GATEWAY 3 PREP SOLUTIONS

Problem 1. Find the x values for which the given curves intersect:
a) $y = x^2 - 1$ and $y = 2x - 2$

We begin by setting the curves equal to each other:

$$x^2 - 1 = 2x - 2$$

Bringing everything to one side, we get

$$x^2 - 2x + 1 = 0$$

which we can factor as

$$(x-1)^2 = 0$$

This is equivalent to $x - 1 = 0$, i.e $x = 1$ is our point of intersection. This is in the domain of both our original curves, so that's our final answer.

b) $y = \frac{2}{x} - x$ and $y = 3 - 2x$

Again set them equal

$$\frac{2}{x} - x = 3 - 2x$$

and we multiply by x to get a quadratic we can tackle:

$$2 - x^2 = 3x - 2x^2$$

Move everything to one side

$$x^2 - 3x + 2 = 0$$

which factors as $(x-1)(x-2) = 0$, so $x = 1$ and $x = 2$ are our candidates. They are both in the domain of our original curves, so we're good.

Note, if one of our x values was zero, we would have to discard it since it is outside the domain of the first curve $\frac{2}{x} - x$ (zero in the denominator). This is why it's important to always check that the solutions you are getting make sense in the context of the problem.

c) $y = \sqrt{x}$ and $y = x$

Set them equal, so $\sqrt{x} = x$, and square both sides to get

$$x = x^2$$

Bringing everything to one side, this is equivalent to

$$x^2 - x = 0$$

and we can factor that as $x(x-1) = 0$, meaning $x = 0$ and $x = 1$ are our solutions (both in the domain of the original curves).

d) $y = \sqrt{3-x}$ and $y = \sqrt{x^2+1}$

Set them equal and square both sides to get rid of the square root. This gives us

$$3 - x = x^2 + 1$$

Rearranging all to one side, we have

$$x^2 + x - 2 = 0$$

which factors as $(x+2)(x-1) = 0$. Then our candidates are $x = -2$ and $x = 1$. You can check that the original curves make sense for both of these x values, so we are good.

e) $y = x + 1$ and $y = \sqrt{2x+10}$

Set them equal and square both sides to get rid of the square root. We get

$$(x+1)^2 = 2x + 10$$

whch after squaring the left hand side (LHS) is the same as

$$x^2 + 2x + 1 = 2x + 10$$

Cancel the $2x$ and move the 10 to the LHS, getting

$$x^2 - 9 = 0$$

but we know how to factor this using the special formula $(a+b)(a-b) = a^2 - b^2$, so this is

$$(x+3)(x-3) = 0$$

so our x values are $x = \pm 3$, both in the domain of the original curves.

Alternately, instead of factoring $x^2 - 9 = 0$, we could have written it as $x^2 = 9$, and apply the square root. We get two solutions, ± 3, as before.

Problem 2. Factor completely:

a) $1 - 16x^4 = \ldots$

This one is similar to the one on our last quiz. We write it as

$$\begin{aligned} 1 - 16x^4 &= 1 - (4x^2)^2 \\ &= (1 + 4x^2)(1 - 4x^2) \\ &= (1 + 4x^2)(1 + 2x)(1 - 2x) \end{aligned}$$

b) $16x^4 - 20x^2 + 4 = \ldots$

This one's a bit tougher. An easy way I like to think about it so I don't get overwhelmed by the exponents:

$$16(x^2)^2 - 20x^2 + 4$$

which looks like a quadratic in x^2. In other words, we can handle this easier if we let $y = x^2$ and try to factor the following first, the go back and subsitute x^2 everywhere we have y. So we want to factor:

$$16y^2 - 20y + 4$$

Now we know it must look like $(??+1)(??+4)$ but since the middle coefficient is negative, it should really be $(??-1)(??-4)$ (another possibility is 2 and 2 in each parenthesis, we'll try that next if this doesn't work).

