Bernoulli Trials

Some probability models arise very often in applications:

- Tossing a single coin (fair or unfair)
- Answering a TRUE/FALSE false at random
- Planting a seed noting if it germinates or not

Q: What do these experiments have in common?

A: They all involve two **mutually exclusive and complementary** outcomes.

Bernoulli Trials

Definition: A **Bernoulli experiment** is a random experiment where the outcome is one of two **mutually exclusive** (and complementary) ways, which can be thought of as **success or failure**.

Examples: male/female, Heads/Tails, life/death, even/odd, nondefective/defective, correct/wrong.

Definition: A sequence of **Bernoulli trials** occurs when a Bernoulli experiment is repeated several **independent** times. The probability of success — say, p — remains the same from one trial to the next.

We usually write p for the probability of success, and $q = 1 - p$ for the probability of failure.

Examples Of Bernoulli Trials

Example: Suppose that 20% of instant lottery tickets are winners. If 5 tickets are purchased, then one possible observed sequence is (0,0,0,1,0), where the fourth ticket is a winner, and the rest are losers. Assuming independence, the probability of this outcome is

$$(0.8)(0.8)(0.8)(0.2)(0.8) = (0.2)(0.8)^4 = 0.08192$$

If we want the probability of getting exactly 1 winning ticket, there are other possible sequences that may arise: 10000, 01000, 00100, 00010, and 00001 are all of them. In fact, there are $\binom{5}{1}$ of them (choose which of the 5 is the winning ticket). Hence the probability of getting exactly one winning ticket out of 5 is

$$\binom{5}{1}(0.2)^1(0.8)^4 = 0.4096$$

Examples Of Bernoulli Trials

Example: The probability of germination of a basil seed is 0.8, and germination is called a success. If we plant 10 seeds, the germination of each seed is independent, and this corresponds to 10 Bernoulli trials with $p = 0.8$ and $q = 0.2$. What is the probability that exactly 7 seeds germinate?

$$P(X = 7) = \binom{10}{7}(0.8)^7(0.2)^3 \approx 0.201$$

Example: Tossing a fair coin 8 times, and Heads is a success. Each toss in is independent, and this corresponds to 8 Bernoulli trials with $p = 0.5$ and $q = 0.5$. What is the probability of observing one or more Heads?

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{8}{0}(0.5)^0(0.5)^8 = 1 - \frac{\binom{8}{0}}{2^8}$$

which is about 99.6%.

Examples Of Bernoulli Trials

Example: A multiple choice exam has 20 questions and 4 choices per question. Assuming each question must be answered (no blanks), this corresponds to 20 Bernoulli trials with $p = 0.25$ and $q = 0.75$.

What is the probability of getting an A (90% or more)?

To get an A, one would need to answer at least 18 questions correctly ($18/20 = 0.90$). Let $X = \#$ correct.

$$\begin{aligned}
P(X \geq 18) &= P(X = 18) + P(X = 19) + P(X = 20) \\
&= \binom{20}{18}(0.25)^{18}(0.75)^2 + \binom{20}{19}(0.25)^{19}(0.75)^1 \\
&\qquad + \binom{20}{20}(0.25)^{20}(0.75)^0 \\
&\approx 0.00000000161
\end{aligned}$$

Examples Of Bernoulli Trials

Example: A multiple choice exam has 20 questions and 4 choices per question. Assuming each question must be answered (no blanks), this corresponds to 20 Bernoulli trials with $p = 0.25$ and $q = 0.75$.

What is the probability of getting a C or better (70% or more)?

To get a C or more, one would need to answer at least 14 questions correctly ($14/20 = 0.70$).

$$\begin{aligned} P(X \geq 14) &= P(X = 14) + \cdots\cdots + P(X = 20) \\ &= \binom{20}{14}(0.25)^{14}(0.75)^{6} + \cdots + \binom{20}{20}(0.25)^{20}(0.75)^{0} \\ &\approx 0.00003 \end{aligned}$$

which is still very small!

