Math 10550 Exam 2 Practice Problems

1. The position of a particle traveling along the real line is given by $p(x) = (x + 2)(x - 1)^3$.
   (a) Find the critical numbers of $p(t)$ and interpret these in terms of the motion of the particle.
   (b) On which intervals is $p(t)$ increasing and decreasing. Interpret these in terms of the motion of the particle.
   (c) Find all local maximum and minimum values of p. Interpret these in terms of the motion of the particle.
   (d) Where is the graph of p concave upward and where is it concave downward? Interpret these in terms of the motion of the particle.
   (e) Find all inflection points of p. Interpret these in terms of the motion of the particle.

2. A box has fixed volume $100\,m^3$. If the height of the box is increasing at a rate of $2\,m/s$ and the width is increasing at a rate of $3\,m/s$, how fast is the surface area changing when the height is $4\,m$ and the base is a square?

3. A skier jumps off a parabolic ramp with the shape $y = x^2$ for $0 \leq x \leq 1$. If the skier maintains a constant horizontal velocity of $20\,m/s$ while on the ramp, what is the maximum height achieved by the skier assuming the acceleration due to gravity is $10\,m/s^2$? How far will the skier travel before returning to the ground ($y = 0$)? (Ignore any air friction.)

4. Let $f(x) = \cos(x)$.
   (a) Find the linearization of f near $x = \frac{\pi}{6}$.
   (b) Find the linearization of f near $x = \frac{\pi}{4}$.
   (c) Use each linearization above to estimate the value of $\cos\left(\frac{\pi}{5}\right)$.
   (d) Argue using the concavity of $\cos(x)$ for which estimate above is closer to the actual value of $\cos\left(\frac{\pi}{5}\right)$.

5. Find the linearization $L(x)$ of $f(x) = \frac{1}{(2+x)^3}$ near $a = -1$. Verify that "up to first order" we have $L(-1 + dx) = f(-1 + dx)$.

6. Let $f(x) = e^x - 3x$. Find the maximum and minimum value of $f(x)$ on the interval $[-1, 5]$.

7. Let $f(t) = \frac{4t}{1+t^2}$. Use derivatives to show where this function is increasing and where it is decreasing. Also show where it is concave up and concave down. List all critical points. List all points of inflection. Sketch the graph of the function from this data. At what t is there a global maximum (if there is one) and at what t is there a global minimum? Indicate clearly on your graph any horizontal or vertical asymptotes.

8. Does there exist a function f such that $f(0) = 3$, $f(9) = 12$, and $f'(x) < 1$ for $0 \leq x \leq 9$?

9. Let f be a function for which $f(3) = 2$ and $-1 \leq f'(x) \leq 4$. Find upper and lower bounds for $f(-3)$.

10. A **fixed point** of a function f is a number a for which $f(a) = a$. Use Rolle's Theorem to show that a differentiable function f with $f'(x) \neq 1$ for all real numbers x has at most one fixed point. *Hint: Interpret fixed points of f in terms of the function $g(x) = f(x) - x$.*

11. Calculate the following limits. If the limits do not exist, state if they converge to ∞ or to $-\infty$. Otherwise if they do not exist, state that they do not exist.
    (a)
    $$\lim_{x\to\infty} \frac{x^2}{e^x}$$

(b)

$$\lim_{x \to 0^+} x \ln(x)$$

(c)

$$\lim_{x \to 0} \frac{\sin(3x)}{\tan(5x)}$$

(d)

$$\lim_{x \to 0} \frac{x}{\cos x + x}$$