Name: Solutions Date: 02/08/2018

M20580 L.A. and D.E. Tutorial
Worksheet 4
Sections 1.8–1.9, 2.1–2.2

1. (a) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and define a transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ by $T(\mathbf{x}) \doteq A\mathbf{x}$. Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.

$$T(\vec{u}) = T\left(\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \\ 15 \end{bmatrix}$$

(b) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a linear transformation. If

$$T(\mathbf{u}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T(\mathbf{v}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad T(\mathbf{w}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Find $T(\mathbf{x})$, where $\mathbf{x} = 2\mathbf{u} + 3\mathbf{v} - \mathbf{w}$.

$$T(\vec{x}) = T(2\vec{u} + 3\vec{v} - \vec{w}) = 2T(\vec{u}) + 3T(\vec{v}) - T(\vec{w})$$

$$= 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 9 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

2. (a) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation such that

$$T\left(\overset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad T\left(\overset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Find the *standard matrix* for T, i.e. find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

$$A = \Big[\underset{\substack{\Uparrow \\ 1^{st}\text{ column}}}{T(\vec{e}_1)} \quad \underset{\substack{\uparrow \\ 2^{nd}\text{ column}}}{T(\vec{e}_2)}\Big] \quad \text{where } \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since we know $T(\vec{e}_1) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, we have $\boxed{A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}}$

standard matrix for T

(b) Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that

$$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_3 \\ x_1 + x_2 + x_3 \end{bmatrix},$$

Find $S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$. Then find the standard matrix for S.

$$S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The standard matrix for S is given by $B = \left[S(\vec{e}_1) \quad S(\vec{e}_2) \quad S(\vec{e}_3)\right]$

$S(\vec{e}_1) = S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as above

$S(\vec{e}_2) = S\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$S(\vec{e}_3) = S\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\boxed{B = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}}$$

3. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Compute $(A+B)(A-B)^T$?

$A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A-B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow (A-B)^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$(A+B)(A-B)^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$

4. Which of the following equations involving 3×3-matrices A, B, C and I_3 (the identity matrix) *could* be *false* for some such matrices A, B, C ?

(a) $(A+B)^2 = A^2 + 2AB + B^2$

(b) $(A+B)C = AC + BC$ ✓ Part c, theorem 2, section 2.1

(c) $(AB)C = A(BC)$ ✓ Part a, theorem 2, section 2.1

(d) $A + B = B + A$ ✓ Part a, theorem 1, section 2.1

(e) $(I_3 + A)(I_3 - A) = I_3 - A^2$

For (e), $(I+A)(I-A) = I \cdot I - IA + AI - AA = I - A + A - A^2 = I - A^2$ ✓

For (a), $(A+B)(A+B) = AA + AB + BA + BB = A^2 + AB + BA + B^2$

Since AB needs not to be BA, $(A+B)^2$ might not be $A^2 + 2AB + B^2 \Rightarrow$ (a) is the correct answer.

5. Find the inverse of the matrix

$$Q = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 1 & 0 \\ 3 & 0 & 7 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 = R_3 - 3R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & -2 \end{array}\right] \xrightarrow{R_1 = R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 0 & 5 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & -2 \end{array}\right]$$

$$\boxed{Q^{-1} = \begin{bmatrix} -7 & 0 & 5 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \end{bmatrix}}$$