ELEMENTS OF CALCULUS: FINAL REVIEW

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1. Precalculus

Problem 1. Find all real solutions to the equations:

a) $x^2 + 3x - 10 = 0$

b) $e^{2x} + 3e^x - 10 = 0$

c) $\ln(t^2 - 3) = 0$

d) $4^{x-2} = 8$.

Problem 2. a) Graph the function $f(x) = -(x-5)^2 + 2$

b) Starting with the graph of $f(x) = \sqrt{x}$, what is the formula of th efunction obtained by a reflection about the y-axis, followed by a vertical stretch by a factor of 2?

c) What is the formula for the function whose graph can be obtained from $y = x^3$ by shifting to the right 1 unit, then reflecting about the x-axis?

Problem 3. Let $f(x) = x^2$. At what point does the line passing through $(2, 2)$ and $(3, 3)$ intersect the graph of $f(x)$?

Problem 4. Consider the function $f(x) = x^2 + 6x + 11$. Complete the square and write it in the form $f(x) = (x + a)^2 + b$. What transformations would one perform on the graph of the basic parabola x^2 to obtain the graph of f?

2. Limits and Continuity

Problem 5. Compute the following limits:

a) $\lim_{x\to 4} \dfrac{x-4}{x^2 - 8x + 3}$

b) $\lim_{x\to 3^-} \dfrac{x-4}{x^2 - 9}$

c) $\lim_{x\to 3^-} \dfrac{\sqrt{5x}(x-3)}{|x-3|}$

d) Suppose that $\lim_{x\to 1} f(x) = 7$, $\lim_{x\to 1} g(x) = 4$, and $\lim_{x\to 1} h(x) = -\infty$. Compute the limit

$$\lim_{x\to 1} \left(f(x) + \frac{1}{g(x) - h(x)} \right)$$

e) $\lim_{x\to 2} \dfrac{x-2}{\sqrt{x^2-4}}$ (Hint: rationalize the denominator)

f) $\lim_{x\to+\infty} \dfrac{7x^9-4x^5+2x-13}{-3x^9+x^8-5x^2+2x}$

g) $\lim_{x\to 4} \left(\dfrac{1}{x-4} - \dfrac{8}{x^2-16}\right)$

h) $\lim_{x\to 0^-} x^{1/4}$

i) $\lim_{x\to\infty} \dfrac{x^4-3x^3+5x+1}{x^5+12x+8}$

j) $\lim_{x\to\infty} \dfrac{x^4-6x+8}{5x^3+8x^4}$

k) $\lim_{x\to\infty} \dfrac{x^8+e^x+1}{5x^8+3e^x+12x^2+5}$

l) $\lim_{x\to\infty} \dfrac{5\ln x+12}{7\ln x+\cos x+6}$

m) $\lim_{x\to\infty} \dfrac{5+e^{-x}+2e^{-2x}}{7+2e^{-x}+3e^{-2x}}$

n) $\lim_{x\to-\infty} \dfrac{1+2e^x+3e^{2x}}{4+5e^x+6e^{2x}+e^{-x}}$

o) $\lim_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x^4+1}$

p) $\lim_{x\to\infty} \dfrac{\sqrt{2x^2+8}}{x+5}$

q) $\lim_{x\to-\infty} \dfrac{\sqrt{\pi x^6+23x+8}}{2x^3+x^2+1}$

Problem 6. You are told that a parabola which "opens up" has roots $x=a$ and $x=b$ (where $a<b$). At what x value is the minimum of the parabola attained?

3. Basic Derivatives

Problem 7.

a) Suppose $f(x)=x^2+1$ and $g(x)=-x^3$. Compute the derivative of $h(x)=\dfrac{f(x)}{g(x)}$.

b) Let $m(x)=x^2$. What is the tangent line to the graph of $f(x)=\dfrac{m(x)+x}{mx+1}$ when $x=0$?

c) Let $f(x)=x-1$. For what x values, if any, does the function $g(x)=\dfrac{f(x)+1}{f(x)}$ have a horizontal tangent? What about a vertical tangent?

d) Let $f(x) = x - 1$. For what x values, if any, does the function $g(x) = \dfrac{f(x) + x^2 + 1}{f(x) + 1}$ have a horizontal tangent? What about a vertical tangent?

e) Find the equation to the tangent AND normal lines to the graph of $f(x) = x^3$ at the point $(1, 1)$.

f) Find the instantaneous rate of change of $y = (x^2 + 3)(2x^3 - 5)$ at $x = 0$.

