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M20580 L.A. and D.E. Tutorial
Worksheet 5
Sections 3.1–3.3

1. Let A be an invertible matrix. Using properties of determinants, show that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$\det(A)\det(A^{-1}) = \det(A \cdot A^{-1}) = \det(I) = 1$

So $\det(A^{-1}) = \dfrac{1}{\det(A)}$

2. Find the determinant of the matrix:

$$A = \begin{array}{c} \begin{array}{cccc} + & - & + & \end{array} \\ \begin{bmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & -4 \\ 0 & 5 & 0 & -6 \end{bmatrix} \end{array}$$

$$\det(A) = 2 \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & -4 \\ 0 & 5 & -6 \end{vmatrix} = 2 \cdot (-1) \cdot (-5) \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix}$$

$$= 10(-18+20)$$

$$= 20$$

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3. Let A and B be 4×4 matrices, with $\det(A) = 5$ and $\det(B) = -1$. Compute:

(a) $\det(AB)$ $= 5 \cdot (-1) = -5$

(b) $\det(5A)$ $= \det(5I_4)\det(A) = 5^4 \cdot 5 = 5^5 = 3125$

(c) $\det(A^T BA)$ $= \det(A)\det(B)\det(A) = 5^2 \cdot (-1) = -25$

(d) $\det(B^5)$ $= (-1)^5 = -1$

(e) $\det(B^{-1}A)$ $= (-1)^{-1} \cdot 5 = -5$

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Caclulate the area of the image of the parallelogram spanned by

$$b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

under the linear transformation T.

S = parallelogram of $\vec{b}_1, \vec{b}_2$

$\text{area}(S) = \left|\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}\right| = 2$

$\det(T) = \begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix} = 4 - 12 = -8$

$\text{area}(T(S)) = |\det(T)| \, \text{area}(S)$

$= 8 \cdot 2$

$= 16$

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5. Use Cramer's rule to compute the solutions of the following systems

$$\begin{aligned} x_1 + x_2 \quad\quad\; &= 3 \\ -3x_1 \quad\quad + 2x_3 &= 0 \\ x_2 - 2x_3 &= 2 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -3 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

$$\det(A) = \begin{vmatrix} 1 & 1 & 0 \\ -3 & 1 & 0 \\ 0 & 1 & -2 \end{vmatrix} = -2 \begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} = -2 \cdot (1-(-3)) = -8$$

$$\det(A_1(\vec{b})) = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & -2 \end{vmatrix} = -2 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = -2$$

$$\det(A_2(\vec{b})) = \begin{vmatrix} 1 & 3 & 0 \\ -3 & 0 & 2 \\ 0 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 \\ -3 & 2 & 0 \\ 0 & 2 & -2 \end{vmatrix} = -2 \begin{vmatrix} 1 & 3 \\ -3 & 2 \end{vmatrix} = -2 \cdot 11 = -22$$

$$\det(A_3(\vec{b})) = \begin{vmatrix} 1 & 1 & 3 \\ -3 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = -(-3) \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 3 \cdot (2-3) = -3$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} \det(A_1(\vec{b})) \\ \det(A_2(\vec{b})) \\ \det(A_3(\vec{b})) \end{bmatrix}$$

$$= -\frac{1}{8} \begin{bmatrix} -2 \\ -22 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} \\ \frac{11}{4} \\ \frac{3}{8} \end{bmatrix}$$