CALCULUS 2
EXAM 1 PRACTICE PROBLEMS

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Contents

1. Inverse Functions

Problem 1. Consider the function $g(x) = \sqrt{4x+4}$

a) Is g a one-to-one function?
b) What is the domain of g?
c) Does g^{-1} exist?
d) What is the domain of g^{-1}?
e) What is the range of g^{-1}?
f) Compute $g^{-1}(4)$.

Problem 2. For the function $f(x) = x^3 + 1$, find $f^{-1}(9)$ and $f^{-1}(28)$.

Problem 3. Find a formula for $f^{-1}(x)$ for the specified functions. Verify your result by checking that $f(f^{-1}(x)) = x$.

a) $f(x) = \dfrac{2x+1}{x-3}$.
b) $f(x) = \sqrt{x-2}$.
c) $f(x) = \dfrac{1}{x-1}$.

Problem 4. Determine $(f^{-1})'(a)$ for the following functions at the specified a value:

a) $f(x) = \sqrt{4x+4}$ at $a = 4$.
b) $f(x) = x^3 + 1$ at $a = 28$.
c) $f(x) = \sqrt{x^3 + 4x + 4}$ at $a = 3$.
d) $f(x) = x^3 + 4x + 6x + 5$ at $a = 5$.
e) $f(x) = \sqrt{x-2}$ at $a = 2$.

Problem 5. Consider a one-to-one function h which satisfies the following:

$$h(10) = 21 \qquad h'(10) = 2 \qquad h^{-1}(10) = 4.5 \qquad h'(4.5) = 3$$

What is $(h^{-1})'(10)$?

2. The Natural Log and Exp Functions

Problem 6. Expand $\ln\left(\dfrac{x^2\sqrt{x^2+1}}{x^2-1}\right)$ using the rules of logarithms.

Problem 7. Combine $\ln(x) + 3\ln(x+1) - \dfrac{1}{2}\ln(x+2)$ into a single logarithm.

Problem 8. Evaluate the integral

$$\int_1^{e^2} \frac{1}{t} dt.$$

Problem 9. Evaluate the limit $\lim_{x\to\infty} \ln\left(\frac{1}{x^2+1}\right)$.

Problem 10. Compute the derivative of the following functions:

a) $\ln|\sqrt[3]{x-1}|$

b) $x^3 \ln(x)$

c) $\sin(\ln x)$

d) $\sin(x)\ln(5x)$

e) $\dfrac{1}{\ln x}$

f) $y = \ln\left(\dfrac{x}{x^2+1}\right)$

g) $y = \ln(x\sqrt{x^2-1})$

h) $y = \ln\sqrt{\dfrac{x+1}{x-1}}$ (without logarithmic differentiation)

i) $y = \sqrt{\dfrac{x+1}{x-1}}$ (with logarithmic differentiation)

Problem 11. Compute the following integrals:

a) $\displaystyle\int \frac{x}{3-x^2}dx$

b) $\displaystyle\int_2^4 \frac{3}{x}dx$

c) $\displaystyle\int \frac{x^2+x+1}{x}dx$

d) $\displaystyle\int_4^9 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

e) $\displaystyle\int \frac{dx}{x\ln(x)}$

f) $\displaystyle\int \frac{\sin(2x)}{1+\cos^2(x)}dx$

g) $\displaystyle\int \frac{\cos(\ln t)}{t}dt$

Problem 12. Compute the derivative of the following functions:

a) $y = e^{\tan(\theta)}$

b) $y = x^3 e^{x^2}$

c) $y = \ln(1 + e^{2x})$

d) $y = \dfrac{e^{2x}}{e^{2x} + 1}$

e) $xe^y + ye^x = 1$

f) $y = x^{e^x}$

g) $y = x^{\sqrt{x}}$

h) $y = x^{x^2}$

Problem 13. Find the inverse of the functions $y = e^{x^3}$ and $y = (\ln x)^5$.

