CALCULUS 2
EXAM 3 PRACTICE PROBLEMS

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Contents

1. p-Series and the Integral Test

Recall that for a positive, continuous (so integrable) function $f(x)$ on $[1, \infty)$, the behaviour of the series $\sum f(n)$ will coincide to the behaviour of the integral

$$\int_*^\infty f(x)dx$$

The p-series is just a special case of this when $f(x) = 1/x^p$.

Problem 1. Derive the p-series result using the integral test, i.e. show that

$$\sum_{n=1}^\infty \frac{1}{x^p} = \begin{cases} \text{convergent} & p > 1 \\ \text{divergent} & p \leq 1 \end{cases}$$

Problem 2. Use the Integral Test to determine which series is convergent or divergent:

$$\text{a) } \sum_{n=1}^\infty ne^{-n} \qquad \text{b) } \sum_{n=1}^\infty \frac{e^n}{1+e^{2n}} \qquad \text{c) } 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots$$

$$\text{d) } \sum_{n=2}^\infty \frac{1}{n \ln n} \qquad \text{e) } \sum_{n=2}^\infty \frac{1}{n(\ln n)^2} \qquad \text{f) } \sum_{n=1}^\infty \frac{e^{-n}}{n^2} \qquad \text{g) } \sum_{n=1}^\infty \frac{(\ln n)^2}{n^2}$$

Problem 3. For which values of p do the following series converge?

$$\text{a) } \sum_{n=2}^\infty \frac{1}{n(\ln n)^p} \qquad \text{b) } \sum_{n=10}^\infty \frac{1}{n \ln n [\ln(\ln n)]^p} \qquad \text{c) } \sum_{n=1}^\infty n(1+n^2)^p \qquad \text{d) } \sum_{n=1}^\infty \frac{\ln n}{n^p}$$

Problem 4. Find all values of c for which the following sum converges

$$\sum_{n=1}^\infty \left(\frac{c}{n} - \frac{1}{n+1} \right)$$

2. The Comparison Tests

Problem 5. Use the regular comparison test to determine if the following series converge or diverge:

$$\text{a)}\ \sum_{n=1}^{\infty} \frac{n+1}{2n^2+n+1} \qquad \text{b)}\ \sum_{n=2}^{\infty} \frac{n^3}{n^4-1} \qquad \text{c)}\ \sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}} \qquad \text{d)}\ \sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$$

$$\text{e)}\ \sum_{n=1}^{\infty} \frac{9^n}{3+10^n} \qquad \text{f)}\ \sum_{n=1}^{\infty} \frac{2+(-1)^n}{n\sqrt{n}} \qquad \text{g)}\ \sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1} \qquad \text{h)}\ \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$$

$$\text{i)}\ \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n e^{-n} \qquad \text{j)}\ \sum_{n=1}^{\infty} \frac{e^{1/n}}{n} \qquad \text{k)}\ \sum_{n=1}^{\infty} \frac{n!}{n^n} \qquad \text{l)}\ \sum_{n=2}^{\infty} \frac{2^n}{n!}$$

$$\text{m)}\ \sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \qquad \text{n)}\ \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \qquad \text{o)}\ \sum_{n=1}^{\infty} \frac{1}{n\cdot\sqrt[n]{n}}$$

Problem 6. Use the limit comparison test (where appropriate) on the series from the previous problem.

Problem 7. Prove the following:

a) If $0 \le a_n \le 1$ and $\sum a_n$ is convergent then $\sum a_n^2$ is also convergent (easy).
b) If $a_n \ge 0$ and $\sum a_n$ is convergent, then $\sum a_n^2$ is also convergent (not as easy).

Problem 8. Suppose $\sum a_n$ and $\sum b_n$ are series of positive terms.

a) Assuming $\sum b_n$ is convergent, prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

then $\sum a_n$ is also convergent. (Hint: what must be true of the tail terms of b_n if the above limit is zero?)

b) Assuming $\sum b_n$ is divergent, prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

then $\sum a_n$ is also divergent.

Problem 9. Consider the sequence of prime numbers $\{p_n\}_{n\ge1} = \{2, 3, 5, 7, 11, 13, \dots\}$. Discuss the convergence or divergence of the series

$$\sum_{n\ge1} \frac{1}{(p_n)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \dots$$

3. Alternating Series

Recall that an alternating series is a series of the form

$$\sum (-1)^n a_n$$

where a_n are strictly positive. The following are examples of alternating series

$$\sum (-1)^{n+3} \frac{1}{3^n} \qquad \sum \frac{\cos(n\pi)}{n^2} \qquad \sum \frac{\sin(n\pi + \pi/2)}{n} \qquad \sum (-e)^{-n}$$

while the following are NOT alternating series

$$\sum \frac{\cos(n)}{n^2} \qquad \sum \frac{\cos(\pi/n)}{n^2} \qquad \sum \frac{(-1)^n}{n+1} \left(\frac{-2}{3}\right)^n$$

