PARTIAL DERIVATIVES AND THE MULTIVARIABLE CHAIN RULE

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LAST TIME

We saw that for a function $z = f(x, y)$ of two variables, we can take the partial derivatives with respect to x or y. For the derivative with respect to x, we have several notations:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = f_x = D_x f$$

and similarly the partial derivative with respect to y:

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f = f_y = D_y f$$

- This can be extended to more than two variables, say $x_1, x_2, \ldots, x_n$.
- When differentiating with respect to one of the variables x_i, treat all the other variables as constants, and take derivative the usual way.

HIGHER ORDER DERIVATIVES

Just as we would with functions of a single variable, we may talk about second, third, fourth, and higher order derivatives for functions of several variables(if they exist). For $z = f(x, y)$, we do:

- differentiate twice with respect to x:
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$
- differentiate twice with respect to y:
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$
- differentiate with respect to x, then with respect to y:
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$
- differentiate with respect to y, then with respect to x:
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

The last two derivatives are called **mixed partial derivatives**.

Warning: In the $\dfrac{\partial^2 f}{\partial y \partial x}$ notation, the denominator read **right to left** gives the correct order of differentiation. In the f_{xy} notation, reading the usual way (left to right) gives the correct order.

Example 1. Find the second partial derivatives of $f(x, y) = 3xy^2 - 2y + 5x^2y^2$.

Solution: The first partials:

$$f_x = 3y^2 + 10xy^2 \qquad\qquad f_y = 6xy - 2 + 10x^2y$$

The second partials:

$$f_{xx} = 10y^2 \qquad f_{yy} = 6x + 10x^2 \qquad f_{xy} = 6y + 20xy \qquad f_{yx} = 6y + 20xy$$

Notice how the mixed second partials are equal. This is not a coincidence! The following theorem discusses when this phenomenon occurs.

Theorem 1. (Clairaunt) For a function $f(x, y)$ if f_{xy} and f_{yx} exist and are continuous on an open disk D in $\mathbb{R}^2$, then for any point $(x, y) \in D$ we have

$$f_{xy}(x, y) = f_{yx}(x, y)$$

This can also be applied to 3 or more variables.

Example 2. For the function $f(x, y, z) = ye^x + x \ln z$, show that

$$f_{xz} = f_{zx} \qquad \text{and} \qquad f_{xzz} = f_{zxz} = f_{zzx}$$

Solution: The relevant first partials are (we don't need f_y)

$$f_x = ye^x + \ln z \qquad f_z = \frac{x}{z}$$

The second partials are (notice the first two are equal, as desired):

$$f_{xz} = \frac{1}{z} \qquad f_{zx} = \frac{1}{z} \qquad f_{zz} = -\frac{x}{z^2}$$

Lastly, third partials (all 3 are equal!):

$$f_{xzz} = -\frac{1}{z^2} \qquad f_{zxz} = -\frac{1}{z^2} \qquad f_{zzx} = -\frac{1}{z^2}$$

Warning: When applying Clairaunt's Theorem to higher order derivatives on a function of several variables, such as $f(x, y)$, it is NOT usually true that f_{xxyy} will equal f_{xyyy}. It is only true that the derivatives associated to any permutation of $xxyy$ will be equal. In other words, each variable must occur the same number of times in the differentiation process. So (assuming the conditions of the theorem are met) we have

$$D_{xyyy} = D_{yxyy} = D_{yyxy} = D_{yyyx}$$

and

$$D_{xxyy} = D_{xyxy} = D_{xyyx} = D_{yxyx} = D_{yyxx} = D_{yxxy}$$

but we can't mix the two together.

THE CHAIN RULE

Recall from single variable calculus that if $y = f(x)$ and $x = g(t)$, where f and g are differentiable, then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

This can be extended to functions of several variables, but there are several cases to consider.

Theorem 2. (Chain Rule, One Independent Variable) Suppose $w = f(x, y)$ is a differentiable function of x and y. If $x = g(t)$ and $y = h(t)$ are both differentiable functions of t, then w is also differentiable with respect to t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

Example 3. Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find dw/dt when $t = 0$.

Solution: By the Chain Rule, we have

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} \\
&= (2xy)(\cos t) + (x^2 - 2y)(e^t) \\
&= 2(\sin t)(e^t)(\cos t) + (\sin^2 t - 2e^t)(e^t) \\
&= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}
\end{aligned}$$

and at $t = 0$, this is $dw/dt = -2$.

Note: For the one independent variable case, we could have substituted in $w = x^2y - y^2$ the functions for x and y to get $w(t) = e^t \sin^2 t - e^{2t}$ and taken the derivative the usual way.

Theorem 2 can be extended to any number of variables. Suppose we have $w = f(x_1, x_2, \ldots, x_n)$, and each **intermediate variable** x_i is a function of a single variable t. Then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial w}{\partial x_2} \cdot \frac{dx_2}{dt} + \cdots + \frac{\partial w}{\partial x_n} \cdot \frac{dx_n}{dt}$$

Another type of composite function is one in which the intermediate variables are themselves functions of more than one variable. The next theorem covers this scenario.