Problem 14. Evaluate the following integrals:

a) $\displaystyle\int_0^1 (x^e + e^x)dx$

b) $\displaystyle\int_0^1 edx$

c) $\displaystyle\int e^x\sqrt{1 + e^x}dx$

d) $\displaystyle\int \frac{e^u}{(1 + e^u)^2}du$

e) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}dx$

f) $\displaystyle\int e^{\sin(\theta)}\cos(\theta)d\theta$

Problem 15. Compute each of the two limits:

$$\lim_{x\to\infty} \frac{x^8 + 5x^3 + 2e^x + 1}{4x^8 + 6x + 3e^x + 9} \qquad\qquad \lim_{x\to\infty} \frac{5x^3 + 2e^{2x} + 1}{6x + 3e^{5x} + 9}$$

3. General Log and Exponential (arbitrary base)

The only things you need to remember for this is

$$\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)} \qquad \text{and} \qquad \frac{d}{dx}a^x = a^x\ln(a)$$

They behave the same as the natural log/exp. Sometimes it is also useful to know the change of base formula

$$\log_b(a) = \frac{\ln(a)}{\ln(b)}$$

Problem 16. Sketch the graph of the following functions:

a) $f(x) = 3\ln x$

b) $\ln(x-3)$

c) $\ln|x|$

d) $e^x + 1$

e) $3e^x + 2$

f) $(0.5)^x$ and 2^x on the same axis

g) e^{-x} and e^x on the same axis

h) 2^x and 10^x on the same axis

i) $\log_2(x)$ and $\log_{10}(x)$ on the same axis

j) 2^{-x} and 10^{-x} on the same axis

4. Exponential Growth/Decay and Compound Interest

Problem 17. The population of Mathland in the year 2000 was 500. The population increases (continuously and steadily) by approximately 10% per year. What is the function $P(t)$ which gives the size of the population after t years? What is $P(0)$? What will the population be in 2050?

Problem 18. The population of Calculand was 700 in the year 2000 ($t = 0$), and 3000 in the year 2010 ($t = 10$). Using the exponential model for population growth, give a general formula for $P(t)$, and use it to estimate the population of Calculand in 2015.

Problem 19. The half-life of the Carbon-14 isotope is approximately 5,730 years. Archaeologists dig up a bowl made of oak and determine that it has only about 40% of the carbon-14 that a similar quantity of living oak has today. Estimate the age of the bowl. (Hint: use exponential decay $m(t) = m_0e^{kt}$ to find k, then solve $0.4 = e^{kt}$ for t).

Problem 20. How long will it take an investment of \$2,000 to double if the investment earns interest at the rate of 6% per year. What if the interest was compounded monthly?

Problem 21. How long will it take an investment of \$5,000 to triple at an interest rate of 4% per year compounded weekly. Assume 52 weeks in a year.

Problem 22. What is the interest rate needed for an investment of \$5,000 to grow to \$6,000 in 3 years if interest is compounded continuously.

Problem 23. Find the interest rate needed for an investment of \$2,000 to double in 5 years if interest is compounded annually.

Problem 24. Find the present value of \$20,000 due in 3 years at an interest rate of 12%/year compounded monthly.

Problem 25. Glen invests \$100,000 in an account yielding 6.6% interest compounded monthly. Being unhappy with the return on his investment, he wishes to reinvest the final amount at the end of the first year into a new account where interest is compounded quarterly. What interest rate should he look for if he wishes to obtain \$130,130 at the end of the third year (i.e. after keeping the money for 2 more years in the second account).