Problem 10. Test the following series for convergence or divergence:

a) $\displaystyle\sum_{n\geq 0} \frac{(-1)^n}{3+5n}$ b) $\displaystyle\sum_{n\geq 1} (-1)^n e^{-n}$ c) $\displaystyle\sum_{n\geq 1} (-1)^n \frac{n^2}{n^2 + \ln n + 1}$ d) $\displaystyle\sum_{n\geq 0} (-1)^n \frac{\sqrt{n}}{2n+3}$

e) $\displaystyle\sum_{n\geq 0} (-\pi)^{-n}$ f) $\displaystyle\sum_{n\geq 0} (-1)^n \arctan(n)$ g) $\displaystyle\sum_{n\geq 1} \frac{\sin(\pi n + \pi/2)}{1 + \ln n}$ h) $\displaystyle\sum_{n\geq 1} (-1)^n \frac{\ln n}{n}$

i) $\displaystyle\sum_{n\geq 1} (-1)^n \sin\left(\frac{\pi}{n}\right)$ j) $\displaystyle\sum_{n\geq 1} (-1)^n \cos\left(\frac{\pi}{n}\right)$ k) $\displaystyle\sum_{n\geq 1} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$

Problem 11. Discuss the convergence or divergence of the following sums:

a) $\displaystyle\sum_{n\geq 0} (-n)^{-n} \cdot n!$ b) $\displaystyle\sum_{n\geq 0} \frac{(-n)^n}{n!}$ c) $\displaystyle\sum_{n\geq 0} \frac{(-2)^n}{n!}$ d) $\displaystyle\sum_{n\geq 0} (-\pi)^{-n} \cdot n!$

Hint: How does $n!$ compare to n^n, and how does $n!$ compare to the exponential function a^n for $a > 1$? We have the following hierarchy of functions when it comes to ranking how quickly they converge to infinity (you should prove this):

$$\log(n) < n^p < a^n < n! < n^n$$

Problem 12. For what values of p do the following series converge?

a) $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n-1}}{n^p}$ b) $\displaystyle\sum_{n\geq 0} \frac{(-1)^n}{n+p}$ c) $\displaystyle\sum_{n\geq 1} (-1)^{n+1} \frac{e^n}{n^p + p^n}$ d) $\displaystyle\sum_{n\geq 1} (-1)^{n-1} \frac{(\ln n)^p}{n}$

4. Absolulte Convergence and the Ratio/Root Tests

Problem 13. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

a) $\sum_{n\geq 0} \frac{(-10)^n}{n!}$ b) $\sum_{n\geq 1} (-1)^{n+1} \frac{2^n}{e^n - 1}$ c) $\sum_{n\geq 1} \frac{(-1)^n}{n^{1/n}}$ d) $\sum_{n\geq 1} \frac{(-1)^{3n+1}}{n^3+1}$

e) $\sum_{n\geq 0} n \left(\frac{2}{3}\right)^n$ f) $\sum_{n\geq 0} \frac{n!}{2017^n}$ g) $\sum_{n\geq 1} \frac{(-1)^n e^{1/n}}{n^2}$ h) $\sum_{n\geq 0} \frac{10^n}{(n+1)4^{2n+1}}$

i) $\sum_{n\geq 1} (-1)^{n+1} \frac{n^2 2^n}{n!}$ j) $\sum_{n\geq 1} (-1)^n \frac{\arctan n}{n^2}$ k) $\sum_{n\geq 1} \frac{n!}{n^n}$ k) $\sum_{n\geq 1} \frac{(-2)^n}{n^n}$

l) $\sum_{n\geq 2} \frac{n}{(\ln n)^n}$ m) $\sum_{n\geq 1} \left(1 + \frac{1}{n}\right)^{n^2}$ n) $\sum_{n\geq 1} \left(1 - \frac{1}{n}\right)^{n^2}$

o) $\sum_{n\geq 1} \left(\frac{n^2+1}{2n^2+1}\right)^n$ p) $\sum_{n\geq 1} \left(\frac{-2n}{n+1}\right)^{5n}$

Problem 14. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ whose terms satisfy the recursions

$$a_1 = 1 \qquad a_{n+1} = \frac{2+\cos n}{\sqrt{n}} a_n$$

and

$$b_1 = 2 \qquad b_{n+1} = \frac{5n+1}{4n+3} b_n$$

Determine whether the two series converge or diverge.

5. Strategy for Testing Series

This section brings together all the series techniques we studied so far.

