CHALLENGE PROBLEMS FOR FINITE MATH STUDENTS

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- Solve the problems of your choice for extra credit.
- You may get a maximum of 5% boost to your final grade for the course.
- For credit, you MUST present your solutions in person, at the blackboard.
- Use of notes during the presentation is NOT allowed.
- The last day to present is Friday, July 28, 2017.

1. Set Theory

Problem 1. (0.5pt) Prove the DeMorgan's Laws for sets:

$$(A \cup B)' = A' \cap B' \quad \text{and} \quad (A \cap B)' = A' \cup B'$$

Problem 2. (0.5pt) The difference of two sets A and B is defined by $A \setminus B = A \cap B'$. Show that for any sets A and B, we have

$$A = (A \cap B) \cup (A \setminus B)$$

2. Counting

Problem 3. (1pt) Give a simple expression for the sum

$$\binom{2016}{0} + \binom{2016}{1} + \binom{2016}{2} + \binom{2016}{3} + \cdots + \binom{2016}{1007} + \binom{2016}{1008}$$

Problem 4. (1pt) Compute the exact value of the alternate sum

$$\binom{2017}{0} - \binom{2017}{1} + \binom{2017}{2} - \binom{2017}{3} + \cdots + \binom{2017}{2016} - \binom{2017}{2017}$$

Problem 5. (1pt) Recall that $\binom{n}{k}$ counts the number of subsets of size k of $\{1, 2, \ldots, n\}$. Show that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Problem 6. (1pt) We learned that the number of subsets of the set $\{1, 2, 3, \dots, n\}$ is equal to 2^n. This means that the set $\{1, 2, 3, \dots, n, \mathbf{n+1}\}$ is going to have $2^{n+1} = 2 \cdot 2^n$ subsets, which is TWICE as many subsets as the original set with just n elements. Can you explain why the number of subsets has doubled? Find a way to construct all the subsets of $\{1, 2, \dots, n, n+1\}$ from the subsets of $\{1, 2, \dots n\}$. It may be helpful to look at small examples and try to spot a pattern.

Problem 7. (2pt) We saw in class that we can perform different operations on sets, like union $A \cup B$, intersection $A \cap B$, and so on. We can also talk about the product of two sets A and B, which we define as

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

The $A \times B$ is a set of all ordered pairs (a, b), where the first coordinate comes from A, and the second coordinate comes from B. For example, the product

$$\{x, y, z\} \times \{1, 2\} = \{\ (x, 1), (x, 2), (y, 1), (y, 2), (z, 1), (z, 2)\ \}$$

contains six elements (the ordered pairs listed above). We can do this for any number of sets. For 3 sets, the product is going to contain ordered triples:

$$A \times B \times C = \{\ (a, b, c) \mid a \in A \text{ and } b \in B \text{ and } c \in C\ \}$$

Suppose you have k finite sets $A_1, A_2, \dots, A_k$, and the number of elements in each is $r_1, r_2, \dots, r_k$, respectively, so $n(A_1) = r_1$, $n(A_2) = r_2$, and so on. How many elements are in the product of all k sets:

$$A_1 \times A_2 \times \cdots \times A_k = \{\ (a_1, a_2, \dots, a_r) \mid a_1 \in A_1 \text{ and } a_2 \in A_2, \text{ and so on }\}$$

Problem 8. (1pt) Consider the arrangement of letters

$$\begin{array}{ccccccccccc}
 & & & & & A & & & & & \\
 & & & & B & & B & & & & \\
 & & & R & & R & & R & & & \\
 & & A & & A & & A & & A & & \\
 & C & & C & & C & & C & & C & \\
A & & A & & A & & A & & A & & A \\
 & D & & D & & D & & D & & D & \\
 & & A & & A & & A & & A & & \\
 & & & B & & B & & B & & & \\
 & & & & R & & R & & & & \\
 & & & & & A & & & & &
\end{array}$$

In how many ways can you spell out ABRACADABRA by starting with the top A, and going downward (diagonally) from one letter to an adjacent letter?

