10350 Tutorial Week 16 - Set 01 Name ______________________

1a. Perform the substitution $u = \cos^2 x - \sin x + 5$ for the integral $\displaystyle\int_0^{\pi/2} \frac{2\cos x \sin x + \cos x}{(\cos^2 x - \sin x + 5)^3}\, dx$.

Be sure to change the integration limits. Do NOT perform the integration. Fill in your answer below:

$$\int_6^4 \underline{\qquad \frac{-1}{u^3} \qquad} \, du$$

Let $u = \cos^2 x - \sin x + 5$. Then $du = -(2\cos x \sin x + \cos x)dx$, and solving for dx gives

$$dx = \frac{-du}{(2\cos x \sin x + \cos x)}$$

We also need to solve for the new bounds. We have

$$u(0) = 6 \text{ (new lower bound)}$$

$$u(\pi/2) = 4 \text{ (new upper bound)}$$

Putting all this together, we get the integral

$$\int_6^4 \frac{-1}{u^3} du$$

1b. Perform the integral you obtained in (a) to evaluate $\displaystyle\int_0^{\pi/2} \frac{2\cos x \sin x + \cos x}{(\cos^2 x - \sin x + 5)^3}\, dx$.

$$\int_6^4 -u^{-3} du = \int_4^6 u^{-3} du = \left.\frac{u^{-2}}{-2}\right|_{u=4}^{u=6} = \frac{-1}{2}(6^{-2} - 4^{-2}) = \frac{5}{288}$$

1c. If $g'(x) = \dfrac{2\cos x \sin x + \cos x}{(\cos^2 x - \sin x + 5)^3}$, find the total change of $g(x)$ over the interval $0 \le x \le \pi/2$.

This is the same as asking to compute $g(\pi/2) - g(0)$, and by the fundamental theorem of calculus this is precisely

$$g(\pi/2) - g(0) = \int_0^{\pi/2} g'(x) dx = \frac{5}{288}$$

by the calculation in parts (a) and (b).

2a. If the instantaneous rate of change of $f(x)$ is given by $\left(\frac{1}{\cos^2 x} + \frac{2}{\csc x}\right)$, find the total change of $f(x)$ over $0 \leq x \leq \pi/4$.

We are given that $f'(x) = \frac{1}{\cos^2 x} + \frac{2}{\csc x}$ (instantaneous rate of change means derivative). By the fundamental theorem of calculus, the total change is

$$f(\pi/4) - f(0) = \int_0^{\pi/4} f'(x)dx = \int_0^{\pi/4} \left(\frac{1}{\cos^2 x} + \frac{2}{\csc x}\right) dx$$

and the integral splits into two integrals

$$\int_0^{\pi/4} \sec^2 x dx + \int_0^{\pi/4} 2 \sin x dx$$

Since the derivative of $\tan x$ is $\sec^2 x$, this is

$$\tan x|_0^{\pi/4} - 2\cos x|_0^{\pi/4} = \left(\tan\frac{\pi}{4} - \tan 0\right) - 2\left(\cos\frac{\pi}{4} - \cos 0\right) = 1 - 0 - 2 \cdot \frac{1}{\sqrt{2}} + 2 = 3 - \sqrt{2}$$

2b. Perform the following integral. If substitution is needed show all steps carefully. $\int \sec(3x)\tan(3x)\,dx \stackrel{?}{=}$

Let $u = 3x$, so $du = 3dx$ and solving for dx gives $dx = \frac{1}{3}du$. Then subsituting in we get the integral

$$\frac{1}{3}\int \sec u \tan u du$$

and since the derivative of $\sec u$ is $\sec u \tan u$, this is precisely

$$\frac{1}{3}\sec u + C$$

Finally, we substitute in $u = 3x$, so the answer is

$$\frac{1}{3}\sec(3x) + C$$

3. Find the anti-derivative $F(x)$ of $f(x) = \sin x \cos^2 x$ such that $F(0) = 2/3$.

We let $u = \cos x$, so $du = -\sin x dx$, which gives us the general antiderivative

$$\int \sin x \cos^2 x dx = -\int u^2 du = -\frac{u^3}{3} + C = -\frac{\cos^3 x}{3} + C$$

