Expected Value

Example: Cassie, in need of extra money, decides to start an illegal gambling house. She devises a game of chance.

The game she proposes is to let the player cast a fair die. If $A = \{1, 2, 3\}$ occurs, the prize is \$1. For $B = \{4, 5\}$, the prize is \$5, and for $C = \{6\}$, the prize is \$11.

Q: How much should Cassie charge for playing the game?

If the game is played a large number of times, 3/6 of the time she pays out \$1, about 2/6 of the time she pays \$5, and about 1/6 of the time she pays \$11.

The approximate average payment is

$$\$1 \cdot \frac{3}{6} + \$5 \cdot \frac{2}{6} + \$11 \cdot \frac{1}{6} = \$4$$

so she should charge more than \$4 to make a profit.

Expected Value

In our example, we intrinsically defined a random variable U to be the payout Cassie makes to the player. The distribution of U was

U (payout)	P(U)
1	3/6
5	2/6
11	1/6

We computed was the **weighted average** for the U values.

Definition: If X is a discrete random variable with values $\{x_1, x_2, \ldots, x_n\}$, and corresponding probabilities are $p(x_1), p(x_2), \ldots, p(x_n)$, the **expected value** of X is

$$\boxed{E(X) = x_1 p(x_1) + x_2 p(x_2) + \cdots + x_n p(x_n)}$$

Expected Value Of A Function

Alternately, let $X =$ the outcome of the die, so

$$P(X) = \frac{1}{6}, \qquad X = 1, 2, 3, 4, 5, 6$$

and $u(X)$ is the payment given by

$$u(X) = \begin{cases} 1, & X = 1, 2, 3 \\ 5, & X = 4, 5 \\ 11, & X = 6 \end{cases}$$

The mathematical expectation of the payment is

$$\begin{aligned} E[u(X)] &= 1\left(\frac{1}{6}\right) + 1\left(\frac{1}{6}\right) + 1\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 11\left(\frac{1}{6}\right) \\ &= 1\left(\frac{3}{6}\right) + 5\left(\frac{2}{6}\right) + 11\left(\frac{1}{6}\right) = 4 \end{aligned}$$

Expected Value Of A Function

If X is a discrete random variable with values $\{x_1, \ldots, x_n\}$, and their corresponding probabilities are $p(x_1), \ldots, p(x_n)$, the **expected value** of the function $u(X)$ is

$$\boxed{E[u(X)] = u(x_1)p(x_1) + u(x_2)p(x_2) + \cdots + u(x_n)p(x_n)}$$

Note: $E(X)$ is the e.v. for the function $u(X) = X$.

We can think of $E[u(X)]$ as a **weighted mean** of $u(X)$.

Expected Value Of A Function

Example: For Cassie's game, compare the expected payout for the following payout schemes:

$$u(X) = \begin{cases} 1, & X = 1, 2, 3 \\ 5, & X = 4, 5 \\ 11, & X = 6 \end{cases} \qquad v(X) = \begin{cases} 0, & X = 1 \\ 1, & X = 2, 3, 4, 5 \\ 20, & X = 6 \end{cases}$$

We already know $E[u(X)] = 4$ from before. For the other payout scheme, the expected payment is going to be

$$E[v(X)] = 0\left(\frac{1}{6}\right) + 1\left(\frac{4}{6}\right) + 20\left(\frac{1}{6}\right) = \frac{24}{6} = 4$$

which is the same as $E[u(X)]$, so Cassie can charge the same amount for playing the game and make a profit, but the grand prize of \$20 may attract more players!

Properties Of Expected Value

The **expected value of a function** satisfies the following properties:

- If c is a constant, then $E(c) = c$.
- If c is a constant and $u(X)$ is a function, then

$$E[c \cdot u(X)] = c \cdot E[u(X)]$$

- For constants c, d and functions u, v, we have

$$E[c \cdot u(X) + d \cdot v(X)] = c \cdot E[u(X)] + d \cdot E[v(X)]$$

The last 2 properties are called **linearity of expectation**.