5. Inverse Trigonometric Functions

Problem 26. Evaluate the following:

$$\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) \qquad \sin^{-1}(\sin\pi) \qquad \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \qquad \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) \qquad \cos\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$\tan^{-1}(1) \qquad \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Problem 27. Use an appropriate triangle to give a formula in terms of x for:

$$\tan(\sin^{-1}(x)) \qquad \cos(\tan^{-1}(x))$$

Problem 28. Prove (using implicit differentiation and appropriate trig identities) the derivatives

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}}$$

Problem 29. Compute the derivatives of the following functions:

a) $\sin^{-1}\sqrt{\cos x}$

b) $\tan^{-1}(\ln x)$

c) $\sin^{-1}(x^2-1)$

d) $\cos^{-1}(x^2-1)$

e) $x\sin^{-1}(x) + \sqrt{1-x^2}$

f) $\tan^{-1}(x-\sqrt{1+x^2})$

g) $\arctan\sqrt{\dfrac{1-x}{1+x}}$

h) $\arcsin(e^x)$

Problem 30. Compute the following integrals:

a) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}dx$

b) $\displaystyle\int_0^{1/2} \frac{1}{1+4x^2}dx$

c) $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{x}{\sqrt{1-x^4}}dx$

d) $\displaystyle\int \frac{1}{x(1+(\ln x)^2)}dx$

e) $\displaystyle\int \frac{\cos^{-1}(x)}{\sqrt{1-x^2}}dx$

f) $\displaystyle\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}dx$

g) $\displaystyle\int \frac{1}{\sqrt{x}(1+x)}dx$

h) $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}dx$

6. L'Hospital's Rule

Problem 31. Evaluate the following limits:

a) $\lim_{x\to 3} \dfrac{x-3}{9-x^2}$

b) $\lim_{x\to 4} \dfrac{x^2-2x-8}{x-4}$

c) $\lim_{x\to -2} \dfrac{x^3+8}{x+2}$

d) $\lim_{t\to 0} \dfrac{e^{2t}-1}{\sin t}$

e) $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$

f) $\lim_{x\to\infty}(\ln x-\sqrt{x})$

g) $\lim_{x\to 0} \dfrac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$

h) $\lim_{x\to 0} \dfrac{e^x-1-x}{x^2}$

i) $\lim_{x\to 1} \dfrac{x^a-1}{x^b-1}$, $b \neq 0$

j) $\lim_{x\to\infty} x\sin(\pi/x)$

k) $\lim_{x\to\infty} x^3 e^{-x^2}$

l) $\lim_{x\to-\infty} x\ln\left(1-\dfrac{1}{x}\right)$

m) $\lim_{x\to 1} \left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right)$

n) $\lim_{x\to 0^+} x^{\sqrt{x}}$

o) $\lim_{x\to\infty} x^{1/x}$

p) $\lim_{x\to\infty} x^{e^{-x}}$

q) $\lim_{x\to 1} \left(\dfrac{9x}{x-1}-\dfrac{9}{\ln x}\right)$

Problem 32. For $0 \leq r \leq 1$ and $t \in (0,\infty)$, show that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$$

Note: this proves that the formula for continuously compound interest is $P_0 e^{rt}$, where P_0 is the initial amount, and r is the interest rate.

7. Integration By Parts

Recall the integration by parts formula:

$$\int uv' = uv - \int u'v$$

Problem 33. Evaluate the following integrals:

a) $\displaystyle\int x\sin(x)dx$

b) $\displaystyle\int x^5\cos(x)dx$ (tabular integration makes this FAST)

c) $\displaystyle\int \ln(x)dx$

d) $\displaystyle\int (\ln(x))^2 dx$

e) $\int \frac{(\ln(x))^2}{x^3} dx$

f) $\int x^4(\ln(x))^2 dx$

g) $\int \arctan(x) dx$

h) $\int e^x \sin(x) dx$

i) $\int x \tan^2(x) dx$

j) $\int x^4 \sin(2x) dx$ (use tabular integration)

k) $\int x^3 e^{-2x} dx$ (use tabular integration)

l) $\int \cos(\ln x) dx$ (make a substitution first)

m) $\int e^{\sqrt{x}} dx$ (make a substitution first)

