10350 Tutorial Week 15 - Set 01 Name ______________________

1. A student wishes to use Newton's method to estimate the value of $\sqrt{3}$ by considering the solution of $x^2 = 3$. If the initial guess $x_0 = 2$, find the values of the next two iterates x_1 and x_2. Fill in your answers below.

Solution: We are trying to find the roots of the equation $f(x) = x^2 - 3 = 0$. We have $f'(x) = 2x$. Recall that Newton's method says that each successive approximation is given by the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We are using the initial guess $x_0 = 2$. The first two iterates are given by:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{2^2 - 3}{2 \cdot 2} = 2 - \frac{1}{4} = \frac{7}{4} = 1.75$$

$$x_2 = x_2 - \frac{f(x_1)}{f'(x_1)} = \frac{7}{4} - \frac{\left(\frac{7}{4}\right)^2 - 3}{2 \cdot \frac{7}{4}} = \frac{7}{4} - \frac{1/16}{7/2} = \frac{7}{4} - \frac{1}{56} = \frac{7 \cdot 14 - 1}{56} = \frac{97}{56} \approx 1.732$$

Answers: $x_1 =$ 1.75 $x_2 =$ 1.732

2. Perform each of the following indefinite integrals. If substitution is needed show all steps carefully.

a. $\displaystyle\int \frac{x^3 - 10x^2 + x - 5}{x^2}\, dx \stackrel{?}{=}$

Solution:

$$\begin{aligned}\int \frac{x^3 - 10x^2 + x - 5}{x^2}\, dx &= \int \left(x - 10 + \frac{1}{x} - 5x^{-2}\right) dx\\ &= \int x\, dx - \int 10\, dx + \int \frac{1}{x}\, dx - \int 5x^{-2}\, dx\\ &= \frac{1}{2}x^2 - 10x + \ln|x| + \frac{5}{x} + C\end{aligned}$$

b. $\displaystyle\int \frac{e^{-x} + e^{x+2}}{e^x}\, dx \stackrel{?}{=}$

Solution:

$$\begin{aligned}\int \frac{e^{-x} + e^{x+2}}{e^x}\, dx &= \int (e^{-2x} + e^2)\, dx\\ &= \int e^{-2x}\, dx + \int e^2\, dx\end{aligned}$$

Now, we can use the substitution $u = -2x$, $du = -2dx$ (so $dx = -\frac{1}{2}du$) to show:

$$\int e^{-2x}\, dx = \int -\frac{1}{2}e^u\, du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-2x} + C.$$

Thus,

$$\begin{aligned}\int \frac{e^{-x} + e^{x+2}}{e^x}\, dx &= \int e^{-2x}\, dx + \int e^2\, dx\\ &= -\frac{1}{2}e^{-2x} + e^2x + C.\end{aligned}$$

3. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{\sqrt{\pi}}{\sqrt{x}} + \sin x$ if $y(\pi) = 3$.

Solution: This is an example of a **separable** differential equation.

$$\begin{aligned}
dy &= \left(\frac{\sqrt{\pi}}{\sqrt{x}} + \sin x\right)\, dx && \text{(Separate the Variables)}\\
\int dy &= \int \left(\frac{\sqrt{\pi}}{\sqrt{x}} + \sin x\right)\, dx && \text{(Integrate Both Sides)}\\
\Rightarrow y &= \int (\sqrt{\pi}\, x^{-1/2} + \sin x)\, dx\\
&= \sqrt{\pi}\,\frac{x^{1/2}}{1/2} - \cos x + C\\
&= 2\sqrt{\pi x} - \cos x + C
\end{aligned}$$

Using our initial condition $y(\pi) = 3$, we can solve for the constant C.

$$\begin{aligned}
y(\pi) = 2\sqrt{\pi^2} - \cos\pi + C &= 3\\
\Rightarrow 2\pi - (-1) + C &= 2\\
\Rightarrow 2\pi + 1 + C &= 3\\
\Rightarrow C &= 2 - 2\pi
\end{aligned}$$

So, $\boxed{y = 2\sqrt{\pi x} - \cos x + 2 - 2\pi}$.

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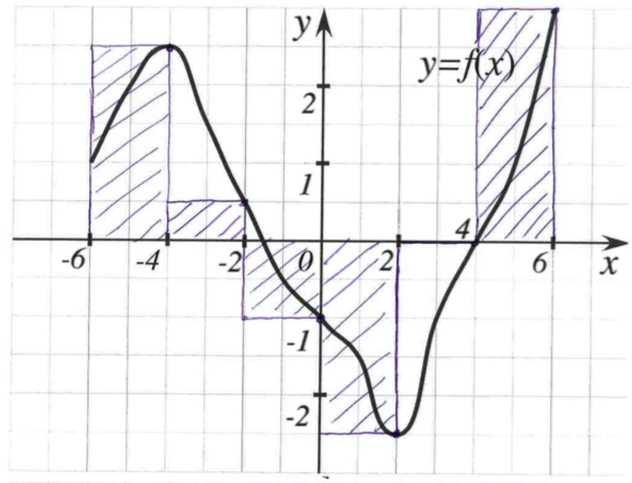
4. Estimate the value of $\int_{-6}^{6} f(x)dx$ using the following method.

a. Right end-point with **6 equal subintervals**.

