**10350 Tutorial Week 12 - Set 01** Name ______________________

Section ______________

**1.** Sketch the graph of $g(x) = \dfrac{x^2}{x^2+1}$ by completing the steps below.

**a.** Find all $x$-intercepts and $y$-intercept of the graph of $g(x)$ whenever possible.

For x-intercept, by solving $g(x) = \frac{x^2}{x^2+1} = 0$, we get $x = 0$. So the coordinate of x-intercept is $(0, 0)$.

For y-intercept, let $x = 0$, and we get $g(0) = 0$. So the coordiate of y-intercept is $(0, 0)$.

**b.** Find coordinates of all critical points, vertical asymptotes, and places where $g(x)$ are undefined.

$g'(x) = \dfrac{2x}{(x^2+1)^2}$

For critical points, take the derivative of $g(x)$. By quotient rule, we have $g'(x) = \frac{2x(x^2+1)-x^2\cdot 2x}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$. Notice that $g'(x)$ is defined everywhere. Then, solve $g'(x) = 0$, we get $x = 0$, and $g(0) = 0$. So (0,0) is a critical point.

Since $g(x) = \frac{x^2}{x^2+1}$, and $x^2 + 1 > 0$ is always true, thus no vertical asymptotes.

Similarly, $g(x)$ is nowhere undefined.

**c.** Determine where $g(x)$ is increasing and where it is decreasing.

From part b, we have $g'(x) = \frac{2x}{(x^2+1)^2}$. Notice $(x^2 + 1)^2 > 0$ everywhere, and $2x < 0$ when $x < 0$, $2x > 0$ when $x > 0$. So it is clear that $g'(x) < 0$ on $(-\infty, 0)$, and $g'(x) > 0$ on $(0, \infty)$.

Thus, $g(x)$ is increasing in $(0, \infty)$, and decreasing in $(-\infty, 0)$.

**d.** Determine the concavity and coordinates of inflection points of $g(x)$.

$g''(x) = \dfrac{-6x^2+2}{(x^2+1)^3}$

First, take the second order derivative of $g(x)$, by quotient rule, we have $g''(x) = \frac{2(x^2+1)^2-2x\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4} = \frac{2(x^2+1)-2x\cdot 2\cdot 2x}{(x^2+1)^3} = \frac{-6x^2+2}{(x^2+1)^3}$.

By solving $g''(x) = 0$, we get $x = \pm\frac{\sqrt{3}}{3}$. Then, we check points in each interval $(-\infty, -\frac{\sqrt{3}}{3})$,$(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$, and $(\frac{\sqrt{3}}{3}, +\infty)$, and get $g''(x) < 0$ in $(-\infty, -\frac{\sqrt{3}}{3})$ and $(\frac{\sqrt{3}}{3}, +\infty)$, and $g''(x) > 0$ in $(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$. Therefore, $g(x)$ is concave up in $(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$, and concave down in $(-\infty, -\frac{\sqrt{3}}{3})$ and $(\frac{\sqrt{3}}{3}, +\infty)$.

Since a inflection point is where $g''(x)$ is defined and concavity changes, so $x = \pm\frac{\sqrt{3}}{3}$, and the coordinates are $(-\frac{\sqrt{3}}{3}, \frac{1}{4})$ and $(\frac{\sqrt{3}}{3}, \frac{1}{4})$.

