Name: SOLUTIONS Date: 12/07/2017

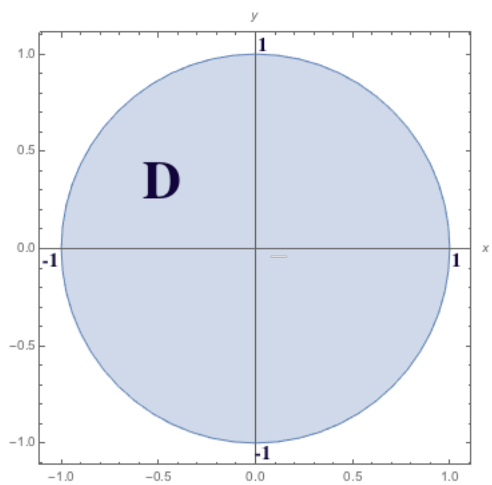
M20550 Calculus III Tutorial
Worksheet 10

1. Compute the area of the part of the paraboloid $z = x^2 + y^2$ which lies inside the cylinder $x^2 + y^2 = 1$.

Solution: Denote S the surface given by the part of the paraboloid $z = x^2 + y^2$ which lies inside the cylinder $x^2+y^2 = 1$. Since the surface S is given by the equation $z = x^2 + y^2$, we can use the following formula to compute the area of S:

$$\begin{aligned}\text{Area}(S) &= \iint_D \sqrt{1 + (z_x)^2 + (z_y)^2}\, dA \\ &= \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA \\ &= \iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA.\end{aligned}$$

Here, D is the projection of S onto the xy-plane. So, D is the unit disk in the xy-plane.



We use polar coordinate to compute the double integral above.

$$\begin{aligned}\iint_D \sqrt{1 + 4(x^2 + y^2)}\, dA &= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{8}\left(\frac{2}{3}\right)\left(1 + 4r^2\right)^{3/2}\Big|_{r=0}^{r=1} d\theta \\ &= \int_0^{2\pi} \frac{1}{12}\left(5^{3/2} - 1\right)\, d\theta \\ &= \frac{\pi}{6}\left(5^{3/2} - 1\right).\end{aligned}$$

So, the area of the given surface is $\dfrac{\pi}{6}\left(5^{3/2}-1\right)$.

Alternatively, if you don't want to remember two formulas for surface area. You can still do this problem by using the formula

$$\text{Area}(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA.$$

In this case, we need a parametrization of S. Since the surface is given by the paraboloid $z = x^2 + y^2$, we can let x and y be the parameters and have $z = x^2 + y^2$. But the surface lies inside the cylinder $x^2 + y^2 = 1$, so x and y lie inside the unit disk $x^2 + y^2 \le 1$ in the xy-plane. So, a parametrization of S is given by

$$\mathbf{r}(x, y) = \left\langle x, y, x^2 + y^2 \right\rangle, \text{ for } (x, y) \in D,$$

where D is the disk centered at $(0, 0)$ with radius 1 in the xy-plane as shown in the picture above.

Then, $\mathbf{r}_x = \langle 1, 0, 2x\rangle$ and $\mathbf{r}_y = \langle 0, 1, 2y\rangle$. So, $\mathbf{r}_x \times \mathbf{r}_y = \langle -2x, -2y, 1\rangle$. Then, $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{1 + 4(x^2 + y^2)}$. And so,

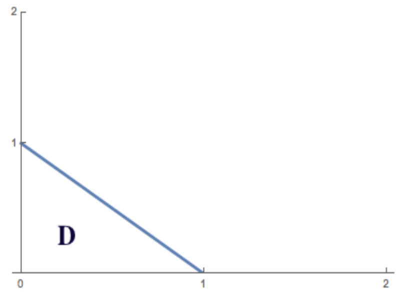
$$\text{Area}(S) = \iint_D |\mathbf{r}_x \times \mathbf{r}_y| \, dA = \iint_D \sqrt{1 + 4(x^2 + y^2)} \, dA = \frac{\pi}{6}\left(5^{3/2} - 1\right) \text{ (as above).}$$

2. Let S be the portion of the graph $z = 4 - 2x^2 - 3y^2$ that lies over the region in the xy-plane bounded by $x = 0$, $y = 0$, and $x + y = 1$. Write the integral that computes $\displaystyle\iint_S \left(x^2 + y^2 + z\right) dS$.

