Name: SOLUTIONS Date: 11/30/2017

M20550 Calculus III Tutorial
Worksheet 9

1. Using the Fundamental Theorem of Line Integrals, evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (y^2 \cos(xy^2) + 3x^2)\,\mathbf{i} + (2xy\cos(xy^2) + 2y)\,\mathbf{j}$ is a conservative vector field and C is any curve from the point $(-1, 0)$ to $(1, 0)$.

Solution: Since we know $\mathbf{F}$ is a conservative vector field, $\mathbf{F} = \nabla f$ for some scalar function $f(x, y)$. So, $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}$. Then, by the fundamental theorem of line integral (FTLI), we have $\int_C \nabla f \cdot d\mathbf{r} = f(1, 0) - f(-1, 0)$. So, let's go about and find the potential function $f(x, y)$ of $\mathbf{F}$ first.

We know $\mathbf{F} = \nabla f$, so $\langle y^2 \cos(xy^2) + 3x^2, 2xy\cos(xy^2) + 2y\rangle = \langle f_x, f_y\rangle$. Thus, we have

$$f_x = y^2 \cos(xy^2) + 3x^2 \tag{1}$$
$$f_y = 2xy\cos(xy^2) + 2y \tag{2}$$

Using equation (1), we have $f = \int (y^2 \cos(xy^2) + 3x^2)\,dx = \sin(xy^2) + x^3 + g(y)$. Now, we need to find $g(y)$ to complete f.

With $f = \sin(xy^2) + x^3 + g(y)$, we compute $f_y = 2xy\cos(xy^2) + g'(y)$. Then from equation (2) above, we must have

$$2xy\cos(xy^2) + g'(y) = 2xy\cos(xy^2) + 2y \implies g'(y) = 2y \implies g(y) = y^2 + C.$$

We only need *a* potential function to apply FTLI, so we can pick $C = 0$. So, a potential function $f(x, y)$ of the vector field $\mathbf{F}$ is

$$f(x, y) = \sin(xy^2) + x^3 + y^2.$$

Finally,

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} &\overset{\text{FTLI}}{=} f(1, 0) - f(-1, 0)\\ &= (\sin 0 + 1^3 + 0^2) - (\sin 0 + (-1)^3 + 0^2)\\ &= 2.\end{aligned}$$

2. Use Green's Theorem to evaluate

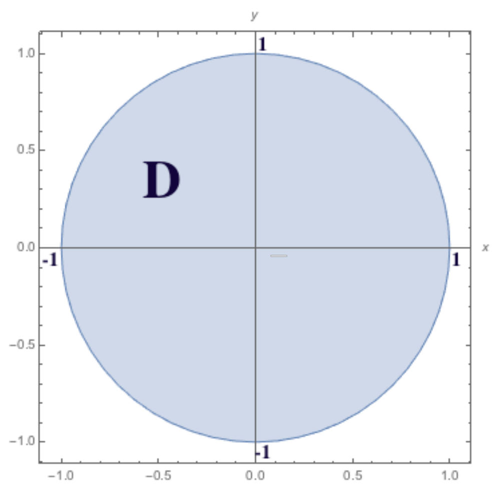
$$\int_C \left(-\frac{y^3}{3} + \sin x\right)\,dx + \left(\frac{x^3}{3} + y\right)\,dy,$$

where C is the circle of radius 1 centered at $(0, 0)$ oriented counterclockwise when viewed from above.

Solution: Let D be the region enclosed by the unit circle C in this problem. By Green's Theorem, we have

$$\int_C \left(-\frac{y^3}{3} + \sin x\right)\,dx + \left(\frac{x^3}{3} + y\right)\,dy = \iint_D x^2 - (-y^2)\,dA.$$

$\left(\text{Here, we have } P = -\frac{y^3}{3} + \sin x \text{ and } Q = \frac{x^3}{3} + y, \text{ so } \frac{\partial P}{\partial y} = -y^2 \text{ and } \frac{\partial Q}{\partial x} = x^2.\right)$
So, instead of computing the line integral $\int_C \left(-\frac{y^3}{3} + \sin x\right)\,dx + \left(\frac{x^3}{3} + y\right)\,dy$, we are going to compute the double integral $\iint_D x^2 + y^2\,dA$, where D is the unit disk as shown below.



