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M20580 L.A. and D.E. Tutorial
Worksheet 6
Sections 4.3–4.6

1. Let $\mathcal{B} = \{(1,1,0),(1,0,1),(0,1,1)\}$. Show that $\mathcal{B}$ is a basis and find the coordinates of the vector $\vec{v} = (a,b,c)$ with respect to $\mathcal{B}$.

> ***Solution:*** One way to see that $\mathcal{B}$ is a basis is to observe that
>
> $$\det \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -2 \neq 0. \tag{1}$$
>
> Let $[(\alpha,\beta,\gamma)]_{\mathcal{B}}$ be the coordinates of (a,b,c) with respect to $\mathcal{B}$. To find α,β,γ, we simply solve the system
>
> $$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{2}$$
>
> which leads to
>
> $$\alpha = \frac{a+b-c}{2}, \ \beta = \frac{a-b+c}{2}, \ \frac{c+b-a}{2}. \tag{3}$$

2. Let $\mathbb{P}_3$ be the set of all polynomials of degree at most 3. We know $\mathbb{P}_3$ is a vector space and $\mathcal{B}_1 = \{1, t, t^2, t^3\}$ is the standard basis for $\mathbb{P}_3$.

(a) Find the coordinates of $3t^2 + t - 1$ relative to the basis $\mathcal{B}_1$.

(b) Let $\mathcal{B}_2 = \{1, 1+t, t+t^2, t^2+t^3\}$. Show that $\mathcal{B}_2$ is a basis for $\mathbb{P}_3$.

(c) Find the coordinates of $3t^2 + t - 1$ relative to the new basis $\mathcal{B}_2$.

Solution: (a) The coordinates are $(-1, 1, 3, 0)$.

(b) We have $\mathcal{B}_2 \subset \mathbb{P}_3$. It is easy to check that $\mathcal{B}_2$ is a linearly independent set, which implies that $\dim \operatorname{span} \mathcal{B}_2 = 4 = \dim \mathbb{P}_3$. So $\mathcal{B}_2$ is also a basis of $\mathbb{P}_3$.

(c) The coordinates are $(1, -2, 3, 0)$, since

$$1(1) - 2(t+1) + 3(t^2+t) = 3t^2 + t - 1. \tag{4}$$

3. Let $C[-\pi, \pi]$ be the vector space of all real-valued continuous functions on $[-\pi, \pi]$. Show that the set of all solutions of the differential equation $y'' + 25y = 0$ is a subspace of $C[-\pi, \pi]$.

Solution: Let $V = \{y \in C[-\pi, \pi] \, : \, y'' + 25y = 0\}$.

(i) Since 0 is a solution to $y'' + 25y = 0$, the zero vector is in V.

(ii) Given two solutions y_1, y_2 of $y'' + 25y = 0$ and a real number c, it is easy to check that

$$(y_1 + y_2)'' + 25(y_1 + y_2) = 0$$

and

$$(cy_1)'' + 25(c \cdot y_1) = 0$$

So V is closed under vector addition and multiplication by scalars, which makes it a vector subspace.

4. Let W be the subset of all polynomials $\mathbf{p}(t)$ in $\mathbb{P}_3$ such that $\mathbf{p}(1) = \mathbf{p}(0)$. Is W a subspace of $\mathbb{P}_3$? If the answer is yes, what is the dimension of W?

Solution: It is easy to check, directly from the definition, that W is a subspace. As for the dimension, if an element of $\mathbb{P}_3$ has coordinates (a, b, c, d), the condition stated becomes

$$a + b + c + d = d, \text{ or } a + b + c = 0. \tag{5}$$

Thus, W has dimension $4 - 1 = 3$.