Solution: First, we need a parametrization of the surface S. Since S is a surface given by the equation $z = 4 - 2x^2 - 3y^2$, we can choose x and y to be the parameters. So,

$$\mathbf{r}(x, y) = \left\langle x, y, 4 - 2x^2 - 3y^2 \right\rangle,$$

and the domain D of the parameters x, y is given by the region in the xy-plane bounded by $x = 0$, $y = 0$, and $x + y = 1$ (see picture below)



Now, $\mathbf{r}_x = \langle 1, 0, -4x \rangle$ and $\mathbf{r}_y = \langle 0, 1, -6y \rangle$. So, $\mathbf{r}_x \times \mathbf{r}_y = \langle 4x, 6y, 1 \rangle$ and $|\mathbf{r}_x \times \mathbf{r}_y| = |\langle 4x, 6y, 1 \rangle| = \sqrt{16x^2 + 36y^2 + 1}$. Thus,

$$\begin{aligned} \iint_S \left(x^2 + y^2 + z\right) dS &= \iint_D x^2 + y^2 + (4 - 2x^2 - 3y^2)|\mathbf{r}_x \times \mathbf{r}_y| \, dA \\ &= \int_0^1 \int_0^{-x+1} (4 - x^2 - 2y^2)\sqrt{16x^2 + 36y^2 + 1} \, dy \, dx. \end{aligned}$$

3. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ and S is a surface given by $x = 2u, y = 2v, z = 5 - u^2 - v^2$, where $u^2 + v^2 \leq 1$. S has downward orientation.

Solution: We have $\mathbf{r}(u, v) = \langle 2u, 2v, 5 - u^2 - v^2 \rangle$, so $\mathbf{r}_u = \langle 2, 0, -2u \rangle$ and $\mathbf{r}_v = \langle 0, 2, -2v \rangle$ and so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 2, 0, -2u \rangle \times \langle 0, 2, -2v \rangle = \langle 4u, 4v, 4 \rangle .$$

Note that $\mathbf{r}_u \times \mathbf{r}_v = \langle 4u, 4v, 4 \rangle$ gives unit normal vectors pointing upward (z-component is positive). But, S has downward orientation so

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -\iint_{u^2+v^2 \leq 1} \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA.$$

Now, $\mathbf{F}\big(\mathbf{r}(u, v)\big) = \langle 2v, -2u, 5 - u^2 - v^2 \rangle$. So

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \left\langle 2v, -2u, 5 - u^2 - v^2 \right\rangle \cdot \langle 4u, 4v, 4 \rangle = 20 - 4u^2 - 4v^2.$$

Thus,

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= -\iint_{u^2+v^2 \leq 1} (20 - 4u^2 - 4v^2) \, dA. \\ &\overset{\text{polar}}{=} -\int_0^{2\pi} \int_0^1 (20 - 4r^2) r \, dr \, d\theta \\ &= -18\pi. \end{aligned}$$

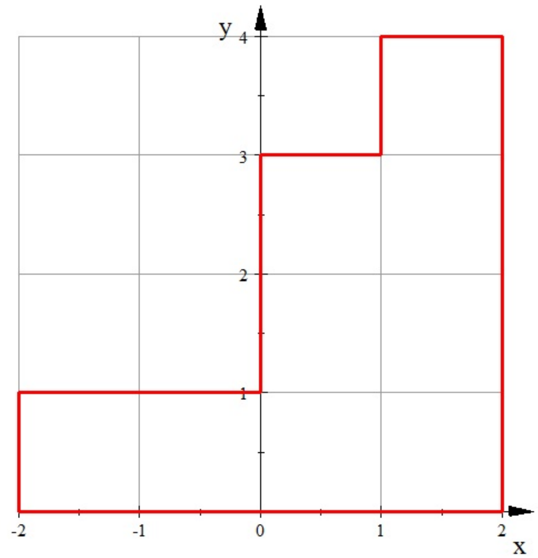
4. Compute the flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the part of the cylinder $x^2 + y^2 = 4$ that lies between the planes $z = 0$ and $z = 2$ with normal pointing away from the origin. Assume the cylinder is closed (it has a top and a bottom).

> ***Solution:*** We want to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$. Since the cylinder is closed, we can use the Divergence Theorem. Notice that
>
> $$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$
>
> and so if we let R be the region enclosed by our cylinder, we get
>
> $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_S \operatorname{div} \mathbf{F}\ dV = \iiint_S 3\ dV = 24\pi$$
>
> which is 3 times the volume 8π.

