CALCULUS 3: STUDY GUIDE

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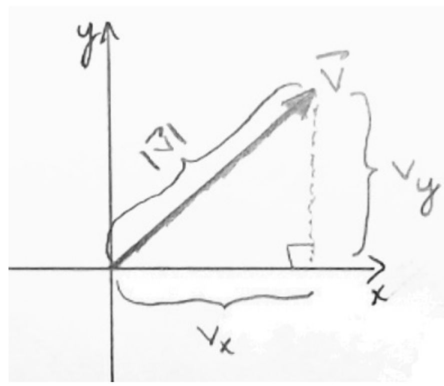
1. Vectors

A **vector** (denoted $\vec{v}$ or $\mathbf{v}$) is basically an ordered pair (in $\mathbb{R}^2$) or ordered triple (in $\mathbb{R}^3$) $\mathbf{v} = \langle v_x, v_y, v_z \rangle$.

We can talk about the **length/size/magnitude of a vector**, given by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$



We can do many operations with vectors, two of them are especially important. For vectors $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ and $\mathbf{v} = \langle v_x, v_y, v_z \rangle$, we can talk about the **dot product**

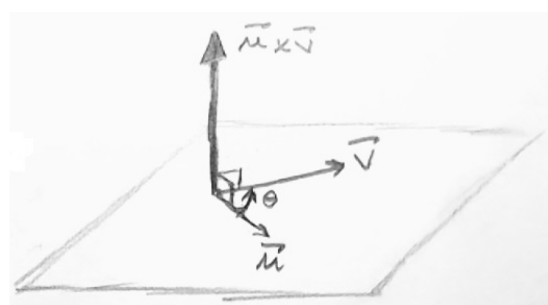
$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z$$

which is a scalar quantity. Geometrically this is related to the cosine of the angle θ between the vectors:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos(\theta)$$

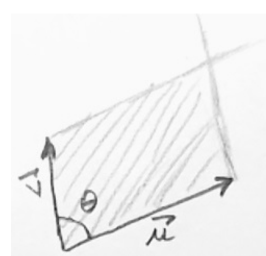
and so if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\mathbf{u} \cdot \mathbf{v} = 0$. We can also talk about the **cross product**

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$



which is a vector quantity that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. The size of the cross product also has a geometric interpretation, and is related to the sine of the angle θ between the vectors:

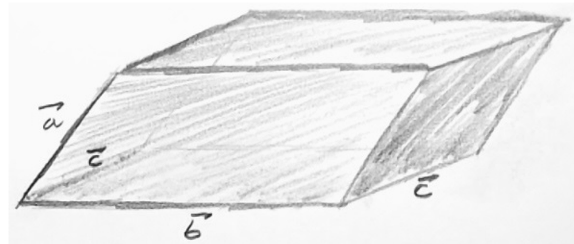
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin(\theta)$$



which means $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ for parallel vectors. Furthermore, the size is equal to the area of the parallelogram spanned by the two vectors.

If we start with 3 vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, they can form 3 adjacent sides of a parallelipiped. The volume of this object can be calculated using the triple product

$$\text{Volume} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

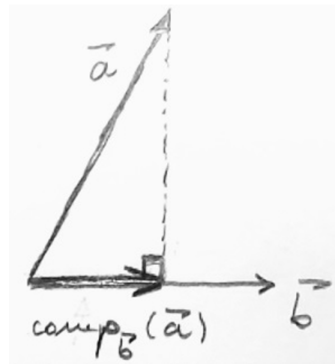


Another area where vector products appear is with **projections**. We have the scalar and vector projections of $\mathbf{a}$ onto $\mathbf{b}$, respectively:

$$\text{comp}_{\mathbf{b}}(\mathbf{a}) = \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|}$$

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} \right) \frac{\mathbf{b}}{|\mathbf{b}|}$$

Notice that here the quantity $\mathbf{b}/|\mathbf{b}|$ is the unit vector in the direction of $\mathbf{b}$.



