WORKSHEET 3 SOLUTIONS

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Problem 1. Compute the following limits:

(a) $\lim_{x\to 4} \dfrac{x-4}{x^2-5x+4}$

Since the limit $\lim_{x\to 4} x^2 - 8x + 3 \neq 0$, we can simply plug in without getting a zero in the denominator. Notice the numerator will then be zero, so

$$\lim_{x\to 4} \frac{x-4}{x^2-8x+3} = 0$$

(b) $\lim_{x\to 3^-} \dfrac{x-4}{x^2-9}$

Factor the denominator as follows:

$$\lim_{x\to 3^-} \frac{x-4}{x^2-9} = \lim_{x\to 3^-} \frac{x-4}{(x+3)(x-3)} = \lim_{x\to 3^-} \frac{x-4}{x+3} \cdot \frac{1}{x-3}$$

and notice the $(x-4)/(x+3) \to -1/6$. However, as $x \to 3^-$, the term $(x-3)$ approaches zero from the negative side, making the second fraction approach $-\infty$, and so

$$\lim_{x\to 3^-} \frac{x-4}{x^2-9} = -\frac{1}{6} \cdot (-\infty) = +\infty$$

(c) $\lim_{x\to 3^-} \dfrac{\sqrt{5x}(x-3)}{|x-3|}$

As $x \to 3^-$, the inside of the absolute value $x - 3 < 0$, so $|x-3| = -(x-3)$. This means that as $x \to 3^-$,

$$\lim_{x\to 3^-} \frac{\sqrt{5x}(x-3)}{|x-3|} = \lim_{x\to 3^-} \frac{\sqrt{5x}(x-3)}{-(x-3)} = \lim_{x\to 3^-} -\sqrt{5x} = -\sqrt{15}$$

Problem 2.

(a) For which values of x is the following function continuous? (Justify your answer)

$$f(x) = \frac{|\cos x| + \sqrt{x-2}}{(x^2-9)(x^2+4)}$$

Let $g(x) = |\sin x| + \sqrt{x-2}$, and let $h(x) = (x^2-9)(x^2+4)$. Then our original function is $f(x) = \dfrac{g(x)}{h(x)}$, and it is defined at all points where g and h are defined, and $h(x) \neq 0$. The domain of g is $[2, \infty)$, and the domain of h is all of $\mathbb{R}$. However $h(x) = 0$ when $x = \pm 3$. Thus the domain of f is $[2, 3) \cup (3, \infty)$.

Since both g and f are continuous everywhere on this domain, f is continuous at every point in its domain as well.

(b) Find a value c that makes the following function continuous everywhere:

$$f(x) = \begin{cases} \frac{\sin(x)\cos(x)}{x} & \text{if } x \neq \frac{\pi}{4} \\ c & \text{if } x = \frac{\pi}{4} \end{cases}$$

We want the limit $\lim_{x\to\pi/4} f(x) = f(\pi/4)$. When computing the limit, we use the top branch, so

$$\lim_{x\to\frac{\pi}{4}} f(x) = \lim_{x\to\frac{\pi}{4}} \frac{\sin(x)\cos(x)}{x} = \frac{\sin \pi/4 \cos \pi/4}{\pi/4} = \frac{\left(\frac{1}{\sqrt{2}}\right)^2}{\frac{\pi}{4}} = \frac{2}{\pi}$$

which means $c = \frac{2}{\pi}$ makes f continuous everywhere.

(c) Suppose we have a continuous function $f(x)$ that satisfies $f(-1) = -1$ and $f(1) = 1$. Can this function have two zeroes inside the interval $(-1, 1)$? Justify. What can you say about the number of zeroes such a function can have inside $(-1, 1)$?

If f were to have two zeroes between -1 and 1, it would have to cross the x-axis twice (or bounce off the x-axis if we have a double root). So f starts out as negative ($f(-1) = -1$), crosses the axis at the first zero, then crosses the axis back at the second zero, ending up in the negative region again. This however contradicts continuity, because $f(1) = 1$ is positive, so it is impossible for f to have two zeroes.