Since $F(0) = 0$, we have $-1/3 + C = 2/3$, which implies $C = 1$, so the antiderivative we seek is

$$F(x) = 1 - \frac{\cos^3 x}{3}$$

4. Find the derivatives of the following functions:

a. $f(x) = \displaystyle\int_e^x e^{\sin(2t)}\, dt$

Let the integral be $g(t) = e^{\sin(2t)}$, so we are looking at an integral of the form

$$f(x) = \int_e^x g(t)dt$$

Now denote by $G(t)$ any antiderivative of $g(t)$, so by the fundamental theorem of calculus we have

$$f(x) = G(t)|_e^x = G(x) - G(e)$$

Then to get the derivative $f'(x)$, we differentiate the right side with respect to x:

$$f'(x) = \frac{d}{dx}\left(G(x) - G(e)\right) = \frac{d}{dx}G(x)$$

and since G is an antiderivative of g, we get our final result

$$f'(x) = \frac{d}{dx}G(x) = g(x) = e^{\sin(2x)}$$

b. $y = \displaystyle\int_{x^2}^1 \frac{t^2+1}{\ln(t^2+1)}\, dt$

Again as in the previosu problem, let g be the integrand $g(t) = \frac{t^2+1}{\ln(t^2+1)}$ and $G(t)$ some antiderivative of g. Then by the fundamental theorem of calculus

$$y(x) = \int_{x^2}^1 g(t) = G(1) - G(x^2)$$

so the derivative with respect to x is

$$y'(x) = \frac{d}{dx}\big(G(1) - G(x^2)\big) = -\frac{d}{dx}G(x^2)$$

which by Chain Rule is

$$y'(x) = -g(x^2)\cdot 2x = -\frac{(x^2)^2+1}{\ln((x^2)^2)+1)}\cdot 2x = -\frac{x^4+1}{\ln(x^4+1)}\cdot 2x$$

5. Referring to the graph of $f(t)$ below, compute the following values or expressions. The dotted line the graph of the tangent line to curve at $t = 2$

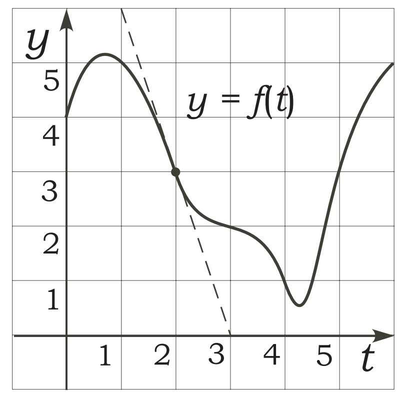
a. Average rate of change of $f(t)$ over $[1, 5]$.

$$\frac{f(5) - f(1)}{5 - 1} = \frac{3 - 5}{4} = \frac{-1}{2}$$

b. Find the linear approximation to the function $f(t)$ at $t = 2$. Estimate $f(1.9)$.

The slope at $x = 2$ is -3 (by the dotted line, it goes down 3 units, over 1 unit), so

$$f(t) \approx -3(t - 2) + f(2) = -3t + 6 + 3 = -3t + 9$$



Recall that the linerization (or linear approximation) is used to approximate a function by a line in the vecinity of some x value (in our case near $x = 2$). The line we are using to approximate is the tangent line, so we could have easily looked at the picture and realize we are just talking about the dotted tangent line at $x = 2$, which has equation $y = 9 - 3x$ (if we were to continue the dotted line, the y-intercept would be up at 9).
To approximate $f(1.9)$, we plug that into our linealization and see that $f(1.9) \approx -3(1.9) + 9 = 3.3$.

