The Binomial Theorem Revisited

How does this pattern continue?

- $(x+y)^0 = 1$
- $(x+y)^1 = x+y$
- $(x+y)^2 = x^2+2xy+y^2$
- $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$
- $(x+y)^4 = ???$

The **Binomial Theorem** says that for any positive integer n and any two real numbers x and y, we can expand

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Q: What happens if we only look at coefficients of $x^a y^b$?

The Binomial Theorem Revisited

Exponent of (x+y)	Coefficients
n = 0	1
n = 1	1 1
n = 2	1 2 1
n = 3	1 3 3 1
n = 4	1 4 6 4 1
n = 5	1 5 10 10 5 1

It turns out that

$$\begin{aligned}(x+y)^4 &= \mathbf{1}x^4y^0 + \mathbf{4}x^3y^1 + \mathbf{6}x^2y^2 + \mathbf{4}x^1y^3 + \mathbf{1}x^0y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

The trianglular array of numbers above is called **Pascal's triangle**. The numbers themselves are the combination numbers $\binom{n}{k}$.

The Binomial Theorem Revisited

Example: Expand $(x+y)^5$ using the Binomial Theorem.

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

Example: If I fully multiply out $(x+y)^{11}$, what's the coefficient of xy^{10}? What about x^7y^4?

From the binomial theorem, the coefficient of xy^{10} is $\binom{11}{10} = \binom{11}{1} = 11$.

Similarly, for x^7y^4 we have $\binom{11}{4} = \dfrac{11!}{4!7!} = 330.$.

Summary of Counting Techniques

We studied a number of counting techniques, and when tackling a counting problem, we often have to use a combination of these principles. The counting principles we have studied are:

- Inclusion-Exclusion Principle
  $$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
- Complement Principle
  $$n(A') = n(U) - n(A)$$
- Multiplication Principle
- Addition Principle (disjoint outcomes)
- Permutations $P(n,k) = \frac{n!}{(n-k)!}$
- Rearrangements with multiplicities $\frac{n!}{r_1! \cdot r_2! \ldots r_k!}$
- Combinations $C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$

Problem Solving Strategy

Often times, a counting problem may have multiple levels/steps, requiring repeated application of one or several principles.

When counting the number of objects in a set, it helps to think of the steps you might take to actually construct such objects (e.g. how do you construct a binary sequence?). It also helps to keep the technique of "overcounting" and the complement principle in mind:

- *What would happen if you remove/change the imposed conditions on the objects?* (e.g. instead of binary strings, what if you used characters from the alphabet? How do you count those?)

Problem Solving Strategy

Example: An experiment consists of rolling a 20 sided die three times. The number on each roll is recorded and written down in the order in which they are observed. How many possible ordered triples of numbers can result from the experiment? (Hint: the triple $(17, 10, 3)$ is not the same result as the triple $(3, 10, 17)$.)

There are 20 ways each roll can come up. The order is important so we can obtain $20 \cdot 20 \cdot 20 = 20^3 = 8000$ triples.

Problem Solving Strategy

Example: (Hoosier Lottery) When you buy a Powerball ticket, you select 5 different white numbers from among the numbers 1 through 59 (order of selection does not matter), and one red number from among the numbers 1 through 35. How many different Powerball tickets can you buy?

First select 5 **distinct** white numbers. Since the order doesn't matter, you have $C(59, 5) = \frac{59!}{5! \cdot 54!} = 5,006,386$ ways.

Then you can pick the red number in $C(35, 1) = 35$ ways.

The total number of tickets is

$$C(59, 5) \cdot C(35, 1) = 5,006,386 \times 35 = 175,223,510.$$

Marble Selection Model

Example: A bag contains 10 different red marbles and 5 different white marbles. You pick 4 marbles from the bag.

(a) How many (different) samples of size 4 are possible?

The order does not matter, neither does the color. We are simply selecting 4 elements from a total of $10 + 5 = 15$ elements. This gives $C(15, 4) = 1,365$ different samples.

(b) How many samples of size 4 consist entirely of red marbles?

The order does not matter. We are selecting 4 elements from a set of 10 (red) elements. This gives $C(10, 4) = 210$ samples.

Marble Selection Model

Example: Distinguishable marbles, 10 red, 5 white.

(c) How many samples have 2 red and 2 white marbles?

We can select 2 labeled red marbles in $C(10, 2)$ ways, and 2 labeled white marbles in $C(5, 2)$ ways. Multiplication principle gives $C(10, 2) \cdot C(5, 2) = 45 \cdot 10 = 450$ samples.

(d) How many samples of size 4 have exactly 3 red marbles?

We can select 3 numbered red marbles in $C(10, 3)$ ways. The last marble MUST be white, and it can be selected in $C(5, 1)$ ways. We get $C(10, 3) \cdot C(5, 1) = 120 \cdot 5 = 600$.

