LAGRANCE MULTIPLIERS

ADRIAN PĂCURAR

LAST TIME

We saw that for a function f of n variables $x_1, x_2, \ldots, x_n$, we can minimize/maximize it along a region defined by the constraint $g = k$ (where k is a constant) by considering the gradient vectors of the two functions. In particular, we look at

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

which gives a system of equations $n+1$ equations and $n+1$ unknowns

$$\begin{cases} \frac{\partial f}{\partial x_1} = \lambda \frac{\partial g}{\partial x_1} \\ \frac{\partial f}{\partial x_2} = \lambda \frac{\partial g}{\partial x_2} \\ \qquad \vdots \\ \frac{\partial f}{\partial x_n} = \lambda \frac{\partial g}{\partial x_n} \\ g(x_1, \ldots, x_n) = k \end{cases}$$

EXAMPLES

Example 1. Min/max the function $f = x^2 + y^2 + z^2$ subject to $x + y + z = 12$.

Solution: Our constraint is $g(x, y, z) = x + y + z = 12$, and the gradient equation $\vec{\nabla} f = \lambda \vec{\nabla} g$ gives rise to the following system:

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = \lambda \\ x + y + z = 12 \end{cases}$$

Since $\lambda = 2x = 2y = 2z$, we immediately get $x = y = z$. By the last constraint, their sum must equal to 12, and so $x = y = z = 4$. Then the function is minimized at $(4, 4, 4)$ with value $3 \cdot 4^2 = 48$. Subject to the given constraint, it does NOT have a maximum. For instance, imagine plugging in $(0, t, 12 - t)$, which satisfies the constraint for any $t \in \mathbb{R}$, and let $t \to \infty$. Then $f \to \infty$.

Example 2. Find points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest and farthest to $(3, 1, -1)$.

Solution: The constraint is the surface on which our domain points are, so the constraint equation is $g = x^2 + y^2 + z^2 = 4$. Now, since we are asked to find points closest and farthest, we are talking about *distance*, so we are trying to maximize $\sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$, but since $\sqrt{}$ is a one-to-one function, it is easier to drop it and look at

$$f = (x-3)^2 + (y-1)^2 + (z+1)^2$$

From $\vec{\nabla} f = \lambda \vec{\nabla} g$, we get the system

$$\begin{cases} 2(x-3) = \lambda 2x \\ 2(y-1) = \lambda 2y \\ 2(z+1) = \lambda 2z \\ x^2 + y^2 + z^2 = 4 \end{cases}$$

and solving for λ in the first 3 equations gives

$$\lambda = \frac{x-3}{x} = \frac{y-1}{y} = \frac{z-1}{z}$$

From expression (1) and (2), and from (1) and (3), we cross multiply to get

$$y(x-3) = x(y-1) \quad \text{and} \quad z(x-3) = x(z+1)$$

so $x = 3y$, $x = -3z$, and thus $y = -z$. We now rewrite the constraint eq in terms of y, to get $(3y)^2 + y^2 + (-y)^2 = 4$, which gives $\boxed{y^2 = \frac{4}{11}, x^2 = \frac{36}{11}, z = -y}$.

We have several possibilities, but recall we are trying to min/max a distance to $(3, 1, -1)$, which has sign pattern $(+, +, -)$, so the closest point will be

$$\left(+\frac{6}{\sqrt{11}}, +\frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \text{(same sign)}$$

while the farthest point will be

$$\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, +\frac{2}{\sqrt{11}}\right) \text{(same sign).}$$

Example 3. Minimize/maximize $f = 2x + 2y + z$ subject to $x^2 + y^2 + z^2 = 9$.

Solution: The system we end up with is

$$\begin{cases} 2 = \lambda 2x \\ 2 = \lambda 2y \\ 1 = \lambda 2z \\ x^2 + y^2 + z^2 = 9 \end{cases}$$

and solving for λ gives

$$\lambda = \frac{1}{x} = \frac{1}{y} = \frac{1}{2z}$$

From the first two, we see that $y = x$, and by mixing any of the first two with the last one gives $x = y = 2z$. To avoid fractions, rewrite the constraint equation in terms of z, to get

$$(2z)^2 + (2y)^2 + z^2 = 9$$

which yields $z = \pm 1$, and so $x = y = \pm 2$. Now, f is a linear combination of x, y, z with positive coefficients, so it will be maximized when we choose the variables to be all positive, and minimized when the variables are chosen to be all negative:

$$f(2,2,1) = 9 \text{ (max)}, \qquad f(-2,-2,-1) = -9 \text{ (min)}.$$

Example 4. Min/max $f = xyz$ on $x^2 + 2y^2 + 3z^2 = 6$.

Solution: The system for this problem is

$$\begin{cases} yz = \lambda 2x \\ xz = \lambda 4y \\ xy = \lambda 6z \\ x^2 + 2y^2 + 3z^2 = 6 \end{cases}$$

and solving for λ gives

$$\lambda = \frac{yz}{2x} = \frac{xz}{4y} = \frac{xy}{6z}$$

Now we cross multiply in groups of two:

From the first two expressions, we get $4y^2z = 2x^2z$, so $2y^2 = x^2$ and $x = \pm\sqrt{2}y$.

From the second and the third, we get $6xz^2 = 4xy^2$, so $3z^2 = 2y^2$.

