Name: SLN Date: 03/22/2018

M20580 L.A. and D.E. Tutorial
Worksheet 7

1. The vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$. What is the corresponding eigenvalue?

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-12+7 \\ 3-6+7 \\ 5-12+5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$$

Thus $\lambda = -2$

2. Let the matrix

$$A = \begin{bmatrix} 1 & 0 & -4 \\ -6 & -1 & 12 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) Find all eigenvalues of A.

$$|A - \lambda I| = \left|\begin{bmatrix} 1-\lambda & 0 & -4 \\ -6 & -1-\lambda & 12 \\ 0 & 0 & -1-\lambda \end{bmatrix}\right| = (-1-\lambda)\Big((1-\lambda)(-1-\lambda) - (-6\cdot 0)\Big)$$

$$= (-1-\lambda)(1-\lambda)(-1-\lambda)$$

$$= (1+\lambda)^2(1-\lambda)$$

Thus $\lambda = \pm \underline{1}$

(b) Find a basis for each eigenspace corresponding to each eigenvalue which you found in part (a). **Make sure you indicate which eigenvalue each subspace basis corresponds to.**

For $\lambda = -1$ $\quad$ Nul$(A+I)$

$$A+I = \begin{bmatrix} 2 & 0 & -4 \\ -6 & 0 & 12 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $x = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

So Eigenspace basis is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

For $\lambda = 1$ $\quad$ Nul$(A-I)$

$$A-I = \begin{bmatrix} 0 & 0 & -4 \\ -6 & -2 & 12 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $x = \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} t$

So basis is $\left\{ \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} \right\}$

(c) Give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$, or if none such exists, explain why. Note: You do **not** need to compute P^{-1}.

Using the Above basis we get

$$P = \begin{bmatrix} 0 & 2 & -1/3 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

since all vectors are linearly independent

Thus $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Thus $A = PDP^{-1}$

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3. Which of the following is NOT an orthogonal set? (Using the standard inner product)

1. $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}$ $\quad (1)(1) + (1)(0) + (1)(-1) = 0$ ✓

2. $\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}-6\\4\end{bmatrix}$ $\quad -12 + 12 = 0$ ✓

3. $\begin{bmatrix}\cos(t)\\\sin(t)\end{bmatrix}, \begin{bmatrix}\sin(t)\\-\cos(t)\end{bmatrix}$ $\quad \sin(t)\cos(t) - \sin(t)\cos(t) = 0$ ✓

4. $\begin{bmatrix}3\\0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}$ $\quad 0 + 0 + 0 + 2 \neq 0$ ✗

5. $\begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\3\end{bmatrix}$ $\quad 0 + 0 + 0 = 0$ ✓

4. Let A be the matrix

$$\begin{bmatrix}\cos(\theta) & \sin(\theta)\\-\sin(\theta) & \cos(\theta)\end{bmatrix}.$$

Find two complex eigenvectors of A.

$$\begin{vmatrix}(\cos(\theta) - \lambda) & \sin\theta\\ -\sin\theta & (\cos\theta - \lambda)\end{vmatrix} = \cos^2\theta - 2\cos\theta\lambda + \lambda^2 + \sin^2\theta =$$

$$= \lambda^2 - 2\cos\theta\lambda + 1 = 0$$

$$=$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$= \cos\theta \pm \sqrt{-\sin^2\theta}$$

$$\boxed{= \cos\theta \pm i\sin\theta}$$

$$= e^{\pm i\theta}$$

$\lambda = \cos\theta + i\sin\theta \qquad A - \lambda = \begin{bmatrix}-i\sin\theta & \sin\theta\\-\sin\theta & -i\sin\theta\end{bmatrix}$

$-i\sin\theta\, x_1 = -\sin\theta\, x_2$

$i x_1 = x_2 \qquad v = \begin{bmatrix}i\\1\end{bmatrix}$

$\lambda = \cos\theta - i\sin\theta \qquad A - \lambda = \begin{bmatrix}i\sin\theta & \sin\theta\\-\sin\theta & i\sin\theta\end{bmatrix}$

$i\sin\theta\, x_1 = -\sin\theta\, x_2$

$i x_1 = -x_2 \qquad v = \begin{bmatrix}-i\\1\end{bmatrix}$