Problem 8. Compute the derivatives for the following:

(a) $(x^2 + x)(x + 1)$

(b) $\dfrac{x^2 - 3x - 10}{x + 2}$

(c) $f(x) = (3 + 4x - x^2)^{1/2}$

(d) $f(x) = \left(\dfrac{x}{1 + x}\right)^5$

(e) $f(x) = 2x^2\sqrt{2 - x}$

(f) $f(x) = (x^2 + 3)^4(2x^3 - 5)^3$

Problem 9. Compute the derivative of the function

$$r(x) = \left[\left(\frac{x + 5}{x^2 - 1}\right)^3 + \left((x + 3)^2 + 9\right)^4\right]^{1/2}$$

4. Implicit Differentiation

Problem 10. Use implicit differentiation to find the following:

(a) dy/dx for $2y^2 = y + x$

(b) dx/dy for $2y^2 = y + x$

(c) $y'(x)$ for $x^2 - xy + y^2 = 3$

(d) $x'(y)$ for $x^2 - xy + y^2 = 3$

Problem 11. Find the tangent line to the graph of $x^2(x^2+y^2) = y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$.

Problem 12. Given $x^2 - y^2 = 36$, find $\dfrac{d^2y}{dx^2}$.

Problem 13. Given that $y(t) = t^3$ and $x(t) = 1-t^2$, differentiate the equation $x^3y+xy^3 = t^2$ with respect to t. Don't simplify.

5. Related Rates

Problem 14. The radius of a circle is changing at the rate of $dr/dt = +2$ cm/s. How fast is the area changing when $r = 2$ cm? Is this increasing or decreasing? What are the units?

Problem 15. A spherical balloon is filling up with air at the rate of 5 cm^3/s. How fast is the radius of the balloon changing when $r = 2$? Is this increasing or decreasing? Use the fact that the volume of a sphere in terms of its radius r is given by $V = \frac{4}{3}\pi r^3$.

Problem 16. The variables x and y are both functions of t and are related by the equation $y = x^2 + 3$. Find dy/dt when $x = 1$, given that $dx/dt = 2$ when $x = 1$.

Problem 17. A pebble is dropped into a calm pond, causing a ripple in the form of a circle. The radius r of the ripple is increasing at a constant rate of 1 ft/sec. When the radius is 4 feet, at what rate is the total area enclosed by the ripple changing?

Problem 18. All edges of a cube are expanding at the same rate, causing the volume of the cube to increase at a rate of 5 cm^3/sec. At what rate are the sides of the cube increasing when the side length is 10 cm? What is the rate of the change of the cube's surface area at that time?

6. Motion Problems

Problem 19. The position of a particle is given by $s(t) = \frac{1}{5}t^5 - 2t^2 + t$, where t is measured in seconds. Is there ever a time during the first second where the particle is at rest?

Problem 20. The position of a particle is given by $s(t) = \pi t^2 - t + 18$. For what (non-negative) value(s) of t is the particle at rest?

Problem 21. The position of a particle is given by $s(t) = e^t - 2t + 18$. For what (non-negative) value(s) of t is the particle at rest?

7. Optimization

Problem 22. Consider the function $f(x) = (x-1)^{1/3} - \dfrac{x}{12}$.

(a) Find all critical points of $f(x)$.

(b) Find the absolute maximum and absolute minimum of $f(x)$ for $0 \leq x \leq 28$.

(c) Find intervals where f is increasing/decreasing.

(d) Find intervals where f is concave up/down.

Problem 23. Find the absolute extrema of the function on the closed interval:

a) $f(x) = x^3 - \dfrac{3}{2}x^2$ on $[-1, 2]$.

b) $g(t) = \dfrac{t^2}{t^2+3}$ on $[-1, 1]$.

c) $h(t) = \dfrac{t}{t+3}$ on $[-1, 6]$.

