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M20580 L.A. and D.E. Tutorial
Worksheet 10
Sections 2.6, 3.1, 3.2, 3.3

1. Given the differential equation,

$$(y - 3x^2 + 4) + (x + 4y^3 - 2y)\frac{dy}{dx} = 0.$$

(a) Determine if the given differential equation is exact

$M = y - 3x^2 + 4$, $N = x + 4y^3 - 2y$

$\Rightarrow M_y = 1$, $N_x = 1$

Since $M_y = N_x$, the D.E. is exact.

(b) Find the solution of the differential equation above

Answer: $yx - x^3 + 4x + y^4 - y^2 = c$

We want to find $\psi(x,y)$ such that

$$\begin{cases} \psi_x = y - 3x^2 + 4 \\ \psi_y = x + 4y^3 - 2y \quad (2) \end{cases} \Rightarrow \psi(x,y) = \int (y - 3x^2 + 4)\, dx \Rightarrow \underbrace{\psi(x,y) = xy - x^3 + 4x + h(y)}_{(*)}$$

Then, from $(*)$, $\psi_y = x + h'(y) \overset{(2)}{=} x + 4y^3 - 2y$

$$\Rightarrow h'(y) = 4y^3 - 2y$$

$$\Rightarrow h(y) = y^4 - y^2$$

Thus, $\psi(x,y) = xy - x^3 + 4x + y^4 - y^2$

So, the solution to the given D.E is:

$\boxed{xy - x^3 + 4x + y^4 - y^2 = c}$ for any constant c.

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2. The differential equation

$$3y^2 - 4x(y^3+1) + xy(2-3xy)y' = 0$$

(a) is exact.

(b) is homogenenous. $\Rightarrow$ not second order, so cannot use the term homogeneous

~~(c)~~ has an integrating factor that is a function of x alone.

(d) has an integrating factor that is a function of y alone.

(e) None of the above.

If you choose (c) or (d), find the integrating factor of the given differential equation.

Let $M = 3y^2 - 4x(y^3+1)$ and $N = xy(2-3xy) = 2xy - 3x^2y^2$

$M_y = 6y - 12xy^2 \neq N_x = 2y - 6xy^2 \Rightarrow$ not exact.

Find integrating factor: compute $M_y - N_x = 6y - 12xy^2 - 2y + 6xy^2 = 4y - 6xy^2 = 2y(2-3xy)$

Note that $\dfrac{M_y - N_x}{N} = \dfrac{2y(2-3xy)}{xy(2-3xy)} = \dfrac{2}{x}$ is a function of x alone

So the integrating factor, μ, is a function of x alone. And $\mu(x)$ satisfies:

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \Leftrightarrow \frac{d\mu}{dx} = \frac{2}{x}\mu \Leftrightarrow \frac{d\mu}{\mu} = \frac{2}{x}dx$$

$$\Leftrightarrow \ln|\mu| = 2\ln|x| \Leftrightarrow e^{\ln|\mu|} = e^{2\ln|x|} \Leftrightarrow |\mu| = x^2 \Leftrightarrow \mu = \pm x^2$$

So, we can choose either $\mu = x^2$ or $\mu = -x^2$ to be the integrating factor for this problem

3. Using the Existence and Uniqueness Theorem for second order linear differential equations, find the maximal interval of existence of the solution to the initial value problem

$$(t^3 - 9t)y'' - 8ty' + (t+4)y = t^2 - 9, \qquad y(2) = 5, \quad y'(2) = -1.$$

$\hookrightarrow t_0 = 2$

$$\Rightarrow y'' \overbrace{- \frac{8t}{t^3 - 9t}}^{p(t)} y' + \overbrace{\frac{t+4}{t^3-9t}}^{q(t)} y = \overbrace{\frac{t^2-9}{t^3-9t}}^{g(t)}$$

$t^3 - 9t = 0 \Rightarrow t(t-3)(t+3) \Rightarrow t = 0$ or $t = -3$ or $t = 3$

So, $p(t)$, $q(t)$, $g(t)$ are continuous on the interval $(-\infty,-3) \cup (-3,0) \cup (0,3) \cup (3,\infty)$

but $t_0 = 2$ so, the interval we're looking for is $(0,3)$

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4. Find the solution of the initial value problem $y'' + 3y' + 2y = 0, \quad y(0) = 1,\ y'(0) = -1$

Answer: $y(t) = e^{-t}$

2nd-order linear eq'n, const coef

Characteristic eq'n: $r^2+3r+2=0 \Leftrightarrow (r+2)(r+1)=0 \Leftrightarrow r=-1,\ r=-2$ (real, different, roots)

So, $y(t) = c_1e^{-t} + c_2e^{-2t}$ $\quad (y'(t) = -c_1e^{-t} - 2c_2e^{-2t})$

$y(0)=1 \Rightarrow c_1 + c_2 = 1$

$y'(0)=-1 \Rightarrow -c_1 - 2c_2 = -1$ +

$-c_2 = 0 \Rightarrow c_1 = 1$

Thus, $\boxed{y(t) = e^{-t}}$

5. (a) Find the general solution to the differential equation $y'' - 4y' + 5y = 0$. (i.e., find $y(t) = c_1y_1(t) + c_2y_2(t)$ where y_1 and y_2 are solutions to the given differential equation. You don't have to find c_1 and c_2.)

characteristic equation: $r^2-4r+5=0 \Rightarrow r = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$

Thus, the general solution is:

$\boxed{y(t) = c_1e^{2t}\cos(t) + c_2e^{2t}\sin(t)}$

(b) If $y(t)$ is the solution to the initial value problem $y'' - 4y' + 5y = 0$, $y(0) = 0,\ y'(0) = 1$ then find $y(\pi/2)$.

Answer: e^{π}

From part (a), $y(t) = c_1e^{2t}\cos(t) + c_2e^{2t}\sin(t)$. Use initial conditions to find c_1, c_2

$y(0) = 0 \Leftrightarrow c_1 = 0 \Rightarrow y(t) = c_2e^{2t}\sin(t)$

$\Rightarrow y'(t) = 2c_2e^{2t}\sin(t) + c_2e^{2t}\cos(t)$

$y'(0) = 1 \Leftrightarrow c_2 = 1$

So, $y(t) = e^{2t}\sin(t)$

And $y(\frac{\pi}{2}) = e^{\pi}\sin(\frac{\pi}{2}) = \boxed{e^{\pi}}$