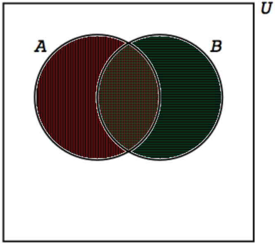
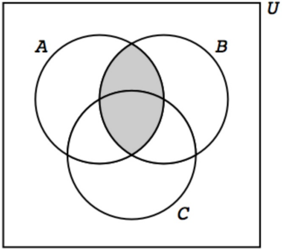
Venn Diagrams

We can visualize subsets of a universal set, and how they interact/overlap, using **Venn diagrams**:



$A \cap B$

$(A' \cup B')'$



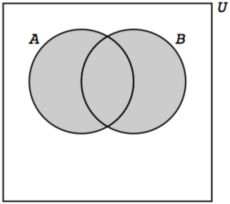
$A \cap B$

$(A' \cup B')'$

On the left, the checkered middle region is $A \cap B$. It is also $(A' \cup B')'$. On the right, the shaded area is $A \cap B$.

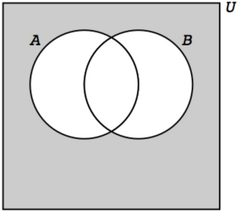
Venn Diagrams

Some more examples:



$A \cup B$

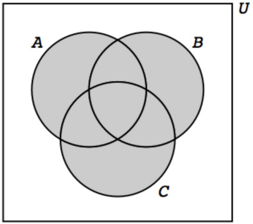
$(A' \cap B')'$



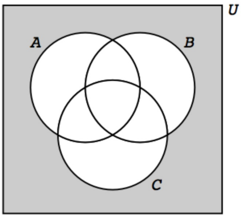
$A' \cap B'$

$(A \cup B)'$

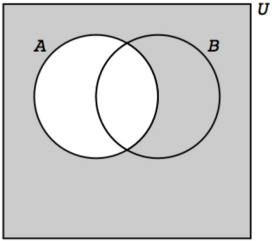
Venn Diagrams



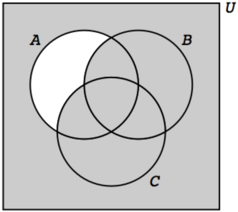
$A \cup B \cup C$
$(A' \cap B' \cap C')'$



$A' \cap B' \cap C'$
$(A \cup B \cup C)'$



A'



$A' \cup B \cup C$
$(A \cap B' \cap C')'$

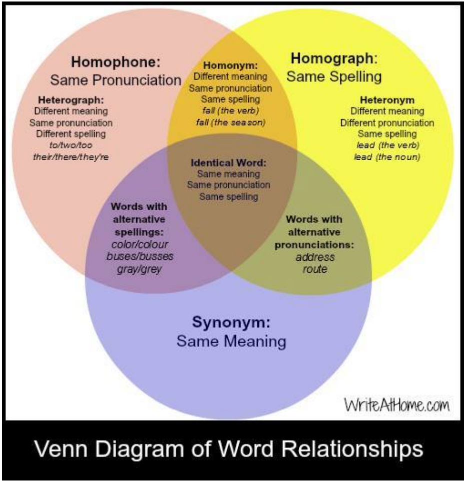
Computing the elements of various subsets

Example: If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$ are subsets of the universal set $U = \{1, 2, 3, \ldots, 10\}$, list the elements of $A' \cup (B \cap C)$.

$A' = \{5, 6, 7, 8, 9, 10\}$
$B \cap C = \{4, 6\}$
$A' \cup (B \cap C) = \{4, 5, 6, 7, 8, 9, 10\}$.

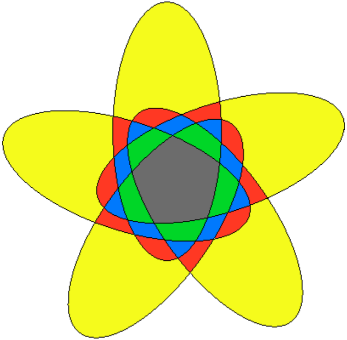
Venn diagrams for presentations

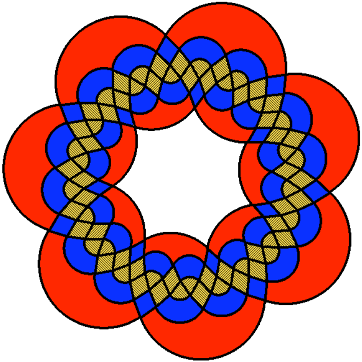
Venn diagrams using two or three sets are often used in presentations.



