Permutations

Example Alan, Cassie, Maggie, Seth and Roger want to take a photo in which three of the five friends are lined up in a row. How many different photos are possible?

AMC	AMS	AMR	ACS	ACR
ACM	ASM	ARM	ASC	ARC
CAM	MAS	MAR	CAS	CAR
CMA	MSA	MRA	CSA	CRA
MAC	SAM	RAM	SAC	RCA
MCA	SMA	RMA	SCA	RAC
ASR	MSR	MCR	MCS	CRS
ARS	MRS	MRC	MSC	CSR
SAR	SMR	RMC	CMS	RCS
SRA	SRM	RCM	CSM	RSC
RSA	MRS	CRM	SMC	SCR
RAS	MSR	CMR	SCM	SRC

60 ways, via an exhaustive (and exhausting!) list.

Permutations

Easier, using multiplication principle:

- 5 choices for the person on the left
- once we've chosen who should stand on the left, we have 4 choices for the position in the middle
- once we've filled both those positions, we have 3 choices for the person on the right

This gives a total of $5 \times 4 \times 3 = 60$ arrangements.

We have computed all **permutations** of the 5 friends, taken 3 at a time. We write

$$P(5,3) = 60$$

Permutations

A **permutation** of **n** objects taken **k** at a time is an arrangement of k of the n total objects *in a specific order*. The number of all permutations of n into k denoted by $P(n, k)$ (read "n Pee k" or "P of n k").

Remember:

- A permutation is an arrangement (or sequence of selections) of objects from a single set.
- Repetitions are not allowed (in our example, the photo AAA is not possible).
- The order in which the elements are arranged/selected is significant (in our example, the photographs AMC and CAM are different).

Permutations

Example: Calculate $P(10, 3)$, the number of permutations of 10 objects, taken 3 at a time.

- We will end up with arrangements of size 3
- There are 10 choices for the first position
- There are 9 choices for the second position
- There are 8 choices for the third (and last) position

$P(10, 3) = 10 \cdot 9 \cdot 8 = 720$. Notice that you multiply 3 numbers, starting with 10.

The General Formula

A general formula, using the multiplication principle, is:

$$\boxed{P(n,k) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdot (n-k+1)}_{k \text{ factors}}.}$$

Note: k is the number of consecutive factors in the product, starting out with the number n (similar to factorials).

Example: Compute the values of $P(5,2)$, $P(11,3)$, and $P(4,4)$.

$P(5,2) = 5 \cdot 4 = 20$

$P(11,3) = 11 \cdot 10 \cdot 9 = 990$

$P(4,4) = 4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$

Permutations

Example: In how many ways can you choose a President, Secretary and Treasurer for a group from 11 candidates, if (1) each candidate is eligible for each position, but (2) no candidate can hold 2 positions?

$P(11, 3) = 11 \cdot 10 \cdot 9 = 990.$

Q: Why can $P(11, 3)$ be used for assigning roles?

Permutations

Example: You have been asked to judge an art contest with 15 entries. In how many ways can you assign 1^{st} , 2^{nd} and 3^{rd} place? (Express your answer as $P(n, k)$ for the appropriate values of n and k, and evaluate.)

$P(15, 3) = 15 \cdot 14 \cdot 13 = 2,730.$

Example: Ten students are to be chosen from a class of 30 and lined up for a photograph. How many such photographs can be taken?

$P(30, 10) = 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21.$

Note: $30 - 10 = 20$ and we stopped at $30 - 10 + 1 = 21$.

$$P(30, 10) = 109,027,350,432,000$$

Factorials

Example: In how many ways can you arrange 5 math books your a shelf? $P(5,5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$

The number $P(n,n) = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$ is denoted by $n!$ (read "n factorial"). It counts the number of ways that n objects can be arranged in a row.

Note: $n!$ gets large *very fast*:

- $1! = 1$
- $2! = 2$
- $3! = 6$
- $4! = 24$
- $5! = 120$
- $6! = 720$
- $7! = 5,040$
- $8! = 40,320$
- $9! = 362,880$
- $10! = 3,628,800$

$59! \approx 10^{80}$ (roughly the number of particles in the universe)

Factorials

We can rewrite our formula for $P(n,k)$ in terms of factorials:

$$\boxed{P(n,k) = \frac{n!}{(n-k)!}.}$$

Example: Find $P(12,5)$.

$$\begin{aligned} P(12,5) &= \frac{12!}{(12-5)!} = \\ &= \frac{12 \cdot 11 \cdots 8 \cdot 7 \cdot 6 \cdots 2 \cdot 1}{7!} \\ &= 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = \\ &= 95{,}040. \end{aligned}$$

Factorials

Example: In how many ways can 10 people be lined up for a photograph?

