CHALLENGE PROBLEMS FOR CALCULUS 2 STUDENTS

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Contents

1. The Gamma Function

Consider the Gamma function given by the integral

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$$

Problem 1.

a) Compute $\Gamma(1)$ and show that for any positive integer n, $\Gamma(n) = (n-1)!$. (Hint: show that Γ satisfies $\Gamma(z+1) = z\Gamma(z)$)

b) For what positive real values of z does the integral make sense? What about negative real values of z? (Hint: think about the convergence of the integral). It turns out that the gamma function is a way to generalize the factorial to non-integer values. But even more, we can extend the factorial to complex numbers! It is defined everywhere except at negative integers, and at zero.

c) Using the fact that $\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$, show that $\Gamma(1/2) = \sqrt{\pi}$.

d) Using the Euler reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z}$$

show the following:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2\pi\sqrt{3}}{3} \qquad \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$$

2. The Log Representation of Arctangent

By considering the composition of inverse functions $\tan(\arctan(x)) = x$ and differentiating implicitly, we found that

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

and so we have a precise integral representation of $\arctan(x)$ for all $x \in \mathbb{R}$ given by

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}dt$$

or, more generally, $\int 1/(1+x^2)dx = \arctan(x) + C$. Many generations of calculus students have crumbled under exam pressure, didn't recognize the antiderivative as arctangent, and tried to compute the integral

$$\int \frac{dx}{1+x^2}$$

by partial fractions. Of course, the polynomial $1 + x^2$ cannot be factored over the real numbers, so they failed. But there is a way to factor the polynomial over the complex numbers, by using $i = \sqrt{-1}$ and writing

$$x^2 + 1 = x^2 - (-1) = (x + \sqrt{-1})(x - \sqrt{-1}) = (x + i)(x - i)$$

Problem 2.

a) Use the method of partial fractions to show that

$$\arctan(x) = \frac{1}{2i}\log\left(\frac{x-i}{x+i}\right) + C$$

b) Show that $C = \pi/2$.

Note: in reality, the situation is more complicated than that, as the logarithm is not the usual $ln(x)$ from calculus of real variables; it is the complex logarithm, which is a multi-valued function. But it turns out that even with all that complication the formula above still holds!

3. The Euler-Mascheroni Constant

Consider the sequence $\{\gamma_1, \gamma_2, \gamma_3, \dots\}$ defined by

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln(n)$$

The first few terms of the sequence are

$$\begin{aligned} \gamma_1 &= 1 \\ \gamma_2 &\approx 0.80685 \\ \gamma_3 &\approx 0.73472 \\ \gamma_4 &\approx 0.69704 \end{aligned}$$

so it would appear that the sequence is decreasing.

Problem 3.

a) Show that γ_n is a decreasing sequence, i.e show that $\gamma_{n+1} - \gamma_n < 0$. (Hint: compute the difference and simplify. Can you argue whether the result is positive or negative?)

b) Argue that the sum is bounded below by zero. (Hint: compare the sum piece with a carefully chosen integral whose value is related to the log piece). We thus have a decreasing sequence that is bounded below, which means it must converge, i.e we have

$$\lim_{n\to\infty} \gamma_n = \gamma$$

The number γ is called the **Euler-Mascheroni constant**, and its value is $\gamma \approx 0.57721566\dots$.

c) Show that γ has the series representation

$$\gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \log\left(\frac{k+1}{k} \right) \right)$$

d) Show that

$$\sum_{k=1}^{n} \frac{1}{k} = \int_0^1 \frac{1-t^n}{1-t} dt \quad \text{and} \quad \ln(n) = \int_0^1 \frac{t^{n-1}-1}{\ln(t)} dt$$

e) Use the result from the previous part together with

$$\ln(n) = \int_0^1 \frac{t^{n-1}-1}{\ln(t)} dt$$

to argue that γ has the integral representation

$$\gamma = \int_0^1 \left(\frac{1}{\ln t} + \frac{1}{1-t} \right) dt = \int_0^1 \left(\frac{1}{\ln(1-t)} + \frac{1}{t} \right) dt$$

Note: the integral representation of $\ln(n)$ mentioned above can be proven using Frullani's Theorem, which states that for a continuously differentiable function $f : [0, \infty] \to \mathbb{R}$, which satisfies $\lim_{x\to\infty} f(x) = 0$, and positive real numbers a and b, we have

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x} dx = f(0) \ln\left(\frac{b}{a} \right)$$

4. Fibonacci Numbers and the Golden Ratio

The Fibonacci are the numbers of the following sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

which is characterized by the fact that every number after the first two is the sum of the two previous two. In other words, we are looking at the sequence F_n given by $F_1 = 0$, $F_2 = 1$, and the recursive formula $F_n = F_{n-1} + F_{n-2}$.

One can take ratio of consecutive Fibonacci numbers and obtain:

$$2/1 = 2, \quad 3/2 = 1.5, \quad 5/3 \approx 1.667, \quad 8/5 = 1.6, \quad 13/8 = 1.625, \quad \dots$$

so it would appear that the ratios converge, hovering around the value 1.6.

