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Worksheet 7 Solutions, Math 10560

1. Find the sum of the following series:

$$\sum_{n=0}^{\infty} \frac{3^n + (-2)^n}{5^n}\ .$$

Solution:
The series is the sum of two geometric series:

$$\sum_{n=0}^{\infty} \frac{3^n + (-2)^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{2}{5}\right)^n.$$

Since $|3/5| = 3/5 < 1$ and $|-2/3| = 2/3 < 1$, both series converge. Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{3^n + (-2)^n}{5^n} &= \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{2}{5}\right)^n \\
&= \frac{1}{1-3/5} + \frac{1}{1-(-2/5)} \\
&= \frac{1}{2/5} + \frac{1}{7/5} \\
&= \frac{5}{2} + \frac{5}{7} \\
&= \frac{35+10}{14} \\
&= \boxed{\frac{45}{14}.}
\end{aligned}$$

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2. Find the sum of the following series:

$$\sum_{n=4}^{\infty}\left[\frac{2n}{n+4}-\frac{2(n+1)}{n+5}\right].$$

Solution: For $n \geq 4$, $a_n = \frac{2n}{n+4} - \frac{2(n+1)}{n+5}$. We compute the partial sum, noticing that all of the terms except the first and last term cancel.

$$\begin{aligned}
S_N &= a_4 + a_5 + \cdots a_{N-1} + a_N \\
&= \left(\frac{8}{8} - \cancel{\frac{10}{9}}\right) + \left(\cancel{\frac{10}{9}} - \cancel{\frac{12}{10}}\right) + \cdots + \left(\cancel{\frac{2(N-1)}{N+3}} - \cancel{\frac{2N}{N+4}}\right) + \left(\cancel{\frac{2N}{N+4}} - \frac{2(N+1)}{N+5}\right) \\
&= 1 - \frac{2(N+1)}{N+5}
\end{aligned}$$

Thus, $\displaystyle\sum_{n=4}^{\infty}\left[\frac{2n}{n+4}-\frac{2(n+1)}{n+5}\right]$ is a telescoping series with sum given by:

$$\begin{aligned}
\sum_{n=4}^{\infty}\left[\frac{2n}{n+4}-\frac{2(n+1)}{n+5}\right] &= \lim_{N\to\infty} S_N \\
&= \lim_{N\to\infty} 1 - \frac{2(N+1)}{N+5} \\
&= 1 - \lim_{N\to\infty} \frac{2N+2}{N+5} \\
&= 1 - \lim_{N\to\infty} \frac{2+2/N}{1+5/N} \\
&= 1 - 2 \\
&= \boxed{-1.}
\end{aligned}$$

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3. State the divergence test:

Solution: The divergence test says that if a sequence $\{a_n\}_{n=1}^{\infty}$ diverges, or converges to a number other than zero, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

What can you say about the following series using the divergence test?

$$(I)\sum_{n=1}^{\infty}\frac{5^n}{n} \qquad (II)\sum_{n=1}^{\infty}\frac{5\cdot(-1)^n}{n} \qquad (III)\sum_{n=1}^{\infty}\frac{(-1)^n(5n+1)}{n}$$

Solution:
(I): In this example, $a_n = \frac{5^n}{n}$. The numerator 5^n grows much faster than the denominator n, so

$$\lim_{n\to\infty} a_n = +\infty.$$

You can show this more precisely by applying L'Hospital's Rule to the function $f(x) = \dfrac{2^x}{x}$.

$$\begin{aligned}\lim_{n\to\infty}\frac{5^n}{n} &= \lim_{x\to\infty}\frac{5^x}{x}\\ &= \lim_{x\to\infty}\frac{5^x(\ln 5)}{1} \qquad \text{(By L'Hospital's Rule)}\\ &= +\infty\end{aligned}$$

Thus, the divergence test tells us that the first series diverges.

(II):Remember that the sequence $\{(-1)^n b_n\}_{n=1}^{\infty}$ converges if and only if $\lim_{n\to\infty}|b_n| = 0$. Now,

$$\lim_{n\to\infty}|b_n| = \lim_{n\to\infty}\frac{5}{n} = 0.$$

In this case, the divergence test is inconclusive, because

$$\lim_{n\to\infty}\frac{5\cdot(-1)^n}{n} = 0.$$

Remember that the divergence test only applies when the sequence $\{a_n\}$ either converges to a number other than zero or diverges.

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(III): We have

$$\lim_{n\to\infty} |b_n| = \lim_{n\to\infty} \frac{5n+1}{n} = \lim_{n\to\infty} 5 + \frac{1}{n} = 5$$

Thus, the alternating sequence $\{\frac{5\cdot(-1)^n}{n}\}_{n=1}^{\infty}$ diverges (as n approaches infinity, the sequence is alternating back and forth between numbers very close to 5 and numbers very close to -5). By the divergence test, $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(5n+1)}{n}$ diverges.

