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Worksheet 10, Math 10560

1. Find a power series representation for the function

$$\frac{x^2}{1+x}$$

in the interval $(-1, 1)$.

Solution:

Recall from our knowledge of geometric series that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{whenever } |x| < 1. \tag{1}$$

Replacing x by $-x$, we get

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{1-(-x)} \\ &= \sum_{n=0}^{\infty} (-x)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^n \qquad \text{whenever } |x| < 1. \end{aligned}$$

Multiplying through by x^2, we obtain

$$\frac{x^2}{1+x} = x^2 \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2} \qquad \text{whenever } |x| < 1.$$

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2. Find a power series representation for the function

$$\frac{x^2}{(1-x^3)^2}$$

in the interval $(-1, 1)$. Hint: Differentiation of a power series may help.

Solution:
Let $f(x) = \frac{x^2}{(1-x^3)^2}$, $g(x) = \frac{1}{1-x^3}$. Then $g'(x) = \frac{3x^2}{(1-x^3)^2} = 3f(x)$. . We can get the desired power series by finding a power series for $g(x)$, performing term by term differentiation, and then multiplying by $\frac{1}{3}$. Replacing x by x^3 in the power series expansion of $\frac{1}{1-x}$ (given as equation 1 in the previous problem), we obtain

$$\begin{aligned} \frac{1}{1-x^3} &= \sum_{n=0}^{\infty} (x^3)^n \qquad \text{whenever } |x^3| < 1. \\ &= \sum_{n=0}^{\infty} x^{3n} \qquad \text{whenever } |x| < 1. \end{aligned}$$

Differentiating, we obtain

$$\begin{aligned} &\frac{d}{dx}\left[\frac{1}{1-x^3}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^{3n}\right] \\ &\Rightarrow \frac{3x^2}{(1-x^3)^2} = \sum_{n=1}^{\infty} 3nx^{3n-1} \\ &\Rightarrow \frac{x^2}{(1-x^3)^2} = \sum_{n=1}^{\infty} nx^{3n-1} \text{whenever } |x| < 1. \end{aligned}$$

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3. a) Write down the Taylor series expansion for $f(x) = \arctan(x)$ about $x = 0$.

Solution: This well-known Taylor Series was discussed in class:

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{for } -1 \leq x \leq 1.$$

b) Compute the following sum. Hint: Use part (a).

$$\sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$$

Solution: We can rewrite this series as

$$\sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1} = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

The trick is to notice that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ is exactly the series above, with $x = 1$ plugged in. So

$$\begin{aligned} 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} &= 4 \arctan(1) \\ &= 4 \cdot \frac{\pi}{4} \\ &= \pi. \end{aligned}$$

Cool Fact: This allows us to write π as an infinite sum of rational numbers.

$$\begin{aligned} \pi &= 4 \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots + \frac{(-1)^n}{2n+1} + \cdots \right] \\ &= 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} + \cdots + \frac{4(-1)^n}{2n+1} + \cdots \end{aligned}$$

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4. Find the radius of convergence and interval of convergence of the following power series. Remember to check the endpoints of your interval.

$$\sum_{n=1}^{\infty} \frac{(-1)^n(4x-1)^n}{2^n\sqrt{n+1}}$$

Solution: We use the ratio test with $a_n = \frac{(-1)^n(4x-1)^n}{2^n\sqrt{n+1}}$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(4x-1)^{n+1}}{2^{n+1}\sqrt{n+2}} \right| \cdot \left| \frac{2^n\sqrt{n+1}}{(-1)^n(4x-1)^n} \right| \\
&= \lim_{n\to\infty} \frac{|4x-1|^{\cancel{n}+1}}{2^{\cancel{n}+1}\sqrt{n+2}} \cdot \frac{\cancel{2^n}\sqrt{n+1}}{\cancel{|4x-1|^n}} \\
&= \lim_{n\to\infty} \frac{|4x-1|}{2}\sqrt{\frac{n+1}{n+2}} \\
&= \frac{|4x-1|}{2} \\
&= \frac{4|x-\frac{1}{4}|}{2} \\
&= 2|x-\frac{1}{4}|
\end{aligned}$$

The series converges when $2|x-\frac{1}{4}| < 1$, or when $|x-\frac{1}{4}| < \frac{1}{2}$. This series has radius of convergence $\frac{1}{2}$ and center $\frac{1}{4}$, and thus converges when $-\frac{1}{4} < x < \frac{3}{4}$. We need to check the endpoint of this interval.

