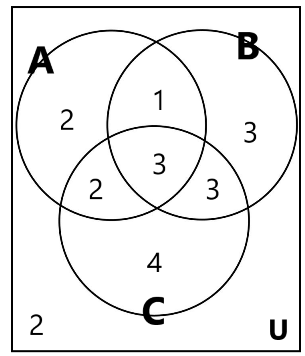
Name: ______________________________ June 30, 2017

FINITE MATH: EXAM 1 SOLUTION

ADRIAN PĂCURAR

Problem 1. Consider the Venn diagram on the right.



a) (2pt) Find $n(U)$.

$$\boxed{n(U) = 2 + 1 + 3 + 2 + 3 + 3 + 4 + 2 = 20}$$

b) (2pt) Find $n(A \cap B)$.

$$\boxed{n(A \cap B) = 1 + 3 = 4}$$

c) (3pt) Find $n(A' \cup B')$.

It is best to use DeMorgan's laws, followed by the complement principle:

$$\boxed{n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B) = 20 - 4 = 16}$$

d) (2pt) Compute the size of $(B \cup C) \setminus A$.

We want the elements inside B or C, but outside A: $\boxed{3 + 3 + 4 = 10}$

e) (2pt) Compute the size of $(A' \cup C')' \cap B$.

Again we use DeMorgan so we are really counting the number of elements inside $(A \cap C) \cap B$, which is the middle region: $\boxed{3}$

f) (2pt) Compute the size of $(A \cup B \cup C)'$.

The easiest way to do this is to use the complement principle. We really want: $\boxed{n(U) - n(A \cup B \cup C) = 2}$ (everything outside the 3 circles).

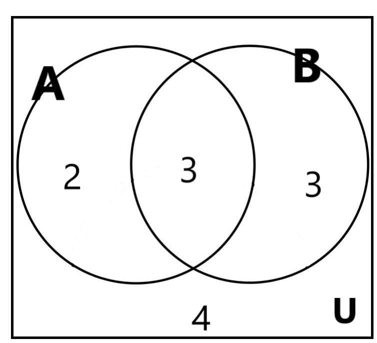
Problem 2. (1pt) How many subsets of size 5 does the set $\{1, 2, 3, \ldots, 13\}$ have?

$$\boxed{\binom{13}{5} = \frac{13!}{5! \cdot 8!}}$$

Problem 3. (4pt) Suppose we have the universal set $U = \{1, 2, 3, \ldots, 11, 12\}$, and consider the following sets:

$$A = \{1, 2, 3, 4, 9\} \qquad \text{and} \qquad B = \{3, 4, 5, 7, 9, 11\}$$

Fill in the Venn diagram shown on the right with the correct **number of elements** for each region.



Problem 4. In our finite math class, we have 7 women and 8 men.

a) (1pt) In how many ways can we arrange everyone in a **single row** for a class photo?

We are arranging everyone in a single row. Male/female doesn't matter, just the total number of people, which is 15. Hence the number of ways is $\boxed{15!}$

b) (3pt) In how many ways can we arrange the students in **two rows** for a class photo, if all the women are in the front row, and all the men are in the back row?

We arrange the two rows separately. The male row can be done in 8! ways, and the female row can be done in 7! ways. Multiplication principle says the total number of ways is $\boxed{7! \cdot 8!}$

Problem 5. A multiple choice exam consists of 25 TRUE/FALSE questions.

a) (1pt) In how many ways can the exam be answered students must answer every question (no questions can are left unanswered)?

For each question we have 2 possible choices. There are 25 questions total, so the number of ways to answer the exam is $\boxed{2^{25}}$

b) (1pt) In how many ways can the exam be answered if students have the option of leaving questions blank.

This time, for each question we have 3 possible choices (true, or false, or leave it blank). There are 25 questions total, so the number of ways to answer the exam is $\boxed{3^{25}}$

c) (2pt) In how many ways can the exam be answered if a student marks exactly 10 of the answers as TRUE, and the remaining answers as FALSE?

We have to decide which of the 25 questions are marked true. In other words, **choose** which of the questions we mark as true. The number of ways to do so is $\boxed{\binom{25}{10}}$

Problem 6. A coin is flipped 10 times in a row, and the result on each flip is recorded.

a) (1pt) How many possible outcomes are there **for a single flip**?

A single flip can come up Heads or Tails, so $\boxed{\text{2 possible outcomes}}$

b) (1pt) How many possible outcomes are there for the 10 flips in a row?

