Name: SOLUTIONS Date: 09/14/2017

M20550 Calculus III Tutorial
Worksheet 3

1. Find an equation of the tangent line to the space curve $\mathbf{r}(t) = \langle 2t^3, 3t, 3t^2 \rangle$ at the point $(-2, -3, 3)$.

> ***Solution:*** First, we want to find t corresponds to the point $(-2,-3,3)$. t corresponds to $(-2,-3,3)$ must satisfy the equations
>
> $$2t^3 = -2, \quad 3t = -3, \quad 3t^2 = 3.$$
>
> From the second equation, we know $t = -1$.
>
> Next, we want to find $\mathbf{r}'(-1)$, the tangent vector at $t = -1$. The derivative of $\mathbf{r}(t)$ is given by $\mathbf{r}'(t) = \langle 6t^2, 3, 6t \rangle$. So the tangent vector at $t = -1$ is $\mathbf{r}'(-1) = \langle 6, 3, -6 \rangle$.
>
> Then, the vector equation of the tangent line at $(-2,-3,3)$ is
>
> $$\langle x, y, z \rangle = \langle -2, -3, 3 \rangle + t \langle 6, 3, -6 \rangle.$$

2. Find the distance from the point $(1, -1, 0)$ to the space curve given by $\mathbf{r}(t) = \left\langle 2t, -t, \sqrt{t} \right\rangle$.

> ***Solution:*** The distance from the point to the curve can be thought of as a function of t in that at each time we can compute the distance from the point to $\mathbf{r}(t)$. This we can write as $D(t) = \sqrt{(2t-1)^2 + (-t-(-1))^2 + (\sqrt{t})^2}$. We would like to minimize this quantity, which we can do by looking for critical points using its derivative. We also note that minimizing $D(t)$ also minimizes $D(t)^2$ and vice versa, so we compute
>
> $$\frac{d}{dt}(D(t)^2) = \frac{d}{dt}(5t^2 - 5t + 2) = 10t - 5.$$
>
> Since there is only one critical point, and the distance is certainly unbounded as t gets large, we reach the minimum distance at $t = \frac{1}{2}$ and see $D(\frac{1}{2}) = \sqrt{\frac{5}{4} - \frac{5}{2} + 2} = \frac{\sqrt{3}}{2}$ is the distance from the point to the curve.

3. Find $\mathbf{r}(t)$ if $\mathbf{r}''(t) = e^t \mathbf{i}$, $\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, and $\mathbf{r}'(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

> ***Solution:***
>
> $$\mathbf{r}'(t) = \int \mathbf{r}''(t)\, dt = \int \left\langle e^t, 0, 0 \right\rangle\, dt = \left\langle e^t, 0, 0 \right\rangle + \mathbf{c}.$$

To find $\mathbf{c}$, we use the information $\mathbf{r}'(0) = \langle 1, 1, 1\rangle$. From the above, we have $\mathbf{r}'(0) = \langle e^0, 0, 0\rangle + \mathbf{c}$. So, $\langle e^0, 0, 0\rangle + \mathbf{c} = \langle 1, 1, 1\rangle \implies \mathbf{c} = \langle 1, 1, 1\rangle - \langle e^0, 0, 0\rangle = \langle 0, 1, 1\rangle$. Thus, we get

$$\mathbf{r}'(t) = \left\langle e^t, 0, 0\right\rangle + \langle 0, 1, 1\rangle \implies \mathbf{r}'(t) = \left\langle e^t, 1, 1\right\rangle.$$

Then

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\, dt = \int \left\langle e^t, 1, 1\right\rangle dt = \left\langle e^t, t, t\right\rangle + \mathbf{d}.$$

To find $\mathbf{d}$, we use the information $\mathbf{r}(0) = \langle 2, 3, 2\rangle$. We have $\mathbf{r}(0) = \langle e^0, 0, 0\rangle + \mathbf{d} = \langle 2, 3, 2\rangle$. So, $\mathbf{d} = \langle 2, 3, 2\rangle - \langle e^0, 0, 0\rangle = \langle 1, 3, 2\rangle$.

Finally, we get

$$\mathbf{r}(t) = \left\langle e^t, t, t\right\rangle + \langle 1, 3, 2\rangle \implies \mathbf{r}(t) = \left\langle e^t + 1, t + 3, t + 2\right\rangle.$$

4. Find the unit tangent vector, the principal unit normal vector, and the unit binormal vectors to the curve $\mathbf{r}(t) = \langle \sin 2t, \cos 2t, 3t^2\rangle$ at $t = \pi$.

Solution: We have $\mathbf{r}(t) = \langle \sin 2t, \cos 2t, 3t^2\rangle$. So

$$\mathbf{r}'(t) = \langle 2\cos 2t, -2\sin 2t, 6t\rangle \implies \mathbf{r}'(\pi) = \langle 2, 0, 6\pi\rangle.$$

$$\mathbf{r}''(t) = \langle -4\sin 2t, -4\cos 2t, 6\rangle \implies \mathbf{r}''(\pi) = \langle 0, -4, 6\rangle.$$

Also,

$$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = \langle 2, 0, 6\pi\rangle \times \langle 0, -4, 6\rangle = \langle 24\pi, -12, -8\rangle = 4\langle 6\pi, -3, -2\rangle.$$

