Basic Counting Techniques

Example: Suppose you have 3 hats, hats A, B and C, and 2 coats, Coats 1 and 2, in your closet. Assuming that you feel comfortable with wearing any hat with any coat, how many different choices of hat/coat combinations do you have? List all combinations.

Hat A with Coat 1	Hat A with Coat 2
Hat B with Coat 1	Hat B with Coat 2
Hat C with Coat 1	Hat C with Coat 2

There are 6 possible hat/coat combinations.

Notice that $3 \cdot 2 = 6$. This is NOT a coincidence!

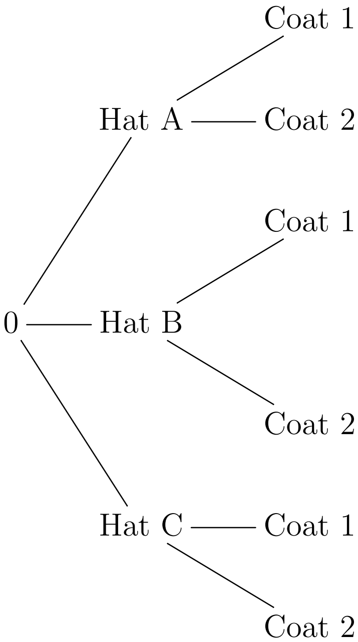
Basic Counting Techniques

We can get some insight into why $3 \cdot 2 = 6$ works by representing all options on a tree diagram. First break the decision making process into two steps here:

Step 1: choose a hat
Step 2: choose a coat

From the starting point 0, we can represent the three choices for step 1 by three branches whose endpoints are labelled by the chosen hat name. From each of these endpoints we draw branches representing the options for step two with endpoints labelled appropriately. The result for the above example is shown on the next slide.

Basic Counting Techniques



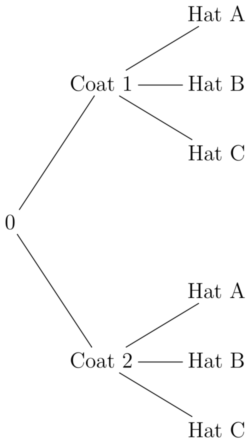
Basic Counting Techniques

Each path on the tree diagram corresponds to a choice of hat and coat. Each of the three branches in step 1 is followed by two branches in step 2, giving us $3 \times 2 = 6$ distinct paths.

If we had m hats and n coats, we would get $m \times n$ paths on our diagram. Of course if the numbers m and n are large, it may be difficult to draw.

Basic Counting Techniques

Here is the problem done with Coats first and then Hats.



The Multiplication Principle

Two-step multiplication principle: Assume that a task can be broken up into a sequence of two consecutive steps. If step 1 can be performed in m ways, and for each of these, step 2 can be performed in n ways, then the task itself can be performed in $m \times n$ ways.

The Multiplication Principle

Example: The South Shore line runs between South Bend Airport and Randolph St. Station in Chicago. There are 20 stations at which it stops along the line. How many one way tickets could be printed, showing a point of departure and a destination? (Assuming you can not depart and arrive at the same station.)

You can start at any of twenty stations. Once this is picked, you can pick any of nineteen destinations. The answer is $20 \cdot 19 = 380$.

If you can get on and off at the same station the answer if $20 \cdot 20 = 400$.

The Multiplication Principle

Example: You want to design a 30 minute workout. For the first 15 minutes, you will work on strength and/or balance choosing from weight training, TRX, Bosu, or bodyweight exercises. For the second 15 minutes, you will choose an aerobic exercise from running, kickboxing, or swimming. How many such workouts are possible?

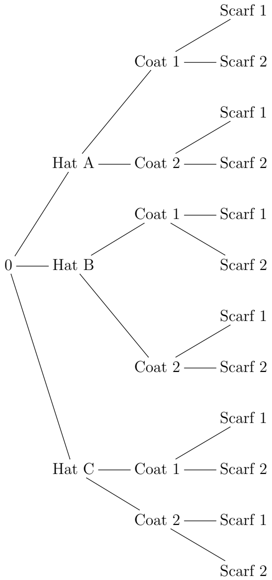
There are 4 things you can do for your first 15 minutes.
There are 3 things you can do for the second 15 minutes.

