**10350 Tutorial Week 07 - Set 01** Name ______________________

Section ______________

**Estimating Derivative of a Function**

Recall there are three ways to estimate the derivative (or the rate of change) of a given function $f(x)$ when we ONLY know a table of values of $f(x)$. We have the:

**Forward Difference formula** **Backward Difference formula**

**Central Difference formula**

Consider a particle moving on a straight line. Its displacement $s(t)$ from a fixed point O on the straight line is given by the table of values below

| $t$ | 0 | 0.5 | 1.0 | 2.0 | 2.5 | 3.0 | 3.5 |
|---|---|---|---|---|---|---|---|
| $s(t)$ | $-4.0$ | $-2.0$ | $-1.0$ | 0 | 1.2 | 1.8 | 2.2 |

**a.** Give all possible estimates for the instantaneous velocity at $t = 0.5$. State which estimate you apply.

$$\text{Forward Difference Estimate: } \frac{f(1.0) - f(0.5)}{1.0 - 0.5} = \frac{-1.0 + 2.0}{0.5} = \frac{1.0}{0.5} = 2.0$$

$$\text{Backward Difference Estimate: } \frac{f(0.5) - f(0)}{0.5 - 0} = \frac{-2.0 + 4.0}{0.5} = \frac{2.0}{0.5} = 4.0$$

$$\text{Central Difference Estimate: } \frac{f(1.0) - f(0)}{1.0 - 0} = \frac{-1.0 + 4.0}{1.0} = 3.0$$

**Caution:** Note that the Central Difference Estimate is only the average of the forwards and backwards estimates if your time intervals are evenly spaced.

**b.** Give all possible estimates for the instantaneous velocity at $t = 3.5$. How does your conclusion differ from Part (a).

$$\text{Backwards Difference Estimate: } \frac{f(3.5) - f(3)}{3.5 - 3} = \frac{2.2 - 1.8}{0.5} = \frac{0.4}{0.5} = 0.8$$

We can't make every estimate in this case because we have no values of s(t) in the table for $t > 3.5$. These values are needed to make a Forward Difference Estimate and a Central Difference Estimate.

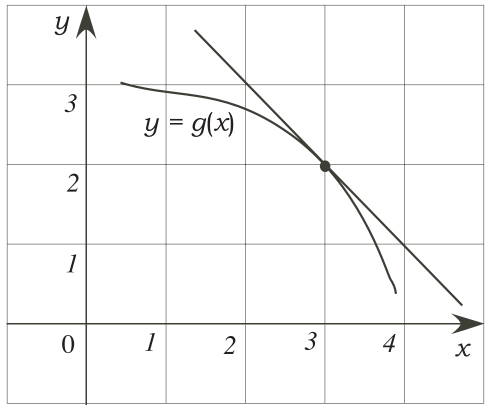
**c.** Give all possible estimates for the instantaneous velocity at $t = 0$. How does your conclusion differ from Parts (a) and (b).

$$\text{Forward Difference Estimate:} \quad \frac{f(0.5) - f(0)}{0.5 - 0} = \frac{-2.0 + 4.0}{0.5} = \frac{2.0}{0.5} = 4.0$$

Note this is the same as the Backward Difference Estimate we used for $t = 0.5$. We can't make Backward or Central Estimates in this case because we have no data for $t < 0$.

**10350 Tutorial Week 07 - Set 01** Name ______________________

Section ______________




**1.** The graph of the function $g(x)$ is given above.

(a) What is the value of $g(3)$? Answer: **g(3)=2**

(b) What is the **instantaneous** rate of change of $g(x)$ at $x = 3$? Answer: **g'(3) = -1**
(Simply find the slope of the tangent line drawn in the picture. )

(c) Find the slope of the graph of $f(x) = \dfrac{e^{g(x)-2}}{x+1}$ at $x = 3$.
**Solution:** We first compute $f'(x)$, noting that we need to apply both quotient rule and chain rule.

$$f'(x) = \frac{(x+1)e^{g(x)-2}g'(x) - e^{g(x)-2} \cdot 1}{(x+1)^2}$$

Next, we plug in $x = 3$ and use our answers from parts $(a)$ and $(b)$ to find f'(3):

$$\begin{aligned} f'(3) &= \frac{4e^{g(3)-2}g'(3) - e^{g(3)-2} \cdot 1}{(4)^2} \\ &= \frac{4e^0 \cdot (-1) - e^0 \cdot 1}{16} \\ &= -\frac{5}{16}. \end{aligned}$$

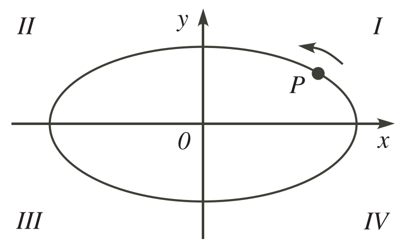
**10350 Tutorial Week 07 - Set 02** Name ______________________________

Section ___________________

**2.** Consider a particle $P$ moving **counterclockwise** around the ellipse

$$x^2 + 4y^2 = 5.$$

Fill in below the sign ($> 0$ or $< 0$) of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in quadrants of the $xy$-plane.