4. Consider the following sequences.

(I) $\left\{(-1)^n \dfrac{n^3+2n-1}{4n^3+1}\right\}_{n=1}^{\infty}$ $\quad (II)$ $\left\{(-1)^n \dfrac{n^2-1}{5^n}\right\}_{n=1}^{\infty}$ $\quad (III)$ $\left\{(-1)^n e^{(1/n)}\right\}_{n=1}^{\infty}$

Determine which ones converge and find $\displaystyle\lim_{n\to\infty} a_n$ for those sequences $\{a_n\}$ that converge.

Solution: We have

$$\{(-1)^n a_n\}_{n=1}^{\infty} \text{ converges if and only if } \lim_{n\to\infty} |a_n| = 0.$$

(I) Let $a_n = \frac{n^3+2n-1}{4n^3+1}$. Note that for all $n \geq 1$, a_n is positive. Now,

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^3+2n-1}{4n^3+1} = \lim_{n\to\infty} \frac{1+2/n^3-1/n^3}{4+1/n^3} = \frac{1}{4} \neq 0.$$

Thus, the sequence $\left\{(-1)^n \frac{n^3+2n-1}{4n^3+1}\right\}_{n=1}^{\infty}$ **diverges**.

(II) We need to compute $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2-1}{5^n}$. We consider the corresponding function $f(x) = \frac{x^2-1}{5^x}$ (defined for all real x) and apply L'Hospital's rule twice (in both cases the limit is in indeterminate form $\frac{\infty}{\infty}$):

$$\begin{aligned}
\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty} \frac{n^2-1}{5^n} \\
&= \lim_{x\to\infty} \frac{x^2-1}{5^x} \\
&\overset{LH}{=} \lim_{x\to\infty} \frac{2x}{5^x(\ln 5)} \\
&\overset{LH}{=} \lim_{x\to\infty} \frac{2}{5^x(\ln 5)^2} \\
&= 0.
\end{aligned}$$

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Thus, the sequence $\left\{(-1)^n \frac{n^2-1}{5^n}\right\}_{n=1}^{\infty}$ **converges**.

(III) Let $a_n = e^{(1/n)}$. Note that $|a_n| = e^{(1/n)}$ and so

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} e^{(1/n)} = e^{\lim_{n\to\infty} 1/n} = 2^0 = 1 \neq 0.$$

Hence, the sequence $\left\{(-1)^n e^{(1/n)}\right\}_{n=1}^{\infty}$ **diverges**.

5. A tank initially contains 100 liters of salt water with 1.5 kilogram of dissolved salt. A well mixed salt water solution containing 3 kilograms of salt per 100 liters is pumped into the tank at a rate of 10 liters per minute. The salt water in the tank is kept thoroughly mixed and is drained at a rate of 5 liters per minute.

(a) Let $y = y(t)$ be the amount of salt in the tank at time t. Give a differential equation relating $\frac{dy}{dt}$ to y.

Solution:

We note that the rate of change of the amount of salt in the tank, denoted $\frac{dy}{dt}$, is given by

$$\frac{dy}{dt} = \text{ rate in - rate out,}$$

where "rate in" denotes the rate at which the salt is entering the tank (in kg/min) and "rate out" denotes the rate at which the salt is exiting the tank (also in kg/min). Now,

$$\begin{aligned} \text{rate in } &= \text{(Concentration of Solution Entering Tank)(Flow Rate In)} \\ &= \frac{3\text{kg}}{100\cancel{L}} \cdot \frac{10\cancel{L}}{\text{min}} \\ &= \frac{3}{10} \text{ kg/min} \end{aligned}$$

Similarly,

$$\text{rate out } = \text{(Conc. of Solution Exiting Tank)(Flow Rate Out).}$$

The difference here is that the concentration of the solution exiting the tank is no longer constant. This concentration in the tank (assuming the tank is evenly mixed) is given by $C(t) = \frac{y(t)}{V(t)}$, where $V(t)$ is the volume in the tank at time t. The solution being added to the tank at a rate of 10 liters per minute, and draining at a rate of 5 liters per minute, so the volume of the tank is increasing at a rate of 5L/min; i.e. $\frac{dV}{dt}$ = 5L/min. So, $V(t) = 5t + C$, where we see that $C = V(0) = 100$. So $C(t) = \frac{y(t)}{5t+100}$ kg/L. Putting this together, we see that

$$\begin{aligned} \text{rate out } &= \frac{y(t)\text{kg}}{5t+100\cancel{\text{L}}} \cdot \frac{5\cancel{\text{L}}}{\text{min}} \\ &= \frac{y(t)}{t+20} \text{ kg/min.} \end{aligned}$$

Thus,

$$\begin{aligned} \frac{dy}{dt} &= \text{ rate in - rate out} \\ &= \frac{3}{10} - \frac{y(t)}{t+20}. \end{aligned}$$

(Units are still kg/min.) We also note that the tank initially contains 1.5 kg of dissolved salt, giving us the initial condition $y(0) = 1.5$. So our differential equation becomes

$$\boxed{\frac{dy}{dt} = \frac{3}{10} - \frac{y(t)}{t+20},\ y(0) = 1.5.}$$

(b) Give a formula for the amount of salt in the tank at time t.