However, f can have any ODD number of zeroes inside $(-1, 1)$.

Problem 3. Suppose that $\lim_{x\to 1} f(x) = 7$, $\lim_{x\to 1} g(x) = 4$, and $\lim_{x\to 1} h(x) = -\infty$. Compute the limit

$$\lim_{x\to 1}\left(f(x) + \frac{1}{g(x) - h(x)}\right)$$

Since $f \to 7$ and $g \to 4$, we have

$$\lim_{x\to 1}\left(f(x) + \frac{1}{g(x) - h(x)}\right) = 7 + \lim_{x\to 1}\frac{1}{4 - h(x)}$$

and $h \to -\infty$ in the denominator, making the entire fraction approach zero. Hence

$$\lim_{x\to 1}\left(f(x) + \frac{1}{g(x) - h(x)}\right) = \lim_{x\to 1} f(x) = 7$$

Problem 4. Suppose that $\dfrac{\sqrt{x^2+9}-3}{2x^2} \le f(x) \le \frac{1}{12}$ for all $x \neq 0$. Compute $\lim_{x\to 0} f(x)$.

The idea is to use the Squeeze/Sandwich theorem. For the lower (left) bound for f, we have

$$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{2x^2} = \lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{2x^2}\cdot\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} = \lim_{x\to 0}\frac{(x^2+9)-3^2}{2x^2(\sqrt{x^2+9}+3)}$$

which becomes

$$\lim_{x\to 0}\frac{x^2}{2x^2(\sqrt{x^2+9}+3)} = \lim_{x\to 0}\frac{1}{2(\sqrt{x^2+9}+3)} = \frac{1}{2\sqrt{9}+6} = \frac{1}{12}$$

The upper (right) bound coincides with the lower bound, so we conclude from the Squeeze Theorem that

$$\lim_{x\to 0} f(x) = \frac{1}{12}$$

Problem 5. Argue without solving for x that there are at least two solutions to the equation $-x^4 + 3x + 2 = 0$. (Hint: use continuity of polynomials and IVT).

Notice that f is defined and continuous everywhere on $\mathbb{R}$ sine it is a polynomial. We have

$$f(-1) = -2$$
$$f(0) = 2$$
$$f(2) = -8$$

so by the IVT there msut be at least one zero of f on $(-1, 0)$, and at least one more zero on $(0, 2)$. Hence f must have at least two zeroes.

Problem 6. Compute the following limits:

(a) $\lim_{x\to\pi^+} \dfrac{\sqrt[3]{\cos x}}{x-\pi}$

As $x \to \pi^+$, the numerator approaches the finite value of $\sqrt[3]{-1} = -1$. However, the denominator $x - \pi$ is positive and approaches zero (from the positive side), so

$$\lim_{x\to\pi^+} \frac{\sqrt[3]{\cos x}}{x-\pi} = \lim_{x\to\pi^+} \frac{-1}{x-\pi} = -\infty$$

(b) $\lim_{x\to\infty} \cos\left(\dfrac{1}{x}\right)$

Since $\lim_{x\to\infty} \frac{1}{x} = 0$, and cosine is continuous at $x = 0$, we have

$$\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1$$

(c) $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^3-x}\right)$

We know that cosine lies between -1 and 1, i.e.

$$-1 \leq \cos\left(\frac{1}{x^3-x}\right) \leq 1$$

and multiplying by x^2 gives us

$$-x^2 \leq x^2\cos\left(\frac{1}{x^3-x}\right) \leq +x^2$$

Since as $x \to 0$, both the lower $(-x^2)$ and upper $(+x^2)$ bounds go to zero, it follows that

$$\lim_{x\to 0} x^2\cos\left(\frac{1}{x^3-x}\right) = 0$$