Examples Of Bernoulli Trials

Example: A multiple choice exam has 20 questions and 4 choices per question. Assuming each question must be answered (no blanks), this corresponds to 20 Bernoulli trials with $p = 0.25$ and $q = 0.75$.

What is the probability of getting a 50% or better?

We need 10 or more correct questions.

$$\begin{aligned} P(X \geq 10) &= P(X = 10) + \cdots\cdots + P(X = 20) \\ &\approx 0.0138 \end{aligned}$$

Note: The probability is so small because p and q are not equal. If they were both 50% (i.e. with TRUE/FALSE questions), the odds would be much better!

The Binomial Distribution

In all these examples, the random variable X was counting the **number of successes** out of n independent Bernoulli experiments.

The probability of success on each individual trial is a constant p, and the probability of failure is $q = 1 - p$.

The distribution of $X = \#$ of successes is called the **Binomial Distribution**, and the probability of X taking the value k is given by

$$\boxed{P(X = k) = \binom{n}{k} p^k q^{n-k}}$$

where $k = 0, 1, 2, \ldots, n$.

Expected Value And Variance

For a single Bernoulli Trial X_1

- $E(X_1) = (0)(1-p) + (1)(p) = p$
- $Var(X_1) = \sigma^2 = p(1-p) = pq$
- $\sigma = \sqrt{p(1-p)} = \sqrt{pq}$

For n indept. Bernoulli Trails $X = X_1 + X_2 + \cdots + X_n$

- $\mu = np$
- $Var(X) = \sigma^2 = np(1-p) = npq$
- $\sigma = \sqrt{np(1-p)} = \sqrt{npq}$

Expected Value And Variance

Example: An observation over a long period of time reveals that, on average, 1 out of 10 items produced by a process is defective. Select 5 items independently from the production line, and test them. Let X denote the number of defective items among the 5.

a) What is the distribution of X?

X is binomial with $n = 5$ and $p = 0.1$.

b) What is $E(X)$ and $Var(X)$?

$$E(X) = np = (5)(0.1) = 0.5$$

$$Var(X) = npq = 5(0.1)(0.9) = 0.45$$

Calculator Use

Example: An unfair coin has $P(H) = 0.6$. If the coin is tossed 15 times, what is the chance of getting at most 8 H?

$$P(X \leq 8) = P(X = 0) + P(X = 1) + \cdots + P(X = 8)$$

which takes a long time to compute term by term. On TI-83 and TI-84 calculators, there is a **cumulative distribution function** for the Binomial which does the adding for you:

$$\textbf{binomcdf}(n, p, k) = P(X = 0) + \cdots + P(X = k)$$

For this example, we use $\text{binomcdf}(15, 0.6, 8) \approx 0.39$.

If we wanted just $P(X = 8)$ instead of the sum from 0 to 8, we use $\textbf{binompdf}(15, 0.6, 8) \approx 0.177$.

Calculator Use

Example: An unfair coin has $P(H) = 0.6$. If the coin is tossed 15 times, what is the chance of getting at least 3 H?

We want 3 or more Heads, which is the complement of 0,1,2 Heads.

$$
\begin{aligned}
P(X \geq 3) &= 1 - P(X < 3) \\
&= 1 - P(X \leq 2) \\
&= 1 - [P(X = 0) + P(X = 1) + P(X = 2)] \\
&= 1 - \text{binomcdf}(15, 0.6, 2) \\
&\approx 0.9997
\end{aligned}
$$

Calculator Use

Example: An unfair coin has $P(H) = 0.6$. If the coin is tossed 15 times, what is the chance of getting between 5 and 10 Heads (inclusive)?

We want $P(5 \leq X \leq 10) = P(X = 5, 6, 7, 8, 9, 10)$.