Solution: Rewriting the equation above as

$$\frac{dy}{dt} + \frac{y(t)}{t+20} = \frac{3}{10}$$

we see that this differential equation has the form of a linear differential equation, with $P(t) = \frac{1}{t+20}$, $Q(t) = \frac{3}{10}$ (you should be able to convince yourself that this equation is not separable). We first compute the integrating factor, $I(t)$,

$$I(t) = e^{\int P(t)\, dt} = e^{\int \frac{1}{t+20}\, dt} = e^{\ln(t+20)} = t + 20.$$

Next we multiply our equation through by $I(t)$

$$\begin{aligned} (t+20)\frac{dy}{dt} + y(t) &= \frac{3}{10}(t+20) \\ \Rightarrow \frac{d}{dt}\left[(t+20)y(t)\right] &= \frac{3}{10}(t+20) \end{aligned}$$

Integrating both sides and solving for $y(t)$ we obtain:

$$\begin{aligned} (t+20)y(t) &= \frac{3}{10}\int (t+20)\ dt \\ \Rightarrow (t+20)y(t) &= \frac{3}{10}\left(\frac{t^2}{2}+20t\right)+C \\ \Rightarrow (t+20)y(t) &= \frac{3}{20}t^2+6t+C \\ \Rightarrow y(t) &= \frac{\frac{3}{20}t^2+6t+C}{t+20} \end{aligned}$$

Next, we use the initial condition $y(0) = 1.5$ to solve for the constant C:

$$\begin{aligned} 1.5 = y(0) &= \frac{C}{20} \\ \Rightarrow C &= 20(1.5) = 30 \end{aligned}$$

Thus,

$$\boxed{y(t) = \frac{\frac{3}{20}t^2+6t+30}{t+20}.}$$

(c) At a given time t, the concentration is given by $C(t) = y(t)/V(t)$, where $V(t)$ is the volume in the tank at time t. Find the limit (algebraically) of $C(t)$ as t goes to infinity. Interpret your answer.

Solution: Using our previous work, we see $y(t) = \frac{\frac{3}{20}t^2+6t+30}{t+20}$, and $V(t) = 5t + 100$.

Thus,

$$\begin{aligned}
\lim_{t\to\infty} C(t) &= \lim_{t\to\infty} \frac{y(t)}{V(t)} \\
&= \lim_{t\to\infty} \frac{\frac{3}{20}t^2 + 6t + 30}{(t+20)(5t+100)} \\
&= \frac{3}{20} \cdot \lim_{t\to\infty} \frac{t^2 + 40t + 200}{5t^2 + 200t + 2000} \\
&= \frac{3}{20} \cdot \lim_{t\to\infty} \frac{1 + 40/t + 200/t^2}{5 + 200/t + 2000/t^2} \\
&= \frac{3}{20} \cdot \frac{1}{5} \\
&= \boxed{\frac{3}{100}\text{kg/L.}}
\end{aligned}$$

As t approaches infinity, the concentration in the tank approaches $\frac{3}{100}$ kg/L. This means the concentration in the tank is approaching the concentration of the solution you are adding to the tank.

(d) Suppose the tank has total capacity of 200L. What is the concentration in the tank at the moment when the tank begins to overflow?

Solution: The tank begins to overflow when $V(t) = 5t + 100 = 200$, i.e. when $5t = 100$, or $t = 20$ minutes. Now,

$$\begin{aligned}
C(t) &= \frac{y(t)}{V(t)} \\
&= \frac{\frac{3}{20}t^2 + 6t + 30}{(t+20)(5t+100)},
\end{aligned}$$

and so,

$$\begin{aligned}
C(20) &= \frac{y(20)}{V(20)} \\
&= \frac{\frac{3}{20}(20)^2 + 6(20) + 30}{(20+20)(200)} \\
&= \frac{210}{40 \cdot 200} \\
&= \frac{21}{800} \\
&= \boxed{0.02625\text{kg/L.}}
\end{aligned}$$

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Extra Problem: Write the following numbers as single fractions. Hint: Write each decimal as a geometric series or a sum of geometric series.

