Name: SOLUTIONS Date: 10/12/2017

M20550 Calculus III Tutorial
Worksheet 6

1. Evaluate the double integral $\iint_R (4-2y)dA$, for $R = [0,1] \times [0,1]$, by identifying it as the volume of a solid.

> ***Solution:*** Notice that $z = f(x,y) = 4 - 2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0,1] \times [0,1] \times [0,4]$ which lies below the plane $z = 4 - 2y$. We can compute this by taking the areas of the rectangular part and the triangular part, and multiplying their sum by the "depth" in the x-direction:
>
> $$\iint_R (4-2y)dA = \left((1)(2) + \frac{(1)(2)}{2}\right)(1) = 3$$

2. Evaluate the iterated integral.

   (a) $\int_0^2 \int_0^\pi r \sin^2\theta \; d\theta dr$

   > ***Solution:*** Since the region of integration is rectangular and the function is separable in θ and r, we can split it as a product of two integrals
   >
   > $$\int_0^2 r \; dr \cdot \int_0^\pi \sin^2\theta \; d\theta = 2 \cdot \int_0^\pi \frac{1}{2}(1 - \cos 2\theta) \; d\theta = \pi$$

   (b) $\iint_R ye^{-xy} dA$ on $R = [0,2] \times [0,3]$

   > ***Solution:*** Notice that the region is rectangular, so the order of integration doesn't matter. However, we cannot separate this as a product of two integrals, since x and y are mixed variables in the function (we can't write it as a product of two functions $f(x)$ times $g(y)$).
   >
   > We could try to integrate with respect to y first, but that would require integration by parts. It turns out it is easier to start with x instead:
   >
   > $$\int_0^3 \int_0^2 ye^{-xy} dxdy = \int_0^3 [-e^{-xy}]_{x=0}^{x=2} dy = \int_0^3 (-e^{-2y} + 1)dy = \frac{1}{2}e^{-6} + \frac{5}{2}$$

3. Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

> ***Solution:*** The cylinder intersects the xy-plane along the line $x = 4$, so in the first octant, the solid lies below the surface $z = 16 - x^2$ and above the rectangle $[0, 4] \times [0, 5]$ in the xy-plane. Then
>
> $$V = \int_0^5 \int_0^4 (16 - x^2)dxdy = \int_0^5 dy \int_0^4 (16 - x^2)dx = 5\left[16x - \frac{1}{3}x^3\right]_0^4 = \frac{640}{3}$$

4. Use polar coordinates to show that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)}dA = \pi$$

and deduce that $\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$.

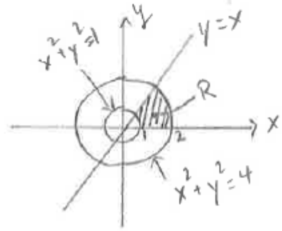
> ***Solution:*** We convert to polar coordinates, remembering that $dx\ dy$ becomes $r\ dr\ d\theta$. For the bounds, notice the original integral covers the entire plane. Thus we have
>
> $$\int_0^{2\pi} \int_0^{\infty} e^{-r^2} r\ dr\ d\theta$$
>
> which now allows us to use u-substitution (which was impossible in the original integral). We take $u = r^2$, so that $du = 2r\ dr$. At the same time we may compute the integral over theta (which is 2π), so we have
>
> $$\pi \int_0^{\infty} e^{-u}du = \pi$$
>
> Now, since the original integrand is a separable function of x and y, i.e. it may be written as a product $e^{-x^2}e^{-y^2}$, and the region of integration is rectangular, our integrals are independent and we may write the original question as
>
> $$\int_{-\infty}^{+\infty} e^{-x^2}dx \cdot \int_{-\infty}^{+\infty} e^{-y^2}dy$$
>
> If we think of y as a dummy variable, we notice that this is the integral we are trying to show equal to $\sqrt{\pi}$, times itself. This proves the desired result, since we have
>
> $$\left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2 = \pi$$
>
> so
>
> $$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

5. Evaluate the given integral.

$$\iint_R \arctan\left(\frac{y}{x}\right)\, dA$$

where $R = \{(x, y) : 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$.

Solution:



Given the geometry of region R, it's best to compute the double integral using polar coordinates.

In polar, we know $dA = r\,dr\,d\theta$ and

$$\arctan\left(\frac{y}{x}\right) = \arctan(\tan\theta) = \theta \quad \left(\text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right).$$

From the picture of the region R, we have $1 \leq r \leq 2$. To find the upper bound for θ, we need to find θ in (I) quad. such that $y = x$. With $y = x$, we have

$r\sin\theta = r\cos\theta \Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$ for θ in (I) quad. So, $0 \leq \theta \leq \frac{\pi}{4}$.