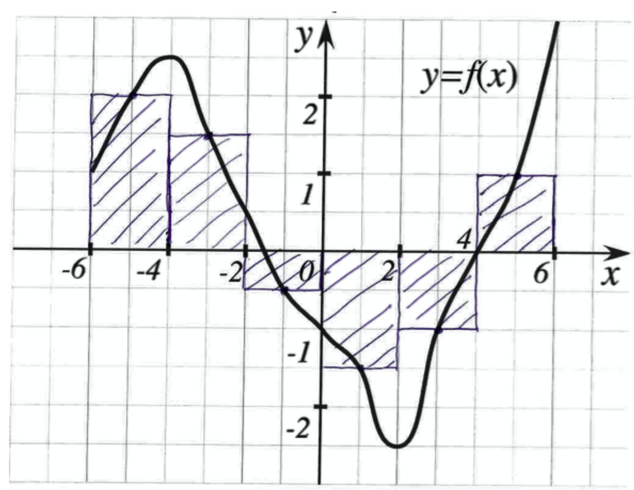
Solution: We first need to divide the interval $[-6, 6]$ into 6 equal subintervals. The length of each of these subintervals is given by $\Delta x = \frac{b-a}{N} = \frac{6-(-6)}{6} = \frac{12}{6} = 2$. The right endpoints are given by $x_i = a + i\Delta x = -6 + 2i$ for i from 1 to 6 (i.e $x_1 = -4, x_2 = -2, ..., x_6 = 6$). This approximation is illustrated by the graphic on the right.



$$\begin{aligned}
\int_{-6}^{6} f(x)d \approx R_6 &= \Delta x \sum_{i=1}^{6} f(x_i) \\
&= 2[f(-4) + f(-2) + f(0) \\
&\quad + f(2) + f(4) + f(6)] \\
&= 2(\cancel{2.5} + 0.5 - 1 - \cancel{2.5} + 0 + 3) \\
&= 2(2.5) \\
&= \boxed{5}
\end{aligned}$$

b. Midpoint Rule with **6 equal subintervals**.

The drawing on the right shows the rectangles used in computing the midpoint rule with 6 equal subintervals, M_6. Again, $\Delta x = 2$, $x_i = a + \Delta x = -6 + 2i$. We are now evaluating the function at the midpoint of each of our subintervals $[x_i, x_{i+1}]$.



$$\begin{aligned}
\int_{-6}^{6} f(x)d \approx M_6 &= \Delta x \sum_{i=0}^{5} f\left(\frac{x_i + x_{i+1}}{2}\right) \\
&= 2[f(-5) + f(-3) + f(-1) \\
&\quad + f(1) + f(3) + f(5)] \\
&= 2(2 + \cancel{1.5} - 0.5 - \cancel{1.5} - \cancel{1} + \cancel{1}) \\
&= 2(1.5) \\
&= \boxed{3}
\end{aligned}$$

5. A particle on a straight line is moving with acceleration function

$$a(t) = \sin(t) + 12t^2 \qquad \text{m/s}^2$$

If initially the particle's velocity is 3 m/s and position is -1 m, find the velocity function and position function. This is the Initial Value Problem

$$s''(t) = a(t) = \sin(t) + 12t^2, \quad s'(0) = v(0) = 3, \quad s(0) = -1.$$

Velocity Function:

$$\begin{aligned} v(t) = s'(t) &= \int a(t)\, dt \\ &= \int \sin(t) + 12t^2\, dt \\ &= -\cos(t) + 4t^3 + C \end{aligned}$$

We use the initial condition v(0) = 3 to solve for the constant C.

$$\begin{aligned} v(0) &= -\cos(0) + 4(0)^3 + C = 3 \\ &\Rightarrow -1 + C = 3 \\ &\Rightarrow \quad C = 4 \end{aligned}$$

So the velocity function is given by $\boxed{v(t) = -\cos(t) + 4t^3 + 4}$.

Position Function:

$$\begin{aligned} s(t) = &= \int v(t)\, dt \\ &= \int -\cos(t) + 4t^3 + 4\, dt \\ &= -\sin(t) + t^4 + 4t + C \end{aligned}$$

We use the initial condition s(0) = -1 to solve for the constant C.

$$v(0) = -\sin(0) + (0)^4 + 4(0) + C = -1 \Rightarrow C = -1$$

So the position function is given by $\boxed{s(t) = -\sin(t) + t^4 + 4t - 1}$.

6. Given that $\int_0^3 f(x)dx = -2$ and $\int_0^9 f(x)dx = 10$, find

a. $\int_3^9 f(x)dx =$

Solution:

$$\int_3^9 f(x)dx = \int_0^9 f(x)dx - \int_0^3 f(x)dx = 10 - (-2) = \boxed{12}$$

b. $\int_0^3 [5 - 2f(x)]dx =$

Solution:

$$\begin{aligned} \int_3^9 f(x)dx &= \int_0^3 5\ dx - 2\int_0^3 f(x)\ dx \\ &= 5x|_0^3 - 2(-2) \\ &= (15 - 0) + 4 \\ &= \boxed{19} \end{aligned}$$