Venn diagrams for presentations

Venn diagrams of more sets are possible, but tend to be confusing as a presentation tool because of the number of possible interactions between sets. The following diagrams show Venn diagrams for 5 sets (left) and for 7 sets (right).





The Inclusion-Exclusion Principle

For any finite set, S, we let $n(S)$ (read "n of S" or "size of S") denote the number of objects in S.

Example: If $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{5, 6, 7, 8, 9, 10\}$ then $n(A) = 7$ and $n(B) = 6$

The Inclusion Exclusion Principle:

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}.$$

Example: Check that this works for A and B from the example above.

$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, so $n(A \cup B) = 10$.
$A \cap B = \{5, 6, 7\}$, so $n(A \cap B) = 3$.

$$10 = 7 + 6 - 3$$

The Inclusion-Exclusion Principle

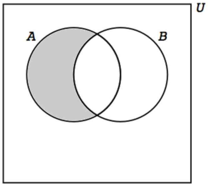
Note: if the sets A and B are disjoint, then $n(A \cap B) = 0$ and so $n(A \cup B) = n(A) + n(B)$.

Formula 1: A and A' are disjoint, so if U is the universe,

$$\boxed{n(A') = n(U) - n(A)}$$

Formula 2: The shaded region below is $A \setminus B$, and $A \setminus B = A \cap B'$ is disjoint from $A \cap B$, so

$$\boxed{n(A \setminus B) = n(A) - n(A \cap B)}$$



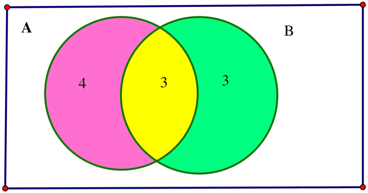
Venn diagrams and Inclusion-Exclusion

We can sometimes use the inclusion-exclusion principle not only as an algebraic tool, but also as a geometric tool to solve a problem by introducing Venn diagrams.

We use a Venn diagram to find the number of elements in each basic region and display how the numbers in each set are distributed among its parts.

Venn diagrams and Inclusion-Exclusion

With $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{5, 6, 7, 8, 9, 10\}$ as above we saw that $n(A \cap B) = 3$, hence the 3 in the region of intersection (the yellow bit).



- Since $n(A) = 7$, Formula 2 says that for the purple region, $n(A \setminus B) = 7 - 3$.
- Similarly, since $n(B) = 6$, Formula 2 says that for the green region, $n(A^c \cap B) = 6 - 3$.
- Note $10 = n(A \cup B) = 4 + 3 + 3$ (or $7 + 6 - 3$ by IE).

More Inclusion-Exclusion

In general, the Inclusion-Exclusion Principle is an equation relating four numbers. Hence if you know three of them, you can find the fourth.

Example: Let X and Y be sets. If $n(X) = 10$, $n(Y) = 12$, and $n(X \cup Y) = 15$, then how many elements are in $X \cap Y$?

IE says $n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$ so

$$15 = 10 + 12 - n(X \cap Y)$$

OR $n(X \cap Y) = 7$.

Also, $X \setminus Y = 3 \quad$ and $\quad Y \setminus X = 5$.

More Inclusion-Exclusion

Example: If $n(A \cup B) = 20$, $n(B) = 10$ and $n(A \cap B) = 5$, how many elements are in A? (solve this using both methods: algebra and Venn diagrams)

Algebraic method:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

so

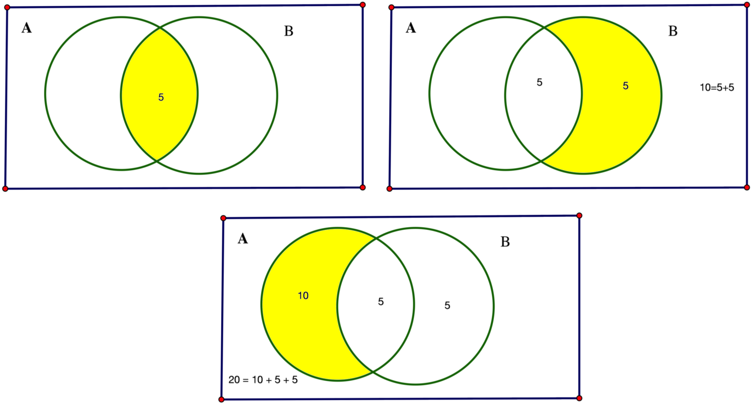
$$20 = n(A) + 10 - 5$$

OR

$$n(A) = 15.$$

More Inclusion-Exclusion

Geometric method:



$$n(A) = 10 + 5 = 15$$

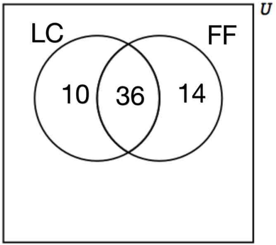
More Inclusion-Exclusion

Example: A survey of a group of students reveals that 60 of them likes at least one of the cereals Frosted Flakes or Lucky Charms. If 50 of them liked Frosted Flakes and 46 of them liked Lucky Charms,