Problem 24. Find the min/max of the following functions

a) $g(x) = -x^2 + 4x + 3$

b) $f(x) = \dfrac{x}{1+x^2}$

c) $f(x) = 9x - \dfrac{1}{x}$ on $[1, 3]$

d) $f(x) = \dfrac{1}{2}x^2 - 2\sqrt{x}$ on $[0, 3]$

e) $g(x) = x\sqrt{4-x^2}$ on its domain

f) $p(x) = xe^{-x^4}$ on its domain

g) $r(x) = e^{-x} \cdot \ln x$ on $[1, \infty)$.

Problem 25. A farmer has 3000 yards of fencing with which to enclose a rectangular piece of land for his chickens. What dimensions should the rectangular enclosure have if he wants to enclose the largest possible area?

Problem 26. A farmer has 3000 yards of fencing with which to enclose a rectangular piece of land for his chickens. One of the sides of the land will be along a river, so no fencing is required for that side. What dimensions should the rectangular enclosure have if he wants to enclose the largest possible area?

Problem 27. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 245,000 square meters in order to provide enough grass for the herd. No fencing is needed along the river. What dimensions will require the least amount of fencing?

Problem 28. A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

Problem 29. Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

Problem 30. A rectangular poster is to contain 24 square inches of print. The margins at the top and bottom of the poster are to be 1.5 inches each, while the left/right margins are to be 1 inch each. What should be the dimensions of the page so that the least amount of paper is used?

Problem 31. Four feet of wire is to be cut in two pieces, and used to make a square and a circle. How much of the wire should be used for the square if we want the maximum total area enclosed by square and circle to be maximized?

8. Exponential and Logarithmic Functions

Problem 32. Consider the function $f(x) = xe^{-x^4}$.

(a) Find all critical points of $f(x)$.

(b) Find the absolute maximum and absolute minimum of $f(x)$ on its entire domain.

(c) Find the inflection points of $f(x)$.

(d) Give a rough sketch of the function.

Problem 33. Where do the curves $y = e^{3x}$ and $y = 3e^x$ intersect?

Problem 34. Solve the equations:

a) $4^x = \left(\frac{1}{16}\right)^{x-7}$

b) $3^{x-x^2} = \frac{1}{9^x}$

c) $\log_4(2x+2) = \frac{1}{2}$

d) $4e^{t-1} = 4$

Problem 35. Sketch the graph of the following functions:

a) $f(x) = 3\ln x$

b) $\ln(x-3)$

c) $\ln|x|$

d) $e^x + 1$

e) $3e^x + 2$

f) $(0.5)^x$ and 2^x on the same axis

g) e^{-x} and e^x on the same axis

h) 2^x and 10^x on the same axis

i) $\log_2(x)$ and $\log_{10}(x)$ on the same axis

j) 2^{-x} and 10^{-x} on the same axis

9. Economics

Problem 36. Suppose the cost for producing a certain item is $C(x) = 400 + 20x$.

a) For the marginal cost function $C'(x)$.

b) Approximate the cost of producing the 11th item.

Problem 37. Suppose we have a demand equation $d(x) = -x + 400$ giving price in terms of quantity demanded.

a) Find the revenue function $R(x)$.
b) Find the marginal revenue function.
c) Compute $R'(50)$. What does this mean?

10. Compound Interest

Problem 38. How long will it take an investment of \$2,000 to double if the investment earns interest at the rate of 6%/year. What if the interest was compounded monthly?

Problem 39. How long will it take an investment of \$5,000 to triple at an interest rate of 4%/year compounded weekly. Assume 52 weeks in a year.

Problem 40. What is the interest rate needed for an investment of \$5,000 to grow to \$6,000 in 3 years if interest is compounded continuously.

Problem 41. Find the interest rate needed for an investment of \$2,000 to double in 5 years if interest is compounded annually.

Problem 42. Find the present value of \$20,000 due in 3 years at an interest rate of 12%/year compounded monthly.

Problem 43. Glen invests \$100,000 in an account yielding 6.6% interest compounded monthly. Being unhappy with the return on his investment, he wishes to reinvest the final amount at the end of the first year into a new account where interest is compounded quarterly. What interest rate should he look for if he wishes to obtain \$130,130 at the end of the third year (i.e. after keeping the money for 2 more years in the second account).