c. The instantaneous rate of change of $p(t) = tf(t)$ at $t = 2$

We are looking for the derivative $p'(t)$ which by power rule is

$$p'(t) = 1 \cdot f(t) + t \cdot f'(t)$$

When evaluated at $t = 2$, this is

$$p'(2) = f(2) + 2f'(2) = 3 + 2(-3) = -3$$

d. The slope to the graph of $Q(t) = \dfrac{f(t)}{t + 1}$ at $t = 2$

We use the quotient rule to get

$$Q'(t) = \frac{f'(t) \cdot (t + 1) - f(t)}{(t + 1)^2}$$

and we evaluate this at $t = 2$ to get

$$Q'(t) = \frac{-3(3) - 3}{3^2} = \frac{-4}{3}$$

6. Let $f(x) = \dfrac{3}{(2x+1)}$. Find all values $x = c$ in the interval $1 \leq x \leq 4$ that satisfy the Mean Value Theorem.

The average rate of change over the given interval is

$$m = \frac{f(4) - f(1)}{4 - 1} = \frac{3/9 - 3/3}{3} = \frac{1}{9} - \frac{1}{3} = \frac{-2}{9}$$

and the derivative of $f(x)$ is

$$f'(x) = -6(2x+1)^{-2}$$

so our function is continuous on $[1, 4]$ and differentiable on $(1, 4)$, meaning the mean value theorem applies (there exists some x-value c inside our interval at which the instantaneous slope of f is equal to the average rate of change over the interval, i.e $f'(c) = m$). To find c, we set the derivative equal to the average rate of change m and solve (here I cancel the negative signs on both sides):

$$\frac{6}{(2c+1)^2} = \frac{2}{9}$$

$$2c + 1 = \pm 3\sqrt{3}$$

so we get two solutions, namely

$$c = \frac{-1 \pm 3\sqrt{3}}{2}$$

However the solution with the negative sign is outside our interval (the numerator is negative), so we only have one possible candidate for c

$$c = \frac{-1 + 3\sqrt{3}}{2}$$

To check that this indeed lies in our interval, notice that

$$1 < \sqrt{3} < 2 \text{ (because } 1 < 3 < 4 \text{ and we apply sqrt)}$$

and multiplying by 3 gives

$$3 < 3\sqrt{3} < 6$$

$$2 < -1 + 3\sqrt{3} < 5$$

and dividing by 2 gives $1 < c < 5/2$ and $5/2$ is less than 4, so c lies in our interval $(1, 4)$ as desired.

7. Find all critical points of the function $f(x) = x - 6 \cdot x^{2/3}$.

We take the derivative, and find it zeroes as well as all the points where it is undefined.

$$f'(x) = 1 - 6 \cdot \frac{2}{3} \cdot x^{-1/3} = 0$$

which yields $x = 64$. Also notice $f'(x)$ is undefined for $x = 0$ (zero in the denominator). Since both 0, and 64 lie in the domain of f, both are critical points.

8 a. Solve the initial value problem: $\dfrac{dy}{dx} = e^{-x} + 3e^{x}$ such that $y(0) = 3$.

$$\int (e^{-x} + 3e^{x})dx = -e^{-x} + 3e^{x} + C$$

and plugging in $x = 0$ this gives $-1 + 3 + C = 3$, so $C = 1$. Hence the solution to the initial value problem is

$$y = -e^{-x} + 3e^{x} + 1$$

8 b. Find also x-intercept of the graph of the function $g(x) = y(x) - 1$ where y is the function you found in part (a).

Finding the x-intercept is equivalent to solving $g(x) = 0$. But this is the same as solving

$$y(x) - 1 = 0$$

$$y(x) = 1$$

$$-e^{-x} + 3e^{x} + 1 = 1$$

$$-e^{-x} + 3e^{x} = 0$$

$$3e^{x} = e^{-x}$$

Now we multiply both sides by e^{x} which yields

$$3e^{2x} = 1$$

$$e^{2x} = \frac{1}{3}$$

$$2x = \ln\frac{1}{3}$$

So our x-intercept is

$$x = \frac{1}{2}\ln\frac{1}{3}$$

We may also write this as $-\ln\sqrt{3}$ or as $\ln\frac{1}{\sqrt{3}}$.