This gives $4 \cdot 3 = 12$ possible workouts.

The Multiplication Principle

Example: If your closet contains 3 hats, 2 coats and 2 scarves. Assuming you are comfortable with wearing any combination of hat, coat and scarf, (and you absolutely need a hat, coat and scarf today), how many different outfits could you select from your closet? Break the decision making process into steps and draw a tree diagram representing the possible choices.

The Multiplication Principle



You have $3 \cdot 2 \cdot 2 = 12$ possible outfits to choose from.

The General Multiplication Principle

If a task can be broken down into R consecutive steps,
Step 1, Step 2, ..., Step R,
and if I can perform Step 1 in m_1 ways,
and for each of these I can perform Step 2 in m_2 ways,
and for each of these I can perform Step 3 in m_3 ways,
and so on, then the task can be completed in

$$m_1 \cdot m_2 \dots m_R$$

different ways.

Note in the previous example, $R = 3$ with $m_1 = 3$, $m_2 = 2$ and $m_3 = 2$.

The General Multiplication Principle

Example: How many License plates, consisting of 2 letters followed by 4 digits are possible? Would this be enough for all the cars in Indiana? (Note that it is not a good idea to try to solve this with a tree diagram).

There are 26 letters and 10 digits so the process is

$$\begin{pmatrix}\text{Pick a}\\ \text{letter}\end{pmatrix}\begin{pmatrix}\text{Pick a}\\ \text{letter}\end{pmatrix}\begin{pmatrix}\text{Pick a}\\ \text{digit}\end{pmatrix}\begin{pmatrix}\text{Pick a}\\ \text{digit}\end{pmatrix}\begin{pmatrix}\text{Pick a}\\ \text{digit}\end{pmatrix}\begin{pmatrix}\text{Pick a}\\ \text{digit}\end{pmatrix}$$

$$26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6,760,000$$

The current population of Indiana is around $6,600,000$, with approximately $5,000,000$ over 18 (census.gov); at roughly one car per adult, this would probably be just enough. Indiana now often uses 3 letters which yields:

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$$

The General Multiplication Principle

Example: A group of students comprised of 7 women and 2 men is to be called at the board, one at a time, to solve difficult Finite Math problems in front of the whole class.

(a) How many ways can they be arranged in one row?

There are 9 students total, so there are

$$9 \cdot 8 \cdot 7 \ldots 3 \cdot 2 \cdot 1 = 9! = 362,880$$

possible ways to do this. Notice that the fact that some of them are men or women doesn't matter in the ordering.

Note: The notation 9! (read "9 factorial") is the falling product $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362,880$.

The General Multiplication Principle

(b) If we have 5 men and 3 women in the class, how many ways can they be arranged for a class photo with the ladies in front row and the guys in the back row?

First row: there are 3 women, so there are $3 \cdot 2 \cdot 1 = 3!$ ways to arrange the first row.

Second row: with 5 men, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ ways to arrange the second row.

Combine rows 1 and 2: The two rows can be arranged independently, so the answer is $3! \cdot 5! = 6 \cdot 120 = 720$ possibilities.

The General Multiplication Principle

Example: How many different 4 letter words (including nonsense words that are not in the dictionary) can you make from the letters of the word

MATHEMATICS

(a) if the letters cannot be repeated (MMMM is not allowed, but MTCS is)?

'MATHEMATICS' has 8 distinct letters

$$\{M, A, T, H, E, I, C, S\}.$$

This gives $8 \cdot 7 \cdot 6 \cdot 5 = 1,680$ possible words.

The General Multiplication Principle

(b) What if letters can be repeated (MMMM is allowed)?

There are still only 8 distinct letters to choose from so the answer is $8 \cdot 8 \cdot 8 \cdot 8 = 8^4 = 4,096$.

