Names ____ Date ____________

Math 10550 Worksheet 1 Solutions

Please work through these questions as a group and hand them in as a group at the end. Each individual should write the method and solutions on their own worksheet to keep for reference and exam review. The aim of this worksheet is you help you integrate the material by having you work on questions that involve several concepts. It is important that everyone in the group understands the material, so I suggest that as you complete the worksheet, you use a mix of brainstorming and discussion of the reasons behind each method you are using. If there is time left over at the end, please pair off and reflect on what concepts were required for each question. If someone has difficulty understanding a solution, they should pair off with someone who understands it. If you have serious difficulty understanding these problems, please talk to your TA about getting help on the material.

1. Consider the function $f(x) = \dfrac{1}{x-1}$ when $x > 1$.

(a) Show that f is one-to-one.

(b) Find $(f^{-1})'(2)$.

(c) Calculate $f^{-1}(x)$ and state the domain and range of f^{-1}.

(d) Calculate $(f^{-1})'(2)$ from the formula in part (c) and check that it agrees with your result from (b).

Solution:

(a) The derivative

$$f'(x) = \frac{-1}{(x-1)^2}$$

is always (strictly) negative when $x > 1$. Thus f is one-to-one. Indeed, because f is decreasing, if $x_1 < x_2$ then $f(x_1) > f(x_2)$. So $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

(b) Note that $f^{-1}(2) = \frac{3}{2}$ since $f(\frac{3}{2}) = \frac{1}{\frac{1}{2}} = 2$. Because f is one-to-one and

$$f'(f^{-1}(2)) = f'(\frac{3}{2}) = -4 \neq 0$$

we may use the formula

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = -\frac{1}{4}$$

(c) We can explicitly find the inverse function f^{-1} by solving the equation $x = \frac{1}{y-1}$ for y.

$$\begin{aligned} x &= \frac{1}{y-1} \\ x(y-1) &= 1 \\ xy - x &= 1 \\ xy &= x+1 \\ y &= \frac{x+1}{x} \end{aligned}$$

Thus $f^{-1}(x) = \frac{x+1}{x}$. The domain of f^{-1} is $(0, \infty)$, the range of f. The range of f^{-1} is $(1, \infty)$, the domain of f.

(d) First we compute the derivative

$$(f^{-1})'(x) = \frac{x - (x+1)}{x^2} = -\frac{1}{x^2}.$$

Now evaluating,

$$(f^{-1})'(2) = -\frac{1}{2^2} = -\frac{1}{4}$$

as expected.

2. The function $f(x) = 6x - \sqrt{x}$ is one-to-one. Find the tangent line to the inverse function $f^{-1}(x)$ at the point $x = 5$.

Solution: Since f is one-to-one, we may use the formula $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$ with $a = 5$, provided that $f'(f^{-1}(a)) \neq 0$.

First, $f^{-1}(5) = 1$, since $f(1) = 6(1) - \sqrt{1} = 5$.

We also know that $f'(x) = 6 - \frac{1}{2\sqrt{x}}$, so $f'(f^{-1}(5)) = f'(1) = 6 - \frac{1}{2\sqrt{1}} = \frac{11}{2} \neq 0$.

Thus $(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{\frac{11}{2}} = \frac{2}{11}$.

The tangent line to the inverse function at the point $x = 5$ is given by

$$\begin{aligned} y - f^{-1}(5) &= [(f^{-1})'(5)](x - 5) \\ y - 1 &= \frac{2}{11}(x - 5) \\ y &= \frac{2}{11}x + \frac{1}{11}. \end{aligned}$$

3. Use Logarithmic differentiation to find the derivative of the function

$$f(x) = x^{7x}.$$

We will discuss the meaning of x^x for all x in class.

Solution: Let $f(x) = x^{7x}$. Then $\ln(f(x)) = \ln(x^{7x}) = 7x \ln x$.