a. $0.\overline{1} = 0.111111111111\ldots$

Solution:

$$\begin{aligned}
0.\overline{1} = 0.111111111\ldots &= 0.1 + 0.01 + 0.001 + 0.0001 + \ldots \\
&= 10^{-1} + 10^{-2} + 10^{-3} + 10^{-4} + \ldots \\
&= \left(\frac{1}{10}\right)^1 + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \left(\frac{1}{10}\right)^4 + \ldots \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n
\end{aligned}$$

This is a geometric series with first term $a = \frac{1}{10}$ and common ratio $r = \frac{1}{10}$. This series converges because $|r| = \frac{1}{10} < 1$. By our formula,

$$\begin{aligned}
0.\overline{1} &= \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n \\
&= \frac{1/10}{1 - 1/10} \\
&= \frac{1/10}{9/10} \\
&= \frac{1}{\cancel{10}} \cdot \frac{\cancel{10}}{9} \\
&= \boxed{\frac{1}{9}.}
\end{aligned}$$

b. $0.\overline{12} = 0.12121212121212\ldots$

Solution: We first observe that $0.\overline{12} = 0.\overline{1} + 0.\overline{01}$, and use part (a). Now

$$\begin{aligned}
0.\overline{01} = 0.01010101\ldots &= 0.01 + 0.0001 + 0.000001 + 0.00000001 + \ldots \\
&= 10^{-2} + 10^{-4} + 10^{-6} + 10^{-8} + \ldots \\
&= \left(\frac{1}{10^2}\right)^1 + \left(\frac{1}{10^2}\right)^2 + \left(\frac{1}{10^2}\right)^3 + \left(\frac{1}{10^2}\right)^4 + \ldots \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{100}\right)^n
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This is a geometric series with first term $a = \frac{1}{100}$ and $r = \frac{1}{100}$. Thus,

$$\begin{aligned}
0.\overline{01} &= \sum_{n=1}^{\infty} \left(\frac{1}{100}\right)^n \\
&= \frac{1/100}{1 - 1/100} \\
&= \frac{1/100}{99/100} \\
&= \frac{1}{\cancel{100}} \cdot \frac{\cancel{100}}{99} \\
&= \frac{1}{99}.
\end{aligned}$$

Now,

$$\begin{aligned}
0.\overline{12} &= 0.\overline{1} + 0.\overline{01} \\
&= \frac{1}{9} + \frac{1}{99} \\
&= \frac{11+1}{99} \\
&= \boxed{\frac{12}{99}.}
\end{aligned}$$

c. $0.\overline{123} = 0.123123123123123\ldots$

Solution: This one is slightly trickier. Notice that

$$\begin{aligned}
0.\overline{123} &= 0.1 + 0.02 + 0.03 + 0.001 + 0.0002 + 0.00003 + \cdots \\
&= \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \frac{1}{10^4} + \frac{2}{10^5} + \frac{3}{10^6} + \cdots \\
&= \left[\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} \cdots\right] + \left[\frac{1}{10^2} + \frac{1}{10^5} + \frac{1}{10^8} + \cdots\right] + 2\left[\frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \cdots\right] \\
&= \left[\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots\right] + \frac{1}{10^2}\left[1 + \frac{1}{10^3} + \frac{1}{10^6} + \cdots\right] + 2\left[\frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \cdots\right] \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n + \frac{1}{10^2}\sum_{n=0}^{\infty} \left(\frac{1}{10^3}\right)^n + 2\sum_{n=1}^{\infty} \left(\frac{1}{10^3}\right)^n
\end{aligned}$$

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Now, we have

$$\begin{aligned}\frac{1}{10^2}\sum_{n=0}^{\infty}\left(\frac{1}{10^3}\right)^n &= \frac{1}{10^2}\cdot\frac{1}{1-1/1000}\\ &= \frac{1}{100}\cdot\frac{1}{999/1000}\\ &= \frac{1}{100}\cdot\frac{1000}{999}\\ &= \frac{10}{999},\end{aligned}$$

and

$$\begin{aligned}2\sum_{n=1}^{\infty}\left(\frac{1}{10^3}\right)^n &= 2\cdot\frac{1/1000}{1-1/1000}\\ &= \frac{2}{999}.\end{aligned}$$

Putting this together, we obtain

$$\begin{aligned}0.\overline{123} &= \sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n + \frac{1}{10^2}\sum_{n=0}^{\infty}\left(\frac{1}{10^3}\right)^n + 2\sum_{n=1}^{\infty}\left(\frac{1}{10^3}\right)^n\\ &= \frac{1}{9}+\frac{10}{999}+\frac{2}{999}\\ &= \frac{111}{999}+\frac{10}{999}+\frac{1}{999}\\ &= \boxed{\frac{123}{999}.}\end{aligned}$$

By now you should notice a pattern emerging. Can you prove it?