Problem 9. (2pt) Select any 8 numbers distinct from $\{1, 2, 3, \dots, 30\}$. Prove that from among the 8 chosen numbers, you can ALWAYS find two disjoint nonempty subsets with the same sum. For example, from among the 8 numbers $\{2, 6, 13, 17, 21, 23, 28, 30\}$, a few such pairs are listed below:

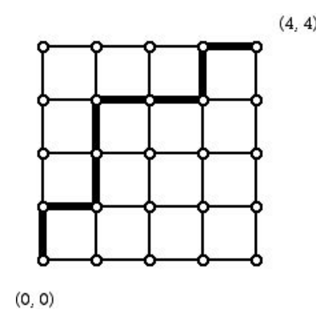
- subsets $\{2, 17\}$ and $\{6, 13\}$ both add up to $2 + 17 = 6 + 13 = 19$
- subsets $\{2, 21\}$ and $\{23\}$ both add up to $2 + 21 = 23$
- subsets $\{2, 6, 13\}$ and $\{21\}$ both add up to 21
- subsets $\{2, 6, 17, 28\}$ and $\{23, 30\}$ both add up to 53

Problem 10. (1pt) What is the minimum number of students in a class in order to be sure that you can find three students whose birthdays are in the same month.

Problem 11. (1pt) When you draw 3 lines such that no 2 are parallel and no 3 intersect at the same point, then you will find that there are 3 points where the lines intersect (draw some examples). Given n lines such that no 2 are parallel and no 3 intersect at the same point, determine the number of points of intersection.

Problem 12. (1pt) Suppose p, q, r are prime numbers. Let $N = p^a q^b r^c$, where the exponents a, b, c are nonnegative integers, so $a, b, c \in \{0, 1, 2, 3, 4, \ldots\}$. What is the number of positive divisors of N?

Problem 13. (1pt) Our taxicab geometry problem can be restated as follows: you start at the origin $(0, 0)$ in the 2-dimensional plane, plan to arrive at (a, b) (where a and b are nonnegative integers) by moving only up and to the right (so always in either the positive x direction or the positive y direction). The image on the right shows an example of such a path going from $(0, 0)$ to $(4, 4)$. Since a is the number of Right steps and b is the number of Up steps, we know the number of paths is given by $C(a + b, a)$, or equivalently by $C(a + b, b)$, which is



$$\frac{(a + b)!}{a! \cdot b!}$$

The taxicab problem can be extended to more than 2 dimensions. Consider 3-dimensional space, where we have 3 axes x, y, z. What is the number of paths from $(0, 0, 0)$ to (a, b, c) (where a, b, c are nonnegative integers) by traveling only in the positive x, positive y, or positive z directions?

3. Probability

Problem 14. (2pt) Recall that two events A and B are **independent** whenever $P(A \cap B) = P(A) \cdot P(B)$. Show that if A and B are independent, then A' and B' are also independent. In other words, show that

$$P(A' \cap B') = P(A') \cdot P(B')$$

Problem 15. (1pt) A drawer contains red socks and black socks. When two socks are drawn at random, the probability that both are red is $\frac{1}{2}$. What is the smallest number of socks in the drawer that will yield this probability?

Problem 16. (2pt) Three prisoners, A, B, and C, with apparently equally good records have applied for parole. The parole board has decided to release two of the three, and the prisoners know this but they do not know which two. A guard (and friend of prisoner A) knows who are to be released.

Prisoner A realizes that it would be unethical to ask the guard if he, A, is to be released, but thinks of asking for the name of *one* prisoner *other than himself* who is to be released. He thinks that before he asks, his chances of release are $\frac{2}{3}$. He also thinks that if the guard says "B will be released", his own chances have now gone down to $\frac{1}{2}$, because either A and B or B and C are to be released (so he is thinking there are now only two possible outcomes, one of which is favorable). And so prisoner A decides not to reduce his chance by asking. However, A is mistaken in his calculations. Explain why.

Problem 17. (1pt) A two-headed coin, a two-tailed coin, and a ordinary coin are placed in a bag. One of the coins is drawn at random and flipped; it comes up "heads". What is the probability that there is a head on the other side of the coin?

Problem 18. (2pt) On average, how many times do you need to roll a die before all six different numbers have turned up?