TEST 3 PRACTICE PROBLEMS

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Recall that in order to **analyze the graph of a function**, you need to:

- Determine the domain and range of the function
- Determine the intercepts, horizontal/vertical/slant asymptotes, and symmetry of the graph
- Locate the x-values where $f'(x)$ and $f''(x)$ are either zero or undefined. Use these results to determine relative (local) extrema, as well as points of inflection.

Problem 1. (Curve Sketching) Analyze and sketch the graph of the following functions

a) $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$

b) $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$

c) $f(x) = \dfrac{x}{\sqrt{x^2 + 2}}$

d) $f(x) = x^4 - 12x^3 + 48x^2 - 64x$

e) $f(x) = \dfrac{\cos x}{1 + \sin x}$

Problem 2. (Optimization Problems)

a) A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

b) Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

c) A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be 1.5 inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?

d) Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. They are to be connected by a snug wire, which touches the ground in a single point, running from ground level to the top of each post. Where should the wire meet the ground so that the least amount of wire is used?

e) Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area?]

f) A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$. What length and width should the rectangle have so that its area is maximum?

We know that Newton's Method is used to approximate zeroes a function $f(x)$ by starting with an initial guess x_1 for the zero, then iterating the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

to obtain better and better approximations for the zero.

Problem 3. (Newton's Method)

a) For $f(x) = x^2 - 5$ and $x_1 = 2.2$, iterate Newton's method twice to approximate the zero of the function. You may use a calculator.

b) Apply Newton's Method to approximate the positive x-value for which the graphs of $f(x) = 2x + 1$ and $g(x) = \sqrt{x+4}$ intersect. Use the initial guess of $x_1 = 1$, and find x_2. [Hint: consider the difference $h(x) = f(x) - g(x)$].

c) Use Newton's Method and the function $f(x) = x^n - a$ to obtain a general rule for approximating $\sqrt[n]{a}$.

d) A fixed point of a function is a value of $x = a$ such that $f(a) = a$. Approximate the fixed point of the function $f(x) = \cos x$ to two decimal places. You may use a calculator.

Recall that F is an **antiderivative** of f on an interval $[a, b]$ when $F'(x) = f(x)$ for all x in $[a, b]$. Any two antiderivatives of the same function f only differ by a constant.

Problem 4. (Antiderivatives)

a) Find the general solution of the differential equation $y'(x) = 2$.

b) $\int \frac{1}{x^3} dx$

c) $\int \sqrt{x} dx$

d) $\int (3x^4 - 5x^2 + x) dx$

e) $\int \frac{x+1}{\sqrt{x}} dx$

f) $\int \frac{\sin x}{\cos^2 x} dx$

One use for antiderivatives is to solve some very basic differential equations. These are equations that involve the derivatives of some function F, where the unknown we are trying to solve is that function F. They look like $dF/dx = f(x)$ which we can rewrite in differential form as $dF = f(x)dx$. One can get a general solution as we saw in Problem 4 by integrating

$$F(x) = \int dF = \int f(x)dx$$

If we are given an initial value $F(x_0) = a$, then we can determine a particular solution.

Problem 5. (Antiderivatives)

a) Find the general solution, then the particular solution that satisfies the initial condition $F(1) = 0$ of the differential equation $F'(x) = \dfrac{1}{x^2}$, $x > 0$.

b) A ball is thrown upward with an initial velocity of 64 feet per second from an initial height of 80 feet. Using -32 feet per second as the acceleration due to gravity, (i) find the position function $s(t)$ giving the height s as a function of time t, and (ii) determine when the ball hits the ground.

c) The rate of growth dP/dt of a population of bacteria is proportional to the square root of t, where P is the population size and t is the time in days. In other words, $dP/dt = k\sqrt{t}$ where k is a constant to be determined.

d) With what initial velocity must an object be thrown upward (from ground level) to reach the top of the Washington Monument (approx 550 feet)? Use -32 ft/s^2 for the acceleration due to gravity.

e) On the Moon, the acceleration due to gravity is -1.6 m/s^2. A stone is dropped from a cliff on the moon and hits the surface of the moon 20 seconds later. How far did it fall? What was its velocity at impact?

Sigma ($\sum$) notation is used as a way to write sums in more compact forms. Suppose we have a sequence of n numbers: $a_1, a_2, a_3, \ldots, a_n$. The sum of these n terms is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation**, a_i is the ith term of the sum, and the lower (starting) and upper (ending) bounds of the sum are 1 and n, respectively.

