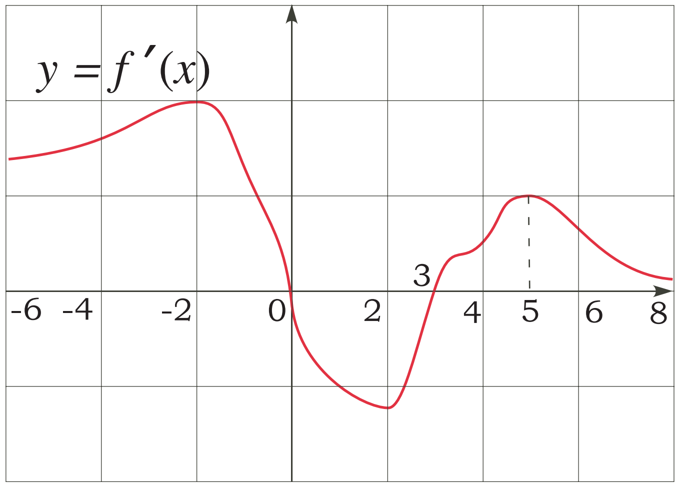
10350 Tutorial Week 11 - Set 01 Name ______________________

Section ______________

1. The graph of the **derivative** of $f(x)$ for $-6 \le x \le 8$ is given below. Answer the following questions:



(a) For which values of x are there the critical points of $f(x)$ for $-6 \le x \le 8$?

$$x = 0, 3$$

(b) Find the values of x for which $f(x)$ is **increasing** for $-6 \le x \le 8$.

$$(-6, 0) \cup (3, 8)$$

(c) Find the values of x for which $f(x)$ is **decreasing** for $-6 \le x \le 8$.

$$(0, 3)$$

(d) Find the value of x for which there is a point of inflection on the graph of $f(x)$ for $-6 \le x \le 8$?

$$x = -2, 2, 5$$

(e) Find the values of x for which the graph of $f(x)$ is **concave up** for $-6 \le x \le 8$.

$$(-6, -2) \cup (2, 5)$$

(f) Find the values of x for which the graph of $f(x)$ is **concave down** for $-6 \le x \le 8$.

$$(-2, 2) \cup (5, 8)$$

2. Find the following limits

a. $\lim_{x\to 0^+} \dfrac{\sin x - x}{\cos(2x) - 1}$.

Evaluating at $x = 0$, we have $\dfrac{\sin(0) - (0)}{\cos(2 \cdot 0) - 1} = \dfrac{0}{0}$. Since the numerator and denominator are differentiable, we can apply L'Hospital's Rule:

$$\lim_{x\to 0^+} \frac{\sin x - x}{\cos(2x) - 1} = \lim_{x\to 0^+} \frac{\cos x - 1}{-2\sin(2x)}.$$

Evaluating at $x = 0$, we have $\dfrac{\cos(0) - 1}{-2cos(0)} = \dfrac{0}{0}$. Applying L'Hospital's Rule again, we obtain:

$$\lim_{x\to 0^+} \frac{\cos x - 1}{-2\sin(2x)} = \lim_{x\to 0^+} \frac{-\sin x}{-4\cos(2x)}.$$

Evaluating at $x = 0$, we have $\lim_{x\to 0^+} \dfrac{-\sin x}{-4\cos(2x)} = \dfrac{-\sin(0)}{-4\cos(2 \cdot 0)} = \dfrac{0}{-4} = 0$.

b. $\lim_{x\to 0^+} (1 + 3x)^{1/x}$.

Recall that $e^{\ln(a)} = a$. Thus we can rewrite the function as follows:

$$(1 + 3x)^{1/x} = e^{\ln(1+3x)^{1/x}} = e^{(1/x)\ln(1+3x)}.$$

We then compute:

$$\lim_{x\to 0^+} (1 + 3x)^{1/x} = \lim_{x\to 0^+} e^{(1/x)\ln(1+3x)} = e^{\lim_{x\to 0^+} (1/x)\ln(1+3x)}$$

Evaluating at $x = 0$, we have $\dfrac{\ln(1 + 3x)}{x} = \dfrac{0}{0}$. Since both functions are differentiable, we can apply L'Hospital's Rule to compute the limit:

$$\lim_{x\to 0^+} \frac{\ln(1 + 3x)}{x} = \lim_{x\to 0^+} \frac{3}{1 + 3x}.$$

Evaluating at $x = 0$, we have $\lim_{x\to 0^+} \dfrac{3}{1 + 3x} = 3$. Therefore the answer is e^3.

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Section ______________

3. The derivative $f'(x)$ of the function $f(x)$ is given below:

$$f'(x) = 3x\sqrt[3]{x-1}.$$

For the following questions you may assume that $f(x)$ is defined for $-\infty < x < \infty$.

a. Find the critical points of $f(x)$.

The critical points are points in the domain where the derivative is zero or undefined. $f'(x) = 3x\sqrt[3]{x-1} = 0$, so $x = 0, 1$ are the critical points.

b. Determine the concavity of $f(x)$ for $-\infty < x < \infty$. Fill your answers in the blanks below.

$$\begin{aligned} f''(x) &= 3(x-1)^{1/3} + 3x \cdot \frac{1}{3}(x-1)^{-2/3} \\ &= \frac{3(x-1)+x}{(x-1)^{2/3}} \\ &= \frac{4x-3}{(x-1)^{2/3}} \end{aligned}$$

The points where concavity can change are the points where the second derivative is zero or undefined and where the first derivative is defined, i.e. $x = \frac{3}{4}, 1$. We then check concavity in each of the intervals $(-\infty, \frac{3}{4})$, $(\frac{3}{4}, 1)$, and $(1, \infty)$ by testing points. We have:

$$f''(0) < 0, \quad f''(\frac{4}{5}) > 0, \quad f''(2) > 0$$

Since positive second derivative means concave up and negative second derivative means concave down, we conclude:

Concave Up: $(\frac{3}{4}, 1) \cup (1, \infty)$ **Concave Down:** $(-\infty, \frac{3}{4})$

c. Find the value of x for which there is a point of inflection on the graph of $f(x)$?

A point of inflection is where the function changes concavity and where the derivative exists, so $x = \frac{3}{4}$.