Partitions

In this section we talk about fully dividing a set into disjoint subsets (or parts), as opposed to just taking out one subset.

Example: Alan, Cassie, Maggie, Seth, Roger, and Beth have volunteered to help at a fund-raising show. One of them will hand out programs at the door, two will run a refreshments stand, and three will help guests find their seats.

In assigning the friends to their duties, we divide or **partition** the set of 6 friends into disjoint subsets of sizes 1, 2, and 3. There are a number of different ways to do this.

Partitions

Below are a few partitions of the set $\{A, B, C, M, R, S\}$:

Prog.	Refr.	Usher
A	CM	SRB
C	AS	MRB
M	CB	ASR
B	SR	ACM
R	CM	SAB

This is not a complete list, it's easy to think of other possible partitions.

We know from experience that it is easier to count the number of such partitions by using our counting principles instead of listing all of them. We can solve this problem easily by breaking the task into steps.

Partitions

Step 1: choose three ushers, $C(6,3) = \dfrac{6!}{3!\ 3!}$

Step 2: choose who runs the refreshments, $C(3,2) = \dfrac{3!}{2!\ 1!}$

Step 3: choose who hands out programs, $C(1,1) = \dfrac{1!}{1!\ 0!}$

Using the multiplication principle, we find that the number of ways that we can split the set of 6 friends into subsets of sizes 3, 2, and 1 is given by:

$$\binom{6}{3} \cdot \binom{3}{2} \cdot \binom{1}{1} = \frac{6!}{3!\ 3!} \cdot \frac{3!}{2!\ 1!} \cdot \frac{1!}{1!\ 0!} = \frac{6!}{3!\cancel{3!}} \cdot \frac{\cancel{3!}}{2!\cancel{1!}} \cdot \frac{\cancel{1!}}{1!0!} = \frac{6!}{3!2!1!}$$

which is equal to 60.

This was example of **ordered partitions**.

Ordered Partitions

Definition: We say a set C is **partitioned** into k nonempty subsets $C_1, C_2, \ldots, C_k$ if:

- The subsets are disjoint.
- $C_1 \cup C_2 \cup \cdots \cup C_k = C$.

The partition in our example is an **ordered partition**. Different subsets of the partition have characteristics that distinguish then from each other (in the example, different partitions were assigned a different task: handing out programs, serving refreshments, and ushing).

Counting Ordered Partitions

The number of ways to partition a set with n elements into k subsets $C_1, \ldots, C_k$, where each subset C_i has r_i elements, is given by

$$\binom{n}{r_1} \cdot \binom{n-r_1}{r_2} \cdots \binom{n-r_1-\ldots-r_{k-1}}{r_k} = \boxed{\frac{n!}{r_1! \cdot r_2! \cdots r_k!}}$$

We have special notation for this. For $n = r_1 + \ldots + r_k$,

$$\boxed{\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! \cdot r_2! \cdots r_k!}}$$

Counting Ordered Partitions

Note: The elements inside each individual subset are not ordered!

Special Case: Partitioning into TWO subsets, one of size k and the other of size $n-k$, is the same as

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Ordered Partitions

Example: In how many ways can the 6 friends $\{A, C, M, S, R, B\}$ be split into groups of two, with each group performing a different task? (e.g. two are assigned to hand out programs, two are assigned to the refreshments stand, and two are assigned as ushers)

This scheme partitions the set of 6 friends into 3 subsets (each of size 2). The subsets are ordered: one subset is assigned to programs, one to refreshments, and one to usher. This gives

$$\binom{6}{2,2,2} = \frac{6!}{2!\,2!\,2!} = \frac{720}{8} = 90$$

possible ways.

Ordered Partitions

Example: In how many ways can a set of 10 people be divided into groups of 5, 3, and 2?

$$\binom{10}{5, 3, 2} = \frac{10!}{5! \cdot 3! \cdot 2!} = 2,520.$$

Example: Evaluate $\binom{7}{3,2,2}$.

$$\binom{7}{3, 2, 2} = \frac{7!}{3! \cdot 2! \cdot 2!} = \frac{5040}{24} = 210$$

Ordered Partitions

Example: A group of 12 new hires at the Electric Car Company will be split into three groups for training. Four will be sent to Dallas, three to Los Angeles, and five to Portland. In how many ways can the group of new hires be divided in this way?