Note that the upper/lower bounds MUST be constants with respect to the index of summation i (it cannot depend on i). However, the lower bound doesn't need to start at 1. Any integer lower than or equal to the upper bound works. Here are a few examples (the first two are different ways of expressing the same sum in sigma notation):

$$\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$$

$$\sum_{i=0}^{5} (i+1) = 1 + 2 + 3 + 4 + 5 + 6$$

$$\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$$

$$\sum_{j=1}^{5} \frac{1}{\sqrt{j}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}}$$

$$\sum_{k=1}^{n} \frac{1}{n}(k^2+1) = \frac{1}{n}(1^2+1) + \frac{1}{n}(2^2+1) + \frac{1}{n}(3^2+1) + \cdots + \frac{1}{n}(n^2+1)$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$

Problem 6. (Sigma Notation)

a) Consider the sequence $\{2, 4, 8, 10, \ldots\} = \{a_1, a_2, a_3, a_4, \ldots\}$. Evaluate $\sum_{i=1}^{4} a_i$.

b) Evaluate $\sum_{j=1}^{3}(2j-1)$

c) Evaluate $\sum_{k=3}^{6} \frac{1}{2}k$

d) What's the difference between $\sum_{n=1}^{5}(2n+3)$ and $\sum_{n=1}^{5} 2n+3$? Compute both.

e) Evaluate $\sum_{n=1}^{4} nx$

f) Use $\sum$ notation to represent $3+6+9+12+\dots$ for 28 terms.

g) Use $\sum$ notation to represent $-3+6-12+24-48+\dots$ for 35 terms.

h) Use $\sum$ notation to represent $8.3+8.1+7.9+7.7+\dots$ for n terms.

i) Represent $7+14+21+28+35+\cdots+105$ using $\sum$ notation.

j) Evaluate $\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000, 10000$. (Find the closed form, then use your calculator)

k) Evaluate $\sum_{k=1}^{4}(k^2 + 1)$

l) Evaluate $\sum_{i=0}^{4}(x^{i+1} - x^i)$. Simplify your result.

m) Evaluate $\sum_{i=0}^{n}(2^{i+1} - 2^i)$. Simplify your result.

n) Evaluate $\sum_{j=1}^{n} \frac{2j+1}{n^2}$. What is the limit as $n \to \infty$ of your result?

Problem 7. (More Sigma Notation)
Use $\sum$ notation to write the sums.

$$\frac{1}{5(1)} + \frac{1}{5(2)} + \frac{1}{5(3)} + \cdots + \frac{1}{5(11)} =$$

$$\frac{9}{1+1} + \frac{9}{1+2} + \frac{9}{1+3} + \cdots + \frac{9}{1+14} =$$

$$\left[7\left(\frac{1}{6}\right) + 5\right] + \left[7\left(\frac{2}{6}\right) + 5\right] + \cdots + \left[7\left(\frac{6}{6}\right) + 5\right] =$$

$$\left[1 - \left(\frac{1}{4}\right)^2\right] + \left[1 - \left(\frac{2}{4}\right)^2\right] + \cdots + \left[1 - \left(\frac{4}{4}\right)^2\right] =$$

$$\left[\left(\frac{2}{n}\right)^3 - \frac{2}{n}\right]\left(\frac{2}{n}\right) + \cdots + \left[\left(\frac{2n}{n}\right)^3 - \frac{2n}{n}\right]\left(\frac{2}{n}\right) =$$

$$\left[2\left(1 + \frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) + \cdots + \left[2\left(1 + \frac{3n}{n}\right)^2\right]\left(\frac{3}{n}\right) =$$

Problem 8. Find a closed formula for the sum of n terms. Use this formula to find the limit as $n \to \infty$. Note: each of these sums represents a definite integral.

a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{24i}{n^2}$

b) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3i}{n}\right)^2 \left(\frac{3}{n}\right)$

Problem 9. Write the following limits of Riemann sums as a definite integral. Evaluate the integral.

a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n^3}(i-1)^2$

b) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{2i}{n}\right)^2 \left(\frac{2}{n}\right)$

c) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{i}{n}\right) \left(\frac{2}{n}\right)$

d) $\lim_{n\to\infty} \sum_{i=1}^{n} \left(1+\frac{3i}{n}\right)^3 \left(\frac{3}{n}\right)$

Problem 10. (Approximating Areas)
Approximate the area under the curve of the given function using left and right endpoints, and the specified number of subintervals.

a) $f(x) = 4x$ on $[-1, 1]$ with $n = 4$ subintervals.

b) $f(x) = 4x$ on $[0, 4]$ with $n = 4$ subintervals.

c) $f(x) = x^2$ on $[0, 4]$ with $n = 4$ subintervals.

d) $f(x) = x^2 + 4x$ on $[0, 4]$ with $n = 4$ subintervals.

