Name: *Solutions* Date: 01/25/2018

**M20580 L.A. and D.E. Tutorial**
**Worksheet 2**
Sections 1.1–1.3

1. (a) Find the general solution of the system of linear equations

$$\begin{array}{rcrcrcrcr} 2x_1 & - & 4x_2 & + & 5x_3 & + & x_4 & = & -3 \\ x_1 & - & 2x_2 & + & 2x_3 & + & x_4 & = & -1 \\ x_1 & - & 2x_2 & + & 3x_3 & & & = & -2 \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & -4 & 5 & 1 & -3 \\ 1 & -2 & 2 & 1 & -1 \\ 1 & -2 & 3 & 0 & -2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & -2 & 2 & 1 & -1 \\ 2 & -4 & 5 & 1 & -3 \\ 1 & -2 & 3 & 0 & -2 \end{array}\right] \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -2 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{cccc|c} 1 & -2 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 = R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variables: $x_1, x_3$

Free variables: $x_2, x_4$

$$x_1 - 2x_2 + 3x_4 = 1 \Rightarrow x_1 = 1 + 2x_2 - 3x_4$$
$$x_3 - x_4 = -1 \Rightarrow x_3 = -1 + x_4$$

General solution:
$$\begin{cases} x_1 = 1 + 2x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = -1 + x_4 \\ x_4 \text{ is free} \end{cases}$$

(b) If the linear system above has infinitely many solutions, give two solutions to the system.

One solution: let $x_2 = 0$, $x_4 = 0$, then $x_1 = 1$ and $x_3 = -1$

$(1, 0, -1, 0)$ is a solution

Another solution: let $x_2 = 1$, $x_4 = 0$, then $x_1 = 3$, $x_3 = -1$

$(3, 1, -1, 0)$ is a solution

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2. *Recall*: Given a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, a ***linear combination*** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is a new vector of the form

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p, \quad \text{for } \textit{some} \text{ scalars } c_1, c_2 \ldots, c_p$$

Suppose

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}4\\7\\1\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}0\\0\\2\end{bmatrix}$$

(a) Give an example of a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

We can do $\vec{v}_1 - \vec{v}_2 + 2\vec{v}_3 = \begin{bmatrix}1\\2\\0\end{bmatrix} - \begin{bmatrix}4\\7\\1\end{bmatrix} + \begin{bmatrix}0\\0\\4\end{bmatrix} = \begin{bmatrix}-3\\-5\\3\end{bmatrix}$

(b) Determine whether the vector $\mathbf{w} = \begin{bmatrix}2\\3\\5\end{bmatrix}$ can be written as a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. If yes, find scalars $a_1, a_2, a_3$ such that $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{w}$.

We want to know if there exist scalars $a_1, a_2, a_3$ such that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{w} \quad \text{or} \quad a_1\begin{bmatrix}1\\2\\0\end{bmatrix} + a_2\begin{bmatrix}4\\7\\1\end{bmatrix} + a_3\begin{bmatrix}0\\0\\2\end{bmatrix} = \begin{bmatrix}2\\3\\5\end{bmatrix} \quad (1)$$

Let's try to find $a_1, a_2, a_3$ :

$$\left[\begin{array}{ccc|c}1&4&0&2\\2&7&0&3\\0&1&2&5\end{array}\right] \sim \left[\begin{array}{ccc|c}1&4&0&2\\0&-1&0&-1\\0&1&2&5\end{array}\right] \sim \left[\begin{array}{ccc|c}1&4&0&2\\0&1&0&1\\0&0&2&4\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&-2\\0&1&0&1\\0&0&1&2\end{array}\right]$$

So, the vector equation (2) has a solution, namely $a_1 = -2$, $a_2 = 1$, $a_3 = 2$

Thus, $\vec{w}$ can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$, where

$$\vec{w} = -2\vec{v}_1 + \vec{v}_2 + 2\vec{v}_3.$$

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3. Fill in the blanks

Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the set of all linear combinations of the vectors $\vec{v_1}, \ldots, \vec{v_p}$

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 3 \\ 8 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$.

(a) Give examples of two vectors that are in the set Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

We need to come up with 2 linear combinations of $\vec{v_1}, \vec{v_2}, \vec{v_3}$

How about $\vec{v_1} + \vec{v_2} + \vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \\ 8 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 2 \end{bmatrix}$

Also, since $\vec{v_1} = 1 \cdot \vec{v_1} + 0 \cdot \vec{v_2} + 0 \cdot \vec{v_3}$, $\vec{v_1}$ is a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$. So, $\vec{v_1}$ is in Span$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$. Similarly, $\vec{v_2}, \vec{v_3}$ are also in Span$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

Moreover, since $\vec{0} = 0 \cdot \vec{v_1} + 0 \cdot \vec{v_2} + 0 \cdot \vec{v_3}$, $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is also in Span$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

(b) How many vectors are there in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

Infitely many

(c) Determine whether the vector $\mathbf{w} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

We want to know if $\vec{w}$ can be written as a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$

i.e. We want to find solution ~~to the~~ associated to the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

← ~~this~~ reveals that the vector eq'n $a_1\vec{v_1} + a_2\vec{v_2} + a_3\vec{v_3}$ doesn't have any solution (inconsistent)

since $0 \neq 3$

Thus, $\vec{w}$ can't be written as a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$

i.e. $\vec{w}$ is not in the Span$\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

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4. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$. Find the value of $h$ such that $\mathbf{w} = \begin{bmatrix} 4 \\ 3 \\ h \\ 0 \end{bmatrix}$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

We want to determine $h$ such that the vector equation

$$a_1\vec{v_1} + a_2\vec{v_2} + a_3\vec{v_3} = \vec{w} \text{ is consistent.}$$

Use matrix tool:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 3 & 0 & 3 \\ -1 & 1 & 1 & h \\ 2 & -2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 3 & 1 & h+4 \\ 0 & -6 & 2 & -8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & h+1 \\ 0 & 0 & 2 & -2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & h+1 \\ 0 & 0 & 0 & -2h-4 \end{array}\right]$$

$\Rightarrow$ in order for the vector equation above to have a solution (consistent), we must have

$$-2h - 4 = 0 \Rightarrow h = -2$$

Conclusion: when $h = -2$, $\vec{w}$ is a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$