We can try $(16y-1)(y-4)$ but notice when foiling we get the term $16y \cdot (-4)$ which is waaay too negative to add up to $-20y$. So let's switch it, and use

$$(y-1)(16y-4)$$

which ends up working, so now we can replace back $y = x^2$ and we have

$$\begin{aligned} 16x^4 - 20x^2 + 4 &= 16y^2 - 20y + 4 \\ &= (y-1)(16y-4) \\ &= (x^2-1)(16x^2-4) \text{ (after replacing back } y = x^2\text{)} \\ &= (x+1)(x-1)(4x+2)(4x-2) \end{aligned}$$

Done!

Problem 3. Express the following expressions in terms of $\ln x$ and $\ln y$:

a)

$$\ln\left(xe^4 \sqrt[3]{\frac{x^6}{y^2}}\right) = \ldots$$

This is just a matter of breaking up the log and being careful with our exponents:

$$\begin{aligned} \ln\left(xe^4 \sqrt[3]{\frac{x^6}{y^2}}\right) &= \ln x + \ln e^4 + \ln\left(\sqrt[3]{\frac{x^3}{y^2}}\right) \\ &= \ln x + 4 + \frac{1}{3}\ln\left(\frac{x^6}{y^2}\right) \\ &= \ln x + 4 + \frac{1}{3}(\ln x^6 - \ln y^2) \\ &= \ln x + 4 + \frac{1}{3}(6\ln x - 2\ln y) \\ &= \ln x + 4 + 2\ln x - \frac{2}{3}\ln y \\ &= 4 + 3\ln x - \frac{2}{3}\ln y \end{aligned}$$

b)

$$\ln\left(xy^3e^2\left(\frac{x^2}{y^3}\right)^{5/7}\right) = \ldots$$

This one is the same as the one before, except we get the 5/7 exponent instead of the 1/3 cube root one, so the fraction is not as pretty. I can do it in tutorial if you really want to see it.

Problem 4. Solve for x in terms of y:

a) $y = \frac{x}{x+2}$

Move the $x + 2$ to the left side to get

$$y(x+2) = x$$

$$xy + 2y = x$$

Now move all the x terms on one side, and everything that doesn't have an x to the other:

$$xy - x = -2y$$

$$x(y-1) = -2y$$

$$x = \frac{-2y}{y-1}$$

b) $y = \frac{x^3-1}{x^3+1}$

This one's very similar to the previous. Move the $x^3 + 1$ to the left, getting

$$y(x^3+1) = x^3 - 1$$

$$x^3 y + y = x^3 - 1$$

$$x^3 y - x^3 = -y - 1$$

$$x^3(y-1) = -y - 1$$

$$x^3 = \frac{-y-1}{y-1}$$

Now after applying the cubed root, we simply get that

$$x = \sqrt[3]{\frac{-y-1}{y-1}}$$

c) $y = \frac{\ln x}{\ln x+2}$

This one is almost identical to the first problem, except instead of x, we have $\ln x$. I won't type it up, but you should end up with

$$\ln x = \frac{-2y}{y-1}$$

and all we need to do to finish this off is get rid of the log by raising e to both sides:

$$x = e^{\frac{-2y}{y-1}}$$

Problem 5. Simplify the following monster expressions:

a)

$$3\left(\frac{x^2+1}{x^2-1}\right)^2 \frac{2x(x^2-1)-2x(x^2+1)}{(x^2-1)^2} = \dots$$

I won't type up the details, but you should end up with (factored form is ok, no need to multiply things out in the numerator):

$$\frac{-12x(x^2+1)^2}{(x^2-1)^4}$$

b)

$$\frac{2x^2(1-x^2)^2 - x^3(2)(1-x^2)(-2x)}{(1-x^2)^4} = \dots$$

You should end up with:

$$\frac{2x^2(1+x^2)}{(1-x^2)^3}$$