Problem 15. Test the series for convergence or divergence:

a) $\displaystyle\sum_{n\geq 1} \frac{n^{2n}}{(1+n)^{3n}}$ b) $\displaystyle\sum_{n\geq 2} \frac{1}{n\sqrt{\ln n}}$ c) $\displaystyle\sum_{n\geq 0} (-1)^n \frac{\pi^{2n}}{(2n)!}$ d) $\displaystyle\sum_{n\geq 0} \frac{n^4}{4^n}$

e) $\displaystyle\sum_{n\geq 1} n\left(\frac{1}{n^3}+\frac{1}{3^n}\right)$ f) $\displaystyle\sum_{n\geq 2} \frac{(-1)^{n-1}}{\sqrt{n}-1}$ g) $\displaystyle\sum_{n\geq 1} \frac{\sin(2n)}{2n}$ h) $\displaystyle\sum_{k\geq 1} \frac{\sqrt[3]{k}}{k\left(\sqrt{k}+1\right)}$

i) $\displaystyle\sum_{n\geq 1} \frac{n!}{e^{n^2}}$ j) $\displaystyle\sum_{n\geq 1} \frac{e^{1/n}}{n^2}$ k) $\displaystyle\sum_{n\geq 1} \frac{2017^n}{2^n+3^n+4^n+\cdots+2016^n}$ k) $\displaystyle\sum_{n\geq 1} \left(\frac{n}{n+1}\right)^{n^2}$

Problem 16. (Challenge) Let $\alpha \in (0,1]$. Compute the value of the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k^\alpha}$$

Hint: treat $\alpha \in (0,1)$ and $\alpha = 1$ separately.

6. Power Series

Problem 17. Find the radius of convergence and the interval of convergence of the series:

$$\text{a)} \sum_{n\geq 1}(-1)^n n x^n \qquad \text{b)} \sum_{n\geq 1}\frac{(-1)^n x^n}{\sqrt[3]{n}} \qquad \text{c)} \sum_{n\geq 0}\frac{x^{3n}}{n!} \qquad \text{d)} \sum_{n\geq 0} n^n x^n$$

$$\text{e)} \sum_{n\geq 1} 2^n n^2 x^n \qquad \text{f)} \sum_{n\geq 1}\frac{x^n}{n^4 4^n} \qquad \text{g)} \sum_{n\geq 0}\frac{(x-2)^n}{n^2+1} \qquad \text{h)} \sum_{n\geq 1}\frac{(-1)^n}{(2n-1)2^n}(x-1)^n$$

$$\text{i)} \sum_{n\geq 1}\frac{(x+\pi)^n}{n^n} \qquad \text{j)} \sum_{n\geq 1}\frac{\sqrt[3]{n}}{8^n}(x+6)^n \qquad \text{k)} \sum_{n\geq 1}\frac{(2x-1)^n}{n^3} \qquad \text{k)} \sum_{n\geq 2}\frac{x^{2n}}{n(\ln n)^2}$$

Problem 18. Suppose a is a real number and b is a positive. Find the radius and interval of convergence of the power series

$$\sum_{n\geq 1}\frac{n}{b^n}(x-a)^n \qquad \text{and} \qquad \sum_{n\geq 2}\frac{b^n}{\ln n}(x-a)^n$$

Problem 19. Suppose $\sum_{n\geq 0} c_n 4^n$ is convergent. What can we say about the series:

$$\sum_{n\geq 0} c_n(-2)^n \qquad \text{and} \qquad \sum_{n\geq 0} c_n(-4)^n$$

Problem 20. Suppose $\sum_{n\geq 0} c_n x^n$ is convergent for $x=-4$, but divergent for $x=6$. What can we say about the convergence or divergence of the following:

$$\text{a)} \sum_{n\geq 0} c_n \qquad \text{b)} \sum_{n\geq 0} c_n 8^n \qquad \text{c)} \sum_{n\geq 0} c_n(-3)^n \qquad \text{d)} \sum_{n\geq 0}(-1)^n c_n 9^n$$

Problem 21. Consider the function f defined by the power series

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \dots$$

so that $c_{2n} = 1$ (the coefficients of even powers of x are 1) and $c_{2n+1} = 2$. Find the interval of convergence of the series and find an explicit formula for f.

Problem 22. Suppose the series $\sum c_n x^n$ has radius of convergence R. What is the radius of convergence of the series $\sum c_n x^{2n}$?

Problem 23. Find the radius of convergence of the series

$$\sum_{n\geq 1}\left(1-\frac{1}{n}\right)^n (x-a)^n$$

7. Representation of Functions as Power Series

This topic relies on manipulating the basic geometric series

$$\sum_{n\geq 0} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \qquad |x| < 1$$

in order to obtain series for new functions.