Marble Selection Model

Example: Distinguishable marbles, 10 red, 5 white.

(e) How many samples of size 4 have at least 3 red?

At least 3 red means we want samples with exactly 3 red, and samples with exactly 4 red marbles (this means addition principle!).

From the last part, there are 600 ways to select samples with exactly 3 red marbles.

We can select 4 red marbles in $C(10, 4)$ way, and 0 white marbles in $C(5, 0) = 1$ ways. This means we have $C(10, 4) \cdot C(5, 0) = 210 \cdot 1 = 210$ ways to do so.

By the addition principle, we have $600 + 210 = 810$ samples (of size 4) with at least 3 red marbles.

Marble Selection Model

Example: Distinguishable marbles, 10 red, 5 white.

(f) How many samples of size 4 contain at least one red marble?
We want (# with exactly 1) + (# with exactly 2) + (# with exactly 3) + (# with exactly 4).

This is

$$\binom{10}{1}\binom{5}{3} + \binom{10}{2}\binom{5}{2} + \binom{10}{3}\binom{5}{1} + \binom{10}{4}\binom{5}{0}$$

which is very annoying to compute. It is much easier to do
(all samples) - (# with zero red)

This is equal to

$$\binom{15}{4} - \binom{10}{0}\binom{5}{4} = 1,365 - 5 = 1360$$

Card Model

Example: Recall that a standard deck of cards has 52 cards. The cards can be classified according to suits (4), rank (13), or colors (2). A poker hand consists of a sample of size 5 drawn from the deck. *Poker problems are often like marble problems, with a hitch or two.*

(a) How many poker hands consist of 2 Aces and 3 Kings?

We can pick aces in $\binom{4}{2}$ ways, and kings in $\binom{4}{3}$ ways.

$$\binom{4}{2} \cdot \binom{4}{3} = 6 \cdot 4 = 24.$$

Card Model

(b) How many poker hands consist of 2 Aces, 2 Kings and a fifth card of a different rank?

You can pick the 2 aces and 2 kings in $\binom{4}{2} \cdot \binom{4}{2} = 6 \cdot 6 = 36$ ways. There are $52 - 8 = 44$ choices left for the card that isn't A or K. This gives $36 \cdot 44 = 1,584$ hands.

(c) How many poker hands have three cards of one rank and two cards of another (aka Full House)?

There are 13 different ranks, and $\binom{4}{3}$ ways to pick 3 cards of one rank.
There are now only 12 choices for the second rank, and then $\binom{4}{2}$ ways to pick 2 cards of that rank. Hence we get

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6 = 3,744.$$

Note: if you decide to pick 2 cards of the first rank instead, the answer doesn't change: $13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{3} = 3,744$.

Card Model

(d) A royal flush is a hand consisting of an Ace, King, Queen, Jack and 10, where all cards are from the same suit. How many royal flushes are possible?

There are 4 suits to choose from, and for each suit, there is only ONE royal flush possible. Hence there are a total of only $4 \cdot 1 = 4$ ways to get a royal flush. It is extremely rare, which is why it is considered the best possible hand.

(e) A flush is a hand consisting of five cards from the same suit (not necessarily consecutive). How many different flushes are possible?

There are 4 suits, and for each of those, there are $\binom{13}{5}$ ways to get all cards of the same suit. This gives a total of $\binom{4}{1} \cdot \binom{13}{5} = 4 \cdot 1,287 = 5,148$ flushes.

Coin Flipping Model

Another useful model is repeatedly flipping a coin. This is especially useful for counting outcomes of a larger experiment involves several repetitions, each with two outcomes.

We will explore probabilities for experiments of this type later when we study the Binomial distribution. We have already used this model in taxi cab geometry.

Example: If I flip a coin 20 times, I get a sequence of Heads (H) and Tails (T).

(a) How many different sequences of heads and tails are possible?

There are 2 choices (H or T) for the first coin, 2 for the second, etc. This gives $\underbrace{2 \cdot 2 \cdots 2}_{\text{20 times}} = 2^{20} = 1,048,576$.

Coin Flipping Model

(b) How may different sequences of heads and tails have exactly 5 heads after 20 flips?

Now we want to keep track of how many H/T are there are in our sequence. This problem is similar to the taxi cab problem. There are 20 positions which need to be filled with either H or T, and 5 of those 20 must be H.

The # of sequences with exactly 5 heads is $C(20, 5) = 15,504$.

Coin Flipping Model

(c) How many different sequences have at most 2 heads?

We can have 0, 1, or 2 heads, but no more.

$$\binom{20}{0} + \binom{20}{1} + \binom{20}{2} = 1 + 20 + 190 = 211$$

(d) How many different sequences have at least three heads?

