Math 10550 Exam 2 Practice Problems

1. The position of a particle traveling along the real line is given by $p(t) = (t+2)(t-1)^3$.

   (a) Find the critical numbers of $p(t)$ and interpret these in terms of the motion of the particle.
   The velocity of the particle is

   $$v(t) = p'(t) = (t-1)^3 + 3(t+2)(t-1)^2 = (4t+5)(t-1)^2.$$

   Thus $p(t)$ has critical points at $t = -\frac{5}{4}$ and $t = 1$, these are times when the particle is not moving.

   (b) On which intervals is $p(t)$ increasing and decreasing. Interpret these in terms of the motion of the particle.
   $p(t)$ is increasing on the intervals $\left(-\frac{5}{4}, 1\right) \cup (1, \infty)$, i.e. the particle has positive velocity and is moving to the right. $p(t)$ is decreasing on the interval $\left(-\infty, -\frac{5}{4}\right)$, i.e. the particle has negative velocity and is moving to the left.

   (c) Find all local maximum and minimum values of p. Interpret these in terms of the motion of the particle.
   p has an absolute minimum value at $t = -\frac{5}{4}$, that is the particle never travels further to the left than $p\left(-\frac{5}{4}\right) = \frac{3}{4} \cdot \frac{-729}{64} = -\frac{2187}{256}$. There is no local maximum value, i.e. given any position of the particle there is always a time where the particle was further to the right.

   (d) Where is the graph of p concave upward and where is it concave downward? Interpret these in terms of the motion of the particle.
   The acceleration of the particle is

   $$a(t) = p''(t) = 4(t-1)^2 + 2(4t+5)(t-1) = (12t+6)(t-1).$$

   $p''(t)$ is positive on the intervals $\left(-\infty, -\frac{1}{2}\right) \cup (1, \infty)$ so the graph of p is concave upward here, this is when the particle has rightward acceleration. $p''(t)$ is negative on the interval $\left(-\frac{1}{2}, 1\right)$ so the graph of p is concave downward here, this is when the particle has leftward acceleration.

   (e) Find all inflection points of p. Interpret these in terms of the motion of the particle.
   p has inflection points at $t = -\frac{1}{2}$ and $t = 1$, there are times when the particle switched between rightward and leftward acceleration.

2. A box has fixed volume $100\,m^3$. If the height of the box is increasing at a rate of $2\,m/s$ and the width is increasing at a rate of $3\,m/s$, how fast is the surface area changing when the height is $4\,m$ and the base is a square?
   When the height is $4\,m$ and the base is a square, the length and width are both $5\,m$. Since $V = \ell wh$ is fixed to be $100\,m^3$, we have

   $$0 = \frac{d\ell}{dt}wh + \ell\frac{dw}{dt}h + \ell w\frac{dh}{dt} = 20\frac{d\ell}{dt} + 60 + 50$$

   so that $\frac{d\ell}{dt} = -\frac{11}{2}\,m/s$. The surface area of the box is $SA = 2\ell w + 2\ell h + 2wh$ and so

   $$\begin{aligned}\frac{dSA}{dt} &= 2\left(\frac{d\ell}{dt}w + \ell\frac{dw}{dt}\right) + 2\left(\frac{d\ell}{dt}h + \ell\frac{dh}{dt}\right) + 2\left(\frac{dw}{dt}h + w\frac{dh}{dt}\right) \\ &= 2(-\frac{55}{2} + 15) + 2(-\frac{44}{2} + 10) + 2(12+10) = -25 - 24 + 44 = -5\,m^2/s.\end{aligned}$$

3. A skier jumps off a parabolic ramp with the shape $y = x^2$ for $0 \le x \le 1$. If the skier maintains a constant horizontal velocity of $20\,m/s$ while on the ramp, what is the maximum height achieved by the skier assuming the acceleration due to gravity is $10\,m/s^2$? How far will the skier travel before

returning to the ground ($y = 0$)? (Ignore any air friction.)
Since the skier is traveling along the parabola $y = x^2$, we have $\frac{dy}{dt} = 2x\frac{dx}{dt} = 40\,m/s$ when the skier exits the ramp. The height of the skier at time t seconds after leaving the ramp is given by $h(t) = h_0 + v_0 t + \frac{1}{2}a_0 t^2 = 1 + 40t - 5t^2$ so that $h'(t) = 40 - 10t$. Thus the skier reaches the maximum height of $81\,m$ after $4\,s$. The height of the skier will be zero when $t = \frac{-40-\sqrt{1600+20}}{-10} \approx 8.0249\,s$ so that the skier travels about $(20\,m/s)(8.0249\,s) = 160.0498\,m$ before returning to the ground.