Warning: In general it is NOT true that for functions u, v, $E(uv) = E(u)E(v)$!

Properties Of Expected Value

Example: Consider the random variable X with distribution

$$P(X) = 1/3 \qquad X = -1, 0, 1$$

Calculate $E(X)$ and $E(X^2)$.

$$E(X) = (-1)\left(\frac{1}{3}\right) + (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) = 0$$

$$\begin{aligned} E(X^2) &= (-1)^2\left(\frac{1}{3}\right) + (0)^2\left(\frac{1}{3}\right) + (1)^2\left(\frac{1}{3}\right) \\ &= \frac{1}{3} + 0 + \frac{1}{3} = \frac{2}{3} \end{aligned}$$

Properties Of Expected Value

Example: Suppose that for a random variable X,

$$E(X) = 3 \qquad \text{and} \qquad E(X^2) = 10$$

Calculate the expected value of $X(5 - X)$.

$$\begin{aligned} E[X(5-X)] &= E(5X - X^2) \\ &= E(5X) - E(X^2) \\ &= 5E(X) - E(X^2) \\ &= (5)(3) - 10 = 5 \end{aligned}$$

Variance Of A Random Variable

For a set of data points $\{x_1, x_2, \ldots, x_n\}$, the variance is given by

$$\sigma^2 = \frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \cdots + (x_n - \mu)^2}{n}$$

and the standard deviation is the square root of σ^2. We can also compute the **variance of a random variable** X with values $\{x_1, \ldots, x_n\}$ and probabilities $p(x_1), \ldots, p(x_n)$:

$$\boxed{\sigma^2(X) = Var(X) = (x_1 - \mu)^2 \cdot p(x_1) + \cdots + (x_n - \mu)^2 \cdot p(x_n)}$$

Notice that this is the same as $Var(X) = E[(X - \mu)^2]$, which can be computed by the formula

$$\boxed{Var(X) = E(X^2) - E(X)^2 = E(X^2) - \mu^2}$$

Variance Of A Random Variable

Example: Suppose that for a random variable X,

$$E(X) = 3 \qquad \text{and} \qquad E(X^2) = 10$$

Compute the variance and the standard deviation of this random variable.

For the variance, we have

$$Var(X) = \sigma^2(X) = E(X^2) - E(X)^2 = 10 - 3^2 = 10 - 9 = 1$$

and the standard deviation is just the square root of the variance, so

$$\sigma(X) = \sqrt{1} = 1$$

Variance Of A Random Variable

Example: Consider the random variable X with distribution

$$P(X) = 1/3 \qquad X = -1, 0, 1$$

Compute $Var(X)$.

First we need the mean, and we saw $\mu = E(X) = 0$.

$$Var(X) = (-1-0)^2\left(\frac{1}{3}\right) + (0-0)^2\left(\frac{1}{3}\right) + (1-0)^2\left(\frac{1}{3}\right) = \frac{2}{3}$$

Alternately, we could have used $E(X^2) = 2/3$ from before:

$$Var(X) = E(X^2) - E(X)^2 = \frac{2}{3} - 0^2 = \frac{2}{3}$$

Insurance

Example: A piece of equipment is insured against early failure. It is known that the probability of failure is 10% during the first year, and 20% during the second or third.

An insurance company will pay $10,000$ if failure occurs during the first year, and $5,000$ if failure occurs during the second or third year. If failure occurs after the first 3 years, no payment will be made.

Calculate the expected value of the payment.

$$10,000(0.1) + 5,000(0.2) + 0(0.7) = 2,000$$

Insurance

Example: Motorcycle insurance is typically bought for a year. Suppose that in your age group, the probability of one or more claims during the one year period is 0.20, and the average repair cost for your bike model is $\$5,000$. Assuming the company breaks even, calculate the **insurance premium**:

a) With zero **deductible**
$(5,000 - 0)(0.20) = 1,000$

b) With a $200 deductible
$(5,000 - 200)(0.20) = 960$

c) With a $500 deductible
$(5,000 - 500)(0.20) = 900$

d) With a $800 deductible
$(5,000 - 800)(0.20) = 840$

Insurance

Example: Let X be the number of days that a certain patient needs to be in the hospital, with distribution

$$P(X) = \frac{5-x}{10}, \qquad X = 1, 2, 3, 4$$

If the patient receives \$200 from an insurance company **for each** of the first 2 days in the hospital, and \$100 **for each day** after the first two days, what is the expected payment for the hospitalization?