Problem 24. Find a power series representation for the given function and state the interval of convergence:

$$\text{a) } \frac{1}{1+x} \quad \text{b) } \frac{1}{3-x} \quad \text{c) } \frac{1}{1-4x^2} \quad \text{d) } \frac{x^2}{9+x^2} \quad \text{e) } \frac{1+x}{1-x} \quad \text{f) } \frac{x^2}{a^3-x^3}$$

$$\text{g) } \frac{x^2}{(1+x)^3} \quad \text{h) } \ln(5-x) \quad \text{i) } x^2 \arctan x^3 \quad \text{j) } \frac{x}{(1+4x)^2} \quad \text{k) } \frac{x^2+x}{(1-x)^3}$$

$$\text{l) } \left(\frac{x}{2-x}\right)^2 \quad \text{m) } \ln\left(\frac{1+x}{1-x}\right) \quad \text{n) } \arctan 2x \quad \text{o) } \ln(x^2+4) \quad \text{p) } \frac{\arctan x}{x}$$

Problem 25. Recall that the Fibonacci sequence is defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with initial conditions $F_0 = 0$ and $F_1 = 1$. Notice that F_n increases, and tends to infinity. Discuss the convergence or divergence of the series

$$\text{a) } \sum_{n=2}^{\infty} (-1)^n \frac{F_n}{F_{n+1}} \qquad \text{b) } \sum_{n=2}^{\infty} (-1)^n \frac{1}{F_n} \qquad \text{c) } \sum_{n=2}^{\infty} \frac{1}{F_n}$$

and find a closed formula for the series

$$S(x) = F_1 x + F_2 x^2 + F_3 x^3 + \cdots = \sum_{n=1}^{\infty} F_n x^n$$

Problem 26. Show that the function defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

satisfies the differential equation $y' = y$, and find a closed formula for f.

Problem 27. Find closed form expressions for the following (assuming x is within the appropriate interval of convergence for those expressions involving x)

$$\text{a) } \sum_{n\geq 1} n x^{n-1} \qquad \text{b) } \sum_{n\geq 1} \frac{n}{2^n} \qquad \text{c) } \sum_{n\geq 0} n(n-1)x^n \qquad \text{d) } \sum_{n\geq 0} \frac{n^2-n}{2^n} \qquad \text{e) } \sum_{n\geq 0} \frac{n^2}{2^n}$$

8. Taylor and Maclaurin Series

For an infinitely differentiable function f, its power series representation centered at a is given by Taylor's formula

$$f(x) = \sum_{n\geq 0} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and if $a = 0$, this is called the Maclaurin series. We also have the Binomial series for any real number r and $|x| < 1$:

$$(1+x)^r = \sum_{n\geq 0} \binom{r}{n} x^n = 1 + \frac{r}{1!}x + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \dots$$

Problem 28. Show that for every nonnegative integer k, the k-th derivative of the Taylor series expansion at a is equal to $f^{(k)}(a)$.

Problem 29.

a) Find the Maclaurin series expansions of e^x, $\cos x$, and $\sin x$.

b) Using the answer from part (a), prove Euler's formula

$$e^{ix} = \cos x + i \sin x$$

where i is the imaginary number $i = \sqrt{-1}$, and deduce Euler's identity $e^{\pi i} = -1$.

c) Prove de Moivre's formula $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

Problem 30. Use the definition of Maclaurin series to find expansions for the following:

a) $(1-x)^{-2}$ b) $\sin(\pi x)$ c) 2^x d) e^{2x} e) $\sqrt{x}$ f) $\sqrt{1+2x}$

Problem 31. Modify already known Maclaurin series to obtain series for the following:

a) $e^x + e^{2x}$ b) $\dfrac{x}{\sqrt{4+x^2}}$ c) $\dfrac{e^x + e^{-x}}{2}$ d) $x^2\ln(1+8x^3)$ e) $\sin^2(x)$ f) xe^{-x}

Problem 32. Find the sum of the series

a) $\displaystyle\sum_{n\geq 0}(-1)^n\frac{x^{4n}}{n!}$ b) $\displaystyle\sum_{n\geq 0}\frac{(-1)^n\cdot\pi^{2n}}{6^{2n}\cdot(2n)!}$ c) $\displaystyle\sum_{n\geq 1}(-1)^{n-1}\frac{3^n}{n5^n}$ d) $\displaystyle\sum_{n\geq 1}(-1)^{n-1}\frac{1}{ne^n}$

e) $\displaystyle\sum_{n\geq 0}(-1)^n\frac{(\ln 2)^n}{n!}$ f) $3 + \dfrac{9}{2} + \dfrac{27}{6} + \dfrac{81}{24} + \dfrac{243}{120} + \dots$

Problem 33. Compute the exact value of the following:

$$\frac{e\cdot e^{1/3}\cdot e^{1/5}\cdot e^{1/7}\dots}{e^{1/2}\cdot e^{1/4}\cdot e^{1/6}\cdot e^{1/8}\dots}$$

Hint: use the rules of the exponents to combine all the exponents into one big sum. What does the sum evaluate to?