**2a.** $\frac{dx}{dt}$ $\underline{\;< \mathbf{0}\;}$ and $\frac{dy}{dt}$ $\underline{\;> \mathbf{0}\;}$ when $P$ is in Quadrant I.

**2b.** $\frac{dx}{dt}$ $\underline{\;< \mathbf{0}\;}$ and $\frac{dy}{dt}$ $\underline{\;< \mathbf{0}\;}$ when $P$ is in Quadrant II.

**2c.** $\frac{dx}{dt}$ $\underline{\;> \mathbf{0}\;}$ and $\frac{dy}{dt}$ $\underline{\;< \mathbf{0}\;}$ when $P$ is in Quadrant III.

**2d.** $\frac{dx}{dt}$ $\underline{\;> \mathbf{0}\;}$ and $\frac{dy}{dt}$ $\underline{\;> \mathbf{0}\;}$ when $P$ is in Quadrant IV.

**2e.** For the same Particle $P$ above, find $\frac{dx}{dt}$ at $(1, -1)$ if $\frac{dy}{dt} = 2$ units per second.

**Solution:** Remember that $x$ and $y$ are both functions of $t$. To stress this point, we can rewrite the equation of our ellipse as $(x(t))^2 + 4(y(t))^2 = 5$. We take the derivative of both sides with respect to t:

$$\frac{d}{dt}\left[(x(t))^2 + 4(y(t))^2\right] = \frac{d}{dt}[5]$$
$$2x \cdot \frac{dx}{dt} + 8y \cdot \frac{dy}{dt} = 0$$
$$x \cdot \frac{dx}{dt} + 4y \cdot \frac{dy}{dt} = 0$$

We plug in the values $x = 1$, $y = -1$, and $\frac{dy}{dt} = 2$, and solve for $\frac{dx}{dt}$.

$$1 \cdot \frac{dx}{dt} + 4(-1)(2) = 0 \Rightarrow \boxed{\frac{dx}{dt} = 8 \text{ units/second}}$$

**10350 Tutorial Week 07 - Set 03** Name ____________________

Section ______________

**3.** A huge spherical snowball is melting such that its radius is **reducing** at a constant rate of 2 cm per minute. At what rate is the volume changing at the instant when the radius of the snowball is 10 cm? $\left(\text{You may leave your answers in terms of } \pi \text{ and use the formula } V = \frac{4}{3}\pi r^3.\right)$

**Solution:** We need to find $\frac{dV}{dt}$ when the radius $r = 10$ cm. We are given that the radius is decreasing at a (constant) rate of 2 cm per minute, that is, $\frac{dr}{dt} = -2$ cm/min. We take the derivative of $V = \frac{4}{3}\pi r^3$ with respect to time.

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

Finally, we evaluate the above when $r = 10$.

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{r=10} &= 4\pi(10 \text{ cm})^2 \cdot (-2\text{cm/min}) \\ &= \boxed{-800\pi \text{ cm}^3/\text{min}} \end{aligned}$$

**10350 Tutorial Week 07 - Set 04** Name ________________

Section ____________

**4.** Consider the curve given by $x^2y^2 - x^4 = 6e^{y-2} - 3$.

**4a.** Find $\dfrac{dy}{dx}$.

**Solution:** Note that now we are considering $y$ to be a function of $x$. We differentiate both sides of the equation above with respect to $x$ and then solve for $\dfrac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}\left[x^2\left(y(x)\right)^2 - x^4\right] &= \frac{d}{dx}\left[6e^{y(x)-2} - 3\right] \\
x^2 \cdot 2y\frac{dy}{dx} + 2xy^2 - 4x^3 &= 6e^{y-2}\frac{dy}{dx} \\
2x^2y\frac{dy}{dx} - 6e^{y-2}\frac{dy}{dx} &= 4x^3 - 2xy^2 \\
(2x^2y - 6e^{y-2})\frac{dy}{dx} &= 4x^3 - 2xy^2 \\
\frac{dy}{dx} &= \frac{4x^3 - 2xy^2}{2x^2y - 6e^{y-2}} \\
&\boxed{\frac{dy}{dx} = \frac{2x^3 - xy^2}{x^2y - 3e^{y-2}}}
\end{aligned}$$

**4b.** Find the equation of the tangent line to the curve given by $x^2y^2 - x^4 = 6e^{y-2} - 3$ at the point $(-1, 2)$.

**Solution:** We first need to find the slope of the tangent line. This is the derivative of the function at the point $P = (-1, 2)$, that is

$$\begin{aligned}
\text{Slope} = \frac{dy}{dx}\Big|_{(-1,2)} &= \frac{2(-1)^3 - (-1)(2)^2}{(-1)^2(2) - 3e^{2-2}} \\
&= \frac{-2+4}{2-3} \\
&= 2/(-1) \\
&= -2.
\end{aligned}$$

Using point-slope form, we find the equation of the tangent line to the curve at the point $(-1, 2)$.

$$y - 2 = -2(x+1) = -2x - 2$$

$$\boxed{y = -2x}$$