4. Let $f(x) = \cos(x)$.

   (a) Find the linearization of f near $x = \frac{\pi}{6}$.

$$L_{\pi/6}(x) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{6}\right)$$

   (b) Find the linearization of f near $x = \frac{\pi}{4}$.

$$L_{\pi/4}(x) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$$

   (c) Use each linearization above to estimate the value of $\cos\left(\frac{\pi}{5}\right)$.

$$L_{\pi/6}\left(\frac{\pi}{5}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{\pi}{5} - \frac{\pi}{6}\right) \approx 0.81367$$
$$L_{\pi/4}\left(\frac{\pi}{5}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\frac{\pi}{5} - \frac{\pi}{4}\right) \approx 0.81818$$

   (d) Argue using the concavity of $\cos(x)$ for which estimate above is closer to the actual value of $\cos\left(\frac{\pi}{5}\right)$.
   Since $f''(x) = -\cos(x)$, we have that $f''(x) < 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. In particular the graph of f is concave downward at both $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$, i.e. the tangent lines at these points lie above the graph of f. It follows that both approximations above are over-estimates and thus the smaller of them, 0.81367, is closer to the actual value of $\cos\left(\frac{\pi}{5}\right)$ $\left(= \frac{1+\sqrt{5}}{4} \approx 0.80902\right)$.

5. Find the linearization $L(x)$ of $f(x) = \frac{1}{(2+x)^3}$ near $a = -1$. Verify that "up to first order" we have $L(-1 + dx) = f(-1 + dx)$.
   We have $f'(x) = -3(2+x)^{-4}$ so $f'(-1) = -3$ and

$$L(x) = f(-1) + f'(-1)(x+1) = 1 - 3(x+1) = -2 - 3x.$$

   Then

$$f(-1+dx) = \frac{1}{\big(2 + (-1+dx)\big)^3} = \frac{1}{(1+dx)^3} \quad \text{and} \quad L(-1+dx) = -2 - 3(-1+dx) = 1 - 3dx,$$

   so we aim to see that up to first order we have

$$\frac{1}{(1+dx)^3} = 1 - 3dx.$$

   We will show that $(1-3dx)(1+dx)^3 = 1$ up to first order. Expanding using the binomial theorem we have

$$\begin{aligned}(1-3dx)(1+dx)^3 &= (1-3dx)\big(1 + 3dx + 3(dx)^2 + (dx)^3\big) \\ &= 1 - 6(dx)^2 - 6(dx)^3 - 3(dx)^4.\end{aligned}$$

   But up to first order we have $(dx)^2 = (dx)^3 = (dx)^4 = 0$ and the result follows.

6. Let $f(x) = e^x - 3x$. Find the maximum and minimum value of $f(x)$ on the interval $[-1, 5]$.
The extreme values will be at either an end point or a critical point. $f'(x) = e^x - 3$. So the critical point will occur when $x = \ln(3)$.

$$f(-1) = e^{-1} + 3 \approx 3.36788 \quad f(5) = e^5 - 15 \approx 133.41316 \quad f(\ln(3)) = 3 - 3\ln(3) \approx .29583$$

By the Extreme Value Theorem: the absolute maximum value is $e^5 - 15$ and the absolute minimum value is $3 - 3\ln(3)$.

7. Let $f(t) = \frac{4t}{1+t^2}$. Use derivatives to show where this function is increasing and where it is decreasing. Also show where it is concave up and concave down. List all critical points. List all points of inflection. Sketch the graph of the function from this data. At what t is there a global maximum (if there is one) and at what t is there a global minimum? Indicate clearly on your graph any horizontal or vertical asymptotes.

$$f'(x) = \frac{4(1+t^2) - (4t)(2t)}{(1+t^2)^2} = \frac{4 - 4t^2}{(1+t^2)^2}$$

So there are critical points at $t = -1, 1$. The function is increasing on $(-1, 1)$, decreasing on $(-\infty, -1)$ and on $(1, \infty)$.

