Math 10550 Exam 3 Practice Problems

1. What is the geometric idea behind Newton's method?
Near a given point a on a curve $y = f(x)$ the tangent line at a gives a close approximation to the points on the curve. In particular, the zero of the tangent line should be close to a zero of the curve $y = f(x)$ if $f(a)$ is close to zero. The linearization of the function f near a is given by $L(x) = f(a) + f'(a)(x - a)$, solving the equation $L(x) = 0$ gives $x = a - \frac{f(a)}{f'(a)}$.

2. Use Newton's method with initial approximation $x_1 = \frac{3}{2}$ to find the first four approximations to the positive root of $f(x) = x^2 - 2$, i.e. find x_2, x_3, and x_4.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{3}{2} - \frac{1/4}{3} = \frac{17}{12} \approx 1.416667$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{17}{12} - \frac{1/144}{17/6} = \frac{577}{408} \approx 1.414216$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \frac{577}{408} - \frac{1/166464}{577/204} = \frac{665857}{470832} \approx 1.414214.$$

Note that x_4 agrees with $\sqrt{2}$ to 11 decimal places.

3. Show that $\arctan(x) = x^2$ has exactly two real solutions ($x = 0$ and some other positive number) by showing that the derivative of $f(x) = \arctan(x) - x^2$ has exactly one real root and this root is positive. The value $x = \frac{\pi}{4}$ is not far from the positive solution, use Newton's method (with a calculator) to approximate the root to within 4 decimal places.
We begin by thinking about $f'(x) = \frac{1}{1+x^2} - 2x$. Note that roots of $f'(x)$ are exactly solutions to the equation $2x^3 + 2x - 1 = 0$. Writing $g(x) = 2x^3 + 2x - 1$ we see that $g(0) < 0$ and $g(1) > 0$, so the Intermediate Value Theorem says $g(x) = 0$ has a solution in the interval $(0, 1)$. But $g'(x) = 6x^2 + 2$ which is never zero, so by the Mean Value Theorem (or Rolle's Theorem) $g(x) = 0$ cannot have more than one solution. It follows that f' has exactly one root and it lies in the interval $(0, 1)$.

Note that $f(1/\sqrt{3}) = \frac{\pi}{6} - \frac{1}{3} > 0$ and $f(1) = \frac{\pi}{4} - 1 < 0$, then the Intermediate Value Theorem says $f(x) = 0$ has a solution, say $x = a$, in the interval $(1/\sqrt{3}, 1)$. Since $f(0) = 0$, the Mean Value Theorem (or Rolle's Theorem) states that $f'(x) = 0$ has a solution in the interval $(0, a)$. If f had another root, this would produce by the same reasoning a second root for f' which we have already seen to be impossible. Thus f has exactly two roots 0 and a (which lies in the interval $(0, 1)$).
Applying Netwon's method for the function f with initial guess $x_1 = \frac{\pi}{4}$ we get

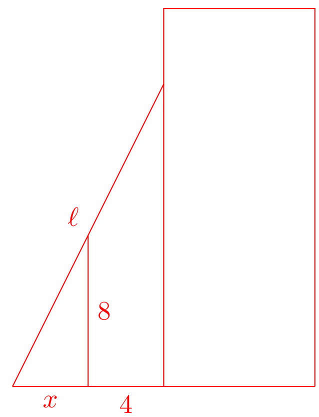
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\arctan(x_i) - x_i^2}{\frac{1}{1+x_i^2} - 2x_i}.$$

This gives

$$x_2 \approx 0.8367716465162773$$
$$x_3 \approx 0.8336181009478392$$
$$x_4 \approx 0.8336061945764655$$
$$x_5 \approx 0.833606194406676.$$

Since x_3 and x_4 agree to 4 decimal places, the actual solution agrees with x_4 to 4 decimal places. Since x_4 and x_5 agree to 9 decimal places, the actual solution agrees with x_5 to 9 decimal places.