8. Trigonometric Integrals

Problem 34. Evaluate the following integrals:

a) $\int \sin^2(x) dx$

b) $\int \cos^3(x) dx$

c) $\int \sin^4(x) dx$

d) $\int \sin^4(x) \cos^5(x) dx$

e) $\int \sin^4(x) \cos^2(x) dx$

f) $\int \tan^6(x) \sec^4(x) dx$

g) $\int \tan^5(x) \sec^7(x) dx$

h) $\int \tan(x) dx$

i) $\int \tan^2(x) dx$

j) $\int \tan^3(x) dx$

k) $\int_0^{\pi/4} \tan^4(x)dx$

l) $\int \sin(3x)\sin(8x)dx$

m) $\int x\sin^2(x^2)dx$

n) $\int x\sec(x)\tan(x)dx$

o) $\int \sec^3(x)dx$ (use integration by parts, $u = \sec(x)$ and $v' = \sec^2(x)$)

9. Trigonometric Substitution

You have three main identities to recognize which substitution to make:

$$\cos^2\theta = 1 - \sin^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \tan^2\theta = \sec^2\theta - 1$$

Problem 35. Evaluate the following integrals:

a) $\int \frac{x^2}{\sqrt{9-x^2}}dx$

b) $\int \frac{\sqrt{x^2-1}}{x^4}dx$

c) $\int_0^a \frac{1}{(a^2+x^2)^{3/2}}dx$

d) $\int_0^{1/2} x\sqrt{1-4x^2}dx$

e) $\int \frac{x+1}{x^2+1}dx$

f) $\int \frac{x^2}{\sqrt{x^2-7}}dx$

g) $\int \frac{1}{\sqrt{x^2+2x+5}}dx$

h) $\int \frac{x^2}{\sqrt{9-4x^2}}dx$

i) $\int \frac{x^2+1}{(x^2-2x+2)^2}dx$

10. Partial Fractions

Here are some examples of the splitting pattern:

$$\frac{*}{(x+1)(x+2)(x-3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-3}$$

$$\frac{*}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x-2)}$$
$$\frac{*}{(x^2+4)^3} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{(x^2+4)^3}$$
$$\frac{*}{(x^4+1)^2(x-3)} = \frac{Ax^3+Bx^2+Cx+D}{x^4+1} + \frac{Ex^3+Fx^2+Gx+H}{(x^4+1)^2} + \frac{I}{(x-3)}$$
$$\frac{*}{(x^4+1)(x^2+8)(x-2)^2} = \frac{Ax^3+Bx^2+Cx+D}{x^4+1} + \frac{Ex+F}{x^2+8} + \frac{G}{x-2} + \frac{H}{(x-2)^2}$$

Problem 36. Show using partial fractions the following chain of equalities:

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + C$$

Problem 37. Evaluate the following integrals:

a) $\displaystyle\int \frac{\sqrt{x+4}}{x}dx$ (make the rationalizing substitution $u = \sqrt{x+4}$ first)

b) $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}dx$

c) $\displaystyle\int_0^1 \frac{2}{2x^2+3x+1}dx$

d) $\displaystyle\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}dx$

e) $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}dx$

f) $\displaystyle\int \frac{x^5+x-1}{x^3+1}dx$ (top degree is bigger, so use polynomial division first!)

g) $\displaystyle\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}dx$ (make the substitution $u = \sqrt[6]{x}$ first!)

h) $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}dx$ (make an appropriate substitution first!)

i) $\displaystyle\int \frac{x^4+9x^2+x+2}{x^2+9}dx$ (simplify first, pay close attention to the numerator/denominator)

j) $\displaystyle\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}dx$ (simplify first)

k) $\displaystyle\int \ln(x^2-x+2)dx$ (use integration by parts first)

l) $\displaystyle\int \frac{4x}{x^3+x^2+x+1}dx$