$10! = \mathrm{Pr}(10, 10)$.

Example: How many three letter words (including nonsense words) can you make from the letters of the English alphabet, if letters cannot be repeated? Express your answer as $P(n, k)$ for the appropriate values of n and k, then evaluate.

$P(26, 3) = 26 \cdot 25 \cdot 24 = 15,600$.

Permutations of objects with some alike

Example: How many 4 letter words can we make by rearranging the letters of the word BEER?

The set $\{B, E, E, R\} = \{B, E, R\}$ but we really have 4 letters to work with (we must use both E's). So we work with the set $\{B, R, E_1, E_2\}$ (pretend for a moment there are two different E_1 and E_2). We arrange them in $4! = 24$ ways:

BRE_1E_2	RBE_1E_2	BE_1RE_2	RE_1BE_2	BE_1E_2R	RE_1E_2B
BRE_2E_1	RBE_2E_1	BE_2RE_1	RE_2BE_1	BE_2E_1R	RE_2E_1B

E_1BE_2R	E_1RE_2B	E_1BRE_2	E_1RBE_2	E_1E_2BR	E_1E_2RB
E_2BE_1R	E_2RE_1B	E_2BRE_1	E_2RBE_1	E_2E_1BR	E_2E_1RB

However words like BRE_1E_2 and BRE_2E_1 are really the same word, $BREE$, since there is no difference between E_1 and E_2 (they are both just E, counted twice). We are **overcounting** by some amount.

Permutations of objects with some alike

BRE_1E_2	RBE_1E_2	BE_1RE_2	RE_1BE_2	BE_1E_2R	RE_1E_2B
BRE_2E_1	RBE_2E_1	BE_2RE_1	RE_2BE_1	BE_2E_1R	RE_2E_1B

E_1BE_2R	E_1RE_2B	E_1BRE_2	E_1RBE_2	E_1E_2BR	E_1E_2RB
E_2BE_1R	E_2RE_1B	E_2BRE_1	E_2RBE_1	E_2E_1BR	E_2E_1RB

Question: By how much did we overcount?

For every word where E_1 comes before E_2, the same word appears again with E_2 coming before E_1. So every word is essentially counted TWICE!

Thus the number of different words we can form by rearranging the letters must be

$$4!/2 = \frac{4!}{2!} = 12 \text{ possible words.}$$

Note: 2! counts the number of ways we can interchange (i.e. permute) the two E's in any given arrangement.

Permutations of objects with some alike

The number of permutations of n objects where r of them are identical is given by

$$\boxed{\frac{n!}{r!}}$$

Note: the number $\frac{n!}{r!}$ is the same as $P(n, n-r)$.

Permutations of objects with some alike

Example: How many words (including nonsense words) can be made from rearrangements of the word ALPACA?

There are 6 letters in ALPACA, and A is repeated 3 times. Pretending for a moment that each A is different, we are looking to permute the set

$$\{A_1, A_2, A_3, L, P, C\}$$

There are 6! ways to do this, but we are **overcounting** by a factor of 3! (the number of possible ways to interchange or permute the indistinguishable copies of A). Hence the number of words is

$$\frac{6!}{3!} = \frac{720}{6} = 120.$$

Permutations of objects with some alike

Example: How many words can be made from rearrangements of the word BANANA?

From the 6 total letters, A is repeated 3 times, N is repeated twice, and B is repeated once. Pretending for a moment that each of these is distinct, we are looking at

$$\{B, A_1, N_1, A_2, N_2, A_3\}$$

and there are 6! permutations of this set. However we are overcounting by the following amounts:

- from the 3 copies of A, by a factor of 3!
- from the 2 copies of N, by a factor of 2!
- from the 1 copy of B, by a factor of 1!

Hence the number of words is:

$$\frac{6!}{1! \cdot 2! \cdot 3!} = 60$$

Permutations of objects with some alike

Suppose you have a collection of n objects, which reduces to k unique objects (once you ignore any repeats). Then the number of permutations of all n objects is

$$\frac{n!}{r_1! \cdot r_2! \cdots r_k!}$$

where r_1 is the number of copies of the first unique object, r_2 is the number of copies of the second unique object, and so on. We call these the **multiplicities** of each object.

Note: The sum of the multiplicities must equal the total number of objects, i.e. $r_1 + r_2 + \cdots + r_k = n$.