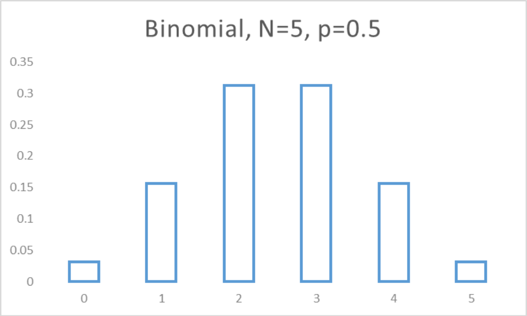
We could do the cumulative probability all the way up to 10, but then we need to subtract the probabilities corresponding to $X = 0, 1, 2, 3, 4$:

$$\begin{aligned} P(5 \leq X \leq 10) &= bcdf(15, 0.6, 10) - bcdf(15, 0.6, 4) \\ &\approx 0.77337 \end{aligned}$$

Binomial Histograms

For a **fair coin** tossed 5 times, the probability distribution has the following histogram:

X	P(X)
0	0.032
1	0.156
2	0.312
3	0.312
4	0.156
5	0.032

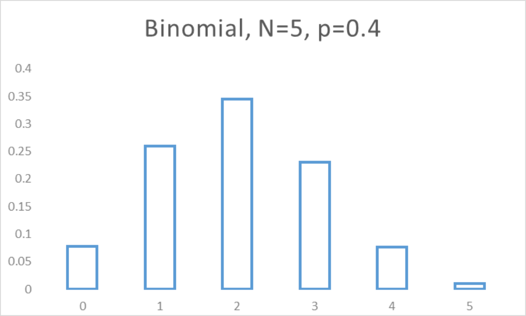


Notice that the probabilities are symmetrical about the mean. $\mu = 2.5$ and $\sigma^2 = 1.25$.

Binomial Histograms

For an **unfair coin** tossed 5 times, the probability distribution changes:

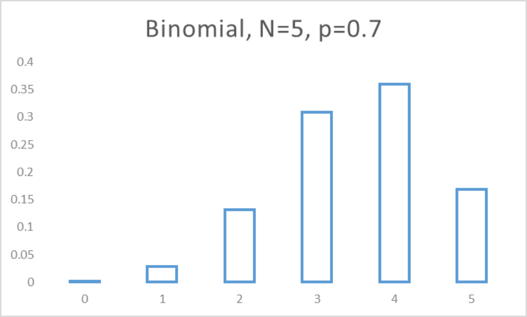
X	P(X)
0	0.0778
1	0.2592
2	0.3456
3	0.2304
4	0.0768
5	0.0102



The probabilities are no longer symmetrical about the mean. $\mu = 2$ and $\sigma^2 = 1.2$. The distribution is **skewed**.

Binomial Histograms

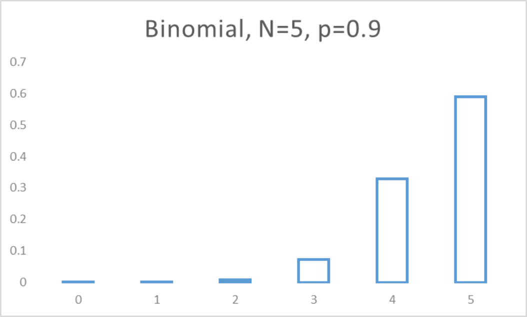
X	P(X)
0	0.0024
1	0.0283
2	0.132
3	0.309
4	0.36
5	0.168