Thus,

$$\iint_R \arctan\left(\frac{y}{x}\right) dA = \int_0^{\pi/4}\int_1^2 \theta\, r\,dr\,d\theta = \int_0^{\pi/4} \frac{1}{2}r^2\theta\Big|_{r=1}^{r=2} d\theta = \int_0^{\pi/4} \frac{3}{2}\theta\,d\theta = \frac{3}{2}\cdot\frac{1}{2}\theta^2\Big|_0^{\pi/4} = \boxed{\frac{3}{64}\pi^2}$$

6. Find the volume of the solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$.

Solution: Let E denote the region given in the question. The volume of the solid is given by

$$\begin{aligned} V &= \iiint_E dV \\ &= \int_0^{2\pi}\int_0^1\int_{r^2}^1 r\,dz\,dr\,d\theta \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^1 zr\big|_{z=r^2}^{z=1}\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 r - r^3\, dr\, d\theta \\
&= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4}\bigg|_0^1\, d\theta \\
&= \int_0^{2\pi} 1/4\, dz \\
&= \frac{\pi}{2}.
\end{aligned}
$$

7. Set up, but do not solve, the integral that gives the volume of the solid region bounded by the paraboloid $z = 3x^2 + 3y^2$ and the cone $z = 4 - \sqrt{x^2 + y^2}$.

Solution: The region of integration will be the interior of the projection of the curve of intersection of $z = 3x^2 + 3y^2$ with $z = 4 - \sqrt{x^2 + y^2}$. Setting the two equal to each other, we have

$$3x^2 + 3y^2 = 4 - \sqrt{x^2 + y^2}$$

and due to the appearence of sums of x^2 and y^2, we choose to convert to polar coordinates. This choice is reinforced by the rotational symmetry of our solid along z-axis. Setting $x = r\cos\theta$ and $y = r\sin\theta$, the equation above becomes

$$3r^2 = 4 - r$$

After rearranging as $3r^2 + r - 4 = 0$, we can factor it

$$(3r + 4)(r - 1) = 0$$

and the only nonnegative solution is $r = 1$. Then our integral should be expressible as an integral over $\theta \in [0, 2\pi]$ and $r \in [0, 1]$. We do top function (cone) minus bottom function (paraboloid), to get

$$\iint_R \left(4 - \sqrt{x^2 + y^2} - (3x^2 + 3y^2)\right) dxdy = \int_0^{2\pi}\int_0^1 (4 - r - 3r^2)r\ dr\ d\theta$$

8. (Optional) Find the maximum value of the function $f(x, y, z) = x + 2y$ on the curve of intersection of the plane $x + y + z = 1$ and the cylinder $y^2 + z^2 = 4$.

Solution: Basically, the problem asks to maximize f subject to two constraints:

$$g(x,y,z) = x + y + z = 1$$
$$h(x,y,z) = y^2 + z^2 = 4$$

We'll do this problem by the method of Lagrange Multipliers: First compute

$$\nabla f(x,y,z) = \langle 1, 2, 0 \rangle$$
$$\nabla g(x,y,z) = \langle 1, 1, 1 \rangle$$
$$\nabla h(x,y,z) = \langle 0, 2y, 2z \rangle$$

We know $\nabla f = \lambda \nabla g + \mu \nabla h$ for some scalars λ, μ. So, along with the two constraints, we have the following system of equations:

$$\begin{cases} 1 & = \lambda \qquad (1) \\ 2 & = \lambda + 2\mu y \qquad (2) \\ 0 & = \lambda + 2\mu z \qquad (3) \\ x + y + z & = 1 \qquad (4) \\ y^2 + z^2 & = 4 \qquad (5) \end{cases}$$

We get $\lambda = 1$ from equation (1). Putting this into equations (2) and (3), we get

$$\begin{cases} 1 & = 2\mu y \\ -1 & = 2\mu z. \end{cases}$$

Adding these two equations, we get $2\mu y + 2\mu z = 0 \implies 2\mu(y + z) = 0$. So, $\mu = 0$ or $y = -z$.

If $\underline{\mu = 0}$, then from equation (2), we have $2 = 1$, a contradiction. So, $\mu \neq 0$.

If $\underline{y = -z}$, then equation (5) yields $2z^2 = 4 \implies z = \pm\sqrt{2}$. So then $y = \mp\sqrt{2}$. And from equation (4), $x = 1 - y - z$. So, $x = 1 - (-\sqrt{2}) - \sqrt{2} = 1$ or $x = 1 - \sqrt{2} - (-\sqrt{2}) = 1$.
So, we obtain the points $(1, -\sqrt{2}, \sqrt{2})$ and $(1, \sqrt{2}, -\sqrt{2})$.

So then,

$$f(1, -\sqrt{2}, \sqrt{2}) = 1 - 2\sqrt{2}$$
$$f(1, \sqrt{2}, -\sqrt{2}) = 1 + 2\sqrt{2}.$$

Thus, the maximum value of f is $1 + 2\sqrt{2}$ on the curve of intersection.

9. (Optional) The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on the ellipse that are nearest and farthest from the origin.

Solution: We need to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ (this corresponds to distance function from origin squared) subject to the two constraints $g = x + y + 2z = 2$ and $h = x^2 + y^2 - z = 0$. Using the gradient equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

we obtain the system

$$\begin{cases} 2x = \lambda + 2\mu x \\ 2y = \lambda + 2\mu y \\ 2z = 2\lambda - \mu \\ x + y + 2z = 2 \\ x^2 + y^2 - z = 0 \end{cases}$$

Solving the equations, we obtain the points $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$. Then we have $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ (which is closest to the origin) and $f(-1, -1, 2) = 6$ (which is farthest from the origin).