Problem 11. (Geometric interpretation of integrals)
Sketch the region corresponding to each definite integral. Then evaluate each integral without using the fundamental theorem.

a) $\int_1^3 2dx$

b) $\int_0^3 (x+2)dx$

c) $\int_{-3}^3 \sqrt{9-x^2}dx$

d) $\int_{-\pi}^{\pi} \sin(x)dx$

e) $\int_{-2}^2 |x|dx$

Problem 12. Suppose a continuous integrable function f satisfies the following:

$$\int_0^5 f(x)dx = 8 \qquad \int_3^{10} f(x)dx = 10 \qquad \int_0^{10} f(x)dx = 21$$

Evaluate the following integrals.

a) $\int_5^5 f(x)dx$

b) $\int_0^5 2f(x)dx$

c) $\int_{10}^3 f(x)dx$

d) $\int_{10}^0 -5f(x)dx$

e) $\int_0^5 [f(x) - 2x]dx$

f) $\int_0^5 \left[2f(x) + \sin\left(\frac{2x}{5}\right)\right] dx$ [Hint: graph the sine part over the given interval]

g) $\int_3^5 f(x)dx$

The technique of u-substitution is useful in integrating functions that arise through differentiation using chain rule. For the indefinite integral (i.e. general antiderivative), this is fairly straightforward:

$$\int f(g(x)) \cdot g'(x)dx = F(g(x)) + C$$

where $g'(x)$ is the piece we normally get from chain rule, and F is an antiderivative of f. For example,

$$\int \frac{2x}{1+x^2}dx = \ln(1+x^2) + C$$

where the $2x$ would correspond to the $g'(x)$ piece, $f(x) = \ln x$, and $g(x) = 1 + x^2$.

To use this technique for definite integrals, we must also change the bounds.

Problem 13. (Substitution)
Determine the following integrals using u-substitution.

a) $\int x(x^2+1)^2dx$

b) $\int \sqrt{2x-1}dx$

c) $\int x\sqrt{2x-1}dx$

d) $\int \sin^2(3x)\cos(3x)dx$

e) $\int_0^1 x(x^2+1)^3 dx$

f) $\int_1^5 \frac{x}{\sqrt{2x-1}} dx$

g) $\int \tan x dx$

Problem 14. (Substitution)
Determine the following integrals using u-substitution.

a) $\int_1^2 2x^2\sqrt{x^3+1}dx$

b) $\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}dx$

c) $\int_0^2 \frac{x}{\sqrt{1+2x^2}}dx$

d) $\int_{-2}^6 x^2\sqrt[3]{x+2}dx$

Problem 15. (Substitution)

a) Without computing the integral, show that $\int_0^1 x^2(1-x)^5 dx = \int_0^1 x^5(1-x)^2 dx$.

b) Show that $\int_0^1 x^a(1-x)^b dx = \int_0^1 x^b(1-x)^a dx$.

c) Show that $\int_0^{\pi/2} \sin^2 x dx = \int_0^{\pi/2} \cos^2 x dx$.

d) If f is continuous and c is a constant, show that $\int_{ca}^{cb} f(x)dx = c\int_a^b f(cx)dx$.

Problem 16. (Graphing antiderivatives)

a) For $f(x) = x(x-1)$, sketch the graph of $g(x) = \int_{-1}^{x} f(t)dt$ where $x \geq -1$.

b) For $f(x) = x^3 - x$, sketch the graph of $g(x) = \int_{-2}^{x} f(t)dt$ where $x \geq -2$.

c) For $f(x) = x - \frac{1}{x}$, sketch the graph of $g(x) = \int_{0.1}^{x} f(t)dt$ where $x \geq 0.1$.

d) For $f(x) = x^2 - 2x + 1$, sketch the graph of $g(x) = \int_{0}^{x} f(t)dt$ where $x \geq 0$.