Math 10550 Final Exam Practice Problems - Part I

1 Functions

1. Sketch the following curves.
   (a) $y = x$, $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$, $y = x^n$ for even $n \geq 2$, $y = x^n$ for odd $n \geq 3$.
   (b) $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$, $y = \frac{1}{x^4}$, $y = \frac{1}{x^5}$, $y = \frac{1}{x^n}$ for even $n \geq 2$, $y = \frac{1}{x^n}$ for odd $n \geq 1$.
   (c) $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$, $y = \sqrt[5]{x}$, $y = \sqrt[n]{x}$ for even $n \geq 2$, $y = \sqrt[n]{x}$ for odd $n \geq 3$.
   (d) $y = \sin(x)$, $y = \cos(x)$, $y = \tan(x)$, $y = \sec(x)$, $y = \csc(x)$, $y = \cot(x)$.
   (e) $y = \arctan(x)$, $y = \arcsin(x)$, $y = \arccos(x)$.
   (f) $y = e^x$, $y = \ln(x)$.

2. Sketch the following curves.
   (a) $y = 3(x+1)^2 - 1$, $y = -(2x+1)^3 + 1$.
   (b) $y = \frac{1}{(x-1)^2(x+2)}$, $y = \frac{x+2}{(x+1)^3(x-3)^2}$.
   (c) $y = \sqrt[4]{3x^2 - 10}$, $y = 2\sqrt[3]{2x-1} + 3$.
   (d) $y = e^{x+2} - 2$, $y = \ln(x-1) + 1$.

2 Limits and Continuity

3. Compute the following limits.
   (a) $\lim_{x\to -3} \frac{x^3+27}{x+3}$
   (b) $\lim_{x\to 0} \frac{\sqrt{4-x^2}-\sqrt{4+x^2}}{x^2}$
   (c) $\lim_{x\to 0} \frac{\sin(4x)}{\tan(5x)}$
   (d) $\lim_{x\to 0} \frac{e^x-x-1}{\arctan(x^2)}$
   (e) $\lim_{x\to 0} \frac{\cos(x)-1}{\sin(x^2)}$

4. Find a and b so that the following function is continuous:

$$f(x) = \begin{cases} x^2 - 4 & \text{if } x < -3; \\ ax + b & \text{if } -3 \leq x < 0; \\ e^x + 2 & \text{if } 0 \leq x. \end{cases}$$

5. Give an example of a single function exhibiting all three types of discontinuity: removable, jump, and infinite.

6. Show that the polynomial $f(x) = -5x^4 + 3x^2 - 5x + 8$ has at least one root (you should not try to find this root).

7. Given positive numbers a and b, show that the equation $\frac{a}{x^3+2x^2-1} + \frac{b}{x^3+x-2} = 0$ has at least one solution in the interval $(-1, 1)$.

3 Derivatives

8. Compute the derivatives of the following functions from the definition.

   (a) $f(x) = x^2 - 3x + 1$

   (b) $g(x) = \sqrt{x}$

   (c) $h(x) = \frac{1}{x-3}$

   (d) $k(x) = \frac{1}{\sqrt{x}}$

   (e) $m(x) = x^{5/3}$

9. Find the tangent lines of the following functions at the indicated points.

   (a) $f(x) = x^3 - x - 3$ at $(2, 4)$

   (b) $g(x) = \ln(2x + e) - 1$ at $(0, 0)$

   (c) $h(x) = \frac{x^2-1}{2x-3}$ at $(1, 0)$

10. Find a, b, c, d so that the following function is differentiable:

$$f(x) = \begin{cases} x^2 + x + 1 & \text{if } x < -1; \\ ax^3 + bx^2 + cx + d & \text{if } -1 \le x < 0; \\ e^x - 1 & \text{if } 0 \le x. \end{cases}$$

11. Show that there do not exist a, b, c making the following function differentiable:

$$f(x) = \begin{cases} \cos(-2x) & \text{if } x < \frac{-\pi}{4}; \\ ax^2 + bx + c & \text{if } \frac{-\pi}{4} \le x < 1; \\ \ln(x) & \text{if } 1 \le x. \end{cases}$$

12. Give an example of a function which fails to be differentiable because of a cusp and one that fails to be differentiable because of a vertical tangent line.

13. Find the derivatives of the following functions.

   (a) $f(x) = \sin(x)\cos(5x)$

   (b) $g(x) = \sqrt{\ln(x-4)}$

   (c) $h(x) = e^{5x^2} \arctan(x)$

14. Use implicit differentiation and that $f(x) = \operatorname{arccot}(x)$ is the inverse of $g(x) = \cot(x)$ to find the derivative of f.

15. Find $\frac{dy}{dx}$ for each curve below.

   (a) $x^2 + y^2 = 4$

   (b) $xy^2 = 7x^4 + 2x + y$

   (c) $e^{xy} = xy + 1$

16. Find the tangent line to the curve $x^2 + xy + y^2 = x - y + 6$ at the point $(2, 1)$.

17. A particle is traveling along the curve $\frac{x^2}{4} + \frac{y^2}{12} = 1$. As the particle passes through the point $(\sqrt{2}, \sqrt{6})$, its velocity in the x-direction is $\sqrt{3}$ units/second, what is the velocity of the particle in the y-direction at that time?

18. Consider a tank of water obtained by revolving the region bounded by $y = x^2$ and $y = 4$ around the y-axis. If water is added to the tank at a rate of 1 m^3/min, how fast is the depth d of water in the tank increasing when $d = 1$?

19. If the vertex angle θ of an isosceles triangle is decreasing at a rate of 1 radian per second and the length ℓ of the legs is increasing at a rate of 3 centimeter per second, how fast is the length of the base changing when $\theta = \frac{\pi}{3}$ and $\ell = \sqrt{6}$?

20. Use linearizations to approximate $\sqrt{3.9}$, $\cos\left(\frac{\pi}{5}\right)$.

21. Use linearizations at $a = 1$ and $a = \sqrt{3}$ to approximate $\arctan\left(\frac{7}{5}\right)$. Argue using concavity for which gives the better approximation.