2. Lines and Planes

The **equation of a line** with direction vector $\mathbf{d} = \langle a, b, c \rangle$ containing the point $\mathbf{r}_0 = (x_0, y_0, z_0)$ can appear in several forms:

$$\mathbf{r}(t) = \langle at + x_0, bt + y_0, ct + z_0 \rangle, \qquad \begin{cases} x(t) = x_0 + at \\ y(t) = y_0 + bt \\ z(t) = z_0 + ct \end{cases}, \qquad \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{d} \cdot t$$

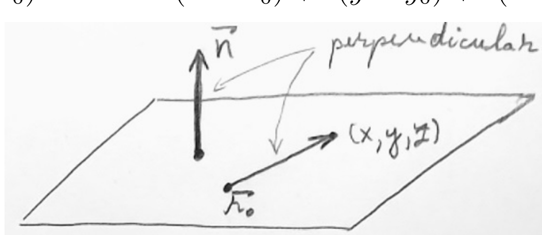
If the direction vector has all nonzero components, we may express this in symmetric form (by solving for the parameter t in 3 different ways):

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Note: $\mathbf{d}$ does not need to be a unit vector. Sometimes we may want to parametrize a **line segment** from $\mathbf{a}$ to $\mathbf{b}$. This can easily be done via the linear interpolation $\mathbf{r}(t) = (1-t)\mathbf{a} + t\mathbf{b}$ for $0 \leq t \leq 1$.

The **equation of a plane** $\mathbf{r} = \langle x, y, z \rangle$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ containing the point $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ can be written as

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \qquad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



It is important to remember the geometry here: the vector $\mathbf{n}$ is perpendicular to the plane, which is useful in solving problems like *finding a plane containing two vectors* (take the cross product to get the normal vector), or *finding a vector containing a line and a point* (use one vector from direction of line, then obtain another vector using the point before taking cross product to get $\mathbf{n}$).

We might also be interested in *finding the angle between two intersecting planes*, in which case we simply find the angle between the normal vectors of the planes.

We can talk about the distance from a point $P = (x_1, y_1, z_1)$ to a plane $ax+by+cz+d = 0$, which is given by

$$D = |\text{comp}_{\mathbf{n}}(\mathbf{b})| = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Similarly, to get the *distance between two nonintersecting planes*, one can first find a point on one of the planes (set the first two variables to anything, then solve for the third to get all 3 coordinates), then use the above formula to compute D.

3. Vector Functions

A **vector function** is nothing but a vector that depends on one or more parameters. For one parameter, we have

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

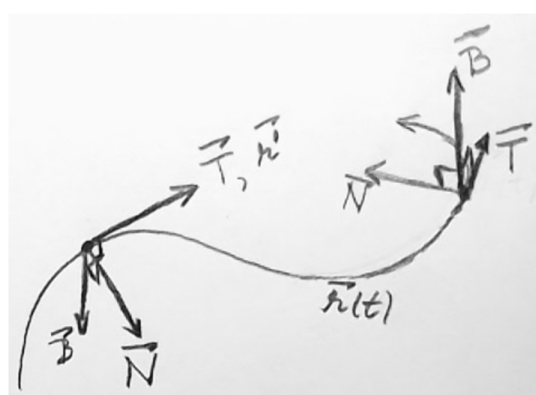
which describes a curve C in $\mathbb{R}^3$. For two parameters, we have

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

which describes a surface S in $\mathbb{R}^3$. In other words, the number of parameters equals the dimension of the object traced out by the vector function.

The **derivative** of a vector function is given by

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$



and it has the property that it is the tangent vector to the original curve C. We can talk about the **unit tangent vector**, which is computed by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|},$$

the **unit normal vector** given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|},$$

and the **binormal vector**

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

Of course, computing these takes a lot of time (because we are dividing by magnitudes, which are functions of t, and this introduces the necessity to use quotient rule when taking subsequent derivatives), and it is easier to use $\mathbf{r}'$ and $\mathbf{r}''$ to find them. This uses the crucial observation that $\mathbf{r}'$ and $\mathbf{r}''$ lie in the same plane determined by $\mathbf{T}$ and $\mathbf{N}$, so:

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{B} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \qquad \mathbf{N} = \mathbf{B} \times \mathbf{T}$$

The **osculating plane** is the plane that locally contains the curve. In other words, it contains $\mathbf{T}$ and $\mathbf{N}$, and so it has direction vector $\mathbf{B}$. But if you are asked to find this, it is easier to use direction vector $\mathbf{r}' \times \mathbf{r}''$ (which has the same direction as $\mathbf{B}$, thus eliminating the need to divide by the size).

