Name: ______________________________ July 14, 2017

FINITE MATH: EXAM 2 SOLUTION

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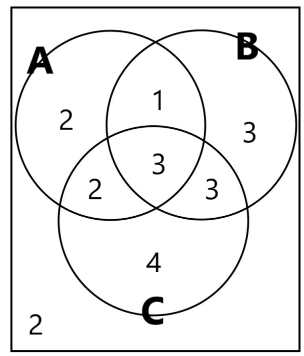
Problem 1. Consider the Venn diagram on the right of a sample space S with **equally likely outcomes**. For each region, the diagram lists the number of outcomes inside that region.



a) (2pt) Find $P(A)$. Simplify the fraction to lowest terms.

$$n(A) = 8, \text{ and } n(S) = 20, \text{ so } \boxed{P(A) = \frac{8}{20} = \frac{2}{5}}$$

b) (2pt) Find $P(B)$. Simplify the fraction to lowest terms.

$$n(B) = 10, \text{ and } n(S) = 20, \text{ so } \boxed{P(B) = \frac{10}{20} = \frac{1}{2}}$$

d) (2pt) Find $P(A \cap B)$. Simplify the fraction to lowest terms.

$$n(A \cap B) = 4, \text{ so } \boxed{P(A \cap B) = \frac{4}{20} = \frac{1}{5}}$$

e) (2pt) Are the events A and B independent?

Notice that $P(A \cap B) = \frac{1}{5}$ and $P(A) \cdot P(B) = \frac{2}{5} \cdot \frac{1}{2} = \frac{1}{5}$, so $\boxed{\text{the events are independent}}$

f) (1pt) Compute $P(A \cup B \cup C)'$.

Notice $(A \cup B \cup C)'$ describes the region outside all the circles, which contains 4 elements. Hence $\boxed{P(A \cup B \cup C)' = \frac{2}{20} = \frac{1}{10}}$

Problem 2. A coin is flipped 10 times in a row, and the resulting H/T sequence is recorded.

a) (1pt) How many possible outcomes are there in the sample space?

There are $\boxed{2^{10}}$ possible outcomes.

b) (2pt) What is the probability of getting exactly 5 Heads?

Since there are 5 heads, there must be 5 tails, so our starting point is the number $(0.5)^5$ (for the 5 heads) times $(0.5)^5$ (for the tails). However, this can occur in $\binom{10}{5}$ ways (the number of possible configurations where the heads appear), so the probability we seek is $\boxed{\binom{10}{5} \cdot (0.5)^5 \cdot (0.5)^5}$

c) (2pt) What is the probability of getting at least one Tail?

We use the complement principle. The chance of getting zero tails is $(0.5)^{10}$ (heads on each of the ten tosses), so the chance of getting at least one tail is $\boxed{1-(0.5)^{10}}$

Problem 3. A classroom is split into two separate groups, G_1 and G_2. There are 3 men and 5 women in G_1, and there are 3 men and 13 women in G_2. The teacher picks one of the two groups at random, with $P(G_1) = 0.3$ and $P(G_2) = 0.7$, then randomly selects a student from that group.

a) (3pt) What is the probability that the chosen student is a woman?

The woman can come from G_1 or G_2, and these are disjoint scenarios. We have $\boxed{P(W) = (0.3)\cdot\frac{5}{8} + (0.7)\cdot\frac{13}{16}}$

b) (1pt) Given that the selected group is G_1, what is the probability that the student is a woman?

Since we are only considering the population from G_1, we have $\boxed{P(W|G_1) = \frac{5}{8}}$

c) (4pt) The selected student is a woman. What is the probability that she is from the first group?

We need to use Bayes' Theorem. We have the reverse probability:

$$\boxed{P(G_1|W) = \frac{P(W\cap G_1)}{P(W)} = \frac{(0.3)\cdot\frac{5}{8}}{(0.3)\cdot\frac{5}{8} + (0.7)\cdot\frac{13}{16}}}$$

Problem 4. (3pt) You have 5 people, $\{A, B, C, D, E\}$, seated in a **single row** at random. What is the probability that person E ends up sitting next to person A? For example, some different such arrangements are BAEDC, BEADC, CDAEB, CDEAB, etc...

There are 5! ways to arrange the 5 people in a row. If we want A,E to sit together, it's best to treat them as a single unit. So we really have to arrange (AE)(B)(C)(D), 4 total objects. There are 4! ways to do so. We still need to account for the fact that AE and EA are different setups, so we multiply by 2. The probability is $\boxed{\frac{2\cdot 4!}{5!} = \frac{2}{5}}$

Problem 5. (2pt) The sample space $S = \{1, 2, 3, 4\}$ contains equally likely outcomes. Consider the events $A = \{1, 2\}$ and $B = \{1, 3\}$. Are A and B independent?

