Name: *Solution* Date: 04/05/2018

M20580 L.A. and D.E. Tutorial
Worksheet 9
Sections 6.5, 1.1, 1.2

1. Let

$$A = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} \end{array}.$$

(a) Use Gram-Schmidt process to find a orthogonal basis for Col A, and use the orthogonal basis you get to find the QR factorization of A.

1. Get the orthogonal basis:

$$v_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} - \frac{15}{9}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

2. The columns of Q should be orthonormal: (Thm 12, pg 359)

$$u_1 = \frac{v_1}{|v_1|} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ 1/3 \end{pmatrix} \text{ and } u_2 = \begin{pmatrix} -1/3 \\ 2/3 \\ -2/3 \end{pmatrix} \text{ already a unit vector}$$

$$\Longrightarrow \boxed{Q = \begin{pmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{pmatrix}}$$

3. Find R. Since we want $A = QR$, $(Q^TQ)^{-1} \cdot (Q^TA) = R$, so we first compute

$$Q^TQ = \begin{pmatrix} 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{pmatrix}\begin{pmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Q^TA = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \boxed{R = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}} \quad \text{(since } Q^TQ \text{ was the identity matrix)}$$

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(b) Use the QR factorization you found in part(a) to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$, where $b = (7, 3, 1)$ (as a column vector).

We use the formula

$$\hat{x} = R^{-1} Q^T b$$

$$R^{-1} = \begin{pmatrix} 1/3 & -5/3 \\ 0 & 1 \end{pmatrix}, \text{ so}$$

$$\hat{x} = \begin{pmatrix} 1/3 & -5/3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$$

$$\boxed{\hat{x} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}}$$

2. Solve the initial value problem

$$\frac{dA}{dt} = 0.05A + 15, \quad A(0) = 0.$$

general solution:

$$\int \frac{dA}{0.05A + 15} = \int dt$$

$$20 \log(0.05A + 15) = t + C_1$$

$$\log(0.05A + 15) = t/20 + C_2$$

$$0.05A + 15 = C_3 e^{t/20}$$

$$0.05A = C_3 e^{t/20} - 15$$

$$A = C e^{t/20} - 300$$

particular sol:

$$t = 0 \Rightarrow C - 300 = 0$$

$$\boxed{A(t) = 300 e^{t/20} - 300}$$

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3. The partial differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

is called *Laplace's equation*, and any solution to this equation is called a *harmonic function*. Determine whether the given function is harmonic.

(a) $f(x, y) = x^2 + y^2$.

$f_x = 2x$ $\quad f_y = 2y$

$f_{xx} = 2$ $\quad f_{yy} = 2$ $\quad \Rightarrow f_{xx} + f_{yy} = 4 \neq 0$

not harmonic

(b) $f(x, y) = x^2 - y^2$.

$f_{xx} = 2$

$f_{yy} = -2$ $\quad \Rightarrow f_{xx} + f_{yy} = 0$, harmonic

(c) (optional) $f(x, y) = e^x \cos y$ and $g(x, y) = e^x \sin y$.

$f_x = f_{xx} = e^x \cos y$

$f_y = -e^x \sin y$ $\quad \Rightarrow f_{xx} + f_{yy} = 0$, harmonic.

$f_{yy} = -e^x \cos y$

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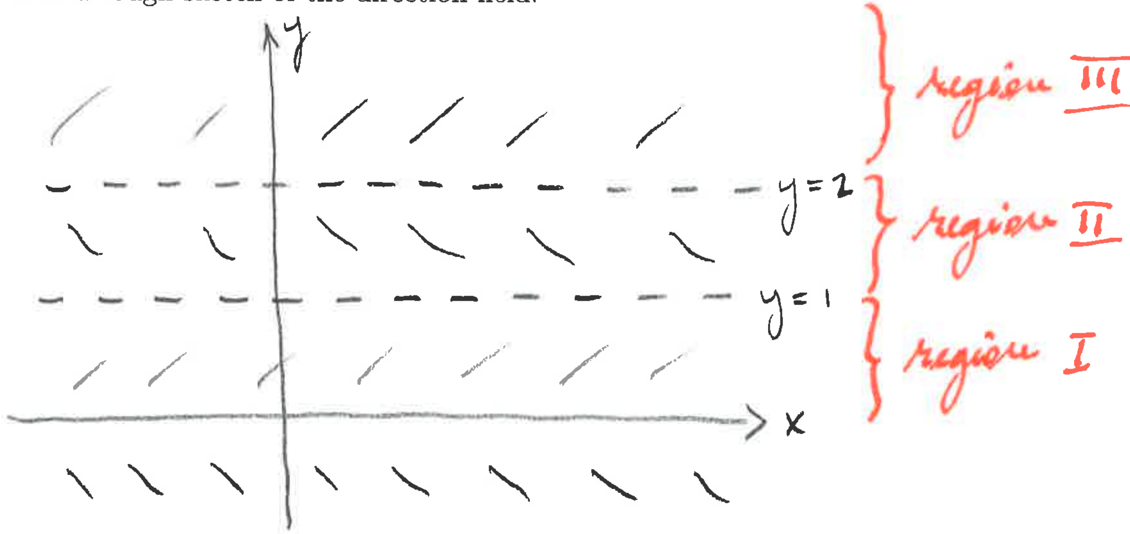
4. Consider the differential equation $\frac{dy}{dx} = y(y-1)(y-2)$, and $\phi(x)$ a solution for various initial conditions.

(a) Give a rough sketch of the direction field.



(b) Without using any integrals, find the general solution $\phi(x)$ for the initial condition $\phi(0) = 1$.

The only solution is $\boxed{\phi(x) = 1}$

(c) Compute $\lim_{x\to\infty} \phi(x)$ for any solution $\phi(x)$ satisfying the given initial condtion:

- $\phi(-1) = 0.5$

we are in region I, so $\boxed{\lim_{x\to\infty} \phi(x) = 1}$

- $\phi(1) = 1.5$

we are in region II, so $\boxed{\lim_{x\to\infty} \phi(x) = 1}$

- $\phi(5) = 3$

region III $\Rightarrow \boxed{\lim_{x\to\infty} \phi(x) = +\infty}$