Quiz 10 Solutions

Q1: Which series below is a power series representation, valid for $-1 < x < 1$, of the function

$$f(x) = \frac{x^2}{1+x^3} \ ?$$

Solution: From our knowledge of geometric series, we know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad \text{for } |x| < 1.$$

Where the condition $|x| < 1$ is the same thing as $-1 < x < 1$. (The series above diverges when $|x| \geq 1$.) Now, making a substitution, we see

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n} = 1 - x^3 + x^6 - x^9 + \cdots \qquad \text{for } |x^3| < 1.$$

Note that $|x^3| < 1$ if and only if $|x| < 1$. Now multiplying through by x^2, we obtain

$$\begin{aligned}
\frac{x^2}{1+x^3} &= x^2 \sum_{n=0}^{\infty} (-1)^n x^{3n} && \text{for } |x| < 1 \\
&= \sum_{n=0}^{\infty} (-1)^n x^{3n} x^2 && \text{for } |x| < 1 \\
&= \sum_{n=0}^{\infty} (-1)^n x^{3n+2} && \text{for } |x| < 1. \\
&(= x^2 - x^5 + x^8 - x^1 1 + \cdots)
\end{aligned}$$

Q2: Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (6x+1)^n}{n+1}.$$

Solution: We compute

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| a_{n+1} \cdot \frac{1}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{\cancel{n}+1} (6x+1)^{\cancel{n}+1}}{n+2} \cdot \frac{n+1}{\cancel{(-1)^n}\cancel{(6x+1)^n}} \right| \\
&= \lim_{n\to\infty} \left| -(6x+1) \cdot \frac{n+1}{n+2} \right| \\
&= \lim_{n\to\infty} |6x+1| \cdot \frac{n+1}{n+2} \\
&= |6x+1| .
\end{aligned}$$

The ratio test says this series converges if $|6x+1| = 6\left|x+\frac{1}{6}\right| < 1$, which is the same as $|x+\frac{1}{6}| < \frac{1}{6}$. Thus, the radius of convergence $R = \frac{1}{6}$.