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M20580 L.A. and D.E. Tutorial
Worksheet 8
Sections 6.1, 6.2, 6.3, 6.4

1.

$$A = \begin{bmatrix}1&0&0\\1&1&0\\0&1&1\\0&0&1\end{bmatrix}$$

Denote $\alpha_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$, $\alpha_2 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$, $\alpha_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, these are the three columns of A. Let $W = Span\{\alpha_1, \alpha_2, \alpha_3\}$.

(a) Find a basis for $W^\perp$.

According to THM 3 in 6.1, $W^\perp$ is the null space of A^T, i.e. the solution set to $A^TX=0$.

$$A^T = \begin{bmatrix}1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix} \xrightarrow{R_2-R_3} \begin{bmatrix}1&1&0&0\\0&1&0&-1\\0&0&1&1\end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&1\end{bmatrix}$$

$\Rightarrow$ A basis for $W^\perp$ can be taken as $\left\{\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\right\}$.

(b) Check your answer in (a), i.e. each vector in your basis for $W^\perp$ is perpendicular to every α_i $(i = 1, 2, 3)$.

Let $\alpha_4 = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$, then $W^\perp = Span\{\alpha_4\}$.

Check: $\alpha_4\cdot\alpha_1 = 1\cdot 1 + (-1)\cdot 1 + 1\cdot 0 + (-1)\cdot 0 = 0$

$\alpha_4\cdot\alpha_2 = 1\cdot 0 + (-1)\cdot 1 + 1\cdot 1 + (-1)\cdot 0 = 0.$

$\alpha_4\cdot\alpha_3 = 1\cdot 0 + (-1)\cdot 0 + 1\cdot 1 + (-1)\cdot 1 = 0$

(c) Use Gram-Schmidt process to find an orthogonal basis for $W = Span\{\alpha_1, \alpha_2, \alpha_3\}$. **You need not normalize your basis.**

$$\gamma_1 = \alpha_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix}$$

$$\gamma_2 = \alpha_2 - \frac{\alpha_2\cdot\alpha_1}{\alpha_1\cdot\alpha_1}\alpha_1 = \begin{bmatrix}0\\1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\1\\0\end{bmatrix}$$

$$\gamma_3 = \alpha_3 - \frac{\alpha_3\cdot\gamma_2}{\gamma_2\cdot\gamma_2}\gamma_2 - \frac{\alpha_3\cdot\gamma_1}{\gamma_1\cdot\gamma_1}\gamma_1$$

$$= \begin{bmatrix}0\\0\\1\\1\end{bmatrix} - \frac{1}{\frac{3}{2}}\cdot\begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\1\\0\end{bmatrix} - \frac{0}{\gamma_1\cdot\gamma_1}\cdot\gamma_1$$

$$= \begin{bmatrix}0\\0\\1\\1\end{bmatrix} + \begin{bmatrix}\frac{1}{3}\\-\frac{1}{3}\\-\frac{2}{3}\\0\end{bmatrix} - \vec{0} = \begin{bmatrix}\frac{1}{3}\\-\frac{1}{3}\\\frac{1}{3}\\1\end{bmatrix}.$$

For the convenience of further calculation, we may take $\left\{ \beta_1 = \gamma_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix},\ \beta_2 = 2\gamma_2 = \begin{bmatrix}-1\\1\\2\\0\end{bmatrix},\ \beta_3 = 3\gamma_3 = \begin{bmatrix}1\\-1\\1\\3\end{bmatrix} \right\}$ as orthogonal basis.

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(d) Let $\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Find the orthogonal projection of β onto W, i.e. $proj_W \beta$, using the orthogonal basis you've found in (c).

$$proj_W \beta = \frac{\beta \cdot \beta_1}{\beta_1 \cdot \beta_1} \beta_1 + \frac{\beta \cdot \beta_2}{\beta_2 \cdot \beta_2} \beta_2 + \frac{\beta \cdot \beta_3}{\beta_3 \cdot \beta_3} \beta_3.$$

where $\left\{ \beta_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \beta_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \ \beta_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix} \right\}$ is the orthogonal basis we've found in (c).

Calculate: $$proj_W \beta = \frac{1}{2} \beta_1 + \frac{-1}{6} \beta_2 + \frac{1}{12} \beta_3$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{6} \\ -\frac{1}{3} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{12} \\ -\frac{1}{12} \\ \frac{1}{12} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

(e) Find the orthogonal projection of β onto $W^{\perp}$, i.e. $proj_{W^{\perp}} \beta$, using an orthogonal basis for $W^{\perp}$.

We know that $proj_{W^{\perp}} \beta = \beta - proj_W \beta$, so you may solve this part using (d). But I suggest you to calculate $proj_{W^{\perp}} \beta$ by again using orthogonal projection formula, so that you can practice the formula again.

$$proj_{W^{\perp}} \beta = \frac{\beta \cdot \alpha_4}{\alpha_4 \cdot \alpha_4} \cdot \alpha_4 \qquad \alpha_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

(f) Using your results of (d) and (e), check that $\beta = proj_W \beta + proj_{W^{\perp}} \beta$. Thus, we get a decomposition of β into two parts, one part is in W, the other part is in $W^{\perp}$.

Check: $$proj_W \beta + proj_{W^{\perp}} \beta = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \beta.$$

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2. Find a **least squares solution** to the system

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ 1 & 1 & -2 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

Note that the columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ of the coefficient matrix A form an **orthogonal** basis for Col A.

$$A^T A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 1 \\ 1 & 1 & -2 \\ 3 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

$$A^T \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$$

$$A^T A x = A^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 15 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = \frac{2}{5} \\ x_2 = 0 \\ x_3 = -\frac{2}{15}. \end{cases}$$

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3. Let

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix}.$$

Use Gram-Schmidt process to find a orthogonal basis for Col A, and use the orthogonal basis you get to find the QR factorization of A.

$$\alpha_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \qquad \alpha_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

Firstly, we orthogonalize $\{\alpha_1, \alpha_2\}$:

Take $\beta_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$,

$$\beta_2 = \cancel{\beta_1 - \beta_2 \cdot \beta_1}\ \alpha_2 - \frac{\alpha_2 \cdot \beta_1}{\beta_1 \cdot \beta_1}\beta_1 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} - \frac{15}{9}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{10}{3} \\ \frac{10}{3} \\ \frac{5}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

So, $Q = [\beta_1 \vdots \beta_2] = \begin{bmatrix} 2 & -\frac{1}{3} \\ 2 & \frac{2}{3} \\ 1 & -\frac{2}{3} \end{bmatrix}$

Suppose $A = QR$, then $Q^TA = Q^TQR$.

$$Q^TA = \begin{bmatrix} 2 & 2 & 1 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 0 & 1 \end{bmatrix}$$

$$Q^TQ = \begin{bmatrix} 2 & 2 & 1 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} 2 & -\frac{1}{3} \\ 2 & \frac{2}{3} \\ 1 & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow R = (Q^TQ)^{-1} \cdot (Q^TA) = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 9 & 15 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{5}{3} \\ 0 & 1 \end{bmatrix}.$$