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Worksheet 2, Math 10560

Times indicate the amount of time that you would be expected to spend on the problem in on an exam.

1. (4 min) Use implicit differentiation to find $\frac{dy}{dx}$ if

$$(\ln 2)(\ln y) = 2^{x+y}.$$

Simplify your answer as much as possible.

Solution: We differentiate both sides with respect to x and solve for $\dfrac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}\left[(\ln 2)(\ln y)\right] &= \frac{d}{dx}\left[2^{x+y}\right] \\
\cancel{(\ln 2)} \cdot \frac{1}{y} \cdot \frac{dy}{dx} &= 2^{x+y} \cdot \cancel{(\ln 2)} \cdot \frac{d}{dx}[x+y] \\
\frac{1}{y} \cdot \frac{dy}{dx} &= 2^{x+y}\left(1 + \frac{dy}{dx}\right) \\
\frac{1}{y} \cdot \frac{dy}{dx} &= 2^{x+y} + 2^{x+y} \cdot \frac{dy}{dx} \\
\frac{1}{y} \cdot \frac{dy}{dx} - 2^{x+y} \cdot \frac{dy}{dx} &= 2^{x+y} \\
\left(\frac{1}{y} - 2^{x+y}\right)\frac{dy}{dx} &= 2^{x+y} \\
\left(\frac{1 - y \cdot 2^{x+y}}{y}\right)\frac{dy}{dx} &= 2^{x+y}
\end{aligned}$$

Hence, $\dfrac{dy}{dx} = \dfrac{y \cdot 2^{x+y}}{1 - y \cdot 2^{x+y}}$.

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2. (4 min) Compute the integral
$$\int_0^{\frac{1}{4}\ln 3} \frac{e^{4x}}{1+e^{4x}}\,dx.$$

Solution: The main idea here is to use u-substitution. This turns an integral we don't know how to solve into a simpler integral that we do. Let $u = 1 + e^{4x}$ (think about why we don't use $u = e^{4x}$), so $du = 4e^{4x}\,dx$, or, equivalently, $\frac{1}{4}du = e^{4x}dx$. Next, we change the bounds of the integration:

$$\begin{aligned} x = 0 &\Rightarrow u = 1 + e^0 = 2, \\ x = \frac{1}{4}\ln 3 &\Rightarrow u = 1 + e^{\ln 3} = 1 + 3 = 4. \end{aligned}$$

Now,

$$\begin{aligned} \int_0^{\frac{1}{4}\ln 3} \frac{e^{4x}}{1+e^{4x}}\,dx &= \frac{1}{4}\int_2^4 \frac{1}{u}\,du \\ &= \frac{1}{4}\left(\ln|u|\Big|_2^4\right) \\ &= \frac{1}{4}(\ln 4 - \ln 2) \\ &= \frac{1}{4}(\ln 2). \end{aligned}$$

3. (2-3 mins) Fill in the blanks in the following:

$5^{\sin x} = e^{\underline{(\sin x)(\ln 5)}}$ $\qquad \ln((\cos x)^{\ln x}) \;=\; \underline{\ln(x)}\; \ln(\cos x)$ $\qquad \log_{10}(x^2) = \underline{2/(\ln 10)}\,\ln x$

Solution: i) We rewrite $5^{\sin x}$, using the fact that $x = e^{\ln x}$ and the power rule for logarithms:

$$5^{\sin x} = e^{\ln(5^{\sin x})} = e^{(\sin x)(\ln 5)}.$$

ii) Using the power rule for logarithms we obtain $\ln((\cos x)^{\ln x}) = \ln(x) \cdot \ln(\cos x)$.
iii) Applying the power rule we get $\log_{10}(x^2) = 2\log_{10} x$. Next we apply the change of base formula $\log_a x = \frac{\ln x}{\ln a}$ to obtain a final answer of $2\frac{\ln x}{\ln 10}$, or $\frac{2}{\ln 10}(\ln x)$.

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4. Differentiate the functions:

(a) (4 mins) $g(u) = (2^{(u+1)^2})^3$

Solution: Note that $g(u) = (2^{(u+1)^2})^3 = 2^{3(u+1)^2}$.

The first way to approach this problem is to use chain rule, noting that $\frac{d}{dx}[a^x] = a^x \ln a$.

$$\begin{aligned} g'(u) &= 2^{3(u+1)^2} \cdot \ln 2 \cdot \frac{d}{du}[3(u+1)^2] \\ &= 2^{3(u+1)^2} \cdot \ln 2 \cdot 6(u+1) \\ &= 6(u+1)(\ln 2) \cdot 2^{3(u+1)^2} \end{aligned}$$

Alternatively, we can use logarithmic differentiation. We take the natural logarithm of both sides and bring down the exponent on the right-hand side:

$$\begin{aligned} \ln(g(u)) &= \ln(2^{3(u+1)^2}) \\ \ln(g(u)) &= 3(u+1)^2 \ln(2) \end{aligned}$$

Next, we differentiate:

$$\begin{aligned} \frac{d}{du}[\ln(g(u))] &= \frac{d}{du}[3(u+1)^2 \ln(2)] \\ \frac{1}{g(u)} \cdot g'(u) &= 6(u+1)(\ln(2)) \\ g'(u) &= 6(u+1)(\ln(2)) \cdot g(u) \\ g'(u) &= 6(u+1)(\ln(2)) \cdot 2^{3(u+1)^2}. \end{aligned}$$

On an exam either approach would be accepted, unless the problem specifically states to use particular method (e.g. logarithmic differentiation).