(c) What if letters cannot be repeated and the word must start with a vowel?

Out of the 8 letters {M, A, T, H, E, I, C, S}, 3 are vowels {A, E, I}, so you have 3 choices for the first vowel.

Once you have done this, you have 7 choices for the second letter, 6 choices for the third letter, and 5 choices for the fourth letter. This gives $3 \cdot 7 \cdot 6 \cdot 5 = 630$ words.

The General Multiplication Principle

A standard deck of 52 cards can be classified according to suits or denominations.

We have 4 suits: Hearts, Diamonds, Clubs, and Spades.

We also have 13 denominations: Kings, Queens, Jacks, 10s, 9s, . . . , 3s, 2s, and Aces.

There are also 2 different colors: red (hearts and diamonds) and black (clubs and spades).

The General Multiplication Principle

Example: Katie and Peter are playing a card game. The dealer will give each one card and the player will keep the card when it is dealt to them.

(a) How many different outcomes can result? $52 \cdot 51$

(b) In how many of the possible outcomes do both players have Hearts? $13 \cdot 12$

(b) In how many of the possible outcomes do both players have a BLACK card? $26 \cdot 25$

The Addition Principle

Recall that the inclusion-exclusion principle says

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

If the sets A and B are **disjoint** then this principle reduces to $n(A \cup B) = n(A) + n(B)$. Thus in counting unions of disjoint sets, we just count the number of elements in each set, and add.

This principle extends easily to more than two disjoint sets. If A_1, A_2, $\ldots A_R$ are disjoint, then

$$n(A_1 \cup A_2 \cup \cdots \cup A_R) = n(A_1) + n(A_2) + \cdots + n(A_R)$$

This is called the **addition principle**.

The Addition Principle

Example: Suppose that we are to choose one student as our class representative. We have 7 men and 8 women currently enrolled. In how many ways can we choose a representative?

Choosing a man or a woman are DISJOINT ways to choose a representative (both can't happen at the same time).

If our representative is male, there are 7 choices to do so. If our representative is female, we have 8 ways to choose one.

In total, by the addition principle, there are $7 + 8 = 15$ possible choices for our representative.

Note: We could have easily just chosen 1 out of 15 students and get the same result!

The Addition Principle

Example: Katie and Peter are playing a card game. The dealer will give each one card and the player will keep the card when it is dealt to them. In how many of the possible outcomes do both players have cards from the same suit?

Method 1: There are four distinct possibilities for the suit they end up with: the players can be dealt two clubs, two diamonds, two heart, or two spades. These are all distinct (i.e. disjoint).

In each of these, there are 13 choices for the first card, and 12 choices for the second. In total, we have

$$(13 \cdot 12) + (13 \cdot 12) + (13 \cdot 12) + (13 \cdot 12) = 624$$

possible outcomes.

The Addition Principle

Method 2: A second approach is that there are 52 ways to pick the first card, which will determine the suit. This leaves us with only 12 ways to pick a second card of the same suit. Again we get $52 \cdot 12 = 624$ possibilities.

Method 3: We can also think of this as a three step process. First pick the suit (4 choices), then pick the first card (13 choices from that suit), and finally pick the second card (only 12 left after dealing the first card). We have

$$4 \cdot 13 \cdot 12 = 624$$

possible outcomes.

The Addition Principle

Example: Suppose you are going to buy a single carton of milk today. You can either (1) buy it on campus when you are at school, (2) at the mall when you go to get a gift for a friend, or (3) in the neighborhood near your apartment on your way home. There are 5 different shops on campus, 2 shops at the mall, and 3 shops in your neighborhood. In how many different shops can you buy the milk?

There are three distinct outcomes. You either buy the milk on campus with 5 choices, OR you buy the milk at the mall with 2 choices, OR you buy the milk in your neighborhood with 3 choices, so there are $5 + 2 + 3 = 10$ different shops you can buy the milk from.

If you answered $5 \cdot 2 \cdot 3$, this is the # of ways to buy one carton of milk from campus, one from the mall, and one from near home (you end up with three cartons).

