Combinations

Example: Five friends, Alan, Cassie, Maggie, Seth and Roger, have won 3 tickets for a concert. They can't afford to buy two more tickets. In how many ways can they choose three people out of the five to go to the concert?

Here is the list of all 60 permutations counted by $P(5, 3)$:

AMC	AMS	AMR	ACS	ACR
ACM	ASM	ARM	ASC	ARC
CAM	MAS	MAR	CAS	CAR
CMA	MSA	MRA	CSA	CRA
MAC	SAM	RAM	SAC	RCA
MCA	SMA	RMA	SCA	RAC
ASR	MSR	MCR	MCS	CRS
ARS	MRS	MRC	MSC	CSR
SAR	SMR	RMC	CMS	RCS
SRA	SRM	RCM	CSM	RSC
RSA	MRS	CRM	SMC	SCR
RAS	MSR	CMR	SCM	SRC

Combinations

This time, the order of selection **doesn't** matter (for example, AMC is the same group as ACM). These 60 possibilities are overcounting by a factor of 6. That leaves us with $60/6 = 10$ possibilities to hand out the 3 tickets:

AMC	AMS	AMR	ACS	ACR
ASR	MSR	MCR	MCS	CRS

With order mattering, there were $P(5,3)$ possibilities, but this overcount needs to be divided by $6 = 3!$ (the # of ways of ordering a group of three people), so the correct count is

$$\frac{60}{3!} = \frac{P(5,3)}{3!} = \frac{5!}{2!3!}.$$

Combinations

We have listed all the **combinations** of 3 out of 5 objects. The number of such combinations is denoted by $C(5,3)$. Sometimes we will write $\binom{5}{3}$ (read "5 choose 3").

Note: this is the same as listing all the subsets of size 3 of the set $\{A, C, M, R, S\}$

Definition: A **combination** of k objects out of n is a selection/subset of k objects chosen from among the n that are available. *The order in which the objects are chosen does not matter.*

Key properties of a combination:

- A combination selects elements from a single set.
- Repetitions are NOT allowed.
- Order is NOT significant.

Combinations

The **number of all possible combinations** of size k out of n is denoted by $C(n, r)$ or $\binom{n}{r}$ (“n choose r”).

In general, this can be computed by taking $P(n, k)$, and dividing by the number of permutations of k (which is $k!$):

$$\boxed{C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}}$$

Example: Evaluate $C(10, 3)$.

$$C(10,3) = \frac{10!}{3! \cdot 7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

Combinations

Example: How many ways are there to choose 7 people from a class of 40 students in order to put together a basketball team?

$$C(40, 7) = \frac{40!}{7! \cdot 33!} = 18,643,560$$

Example: In a soccer tournament with 15 teams, each team must play each other team exactly once. How many matches must be played?

Each match consists of two teams. The # of matches is

$$C(15, 2) = \frac{15!}{2! \cdot 13!} = \frac{15 \cdot 14}{2 \cdot 1} = 15 \cdot 7 = 105$$

Combinations

Example: A poker hand consists of five cards dealt at random from a standard deck of 52. How many different poker hands are possible?

$$C(52, 5) = \frac{52!}{5! \cdot 47!} = 2,598,960$$

Example: A standard deck of cards consists of 13 hearts, 13 diamonds, 13 spades, and 13 clubs. How many poker hands consist entirely of clubs?

$$C(13, 5) = \frac{13!}{5! \cdot 8!} = 1,287$$

Combinations

Example: How many poker hands consist of red cards only?

There are two red suits (hearts and diamonds), so we have a total of 26 red cards to pick from.

$$C(26, 5) = \frac{26!}{5! \cdot 21!} = 65,780$$

Example: How many poker hands consist of 2 kings and 3 queens?

There are 4 kings and 4 queens. We can select the 2 kings in $C(4, 2)$ ways, and the 3 queens in $C(4, 3)$ ways. Then, by the multiplication principle, we get a total of $C(4, 2) \cdot C(4, 3) = 6 \cdot 4 = 24$ such poker hands.

Combinations

Example: (Quality Control) A factory produces light bulbs and ships them in boxes of 50 to their customers. A quality control inspector checks a box by taking out a sample of size 5 at random, and checking if any of those 5 bulbs are defective. If at least one defective bulb is found, the box is not shipped. Otherwise the box is shipped. How many different samples of size five can be taken from a box of 50 bulbs?

$$C(50, 5) = 2,118,760$$

Combinations

Example: (Quality Control) If a box of 50 light bulbs contains 20 defective light bulbs and 30 non-defective light bulbs, how many samples of size 5 can be drawn from the box so that all of the light bulbs in the sample are good?

$$C(30, 5) = 142,506$$

Note: The total number of samples was 2,118,760. Compare that with the number of samples in which all the light bulbs are good, 142,506, a much smaller number!

Problems using a mixture of counting principles

Example: How many poker hands have at least two kings?

At least two kings means we can have 2, 3, or 4 kings.

There are $C(4,2)$ ways to get 2 kings and $C(48,3)$ ways to fill out the hand. That gives $C(4,2) \cdot C(48,3)$ hands with exactly 2 kings.

There are $C(4,3) \cdot C(48,2)$ hands with exactly 3 kings.

There are $C(4,4) \cdot C(48,1)$ hands with exactly 4 kings.

The number of hands with at least 2 kings is:

$$\binom{4}{2}\binom{48}{3} + \binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}$$

which is $6 \times 17,296 + 4 \times 1,128 + 1 \times 48 = 108,336$.