This is an ordered partition problem (the 12 new hires are divided into 3 disjoint subsets which can be distinguished by their corresponding assignments).

$$\binom{12}{4,3,5} = \frac{12!}{5! \cdot 4! \cdot 3!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 3 \cdot 2} = 27,720$$

Ordered Partitions

Example: I want to split the class of 15 students into three groups with five students in each group. One group will discuss the last WebAssign assignment, a second group will review for Friday's exam, and a third group will practice for the next quiz. In how many ways can I form the groups?

$$\binom{15}{5,5,5} = \frac{15!}{(5!)^3} = 756,756$$

Q: What if I wanted to split the 15 students into 3 groups of 5, but they all work on studying for Friday's exam?

Unordered Partitions

A partition is **unordered** when no distinction is made between subsets of the same size (the order of the subsets does not matter).

We use the **overcounting technique** to find a formula for the number of unordered partitions.

- First find the number of ordered partitions
- Divide this by the number of ways orderings of the k subsets, which is $k!$

Example: If we are splitting the 15 friends into groups 5, and they are working on the same task, the number of ways to do so is:

$$\frac{1}{3!}\binom{15}{5,5,5} = \frac{15!}{3! \cdot (5!)^3} = \frac{756,756}{6} = 126,126$$

Unordered Partitions

Example: Our group of 6 friends $\{A, C, M, S, R, B\}$ have signed up to distribute fliers in the neighborhood. The person who hired them doesn't care how they do this but wants groups of two. How many ways can they divide up?

Since we are doing groups of two, we end up with 3 groups. Each group is performing the same task, so we don't care about the order, which is why we need to divide by 3!

$$\frac{1}{3!}\binom{6}{2,2,2} = \frac{1}{3!} \cdot \frac{6!}{2! \cdot 2! \cdot 2!} = 15 \text{ possible pairings.}$$

Counting Unordered Partitions

The number of ways in which a set of n elements can be partitioned into k **unordered subsets** of r elements each is given by

$$\boxed{\frac{1}{k!}\binom{n}{r, r, \ldots, r} = \frac{1}{k!}\frac{n!}{r! \cdot r! \cdots r!} = \frac{n!}{k!(r!)^k}}$$

Note: The subsets must all have the same size (so $kr = n$), otherwise we would be able to distinguish them by their size, which results in an ordered arrangement!

Example: In how many ways can a set with 12 elements be divided into four unordered subsets of equal sizes?

The size of each subset must be 3, so the number of ways is

$$\frac{12!}{4! \cdot (3!)^4} = 15,400$$

Unordered Partitions

Example: There are 32 competitors in a fencing tournament. In how many ways can they be paired up for the matches in the first round?

There are 16 matches total, so we partition a set of size 32 into 16 subsets of equal sizes (2). The number of ways is

$$\frac{32!}{16! \cdot (2!)^{16}} = 191,898,783,962,510,625$$

Q: In how many ways can the winners of the first round be paired up for the matches in the second round?

We have 16 winners, split into 8 subsets of size 2:

$$\frac{16!}{8! \cdot (2!)^{8}} = 2,027,025$$

Unordered Partitions

Example: Find the number of partitions of a 20 element set into subsets of sizes 2, 2, 2, 4, 4, 3, 3. No distinction will be made between subsets except for their size.

We start with the number of ordered partitions, which is

$$\frac{20!}{2! \cdot 2! \cdot 2! \cdot 4! \cdot 4! \cdot 3! \cdot 3!}.$$

For unordered partitions, we divide by 3! (since we have 3 subsets of size 2), 2! (since we have 2 subsets of size 4), and another 2! (since we have 2 subsets of size 3):

$$\frac{1}{3! \cdot 2! \cdot 2!} \times \frac{20!}{(2!)^3 \cdot (4!)^2 \cdot (3!)^2} = 611,080,470,000$$

Unordered Partitions

Example: A math teacher wishes to split a class of 30 students into groups. All groups will work on the same problem. Five groups will have 4 students, two groups will have 3 students, and two groups will have 2 students. In how many ways can the teacher assign students to the groups?

Ordered partitions (if each group was assigned to work on a different problems): $\frac{30!}{(4!)^5 \cdot (3!)^2 \cdot (2!)^2}$.

Since the groups are working on the same problem, the only distinguishing factor is their size. Then the number of unordered partitions is:

$$\frac{1}{5! \cdot 2! \cdot 2!} \times \frac{30!}{(4!)^5 \cdot (3!)^2 \cdot (2!)^2}$$