(b) (4 mins) $f(x) = (\tan x)^{\ln x}$

Solution: We will use logarithmic differentiation.

First take natural logarithm in both sides to bring down the exponent in the right hand side:

$$\begin{aligned} \ln(f(x)) &= \ln((\tan x)^{\ln x}) \\ \ln(f(x)) &= \ln(x) \cdot \ln(\tan x). \end{aligned}$$

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Next, we differentiate, remembering to use product rule on the right-hand side:

$$\begin{aligned}
\frac{d}{dx}\left[\ln(f(x))\right] &= \frac{d}{dx}\left[\ln(x)\ln(\tan x)\right] \\
\frac{1}{f(x)} \cdot f'(x) &= \frac{1}{x} \cdot \ln(\tan x) + \frac{1}{\tan x} \cdot \sec^2 x \cdot \ln x \\
f'(x) &= f(x)\left(\frac{\ln(\tan x)}{x} + \frac{\sec^2 x \cdot \ln x}{\tan x}\right) \\
f'(x) &= (\tan x)^{\ln x}\left(\frac{\ln(\tan x)}{x} + \frac{\sec^2 x \cdot \ln x}{\tan x}\right).
\end{aligned}$$

We note that:

$$\begin{aligned}
\frac{\sec^2 x}{\tan x} &= \frac{1/\cos^2 x}{\sin x/\cos x} \\
&= \frac{1}{\cos^{\cancel{2}} x} \cdot \frac{\cancel{\cos x}}{\sin x} \\
&= \frac{1}{\cos x \sin x}.
\end{aligned}$$

So an equivalent answer would be $f'(x) = (\tan x)^{\ln x}\left(\frac{\ln(\tan x)}{x} + \frac{\ln x}{\sin x \cos x}\right)$.

5. (3 mins) Evaluate the indefinite integral

$$\int x e^{x^2+5}\, dx.$$

Solution: Use u-substitution with $u = x^2 + 5$, $du = 2x\, dx$. Thus,

$$\begin{aligned}
\int x e^{x^2+5}\, dx &= \frac{1}{2}\int e^u\, du \\
&= \frac{1}{2}e^u + C \\
&= \frac{1}{2}e^{x^2+5} + C.
\end{aligned}$$

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6. (4 min) A bacteria culture contains 300 cells initially and grows at a rate proportional to its size (grows exponentially). After 5 hours the population has increased to 600. When will the population reach 6,000?

Solution: Let $y(t)$ = population of bacteria after t hours. Then $y(0) = 300$ and exponential growth formula gives:

$$y(t) = y(0)e^{kt} = 300e^{kt}$$

where k is some constant real number. Moreover, $k > 0$ because we have a growth model.

First, we need to find the value of k. Note that we are given $y(5) = 600$, hence we have:

$$600 = y(5) = 300e^{2k} \Rightarrow 2 = e^{5k}.$$

To solve for k we need to apply natural logarithm to both sides to bring down the exponent on the right-hand side:

$$\ln 2 = \ln e^{5k} = 5k \ln e = 5k \Rightarrow k = \frac{1}{5}\ln 2.$$

Hence,

$$y(t) = 300e^{(\frac{1}{5}\ln 2)t}.$$

(An alternative way of writing $y(t)$ is as follows:

$$y(t) = 300e^{(\frac{1}{5}\ln 2)t} = 300e^{(\frac{t}{5})\ln 2} = 300e^{\ln(2^{t/5})} = 300(2)^{t/5}.$$

This agrees with the fact that the population is doubling every 5 hours.)
Now, the question asks us to find for what value of $t = t_0$ we have $y(t_0) = 9000$. That is:

$$9000 = y(t_0) = 300e^{(\frac{1}{5}\ln 2)t_0} \Rightarrow 30 = e^{(\frac{1}{5}\ln 2)t_0}$$

and now we solve for t_0. AGAIN, since t_0 is in the exponent of the right hand side, we have to apply natural logarithm to both sides:

$$\ln 30 = \ln e^{(\frac{1}{5}\ln 2)t_0}$$
$$\ln 30 = \left(\frac{1}{5}\ln 2\right) t_0$$
$$t_0 = \frac{5\ln 30}{\ln 2}$$

So, after $\frac{5\ln 30}{\ln 2} \approx 24.5$ hours the population will reach 6000.