**e.** Find all asymptotes and limit at infinity whenever applicable. Check for any symmetry.

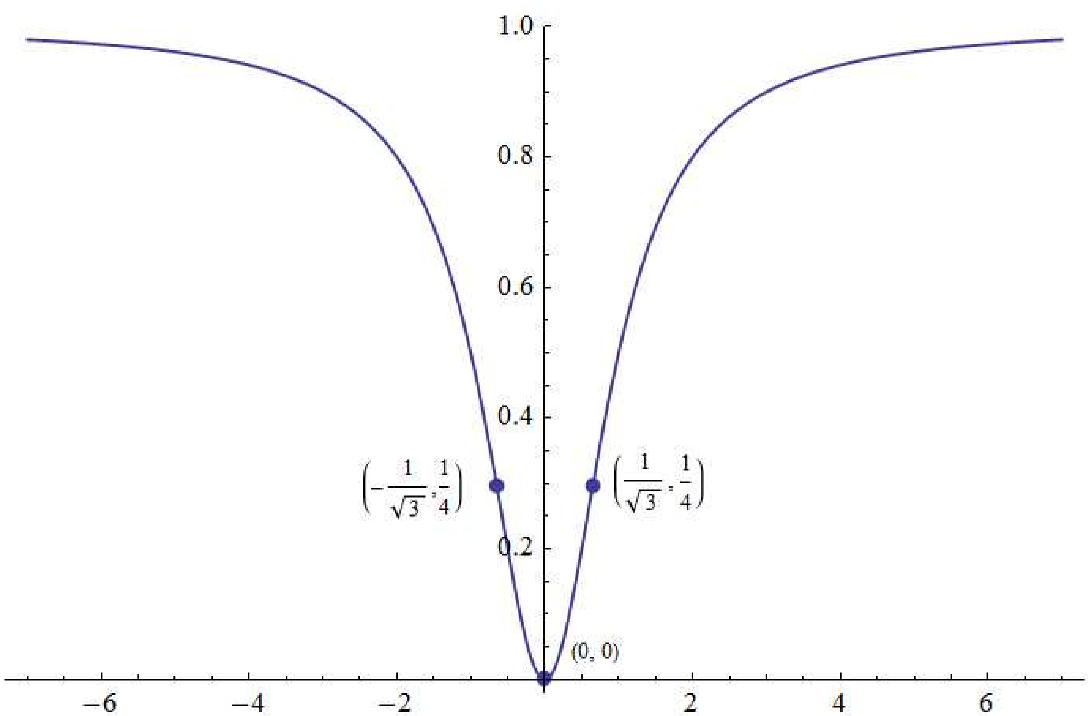
We have $\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty} \frac{x^2}{x^2+1} = 1$ by L'Hospital's rule. So, y=1 is a horizontal asymptote.

For symmetry, we have $g(-x) = \frac{(-x)^2}{(-x)^2+1} = \frac{x^2}{x^2+1} = g(x)$.

**f.** Find the range of the function $g(x)$.

Consider all critical points and limit at infinity, we have $g(0) = 0$, and $\lim_{x\to\pm\infty} g(x) = 1$. So, the range of the function should be $[0, 1)$.

**g.** Sketch the graph below labeling all important features. Your picture should be large and clear.



**10350 Tutorial Week 12 - Set 02** Name ______________ Section ________

**2a.** Find the absolute (global) maximum and minimum of $f(x) = \dfrac{4}{(x^2-1)^2+1}$ on the interval $[0, 2]$.

Hint: $f(x) = 4[(x^2-1)^2+1]^{-1}$

By chain rule, we have $f'(x) = 4(-1)[(x^2-1)^2+1]^{-2} \cdot 2(x^2-1) \cdot 2x = \frac{-16x(x^2-1)}{[(x^2-1)^2+1]^2}$. Notice $f'(x)$ is defined in $[0, 2]$. Then, solve $f'(x) = 0$, we get $x = 0, \pm 1$. Since $-1 \notin [0, 2]$, so critical points are x=0,1.

Plug critical points and endpoints in f(x), we have $f(0) = 2$, $f(1) = 4$, and $f(2) = \frac{2}{5}$. Therefore, the global max is $f(1) = 4$, and global min is $f(2) = \frac{2}{5}$.

**2b.** Using the steps below, find the global maximum and minimum of $f(x) = \dfrac{4}{(x^2-1)^2+1}$ on $[0, \infty)$.

**Step 1:** Find all critical points in the domain of $f(x)$ and the values of $f(x)$ there. Classify them using first derivative test.

By part 2a, we have $f'(x) = \frac{-16x(x^2-1)}{[(x^2-1)^2+1]^2}$, and all critical points in the domain are $x = 0, 1$. And $f(0) = 2$, $f(1) = 4$.

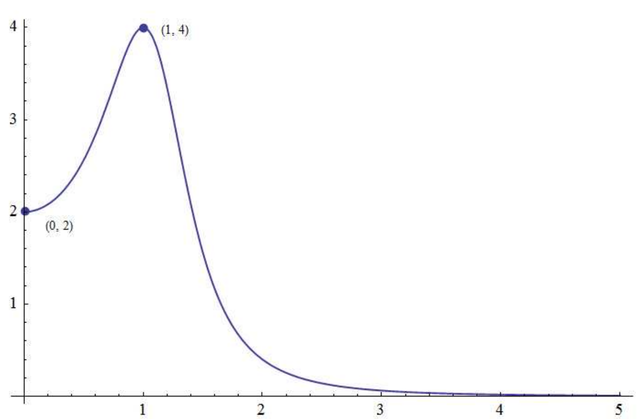
Using first derivative text, we get $f'(x) > 0$ in $(0, 1)$, and $f'(x) < 0$ in $(-1, 0)$ and $(1, \infty)$. So $f(x)$ is increasing in $(0, 1)$, and decreasing in $(-1, 0)$ and $(1, \infty)$. Thus, $x = 0$ is a local min, and $x = 1$ is a local max.