5. Let S be the surface defined as $z = 4 - 4x^2 - y^2$ with $z \geq 0$ and oriented upward. Let $\mathbf{F} = \langle x - y, x + y, ze^{xy} \rangle$. Compute $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$. (*Hint*: use one of the theorems you learned in class.)

> ***Solution:*** This question uses Stokes' theorem: S is a surface with boundary, and we are taking the flux integral of the curl of $\mathbf{F}$.
>
> The boundary of S is given by $z = 0, 4x^2 + y^2 = 4$, and since S is oriented with upward orientation, the boundary of S has counterclockwise orientation when viewed from above. Thus, a parametrization of the boundary is given by
>
> $$\mathbf{r}(t) = \langle \cos t, 2\sin t, 0 \rangle, 0 \leq t \leq 2\pi.$$
>
> Thus, by Stokes' Theorem, we have
>
> $$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} \langle \cos t - 2\sin t, \cos t + 2\sin t, 0 \rangle \cdot \langle -\sin t, 2\cos t, 0 \rangle\, dt \\ &= \int_0^{2\pi} \left(-\sin t \cos t + 2\sin^2 t + 2\cos^2 t + 4\sin t \cos t \right) dt \\ &= \int_0^{2\pi} (2 + 3\sin t \cos t)\, dt = \left(2t + \frac{3}{2}\sin^2 t \right) \Big|_0^{2\pi} \\ &= 4\pi. \end{aligned}$$

6. Evaluate $\int_C (x^4y^5 - 2y)dx + (3x + x^5y^4)dy$ where C is the curve below and C is oriented in clockwise direction.



Solution: This problem uses Green's theorem. One main clue is the shape of the curve C (it has 8 pieces!). Let D be the region enclosed by the curve C. And since the orientation of C is clockwise, instead of counterclockwise, we have

$$\begin{aligned}
\int_C (x^4y^5 - 2y)dx + (3x + x^5y^4)dy &= -\iint_D \left[(3 + 5x^4y^4) - (5x^4y^4 - 2)\right] dA \\
&= -\iint_D 5\,dA \\
&= -5\iint_D 1\,dA \\
&= -5 \cdot \text{Area}(D) \\
&= -5 \cdot 9 \\
&= -45.
\end{aligned}$$

7. Let S be the boundary surface of the region bounded by $z = \sqrt{36 - x^2 - y^2}$ and $z = 0$, with outward orientation. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} - 2yz\mathbf{k}$.

Solution: This is a closed surface, so the divergence theorem works nicely here.

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(-2yz) = 1 + 2y - 2y = 1$$

Call the solid H (since it's half of a ball). So, the divergence theorem gives

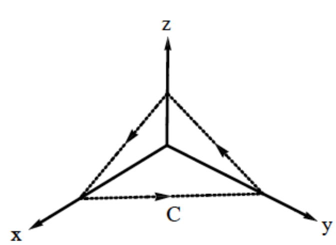
$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_H 1\,dV = \text{Volume}(H)$$

The solid H is half of the ball of radius 6, and so its volume is

$$\text{Volume}(H) = \frac{1}{2}\left(\frac{4}{3}\pi(6)^3\right) = \frac{2}{3}(216\pi) = 144\pi.$$

8. Let C be the boundary curve of the part of the plane $x + y + 2z = 2$ in the first octant. C has counterclockwise orientation when viewing from above. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle e^{\sin x^2}, z, 3y\rangle$.

Solution:



Note: To compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ directly, we need to do 3 integrals since C consists of 3 pieces. But, because we know C is the *boundary curve* of the surface $x + y + 2z = 2$ in the first octant, we can try to use Stokes' Theorem.

By Stokes' Theorem,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S},$$

where S is the part of the plane $x+y+2z = 2$ in the first octant. Since the equation $x + y + 2z = 2$ or $z = 1 - \frac{1}{2}x - \frac{1}{2}y$ determines S, a parametrization of S is given by

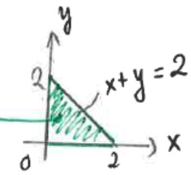
$$\mathbf{r}(x, y) = \left\langle x, y, 1 - \tfrac{1}{2}x - \tfrac{1}{2}y\right\rangle, \text{ where } (x, y) \in D.$$

The domain D is given by the projection of S onto the xy-plane.