We could use

$$\binom{20}{3} + \binom{20}{4} + \binom{20}{5} + \cdots + \binom{20}{19} + \binom{20}{20}$$

but which is very annoying to compute. It is better to do

$$2^{20} - \left[\binom{20}{0} + \binom{20}{1} + \binom{20}{2}\right] = 1,048,576 - 211 = 1,048,365$$

How many Ginos Pizzas are there?

In its "Build your own pizza" option, Ginos offers a choice of 3 crusts, 6 sauces, 8 cheeses, 9 meats, 20 veggies and 6 finishes. How many different pizzas could you make, by choosing a crust, then a sauce (or perhaps no sauce), then some selection of cheeses, then some selection of meats, then some selection of veggies, then some selection of finishes?

Think about assembling the order, item-by-item. First select the crust (3 options), then add the sauce (7 options, because no sauce is an option). So far we have $3 \cdot 7 = 21$.

You have 2^8 cheese options, then 2^9 meat options, then 2^{20} veggie options, and finally 2^6 finishing options.

The total number of pizzas is
$21 \cdot 2^8 \cdot 2^9 \cdot 2^{20} \cdot 2^6 \approx 1.8 \times 10^{14}$.

How many Ginos Pizzas are there?

More realistically: how many different pizzas could you make, by choosing a crust, at most one sauce, at most three cheeses, at most two meats, at most five veggies, and at most two finishes?

First you select the crust (3 options), then you add the sauce (7 options). So far we have $3 \cdot 7 = 21$.

Cheese: $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} = 1 + 8 + 28 + 56 = 93$ options.

Meat: $\binom{9}{0} + \binom{9}{1} + \binom{9}{2} = 1 + 9 + 36 = 46$ options.

Veggie: $\binom{20}{0} + \binom{20}{1} + \binom{20}{2} + \binom{20}{3} + \binom{20}{4} + \binom{20}{5} = 21,700$.

Finishes: $\binom{6}{0} + \binom{6}{1} + \binom{6}{2} = 1 + 6 + 15 = 22$ options.

Total: $21 \cdot 93 \cdot 46 \cdot 21,700 \cdot 22 = 42,888,661,200 = \text{A LOT}$.

How many Ginos Pizzas are there?

More realistically still: choose a crust, at most one sauce, at most three cheeses, at most two meats, at most five veggies, and at most two finishes. HOWEVER you must choose EITHER a sauce OR at least one cheese (can't go both sauce-less and cheese-less), AND you must choose at least one meat OR at least one veggie (can't go topping-less)?

Sauce (7) and cheese (93): $7 \cdot 93 - 1 = 650$ options (-1 for the sauce-less cheese-less option that we shouldn't count).

Meat(45) and Veggie(21,700): $45 \cdot 21,700 - 1 = 976,499$.

Crust: 3 options.

Finishings: 22 options.

Grand Total: $3 \cdot 650 \cdot 976,499 \cdot 22 = 41,891,807,100$.

One pizza a day, for every currently enrolled ND student, for the next 13,000+ years!

Extra Problems

Example: (a) How many different words (including nonsense) can you make by rearranging the letters of the word EFFERVESCENCE?

The multiplicities for each different letter are

$$E \mapsto 5 \quad F \mapsto 2 \quad R \mapsto 1 \quad V \mapsto 1 \quad S \mapsto 1 \quad C \mapsto 2 \quad N \mapsto 1$$

We have $5 + 2 + 1 + 1 + 1 + 2 + 1 = 13$ letters total (with some repeating). The number of words is given by:

$$\frac{13!}{2! \cdot 5! \cdot 1! \cdot 1! \cdot 1! \cdot 2! \cdot 1!} = 12,972,960$$

Extra Problems

(b) How many different 4 letter words (including nonsense) can be made from the letters of the word EFFERVESCENCE, if letters cannot be repeated?

There are 7 distinct letters so if repetitions are not permitted the answer is $P(7,4) = \frac{7!}{(7-4)!} = 840$.

(c) What if letters can be repeated?

There are $7 \cdot 7 \cdot 7 \cdot 7 = 7^4 = 2401$ words.

Note: Do not confuse this with the following MUCH harder problem: given 13 tiles with the letters in EFFERVESCENCE, how many 4 letter words can be produced by adjoining 2 tiles?

Extra Problems

Example: The ND Model UN Club has 20 members: 5 seniors, 4 juniors, 2 sophomores, and 9 freshmen.

(a) In how many ways can the club select a president, a secretary and a treasurer if every member is eligible for each position and no member can hold two positions?

20 members, 3 positions to fill, so $P(20, 3)$. Note you are selecting an ordered subset of 3 distinct elements (the order determines who gets what position).