$P(A) = 1/2$ and $P(B) = 1/2$. Also, $A \cap B = \{1\}$ which occurs with probability $1/4$, and this is equal to the product of $P(A)$ and $P(B)$. Hence $\boxed{\text{the events are independent.}}$

Problem 6. The probabilities for an unbalanced six-sided die are given below:

Outcome	1	2	3	4	5	6
Probability	0.1	0.1	0.3	0.2	0.1	0.2

You **roll two** of these unbalanced dice, and observe the sum of the numbers that come up. For example, (2,1) and (1,2) are different outcomes for which the sum is 3.

a) (2pt) What is the probability that the sum of the two numbers is 1?

We can never get a sum of 1 when rolling two of these dice, so the probability is $\boxed{\text{zero.}}$

b) (2pt) What is the probability that the sum of the two numbers is 2?

The only way this can happen is when we roll $(1, 1)$. Each roll in independent, so the probability is $\boxed{P(1, 1) = (0.1)(0.1) = 0.01}$

c) (3pt) What is the probability that the sum of the two numbers is 4?

The outcomes which correspond to a sum of 4 are (1,3), (2,2), and (3,1). The probability we want is $\boxed{P(1, 3) + P(2, 2) + P(3, 1) = (0.1)(0.3) + (0.1)(0.1) + (0.3)(0.1) = 0.07}$

d) (3pt) What is the probability that the sum of the two numbers is 7?

The outcomes which sum to 7 are (1,6), (2,5), (3,4), (4,3), (5,2), (6,1). The probability we want is $\boxed{(0.1)(0.2) + (0.1)(0.1) + (0.3)(0.2) + (0.2)(0.3) + (0.1)(0.1) + (0.2)(0.1)}$ which is 0.18.

Problem 7. A child forms **3-letter words** by picking letters from $\{A, B, C, D, E\}$ at random. **Letters may be repeated**.

a) (1pt) What is the probability that the word starts with A? Simplify your answer.

$$\boxed{\frac{1 \cdot 5 \cdot 5}{5 \cdot 5 \cdot 5} = \frac{1}{5}}$$

b) (1pt) What is the probability that "DA" appears somewhere in the word? (the other letter can be anything)

DA can appear at the beginning (DA$*$), or at the end ($*$DA), and these are disjoint scenarios. The probability we want is $\boxed{\frac{1 \cdot 1 \cdot 5}{5 \cdot 5 \cdot 5} + \frac{5 \cdot 1 \cdot 1}{5 \cdot 5 \cdot 5} = \frac{2}{25}}$

c) (1pt) What is the probability that the word begins with a consonant, has a vowel for the middle letter, and ends with consonant?

$$\boxed{\frac{3 \cdot 2 \cdot 3}{5 \cdot 5 \cdot 5}}$$

Problem 8. A mathematics professor assigns two problems for homework and knows that the probability of a student solving the first problem is 0.50, the probability of solving the second is 0.60, and the probability of solving both is 0.30.

a) (1pt) Are the events independent?

Notice that $(0.5)(0.6) = 0.3$, so $\boxed{\text{the events are independent.}}$

b) (2pt) A randomly chosen student has solved the second problem. What is the probability he also solves the first problem?

Due to independence, we have $\boxed{P(P_1|P_2) = P(P_1) = 0.5}$

c) (2pt) A randomly chosen student has solved the first problem. What is the probability she also solves the second problem?

Again, due to independence we have $\boxed{P(P_2|P_1) = P(P_2) = 0.6}$

Problem 9. (4pt) A child has 1 Red and 4 White marbles in his left pocket, and 2 Red and 2 White marbles in his right pocket. He transfers a marble (at random) from his left pocket to his right pocket. After the transfer, he picks (at random) a marble from his right pocket. What is the probability of picking a White marble from his right pocket?

There are two disjoint scenarios. First, it's possible that the transfered marble is red, giving us $(1/5)(3/5)$ (as now we have 3 red marbles in the right pocket to choose from, and 5 total marbles instead of 4). Second, it's possible that the transfered marble is white, giving us $(4/5)(2/5)$. The probability of picking a red marble is going to be $\boxed{P(R) = \frac{1}{5} \cdot \frac{3}{5} + \frac{4}{5} \cdot \frac{2}{5} = 0.44}$

Problem 10. (2pt) A crate contains 20 total apples, 4 of which are spoiled. You select 4 apples at random. What is the probability that at least one apple is bad?

The easiest method is to use the complement principle: $\boxed{1 - P(\text{all good}) = 1 - \frac{\binom{16}{4}}{\binom{20}{4}} \approx 0.624}$

Problem 11. (2pt) A child has eight cards numbered $\{2, 3, 4, \ldots, 9\}$. He creates **4-digit** numbers by randomly picking 4 of the 8 cards, and arranging them in some order. What is the probability that the number he obtains is smaller than 5000?