From the first and the third, we get $6yz^2 = 2yx^2$, so $3z^2 = x^2$. Writing the constraint in terms of x (easiest choice) yields

$$x^2 + (x^2) + (x^2) = 6 \iff x = \pm\sqrt{2}$$

Then solving for the other variables, $y = \pm 1$ and $z = \pm\sqrt{2/3}$. This means we have 8 possible points to test, depending on the sign choice:

Point	Value
$(\sqrt{2}, 1, \sqrt{\frac{2}{3}})$	$\frac{2}{\sqrt{3}}$(max)
$(-\sqrt{2}, -1, \sqrt{\frac{2}{3}})$	$\frac{2}{\sqrt{3}}$(max)
$(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}})$	$\frac{2}{\sqrt{3}}$(max)
$(\sqrt{2}, -1, -\sqrt{\frac{2}{3}})$	$\frac{2}{\sqrt{3}}$(max)
$(-\sqrt{2}, 1, \sqrt{\frac{2}{3}})$	-$\frac{2}{\sqrt{3}}$(min)
$(\sqrt{2}, -1, \sqrt{\frac{2}{3}})$	-$\frac{2}{\sqrt{3}}$(min)
$(\sqrt{2}, 1, -\sqrt{\frac{2}{3}})$	-$\frac{2}{\sqrt{3}}$(min)
$(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}})$	-$\frac{2}{\sqrt{3}}$(min)

LAGRANCE METHOD FOR TWO CONSTRAINTS

Sometimes we may want to minimize/maximize a function $f(x, y, z)$ which subject to more than one constraint. In the case when we have two constraints, say $g(x, y, z) = k$ and $h(x, y, z) = l$ (where k and l are both constants), we need to introduce *two free variables*, say λ and μ, and solve the system induced by

$$\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h$$

which would look like

$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} + \mu \frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} + \mu \frac{\partial h}{\partial y} \\ \frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} + \mu \frac{\partial h}{\partial z} \\ g(x, y, z) = k \\ l(x, y, z) = l \end{cases}$$

The geometric interpretation of this would be to try to min/max f restricted to not just a single surface (one constraint), but to the *intersection of surfaces* determined by $g = k$ and $h = l$. This method generalizes to any number of constraints in theory, but then the system gets more difficult (or impossible) to solve analytically.

Example 5. Maximize $f = x + 2y + 3z$ on the intersection of the plane $x - y + z = 1$ with the cylinder $x^2 + y^2 = 1$ (the intersection will be an ellipse).

Solution: Notice that the constraints we are working with are $g = x - y + z = 1$ and $h = x^2 + y^2 = 1$. Then $\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h$ translates to the system

$$\begin{cases} 1 = \lambda + 2x\mu \\ 2 = -\lambda + 2y\mu \\ \boxed{3 = \lambda} + 0z\mu \\ x - y + z = 1 \\ x^2 + y^2 = 1 \end{cases}$$

Luckily we are able to solve for $\lambda = 3$ right away, and we plug it into the first two equations

$$-2 = 2x\mu \qquad \text{and} \qquad 5 = 2y\mu$$

which allows us to write μ in two different ways

$$\mu = \frac{-1}{x} = \frac{5}{2y}$$

so $-2y = 5x$. Since we have a relation between x and y, and the last constraint only depends on these variables, we solve for y and square it to get $y^2 = \frac{25}{4}x^2$, and plugging in gives

$$x^2 + \frac{25}{4}x^2 = 1$$

or $x^2 = \frac{4}{29}$, which allows us to solve for x and y:

$$x = \pm\frac{2}{\sqrt{29}}, \qquad y = \pm\frac{5}{\sqrt{29}}$$

This leaves us with a lot of possibilities to test, but if we turn out attention to the first constraint, we find $z = 1 - x + y$. Recall that we want to maximize $x + 2y + 3z$, which is the same as

$$-2x + 4y + 3$$

once we substitute in for z. To make this quantity as large as possible, we must choose the negative solution for x (since x has a negative coefficient, making the product positive) and the positive solution for y. Then the max is

$$\max(f) = (-2x + 4y + 3)\big|_{(-2/\sqrt{29},+5/\sqrt{29})} = \frac{4}{\sqrt{29}} + \frac{25}{\sqrt{29}} + 3 = \boxed{3 + \sqrt{29}}.$$

Example 6. Min/max $f = xy + yz$ subject to $xy = 1$ and $y^2 + z^2 = 1$.

Solution: The system we obtain is

$$\begin{cases} y = \lambda y \\ x + z = \lambda x + 2\mu y \\ y = 2\mu z \\ xy = 1 \\ y^2 + z^2 = 1 \end{cases}$$

From the first equation, either $y = 0$ or $\lambda = 1$, but $y \neq 0$ since $xy = 1$ is nonzero, so $\boxed{\lambda = 1}$. Substitutiing into the second equation, and canceling the x gives

$$z = 2\mu y$$

which we combine with the third to solve for μ in two different ways:

$$\mu = \frac{z}{2y} = \frac{y}{2z}$$

If we cross multiply, we see that $y^2 = z^2$, and using the constraints we get

$$\boxed{y = \pm\frac{1}{\sqrt{2}}}, \qquad \boxed{z = \pm\frac{1}{\sqrt{2}}}, \qquad \text{and} \qquad \boxed{x = \frac{1}{y} = \pm\sqrt{2}}$$

Checking the possible points, we find that the min/max of f will be

$$f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{2} \qquad \text{and} \qquad f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

Note that we could have alternately solved for μ using $z = 2\mu y = 2\mu(2\mu z)$ (since we found $y = 2\mu z$), and this would have been $4\mu^2 = 1$, or $2\mu = \pm 1$, giving us the relation $z = \pm y$, and then we could have continued the rest of the way using the constrants as we did above.

Exercise: Show that a closed rectangular box with fixed surface area A and maximal volume must be a cube.