Then,

$$\mathbf{T}(\pi) = \frac{\mathbf{r}'(\pi)}{|\mathbf{r}'(\pi)|} = \frac{\langle 2, 0, 6\pi\rangle}{|\langle 2, 0, 6\pi\rangle|} = \frac{1}{\sqrt{4 + 36\pi^2}}\langle 2, 0, 6\pi\rangle.$$

$$\mathbf{B}(\pi) = \frac{\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)}{|\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)|} = \frac{4\langle 6\pi, -3, -2\rangle}{4\,|\langle 6\pi, -3, -2\rangle|} = \frac{1}{\sqrt{13 + 36\pi^2}}\langle 6\pi, -3, -2\rangle.$$

$$\begin{aligned}
\mathbf{N}(\pi) = \mathbf{B}(\pi) \times \mathbf{T}(\pi) &= \frac{1}{\sqrt{13 + 36\pi^2}}\langle 6\pi, -3, -2\rangle \times \frac{1}{\sqrt{4 + 36\pi^2}}\langle 2, 0, 6\pi\rangle \\
&= \frac{1}{\sqrt{13 + 36\pi^2}}\frac{1}{\sqrt{4 + 36\pi^2}}\langle 6\pi, -3, -2\rangle \times \langle 2, 0, 6\pi\rangle \\
&= \frac{1}{\sqrt{13 + 36\pi^2}}\frac{1}{\sqrt{4 + 36\pi^2}}\left\langle -18\pi, -4 - 36\pi^2, 6\right\rangle.
\end{aligned}$$

5. Find the equation for the normal and osculating planes to the curve $\mathbf{r}(t) = 2\cos(3t)\mathbf{i} + t\mathbf{j} + 2\sin(3t)\mathbf{k}$ at the point $(-2, \pi, 0)$.

Solution: First, we note that t corresponds to the point $(-2, \pi, 0)$ is $t = \pi$ since $\mathbf{r}(t) = \langle 2\cos(3t), t, 2\sin(3t)\rangle = \langle -2, \pi, 0\rangle$ implies $t = \pi$ by looking at the second component.

A normal vector of the normal plane at $t = \pi$ is $\mathbf{r}'(\pi)$. We have

$$\mathbf{r}'(t) = \langle -6\sin(3t), 1, 6\cos(3t)\rangle \implies \mathbf{r}'(\pi) = \langle 0, 1, -6\rangle .$$

So, the normal plane at the point $(-2, \pi, 0)$ is given by

$$\langle 0, 1, -6\rangle \bullet \langle x, y, z\rangle = \langle 0, 1, -6\rangle \bullet \langle -2, \pi, 0\rangle \implies y - 6z = \pi.$$

A normal vector of the osculating plane at $t = \pi$ is $\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)$. We have, $\mathbf{r}''(t) = \langle -18\cos(3t), 0, -18\sin(3t)\rangle$ and so $\mathbf{r}''(\pi) = \langle 18, 0, 0\rangle$. Then,

$$\mathbf{r}'(\pi) \times \mathbf{r}''(\pi) = \langle 0, 1, -6\rangle \times \langle 18, 0, 0\rangle = 18\,\langle 0, 1, -6\rangle \times \langle 1, 0, 0\rangle = 18\,\langle 0, -6, -1\rangle .$$

So, we can take $\langle 0, 6, 1\rangle$ to be a normal vector for this osculating plane. And the equation is

$$\langle 0, 6, 1\rangle \bullet \langle x, y, z\rangle = \langle 0, 6, 1\rangle \bullet \langle -2, \pi, 0\rangle \implies 6y + z = 6\pi.$$

6. Find the length of the curve $\mathbf{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3\right\rangle$ from $(0, 0, 0)$ to $\left(2, 1, \frac{1}{3}\right)$.

Solution:

First, we need the derivative:

$$\mathbf{r}'(t) = \left\langle 2, 2t, t^2\right\rangle$$

and its magnitude

$$|\mathbf{r}'(t)| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2 + t^2)^2} = 2 + t^2 \quad \text{since } 2 + t^2 > 0.$$

And now, the point $(0, 0, 0)$ corresponds to $t = 0$ and the point $\left(2, 1, \frac{1}{3}\right)$ correspond to $t = 1$. Then, we have the length of $\mathbf{r}$ is

$$L = \int_{t=0}^{t=1} |\mathbf{r}'(t)|\, dt = \int_0^1 (2 + t^2)dt = 2t + \frac{1}{3}t^3\Big|_0^1 = \frac{7}{3} - 0 = \frac{7}{3}.$$

7. A particle moves with position function $\mathbf{r}(t) = \langle \cos t, \sin t, \cos^2 t \rangle$. Find the tangential and normal components of acceleration when $t = \pi/4$.

Solution: We have

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, -2\cos t \sin t \rangle = \langle -\sin t, \cos t, -\sin(2t) \rangle$$

$$\implies \mathbf{r}'(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1 \right\rangle,$$

$$\mathbf{r}''(t) = \langle -\cos t, -\sin t, -2\cos(2t) \rangle \implies \mathbf{r}''(\pi/4) = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle.$$

And so

$$a_T = \frac{\mathbf{r}'(\pi/4) \bullet \mathbf{r}''(\pi/4)}{|\,\mathbf{r}'(\pi/4)\,|} = 0.$$

We know $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$. Since $a_T = 0$, we get $\mathbf{a} = a_N\mathbf{N}$. So,

$$|\mathbf{a}| = a_N|\mathbf{N}| = a_N \cdot 1 = a_N.$$

Thus,

$$\begin{aligned} a_N = |\mathbf{a}| &= |\,\mathbf{r}''(\pi/4)\,| \\ &= \left| \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle \right| \\ &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \\ &= 1 \end{aligned}$$