Problems using a mixture of counting principles

Example: In the Notre Dame Juggling Club, there are 5 graduate students and 7 undergraduates. Student Activities Office will fund 5 people to attend, as long as at least three are undergraduates. In how many ways can 5 people be chosen to go to the performance so that the funding will be granted?

Break the problem up, by the # of undergrads attending:

- 3 undergrads: $C(7,3) \cdot C(5,2) = 35 \cdot 10 = 350$
- 4 undergrads: $C(7,4) \cdot C(5,1) = 35 \cdot 5 = 175$
- 5 undergrads: $C(7,5) \cdot C(5,0) = 21 \cdot 1 = 21$

The number is $350 + 175 + 21 = 546$.

Warning: $C(7,3) \cdot C(9,2) = 1,260$ is this NOT corect!

Problems using a mixture of counting principles

Example: Gino's Pizza Parlor offers 3 three types of crust, 2 types of cheese, 4 veggie toppings, and 3 meat toppings. Pat always chooses 1 type of crust, 1 type of cheese, 2 veggie toppings, and 2 meat toppings. How many different pizzas can Pat create?

Apply the multiplication principle:

$$C(3,1) \cdot C(2,1) \cdot C(4,2) \cdot C(3,2) = 3 \cdot 2 \cdot 6 \cdot 3 = 108$$

Example: How many subsets of a set of size 5 have at least 4 elements?

Apply the addition principle. We can have 4 or 5 elements, so $C(5,4) + C(5,5)$ subsets.

Special Cases and Formulas

- From the formula $C(n,k) = \frac{n!}{k!(n-k)!}$, notice that $C(n,k) = C(n, n-k)$. Choosing k out of n elements to make a subset is the SAME as deciding which $n-k$ elements do not belong to the subset.
- $C(n,0) = 1$. There is exactly one subset with zero elements (the empty set $\emptyset$). For the formula to always hold, we want $\frac{n!}{0!(n-0)!} = 1$, so we define $0! = 1$.
- $C(n,1) = n = \frac{n!}{1!(n-1)!}$. There are n ways to choose a single element (subset of size one) out of n.
- $C(n,n) = 1$.

How many subsets does a set have?

A set of size 1, say $\{1\}$, has 2 subsets:

$$\emptyset \text{ and } \{1\}.$$

A set of size 2, say $\{1, 2\}$, has 4 subsets:

$$\emptyset,\ \{1\},\ \{2\} \text{ and } \{1, 2\}.$$

A set of size 3, say $\{1, 2, 3\}$, has 8 subsets:

$$\emptyset,\ \{1\},\ \{2\},\ \{3\},\ \{1, 2\},\ \{1, 3\},\ \{2, 3\} \text{ and } \{1, 2, 3\}.$$

In general, a set of size n has 2^n subsets. We can choose a subset by going through each element in turn, and deciding whether it is in the subset of not (2 choices for each element). By the multiplication principle we get a total of $\underbrace{2 \times 2 \times \ldots \times 2}_{n \text{ times}} = 2^n$ possible subsets.

How many subsets does a set have?

A set of size n has $\binom{n}{0}$ subsets of size 0, $\binom{n}{1}$ subsets of size 1, $\binom{n}{2}$ subsets of size 2, and so on. In all, it has

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

possible subsets by the addition principle. Therefore

$$\boxed{\begin{aligned} 2^n &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} \\ 2^n &= C(n,0) + C(n,1) + C(n,2) + \cdots + C(n,n) \end{aligned}}$$

Example: A set has ten elements. How many subsets does it have? $2^{10} = 1024$ subsets.

How many subsets does a set have?

Example: A set has ten elements. How many of its subsets have size at least 2?

$\binom{10}{2} + \binom{10}{3} + \binom{10}{4} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}$

This is incredibly annoying to compute. It is easier instead to use the complement principle: count the total number of subsets, and remove those that have sizes 0 and 1:

$$2^{10} - \left[\binom{10}{0} + \binom{10}{1}\right] = 1024 - (1 + 10) = 1013$$

Example: How many different tips could you leave at a restaurant, if you have a quarter, a \$1 coin, a \$5 bill, and a \$10 bill in your wallet?

You can leave any subset of your money. You have 4 items so there are $2^4 = 16$ possible tips.

The Binomial Theorem

How does this pattern continue?

- $(x+y)^0 = 1$
- $(x+y)^1 = x + y$
- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

The **Binomial Theorem** says that for any positive integer n and any two real numbers x and y, we can expand

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

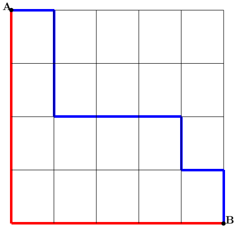
Example: If I fully multiply out $(x+y)^{11}$, what's the coefficient of x^4y^7? From the bin. theorem, it is $\binom{11}{4} = 330$.

Taxi Cab Geometry revisited

We saw that the number of taxi cab routes (only traveling south or east) from A to B is the number of permutations of SSSSEEEEE, which is

$$\frac{9!}{4!5!} = C(9,4) = C(9,5).$$

For instance, routes SSSSEEEEE (red) and ESSEEESES (blue) are drawn below.



# of routes = # of ways to choose 4 objects from a set of 9 objects. Each route has 9 steps total. Pick which 4 of them will be S's, and make the remaining 5 steps all E's.