Using polar coordinates,

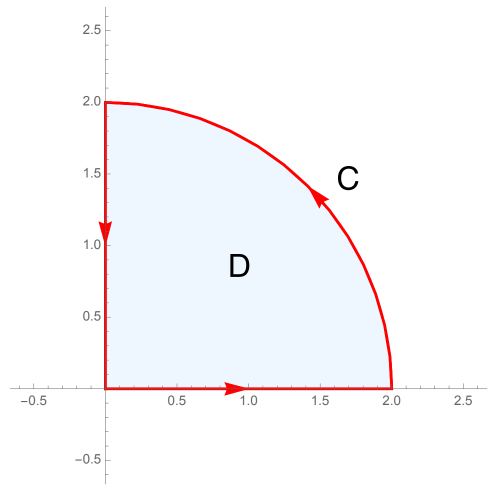
$$\iint_D x^2 + y^2\,dA = \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = 2\pi\left(\frac{1}{4}\right) = \frac{\pi}{2}.$$

Hence,

$$\int_C \left(-\frac{y^3}{3} + \sin x\right)\,dx + \left(\frac{x^3}{3} + y\right)\,dy = \frac{\pi}{2}.$$

3. A particle starts at the origin $(0,0)$, moves along the x-axis to $(2,0)$, then along the curve $y = \sqrt{4 - x^2}$ to the point $(0,2)$, and then along the y-axis back to the origin. Find the work done on this particle by the force field $\mathbf{F}(x,y) = y^2\,\mathbf{i} + 2x(y+1)\,\mathbf{j}$.

Solution: First we note that the curve C (drawn below) is a positively oriented, piecewise-smooth, simple closed curve in the plane. Let D be the region bounded by C.



The components of the vector field, $P = y^2$ and $Q = 2x(y+1)$, have continuous partial derivatives on an open region containing D (namely, all of $\mathbb{R}^2$). We may apply Green's Theorem:

$$\int_C P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA.$$

Note that we have $\frac{\partial Q}{\partial x} = 2(y+1) = 2y+2$ and $\frac{\partial P}{\partial y} = 2y$. Finally, we compute the work done on the particle by the force field.

$$\begin{aligned} W = \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C y^2\,dx + 2x(y+1)\,dy \\ &\overset{Green}{=} \int\int_D (2y+2-2y)\,dA \\ &= 2\int\int_D dA \end{aligned}$$

Note that this is just twice the area of the region D. We may compute this as a double integral using polar coordinates $\left(W = 2\int_0^{\pi/2}\int_0^2 r\,dr\,d\theta\right)$ or by using the formula for the area of a circle. Thus,

$$W = 2(\text{Area of } D) = 2\left(\frac{\pi\cdot 2^2}{4}\right) = 2\pi.$$

4. (a) Compute div $\mathbf{F}$, where $\mathbf{F} = \langle e^y, zy, xy^2 \rangle$.

   (b) Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle xyz, -y^2z, yz^2 \rangle$? Why?

Solution: (a) div $\mathbf{F} = \dfrac{\partial}{\partial x}(e^y) + \dfrac{\partial}{\partial y}(zy) + \dfrac{\partial}{\partial z}(xy^2) = 0 + z + 0 = z$

(b) For this problem, we need to remember the fact

$$\text{div curl } \mathbf{F} = 0 \quad \text{for any vector field } \mathbf{F}.$$

If there is a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that curl $\mathbf{G} = \langle xyz, -y^2z, yz^2 \rangle$ then by the fact above, $\mathbf{G}$ would satisfy the rule

$$\text{div}\,\text{curl}\,\mathbf{G} = 0 \quad \text{or} \quad \text{div}\left\langle xyz, -y^2z, yz^2 \right\rangle = 0.$$

But,

$$\text{div}\left\langle xyz, -y^2z, yz^2 \right\rangle = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(-y^2z) + \frac{\partial}{\partial z}(yz^2) = yz - 2yz + 2yz = yz \neq 0.$$

Thus, there is no such $\mathbf{G}$.

5. Write an equation of the tangent plane to the parametric surface

$$x = u^2 + 1, \quad y = v^3 + 1, \quad z = u + v,$$

at the point $(5, 2, 3)$.

Solution: The surface is given by the vector equation $\mathbf{r}(u, v) = \langle u^2 + 1, v^3 + 1, u + v \rangle$. So, a normal vector to the tangent plane at $(5, 2, 3)$ is given by $\mathbf{r}_u \times \mathbf{r}_v$ at the point $(5, 2, 3)$.