(a) How many of them liked both cereals?

Given: $n(FF) = 50$, $n(LC) = 46$, $n(FF \cup LC) = 60$ so $n(FF \cap LC) = 50 + 46 - 60 = 36$.

(b) Draw a Venn diagram showing the results of the survey.



(c) How many students liked Frosted Flakes but did not like Lucky Charms? 14

More Inclusion-Exclusion

Example: A survey of 70 students revealed that 64 of them liked to learn visually. How many of them did not like to learn visually? By formula Formula 1, $70 - 64 = 6$.

Example: 68 students were interviewed about their music preferences. 66 of them liked at least one of the music types, Rap, Classical and Eighties. How many didn't like any of the above music types?
Given: $n(R \cup C \cup E) = 66$ and $n(U) = 68$, so number who didn't like any of the above music types is

$$n\big((R \cup C \cup E)'\big) = 68 - 66 = 2.$$

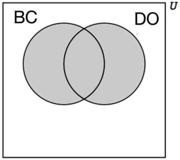
More Inclusion-Exclusion

Example : In a survey of 70 students on Movie preferences, the students were asked whether they liked the movies "The Breakfast Club" and "Ferris Bueller's Day Off". (All students had seen both movies and the only options for answers were like/dislike.) 50 of the students said they liked "The Breakfast Club" and 25 of them said they didn't like "Ferris Bueller's Day Off". All students liked at least one of the movies.

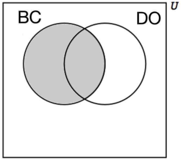
(a) How many students said they liked both movies?

(b) Display the survey results on a Venn diagram.

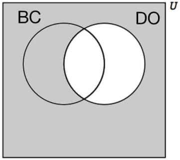
More Inclusion-Exclusion



70 Figure 1



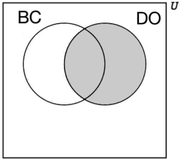
50 Figure 2



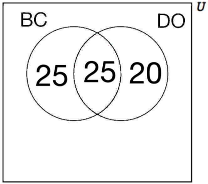
25 Figure 3

Since everyone liked at least one movie, $n(U) = 70$.

From Figure 3 we see that $n(U) - n(DO') = 45$ students did like “Ferris Bueller’s Day Off”.



45

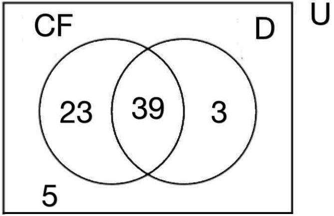


The number of students who liked both movies is 25.

More Inclusion-Exclusion

Example: In a survey of a group of 70 viewers, 62 liked the movie "The Hunger Games: Catching Fire", 42 liked the movie "Divergent" and 39 liked both movies.

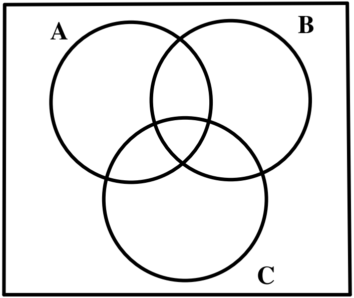
(a) Represent this information in a Venn diagram.



(b) Use the Venn diagram to find how many of those surveyed did not like either movie. 5

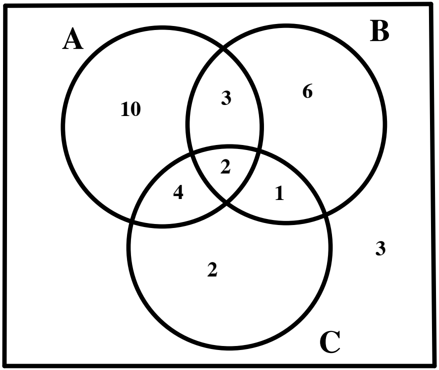
Venn Diagrams of 3 sets

A Venn diagram of 3 sets divides the universal set into 8 non-overlapping regions. We can sometimes use partial information about numbers in some of the regions to derive information about numbers in other regions or other sets.



