Name: Date:

Worksheet 9, Math 10560

1. (a) Give a definition of conditional convergence.

Solution: A series $\sum a_n$ is called *conditionally convergent* if the series is convergent, but not absolutely convergent (i.e. $\sum |a_n|$ does not converge).

(b) Which series below conditionally converges (justify your answer)?

Solution: Only (i) is conditionally convergent. See below for explanations.

i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

Solution: This is an alternating series with $b_n = \frac{1}{\sqrt{n}} > 0$. Applying the Alternating Series Test, we see that

$$b_{n+1} = \frac{1}{\sqrt{n+1}} \leq b_n = \frac{1}{\sqrt{n}} \text{ for all } n \geq 1$$

and

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Thus, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges. However, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-series test with $p = \frac{1}{2}$, and so the series does not converge absolutely. We conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ *conditionally converges.*

ii) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 7^n}{\sqrt{n}}$

Solution: Note that

$$\begin{aligned}
\lim_{n\to\infty} \frac{7^n}{\sqrt{n}} &= \lim_{n\to\infty} \frac{7^n \ln 7}{\frac{1}{2}n^{-\frac{1}{2}}} \\
&= \lim_{n\to\infty} 14 \cdot 7^n \sqrt{n} \\
&= \infty
\end{aligned}$$

Name: Date:

Thus, $\lim_{n\to\infty}(-1)^{n-1}\frac{7^n}{\sqrt{n}}$ does not exist. By the Divergence Test we see that this series diverges.

iii) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}$

Solution:
This series *diverges* because it is a geometric series with $r=-1$.

iv) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$

Solution:
The series $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges and so the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ is *absolutely convergent.*

2. (a) Say whether or not the following series are convergent and how you arrived at your conclusion including which test you used and how it applies.

i) $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{n^2+e^n}$

Solution: First we check the Divergence Test:

$$\lim_{n\to\infty}\frac{e^n}{n^2+e^n} = \lim_{n\to\infty}\frac{1}{\frac{n^2}{e^n}+1} = 1 \neq 0$$

Therefore, by the Divergence Test, $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{n^2+e^n}$ diverges.

ii) $\displaystyle\sum_{n=2}^{\infty}\frac{\cos(n^n)}{n^4+1}$

Solution: Note that $|\cos(n^n)|\leq 1$. Hence, we have the inequalities:

$$\left|\frac{\cos(n^n)}{n^4+1}\right| = \frac{|\cos(n^n)|}{|n^4+1|} \leq \frac{1}{n^4+1} \leq \frac{1}{n^4}$$

Name: Date:

Now, since $\sum_{n=2}^{\infty} \frac{1}{n^4}$ converges (because it is a p-series with $p = 4 > 1$), then by the Comparison Test, $\sum_{n=2}^{\infty} \left| \frac{\cos(n^n)}{n^4+1} \right|$ converges. Therefore, $\sum_{n=2}^{\infty} \frac{\cos(n^n)}{n^4+1}$ converges absolutely and so it converges.

iii) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(n-1)!}$

Solution: This is a typical problem for which the Ratio Test is a good one to try first:

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{2^{n+1}}{(n+1-1)!}}{\frac{2^n}{(n-1)!}} = \frac{2^{n+1}}{n!} \cdot \frac{(n-1)!}{2^n} = \frac{2}{n}.$$

Hence,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2}{n} = 0 < 1.$$

Therefore, by the Ratio Test, $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{(n-1)!}$ is *absolutely convergent.*

iv) $\sum_{n=1}^{\infty} \left(\frac{n^2+n}{2n^2+1} \right)^n$

Solution: This is a typical problem for which the Root Test is a good one to try first:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{n^2+n}{2n^2+1} \right)^n \right|} = \lim_{n\to\infty} \frac{n^2+n}{2n^2+1} = \frac{1}{2} < 1$$

Therefore, by the Root Test, $\sum_{n=1}^{\infty} \left(\frac{n^2+n}{2n^2+1} \right)^n$ converges absolutely and so it converges.

Name: Date:

3. Find the radius of convergence and interval of convergence of the following power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{\sqrt{n}}$.

Solution: We apply the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{|(-1)^n(x-2)^{n+1}/\sqrt{n+1}|}{|(-1)^{n-1}(x-2)^n/\sqrt{n}|} \\
&= \lim_{n\to\infty} |x-2| \cdot \sqrt{\frac{n}{n+1}} \\
&= |x-2|
\end{aligned}$$

The ratio test says that the power series will converge if $|x-2| < 1$. Thus the radius of convergence is given by $R = 1$, and the interval of convergence is $1 < x \leq 3$.