When $x = -\frac{1}{4}$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n(4x-1)^n}{2^n\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{2^n\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{\cancel{2^n}(-1)^{2n}}{\cancel{2^n}\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}.$$

This series diverges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. (Let $a_n = \frac{1}{\sqrt{n+1}}$, $b_n = \frac{1}{\sqrt{n}}$ and show $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$. Then since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-series test, so does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$.)

When $x = \frac{3}{4}$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n(4x-1)^n}{2^n\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{\cancel{2^n}(-1)^n}{\cancel{2^n}\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}.$$

We have $b_n = \frac{1}{\sqrt{n+1}} > 0$. Furthermore,

- $b_{n+1} = \frac{1}{\sqrt{n+1}} \le \frac{1}{\sqrt{n}} = b_n$ for all $n \ge 1$,

- $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0.$

Thus, the series converges by the Alternating Series Test. The inteval of convergence is $(-\frac{1}{4}, \frac{3}{4}]$.

5. a) Find the 6th Taylor Polynomial of $f(x) = \sin(x)$ about $a = \frac{\pi}{2}$.

Solution: By definition, the 6th Taylor Polynomial of $f(x)$ about $x = a$ is given by

$$\begin{aligned} T_6(x) =& f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 \\ &+ \frac{f^{(4)}(a)}{4!}(x-a)^4 + \frac{f^{(5)}(a)}{5!}(x-a)^5 + \frac{f^{(6)}(a)}{6!}(x-a)^6. \end{aligned}$$

Here we have $a = \frac{\pi}{2}$ and we know $f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$. We compute the first 6 derivatives:

$$\begin{aligned}
f'(x) &= \cos(x) & &\Rightarrow f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0 \\
f''(x) &= -\sin(x) & &\Rightarrow f''\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1 \\
f^{(3)}(x) &= -\cos(x) & &\Rightarrow f^{(3)}\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) = 0 \\
f^{(4)}(x) &= \sin(x) & &\Rightarrow f^{(4)}\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \\
f^{(5)}(x) &= \cos(x) & &\Rightarrow f^{(5)}\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0 \\
f^{(6)}(x) &= -\sin(x) & &\Rightarrow f^{(6)}\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1
\end{aligned}$$

So,

$$\begin{aligned} T_6(x) &= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 \\ &= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x - \frac{\pi}{2}\right)^6. \end{aligned}$$

b) Write down the Taylor series expansion of $\sin(x)$ about $\frac{\pi}{2}$. Write out at least the first four terms for each series required below in addition to the general formula for the nth term; for example

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

Solution: Most of the work was done in part a. The key thing to recognize here is that the values $1, 0, -1, 0$ repeat. For all odd values of n, we have $f^{(n)}\left(\frac{\pi}{2}\right) = 0$. Furthermore, if n is even we have $n = 2k$ and

$$\begin{aligned} f^{(2k)}\left(\frac{\pi}{2}\right) &= \begin{cases} 1 & \text{if } k \text{ is even (i.e. if } 2k \text{ is divisible by 4);} \\ -1 & \text{if } k \text{ is odd.} \end{cases} \\ &= (-1)^k \end{aligned}$$

So, our general term is of the form $\frac{(-1)^k \left(x - \frac{\pi}{2}\right)^{2k}}{(2k)!}$. Thus,

$$\begin{aligned} \sin(x) &= 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{24}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{720}\left(x - \frac{\pi}{2}\right)^6 + \cdots + \frac{(-1)^k \left(x - \frac{\pi}{2}\right)^{2k}}{(2k)!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \left(x - \frac{\pi}{2}\right)^{2k}}{(2k)!} \end{aligned}$$