Venn Diagrams of 3 sets

Example: The following Venn diagram shows the number of elements in each region for the sets A, B and C which are subsets of the universal set U.



Find the number of elements in each of the following sets:

(a) $A \cap B \cap C$ $\quad 2$

(b) B' $\quad 3+2+4+10=19$

(c) $A \cap B$ $\quad 3+2=5$

(d) C $\quad 2+4+2+1=9$

(e) $B \cup C$

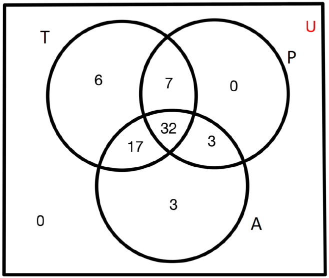
$2+4+2+1+3+6=18$

Venn Diagrams of 3 sets

Example: In a survey of a group of 68 Finite Math students, 62 liked the movie "Terminator", 42 liked the movie "Predator", and 55 liked the movie "Alien". 32 of them liked all 3 movies, 39 of them liked both "Terminator" and "Predator", 35 of them liked both "Predator" and "Alien" and 49 of them liked both "Terminator" and "Alien". Represent this information on a Venn Diagram.

Given information:

$n(U) = 68, \quad n(T) = 62,$
$n(P) = 42, \quad n(A) = 55,$
$n(T \cap P) = 39,$
$n(A \cap P) = 35,$
$n(T \cap A) = 49,$
$n(T \cap P \cap A) = 32.$

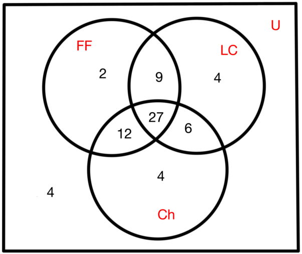


Venn Diagrams of 3 sets

Exercise: In a survey of a group of 68 Finite Math students, 50 said they liked Frosted Flakes, 49 said they liked Cheerios and 46 said they liked Lucky Charms. 27 said they liked all three, 39 said they liked Frosted Flakes and Cheerios, 33 said they liked Cheerios and Lucky Charms and 36 said they liked Frosted Flakes and Lucky Charms. Represent this information on a Venn Diagram. How many didn't like any of the cereals mentioned?

Given information:

$n(U) = 68, \quad n(FF) = 50,$
$n(Ch) = 49, \quad n(LC) = 46,$
$n(FF \cap Ch) = 39,$
$n(LC \cap Ch) = 33,$
$n(FF \cap LC) = 36,$
$n(FF \cap Ch \cap LC) = 27.$

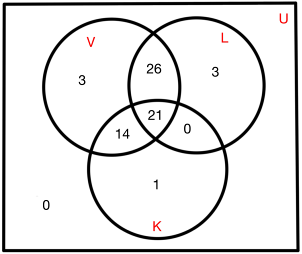


Venn Diagrams of 3 sets

Exercise: The results of a survey of 68 Finite Math students on learning preferences were as follows: 64 liked to learn visually, 50 liked learning through listening and 36 liked learning kinesthetically. 21 liked using all three channels, 47 liked to learn visually and through listening, 35 liked to learn both visually and kinesthetically, 21 liked to learn through listening and kinesthetically. How many preferred only visual learning?

Given information:

$n(U) = 68$, $n(V) = 64$,
$n(L) = 50$, $n(K) = 36$,
$n(V \cap L) = 47$;
$n(V \cap K) = 35$;
$n(L \cap K) = 21$.
$n(V \cap L \cap K) = 21$.

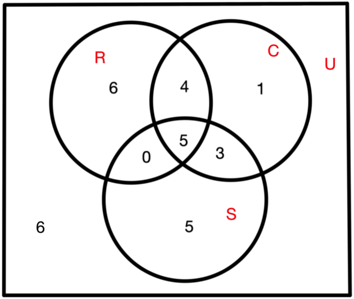


Old exam questions for review

Problem 1: In a group of 30 people, 15 run, 13 swim, 13 cycle, 5 run and swim, 8 cycle and swim, 9 run and cycle, and 5 do all three activities. How many of the 30 people neither run nor cycle?