One can interpret $\mathbf{r}(t)$ as the position of a particle at time t. Then we can talk about velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$, which are simply

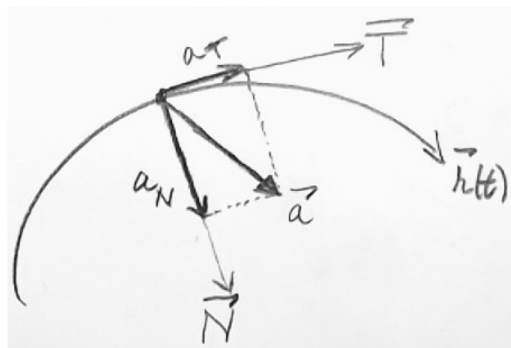
$$\mathbf{v}(t) = \mathbf{r}'(t) \qquad \mathbf{a}(t) = \mathbf{r}''(t)$$

For this application, we can ask about the **tangential and normal components of acceleration**, which can be computed via

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|}$$

I showed during a review session the geometric reasons behind these formulas.



Another question we can ask is the **arc length** of a curve C for $a \leq t \leq b$. This is computed via

$$L = \int_a^b |\mathbf{r}'(t)| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

while the **arc length function** is very similar:

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

4. Functions of Several Variables

For most of the course we studied functions of 2 variables $z = f(x, y)$, as these have a nice visualization as a 2-dimensional surface in $\mathbb{R}^3$. The domain is some subset of the xy-plane.

We can talk about **level curves**, which are basicaly curves in the domain that satisfy $f(x, y) = k$ (where k is a constant). The geometric interpretation is this: the level curve is the set of points (x, y) in the domain that give that give the same value/height when plugged into f.

We discussed **limits and continuity**, and I showed in tutorial and in previous reviews some strategy for approaching this. Review #2 has a lot of problems for you to practice on.

5. Partial Derivatives, Gradient, Directional Derivatives

Partial derivatives are fairly straight forward. For the derivative with respect to x, we have several notations:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = f_x = D_x f$$

and similarly the partial derivative with respect to y.

One application is finding the **tangent plane**, and for this we need to remember that the **gradient vector**

$$\nabla f = \langle f_x, f_y \rangle$$

is always perpendicular to the surface. However, if we try to find the tangent plane, we will use the following form for the gradient:

$$\langle f_x, f_y, -1 \rangle$$

The -1 comes from the fact that we really need 3 components to describe a vector that is perpendicular to a surface in $\mathbb{R}^3$. Clearly $\langle f_x, f_y \rangle$ only has two dimensions, which is not enough. To fix this, instead of doing $z = f(x, y)$, we consider the surface $f(x, y) - z = 0$ as a level curve of the function $F(x, y, z) = f(x, y) - z$, and take the gradient of F instead.

A more complex topic that arises from partial derivatives is the **chain rule**. For this, please review my notes on the chain rule (available on the tutorial page, under Week 5).

