10350 Tutorial Week 03 - Set 01 Name ______________________________

Section ____________________

1. Consider the piecewise defined function:

$$f(x) = \begin{cases} x-1 & \text{if } x < 0 \\ x^2+1 & \text{if } x \geq 0 \end{cases}$$

Find the following limits:

1a. $\lim\limits_{x\to 0^-} \dfrac{f(x)-5}{x-2} \stackrel{?}{=}$

Since we are taking the limit from the left, we are in the $x < 0$ region of the domain, so in computing the limit, we use the $x-1$ branch of our function:

$$\lim_{x\to 0^-} \frac{f(x)-5}{x-2} = \lim_{x\to 0} \frac{(x-1)-5}{x-2} = \frac{(0-1)-5}{0-2} = \frac{-6}{-2} = 3$$

1b. $\lim\limits_{x\to 2} \dfrac{f(x)-5}{x-2} \stackrel{?}{=}$

This time we are taking the limit as $x \to 2$, so we fall in the $x \geq 0$ region of our domain. Hence we use x^2+1 in computing the limit:

$$\lim_{x\to 2} \frac{f(x)-5}{x-2} = \lim_{x\to 2} \frac{(x^2+1)-5}{x-2} = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4$$

1c. Is $f(x)$ continuous at $x = 0$? Use limit to explain your conclusion.

We need to compute the one-sided limits for $f(x)$ first. We have:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0}(x-1) = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0}(x^2+1) = +1$$

Furthermore, $f(0) = 0^2 + 1 = 1$.

Since the one-sided limits are distinct, our function is NOT continuous. In fact, the limit as $x \to 0$ does not exist. For continuity, we need the limit to exist, AND to coincide with the function value. In other words, we'd need

$$\lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-} f(x)$$

1d. Circle the following properties that apply to $f(x)$ at $x = 0$.

Continuous Jump Discontinuity Removable Discontinuity

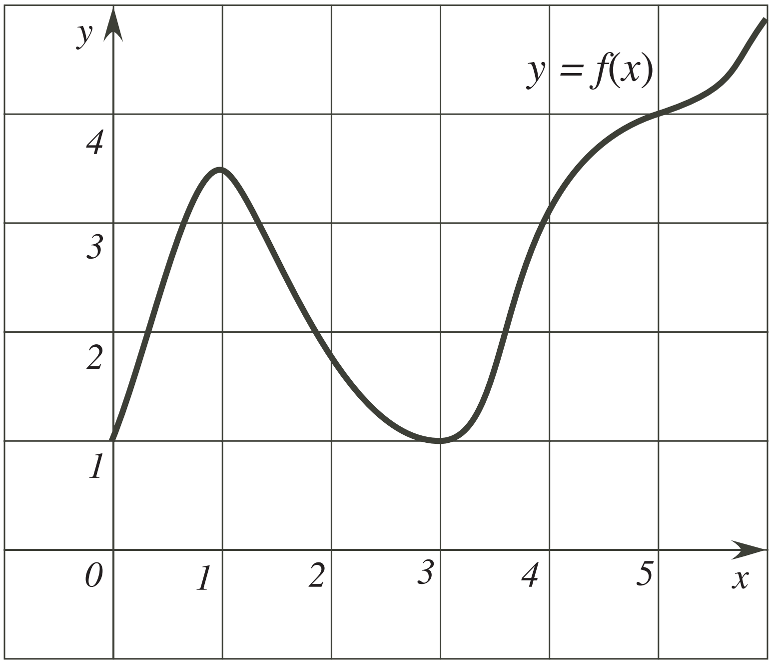
Left Continuous Right Continuous

We have a **jump discontinuity** since the one-sided limits are distinct. However, notice f is **right continuous** because $\lim_{x\to 0^+} f(x) = 1 = f(0)$, so the right-sided limit is equal to 1, and this matches the function value $f(0) = 1$.

10350 Tutorial Week 03 - Set 02 Name ______________________

Section ______________

2.



2a. Find the **average** rate of change of the function $f(x)$ over the interval $[3, 5]$.

We simply take

$$\frac{f(5) - f(3)}{5 - 3} = \frac{4 - 1}{2} = \frac{3}{2}$$

2b. Find the **instantaneous** rate of change of the function $f(x)$ at $x = 3$.

We look at the graph at $x = 3$, and notice this is a minimum (it's a valley), which means the slope of the tangent line is zero. So the instantaneous rate of change (aka slope of tangent) at $x = 3$ is **zero**.

2c. Is the **instantaneous** rate of change of the function $f(x)$ at $x = 4$ positive or negative?

Again we look at the graph, and notice that our function is **increasing** (tangent has positive slope) at $x = 4$. Hence the rate of change is **positive**.

2d. Order the **instantaneous** rates of change of the function $f(x)$ at $x = 1, 2, 4$ and 5 from smallest to largest in value.

(Smallest rate) $x = \underline{2}$; $x = \underline{1}$; $x = \underline{5}$; $x = \underline{4}$ (Greatest rate)

Notice at $x = 2$, the function is decreasing, so the slope is negative. At $x = 1$, we have a horizontal tangent (it's a max), so the slope is zero. The tangent lines to the curve at $x = 4$ and $x = 5$ have positive slope (the function is increasing), so to determine which one is has bigger slope, notice that the graph is steeper at $x = 4$, so that will be the bigger slope of the two. Hence the order is $2, 1, 5, 4$.

3. Consider an account with principle is \$2000 paying interest at an annual rate of 4% compounded **quarterly**.

3a. Find the balance of the account after 8 years. Simplify as far as possible and leave your answer in the form $k \cdot a^b$.

Recall the formula for compound interest is

$$P(t) = P_0 \cdot \left(1 + \frac{r}{n}\right)^{nt}$$

In our problem, $P_0 = 2000$ (the initial amount), $r = 0.04$ and $n = 4$ (we are compounding quarterly, so 4 times per year). We are interested in the amount after $t = 8$ years, so we have

$$2000 \cdot \left(1 + \frac{0.04}{4}\right)^{4 \cdot 8} = 2000 \cdot (1.01)^{32}$$

3b. How long will it take the balance of the account to increase 8 fold?

We want to solve for x in the equation

$$8 \cdot 2000 = 2000 \cdot \left(1 + \frac{0.04}{4}\right)^{4 \cdot x}$$

We begin by canceling the 2000 and taking natural log of both sides to isolate the exponent:

$$8 = 1.01^{4x}$$

$$\ln(1.01^{4x}) = \ln 8$$

Using rules of log, the $4x$ comes in front:

$$4x \ln 1.01 = \ln 8$$

$$4x = \frac{\ln 8}{\ln 1.01}$$

Finally we divide by 4

$$x = \frac{1}{4} \cdot \frac{\ln 8}{\ln 1.01} \simeq 52 \text{ years}$$