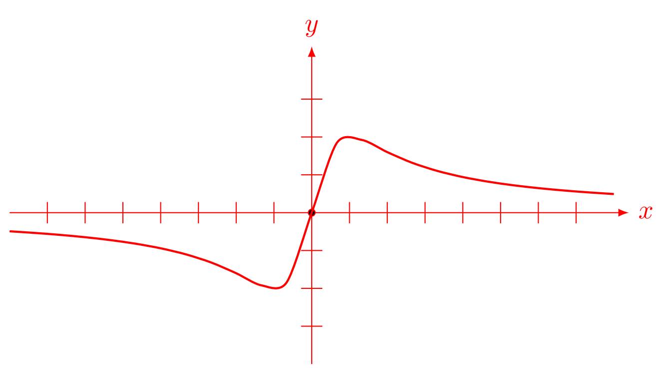
The second derivative is

$$\begin{aligned} f''(t) &= \frac{-8t(1+t^2)^2 - (4-4t^2)\big[2(1+t^2)(2t)\big]}{(1+t^2)^4} = (1+t^2)\frac{-8t(1+t^2) + 4t(4t^2-4)}{(1+t^2)^4} \\ &= \frac{-24t + 8t^3}{(1+t^2)^3} = 8t\frac{t^2-3}{(1+t^2)^3}. \end{aligned}$$

Thus:

- $f'' > 0$ *on* $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$
- $f'' < 0$ *on* $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

So the graph is concave up on $(-\sqrt{3}, 0)$ and on $(\sqrt{3}, \infty)$, concave down on $(-\infty, -\sqrt{3})$ and on $(0, \sqrt{3})$ so that $-\sqrt{3}, 0, \sqrt{3}$ are all points of inflection.
There are no vertical asymptotes since the function is continuous and always defined. The horizontal asymptote at $-\infty$ is 0, approached from below, and the horizontal asymptote at ∞ is 0 approached from above. The global minimum is at -1 and is $-4/2 = -2$ and the global maximum is at 1 and is 2.



8. Does there exist a function f such that $f(0) = 3$, $f(9) = 12$, and $f'(x) < 1$ for $0 \le x \le 9$?
By the Mean Value Theorem, there would exist a point $c \in (0, 9)$ so that $f'(c) = \frac{f(9)-f(0)}{9-0} = \frac{9}{9} = 1$. But we assumed that the derivative is always strictly less than 1 so no such function can exist.

9. Let f be a function for which $f(3) = 2$ and $-1 \leq f'(x) \leq 4$. Find upper and lower bounds for $f(-3)$.
By the Mean Value Theorem, there is a point $c \in (-3, 3)$ so that

$$f'(c) = \frac{f(3) - f(-3)}{3 - (-3)} = \frac{2 - f(-3)}{6} \quad \text{or} \quad f(-3) = 2 - 6f'(c).$$

Since $-1 \leq f'(c) \leq 4$, we get $-22 \leq f(-3) \leq 8$.

10. A **fixed point** of a function f is a number a for which $f(a) = a$. Use Rolle's Theorem to show that a differentiable function f with $f'(x) \neq 1$ for all real numbers x has at most one fixed point. *Hint: Interpret fixed points of f in terms of the function $g(x) = f(x) - x$.*
A fixed point for f is exactly a root of g. If g had more than one root, say a and b, then Rolle's Theorem would guarantee the existence of a number $c \in (a, b)$ for which $g'(c) = 0$. But $g'(x) = f'(x) - 1$ and so we would have $g'(c) = f'(c) - 1$ or $f'(c) = 1$, which is impossible by assumption. Thus g can have at most one root, i.e. f can have at most one fixed point.

11. Calculate the following limits. If the limits do not exist, state if they converge to ∞ or to $-\infty$. Otherwise if they do not exist, state that they do not exist.

(a)

$$\lim_{x\to\infty} \frac{x^2}{e^x}$$

We use L'Hôpital's rule (since numerator and denominator both approch ∞) twice:

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

(b)

$$\lim_{x\to 0^+} x\ln(x)$$

We use L'Hôpital's Rule (since numerator and denominator both approch ∞), but first we have to make a denominator out of the x:

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

(c)

$$\lim_{x\to 0} \frac{\sin(3x)}{\tan(5x)}$$

Numerator and denominator both approach 0 so we use L'Hôpital's rule.

$$\lim_{x\to 0} \frac{\sin(3x)}{\tan(5x)} = \lim_{x\to 0} \frac{3\cos(3x)}{5\sec^2(5x)} = 3/5.$$

(d)

$$\lim_{x\to 0} \frac{x}{\cos x + x}$$

This is continuous at 0, so we get the limit by plugging in 0, which gives $0/1 = 0$. Notice that L'Hôpital's rule does not apply (since the denominator does not approach 0), and would give the wrong answer if used here.