4. A fence 8 feet tall runs parallel to a tall building at a distance of 4 feet from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building? [Hint: Use similar triangles.]
Consider the following picture where we aim to determine the smallest possible ℓ:



By similarity of triangles and the Pythagorean Theorem, $\frac{\ell}{x+4} = \frac{\sqrt{x^2+64}}{x}$ *or*

$$\ell(x) = \frac{x+4}{x}\sqrt{64+x^2}.$$

Taking the derivative with respect to x *we get*

$$\ell'(x) = \frac{-4}{x^2}\cdot\sqrt{64+x^2} + \frac{x+4}{x}\cdot x(64+x^2)^{-1/2} = \frac{-4(64+x^2)+x^2(x+4)}{x^2\sqrt{64+x^2}} = \frac{-256+x^3}{x^2\sqrt{64+x^2}}.$$

Thus the critical points of $\ell(x)$ *are* $x = 0$ *and* $x = \sqrt[3]{256}$*. Since* $\ell \to \infty$ *as* $x \to 0$*, the minimum value of* ℓ *must occur when* $x = \sqrt[3]{256}$*, i.e.* $\ell \approx 16.65$ *feet.*

5. The best size medium box for UPS shipping has to have dimensions subject to the following restriction

$$l + 2w + 2h = 130,$$

where l is the length, w is the width, and h is the height. Assume $h = w$ and find the length and width which give the largest volume subject to this restriction.
The volume of the box is given by $V = lwh = lw^2$*. Since* $l = 130 - 4w$*, we aim to maximize the function* $V(w) = (130-4w)w^2$ *with* $V'(w) = -4w^2 + 2(130-4w)w = 4w(65-3w)$*. Since* $V'(w) = 0$ *for* $w = 0$ *and* $w = \frac{65}{3}$*, we must compare* $V(0) = 0$*,* $V\left(\frac{65}{3}\right) = \frac{549250}{27} \approx 20342.5926$*, and* $V\left(\frac{65}{2}\right) = 0$*. Since* $V\left(\frac{65}{3}\right)$ *is largest, we see that* $l = \frac{130}{3}$ *and* $w = h = \frac{65}{3}$ *give the largest volume.*

6. A piece of wire 12cm long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is

   (a) a maximum?

   (b) a minimum?

Hint 1: The function you want to maximize/minimize should be defined on the closed interval [0, 12].
Hint 2: Begin by finding an equation for the area of an equilateral triangle given the length of a side.
An equilateral triangle with side length t *has height* $\frac{\sqrt{3}}{2}t$ *and base* t*, so its area is* $\frac{\sqrt{3}}{4}t^2$*. Now suppose the square has side length* s *and the triangle has side length* t*. Thus, adding the perimeters of the square and triangle, we obtain the constraint* $12 = 4s + 3t$*. The total area is given by* $A = s^2 + \frac{\sqrt{3}}{4}t^2$ *or* $A(t) = \left(3 - \frac{3}{4}t\right)^2 + \frac{\sqrt{3}}{4}t^2$ *since* $s = 3 - \frac{3}{4}t$*. Taking the derivative with respect to* t *we find* $A'(t) = -\frac{3}{2}\left(3 - \frac{3}{4}t\right) + \frac{\sqrt{3}}{2}t$ *which is zero for* $t = \frac{36}{9+4\sqrt{3}}$*. Since* $A(0) = 9$*,* $A\left(\frac{36}{9+4\sqrt{3}}\right) \approx 3.9147$*, and* $A(4) = 4\sqrt{3} \approx 6.9282$*, the Extreme Value Theorem states* A *has a maximum when the side length of the triangle is zero, i.e. the square has side length 3, and that* A *has a minimum when the side length of the triangle is* $\frac{36}{9+4\sqrt{3}}$*.*

