CALCULUS 3: EXAM 2 REVIEW

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Contents

Fun Problem: Show that the n-dimensional cube in $\mathbb{R}^n$ does not have any obtuse angles. In other words, if you pick 3 vertices or corners of the cube, the triangle determined by these is not obtuse. (Hint: think about vectors and angles).

1. Limits and Continuity

Problem 1. Find the limit (if it exists), or explain why the limit does not exist.

$$(a)\ \lim_{(x,y)\to(1,-1)} \frac{x^2y}{1+xy^2} \qquad (b)\ \lim_{(x,y)\to(0,0)} \frac{1}{x+y} \qquad (c)\ \lim_{(x,y)\to(0,0)} \frac{1}{x^2y^2}$$

$$(d)\ \lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}} \qquad (e)\ \lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1} \qquad (f)\ \lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+1)(y^2-1)}$$

$$(g)\ \lim_{(x,y)\to(0,1)} \frac{x^2}{y^2-1} \qquad (h)\ \lim_{(x,y,z)\to(0,0,0)} \frac{xy+xz+yz}{x^2+y^2+z^2} \qquad (i)\ \lim_{(x,y,z)\to(0,0,0)} \frac{x^2+2y^2+3z^2}{x^2+y^2+z^2}$$

Problem 2. Use polar coordinates to find the limit, or explain why the limit does not exist.

$$(a)\ \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} \qquad (b)\ \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} \qquad (c)\ \lim_{(x,y)\to(0,0)} \frac{x^2+xy+y^2}{x^2+y^2}$$

$$(d)\ \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} \qquad (e)\ \lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) \qquad (f)\ \lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$$

Problem 3. Discuss the limit/continuity of the function at the origin.

$$(a)\ f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2} \qquad (b)\ f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$$

If not continuous, define $f(0,0)$ such that f becomes continuous at the origin.

Problem 4. Determine the value of the constant c so that

$$g(x,y) = \begin{cases} \dfrac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ c & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous.

Problem 5. Argue that the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(\vec{v}) = (2\hat{\imath} - 3\hat{\jmath} + \hat{k}) \cdot \vec{v}$ is continuous.

Problem 6. Argue that the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(\vec{v}) = (6\hat{\imath} - 5\hat{k}) \times \vec{v}$ is continuous in each of its 3 components.

2. Partial Derivatives

Problem 7. Find a general formula for $\partial f / \partial x_i$, or evaluate it at the specified point.
(a) $f(x_1, \ldots, x_n) = x_1 + x_2^2 + x_3^3 + \cdots + x_n^n$ at the point $(1, 1, \ldots, 1)$.
(b) $u = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.
(c) $g = \sin(x_1 + 2x_2 + \cdots + nx_n)$ at the origin.

Problem 8. Consider the function $F(x,y,z) = 2x^3y + xz^2 + y^3z^5 - 7xyz$.
(a) Find F_{xx}, F_{yy}, and F_{zz}.
(b) Calculate the mixed partials F_{xy}, F_{yx}, F_{xz}, F_{zx}, F_{yz}, and F_{zy}, and verify that Clairaut's Theorem holds.
(c) Is $F_{xyx} = F_{xxy}$? How could you predict this without resorting to calculation?
(d) Is $F_{xyz} = F_{yzx}$? What about $F_{xxz} = F_{xzz}$?

Problem 9. The partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

is called **Laplace's equation**, and any function that satisfies this is called a **harmonic function**. Determine whether the given function is harmonic.
(a) $f(x,y) = x^2 + y^2$.
(b) $f(x,y) = x^2 - y^2$.
(c) $f(x,y) = e^x \cos y$ and $g(x,y) = e^x \sin y$.
(d) $f(x,y) = \ln(x^2 + y^2)$ defined on $\mathbb{R}^2 - \{0\}$.

It turns out we can generalize Laplace's equation to functions of n variables:

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 1$$

Discuss the harmonicity of the 3-variable functions $x^2 + y^2 - 2z^2$ and $x^2 - y^2 + z^2$.

