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M20550 Calculus III Tutorial
Worksheet 8

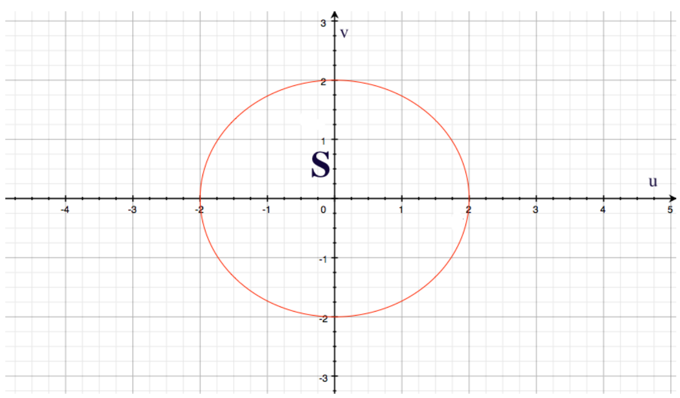
1. Compute $\displaystyle\iint_R \frac{1}{2}\, dA$ where R is the region bounded by $2x^2 + 2xy + y^2 = 8$ using the change of variables given by $x = u + v$ and $y = -2v$.

Solution: We know R is the region bounded by $2x^2 + 2xy + y^2 = 8$. Using the transformation $x = u + v$ and $y = -2v$, the boundary $2x^2 + 2xy + y^2 = 8$ will turn into

$$2(u+v)^2 + 2(u+v)(-2v) + (-2v)^2 = 8.$$

$$\implies u^2 + v^2 = 4.$$

So, the transformation of R, denote S, is the region bounded by the circle $u^2 + v^2 = 4$ in the uv-plane.



Before proceeding to compute the double integral, we need to find the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = (1)(-2) - (1)(0) = -2.$$

Thus,

$$\begin{aligned}\iint_R \frac{1}{2}\,dA &= \iint_S \frac{1}{2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dA \\ &= \int_0^{2\pi}\int_0^2 \frac{1}{2}|-2|r\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}r^2\Big|_{r=0}^{r=2}\,d\theta \\ &= \int_0^{2\pi} 2\,d\theta \\ &= 4\pi.\end{aligned}$$

2. Let R be the parallelogram enclosed by the lines $x+3y=0$, $x+3y=2$, $x+y=1$, and $x+y=4$. Evaluate the following integral by making appropriate change of variables

$$\iint\limits_R \frac{x+3y}{(x+y)^2}\,dA.$$

Solution: Observe the set of equations:

$$x+3y=0 \qquad\qquad x+3y=2$$
$$x+y=1 \qquad\qquad x+y=4$$

So, if we let

$$u = x+3y \quad \text{ and } v = x+y,$$

then the transformation of R, denote S, is given by the region bounded by the lines

$$u=0 \qquad\qquad u=2$$
$$v=1 \qquad\qquad v=4$$

So, S is the region bounded by the rectangle $[0,2]\times[1,4]$ in the uv-plane.

Next, we need to compute the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}.$$

In order to compute these partials, we need to write x and y in terms of u and v. We have

$$\begin{aligned} x + 3y &= u \quad (eq\ 1) \\ x + y &= v \quad (eq\ 2) \end{aligned}$$

$(eq\ 1) - (eq\ 2)$ is equivalent to $2y = u - v \implies y = \frac{1}{2}u - \frac{1}{2}v$. And $(eq\ 1) - 3(eq\ 2)$ gives $-2x = u - 3v \implies x = -\frac{1}{2}u + \frac{3}{2}v$. So,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}.$$

Note that since $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(x,y)}^{-1}$, we could have solved for the latter Jacobian instead and taken its reciprocal since it was a bit faster to compute in this case.

And so, we get

$$\begin{aligned} \iint\limits_R \frac{x+3y}{(x+y)^2}\, dA &= \iint\limits_S \frac{u}{v^2} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA \\ &= \int_1^4 \int_0^2 \frac{u}{v^2} \left|-\frac{1}{2}\right| du\, dv \\ &= \int_1^4 \frac{1}{4}u^2 v^{-2} \Big|_{u=0}^{u=2} dv \\ &= \int_1^4 v^{-2}\, dv \\ &= -\frac{1}{v}\Big|_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}. \end{aligned}$$

3. Evaluate the line integral $\int_C (z - 2xy)\, ds$ along the curve C given by $\mathbf{r}(t) = \langle \sin t, \cos t, t\rangle$, $0 \le t \le \frac{\pi}{2}$.