Theorem 3. (Chain Rule, Two Independent Variables) Suppose $w = f(x, y)$ is a differentiable function of x and y. If $x = g(s, t)$ and $y = h(s, t)$ such that the first partials

$$\frac{\partial x}{\partial s}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial s}, \text{ and } \frac{\partial y}{\partial t}$$

all exist, then $\partial w/\partial s$ and $\partial w/\partial t$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} \qquad \text{and} \qquad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Note: In general, consider $w = f(x_1, x_2, \ldots, x_n)$, where each intermediate variable x_i is a function of the independent variables $t_1, \ldots, t_k$. Then the partial derivative of w with respect to t_i is given by

$$\frac{\partial w}{\partial t_i} = \frac{\partial w}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial w}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial w}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

Example 4. Consider the function $w = xy + yz + xz$. Calculate $\partial w/\partial s$ and $\partial w/\partial t$ on the surface $\mathbf{r}(s,t) = \langle s\cos t, s\sin t, t\rangle$ when $s = 1$ and $t = 2\pi$.

Solution: By extending the result of Theorem 3, the partial with respect to s becomes

$$\begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}\\ &= (y+z)(\cos t) + (x+z)(\sin t) + (y+x)(0)\\ &= (y+z)(\cos t) + (x+z)(\sin t)\end{aligned}$$

and at $(s,t) = (1, 2\pi)$, we have $x = 1, y = 0, z = 2\pi$, so $\partial w/\partial s = 2\pi$.

For the partial with respect to t, we get

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}\\ &= (y+z)(-s\sin t) + (x+z)(s\cos t) + (y+x)(1)\end{aligned}$$

and for $(s,t) = (1, 2\pi)$ this becomes $\partial w/\partial t = 2 + 2\pi$.

IMPLICIT DIFFERENTIATION

Suppose x and y are related by the formula $F(x,y) = 0$. Typically we find the derivative dy/dx by differentiating implicitly, then solving for the dy/dx term, which can sometimes be annoying (a lot of writing). The Chain Rule provides a nicer alternative.

Consider the function $w = F(x,y)$. Taking derivative with respect to x, the Chain Rule gives

$$\frac{dw}{dx} = F_x\frac{dx}{dx} + F_y\frac{dy}{dx}$$

Now, since x and y were related by $F = 0$, the above must equal to zero, so

$$F_x + F_y\frac{dy}{dx} = 0$$

and solving for dy/dx gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

A similar argument can be made for $F(x,y,z) = 0$ implicitly relating variables x, y, z, or any number of variables. The following theorem states the precise result.

Theorem 4. (Chain Rule, Implicit Differentiation)

(1) If the equation $F(x,y) = 0$ defines y implicitly as a differentiable function of x, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}, \quad F_y \neq 0$$

(2) If the equation $F(x,y,z)$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}, \quad F_z \neq 0$$

Note: We can use the same idea to find $\partial y/\partial z$, which would be $-F_z/F_y$. Similarly, we may be interested in finding $\partial x/\partial y = -F_y/F_x$.

Example 5. Find dy/dx for the curve $y^3 + y^2 - 5y - x^2 + 4 = 0$.

Solution: We begin by finding the first partials of $F(x, y) = y^3 + y^2 - 5y - x^2 + 4$:

$$F_x = -2x \qquad F_y = 3y^2 + 2y - 5$$

Thus

$$\frac{dy}{dx} = -\frac{-2x}{3y^2 + 2y - 5} = \frac{2x}{3y^2 + 2y - 5}$$

(Notice how much faster this is than the usual method.)

Example 6. Find $\partial z/\partial x$ and $\partial z/\partial y$ for the surface $3x^2z - x^2y^2 + 2z^3 = 3yz + 5$.

Solution: We look at the function $F(x, y, z) = 3x^2z - x^2y^2 + 2z^3 - 3yz - 5$ (technically we could omit the constant since its derivative will be zero). The first partials are

$$F_x = 6xz - 2xy^2 \qquad F_y = -2x^2y + 3z \qquad F_z = 3x^2 + 6z^2 - 3y$$

and by Theorem 4, we have

$$\frac{\partial z}{\partial x} = \frac{2xy^2 - 6xz}{3x^2 + 6z^2 - 3y} \qquad \text{and} \qquad \frac{\partial z}{\partial y} = \frac{2x^2y - 3z}{3x^2 + 6z^2 - 3y}$$

Exercise: We know that the magnitude of the gravitational force between two point masses is inversely proportional to the square of the distance between them. In fact, it is equal to

$$F_g = G\frac{m_1 \cdot m_2}{r^2}$$

where G is the gravitational constant, and m_1, m_2 are the masses of the two objects. A similar relationship is found for the electric force and the magnetic force.

Suppose one point mass is fixed at the origin, another travels along the parabolic curve $\mathbf{C}(t) = \langle 2t, 1 - t^2, t^2 \rangle$, and consider the simpler function

$$F(r) = \frac{1}{r^2}$$

where r is the distance between the two charges. Find dF/dt. Can you find a point in space/time when the force is maximum? What can you say about the distance between the two charges at that point?