Problem 10. (a taste of complex analysis) Suppose $w = x + iy$ is a complex number, where $x, y \in \mathbb{R}$ and $i^2 = -1$. We can talk about functions of a complex variable $f : \mathbb{C} \to \mathbb{C}$, where the input is a complex number, and the output is again a complex number.

Consider the function $f(w) = w^2$, which takes the number w and multiplies it to itself. Explicitly, this is given by

$$f(w) = w^2 = (x + iy)(x + iy) = (x^2 - y^2) + i(2xy) = u(x, y) + iv(x, y)$$

where $u = x^2 - y^2$ and $v = 2xy$ (the real and imaginary parts of f, respectively). This can be viewed as a vector-valued function $g(x, y) = \langle u(x, y), v(x, y)\rangle = \langle x^2 - y^2, 2xy\rangle$.

The complex analysis analog of a differentiable function is called a **holomorphic function**. One can check for holomorphicity of $f = u + iv$ by making sure that the real and imaginary parts $u(x, y)$ and $v(x, y)$ satisfy the **Cauchy-Riemann equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Use the Cauchy-Riemann equations to show that the complex function $f(w) = w^2$ is holomorphic.

Problem 11. Consider the function $u(x, y, z) = e^{ax+by+cz}$, where $a^2 + b^2 + c^2 = 1$. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = u$$

Can you generalize this result for $u = e^{a_1x_1+a_2x_2+\cdots+a_nx_n}$ with $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$?

3. Chain Rule

Problem 12. Redo all the examples from the chain rule lecture notes. After that, do all the "optional" chain rule problems from Worksheet 5.

Problem 13. (given as an exercise for the chain rule lecture) We know that the magnitude of the gravitational force between two point masses is inversely proportional to the square of the distance between them. In fact, it is equal to

$$F_g = G\frac{m_1 \cdot m_2}{r^2}$$

where G is the gravitational constant, and m_1, m_2 are the masses of the two objects. A similar relationship is found for the electric force and the magnetic force.

Suppose one point mass is fixed at the origin, another travels along the parabolic curve $\mathbf{C}(t) = \langle 2t, 1 - t^2, t^2\rangle$, and consider the simpler function modeling the gravitational force

$$F(r) = \frac{1}{r^2}$$

where r is the distance between the two charges. Find dF/dt. Can you find a point in space/time when the force is maximum? What can you say about the distance between the two charges at that point?

Problem 14. Suppose x and t are independent variables, $a \in \mathbb{R}$, and f, g are differentiable. Show that any function of the form $z = f(x + at) + g(x - at)$ satisfies the **wave equation**

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Problem 15. A function f is called **homogeneous of degree** n if it satisfies the equation $f(tx, ty) = t^n f(x, y)$, where $n \in \mathbb{N}$ and f has continuous second-order partials.

(a) Verify that $f(x, y) = x^2 y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

(b) Show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

(Hint: differentiate $f(tx, ty)$ with respect to t.)

Problem 16. Recall that if we have variables x, y, z related implicitly by $F(x, y, z) = 0$, then we can avoid using the longer method of implicit differentiation, and use the chain rule to compute various derivatives (see lecture notes on chain rule).

(a) Show that if x, y, z are related by $F(x, y, z) = 0$, then we obtain the following relation between second order partials:

$$\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) = -1$$

(b) Verify that this holds for the ellipsoid $ax^2 + by^2 + cz^2 = d$.

Problem 17. Let $\vec{x} = \langle x_1, x_2, \ldots x_n\rangle$ be any vector in $\mathbb{R}^n$, and for $i = 1, \ldots, n$, define

$$u_i = \frac{x_i}{\left(\sum_{i=1}^n x_i^2\right)^{1/2}}$$

Show that the vector $\vec{u} = \langle u_1, u_2, \ldots, u_n\rangle$ lies on the unit hypersphere in $\mathbb{R}^n$.