Solution: $\int_C (z - 2xy)\, ds$ is a line integral with respect to arc length (because of the ds at end). Since $\mathbf{r}(t) = \langle \sin t, \cos t, t\rangle$, we get $x(t) = \sin t$, $y(t) = \cos t$, $z(t) = t$. So, $z - 2xy = t - 2\sin t\cos t$. And $\mathbf{r}'(t) = \langle \cos t, -\sin t, 1\rangle$. So,

$$ds = |\mathbf{r}'(t)|dt = \sqrt{(x')^2 + (y')^2 + (z')^2}\,dt = \sqrt{\cos^2 t + (-\sin t)^2 + 1^2}\,dt = \sqrt{2}\,dt.$$

Thus, for $0 \le t \le \dfrac{\pi}{2}$,

$$\begin{aligned}\int_C (z - 2xy)\,ds &= \int_0^{\pi/2} (t - 2\sin t\cos t)\,\sqrt{2}\,dt \\ &= \sqrt{2}\left[\frac{1}{2}t^2 - \sin^2 t\right]_0^{\pi/2} \\ &= \sqrt{2}\left[\frac{\pi^2}{8} - 1\right].\end{aligned}$$

4. Find $\displaystyle\int_C 2xy^3\,ds$ where C is the upper half of the circle $x^2 + y^2 = 4$.

Solution: First, let's parametrize the curve C. C is the upper half of the circle $x^2 + y^2 = 4$. So, we can let

$$x(t) = 2\cos t, \qquad y(t) = 2\sin t \qquad \text{for } 0 \le t \le \pi.$$

Then, $x'(t) = -2\sin t$ and $y'(t) = 2\cos t$. Therefore,

$$ds = \sqrt{(x')^2 + (y')^2}\,dt = \sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt = \sqrt{4\sin^2 t + 4\cos^2 t}\,dt = 2\,dt.$$

Thus, for $0 \le t \le \pi$,

$$\begin{aligned}\int_C 2xy^3\,ds &= \int_0^{\pi} 2\,(2\cos t)\,(2\sin t)^3\;2\,dt \\ &= \int_0^{\pi} 64\left(\sin^3 t\right)(\cos t)\;dt \\ &= 16\left[\sin^4 t\right]_0^{\pi} \\ &= 0.\end{aligned}$$

5. Calculate the line integral $\displaystyle\int_C (y^2 + x)\,dx + 4xy\,dy$ where C is the arc of $x = y^2$ from $(1, 1)$ to $(4, 2)$.

Solution: First, we need to parametrize the curve C. Since C is a part of the curve $x = y^2$, we can let $y = t$; then we have $x = t^2$. Moreover, since the curve C is the part from $(1, 1)$ to $(4, 2)$, we get $1 \leq y \leq 2$. So, we have $1 \leq t \leq 2$. Thus, a parametrization of C is as follows:

$$x(t) = t^2, \qquad y(t) = t \qquad \text{for } 1 \leq t \leq 2.$$

Now, $\int_C (y^2 + x)\,dx + 4xy\,dy$ is a line integral with respect to x and y because we see the dx and dy. Here,

$$dx = x'(t)\,dt = 2t\,dt \quad \text{and} \quad dy = y'(t)\,dt = 1\,dt.$$

So, for $1 \leq t \leq 2$,

$$\begin{aligned}\int_C (y^2 + x)\,dx + 4xy\,dy &= \int_1^2 \left[\left(t^2 + t^2\right)2t + 4(t^2)(t)\right] dt \\ &= \int_1^2 8t^3\,dt \\ &= \left[2t^4\right]_1^2 \\ &= 2^5 - 2 = 30.\end{aligned}$$

6. Compute $\int_C x^2\,ds$ where C is the intersection of the surface $x^2 + y^2 + z^2 = 4$ and the plane $z = \sqrt{3}$.