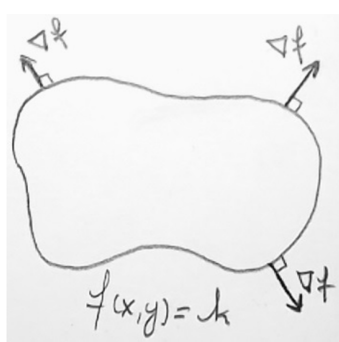
We know one of the properties of the gradient $\nabla f = \langle f_x, f_y \rangle$ is that **it points in the direction of maximum change**: if you are sitting on the hilly surface given by $z = f(x, y)$ and travel in the direction indicated by $\langle f_x, f_y \rangle$, then you will experience a maximum change in height. But we can also talk about traveling in other directions, and ask how much will the height change! This is the topic of **directional derivatives**. Suppose you travel in the direction of unit vector $\mathbf{u} = \langle a, b \rangle$ (always a unit vector!). Then the directional derivative of f in that direction is

$$D_{\mathbf{u}}(f) = \nabla f \cdot u = f_x a + f_y b$$

This means that the maximum value of the directional derivative is precisely the size of the gradient vector, as we would have to travel in the direction of $\mathbf{u} = \nabla f / |\nabla f|$ (since it must be a unit vector, and

$$\text{max change} = D_u(f) = \nabla f \cdot \frac{\nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$$

Relating $\vec{\nabla} f$ to level curves $f(x, y) = k$, the gradient will be perpendicular to the level curve, as show below:



6. Optimization

There are basically two types of min/max problems. The first is min/max without a constraint, in which case we set $\nabla f = 0$, and solve for critical points to find local min/maxes, and possible saddle points (the analog here is local min/max and inflection points from single variable). We look at the quantity

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

since usually we have $f_{xy} = f_{yx}$. If $D < 0$, then we have a saddle point. If $D > 0$, we need to look at the sign of f_{xx}. If it's positive, it is a min, otherwise it is a max. An easy way to remember this is to go back to single variable: positive second derivative = concanve up = local min, and viceversa.

The second type of problem is min/max f on a restricted domain $g(x, y) = c$, in which case we try to solve the system given by $\nabla f = \lambda \nabla g$, together with the constraint $g(x, y) = c$. See my notes on Lagrange multipliers for examples on how to do this, as well as Worksheet 5 and 6.

Of course, you might encounter a problem that asks to min/max f inside or outside $g(x, y) = c$ (or this constraint may be given by some inequality instead of having an equal sign), in which case you need to do both methods.

7. Multiple Integrals

We learned how to compute double or triple integrals over a 2-dimensional or 3-dimensional region, respectively:

$$\iint_R f(x, y)dA \qquad\qquad \iiint_R f(x, y, z)dV$$

and saw that dA or dV has various forms, depending on the coordinate system we use.

Differential	Cartesian	Cylindrical	Spherical
dA	$dx\ dy$	$r\ dr\ d\theta$	N/A
dV	$dx\ dy\ dz$	$rdz\ dr\ d\theta$	$\rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$

Using multiple integrals, we can compute things like the **average value of a function**. Here are the formulas for 1 and 2 dimensions:

$$\frac{1}{b-a}\int_a^b f(x)dx \qquad\qquad \frac{1}{\text{Area}(R)}\iint_R f(x, y)dA$$

Notice how the notion of length in $\mathbb{R}$ becomes area in $\mathbb{R}^2$ (and would become volume in $\mathbb{R}^3$).

One application is finding **center of mass** for a thin plate R (in $\mathbb{R}^2$). We need the total mass m, which is just

$$m = \iint_R \rho dA$$

and also the moments (my notation is a little different from the book):

$$M(x) = \iint_R x\rho dA \qquad M(y) = \iint_R y\rho dA$$

so then the center of mass is the point $\left(\frac{M(x)}{m}, \frac{M(y)}{m}\right)$. In order to compute this for a solid in $\mathbb{R}^3$, we would need to do triple integral over the solid, and compute one additional z-moment, but the idea is the same.

Some simpler computations include simply finding the area or volume of R:

$$\text{Area} = \iint_R dA \qquad\qquad \text{Volume} = \iiint_R dV$$

If you remember these, it will provide a good starting point for many problems, and the rest is just finding bounds, and computing dA or dV (depending on which coordinate system you're working with).

