Independence

Example: A card is drawn at random from a standard deck of cards. Last time we considered the following events:

- H was the event that a heart is drawn
- R was the event that a red card is drawn

We found that

$$P(H|R) = \frac{1}{2} \neq P(H) = \frac{1}{4}$$

Since $P(H|R) \neq P(H)$, the event R has some influence on event H, since prior knowledge about R changes the probability. We say that H and R are **dependent**.

Independence

For the same experiment, consider the following events:

- F = the event that a face card (K,Q,J) is drawn
- R = the event that a red card is drawn

This time,

$$P(F|R) = \frac{6}{26} \text{ is equal to } P(F) = \frac{12}{52}$$

$P(F)$ is not influenced by the prior knowledge that the card is red. In this case, we say that the events F and R are **independent**.

Independence

Definition: Two events A and B are **independent** if

$$\boxed{P(A|B) = P(A)}$$

This is equivalent to the following:

$$\boxed{P(B|A) = P(B)}$$

and

$$\boxed{P(A \cap B) = P(A)P(B)}$$

For independent events, the chance that one will occur is not influenced in any way by the knowledge that the other has occurred.

Independence

The following are examples of **independent** events:

- Roll a fair six sided die twice and observe the pair of numbers that come up on each roll. Getting a six on the first roll does not affect the outcome of the second roll.
- Picking marbles from a bag, one at a time (with replacement), and observing the colour of each marble.

The following are examples of **dependent** events:

- Picking marbles from a bag, one at a time (without replacement), and observe the colour of each marble.
- Picking two students at random from the class for the role of class president and class treasurer.

Marbles In A Bag

Example: A bag has 6 red marbles and 4 blue marbles. I draw a marble at random from the bag and replace it, then I draw a second marble. What is the probability that at least one of the marbles is blue?

The two draws are independent. We have:
P(blue on the first) = 0.4
P(blue on the second) = 0.4
P(blue on both) = (0.4)(0.4) = 0.16 (this is where we use independence).

By IE principle, the probability of at least one blue is

$$0.4 + 0.4 - 0.16 = 0.64$$

Unconnected events

Example: The national soccer team of Romania has no known connection to the Notre Dame Lacrosse team. The chances that Romania their first game at the next world cup is 0.7, while the chances that the Notre Dame Lacrosse team will win their next game is 0.999.

It is reasonable to assume that the events R (Romania wins) and L (ND Lacrosse wins) are independent. Based on this assumption, the probability that both teams will win their next games is:

$$P(R \cap L) = P(R) \cdot P(L) = 0.7 \cdot 0.999 = 0.6993$$

Union of Independent Events

If A and B are independent, we can use the identity $P(A \cap B) = P(A) \cdot P(B)$ in our Inclusion-Exclusion principle:

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)}$$

Example: If E and F are independent events with $P(E) = 0.2$ and $P(F) = 0.4$, what is $P(E \cup F)$?

Since the events are *independent*, we have

$$P(E \cup F) = 0.2 + 0.4 - (0.2)(0.4) = 0.2 + 0.4 - 0.08 = 0.52$$

Union of Independent Events

Example: Draw a card at random from a deck of cards, and then draw a second card at random from a **different** deck of cards. What is the probability that both cards will be aces?

The probability of drawing an ace from a single deck is $\frac{4}{52} = \frac{1}{13}$.

Since we are using different decks, drawing an ace as the first card (event A_1) is **independent** from drawing an ace as the second card (event A_2).

Another way to see independence is by looking at $P(A_2|A_1) = \frac{4}{52} = P(A_2)$. Hence

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2) = \frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$$

Union of Independent Events

Note: If E and F are independent events, then their complements E' and F' are also independent. (Extra Credit!)

Example: Mary is taking a multiple choice quiz with two questions. Each question has 5 answer choices. Mary has no idea what the right answers are, so she guesses.

a) What are the chances that she gets both questions wrong?

$$\frac{4}{5} \cdot \frac{4}{5} = \frac{16}{25} = 64\%$$

Union of Independent Events

Example: Mary is taking a multiple choice quiz with two questions. Each question has 5 answer choices. Mary has no idea what the right answers are, so she guesses.

(b) What are the chances that she gets at least one question right?

The easiest way to do this is by using the complement principle: $100\% - 64\% = 36\%$.