(a) 8 (b) 10 (c) 9 (d) 12 (e) 11

Given information:
$n(U) = 30,\quad n(R) = 15,$
$n(S) = 13,\quad n(C) = 13,$
$n(R \cap S) = 5$, $n(C \cap S) = 8,$
$n(R \cap C) = 9,$
$n(R \cap C \cap S) = 5.$

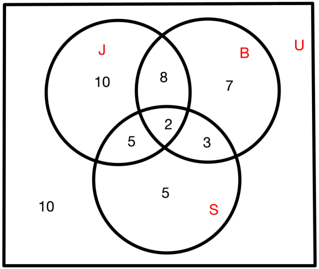


Answer is $6 + 5 = 11$ or (e)

Old exam questions for review

Problem 2: Out of 50 students who exercise regularly, 25 jog, 20 bodybuild and 15 swim. 10 bodybuild and jog, 5 bodybuild and swim, 7 jog and swim and 2 people do all three. How many students do not do any of these activities?

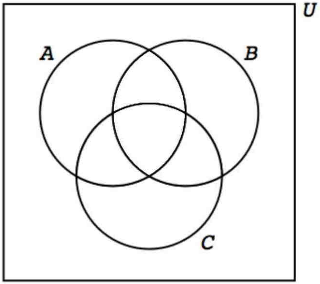
(a) 10 $\quad$ (b) 15 $\quad$ (c) 4 $\quad$ (d) 0 $\quad$ (e) 2



Answer is 10 or (a)

Worked homework problem

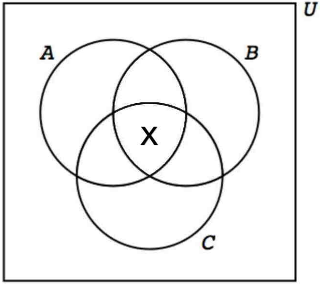
Here is an example of a type of problem from the homework. Given 3 subsets A, B and C of a universal set U, suppose $n(U) = 68$; $n(A \cup B \cup C) = 64$; $n(A) = 50$; $n(B) = 49$; $n(C) = 46$. $n(A \cap B) = 39$; $n(C \cap B) = 33$; $n(A \cap C) = 36$. Fill in the Venn diagram.



We do not know $n(A \cap B \cap C)$ or this would be just another example of earlier problems.

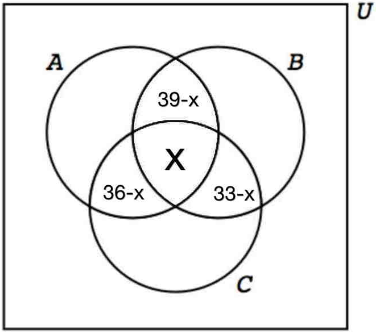
Worked homework problem

Denote $n(A \cap B \cap C)$ by x.



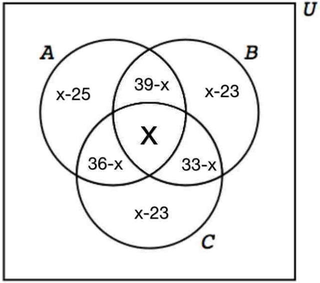
Worked homework problem

Now work out the double intersections. We were given that
$n(A \cap B) = 39, \quad n(C \cap B) = 33, \quad n(A \cap C) = 36.$



Worked homework problem

Now work out the rest of the sets, given that $n(A) = 50$, $n(B) = 49$, $n(C) = 46$.



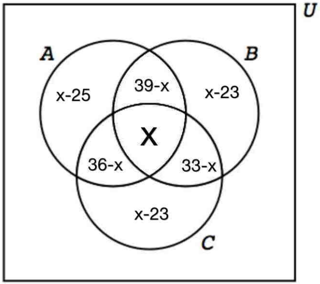
For example, if y denotes the part of A outside of $B \cup C$,

$$50 = y + (39 - x) + (36 - x) + x = y + 75 - x$$

i.e. $50 = y + 75 - x$, so $y = x - 25$. The others are similar.

Worked homework problem

Add all the pieces together and use $n(A \cup B \cup C) = 64$.



$$\begin{aligned} n(A \cup B \cup C) &= x + (39 - x) + (36 - x) + (33 - x) \\ &\quad + (x - 25) + (x - 23) + (x - 23) \\ &= x + (108 - 71) = x + 37 = 64 \end{aligned}$$

so $x = 27$. For the outside, $n(A \cup B \cup C)' = 68 - 64 = 4$.

Worked homework problem

Finally, plugin $x = 27$ to get the values for all the inner regions.