Math 10560 Quiz 3 Solutions

1. Evaluate the definite integral

$$\int_{-1}^{1} \arctan(x)dx$$

Note: $\arctan(x) = \tan^{-1}(x)$.

Solution: The long way to solve this problem is using integration by parts. The indefinite integral, when we let $u = \arctan(x)$ and $v' = 1$, becomes

$$\int \arctan(x)dx = x\arctan(x) - \int \frac{x}{1+x^2}dx = x\arctan(x) - \frac{1}{2}\ln|1+x^2| + C$$

Evaluating the antiderivative between -1 and 1 gives us

$$\left(\arctan(1) - \frac{1}{2}\ln(2)\right) - \left(-\arctan(-1) - \frac{1}{2}\ln(2)\right) = \frac{\pi}{4} - \frac{1}{2}\ln 2 + \frac{-\pi}{4} + \frac{1}{2}\ln 2 = 0$$

The short way to solve the problem is to realize that $\arctan(x)$ is an odd function, i.e $\arctan(x) = -\arctan(-x)$, meaning whatever (signed) area we accumulate on the interval $[-1, 0]$ (half our domain of integration) will be the negative of the area accumulated from $[0, 1]$, so they cancel out and the net area is zero.

2. Evaluate the limit

$$\lim_{x\to 0^+} \frac{1}{x^x}$$

Solution: Let $L = \lim_{x\to 0^+} x^{-x}$. We are looking to find the value of L. Assuming the limit exists, we proceed by taking natural log of both sides:

$$\ln L = \ln\left(\lim_{x\to 0^+} x^{-x}\right)$$

and since ln is a continuous function, it commutes with the limit to give us

$$\begin{aligned}\ln L &= \lim_{x\to 0^+} \ln(x^{-x})\\ &= \lim_{x\to 0^+} -x\ln(x)\end{aligned}$$

which is the $0\cdot\infty$ case for L'Hopital. We can turn this into ∞/∞ by bringing the x in the denominator as $1/x$:

$$\ln L = \lim_{x\to 0^+} \frac{\ln x}{-1/x}$$

and now we apply L'Hopital to get

$$\begin{aligned}\ln L &= \lim_{x\to 0^+} \frac{1/x}{1/x^2}\\ &= \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{x^2}{1} = 0\end{aligned}$$

so we obtain the relation $\ln L = 0$. Hence our limit $L = 1$.