Alternately, we can do $P(R, W) + P(W, R) + P(R, R)$:

$$\frac{1}{5} \cdot \frac{4}{5} + \frac{4}{5} \cdot \frac{1}{5} + \frac{1}{5} \cdot \frac{1}{5} = \frac{4 + 4 + 1}{25} = \frac{9}{25}$$

Multiple Independent Events

Given a collection $\{E_1, \ldots, E_n\}$ of independent events

$$P(E_1 \cap E_2 \cap \cdots \cap E_n) = P(E_1) \cdot P(E_2) \cdots P(E_n).$$

Example: There were 3 questions on Mary's quiz, and Mary makes a random guess for each question.

a) what are the chances that she gets all three correct?

$$\frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} = \frac{1}{125} = 0.008$$

b) What are the chances that she gets all three wrong?

$$\frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} = \frac{64}{125} = 0.512$$

Reliability Theory

A new phone has 4 independent electronic components of type B. Suppose each such component has a probability of 0.01 of failure within 10 years. What are the chances that at least one of these components will last more than 10 years?

Since components are independent, the failure of one does not influence the rest. We use the complement prinnciple. First compute the probability of all failing within 10 years:

$$0.01^4 = 0.00000001$$

We want the complementary probability:

$$1 - 0.00000001 = 0.99999999$$

Reliability Theory

b) If the probability of failure within 10 years for a component is 0.2, how many components should the manufacturer use in order to ensure that at least one will be operating after 10 years with probability 0.99?

The probability for at least one out of n components lasting more than 10 years is $1 - 0.2^n$.

We need the smallest integer n such that this probability is bigger than 0.99.

n	$1 - 0.2^n$
1	0.80
2	0.96
$\rightarrow$ 3	0.992
4	0.9984
5	0.99968

Repeating a trial many times

Example: A basketball player takes 4 independent free throws with a probability of .7 of getting a basket on each shot. Find the probability that he gets exactly 2 baskets.

B = gets a basket, M = misses. Any arrangement of $BBMM$ corresponds to him getting exactly 2 baskets. The probability is going to be

$$\frac{4!}{2! \cdot 2!} \cdot 0.7^2 \cdot (1 - 0.7)^2 = 0.2646$$

Checking for Independence

If any of the formulas

$$P(E \cap F) = P(E) \cdot P(F)$$

$$P(E|F) = P(E) \qquad\qquad P(F|E) = P(F)$$

hold true, then the other two are automatically true and E and F are independent.

To verify that two events are independent, we **only need to check one of the above 3 formulas**. We choose the most suitable one, depending on the information we are given.

Checking for Independence

Example: At a certain university, 50% of all students regularly attend football games and 60% of the first year students regularly attend football games. Choose a student at random, and consider the events:

- G = *the student attends football games regularly*
- F = *the student is a first year student*

We are given $P(G) = 0.5$ and $P(G|F) = 0.6$. The probability $P(F)$ is not given, but it can't be zero (there are first year students). Since $P(G|F) \neq P(G)$ and $P(F) \neq 0$, the events are not independent.

Example: If $P(E) = 0.3$, $P(F) = 0.4$, and $P(E \cap F) = .2$, are E and F independent? Since $0.3 \cdot 0.4 = 0.12 \neq 0.2$, these events are not independent.

Checking for Independence

Example: 300 students were asked if they thought that their online homework for Elvish 101 was too easy. The results are shown in the table below.

	Yes (Y)	No	Neutral (Ne)
Male (M)	75	39	36
Female	91	16	43

- M = *an individual selected at random is male*
- Ne = *an individual selected at random says "Neutral"*
- Y = *an individual selected at random says "Yes"*

Checking for Independence

	Yes (Y)	No	Neutral (Ne)
Male (M)	75	39	36
Female	91	16	43

a) What is $P(Ne)$? $P(Ne) = \frac{79}{300} \approx 0.263.$

b) What is $P(Ne|M)$?

There are 150 males, so $P(Ne|M) = 36/150 = 0.24.$

c) Are the events Ne and M independent?

$P(Ne|M) = 0.24$ and $P(Ne) = 0.263$ equal? No, so Ne and M are not independent. **BUT:** The values are very close! You might infer that there is no statistically significant difference between the two.

Mutually Exclusive Events And Independence

Recall that E and F are **mutually exclusive** if $P(E \cap F) = 0$.

d) Are the events Y and Ne from the previous example mutually exclusive?

Yes, since you cannot answer both yes and neutral at the same time.

Mutually exclusive events are typically **not** independent. Knowing that E occurs tells you **for certain** that F can't occur. The exception occurs when one of the event has probability zero, but we are rarely interested in such events, so we rarely encounter events which are both mutually exclusive and independent.

A Warning About Assuming Independence

Warning: sometimes our assumptions that seemingly unrelated events are independent can be wrong.

For an example where independence was assumed leading to serious consequences, see the reference to the trial of Sally Clark in the following video:

`Ted Talks: How Statistics Fool Juries`