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M20550 Calculus III Tutorial
Worksheet 2

1. Find an equation of the plane passes through the point $(1, 1, -7)$ and perpendicular to the line $x = 1 + 4t$, $y = 1 - t$, $z = -3$.

> ***Solution:*** To write an equation of a plane, we need one point on the plane and a normal vector (a vector that is perpendicular to the plane).
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> In this problem, we have the point $(1, 1, -7)$ on the plane. Now, we need to find a normal vector. We know our plane is perpendicular to the line $x = 1 + 4t$, $y = 1 - t$, $z = -3$. So, the parallel vector to this line, which is $\mathbf{v} = \langle 4, -1, 0 \rangle$, can be used as the normal vector to our plane.
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> Finally, an equation of the plane with normal vector $\langle 4, -1, 0 \rangle$ passing through $(1, 1, -7)$ is given by
>
> $$\begin{aligned} \langle 4, -1, 0 \rangle \cdot \langle x, y, z \rangle &= \langle 4, -1, 0 \rangle \cdot \langle 1, 1, -7 \rangle \\ \implies 4x - y &= 3. \end{aligned}$$

2. Let ℓ be the line of intersection of the planes given by equations $x - y = 1$ and $x - z = 1$. Find an equation for ℓ in the form $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$.

> ***Solution:*** To write an equation of the line ℓ, we need to find one point on ℓ and a parallel vector to ℓ.
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> Since ℓ is the line of intersection of two planes, to find a point on ℓ, we need to find a point that contained in both planes. A point on both planes can be found by setting $x = 1$, so $y = z = 0$. And we get the point $(1, 0, 0)$ on ℓ.
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> A normal vector for the first plane is $\langle 1, -1, 0 \rangle$ and a normal vector for the second plane is $\langle 1, 0, -1 \rangle$. A parallel vector of ℓ is a vector perpendicular to the normal vectors of both planes. Thus, a parallel vector of ℓ is given by
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> $$\langle 1, -1, 0 \rangle \times \langle 1, 0, -1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle .$$
>
> Hence, the vector equation of ℓ is
>
> $$\mathbf{r}(t) = \langle 1, 0, 0 \rangle + t \langle 1, 1, 1 \rangle .$$

Another way to solve this problem is just to consider the following, ℓ is the set of points that satisfy both

$$\begin{aligned} x - y &= 1 \\ x - z &= 1 \end{aligned}$$

which is equivalent to the set of points which satisfy

$$x - 1 = y = z$$

which is the cartesian equation for the line ℓ, the to go from the cartesian equation to the vector equation, we just set

$$\begin{aligned} x - 1 &= t \\ y &= t \\ z &= t \end{aligned}$$

and this system of equations is equivalent to the system

$$\begin{aligned} x &= 1 + t \\ y &= t \\ z &= t \end{aligned}$$

and this gives us that a vector equation for ℓ is given by;

$$\mathbf{r}(t) = (1, 0, 0) + t\langle 1, 1, 1\rangle.$$

3. How many times does a particle traveling along the curve $\mathbf{r}(t) = \langle t^2 + 1, 2t^2 - 1, 2 - 3t^2\rangle$ hit the plane
$2x + 2y + 3z = 3$? What is the point(s) of intersection?

Solution: (a) We have $\mathbf{r}(t) = \langle t^2 + 1, 2t^2 - 1, 2 - 3t^2\rangle$. So the x, y, z-coordinates of the particle are given by:

$$x = t^2 + 1, \qquad y = 2t^2 - 1, \qquad z = 2 - 3t^2.$$

If the particle hits the plane, the x, y, z-coordinates of the particle have to satisfy the equation $2x + 2y + 3z = 3$. Thus, we get the equation

$$\begin{aligned} 2(t^2+1) + 2(2t^2-1) + 3(2-3t^2) &= 3 \\ 2t^2 + 2 + 4t^2 - 2 + 6 - 9t^2 &= 3 \\ -3t^2 + 6 &= 3 \\ t^2 &= 1 \\ t &= 1 \quad \text{or} \quad t = -1 \end{aligned}$$

Thus, the particle hits the plane twice. And with $t = 1$, we get $x = 1^2 + 1 = 2$, $y = 2(1)^2 - 1 = 1$, $z = 2 - 3(1)^2 = -1 \implies (2, 1, -1)$.
With $t = -1$, $x = (-1)^2 + 1 = 2$, $y = 2(-1)^2 - 1 = 1$, $z = 2 - 3(-1)^2 = -1 \implies (2, 1, -1)$. So, we only have one point of intersection, that is $(2, 1, -1)$.

4. Let P be a plane with normal vector $\langle -2, 2, 1\rangle$ passing through the point $(1, 1, 1)$. Find the distance from the point $(1, 2, -5)$ to the plane P.

Solution: Let's make a vector $\mathbf{b}$ from the point $(1, 1, 1)$ to the point $(1, 2, -5)$:

$$\mathbf{b} = \langle 1-1, 2-1, -5-1\rangle = \langle 0, 1, -6\rangle .$$

Then, the distance D from the point $(1, 2, -5)$ to the plane P is given by

$$D = |\text{comp}_{\mathbf{n}}\mathbf{b}| = \frac{|\mathbf{n} \bullet \mathbf{b}|}{|\mathbf{n}|} = \frac{|\langle -2, 2, 1\rangle \bullet \langle 0, 1, -6\rangle|}{|\langle -2, 2, 1\rangle|} = \frac{|-4|}{\sqrt{(-2)^2 + 2^2 + 1^2}} = \frac{4}{3}.$$

5. Find an equation of the plane that passes through the point $(1, 2, 3)$ and contains the line $\frac{1}{3}x = y - 1 = 2 - z$.