7. Show that $F(x) = \ln|\sec(x) + \tan(x)|$ is an antiderivative of $f(x) = \sec(x)$.

$$F'(x) = \frac{\sec(x)\tan(x) + \sec^2(x)}{\sec(x) + \tan(x)} = \sec(x).$$

8. Approximate $\int_0^\pi \sin(x)\,dx$:

   (a) by a left-hand Riemann sum with 3 rectangles;

$$\int_0^\pi \sin(x)\,dx \approx \sin(0)\cdot\frac{\pi}{3} + \sin\left(\frac{\pi}{3}\right)\cdot\frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right)\cdot\frac{\pi}{3} = \frac{\sqrt{3}\pi}{3}.$$

   (b) by a right-hand Riemann sum with 4 rectangles;

$$\int_0^\pi \sin(x)\,dx \approx \sin\left(\frac{\pi}{4}\right)\cdot\frac{\pi}{4} + \sin\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{4} + \sin\left(\frac{3\pi}{4}\right)\cdot\frac{\pi}{4} + \sin(\pi)\cdot\frac{\pi}{4} = \frac{\sqrt{2}\pi}{4} + \frac{\pi}{4}.$$

   (c) using midpoints and 3 rectangles.

$$\int_0^\pi \sin(x)\,dx \approx \sin\left(\frac{\pi}{6}\right)\cdot\frac{\pi}{3} + \sin\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{3} + \sin\left(\frac{5\pi}{6}\right)\cdot\frac{\pi}{3} = \frac{2\pi}{3}.$$

9. Write a limit defining $\int_1^3 xe^x\,dx$, your answer should only involve i's' and n's. Do not attempt to calculate this limit.

$$\int_1^3 xe^x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)e^{1+\frac{2i}{n}}\cdot\frac{1}{n}$$

10. In this question you will need to use the following summation formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

Use the definition of the definite integral to compute each of the following, afterward compute each integral using the Fundamental Theorem of Calculus, Part II to check your answers:

   (a) $\int_0^2 x\,dx$

$$\int_0^2 x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2i}{n}\cdot\frac{2}{n} = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{4}{n^2}\cdot\frac{n(n+1)}{2} = 2.$$

   (b) $\int_0^2 x^2\,dx$

$$\int_0^2 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2\cdot\frac{2}{n} = \lim_{n\to\infty}\frac{8}{n^3}\sum_{i=1}^{n} i^2 = \lim_{n\to\infty}\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{8}{3}.$$

   (c) $\int_0^2 x^3\,dx$

$$\int_0^2 x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^3\cdot\frac{2}{n} = \lim_{n\to\infty}\frac{16}{n^4}\sum_{i=1}^{n} i^3 = \lim_{n\to\infty}\frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4} = 4.$$

(d) $\int_1^2 (2x^3 - x^2 + 5x - 4)\, dx$

$$\begin{aligned}
&\int_1^2 (2x^3 - x^2 + 5x - 4)\, dx \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\left(1+\frac{i}{n}\right)^3 - \left(1+\frac{i}{n}\right)^2 + 5\left(1+\frac{i}{n}\right) - 4 \right) \cdot \frac{1}{n} \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^{n} \left(2\left(1+3\frac{i}{n}+3\frac{i^2}{n^2}+\frac{i^3}{n^3}\right) - \left(1+2\frac{i}{n}+\frac{i^2}{n^2}\right) + 5\left(1+\frac{i}{n}\right) - 4 \right) \cdot \frac{1}{n} \\
&\quad = \lim_{n\to\infty} \left(\frac{2}{n^4}\sum_{i=1}^{n} i^3 + \frac{5}{n^3}\sum_{i=1}^{n} i^2 + \frac{9}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}\sum_{i=1}^{n} 1 \right) \\
&\quad = \lim_{n\to\infty} \left(\frac{2}{n^4}\cdot\frac{n^2(n+1)^2}{4} + \frac{5}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{9}{n^2}\cdot\frac{n(n+1)}{2} + \frac{2}{n}\cdot n \right) \\
&\quad = \frac{1}{2} + \frac{5}{3} + \frac{9}{2} + 2 = \frac{26}{3}.
\end{aligned}$$