We have 2 choices for each of the 10 flips, which gives $\boxed{2^{10} \text{ possible outcomes}}$

c) (2pt) After 10 flips, in how many ways can you get exactly 5 Heads and 5 Tails?

We need to decide where the 5 heads go (out of 10 possible places). There are $\boxed{\binom{10}{5} \text{ possible outcomes}}$

d) (2pt) After 10 flips, in how many ways can you get **at least one** Tail?

The easiest way to do this is with the complement principle. From all the possible outcomes, we remove the possibility of getting zero Tails:

$\boxed{2^{10} - \binom{10}{0} \text{ possible outcomes}}$

Problem 7. Dana decides to plant either a rose bush or a small tree in her back yard. Home Depot has 4 varieties of rose bushes, and 7 varieties of small trees.

a) (1pt) In how many ways can she select **a single plant**?

She is picking one plant, which can be either a rose bush or a small tree (and these are disjoint possibilities). The number of ways she can do this is $\boxed{4 + 7 = 11}$

b) (1pt) In how many ways can she select **one plant of each type** (so she ends up with 2 plants total?

She is picking two plants, one from each category. The number of ways she can do this is $\boxed{4 \cdot 7 = 28}$

Problem 8. (2pt) A classroom is split into two separate groups (they don't necessarily have equal sizes). The teacher selects two students, one from each group, and the number of ways in which he can select the students is 35. How many students are in each group?

Notice that $35 = 5 \cdot 7$, so the sizes of the two groups must be $\boxed{\text{5 students and 7 students}}$

Problem 9. (2pt) In how many different ways can you rearrange the letters of the word CALCULUS?

First we need to figure out how many times each letter appears. C appears 2 times, A appears 1 time, L appears 2 times, U appears 2 times, S appears 1 time. The number of possible rearrangements is

$$\boxed{\frac{8!}{2! \cdot 1! \cdot 2! \cdot 2! \cdot 1!}}$$

Problem 10. A child forms **3-letter words** using letters from the word CALCULUS. Letters may be repeated.

a) (1pt) How many different words can be formed if letters may be repeated?

This is NOT a rearrangement problem, so the multiplicities of the letters don't matter. We really only have 5 letters to work with, $\{C, A, L, U, S\}$. That leaves us with 5 choices for each position, so $\boxed{5 \cdot 5 \cdot 5 = 125}$

b) (1pt) How many different words begin with 'C'?

Our word must look like "C $* *$", so $\boxed{1 \cdot 5 \cdot 5 = 25}$

c) (1pt) How many different words begin with a vowel?

Letters A and U are the only available vowels. $\boxed{2 \cdot 5 \cdot 5 = 50}$

d) (1pt) How many different words end with a vowel?

Letters A and U are the only available vowels. $\boxed{5 \cdot 5 \cdot 2 = 50}$

e) (3pt) How many different words begin with a vowel or end with a vowel, **but not both**?

We need to treat the disjoint possibilities of starting/ending with a vowel separately. If we start with a vowel, we have $2 \cdot 5 \cdot 3$ possibilities (as the last letter MUST be a consonant, and there are 3 of them). If we end with a vowel, we have $3 \cdot 5 \cdot 2$ as the first letter must be a consonant. The total is $\boxed{2 \cdot 5 \cdot 3 + 3 \cdot 5 \cdot 2 = 30 + 30 = 60}$

f) (1pt) How many 3-letter words can be formed if letters may NOT be repeated?

$\boxed{5 \cdot 4 \cdot 3 = 60}$

Problem 11. (3pt) In how many different ways can 7 people be seated at a **round table**, if any rotation of the table is considered to be the same arrangement?

We seat people at the table one by one. For the first person we have 7 choices, for the second person we have 6, for the third we have 5, and so on. This gives 7!, but that's an overcount. We need to divide by the number of possible rotations. A circle of 7 people has 7 different rotations, so the correct count is $\boxed{\frac{7!}{7} = 6!}$

Problem 12. (3pt) In how many ways can 8 different pairs of shoes be put on display **in a single row** if the pairs are to appear in the correct order (left shoe before right shoe, not the other way around)?

Since we want the pairs to be in order, we can treat each pair of shoes as a whole unit (with no way to interchange the two shoes in a pair). We have 8 units total, so the number of arrangements is $\boxed{8!}$

Problem 13. (3pt) Use the Binomial Theorem and Pascal's triangle to expand the product $(x + y)^6$. **Compute the coefficients** (don't leave them in "choose" form).