**Step 2:** Find the values of $f(x)$ at the end-points (if any) of its domain. $f(0) = 2$

**Step 3:** If end-point not included, or $\pm\infty$, find all limits of $f(x)$ towards end of interval.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{4}{(x^2-1)^2+1} = \frac{4}{+\infty} = 0.$$

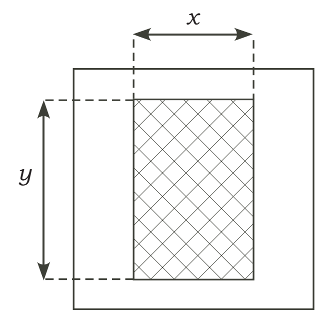
**Step 4:** Give a schematic sketch (ignore concavity) of the graph of $f(x)$ clearly indicating where the global maximum and minimum are. State the global maximum and minimum of $f(x)$ on $[0, \infty)$ if any. Find the range of the function $f(x)$ for $x$ in $[0, \infty)$.




Consider all critical points, endpoints, and asymptotes, we have the global max is $f(1) = 4$, and non global min. The range of the function for $x$ in $[0, \infty)$ is $(0, 4]$.

**3.** A graphic artist designing a poster for commercial use is instructed to have **one** inch margins top and bottom, and **two** inches along each side around the printed portion of the poster. It is further specified that the total area of the printed portion is 98 square inches. Suppose the width of the printed portion is $x$ inches and the length of the poster is $y$ inches. To cut paper cost, the total area of the poster (margin and printed portion) is to be minimized.

**a.** Write down the relation between $x$ and $y$.




Since the total area of the printed portion is 98 square inches, we have $xy = 98$, thus $y = \frac{98}{x}$.

**b.** Find the function $A(x)$ you should minimize?

The length of the whole poster is $y + 2$, and the width of the whole poster is $x + 4$.

Thus, $A(x) = (x + 4)(y + 2) = (x + 4)(\frac{98}{x} + 2) = 2x + \frac{392}{x} + 106$.

**c.** What is the range of $x$ on which you should minimize $A(x)$? Is this a closed and bounded interval?

By $x \geqslant 0$, and $y = \frac{98}{x} \geqslant 0$, notice that $x$ cannot equal to 0 such that $\frac{98}{x}$ makes sense, we have the range of $x$ is $(0, \infty)$. It is not closed and bounded.

**d.** Find the value of $x$ that minimizes the total area of the poster and give the area of the poster.

Take the derivative of $A(x)$, we have $A'(x) = 2 - \frac{392}{x^2}$. Solve $A'(x) = 0$ in $(0, \infty)$, we have $x = 14$ is a critical point.

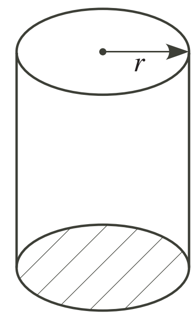
Then, by first derivative test in $(0, 14)$ and $(14, \infty)$, we find that $A'(x) < 0$ in $(0, 14)$, and $A'(x) > 0$ in $(14, \infty)$. So, $A(x)$ is decreasing in $(0, 14)$, and increasing in $(14, \infty)$.

Therefore, $x = 14$ is the value minimizes the total area of the poster and $A(14) = 162$.

**4.** A cylindrical container with an **open** top is to be made with two different kind of material. The curve side is constructed with material of density $4$ lb/ft$^2$ and the circular bottom is constructed with material of $5$ lb/ft$^2$. Answer the questions below if the volume of the cylindrical container is $10\pi$ ft$^3$.

(The curved surface area of a cylinder is $2\pi rh$. Volume of a cylinder is $\pi r^2 h$)

(a) Write down the total weight of the container $W(r)$ in terms of its radius $r$.




Since the volume of the cylinder is $10\pi$ ft$^3$, we have $\pi r^2 h = 10\pi$. So $h = \frac{10}{r^2}$.
Then, we have $W(r) = 4 \cdot 2\pi rh + 5\pi r^2 = \frac{80\pi}{r} + 5\pi r^2$.

(b) Write down the range of the possible values of $r$: $(0, \infty)$.

(c) Using calculus, find the radius $r$ that minimizes the weight $W$ of the container. You must show appropriate checks to justify that your answer makes $W$ minimum.

First, take the derivative $W'(r) = \frac{-80\pi}{r^2} + 10\pi r$. It is defined in $(0, \infty)$. Then, solve $W'(r) = 0$ in $(0, \infty)$, we get the critical point $x = 2$.

By first derivative test in $(0, 2)$ and $(2, \infty)$, $W'(r) < 0$ in $(0, 2)$ and $W'(r) > 0$ in $(2, \infty)$. So, $W(r)$ is decreasing in $(0, 2)$ and increasing in $(2, \infty)$. Thus, $r = 2$ is the radius that minimizes $W$, and $W(2) = 60\pi$ lb.