Note: if each of the n objects appears only once (no multiple copies), then each $r_i = 1$ and the denominator reduces to $1! \cdot 1! \cdots 1! = 1$, giving us $n!$ as before.

Permutations of objects with some alike

Example: How many words can be made from rearrangements of the letters of the word BOOKKEEPER?

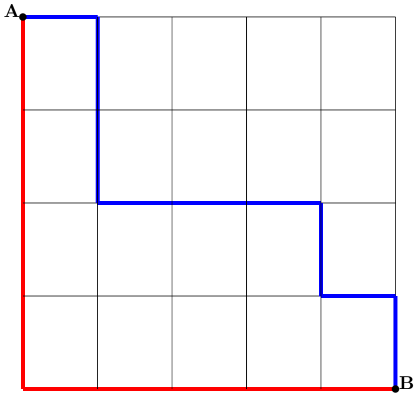
There are 10 letters in BOOKKEEPER. In alphabetical order, they are B, E, K, O, P, R. Their multiplicities are 1, 3, 2, 2, 1, 1, respectively. This means we have

$$\frac{10!}{1! \cdot 3! \cdot 2! \cdot 2! \cdot 1! \cdot 1!} = 151,200 \text{ rearrangements.}$$

Note: the total number of letters in the original word (this is the value of n) is the sum of the multiplicities of distinct letters: $10 = 1 + 3 + 2 + 2 + 1 + 1$.

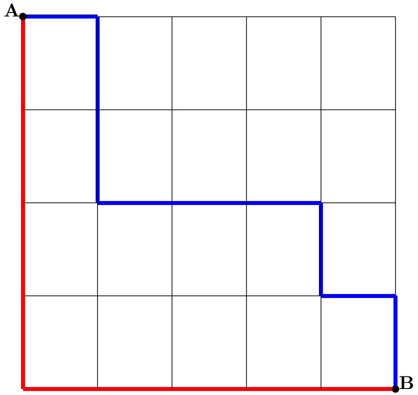
Taxi cab geometry

In how many ways can a taxi drive from A to B, if you can only travel eastward or southward?



Two possible routes are shown: SSSSEEEEE (in red) and ESSEEEESES (in blue).

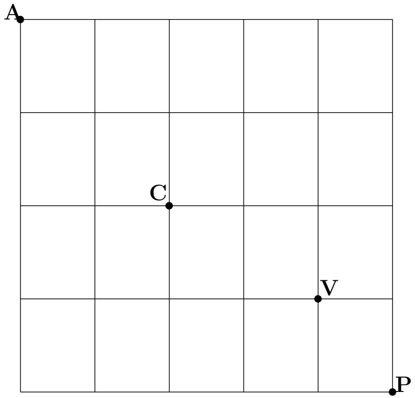
Taxi cab geometry



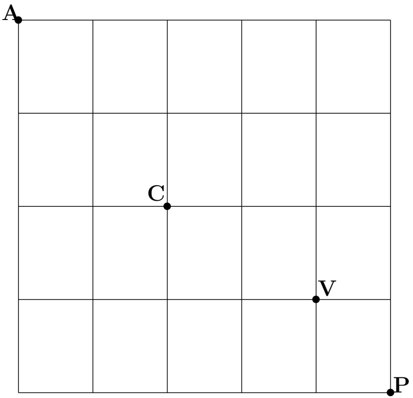
To get from A to B, the taxi must travel southward (S) **four** times, and eastward (E) **five** times. In totat, such paths must use 4 S's and 5 E's. Any rearrangement of SSSSEEEEE gives a valid route, and there are $\dfrac{9!}{4!5!}$ total.

Taxi cab geometry

Example: A streetmap of Mathville is given below. You arrive at the Airport (A) and wish to take a taxi to Pascal's house at P. The taxi driver, being of the honest sort, will take a route from A to P with no backtracking, always traveling south or east.



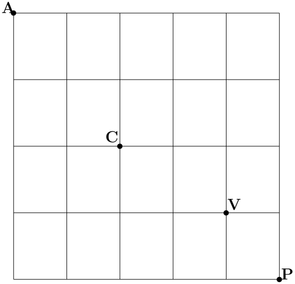
Taxi cab geometry



(a) How many such routes are possible from A to P?

You must 4 blocks south and 5 blocks east (a total of 9), so you have $\dfrac{9!}{4! \cdot 5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} = 9 \cdot 2 \cdot 7 = 126$ routes.

