\end{aligned}$$

For the second term of the integral, we use u-substitution with $u = x^2 + 9 \Rightarrow du = 2x\,dx$. Then $\displaystyle\int \frac{x}{x^2+9} = \int \frac{1}{2u}\,du = \frac{1}{2}\ln|u| + C$.

For the third term, we note that $\displaystyle\int \frac{1}{x^2+9}\,dx = \int \frac{1}{9((\frac{x^2}{9})+1)}\,dx = \frac{1}{9}\int \frac{1}{(\frac{x}{3})^2+1}\,dx$. Then we use u-substitution with $u = \dfrac{x}{3} \Rightarrow du = \dfrac{1}{3}\,dx$; this gives us: $\displaystyle\frac{1}{3}\int \frac{1}{u^2+1}\,du = \frac{1}{3}\tan^{-1}(u) + C = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

3. Determine the following limit:
$$\lim_{x\to\infty} \frac{1}{x(e^{\frac{1}{x}}-13)}$$

Solution:

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x(e^{\frac{1}{x}}-13)} &= \lim_{x\to\infty} \frac{1/x}{(e^{\frac{1}{x}}-13)} \\ \text{(L'Hospital)} &= \lim_{x\to\infty} \frac{-1/x^2}{-\frac{1}{x^2}e^{1/x}} \\ &= \lim_{x\to\infty} \frac{1}{e^{1/x}} \\ &= 1.\end{aligned}$$

4. Evaluate the integral
$$\int_0^{\pi/4} x\sin(4x)dx.$$

Solution:

Set $u = x$ and $v = -\frac{1}{4}\cos(4x)$, then $\mathrm{d}v = \sin(4x)\mathrm{d}x$. Use integration by parts.

$$\begin{aligned}\int_0^{\pi/4} x\sin(4x)\,\mathrm{d}x &= -\frac{1}{4}x\cos(4x)\Big|_0^{\pi/4} + \frac{1}{4}\int_0^{\pi/4}\cos(4x)\,\mathrm{d}x \\ &= -\frac{1}{4}x\cos(4x)\Big|_0^{\pi/4} + \frac{1}{16}\sin(4x)\Big|_0^{\pi/4} \\ &= \frac{\pi}{16}\end{aligned}$$

5. A sample of a Cobalt-60 has an initial mass of 6 grams. Let $M(t)$ denote the mass of the sample after t days, $M(t)$ decreases at a rate that is proportional to the amount of the substance present at time t. That is
$$M'(t) = kM(t).$$
Cobalt-60 has a half-life of 1925 days.

(a) Give a formula for $M(t)$. (Solve for all unknown constants).

(b) How long (how many days) will it take for the sample to decrease from 6 grams to 1 gram?

Solution:

For part (a), we recognize from the condition $M'(t) = kM(t)$ that $M(t)$ should be an exponential (decay) function (think about the derivatives of exponentials and what they look like), so it will have the form
$$M(t) = M_0e^{-kt}$$

where $M_0 = 6$ is the initial mass (in grams), t is time (in years), and k is a constant to be determined. Since we are told the half life is 1925 days, we have

$$0.5M_0 = M_0 e^{-k(1925)} \iff \ln(0.5) = -k \cdot 1925$$

which gives $k = -\dfrac{\ln 0.5}{1925}$. Hence our function is

$$M(t) = 6e^{\frac{\ln(0.5)}{1925} \cdot t}$$

For part (b), we need to solve for the value of t in

$$1 = 6e^{\frac{\ln(0.5)}{1925} \cdot t} \iff \ln(1/6) = \frac{\ln(1/2)}{1925} \cdot t \iff t = 1925\frac{\ln(1/6)}{\ln(1/2)}$$

or alternately

$$t = 1925\frac{\ln(6)}{\ln(2)}.$$

6. Compute

$$\int \frac{1}{\sqrt{x^2 - 14x + 50}}\, dx$$

Solution:

Regular substitution doesn't work here: if we let $u = x^2 - 14x + 50$, there is no factor on top that would be a scalar multiple of the differential $du = (2x - 14)dx$. Instead we try trigonometric substitution, but to figure out the appropriate substitution, we must first complete the square in the denominator:

$$\int \frac{dx}{\sqrt{x^2 - 14x + 50}} = \int \frac{dx}{\sqrt{x^2 - 2 \cdot 7x + 49 + 1}} = \int \frac{dx}{\sqrt{(x - 7)^2 + 1}}$$

and $(x - 7)^2 + 1$ resembles the trig identity $\tan^2(x) + 1 = \sec^2(x)$, so we let $x - 7 = \tan(\theta)$, which gives $dx = \sec^2(\theta)d\theta$, and the entire denominator becomes

$$\sqrt{(x - 7)^2 + 1} = \sqrt{\tan^2(x) + 1} = \sqrt{\sec^2(x)} = \sec(\theta)$$

After the substitution, our original integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2 - 14x + 50}} &= \int \frac{\sec^2(\theta)d\theta}{\sec(\theta)} \\ &= \int \sec(\theta)d\theta \\ &= \ln|\sec(\theta) + \tan(\theta)| + C\end{aligned}$$

Now, since $x-7=\tan\theta$, the $\sec(\theta)$ term inside the log can be expressed as a function of x by using an appropriate right triangle whose legs have lengths $x-7$ and 1, respectively. Hence our integral equals:

$$\int \frac{dx}{\sqrt{x^2-14x+50}} = \ln\left|\sqrt{x^2-14x+50}+x-7\right|+C$$

7. Compute

$$\int \sqrt{3-2x-x^2}\,dx.$$

Solution: First we complete the square: $3-2x-x^2 = 4-(x+1)^2$. Then we can use u-substitution with $x+1=2\sin\theta \Rightarrow dx = 2\cos\theta\,d\theta$:

$$\begin{aligned}
\int \sqrt{4-(x+1)^2}\,dx &= \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta \\
&= 4\int \cos^2\theta\,d\theta \\
&= 2\int (1+\cos(2\theta))\,d\theta \\
&= 2\theta+\sin 2\theta + C \\
&= 2\theta + 2\sin(\theta)\cos(\theta)+C.
\end{aligned}$$

Then using a right triangle, we can solve for the trig functions in terms of x:

$$\begin{aligned}
2\theta+2\sin\theta\cos\theta+C &= 2\sin^{-1}\left(\frac{x+1}{2}\right)+2\frac{x+1}{2}\frac{\sqrt{3-2x-x^2}}{2}+C \\
&= 2\sin^{-1}\left(\frac{x+1}{2}\right)+\frac{x+1}{2}\sqrt{3-2x-x^2}+C.
\end{aligned}$$

Formula Sheet

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x-y) + \sin(x+y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\int \sec\theta = \ln|\sec\theta + \tan\theta| + C$$