The Addition Principle

Example: You wish to photograph 5 children on a soccer team. You line the children up in a row, but Malcom doesn't want to be in the middle. He insists on standing at the end of the row (either end will do). If this is the only restriction, in how many ways can you line the children up for the photograph?

There are two distinct possibilities: Malcom is on the left or Malcom is on the right. For each of those choices, there are $4!$ ways to arrange the remaining children. Hence you have $4! + 4!$ possible lineups.

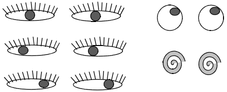
Emoticons

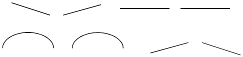
Example: How many emoticons can you make?

You are given 5 pairs of eyes, 4 sets of eyebrows, 2 noses, 5 mouths and 7 hairstyles to choose from. How many possible emoticons can you make using combinations of these features, if each emoticon you make has a pair of eyes, a pair of eyebrows, a nose, a mouth, and one of the given hairstyles?

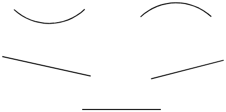
Emoticons

Noses





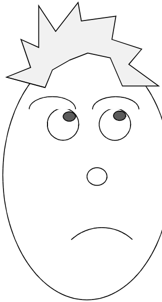
Eyebrows



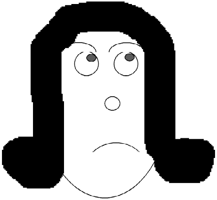
Mouths

Emoticons

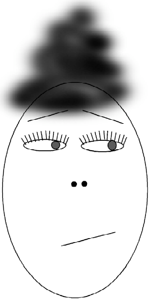
Here is an example of 3 emoticons with the features given!



I don't want a Lisa Simpson Hairdo!



If you say "Multiplication Principle" one more time.....



How many roads must a face walk down.....

$5 \cdot 4 \cdot 2 \cdot 5 \cdot 7 = 1,400$ emoticons.

Old Exam Questions For Review

Problem: Adrian likes to play mind games with his Give-Card Draw opponents by revealing his poker hand one card at a time (there are 5 cards total in a hand). In how many ways can he reveal his cards in this way?

$(a)\ 2^5 \qquad (b)\ 5 \qquad (c)\ 5^2 \qquad (d)\ 120 \qquad (e)\ 100$

There are 5 choices for the first card, 4 choices for the second, 3 choices for the third, and so on. This gives $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 120$ arrangements.

If you want to know how many ways you can lay out a deck of cards, the answer is 52! (easy to write down) or

$80,658,175,170,943,878,571,660,636,856,403,766,975,289,505,440,883,277,824,000,000,000,000$

Old Exam Questions For Review

Problem: Bruno's pizza joint offers a mix and match pizza on its menu. There are 4 different meats to choose from, 5 different vegetables, 4 different types of cheese, and 2 different types of crust. How many different types of pizza can be made by choosing 1 type of meat, 1 vegetable, 1 cheese, and 1 crust?

(a) 80 (b) 4 (c) 20 (d) 160 (e) 49

You can make $4 \cdot 5 \cdot 4 \cdot 2 = 160$ different pizzas.

Old Exam Questions For Review

Problem: Bruno's pizza joint offers a mix and match pizza on its menu. There are 4 different meats to choose from, 5 different vegetables, 4 different types of cheese, and 2 different types of crust. *You can choose to go vegetarian.* How many different types of pizza can be made by choosing 1 type of meat (or no meat), 1 veggie, 1 cheese, and 1 crust?

Vegetarian or carnivore are disjoint outcomes for the pizza.

If you go carnivore, you can make $4 \cdot 5 \cdot 4 \cdot 2 = 160$ different pizzas. If you go vegetarian, you can make $5 \cdot 4 \cdot 2 = 40$ pizzas.

By the addition principle, you can make a total of $160 + 40 = 200$ pizzas.

Q: What would the quantity $160 \cdot 40$ count?