Differentiating both sides of the logarithmic equation with respect to x, we have

$$\frac{f'(x)}{f(x)} = 7 \cdot \ln x + 7x \cdot \frac{1}{x} = 7 \ln x + 7.$$

Finally, we solve the above equation for f':

$$f'(x) = (f(x))(7 \ln x + 7) = x^{7x}(7 \ln x + 7).$$

4. Evaluate the integrals

$$\int_e^9 \frac{dx}{2x\sqrt{\ln x}}, \qquad \int_1^{e^{\pi/2}} \frac{\cos(\ln x)}{x} dx, \qquad \int \frac{2x^3-5}{x^4-10x+7} dx$$

Solution:

(a) Let $u = \ln x$. Then $du = \frac{dx}{x}$. When we perform the substitution (making sure to change the limits of integration appropriately) we have

$$\begin{aligned} \int_e^9 \frac{dx}{2x\sqrt{\ln x}} = \int_{\ln e}^{\ln 9} \frac{du}{2\sqrt{u}} &= \left[\sqrt{u}\,\right]_1^{\ln 9} \\ &= \sqrt{\ln 9} - 1. \end{aligned}$$

(b) Again, let $u = \ln x$. This substitution transforms the original integral as follows: $\int_0^{\pi/2} \cos u du$

$$\begin{aligned} \int_1^{e^{\pi/2}} \frac{\cos(\ln x)}{x} dx &= \int_0^{\pi/2} \cos u \, du \\ &= [\sin u]_0^{\pi/2} = 1 - 0 = 1. \end{aligned}$$

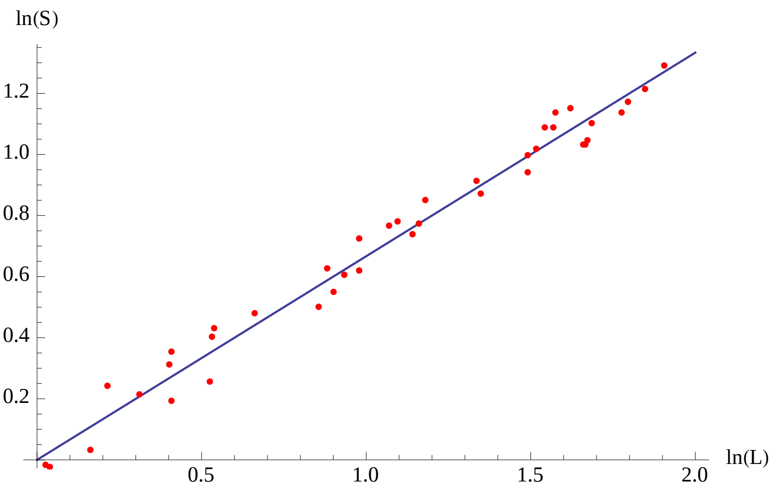
(c) Let $u = x^4 - 10x + 7$. Then $du = 4x^3 - 10 \ dx = 2(2x^3 - 5) \ dx$. Applying this substitution yields

$$\begin{aligned} \int \frac{2x^3-5}{x^4-10x+7} dx &= \frac{1}{2} \int \frac{1}{u} \ du \\ &= \frac{1}{2} \ln|u| + C \\ &= \frac{1}{2} \ln|x^4 - 10x + 7| + C. \end{aligned}$$

5. Common use for logarithms for managing and interpreting data. Suppose you collected data on 40 different types of animals and record a pair of numbers, the length of their front leg, L, and their walking stride length, S, for each. Suppose you plotted the logarithms of observations on a Cartesian plane and you found that the observations lie very close to the graph of the function

$$\ln(S) = \frac{2\ln(L)}{3}$$

for the range of data in question.



Given that the natural logarithm is a one to one function, can you give a function which roughly models the relationship between walking stride length, S, and leg length, S, for animals (a model of the type $S = f(L)$)?

Solution: We solve the equation $\ln(S) = \frac{2\ln(L)}{3}$ for S:

$$\begin{aligned} e^{\ln(S)} &= e^{\frac{2\ln(L)}{3}} \\ S &= e^{\frac{2\ln(L)}{3}} \\ S &= e^{\ln(L^{\frac{2}{3}})} \\ S &= L^{\frac{2}{3}}. \end{aligned}$$