The Pascal triangle is going to be

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

and if we continue this all the way to the correct row, the coefficients are 1 6 15 20 15 6 1, so the expanded product is going to be

$$\boxed{(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6}$$

Problem 14. (3pt) A 4-digit PIN number is formed using the digits $\{1, 2, 3, 4, 5\}$. How many numbers can be formed if the first 2 digits may be repeated, but the last 2 digits must be unique?

This one is a little tricky. We need to consider two possibilities. First, it's possible for the initial 2 digits to be the same, and this can be done in $5 \cdot 1 \cdot 4 \cdot 3$ ways (second digit is equal to the first, so only one choice there, while the last to have 4 and 3 choices as they can't be repeats of what came before). Second, it's possible for the initial 2 digits to be different, and this can be done in $5 \cdot 4 \cdot 3 \cdot 2$ ways. By the addition principle, we have $\boxed{5 \cdot 1 \cdot 4 \cdot 3 + 5 \cdot 4 \cdot 3 \cdot 2 = 60 + 120 = 180}$

Problem 15. (3pt) A die is rolled five times, and the **sum** of the two numbers is recorded. In how many ways can one get a sum of 6?

The possible ways to get a sum of 6 are: $1+5$, $2+4$, $3+3$, $4+2$, $5+1$. So we have $\boxed{\text{5 ways total}}$

Problem 16. Suppose you select **six cards** from a standard deck of 52 cards.

a) (1pt) How many different 6-card samples are possible?

We are selecting 6 objects out of 52, so $\boxed{\binom{52}{6}}$

b) (1pt) How many different 6-card samples contain the four Queens?

There are only four Queens in a deck, and our 6 card sample must have all of them. Then only 2 more cards are needed to complete our sample, and those are picked out of the 48 remaining cards in a deck: $\boxed{\binom{48}{2}}$

c) (3pt) How many different 6-card samples contain 3 Hearts and 3 Black cards?

There are 13 hearts total to choose from. Since hearts are red, we have 26 black cards available once we choose the hearts: $\boxed{\binom{13}{3} \cdot \binom{26}{3}}$

d) (3pt) How many different 6-card samples contain at least one red card?

The simplest strategy is to take all possible 6 card samples, and remove those that contain no red (i.e all black) cards. This is $\boxed{\binom{52}{6} - \binom{26}{0}\binom{26}{6}}$, where the second term counts the number of ways to get 0 red cards and 6 black cards.

Problem 17. (3pt) Billy has 12 baseball cards, and Scottie only has 2. Billy owes Scottie \$5, but instead Scottie would prefer to trade 2 of his own cards four 4 of Billy's cards. In how many ways can the trade be made?

Notice that Scottie is giving up all of his 2 cards, and there is only one way to do so. The number of ways for the trade to happen is equal to the number of ways to choose 4 cards from Billy's collection, $\boxed{\binom{12}{4}}$

Problem 18. (3pt) How many positive factors/divisors does the number 3,888 have? It may be useful to know that the prime factorization of 3,888 is $2^4 \cdot 3^5$.

Any positive divisor must be of the form $2^a 3^b$, where $a \in \{0, 1, 2, 3, 4\}$ and $b \in \{0, 1, 2, 3, 4, 5\}$. Since there are 5 choices for a and 6 choices for b, the number of positive divisors is $\boxed{5 \cdot 6 = 30}$

Problem 19. (3pt) A child is making 3-letter words using letters from the English alphabet. Letters can be repeated. How many 3-letter words will contain the letter 'A' exactly once?

Since the letter "A" appears exactly once, first decide where in the 3 letter sequence it appears. This can be done in $\binom{3}{1}$ ways. Then fill in the rest of the letters in $25 \cdot 25$ ways (we lose the ability to reuse letter "A" since it appears only once). We get $\boxed{\binom{3}{1} 25^2 = 1,875}$

Problem 20. (2pt) How many different phone numbers are there that begin with area code 574? A standard number has the format (574) xxx-xxxx, and any of the digits $\{0, 1, 2, \ldots, 9\}$ may be used for the rest of the number (digits may be repeated).

The are code is fixed. We just need to fill out the remaining 7 digits of the phone number, and each digit has 10 possible values, giving us $\boxed{10^7}$ possible phone numbers.