The expected payment is going to be

$$\$200P(1) + \$400P(2) + \$500P(3) + \$600P(4)$$

and using the distribution for X, we have

$$(200)\left(\frac{4}{10}\right) + (400)\left(\frac{3}{10}\right) + (500)\left(\frac{2}{10}\right) + (600)\left(\frac{1}{10}\right) = \$360$$

Gambling

Example: In a state lottery, a 3-digit integer is selected at random. A player bets \$1 on a particular number, and if that number is selected, the payoff is \$500 minus the \$1 paid for the ticket. If X is the payoff to the player (-1 or 499), find $E(X)$.

The number of 3-digit integers is the number of integers between 100 and 999 (inclusive): $999 - 100 + 1 = 900$. Of those 900 different numbers, each having an equal probability of being selected.

$$E(X) = (-1)\left(\frac{899}{900}\right) + (499)\left(\frac{1}{900}\right) \approx -0.444$$

so the player loses about 44 cents on average.

Gambling: Roulette

Example: A roulette wheel in U.S. has 38 numbers (18 Red, 18 Black, and 2 Green). In France, it has only 37 numbers (18R, 18B, 1G). A ball is rolled around the wheel and ends up in one of the slots with equal probability.

A player bets \$1 on red. He wins \$1 if the ball ends up in a red slot (and his \$1 bet is returned), and losese \$1 otherwise. Find the expected value of this game to the player in U.S. and in France.

U.S.: $(1)(18/38) + (-1)(20/38) \approx -0.0526$, so the player loses about 5 cents on average.

France: $(1)(18/37) + (-1)(19/37) \approx -0.027$, so the player loses about 3 cents on average.

Gambling: Craps

Example: In the gambling game craps, the player wins \$1 with probability 0.49293, and loses \$1 with probability 0.50707 for each \$1 bet. What is the expected value of the game to the player?

The expected gain is going to be

$$(-1)(0.50707) + (1)(49293) = -0.01414$$

so the player loses about 1 cent on average.

More Examples:

Example: A school has 20 classes: 16 with 25 students in each, three with 100 students in each, and one with 300 students, for a total of 1000 students.

a) What is the average class size?

$$(25)\left(\frac{16}{20}\right) + (100)\left(\frac{3}{20}\right) + (300)\left(\frac{1}{20}\right) = 50$$

b) A student is randomly selected out of 1000, and X is equal to the size of the class to which the student belongs. Find the distribution of X, and $E(X)$.

Possible values for X: $\{25, 100, 300\}$.

X	P(X)
25	$400/1000 = 40\%$
100	$300/1000 = 30\%$
300	$300/1000 = 30\%$

$$E(X) = (25)(0.40) + (100)(0.30) + (300)(0.30)$$

$$E(X) = 130$$

More Examples:

Example: Select at random an integer from $\{1, 2, 3, 4, 5\}$, and suppose the payment is equal to the reciprocal of the number (e.g. if 3 is selected the payment is 1/3). Find the expected payment.

$$\left(\frac{1}{1}\right)\left(\frac{1}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{5}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{5}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{5}\right) + \left(\frac{1}{5}\right)\left(\frac{1}{5}\right) \approx 0.457$$

Example: (Calculus) Select at random an integer from $\{1, \ldots, n\}$, and suppose the payment is equal to the reciprocal of the number (e.g. if 3 is selected the payment is 1/3). Find a formula for the expected payment. What if $n \to \infty$?

$$\left(\frac{1}{1}\right)\left(\frac{1}{n}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{n}\right) + \cdots + \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k} \to \int_0^1 \frac{dx}{x} = \infty$$