Since the numbers are written on cards, it's not possible to have repeats. The size of the sample space is $8 \cdot 7 \cdot 6 \cdot 5$. The only way to get a number less than 5000 is if the beginning digit is smaller than 5, so the beginning digit can only be $\{2, 3, 4\}$ (3 choices). The remaining 3 digits can be anything. The probability we want is going to be $\boxed{\frac{3 \cdot 7 \cdot 6 \cdot 4}{8 \cdot 7 \cdot 6 \cdot 5} = \frac{3}{8} = 0.375}$

Problem 12. (4pt) An **unfair coin** has probability of Heads equal to $P(H) = 0.6$. If you toss the coin 10 times, what is the probability of getting exactly 5 Tails?

Since we want 5 tails, the rest have to be 5 heads. This gives us $(0.4)^5 \cdot (0.6)^5$. However, we need to multiply by the number of ways in which 5 tails and 5 heads can appear, which is $\binom{10}{5}$. The probability then becomes $\boxed{\binom{10}{5} \cdot (0.4)^5 (0.6)^5 \approx 0.20}$

Problem 13. (2pt) A sample space contains 100 equally likely outcomes. Given that

$$n(E \setminus F) = 40 \qquad n(F \setminus E) = 10 \qquad n(E \cap F) = 10$$

are the events E and F independent?

Notice that $n(E) = 40 + 10 = 50$, and similarly $n(F) = 20$. This means $P(E) = 0.5$ and $P(F) = 0.2$. When we multiply the two, we get 0.1, and this is exactly $P(E \cap F)$, so $\boxed{\text{the events are independent.}}$

Problem 14. Three inspectors look at a critical component of a rocket. Their probabilities for finding a defect are 0.95, 0.90, and 0.80. Each inspector operates independently of the rest.

a) (2pt) What is the probability that **all three** inspectors find a defect?

Using independence, we have $\boxed{(0.95)(0.90)(0.80) = 0.684}$

b) (3pt) What is the probability that **none** of the inspectors find a defect?

We just use the complement principle for each inspector independently, and get $\boxed{(1-0.95)(1-0.90)(1-0.80) = (0.05)(0.10)(0.20) = 0.001}$

c) (3pt) What is the probability that **only one** of the inspectors finds a defect (and the other two don't)?

There are 3 disjoint scenarios. Either the first inspector finds a defect (and the other 2 don't), or the second finds a defect (and the other 2 don't), or the third. Using the addition principle, we have $\boxed{(0.95)(0.10)(0.20) + (0.05)(0.90)(0.20) + (0.05)(0.10)(0.80)}$

Problem 15. (4pt) A bag contains 3 Red and 2 White marbles. A second bag contains an unknown number of Red marbles, and 1 White marbles:

$$\underbrace{\boxed{3R \quad 2W}}_{\text{Bag 1}} \qquad\qquad \underbrace{\boxed{?R \quad 1W}}_{\text{Bag 2}}$$

A marble is drawn at random from each bag, and the probability of getting two marbles of the same color is 8/15. How many red marbles are in the second bag?

Let x be the (unknown) number of red marbles in the second bag. Drawing 2 marbles of the same color can happen in 2 ways: either both are red, or both are white. The probability of drawing both red is $\frac{3}{5} \cdot \frac{x}{x+1}$, and the probability of drawing both white is $\frac{2}{5} \cdot \frac{1}{x+1}$. This gives the equation

$$\frac{3}{5} \cdot \frac{x}{x+1} + \frac{2}{5} \cdot \frac{1}{x+1} = \frac{8}{15}$$

and all we need to do is solve for x. First we multiply both sides by 5 and combine the two fractions to get

$$\frac{3x+2}{x+1} = \frac{8}{3}$$

Next we cross multiply and obtain

$$9x + 6 = 8x + 8$$

Finally we are able to solve for x, so the number of red marbles in the second bag is $\boxed{x = 2}$

Problem 16. (3pt) Suppose $P(A) = 0.6$ and $P(A \cap B) = 0.3$. If the events A and B are known to be independent, what is $P(B)$?

Since the events are independent, $P(A)P(B) = P(A \cap B)$, so

$$\boxed{P(B) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.6} = 0.5}$$

Problem 17. (3pt) A single card is drawn at random from six different decks (so you end up with six cards total). What is the probability that all six cards are different?

The number of possible outcomes is 52^6, and we are interested in the scenario of all cards being different, so the probability we seek is $\boxed{\dfrac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47}{52^6} \approx 0.761}$

Problem 18. (2pt) A small grocery store has 7 cartons of milk left, 2 of which are sour. If you are going to buy the **second** carton of milk sold that day at random, what is the probability of selecting a sour carton of milk?

There are two disjoint scenarios, and they depend on what happens with the buyer that comes before you. It may be the case that the buyer before gets a good carton of milk, and you get a sour one: $(5/7)(2/6)$. It may also be the case that the buyer before you gets a sour milk, and you get a sour milk: $(2/7)(1/6)$. Hence the probability we want is

$$\boxed{\frac{5}{7} \cdot \frac{2}{6} + \frac{2}{7} \cdot \frac{1}{6} \approx 0.286}$$