S: part of the plane $x + y + 2z = 2$ in the first octant

↳ $\vec{r}(x,y) = \langle x, y, 1 - \frac{1}{2}x - \frac{1}{2}y \rangle$, $(x,y) \in D$

projection of S onto the xy-plane



Now, $\vec{r}_x = \langle 1, 0, -\frac{1}{2} \rangle$

$\vec{r}_y = \langle 0, 1, -\frac{1}{2} \rangle$ $\Rightarrow \vec{r}_x \times \vec{r}_y = \langle \frac{1}{2}, \frac{1}{2}, 1 \rangle$

↳ upward orientation for S ✓

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{\sin x^2} & z & 3y \end{vmatrix} = \langle 2,0,0 \rangle$$

So, $\operatorname{curl}\vec{F}(\vec{r}(x,y)) = \langle 2,0,0 \rangle$

Thus, $$\iint_D \operatorname{curl}\vec{F} \cdot (\vec{r}_x \times \vec{r}_y)\, dA = \iint_D \langle 2,0,0 \rangle \cdot \langle \tfrac{1}{2}, \tfrac{1}{2}, 1 \rangle\, dA = \iint_D 1\, dA = \text{Area}(D) = \boxed{2}$$

9. *(A Challenging Problem)* Evaluate

$$\int_C (y^3 + \cos x)dx + (\sin y + z^2)dy + x\ dz$$

where C is the closed curve parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t, \sin 2t \rangle$ with counterclockwise direction when viewed from above. (*Hint*: the curve C lies on the surface $z = 2xy$.)

Solution: If you rewrite this integral as $\int_C \mathbf{F} \cdot d\mathbf{r}$ and note that the curve C lies in $\mathbb{R}^3$ and not in the plane (otherwise we'd use Green's theorem), we see that Stokes' theorem applies to it. The hint provides the surface to fill in the curve with.

First, we need to parametrize the surface $z = 2xy$:

$$\mathbf{p}(x,y) = \langle x, y, 2xy \rangle, \quad (x,y) \in D = \left\{(x,y) | x^2 + y^2 \leq 1\right\}$$

as the parametrization. C has counterclockwise orientation when viewed from above, so this means that the surface, call it S, we fill it in with must have upward orientation.

$$\begin{aligned} \mathbf{p}_x &= \langle 1, 0, 2y \rangle \\ \mathbf{p}_y &= \langle 0, 1, 2x \rangle \\ \mathbf{p}_x \times \mathbf{p}_y &= \langle -2y, -2x, 1 \rangle \end{aligned}$$

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Notice that $\mathbf{p}_x \times \mathbf{p}_y$ points upward, since the $\hat{k}$-component is positive, so this is the correct choice for the orientation. Now, we need the curl of $\mathbf{F}$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 + \cos x & \sin y + z^2 & x \end{vmatrix} = \left\langle -2z, -1, -3y^2 \right\rangle$$

Finally, we apply Stokes' Theorem

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\text{curl } \mathbf{F}) \cdot d\mathbf{S} \\
&= \iint_D (\text{curl } \mathbf{F}) \cdot (\mathbf{p}_x \times \mathbf{p}_y) dA \\
&= \iint_D \left\langle -4xy, -1, -3y^2 \right\rangle \cdot \langle -2y, -2x, 1 \rangle \, dA \\
&= \iint_D \left(8xy^2 + 2x - 3y^2\right) dA \\
&= \int_0^{2\pi} \int_0^1 \left(8r^3 \cos\theta \sin^2\theta + 2r\cos\theta - 3r^2 \sin^2\theta\right) r \; dr d\theta \\
&= \int_0^{2\pi} \left(\frac{8}{5} r^5 \cos\theta \sin^2\theta + \frac{2}{3} r^3 \cos\theta - \frac{3}{4} r^4 \sin^2\theta\right)\Bigg|_0^1 d\theta \\
&= \int_0^{2\pi} \left(\frac{8}{5} \cos\theta \sin^2\theta + \frac{2}{3}\cos\theta - \frac{3}{4}\sin^2\theta\right) d\theta \\
&= \int_0^{2\pi} \left(\frac{8}{5} \cos\theta \sin^2\theta + \frac{2}{3}\cos\theta - \frac{3}{4}\left(\frac{1-\cos 2\theta}{2}\right)\right) d\theta \\
&= \left(\frac{8}{15}\sin^3\theta + \frac{2}{3}\sin\theta - \frac{3\theta}{8} + \frac{3}{16}\sin 2\theta\right)\Bigg|_0^{2\pi} \\
&= -\frac{3}{4}\pi
\end{aligned}$$