One challenging topic that arises is **change of variable** in an integral. Sometimes the region R in the xy-plane is difficult to integrate over, so instead we change it via a transformation T to a new region in a new uv-plane. Then

$$\iint_R f(x,y)dxdy = \iint_{T(R)} f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

I went over this at the board in tutorial and mentioned the relationship:

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \cdot \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 1$$

This can save some time if a situation arises where you don't need to solve for $x(u,v)$ and $y(u,v)$. You can simply use $u(x,y)$ and $v(x,y)$ to compute the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ by taking the reciprocal of $\frac{\partial(u,v)}{\partial(x,y)}$. **Warning:** the Jacobian is always taken to have a positive sign, even if the determinant somehow ends up being negative.

8. Vector Fields and Line Integrals

A **vector field** is similar to a vector function. The value of the vector depends on the coordinates/location (x,y,z) (in $\mathbb{R}^3$). Usually we break this up into components:

$$\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

where P, Q, R are scalar functions. If we have a vector field in $\mathbb{R}^2$, then $R = 0$, so we can simply specify it using two components $\mathbf{F} = \langle P, Q\rangle$.

The idea of a **field** arises from physics, where we talk about gravitational, magnetic, electric fields, which act on particles and give rise to their corresponding forces. Hence we can talk about the **work done by a field F** as a particle moves along a curve C. This is a **line integral**

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a parametrization of the curve C, and differential $d\mathbf{r} = \mathbf{r}'(t)dt$. Hence, incomputing the work over $a \le t \le b$, we are actually doing

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \langle x'(t), y'(t), z'(t)\rangle dt$$

which is just a single dimensional integral in the parameter t. Some notations for $d\mathbf{r}$:

$$d\mathbf{r} = \langle dx, dy, dz \rangle = \langle x'(t), y'(t), z'(t) \rangle dt$$

Because of this, one may see line integrals in the form

$$\int_C Pdx + Qdt + Rdz$$

but this is exactly $\int_C \mathbf{F} \cdot d\mathbf{r}$, since

$$\int_C Pdx + Qdt + Rdz = \int_C \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle = \int_C \mathbf{F} \cdot d\mathbf{r}$$

If we want to traverse C in the opposite direction (which we denote by $-C$), we have

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$$

One special type of field is a **conservative field**, which is a field $\mathbf{F}$ that arises as a gradient of some scalar function f. Thus if we can find f such that

$$\mathbf{F} = \nabla f$$

then we know for sure that $\mathbf{F}$ is conservative. Every conservative field is of this form.

One nice property of conservative fields $F = \nabla f$ is that the integral is **path independent**, so the value of the integral only depends on the starting point A and ending point B of the curve:

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$$

This is called the **fundamental theorem of line integrals**. To make an analogy to the fundamental theorem from single variable calculus, one can think of f as an "antiderivative" of ∇f, and the gradient ∇f as a derivative of a function. This analogy also works when we think about tangent lines to a curve, and their generalization to surfaces: tangent planes (we end up using the gradient to find these).

Path independence also means that if C_1 and C_2 are two curves with the same initial and final starting point $A \to B$, then

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r} = f(B) - f(A)$$

Clairaunt's theorem easily gives the following property of a conservative vector field $\mathbf{F} = \langle P, Q \rangle$:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Furthermore, the converse is also true: if the above relationship holds true for all points throughout some domain D, then $\mathbf{F}$ is conservative on that domain!

9. Div, Curl, Surface Integrals

For a vector field $\mathbf{F}$, we have two operations that are important. First is the **divergence** of the field,

$$\mathrm{div}(\mathbf{F}) = \nabla \cdot F$$

which is a scalar quantity, and the **curl** of the field, given by

$$\mathrm{curl}(\mathbf{F}) = \nabla \times F$$

which is a vector quantity, where ∇ is the operator

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

If F is conservative, then its curl must always be the zero vector:

$$\mathrm{curl}(\nabla f) = \mathbf{0}$$

Also, for ANY field $\mathbf{F}$, we have

$$\mathrm{div}\ \mathrm{curl}\ \mathbf{F} = 0$$

We can also talk about **surface integrals** over a surface S parametrized by $\mathbf{r}(u, v)$. We can integrate a scalar function over a surface S by

$$\iint_S f\ dS$$

where $dS = |\mathbf{r}_u \times \mathbf{r}_v| dudv$, or we can compute the **flux** of a field $\mathbf{F}$ over the surface

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v) dudv$.

