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## Worksheet 8, Math 10560

1. (a) State the comparison test.

**Solution:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n$, then $\sum a_n$ is also convergent;

ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n$, then $\sum a_n$ is also divergent;

(b) State the limit comparison test:

**Solution:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

Where $c$ is a positive number, then either both series converge or both diverge.

(c) For what values of $p$ does the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge/diverge?

**Solution:** If $p > 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge and if $p \leq 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverge.

(d) Use the comparison test or the limit comparison test to determine which of the following series are convergent? ( Say which test you are using, what known series you are comparing to and show the work in making your conclusion of converges/diverges. )

i) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}\left(\frac{2}{3}\right)^n$

**Solution:** Since $\displaystyle\frac{\ln n}{n}\left(\frac{2}{3}\right)^n \leq \left(\frac{2}{3}\right)^n$, and $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = 2$. Apply the comparison test we get that $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}\left(\frac{2}{3}\right)^n$ converges.

ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(4n)^2 + n + 1}$ .

**Solution:** Since $\displaystyle\lim_{n\to\infty} \frac{n^2}{(4n)^2 + n + 1} = \frac{1}{16}$ and by (c) we know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Hence the limit comparison test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(4n)^2 + n + 1}$

converges.

iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{3n^2+7}}$

**Solution:** Since $\displaystyle\lim_{n\to\infty} \frac{n}{\sqrt[3]{3n^2+7}} = \infty$, it diverges following from the divergence test.

iv) $\displaystyle\sum_{n=100}^{\infty} \sin^2\left(\frac{\pi}{n^2}\right)$

**Solution:** We want to show that

$$\lim_{n\to\infty} \frac{\left(\frac{\pi}{n^2}\right)^2}{\sin^2(\frac{\pi}{n^2})} = 1$$

Then since $\displaystyle\sum_{n=100}^{\infty} \left(\frac{\pi}{n^2}\right)^2$ is convergent, follow from the limit comparison test we know that $\displaystyle\sum_{n=100}^{\infty} \sin^2\left(\frac{\pi}{n^2}\right)$ is also convergent. To compute the limit, let's set $f(x) = x^2$ and $g(x) = \sin^2(x)$. Then $f(\frac{\pi}{n^2}) = \left(\frac{\pi}{n^2}\right)^2$ and $g(\frac{\pi}{n^2}) = \sin^2(\frac{\pi}{n^2})$, Using L'Hospital rule, we have

$$\lim_{x\to 0} \frac{x^2}{\sin^2(x)} = \lim_{x\to 0} \frac{2x}{2\sin(x)\cos(x)} = \lim_{x\to 0} \frac{2}{2\cos^2(x) - 2\sin^2(x)} = 1$$

Hence

$$\lim_{n\to\infty} \frac{\left(\frac{\pi}{n^2}\right)^2}{\sin^2(\frac{\pi}{n^2})} = \lim_{x\to 0} \frac{x^2}{\sin^2(x)} = 1$$

2. (a) State the alternating series test:

**Solution:** If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

where $b_n > 0$ satisfies

- $b_n$ is decreasing i.e. $b_{n+1} \leq b_n$
- $\lim_{n\to\infty} b_n = 0$,

then the series converges.

(b) Can the alternating series test be used to show that a series diverges?

**Solution:** No.

(c) Can you conclude that any of the series shown below converges using the alternating series test? (if so give details)

i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

**Solution:** Let $b_n = \frac{1}{\sqrt{n}}$. Then $b_n > 0$, $b_n$ is decreasing and $\lim_{n\to\infty} b_n = 0$. Hence by the alternating series test the series converges.

ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}e^n}{\sqrt{n}}$

**Solution:** Let $b_n = \frac{e^n}{\sqrt{n}}$. Then $b_n > 0$ but increasing and goes to infinity. Hence we can not use the alternating series test to say anything about the convergence. On the other hand, we can apply the divergence test and conclude that it diverges.

iii) $\displaystyle\sum_{n=14}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^3+n}}$

**Solution:** Let $b_n = \frac{1}{\sqrt{n^3+n}}$. Then, $b_n > 0$ and $b_n$ is decreasing. Moreover, $\lim_{n\to\infty} b_n = 0$. Hence, by the alternating series test the series converges.

iv) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n 2^n}{(n-1)!}$

**Solution:** Let $b_n = \frac{2^n}{(n-1)!}$, then $b_n > 0$. To see $b_n$ is decreasing, we can compute the ratio:

$$\frac{b_{n+1}}{b_n} = \frac{\frac{2^{n+1}}{(n+1-1)!}}{\frac{2^n}{(n-1)!}} = \frac{2^{n+1}}{n!} \cdot \frac{(n-1)!}{2^n} = \frac{2}{n} < 1$$

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This also implies that

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{b_n}{b_{n-1}}\frac{b_{n-1}}{b_{n-2}}\cdots\frac{b_4}{b_3}b_3 \\
&= \lim_{n\to\infty} \frac{2}{n-1}\frac{2}{n-2}\cdots\frac{2}{3}\frac{8}{2} \\
&< \lim_{n\to\infty} \left(\frac{2}{3}\right)^{n-3} 4 = 0.
\end{aligned}$$

Therefore, by the Alternating Series Test, $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n 2^n}{(n-1)!}$ covnerges.

3. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{n^2}$ is an alternating series which satisfies the conditions of the alternating series test. Use the Alternating Series Estimation Theorem to determine the smallest $k$ on the list below so that the $k$-th partial sum is within $\frac{1}{100}$ of the actual sum.

**Solution:** Here $b_n = \frac{4}{n^2}$. Let $S_k$ be the $k$th partial sum of the series and $S$ be the "actual" sum of the series. Then the error is $R_k = S - S_k$. Now we want to find the smallest value of $k$ such that $|R_k| \le \frac{1}{100}$.

Using Alternating Series Estimation Theorem we see that $|R_k| \le b_{k+1}$. Now, we make

$$b_{k+1} \le \frac{1}{100}.$$

because this automatically gives us $|R_k| \le \frac{1}{100}$.

Now,

$$\begin{aligned}
b_{k+1} \le \frac{1}{100} &\Rightarrow \frac{4}{(k+1)^2} \le \frac{1}{100} \\
&\Rightarrow 400 \le (k+1)^2 \\
&\Rightarrow 20 \le k+1 \\
&\Rightarrow 19 \le k.
\end{aligned}$$

This implies that $k \ge 19$.