(b) In how many ways can the club choose a group of 5 members to attend the next Model UN meeting in Washington.

$C(20, 5)$ ways (the subsets of all the members which have 5 elements, but the order isn't important).

Extra Problems

Example: The ND Model UN Club has 20 members: 5 seniors, 4 juniors, 2 sophomores, and 9 freshmen.

(c) In how many ways can the club choose a group of 5 members to attend the next Model UN meeting in Washington if all members of the group must be freshmen?

$C(9, 5)$ ways, since you now must select your subset from the set of 9 freshmen.

Extra Problems

Example: The ND Model UN Club has 20 members: 5 seniors, 4 juniors, 2 sophomores, and 9 freshmen.

(d) In how many ways can the group of 5 be chosen if there must be at least one member from each class?

There are 5 ways to select a senior, 4 ways for a junior, 2 ways for a sophomore, and 9 ways for a freshman. This gives $5 \cdot 4 \cdot 2 \cdot 9 = 360$ ways to select a subset with 4 elements containing one member of each class.

There are $20 - 4 = 16$ members left, and you may choose any of them to complete the group. This gives a total of $\frac{360 \cdot 16}{2} = 2,880$ possible groups.

Q: Why did we divide by 2? Two members are from the same year, and we don't care about the order.

Extra Problems

Method 2:

Because there are 5 members in the subset and 4 classes, exactly one class occurs twice.

If we have 2 seniors, these can be selected in $\binom{5}{2}$ ways and the rest filled out with 1 junior, 1 sophomore, and 1 freshman: $\binom{5}{2} \cdot 4 \cdot 2 \cdot 9 = 720$ ways.

If 2 juniors, selected them in $\binom{4}{2}$ ways, and fill the rest with 1 senior, 1 sophomore and 1 freshman:$5 \cdot \binom{4}{2} \cdot 2 \cdot 9 = 540$.

If 2 sophomores, $5 \cdot 4 \cdot \binom{2}{2} \cdot 9 = 180$.

If 2 freshmen, $5 \cdot 4 \cdot 2 \cdot \binom{9}{2} = 1,440$.

Grand Total: $720 + 540 + 180 + 1,440 = 2,880$, as before.

Extra Problems

Example: Harry Potter's closet contains 12 numbered brooms: 8 are Comet 260's (numbered 1 - 8) and 4 are Nimbus 2000's (Numbered 9-12). Harry, Ron, George and Fred want to sneak out for a game of Quidditch in the middle of the night. They don't want to turn on the light in case Snape catches them. They reach in the closet and pull out a sample of 4 brooms.

Extra Problems

Example: 8 Comet brooms (numbered 1 - 8) and 4 are Nimbus brooms (Numbered 9-12), take out a sample of 4 brooms.

(a) How many different samples are possible?

This is not a well-defined question. Do you want to know how many different sets of brooms you can get or do you want to know how many ways there are if we keep track of which broom Harry gets, which one Ron gets, and so on.

In other words, do you want **subsets** or **ordered subsets**?

For subsets, the answers is $C(12, 4) = 495$, while for ordered subsets (keeping track who gets what) the answer is $P(12, 4) = 11,880$.

Extra Problems

Example: 8 Comet brooms (numbered 1 - 8) and 4 are Nimbus brooms (Numbered 9-12), take out a sample of 4 brooms.

(b) How many samples have only Comet 260's in them?
Can be $C(8, 4)$ or $P(8, 4)$.

(c) How many samples have exactly one Comet 260 in them?

The unordered version: $\binom{8}{1} \cdot \binom{4}{3} = 8 \cdot 4 = 32$.

The ordered version: take the unordered set of 4 elements, and order them (there are $4! = 24$ ways to do this). This gives $32 \cdot 24 = 768$.

Extra Problems

Example: 8 Comet brooms (numbered 1 - 8) and 4 are Nimbus brooms (Numbered 9-12), take out a sample of 4 brooms.

(d) How many samples have at least 3 Comet 260's?

We need to count the samples that have exactly 3, and the samples that have exactly 4, then add the two answers.

For exactly 3 Comet 260's we have:

Unordered: $\binom{8}{3} \cdot \binom{4}{1}$ Ordered: $4! \cdot \binom{8}{3} \cdot \binom{4}{1}$

For exactly 4 Comet 260's we have:

Unordered: $\binom{8}{4} \cdot \binom{4}{0}$ Ordered: $4! \cdot \binom{8}{4} \cdot \binom{4}{0}$

The number of samples with at least 3 Comet 260's is:
Unordered: $\binom{8}{3} \cdot \binom{4}{1} + \binom{8}{4} \cdot \binom{4}{0}$ Ordered: multiply by 4!