Problem 44. The same amount of money is invested in two different accounts. Account A pays simple interest at a rate of 5% per year, while account B pays compound interest at the same rate (compounded annually).

a) Write a formula for the amount of money $A(t)$ in the first account after t years. Do the same for the amount of money $B(t)$ in the second account.
b) How long does it take for the amount in the first account to double in size?
c) Which is bigger: $A(t)$ or $B(t)$? In other words, if the same amount of time passes, which type of interest offers a bigger return?
d) Suppose we have three accounts with each offering the same rate, but one offers simple interest, the second offers compound interest, and the third offers continuously compounded interest. Given the same starting principal amount, which account grows the slowest? Which account grows the fastest?

11. Derivatives of Log and Exponential

Problem 45. Compute dy/dx of the following:

a) $y = \ln\left(\dfrac{x}{x^2+1}\right)$

b) $y = \ln(x\sqrt{x^2-1})$

c) $y = \ln\sqrt{\dfrac{x+1}{x-1}}$ (without logarithmic differentiation)

d) $y = \sqrt{\dfrac{x+1}{x-1}}$ (with logarithmic differentiation)

e) $4xy + \ln(x^2y) = 7$

f) $y = e^{-8x}$

g) $y = x^3e^{x^2}$

h) $y = \ln(1+e^{2x})$

i) $y = \dfrac{e^{2x}}{e^{2x}+1}$

j) $xe^y + ye^x = 1$

k) $y = x^{e^x}$

l) $y = x^{\sqrt{x}}$

m) $y = x^{x^2}$

12. Integration

Problem 46. Compute the following indefinite integrals:

a) $\displaystyle\int \sqrt{2}dx$

b) $\displaystyle\int \frac{1}{x^4}dx$

c) $\displaystyle\int 2x^5dx$

d) $\displaystyle\int 3r^{-2/3}dr$

e) $\displaystyle\int (x^2+x+x^{-3})dx$

f) $\displaystyle\int (1+t+e^t)dt$

g) $\displaystyle\int \frac{x^4-1}{x^2}dx$

h) $\displaystyle\int 5e^{5y}dy$

i) $\int e^{2x-1}dx$

j) $\int x^2 e^{2x^3+4}dx$

k) $\int e^x(e^x+5)^2dx$

l) $\int \frac{5-e^x}{e^{2x}}dx$

m) $\int \frac{2x+1}{x^2+x}dx$

n) $\int \frac{2e^{2x}}{1+e^{2x}}dx$

o) $\int xe^{ax^2}dx$ (where a is a positive constant)

p) $\int \frac{-2x}{(x^2+1)^2}dx$

Problem 47. Find the solution (general or particular) to the differential equations:

a) $f'(x) = 6,\ f(0) = 8$

b) $\frac{dy}{ds} = 10s - 12s^3,\ y(3) = 2$

c) $y'' = 2,\ y'(0) = 5,\ y(2) = 10$

d) $f''(x) = x^2,\ f'(0) = 8,\ f(0) = 4$

e) $f''(x) = x^{-3/2},\ f'(4) = 2,\ f(0) = 0$

f) $\frac{dy}{dx} = 4x + \frac{4x}{\sqrt{16-x^2}}$

g) $\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}}$

h) $\frac{dy}{dx} = \frac{x+1}{(x^2+2x-3)^2}$

i) $\frac{dy}{dx} = \frac{x-1}{\sqrt{x^2-8x+1}}$

Problem 48. Use substitution to compute the following indefinite integrals:

a) $\int x(x^2+1)^2dx$

b) $\int \sqrt{2x-1}dx$

c) $\int x\sqrt{3x-1}dx$

d) $\int 3(3x-1)^4dx$

e) $\int (2x+1)(x^2+x)dx$

f) $\int 3x^2\sqrt{x^3-2}dx$

g) $\int \frac{-4x}{(1-2x)^2}dx$

h) $\int \frac{3x^2}{x^3+1}dx$

i) $\int 2(2x^3+x)e^{x^4+x^2+1}dx$

j) $\int x(x^2+1)^2dx$, compare to $\int (x^2+1)^2dx$

k) $\int \frac{x}{\sqrt{2x-1}}dx$

l) $\int \frac{x^3}{1+x^4}dx$

m) $\int \frac{6x^2}{4x^3-9}dx$

n) $\int \frac{1}{\sqrt{2x}}dx$

o) $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$

p) $\int \frac{1+e^x}{x+e^x}dx$

q) $\int \frac{x}{\sqrt[3]{5x^2}}dx$

13. Definite Integrals, Areas, Motion

Problem 49. Use 4 rectangles to approximate the area of the region below $f(x) = -x^2+5$ and the x-axis, between $x = 0$ and $x = 2$. Do this in three different ways: using right endpoints, left endpoints, and midpoints.

Problem 50. Suppose that for a certain continuous function f,

$$\int_1^3 f(x)dx = 5 \quad \int_3^5 f(x)dx = -2 \quad \int_5^6 f(x)dx = 1.5 \quad \int_6^7 f(x)dx = -1 \quad \int_7^{10} f(x)dx = 3$$

Compute $\int_1^6 f(x)dx$, $\int_3^7 f(x)dx$, and $\int_6^{10} f(x)dx$.

Problem 51. Evaluate the following integrals

a) $\displaystyle\int_1^e \frac{1}{x}dx$

b) $\displaystyle\int_1^4 3\sqrt{x}dx$

c) $\displaystyle\int_{-1}^2 4xdx$

d) $\displaystyle\int_1^3 x^3dx$

e) $\displaystyle\int_1^2 \left(\sqrt{x} + \frac{2}{\sqrt{x}}\right) dx$

f) $\displaystyle\int_1^2 \left(1 + \frac{1}{u} - \frac{1}{u^2}\right) du$

g) $\displaystyle\int_0^4 6x\sqrt{25 - x^2}dx$

h) $\displaystyle\int_0^4 \sqrt{16 - x^2}dx$

i) $\displaystyle\int_0^5 \frac{x}{\sqrt{x^2 + 1}}dx$

j) $\displaystyle\int_0^1 x^2(1 - x)^{1/2}dx$

k) $\displaystyle\int_1^9 \frac{1}{\sqrt{x}(1 + \sqrt{x})^2}dx$

l) $\displaystyle\int_{-1000\pi}^{1000\pi} xe^{-x^{16}}dx$

Problem 52. Without computing the integral, show that

$$\int_0^1 x^2(1 - x)^5dx = \int_0^1 x^5(1 - x)^2dx$$

Problem 53. For a and b positive integers, show that

$$\int_0^1 x^a(1 - x)^bdx = \int_0^1 x^a(1 - x)^bdx$$

Problem 54. Compute the average value of the following:

a) $\dfrac{1}{x}$ on $[1, e]$.

b) e^{-x} on $[0, 4]$.

c) xe^{x^2} on $[0, 2]$.

d) $\dfrac{\ln x}{x}$ on $[1, 2]$.

e) x^2 on $[0, 1]$ and on $[0, 2]$.

Problem 55. Find the following areas:

a) between $y = -(x-2)^2 - 2$ and the x-axis from $x = -1, 3$.

b) between $y = x$ and $y = x^2$ on $[1, 3]$.

c) between $y = x$ and $y = x^9$ from $x = -1, 1$.

d) enclosed by $y = x^2 + 1$ and $y = -x^4 + 3$.

e) enclosed by $y = x^5 - x$ and the x-axis.

f) enclosed by $y = x^2$ and $y = \sqrt{x}$.

g) enclosed by $y = 3x^3 - x^2 - 10x$ and $y = -x^2 + 2x$.

Problem 56. An empty chemical storage tank is filled with salt water at a rate of $r(t) = 180 - 3t$ liters per second, for t between 0 and 60 seconds. How much salt water is flowing in the first 20 seconds? What is the capacity of the tank?

Problem 57. The velocity of a particle is given by $v(t) = t^3 - 10t^2 + 29t - 20$ ft/sec. What is the displacement from $t = 1$ to $t = 5$? What is the total distance traveled from $t = 1$ to $t = 5$?