First, $\mathbf{r}_u = \langle 2u, 0, 1 \rangle$ and $\mathbf{r}_v = \langle 0, 3v^2, 1 \rangle$. Now, we want to find (u, v) corresponds to the point $(x, y, z) = (5, 2, 3)$. So, we want to find (u, v) that satisfies:

$$5 = u^2 + 1, \quad 2 = v^3 + 1, \quad 3 = u + v.$$

$2 = v^3 + 1$ implies $v = 1$. So, $3 = u + v \implies 3 = u + 1 \implies u = 2$. And we see that $u = 2$ satisfies the equation $5 = u^2 + 1$. Thus, $(u, v) = (2, 1)$ gives the points $(x, y, z) = (5, 2, 3)$.

Now, with $u = 2$ and $v = 1$, we have $\mathbf{r}_u = \langle 4, 0, 1 \rangle$ and $\mathbf{r}_v = \langle 0, 3, 1 \rangle$. So, $\mathbf{r}_u \times \mathbf{r}_v = \langle 4, 0, 1 \rangle \times \langle 0, 3, 1 \rangle = \langle -3, -4, 12 \rangle$. So, $\langle -3, -4, 12 \rangle$ can be chosen as a normal vector to the tangent plane at the point $(5,2,3)$. And so, an equation of this tangent plane is given by

$$\begin{aligned}\langle -3,-4,12\rangle \bullet \langle x,y,z\rangle &= \langle -3,-4,12\rangle \bullet \langle 5,2,3\rangle \\ \implies -3x-4y+12z &= 13.\end{aligned}$$

6. Write the integral that computes the surface area of the surface S parametrized by $\mathbf{r}(u,v)=\langle u^2\cos v, u^2\sin v, v\rangle$, where $0\le u\le 1$ and $0\le v\le \pi$.

Solution: The area of the surface S is given by

$$\text{Area}(S)=\iint_D |\mathbf{r}_u\times\mathbf{r}_v|\;dA.$$

where D is the region given by $0\le u\le 1$ and $0\le v\le\pi$. With $\mathbf{r}(u,v)=\langle u^2\cos v, u^2\sin v, v\rangle$, we have $\mathbf{r}_u=\langle 2u\cos v, 2u\sin v, 0\rangle$ and $\mathbf{r}_v=\langle -u^2\sin v, u^2\cos v, 1\rangle$. Then

$$\mathbf{r}_u\times\mathbf{r}_v=\langle 2u\cos v,2u\sin v,0\rangle\times\left\langle -u^2\sin v,u^2\cos v,1\right\rangle=\left\langle 2u\sin v,-2u\cos v,2u^3\right\rangle.$$

So,

$$\begin{aligned}|\mathbf{r}_u\times\mathbf{r}_v| &= \left|\left\langle 2u\sin v,-2u\cos v,2u^3\right\rangle\right| \\ &= \sqrt{(2u\sin v)^2+(-2u\cos v)^2+(2u^3)^2} \\ &= \sqrt{4u^2+4u^6} \\ &= \sqrt{4u^2(1+u^4)}=2u\sqrt{1+u^4}.\end{aligned}$$

Finally,

$$\text{Area}(S)=\iint_D|\mathbf{r}_u\times\mathbf{r}_v|\;dA=\iint_D 2u\sqrt{1+u^4}\,dA=\int_0^1\int_0^\pi 2u\sqrt{1+u^4}\,dv\,du.$$

7. Compute the surface integral $\displaystyle\iint_S (x+y+z)\;dS$, where S is a surface given by $\mathbf{r}(u,v)=\langle u+v,u-v,1+2u+v\rangle$ and $0\le u\le 2,\ 0\le v\le 1$.

Solution: First, we know

$$\iint_S (x+y+z)\;dS=\iint_D\Big[(u+v)+(u-v)+(1+2u+v)\Big]|\mathbf{r}_u\times\mathbf{r}_v|\,dA,$$

where D is the domain of the parameters $u,\ v$ given by $0 \leq u \leq 2,\ 0 \leq v \leq 1$.

We have $\mathbf{r}_u = \langle 1, 1, 2 \rangle$ and $\mathbf{r}_v = \langle 1, -1, 1 \rangle$. Then,
$\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 1, 2 \rangle \times \langle 1, -1, 1 \rangle = \langle 3, 1, -2 \rangle$. So,

$$|\mathbf{r}_u \times \mathbf{r}_v| = |\,\langle 3, 1, -2 \rangle\,| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}.$$

Thus,

$$\begin{aligned}\iint_S (x + y + z)\ dS &= \int_0^1 \int_0^2 (4u + v + 1)\sqrt{14}\ du\ dv \\ &= 11\sqrt{14}.\end{aligned}$$