PRACTICE QUIZ 14 SOLUTIONS

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Time: 15 min
Time to beat: ? min

Problem 1. Compute the limit $\lim_{x\to\infty} \frac{\sin x}{x}$.

We know that the sine function oscillates between -1 and 1, i.e

$$-1 \leq \sin x \leq 1$$

and since we are computing a limit as $x \to +\infty$, we can assume x is positive. Then dividing everything by x doesn't change the inequality sign, so

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

Now both the left and the right functions ($-1/x$ and $1/x$) go to zero as $x \to \infty$, so by the Squeeze Theorem

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

Note this is the rigorous way of doing it. In practice, I always just think of $\sin(x)$ as something between -1 and 1, so dividing that by x is like having $\frac{C}{x}$ where I treat C as a constant. Then clearly the limit as $x \to \infty$ must be zero.

Problem 2. Compute $\lim_{x\to\infty} \frac{2-\cos x}{x+3}$.

We know

$$-1 \leq \cos x \leq +1$$

We somehow want the middle term to first look like $2 - \cos x$ then divide by $x + 3$, which will give us our original function in the middle. First multiply by -1 so the order of the inequality switches

$$1 \geq -\cos x \geq -1$$

and add 2 to all sides

$$3 \geq 2 - \cos x \geq 1$$

which we may as well write as

$$1 \leq 2 - \cos x \leq 3$$

Now in our limit $x \to +\infty$, so we assume x is positive, and so is $x + 3$, which allows us to divide by $x + 3$ without switching inequalities, which gives us

$$\frac{1}{x+3} \leq \frac{2-\cos x}{x+3} \leq \frac{3}{x+3}$$

Finally since both the left and the right functions go to zero as $x \to \infty$, the Squeeze Theorem tells us that

$$\lim_{x\to\infty} \frac{2-\cos x}{x+3} = 0$$

This is how we do it rigorously. But in practice it helps to think of $\cos x$ as something between -1 and 1, so we can treat the numerator as some constant. Dividing that by $x+3$ means the limit has to be zero as $x \to \infty$, because it's as if we had $\frac{C}{x+3}$ for some constant C.

Problem 3. Compute $\lim_{x\to\infty} \frac{\cos^2 x}{3-2x}$.

We know

$$-1 \leq \cos(x) \leq +1$$

which we may as well write in terms of absolute value as

$$|\cos(x)| \leq 1$$

Now squaring both sides preserves the inequality, so

$$|\cos(x)|^2 \leq 1^2 = 1$$

and by the properties of absolute value this is

$$|\cos^2(x)| \leq 1$$

but squaring something is always nonnegative so we can drop the absolute value and just write

$$0 \leq \cos^2(x) \leq 1$$

Now as $x \to +\infty$, the denominator $3-2x$ is negative so dividing by it switches our inequalities, giving us

$$\frac{0}{3-2x} \geq \frac{\cos^2(x)}{3-2x} \geq \frac{1}{3-2x}$$

which is the same as

$$\frac{1}{3-2x} \leq \frac{\cos^2(x)}{3-2x} \leq 0$$

but since $\lim_{x\to\infty} \frac{1}{3-2x} = 0$, and the right side is also zero, the Squeeze Theorem tells us that

$$\lim_{x\to\infty} \frac{\cos^2 x}{3-2x} = 0$$

This is how we do it rigorously. In practice, I know cosine is between -1 and 1, so its square must be between 0 and 1. Then it's as if I'm taking the limit of $\frac{C}{3-2x}$ for some constant C between 0 and 1, but that's zero as $x \to \infty$.

Problem 4. Compute the left-sided limit $\lim_{x\to 0^-} x^3 \cos\left(\frac{2}{x}\right)$.

We've seen a few solutions on how to do it the formal way, being careful with inequalities and such. Let's do this one the informal way. I know that no matter what I plug into my cosine function, it will be something between -1 and 1 (same is true for sine).

So I can think of my limit as $\lim_{x\to 0^-} Cx^3$ for some constant C between 0 and 1. But $x^3 \to 0$, so the limit has to be zero.

Alternately, because of what we know about cosine, we have

$$(-1)\cdot|x^3| \le \left|x^3\cos\left(\frac{2}{x}\right)\right| \le (+1)\cdot|x^3|$$

but the left and right functions both go to zero as $x \to 0^-$. So my limit is zero.

Problem 5. Find $\lim_{x\to\infty} \frac{x^2(2+\sin^2 x)}{x+100}$.

The $2+\sin^2 x$ term in the numerator oscillates between 2 and 3 (because the sine squared is between 0 and 1), so really we are looking at a limit of the form (C is a positive constant between 2 and 3):

$$\lim_{x\to\infty} \frac{Cx^2}{x+100} = +\infty$$

so our original limit has to be $+\infty$ (i.e. does not exist).