4. Problems Involving The Gradient

Recall that the gradient ∇f can be used to compute the tangent plane to a surface S in $\mathbb{R}^3$ defined by $f(x, y, z) = c$. For a point $(x_0, y_0, z_0) \in S$, this is given by

$$\nabla f \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

Problem 18. The equation $x^2 + y^2 + z^2 + w^2 = 4$ defines a **hypersphere of radius 2** in $\mathbb{R}^4$. Determine the hyperplane tangent to the hypersphere at the point $(-1, 1, 1, -1)$.

Note that this will be a linear equation of 4 variables, meaning it is an affine copy of $\mathbb{R}^3$ sitting in $\mathbb{R}^4$. This is similar to how a linear equation of 3 variables determines a plane in $\mathbb{R}^3$, but any such plane is just an affine copy of $\mathbb{R}^2$.

Problem 19. Give an equation for the normal line to the surface defined by $e^{xy}+e^{xz}-2e^{yz}=0$ at the point $(-1,-1,-1)$.

Problem 20. Find an equation to the hyperplane tangent to the $(n-1)$-dimensional ellipsoid

$$x_1^2+2x_2^2+3x_3^2+\cdots+nx_n^2=\frac{n(n+1)}{2}$$

at the point $(-1,-1,\ldots,-1)\in\mathbb{R}^n$.

Problem 21. Find an equation to the tangent hyperplane to the $(n-1)$-dimensional sphere

$$x_1^2+x_2^2+x_3^2+\cdots+x_n^2=1$$

at the point $(1/\sqrt{n},1/\sqrt{n},\ldots\ldots,1/\sqrt{n})\in\mathbb{R}^n$.

5. Min/Max and Lagrange Multipliers

There are basically two types of min/max problems. The first is min/max without a constraint, in which case we set $\nabla f=0$, and solve for critical points to find local min/maxes, and possible saddle points (the analog here is local min/max and inflection points from single variable). We look at the quantity

$$D=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix}=f_{xx}f_{yy}-(f_{xy})^2$$

since usually we have $f_{xy}=f_{yx}$.

The second type of problem is min/max f on a restricted domain $g(x,y)=c$, in which case we try to solve the system given by $\nabla f=\lambda\nabla g$, together with the constraint $g(x,y)=c$. See my notes on Lagrange multipliers for examples on how to do this, as well as Worksheet 5 and 6.

Of course, you might encounter a problem that asks to min/max f inside or outside $g(x,y)=c$ (or this constraint may be given by some inequality instead of having an equal sign), in which case you need to do both methods.

Problem 22. A unit "cube" in $\mathbb{R}$ is just an interval of length 1, say $[0,1]$. In $\mathbb{R}^2$, this becomes a square, with corners such as $(0,0),(0,1),(1,0),(1,1)$. In $\mathbb{R}^3$, the unit cube is an actual cube in the classical/geometrical sense, with side length 1. Calculate the length of the largest diagonal of the n-dimensional cube in $\mathbb{R}^n$.

Problem 23. Maximize the given function subject to the given constraint

(a) $f=x+y$ on $x^2+y^2=1$.

(b) $f=x+y+z$ on $x^2+y^2+z^2=1$.

(c) $f=x+y+z+w$ on the 4-dimensional sphere $x^2+y^2+z^2+w^2=1$.

(d) $f=x_1+x_2+\cdots+x_n$ on the n-dimensional sphere $x_1^2+x_2^2+\cdots+x_n^2=1$.

(e) $f=x^6+y^6+z^6$ on $x^2+y^2+z^2=6$.

Problem 24. Find the highest and lowest points on the ellipse obtained by intersecting the paraboloid $z = x^2 + y^2$ with the plane $x + y + 2z = 2$.

Problem 25. Heron's formula for the area of a triangle whose sides have lengths x, y, and z is given by

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

where $s = \frac{1}{2}(x + y + z)$ is the semiperimeter of the triangle. Use this to show that, for a fixed perimeter P, the triangle with largest area must be equilateral.

Problem 26. Show that a rectangular box of fixed surface area A and maximal volume must be a cube. Calculate the side length and volume in terms of A.

Problem 27. The cylinder $x^2 + y^2 = 4$ and the plane $2x + 2y + z = 2$ intersect in an ellipse. Find the points on the ellipse that are nearest to and farthest from the origin.

