10350 Tutorial Week 10 - Set 01 Name ______________________

Section ______________

1a. Find all critical points of $f(x) = (x-1)^{1/3} - \dfrac{x}{12}$.

First we take the derivative and set it equal to zero

$$f'(x) = \frac{1}{3}(x-1)^{-2/3} - \frac{1}{12} = 0$$

$$(x-1)^{-2/3} = \frac{1}{4}$$

$$(x-1)^{2/3} = 4$$

$$(x-1)^2 = 4^3 \ (= 64)$$

$$x - 1 = \pm 8$$

So two of the critical points are $x = -7, 9$. However other candindates for critical points include those x values where the derivative f' does not exist. This occurs at $x = 1$ as we get a zero in the denominator (the exponent of $x - 1$ is negative, so it is in the denominator). Notice $f(1) = -\frac{1}{12}$ so this point is in the domain, so we include it.

Finally, our critical points are $x = -7, 1, 9$.

1b. Find the absolute maximum and absolute minimum of $f(x) = (x-1)^{1/3} - \dfrac{x}{12}$ for $0 \leq x \leq 28$.

We evaluate the function at the critical points which are in the given interval $[0, 28]$, as well as at the end points of the interval. We get:

$$f(0) = -1$$

$$f(1) = -1/12$$

$$f(9) = 5/4$$

$$f(28) = 2/3$$

Notice: we omitted $x = -7$ as it does not belong to the given interval. So the absolute min is -1 (at $x = 0$) and the absolute max is $5/4$ (at $x = 9$).

10350 Tutorial Week 10 - Set 02 Name ______________________

Section ______________

2a. Find all critical points of $g(x) = xe^{-2x^2}$.

We take the derivative and set it equal to zero. By product rule we have

$$f'(x) = e^{-2x^2} + xe^{-2x^2}(-4x) = 0$$

which is the same as

$$e^{-2x^2} = 4x^2 e^{-2x^2}$$

and after canceling the exponential this becomes

$$4x^2 = 1$$

so the critical points are $x = \pm\frac{1}{2}$. There are no x values for which the derivative is undefined, so these are the only two critical points.

2b. Find the absolute maximum and absolute minimum of $g(x) = xe^{-2x^2}$ for $0 \leq x \leq 1$.

Notice $x = -1/2$ is outside the given interval $[0, 1]$, so we are only interested in the following:

$$f(0) = 0$$

$$f(1/2) = \frac{1}{2\sqrt{e}}$$

$$f(1) = \frac{1}{e^2}$$

It is clear the minimum is $f(0) = 0$, but what about the max? We could use a calculator, but if we don't have one handy, use the fact that $e \approx 2.71$, so $2 < e$. Multiply both sides by $\sqrt{e}$, and you get $2\sqrt{e} < e\sqrt{e}$. Using the fact that $\sqrt{e} < e$ (true for any number greater than 1), we get

$$2\sqrt{e} < e\sqrt{e} < e \cdot e$$

In other words,

$$2\sqrt{e} < e^2$$

so then the reciprocals satisfy

$$\frac{1}{2\sqrt{e}} > \frac{1}{e^2}$$

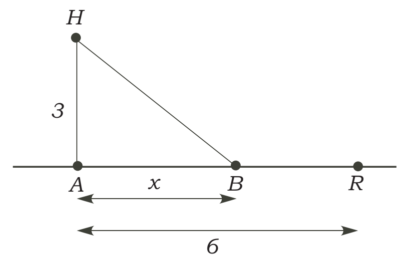
and our max has to be $f(1/2) = \frac{1}{2\sqrt{e}}$.

10350 Tutorial Week 10 - Set 03 Name ____________________

Section ____________

3. A house H is located in the woods, 3 miles from the nearest point, A, on a straight road. A restaurant, R, is located 6 miles down the road from A. Jack can ride his bike 10 miles per hour in the woods and 20 miles per hour along the road. He decides to ride the bike through the woods to some intermediate point B, x miles from A, and then ride along the road to R. Since he is starving, he wants to minimize his time T.