Problem 4.

a) Show the following:

$$\phi = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

The number you found is the **Golden Ratio**, and its value is $\phi \approx 1.61803399$ and it appears in the most unexpected places - even in nature!

b) Show that the number $\psi = \frac{1-\sqrt{5}}{2}$ satisfies

$$\psi = 1 - \phi = -\frac{1}{\phi}$$

and that both ϕ and ψ are solutions to the equations

$$x^2 = x + 1 \quad \text{and} \quad x^n = x^{n-1} + x^{n-2}$$

so that the powers of ϕ and ψ satisfy the Fibonacci recurrence.

c) In the previous part you showed that

$$\psi^n = \psi^{n-1} + \psi^{n-2} \quad \text{and} \quad \phi^n = \phi^{n-1} + \phi^{n-2}$$

Define for any $a, b \in \mathbb{R}$ the sequence $C_n = a\phi^n + b\psi^n$ and show that it satisfies

$$C_n = C_{n-1} + C_{n-2}$$

then carefully choose a and b so that $C_0 = 0$, $C_1 = 1$. This means C_n is precisely the Fibonacci sequence.

Congratulations! You just showed that for $n \geq 0$,

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

which is a very surprising result - even though the formula involves $\sqrt{5}$, an irrational number, the formula will always yield an integer!

Problem 5. Prove that

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}$$

5. Trigonometric Sums

Recall the subtraction formula for tangent:

$$\tan(a-b) = \frac{\tan(a) - \tan(b)}{1 + \tan(a)\tan(b)}$$

Problem 6.

a) Show that

$$\tan^{-1} u - \tan^{-1} v = \tan^{-1}\left(\frac{u-v}{1+uv}\right)$$

where $\tan^{-1}$ is the arctangent function.

b) For simplicity, write $a_k = \tan^{-1}(k)$, and show that

$$\frac{1}{1+k+k^2} = \tan(a_{k+1} - a_k)$$

c) Show that

$$\sum_{k=0}^{n} \tan^{-1}\left(\frac{1}{1+k+k^2}\right) = \tan^{-1}(n+1)$$

d) Evaluate the infinite sum

$$\sum_{k=0}^{\infty} \tan^{-1}\left(\frac{1}{1+k+k^2}\right)$$

or argue that it is divergent.

Problem 7.

a) Using similar ideas to the above, prove that

$$\sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{2k^2}\right) = \tan^{-1}\left(\frac{n}{n+1}\right)$$

b) Evaluate the infinite sum

$$\sum_{k=1}^{\infty} \tan^{-1}\left(\frac{1}{2k^2}\right)$$

or argue that it is divergent.

Problem 8.

a) Justify in detail each step in the computation

$$\frac{\tan a}{\cos 2a} = \frac{\tan a(1+\tan^2 a)}{1-\tan^2 a} = \frac{2\tan a - \tan a(1-\tan^2 a)}{1-\tan^2 a} = \tan 2a - \tan a$$

b) Compute the sum

$$\frac{\tan 1}{\cos 2} + \frac{\tan 2}{\cos 4} + \frac{\tan 4}{\cos 8} + \cdots + \frac{\tan 2^n}{\cos 2^{n+1}}$$

6. Some Telescoping Sums

Problem 9. Show that

$$\sum_{k=1}^{n} k!(k^2+k+1) = (n+1)!(n+1)+1$$

Problem 10. Suppose that $a_1, a_2, a_3, \ldots$ is an infinite arithmetic sequence with common difference d (i.e. the difference between consecutive terms $a_{k+1} - a_k = d$).

a) Show that the sequence of partial sums is given by

$$\sum_{k=1}^{n} \frac{1}{a_k a_{k+1}} = \frac{n}{(a_1+nd)a_1}$$

b) Compute the infinite sum

$$\sum_{k=1}^{\infty} \frac{1}{a_k a_{k+1}}$$

or argue that it is divergent.

c) Construct an arithmetic sequence $\{b_n\}$ (i.e give a starting value and common difference) that satisfies

$$\sum_{k=1}^{\infty} \frac{1}{b_k b_{k+1}} = 21$$

Problem 11. Consider the Fibonacci sequence with slightly different starting values, namely $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$, so it is pretty clear that $\lim_{n\to\infty} F_n = \infty$. Note that this yields the same sequence as before, without the starting zero. Sometimes it is useful to think of the recursion as

$$F_n = F_{n+1} - F_{n-1}$$

a) Compute the infinite sum

$$\sum_{n=2}^{\infty} \frac{F_n}{F_{n-1}F_{n+1}}$$

or argue that it is divergent.

b) Compute the infinite sum

$$\sum_{n=2}^{\infty} \frac{1}{F_{n-1}F_{n+1}}$$

or argue that it is divergent.

7. A curious identity: $e^{i\pi} = -1$

Problem 12.

a) Compute the McLaurin series for $\sin(x)$, $\cos(x)$, and e^x, and prove that:

$$e^{ix} = \cos(x) + i\sin(x)$$

where i is the imaginary number $i = \sqrt{-1}$.

b) Prove that for any real number x and any integer n, we have

$$(\cos x + i\sin x)^n = \cos(nx) + i\sin(nx)$$

The formula in part (a) is called **Euler's formula**, and the more general formula from part (b) is called **DeMoivre's formula**. Using DeMovire's, one can derive a multitude of trigonometric identities. For example, when $n = 2$, we have

$$(\cos x + i\sin x)^2 = \cos(2x) + i\sin(2x)$$

Squaring the LHS gives

$$\cos^2 x + 2i\cos x\sin x - \sin^2 x = \cos(2x) + i\sin(2x)$$

and equating the real and imaginary parts yields the familiar double-angle formulas:

$$\begin{aligned}\cos(2x) &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1\\ \sin(2x) &= 2\sin x\cos x\end{aligned}$$

Problem 13. Show that

$$\begin{aligned}\cos(3x) &= 4\cos^3 x - 3\cos x\\ \sin(3x) &= 3\sin x - 4\sin^3 x\end{aligned}$$

Problem 14. Derive formulas for $\cos(5x)$ and $\sin(5x)$.

Problem 15. Prove that $e^{i\pi} = -1$.