Solution: The intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = \sqrt{3}$ is the circle

$$x^2 + y^2 + \left(\sqrt{3}\right)^2 = 4, \quad z = \sqrt{3}$$

$$\text{or simply} \quad x^2 + y^2 = 1, \quad z = \sqrt{3}.$$

Thus, a parametrization of C could be

$$\mathbf{r}(t) = \left\langle \cos t, \sin t, \sqrt{3} \right\rangle \quad \text{for } 0 \leq t \leq 2\pi.$$

Then, $\mathbf{r}'(t) = \langle -\sin t, \cos t, 0\rangle \implies |\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1.$

So $ds = |\mathbf{r}'(t)|\, dt = 1\, dt$. Finally, for $0 \le t \le 2\pi$,

$$\begin{aligned}
\int_C x^2\, ds &= \int_0^{2\pi} \left(\cos^2 t\right)\, dt \\
&= \int_0^{2\pi} \frac{1}{2}\left(1+\cos 2t\right) dt \\
&= \frac{1}{2}\left[t + \frac{1}{2}\sin(2t)\right]_0^{2\pi} \\
&= \pi.
\end{aligned}$$

7. Determine whether or not the following vector fields are conservative:
   (a) $\mathbf{F} = (3+2xy)\,\mathbf{i} + (x^2-3y^2)\,\mathbf{j}$
   (b) $\mathbf{F} = \mathbf{i} + \sin z\ \mathbf{j} + y\cos z\ \mathbf{k}$

Solution: (a) Since $\mathbf{F}$ is a vector field on $\mathbb{R}^2$, we use the criterion $\dfrac{\partial P}{\partial y} \overset{?}{=} \dfrac{\partial Q}{\partial x}$ to see if $\mathbf{F}$ is conservative or not. We have $\mathbf{F} = \langle 3+2xy, x^2-3y^2\rangle$. So, $P = 3+2xy$ and $Q = x^2-3y^2$ and

$$\frac{\partial P}{\partial y} = 2x = \frac{\partial Q}{\partial x}.$$

Since $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, $\mathbf{F}$ is a conservative vector field on $\mathbb{R}^2$.

(b) Since $\mathbf{F}$ is a vector field on $\mathbb{R}^3$ of the form $\langle P, Q, R\rangle$, we will need to check three separate equations, namely:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and} \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

In the first equation, both partial are equal to zero, so they are equal to each other. In the second equation, we also get zero for both partials. For the third equation, $\partial Q/\partial z = \cos z = \partial R/\partial y$. Hence the field is conservative.

We will see in class that this is equivalent to using the criterion curl $\mathbf{F} \overset{?}{=} \mathbf{0}$ to see if $\mathbf{F}$ is conservative or not. We have $\mathbf{F} = \langle 1, \sin z, y\cos z\rangle$. And

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & \sin z & y\cos z \end{vmatrix} = \langle \cos z - \cos z, 0, 0\rangle = \langle 0,0,0\rangle = \mathbf{0}.$$

Since curl $\mathbf{F} = \mathbf{0}$, $\mathbf{F}$ is a conservative vector field on $\mathbb{R}^3$.

8. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle y + yz\sin xy, y + xz\sin xy, y - \cos xy\rangle$ and C is given as the path traced out by $\mathbf{r}(t) = \langle 0, 4\sin t, 3\cos t + 2\rangle$ from $t = 0$ to 4π, i.e. a circle traced around twice.

Solution: We first notice that $\mathbf{F} = \langle y, y, y\rangle - \nabla f$, where $f = z\cos xy$. So we have $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \langle y, y, y\rangle \cdot d\mathbf{r} - \oint_C \nabla f \cdot d\mathbf{r}$, but the latter integral is zero by the Fundamental Theorem of Line Integrals. We see $d\mathbf{r} = \langle 0, 4\cos t, -3\sin t\rangle\, dt$, so our integral is now given by

$$\int_0^{4\pi} \langle 4\sin t, 4\sin t, 4\sin t\rangle \cdot \langle 0, 4\cos t, -3\sin t\rangle\, dt = \int_0^{4\pi} 16\sin t\cos t\; dt - \int_0^{4\pi} 12\sin^2 t\; dt$$

We see the first integral is 0 by periodicity, so we are left with

$$-12\int_0^{4\pi} \sin^2 t\; dt = -6\int_0^{4\pi} (1 - \cos 2t) dt = -24\pi + 6\int_0^{4\pi} \cos 2t\; dt = -24\pi$$

again by periodicity.