6. Multiple Integrals

Problem 28. Sketch the region of integration, then evaluate the iterated integral.

$$(a)\ \int_0^2 \int_0^{y^2} y\, dx\, dy \qquad (b)\ \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3\, dy\, dx \qquad (c)\ \int_{-1}^1 \int_0^{\sqrt{1-y^2}} 3\, dx\, dy$$

$$(d)\ \int_0^1 \int_0^x \sqrt{1-x^2}\, dy\, dx \qquad (e)\ \int_1^\infty \int_0^{1/x} y\, dy\, dx \qquad (f)\ \int_0^2 \int_x^2 x\sqrt{1+y^3}\, dy\, dx$$

(Hint: for the last one, you need to switch the order of integration.)

Problem 29. Sketch the region, switch the order of integration, then evaluate.

$$(a)\ \int_0^1 \int_{2x}^2 4e^{y^2}\, dy\, dx \qquad (b)\ \int_0^1 \int_y^1 \sin(x^2)\, dx\, dy \qquad (c)\ \int_0^2 \int_{y^2}^4 \sqrt{x}\sin x\, dx\, dy$$

Problem 30. Evaluate $\iint_D e^{x^2}\, dA$, where D is the triangular region with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

Problem 31. Find the volume of the given solid.

(a) Under the surface $z = 2x + y^2$ and above the region bounded by the curves $x = y^2$ and $x = y^3$.

(b) Enclosed by the paraboloid $z = x^2 + 3y^2$ and the planes $x = 0$, $y = 1$, $y = x$, and $z = 0$.

(c) Bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$.

(d) Enclosed by the cylinders $z = x^2$, $y = x^2$, and the planes $z = 0$ and $y = 4$.

(e) Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$.

Problem 32. Evaluate the integral.

(a) $\iint_D xy\ dA$ over the disk D centered at the origin and radius 3.

(b) $\iint_R \sqrt{4-x^2-y^2}\ dA$ over $R = \{(x,y) : x^2+y^2 \le 4, x \ge 0\}$.

(c) $\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\ dy\ dx$.

(d) $\int_0^a \int_{-\sqrt{a^2-y^2}}^{0} x^2 y\ dx\ dy$.

Problem 33. Find the volume of the given solid.

(a) Under the cone $z = \sqrt{x^2+y^2}$ and above the disk $x^2+y^2 \le 4$.

(b) Enclosed by the hyperboloid $-x^2-y^2+z^2 = 1$ and the plane $z = 2$.

(c) Above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$.

(d) Bounded by the paraboloids $y = 3x^2+3z^2$ and $y = 4-x^2-z^2$.

(e) Inside both the cylinder $x^2+y^2 = 4$ and the ellipsoid $4x^2+4y^2+z^2 = 64$.

Problem 34. In tutorial we learned that if $R = [a,b]\times[c,d]$, f is continuous on $[a,b]$, and g is continuous on $[c,d]$, then we can split the double integral as a product

$$\iint_R f(x)g(y)\ dA = \left(\int_a^b f(x)dx\right)\left(\int_c^d g(y)dy\right)$$

Prove this.

Problem 35. Find the volume of the solid in the first octant bounded above by the plane determined by points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$, where $a,b,c > 0$. Your answer should be $abc/6$, so the plane cuts this down a sixth of the volume of the rectangular parallelipiped $[0,a]\times[0,b]\times[0,c]$.

Problem 36. We computed in tutorial the value of the **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

The following variation plays a very important role in probability theory, and it is intimately connected to the **normal distribution** with mean zero and standard deviation σ:

$$\int_{-\infty}^{\infty} e^{-x^2/(2\sigma^2)} dx = \sigma\sqrt{2\pi}$$

Prove this result.

Note: Other topics that may appear on this exam are *Mass, Centers of Mass, and Moments*, and possibly *Triple Integrals*. Email the instructor after the break to get a better idea about this. Just because you don't see a topic here, it doesn't mean it's not covered on the exam (as a TA, I don't get to see the exam until the day of).