$\mu = 3.5$ $\sigma^2 = 1.05$

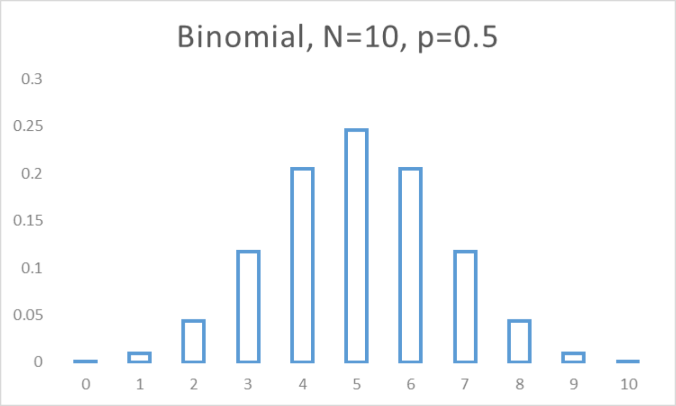
Binomial Histograms

X	P(X)
0	0.00001
1	0.00045
2	0.0081
3	0.0729
4	0.3281
5	0.5905



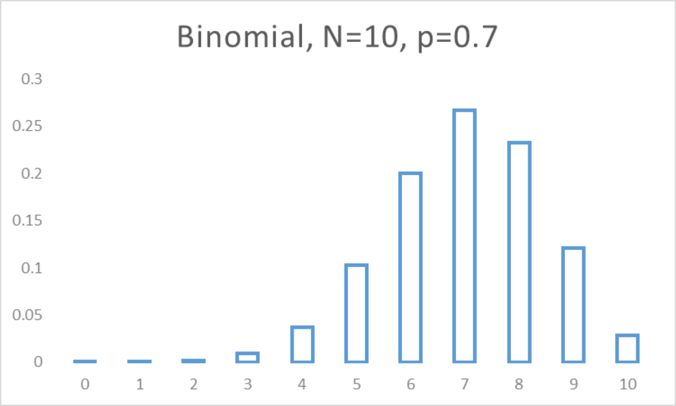
$\mu = 4.5$ $\sigma^2 = 0.45$

Binomial Histograms



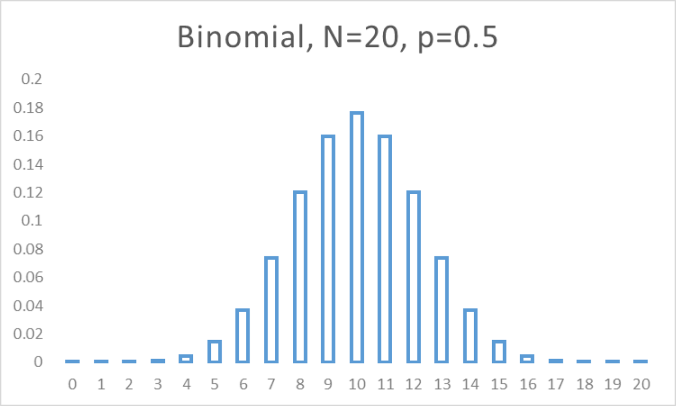
$$\mu = 5 \qquad \sigma^2 = 2.5$$

Binomial Histograms



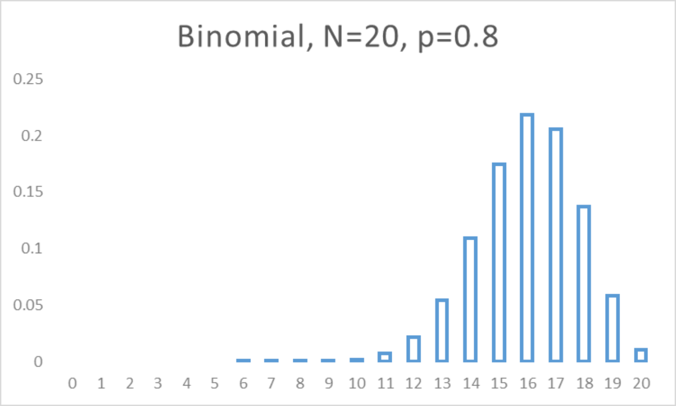
$$\mu = 7 \qquad \sigma^2 = 2.1$$

Binomial Histograms



$\mu = 10$ $\sigma^2 = 5$

Binomial Histograms



$$\mu = 16 \qquad\qquad \sigma^2 = 3.2$$