Some Extra Old Exam Questions

4. Find $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n-1}}$.
   (a) 20
   (b) $\frac{4}{5}$
   (c) $\frac{5}{4}$
   (d) 4
   (e) 5

Solution: The correct answer is **(a)**.

$$\frac{2^{2n}}{5^{n-1}} = \frac{4^n}{5^{n-1}} = 4 \cdot \frac{4^{n-1}}{5^{n-1}} = 4 \cdot \left(\frac{4}{5}\right)^{n-1}$$

Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n-1}} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{5}\right)^{n-1}$ is a geometric series with ratio $r = \frac{4}{5} < 1$. Hence, it converges and its sum is:

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{5^{n-1}} = \sum_{n=1}^{\infty} 4 \cdot \left(\frac{4}{5}\right)^{n-1} = \frac{4}{1-\frac{4}{5}} = \frac{4}{\frac{1}{5}} = 20.$$

Name: Date:

5. Test the following series for absolute convergence, conditional convergence or divergence:

$$(1)\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}; \qquad (2)\sum_{n=1}^{\infty}\frac{(-1)^n}{(1.2)^n}; \qquad (3)\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1.2}}.$$

(a) (1) converges conditionally, (2) and (3) converge absolutely

(b) (1) and (2) converge conditionally, (3) converges absolutely

(c) (1) and (2) converge absolutely, (3) converges conditionally

(d) (1) and (3) converge absolutely, (2) converges conditionally

(e) (1) converges absolutely, (2) and (3) converge conditionally.

Solution: The correct solution is **(a)**: (1) converges conditionally, (2) and (3) converge absolutely.

(1) The series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} = -\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$. By the Alternating Series Test (see 1b part i.), this series converges even though the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ does not. Thus (1) *converges conditionally*. This reduces our choices to (a) and (b).

(2) The series $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{(1.2)^n}\right| = \sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^n$ converges by the Geometric Series Test, so (2) is *absolutely convergent.*

(3) The series $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^{1.2}}\right| = \sum_{n=1}^{\infty}\frac{1}{n^{1.2}}$ converges by the *p*-series test and so (3) is *absolutely convergent.*

Name: Date:

6. Find the center a and the radius of convergence R for

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+2}} \left(\frac{x+1}{2}\right)^n .$$

(a) $a = -1,\ R = 2$

(b) $a = -\frac{1}{2},\ R = 2$

(c) $a = 1,\ R = 2$

(d) $a = -1,\ R = 1$

(e) $a = -1,\ R = \frac{1}{2}$

Solution: The correct answer is **(a)**.

We write the above power series in the form $\displaystyle\sum_{n=1}^{\infty} c_n(x-a)^n$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+2}} \left(\frac{x+1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+2}} \left(\frac{1}{2}(x+1)\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n\sqrt{n^2+2}}(x+1)^n$$

So, we get that $c_n = \dfrac{(-1)^n}{2^n\sqrt{n^2+2}}$ and $a = -1$. Then, to get the radius of convergence R we compute:

$$R = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n\to\infty} \frac{\frac{1}{2^n\sqrt{n^2+2}}}{\frac{1}{2^{n+1}\sqrt{(n+1)^2+2}}} = \lim_{n\to\infty} \frac{2^{n+1}\sqrt{(n+1)^2+2}}{2^n\sqrt{n^2+2}}$$

$$= \lim_{n\to\infty} 2\cdot\sqrt{\frac{n^2+2n+3}{n^2+2}} = 2\cdot\sqrt{\lim_{n\to\infty}\left(\frac{n^2+2n+3}{n^2+2}\right)} = 2\cdot\sqrt{1} = 2$$

Therefore, $a = -1, R = 2$.

Name: Date:

7. Consider the following series

$$(I) \quad \sum_{n=1}^{\infty} \frac{n \cdot 3^n}{(n+1)!} \qquad\qquad (II) \quad \sum_{n=1}^{\infty} \left(\frac{e^n}{2e^n+1}\right)^n.$$

Which of the following statements is true?

(a) They both converge.

(b) They both diverge.

(c) (I) converges and (II) diverges.

(d) (I) diverges and (II) converges.

(e) The root test is inconclusive when applied to (II).

Solution: The correct answer is **(a)**: they both converge.
To see if (I) converges, we apply the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)\cdot 3^{n+1}/(n+2)!}{n \cdot 3^n/(n+1)!} \\
&= \lim_{n\to\infty} 3 \cdot \frac{n+1}{n} \cdot \frac{(n+1)!}{(n+2)!} \\
&= \lim_{n\to\infty} 3 \cdot \frac{n+1}{n} \cdot \frac{\cancel{(n+1)!}}{(n+2)\cdot \cancel{(n+1)!}} \\
&= \lim_{n\to\infty} \frac{3n+3}{n^2+2n} \\
&= 0 < 1
\end{aligned}$$

Thus, by the Ratio Test the series $\sum_{n=1}^{\infty} \frac{n \cdot 3^n}{(n+1)!}$ converges absolutely, and hence is convergent.

Name: Date:

For (II), we apply the Root Test:

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{e^n}{2e^n+1}\right)^n\right|} &= \lim_{n\to\infty}\frac{e^n}{2e^n+1}\\ &= \lim_{n\to\infty}\frac{1}{2+e^{-n}}\\ &= \frac{1}{2+0}\\ &= \frac{1}{2} < 1\end{aligned}$$

Thus, by the Root Test, the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{e^n}{2e^n+1}\right)^n$ converges absolutely, and hence is convergent.