11. Find an antiderivative of $f(x) = \frac{1}{x^4}$ and apply the Fundamental Theorem of Calculus, Part II for the definite integral $\int_{-1}^{1} \frac{dx}{x^4}$. Explain why your answer does not make sense and why the Fundamental Theorem of Calculus should not be applied in this situation.
An antiderivative of $f(x)$ is the function $F(x) = -\frac{1}{3x^3}$. Naively applying the Fundamental Theorem of Calculus, Part II gives

$$\int_{-1}^{1} \frac{dx}{x^4} = F(1) - F(-1) = -\frac{1}{3} + \frac{1}{-3} = -\frac{2}{3}.$$

But notice that $f(x)$ is always positive on the interval $[-1,1]$ (when it is defined), so the area cannot possibly be negative. The problem comes from applying FTCII to the function $f(x)$ which is not continuous on the interval $[-1,1]$.

12. Compute $g'(x)$ for each of the following:

(a) $g(x) = \int_{x+2}^{0} \sin^2(t)\, dt$
If $F(x)$ is an antiderivative of $\sin^2(x)$ then $g(x) = F(0) - F(x+2)$. Taking derivatives gives

$$g'(x) = -F'(x+2) = -\sin^2(x+2).$$

(b) $g(x) = \int_{\cos(x)}^{e^x} \frac{t^2}{\cos(t)+\sin(t)}\, dt$
If $F(x)$ is an antiderivative of $\frac{x^2}{\cos(x)+\sin(x)}$ then $g(x) = F(e^x) - F(\cos(x))$. Taking derivatives gives

$$g'(x) = e^x F'(e^x) + \sin(x)F'(\cos(x)) = e^x \frac{e^{2x}}{\cos(e^x)+\sin(e^x)} + \sin(x)\frac{\cos^2(x)}{\cos(\cos(x))+\sin(\cos(x))}.$$

(c) $g(x) = \int_{\arctan(\ln(x))}^{0} \frac{t}{t^3+1}\, dt$
If $F(x)$ is an antiderivative of $\frac{x}{x^3+1}$ then $g(x) = F(0) - F(\arctan(\ln(x)))$. Taking derivatives gives

$$g'(x) = -F'(\arctan(\ln(x)))\cdot\frac{1}{1+\ln^2(x)}\cdot\frac{1}{x} = -\frac{\arctan(\ln(x))}{\arctan^3(\ln(x))+1}\cdot\frac{1}{1+\ln^2(x)}\cdot\frac{1}{x}.$$

13. Suppose that $f(x)$ is a continuous, **odd** function satisfying the following:

$$\int_{-3}^{5} f(x)\,dx = \pi \qquad \int_{5}^{8} f(x)\,dx = e.$$

Find $\int_3^8 f(x)\,dx$.

Since f is odd, we have $\int_{-3}^{3} f(x)\,dx = 0$ and so

$$\int_3^8 f(x)\,dx = \int_{-3}^{3} f(x)\,dx + \int_3^8 f(x)\,dx = \int_{-3}^{8} f(x)\,dx = \int_{-3}^{5} f(x)\,dx + \int_5^8 f(x)\,dx = \pi + e.$$

14. Compute the following indefinite integrals:

(a) $\int \frac{2x^3+4}{x^4+4x^2}\,dx$

Here we use the substitution $u = x^4 + 4x^2$ so that $du = (4x^3 + 8x)dx$ and the integral becomes

$$\int \frac{2x^3+4}{x^4+4x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^4+4x^2) + C.$$

(b) $\int e^{2x}\sqrt{4-e^{2x}}\,dx$

Here we use the substitution $u = 4 - e^{2x}$ so that $du = -2e^{2x}dx$ and the integral becomes

$$\int e^{2x}\sqrt{4-e^{2x}}\,dx = -\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(4-e^{2x})^{3/2} + C.$$

(c) $\int \frac{\arctan(x)}{x^2+1}\,dx$

Here we use the substitution $u = \arctan(x)$ so that $du = \frac{1}{x^2+1}dx$ and the integral becomes

$$\int \frac{\arctan(x)}{x^2+1}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\arctan^2(x) + C.$$

15. Define $\ln(x) = \int_1^x \frac{1}{t}\,dt$.

(a) Use the substitution $u = \frac{t}{x}$ to show that $\int_x^{xy} \frac{1}{t}\,dt = \int_1^y \frac{1}{u}\,du$ and conclude by observing the well-known logarithm identity $\ln(xy) = \ln(x) + \ln(y)$.