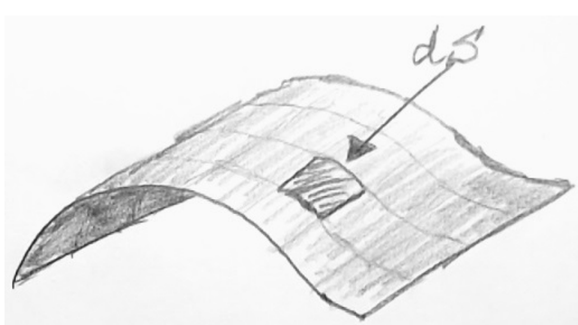
To compute the **area of a surface** S, we simply do

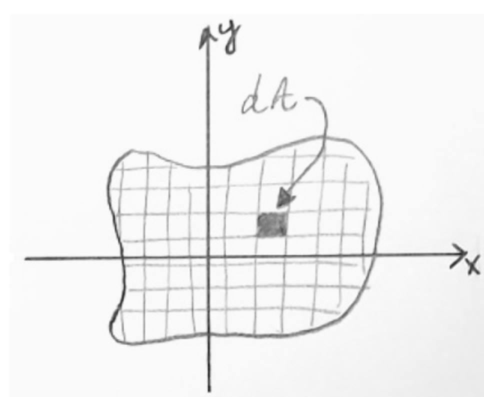
$$\mathrm{Area}(S) = \iint_S dS$$

Notice the similarity to the area formula for a region R in the xy-plane:

$$\mathrm{Area}(R) = \iint_R dA$$

The only difference is that dS is more complicated:



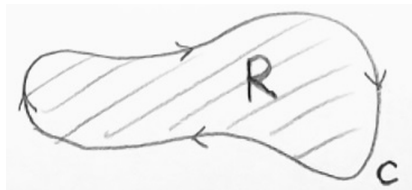


10. Green, Gauss, Stokes

Theorem 1. (Green) If C is a positively oriented (counterclockwise), piecewise-smooth, simple closed curve, and R is the region enclosed by C, then for $\mathbf{F} = \langle P, Q \rangle$ we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

This makes it clear that if $\mathbf{F}$ is conservative, the integral over a closed curve is always zero.

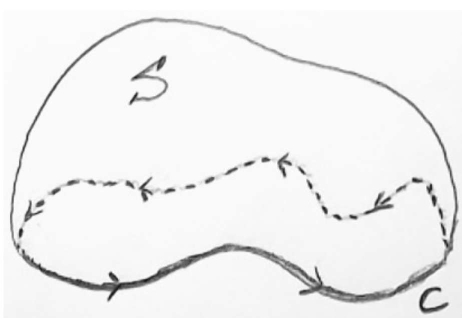


Green's theorem works well if C is entirely contained in the xy-plane. But it is possible to have curves that cannot be contained in a plane: they lie on more complicated surfaces in $\mathbb{R}^3$. Stokes' theorem helps us get over this obstacle, and generalizes Green.

Theorem 2. (Stokes) If S is an oriented piecewise-smooth surface with a simple, closed, piecewise-smooth boundary C with positive (counterclockwise) orientation, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$$

Notice that the $\mathbf{k}$ component of the curl is precisely $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, which appears in Green. This is not an accident: Stokes is a generalization of Green.



Theorem 3. (Gauss/Divergence) Suppose S is a closed surface with positive (outward) orientation which encloses a simple solid region E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\mathbf{F}) dV$$

We see this theme of relating an integral over some surface/solid/region D to an integral over its boundary ∂D. This can be generalized to n-dimensional space, on spaces called manifolds. In fact, the fundamental theorem of calculus, Green, and Divergence theorem above are ALL manifestations of the generalized Stokes theorem (see Wikipedia):

$$\int_{\partial D} \omega = \int_D d\omega$$