Solution: For this problem, in order to find a normal vector of the plane, we first need to find two vectors on the plane then take their cross product.

One vector that lies on the plane is a parallel vector of the line $\frac{1}{3}x = y - 1 = 2 - z$ (because this line is contained in the plane). Note that $\frac{1}{3}x = y - 1 = 2 - z \iff \frac{x-0}{3} = \frac{y-1}{1} = \frac{z-2}{-1}$. So, a parallel vector of this line is $\mathbf{v_1} = \langle 3, 1, -1\rangle$. Thus, we have $\mathbf{v_1} = \langle 3, 1, -1\rangle$ lies on the plane.

To get another vector on the plane, we take one point on the line and make a vector with the point on the plane $(1, 2, 3)$. One point on the line $\dfrac{x-0}{3} = \dfrac{y-1}{1} = \dfrac{z-2}{-1}$ is $(0, 1, 2)$. So, we get the second vector $\mathbf{v_2}$ on the plane, $\mathbf{v_2} = \langle 1-0, 2-1, 3-2 \rangle = \langle 1, 1, 1 \rangle$.

Then, a normal vector is given by

$$\mathbf{v_1} \times \mathbf{v_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \langle 2, -4, 2 \rangle .$$

So, the equation of the required plane is

$$\begin{aligned} \langle 2, -4, 2 \rangle \bullet \langle x, y, z \rangle &= \langle 2, -4, 2 \rangle \bullet \langle 1, 2, 3 \rangle \\ \implies 2x - 4y + 2z &= 0 \\ \implies x - 2y + z &= 0 \end{aligned}$$

6. Find a vector function that represents the curve of intersection of the cylinder $x^2+y^2 = 9$ and the plane $x + y - z = 5$.

Solution: To find a vector function that represents the curve of intersection, we need to be able to describe x, y, z in terms of t for this curve.

On the xy-plane, $x^2 + y^2 = 9$ represents a circle centers at the origin with radius 3. So, we can write the parametric equations for this circle as follows:

$$x = 3\cos t, \qquad y = 3\sin t, \qquad 0 \le t \le 2\pi.$$

And from the equation of the plane, we get

$$z = x + y - 5 \implies z = 3\cos t + 3\sin t - 5, \qquad 0 \le t \le 2\pi.$$

So, a vector function that represents the curve of intersection is given by

$$\mathbf{r}(t) = (3\cos t)\,\mathbf{i} + (3\sin t)\,\mathbf{j} + (3\cos t + 3\sin t - 5)\,\mathbf{k}, \qquad 0 \le t \le 2\pi.$$

7. Give a vector valued function that describes the position of a particle that starts at the point $(0, 1)$ at time $t = 0$ and then moves along the unit circle in the xy-plane clockwise.

Solution: Observe that if we take our usual parametrization of the unit circle, $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$, and reflect through the line $y = x$, we get the motion we wish to describe. Then recall that reflection through the line $y = x$ is given by the map $(a, b) \mapsto (b, a)$. So one solution is the function

$$\boldsymbol{\phi}(t) = \langle \sin(t), \cos(t) \rangle.$$

Note: there are many solutions to this problem, $\boldsymbol{\phi}(ct) = \langle \sin(ct), \cos(ct) \rangle$ for any positive value of c would work also. (These just represent the particle moving at different speeds.)

8. Imagine a wheel of unit radius rolling from left to right along the x-axis in the xy-plane with a constant angular velocity of $\frac{1rad}{sec}$. Let p be the point on the wheel that has coordinates $(0, 0)$ at time $t = 0$. Find a vector valued function that describes the position of p at time t. What if the wheel had radius a? (The curve traced out by the motion of this point is called a cycloid.)

Solution: Notice that our function, $\mathbf{r}(t)$, may be written as a sum of three functions $\langle c(t), 0 \rangle + \langle 0, 1 \rangle + \boldsymbol{\alpha}(t)$, where $c(t)$ is the x-component of the center of the wheel at time t and $\boldsymbol{\alpha}(t) = -\boldsymbol{\phi}(t)$, where $\boldsymbol{\phi}$ was our solution to the previous problem, i.e. $\boldsymbol{\alpha}(t) = \langle -\sin(t), -\cos(t) \rangle$. This can be seen geometrically by drawing a picture of the wheel at some time t, and drawing the three vectors in the sum above in such a way that the initial point of each one is the terminal point of the last, with the first one positioned so that its initial point is at the origin. The stipulation that the angular velocity of the wheel is 1 radian per second gives us that $c(t) = t$. So, $\mathbf{r}(t) = \langle t - \sin(t), 1 - \cos(t) \rangle$. If the radius of the wheel is a then by similar reasoning we get $\mathbf{r}(t) = \langle at - a\sin(t), a - a\cos(t) \rangle$.
The following link will bring you to a page about this problem on the site math-stackexchange that has a nice animation and some good solutions, please have a look;

https://math.stackexchange.com/questions/133604/how-to-find-the-parametric-equation-of-a-cycloid

The solution by Robert Israel is particularly good.