Taxi cab geometry

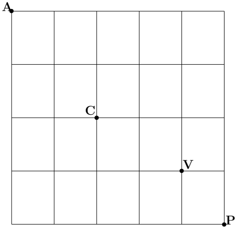


(b) If you insist on stopping off at the Combinatorium at C, how many routes can the taxi driver take from A to P?

This is really two taxicab problems combined with the Multiplication Principle. In words, there are a total of (# of paths from A to C) $\times$ (# of paths from C to P).

$$\left(\frac{4!}{2! \cdot 2!}\right) \times \left(\frac{5!}{2! \cdot 3!}\right) = 6 \times 10 = 60$$

Taxi cab geometry



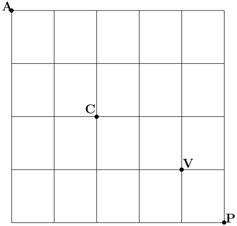
(c) If wish to stop off at both the Combinatorium (C) and the Vennitarium (V), how many routes can the taxi driver take?

There are three taxicab problems to consider. The # of paths is (# A to C) × (# C to V) × (# V to P). This is

$$\left(\frac{4!}{2! \cdot 2!}\right) \times \left(\frac{3!}{1! \cdot 2!}\right) \times \left(\frac{2!}{1! \cdot 1!}\right) = 6 \times 3 \times 2 = 36$$

possible routes from A to P by stopping first at C, then V.

Taxi cab geometry



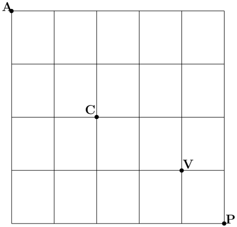
(d) If you wish to stop at either C or V (at least one, OR both), how many routes can the taxi driver take?

We need to use the Inclusion-Exclusion Principle.

$C = \{$all paths from A to P that go through C$\}$
$V = \{$all paths from A to P that go through V$\}$

We want is $n(C \cup V)$ since $C \cup V$ is the set of all paths which go through C, or V, or both.

Taxi cab geometry



$C = \{$all paths from A to P that go through C$\}$
$V = \{$all paths from A to P that go through V$\}$

We want is $n(C \cup V)$ since $C \cup V$ is the set of all paths which go through C or V.

$$\boxed{n(C \cup V) = n(C) + n(V) - n(C \cap V)}$$

We saw that $n(C) = \left(\frac{4!}{2! \cdot 2!}\right) \times \left(\frac{5!}{2! \cdot 3!}\right) = 6 \times 10 = 60.$

We compute $n(V) = \left(\frac{7!}{3! \cdot 4!}\right) \times \left(\frac{2!}{1! \cdot 1!}\right) = \frac{7 \cdot 6 \cdot 5}{6} \times 2 = 70.$

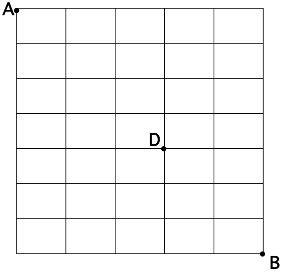
We still need $n(C \cap V)$, but this is the set of all paths which go through both C and V and we already computed this: $n(C \cap V) = \left(\frac{4!}{2! \cdot 2!}\right) \times \left(\frac{3!}{1! \cdot 2!}\right) \times \left(\frac{2!}{1! \cdot 1!}\right) = 6 \times 3 \times 2 = 36.$

Hence

$$n(C \cup V) = 60 + 70 - 36 = 94.$$

Taxi cab geometry

Example: Christine, on her morning run, wants to get from point A to point B.

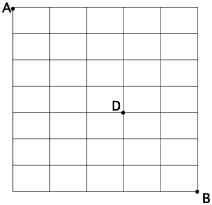


(a) How many routes with no backtracking can she take?

(b) How many of those routes go through the point D?

(c) Christine is deathly afraid of dogs and wants to avoid the Doberman at D. How many routes can she take?

Taxi cab geometry



(a) How many routes with no backtracking can she take?

(b) How many of those routes go through the point D?

(c) Christine is deathly afraid of dogs and wants to avoid the Doberman at D. How many routes can she take?

Let U be the set of all paths from A to B, and D be those paths that go through point D.

(a) $n(U) = \dfrac{(5+7)!}{5! \cdot 7!} = 792$

(b) $n(D) = \dfrac{(3+4)!}{3! \cdot 4!} \times \dfrac{(2+3)!}{2! \cdot 3!} = 35 \times 10 = 350$

(c) $n(D') = n(U) - n(D) = 792 - 350 = 442$