3a. Find time T in terms of x.



First recall that $v = \frac{d}{t}$ (velocity is distance over time). Since we are interested in the time, we must take $t = \frac{d}{v}$, i.e divide the distance by the velocity.

The total time is the time it takes to travel HB plus the time it takes to travel BR. For the time over HB we use the Pythagorean theorem to get the distance $\sqrt{x^2 + 3^2}$, and since the velocity is 10, the time

$$T_{HB}(x) = \frac{\sqrt{x^2+9}}{10} = \frac{1}{10}(x^2+9)^{1/2}$$

For the time over BR, the distance is $6 - x$, and since the velocity is 20, we have

$$T_{BR}(x) = \frac{6-x}{20} = \frac{1}{10}(3 - x^2)$$

Then the total time $T(x)$ is the sum of the two, which after we factor out 1/10 from both is

$$T(x) = \frac{1}{10}\left[(x^2+9)^{1/2} + 3 - \frac{x}{2}\right]$$

Note: you don't need to necessarily factor 1/10 out as I did. I am doing this to make taking the derivative later on easier.

3b. What are the possible values of x on which you should minimize $T(x)$? Is this a closed and bounded interval?

$T(x)$ makes sense for $0 \le x \le 6$, which is the closed bounded interval $[0, 6]$.

3c. Find x that minimizes the time T.

We take the derivative and set it equal to zero:

$$T'(x) = \frac{1}{10}\left[\frac{2x}{2\sqrt{x^2+9}} - \frac{1}{2}\right] = 0$$

$$\frac{2x}{2\sqrt{x^2+9}} = \frac{1}{2}$$

$$\sqrt{x^2+9} = 2x$$

$$x^2 + 9 = 4x^2$$

$$3x^2 = 9$$

$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

but since we are only interested in $x \in [0, 6]$, the critical point in question is $x = +\sqrt{3}$.

To figure out if this is really the min, we can plug in 1 (which is to the left of $\sqrt{3}$) and 2 (which is to the right) into the derivative, and using a calculator note that

$$T'(1) < 0$$

while

$$T'(2) > 0$$

so the function $T(x)$ is decreasing before $\sqrt{3}$ and increasing after, making $x = \sqrt{3}$ indeed a minimum.

10350 Tutorial Week 10 - Set 04 Name ______________________

Section ______________

4. Consider the function $f(x) = x\ln(x^2)$.

4a. Find the derivative of $f(x)$.

Using the product rule we have

$$f'(x) = \ln(x^2) + x \cdot \frac{1}{x^2} \cdot 2x = 2\ln x + 2$$

(equivalently $\ln(x^2) + 2$)

4b. Using Q4(a), verify that the function $f(x) = x\ln(x^2)$ satisfies the hypotheses of the Mean Value Theorem on $[1, e]$. Explain clearly in words.

Notice f is defined on $[1, e]$, and continuous on that interval. Furthermore the derivative f' is defined on $(1, e)$, so f is differentiable on $(1, e)$. Thus the hypotheses for MVT are satisfied.

4c. Find all numbers c that satisfies the conclusion of the Mean Value Theorem for $f(x) = x\ln(x^2)$ on $[1, e]$.

By the MVT, we need all $x \in (1, e)$ such that

$$f'(x) = \frac{f(e) - f(1)}{e - 1} = \frac{2e - 0}{e - 1} = \frac{2e}{e - 1}$$

In other words we need to solve

$$\begin{aligned} 2\ln x + 2 &= \frac{2e}{e-1} \\ 2\ln x &= \frac{2e}{e-1} - 2 \\ 2\ln x &= \frac{2e - 2e + 2}{e - 1} \\ \ln x &= \frac{1}{e-1} \end{aligned}$$

This gives us $c = e^{1/(e-1)}$. We still need to check this is in the interval $(1, e)$ which can be easily done by calculator.