For $u = \frac{t}{x}$ we have $du = \frac{1}{x}dt$ and when $t = x$ we have $u = 1$ while for $t = xy$ we have $u = y$. It follows that the integral becomes

$$\int_x^{xy} \frac{1}{t}\,dt = \int_1^y \frac{1}{xu}x du = \int_1^y \frac{1}{u}\,du.$$

The following then yields the desired identity:

$$\ln(xy) = \int_1^{xy} \frac{1}{t}\,dt = \int_1^x \frac{1}{t}\,dt + \int_x^{xy} \frac{1}{t}\,dt = \int_1^x \frac{1}{t}\,dt + \int_1^y \frac{1}{u}\,du = \ln(x) + \ln(y).$$

(b) Use the substitution $u = \sqrt[n]{t}$ to show that $\int_1^{x^n} \frac{1}{t}\,dt = n\int_1^x \frac{1}{u}\,du$ and conclude by observing the well-known logarithm identity $\ln(x^n) = n\ln(x)$.

For $u = \sqrt[n]{t}$ we have $du = \frac{1}{n}t^{-(n-1)/n}dt$ or $dt = nu^{n-1}du$ and when $t = 1$ we have $u = 1$ while for $t = x^n$ we have $u = x$. It follows that the integral becomes

$$\int_1^{x^n} \frac{1}{t}\,dt = \int_1^x \frac{1}{u^n}nu^{n-1}du = n\int_1^x \frac{1}{u}\,du.$$

The following then yields the desired identity:

$$\ln(x^n) = \int_1^{x^n} \frac{1}{t}\, dt = n \int_1^{x} \frac{1}{u}\, du = n \ln(x).$$

16. Compute the following definite integrals in two ways, by apply the results of Exercise 14 along with Part II of the Fundamental Theorem of Calculus and by making appropriate substitutions in the bounds:

    (a) $\int_1^2 \frac{2x^3+4}{x^4+4x^2}\, dx$
    Using the substitution $u = x^4 + 4x^2$ *from above we have* $du = (4x^3 + 8x)dx$ *and the integral becomes*

    $$\int_1^2 \frac{2x^3+4}{x^4+4x^2}\, dx = \frac{1}{2}\int_5^{32} \frac{1}{u}\, du = \left[\frac{1}{2}\ln|u|\right]_5^{32} = \frac{1}{2}\ln(32) - \frac{1}{2}\ln(5).$$

    (b) $\int_0^{\ln(2)} e^{2x}\sqrt{4-e^{2x}}\, dx$
    Using the substitution $u = 4 - e^{2x}$ *from above we have* $du = -2e^{2x}dx$ *and the integral becomes*

    $$\int_0^{\ln(2)} e^{2x}\sqrt{4-e^{2x}}\, dx = -\frac{1}{2}\int_3^{0} \sqrt{u}\, du = \left[-\frac{1}{3}u^{3/2}\right]_3^{0} = \frac{1}{3}(3)^{3/2} = \sqrt{3}.$$

    (c) $\int_{-1}^{1} \frac{\arctan(x)}{x^2+1}\, dx$
    Using the substitution $u = \arctan(x)$ *from above we have* $du = \frac{1}{x^2+1}dx$ *and the integral becomes*

    $$\int_{-1}^{1} \frac{\arctan(x)}{x^2+1}\, dx = \int_{-\pi/4}^{\pi/4} u\, du = 0.$$