ELEMENTS OF CALCULUS: EXAM 3 REVIEW

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1. Optimization

Problem 1. Find the absolute extrema of the function on the closed interval:

a) $f(x) = x^3 - \frac{3}{2}x^2$ on $[-1, 2]$.

b) $g(t) = \dfrac{t^2}{t^2+3}$ on $[-1, 1]$.

c) $h(t) = \dfrac{t}{t+3}$ on $[-1, 6]$.

Problem 2. Find the min/max of the following functions

a) $g(x) = -x^2 + 4x + 3$

b) $f(x) = \dfrac{x}{1+x^2}$

c) $f(x) = 9x - \dfrac{1}{x}$ on $[1, 3]$

d) $f(x) = \frac{1}{2}x^2 - 2\sqrt{x}$ on $[0, 3]$

e) $g(x) = x\sqrt{4-x^2}$ on its domain

f) $p(x) = xe^{-x^4}$ on its domain

g) $r(x) = e^{-x} \cdot \ln x$ on $[1, \infty)$.

Problem 3. A farmer has 3000 yards of fencing with which to enclose a rectangular piece of land for his chickens. What dimensions should the rectangular enclosure have if he wants to enclose the largest possible area?

Problem 4. A farmer has 3000 yards of fencing with which to enclose a rectangular piece of land for his chickens. One of the sides of the land will be along a river, so no fencing is required for that side. What dimensions should the rectangular enclosure have if he wants to enclose the largest possible area?

Problem 5. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 245,000 square meters in order to provide enough grass for the herd. No fencing is needed along the river. What dimensions will require the least amount of fencing?

2. Exponential and Logarithmic Functions

Problem 6. The position of a particle is given by $s(t) = e^x - 2t + 18$. For what (non-negative) value(s) of t is the particle at rest?

Problem 7. Consider the function $f(x) = xe^{-x^4}$.
(a) Find all critical points of $f(x)$.

(b) Find the absolute maximum and absolute minimum of $f(x)$ on its entire domain.

(c) Find the inflection points of $f(x)$.

(d) Give a rough sketch of the function.

Problem 8. Where do the curves $y = e^{3x}$ and $y = 3e^x$ intersect?

Problem 9. Solve the equations:
a) $4^x = \left(\frac{1}{16}\right)^{x-7}$

b) $3^{x-x^2} = \frac{1}{9^x}$

c) $\log_4(2x+2) = \frac{1}{2}$

d) $4e^{t-1} = 4$

Problem 10. Sketch the graph of the following functions:
a) $f(x) = 3\ln x$

b) $\ln(x-3)$

c) $\ln|x|$

d) $e^x + 1$

e) $3e^x + 2$

f) $(0.5)^x$ and 2^x on the same axis

g) e^{-x} and e^x on the same axis

h) 2^x and 10^x on the same axis

i) $\log_2(x)$ and $\log_{10}(x)$ on the same axis

j) 2^{-x} and 10^{-x} on the same axis

3. Compound Interest

Problem 11. How long will it take an investment of \$2,000 to double if the investment earns interest at the rate of 6%/year. What if the interest was compounded monthly?

Problem 12. How long will it take an investment of \$5,000 to triple at an interest rate of 4%/year compounded weekly. Assume 52 weeks in a year.

Problem 13. What is the interest rate needed for an investment of \$5,000 to grow to \$6,000 in 3 years if interest is compounded continuously.

Problem 14. Find the interest rate needed for an investment of \$2,000 to double in 5 years if interest is compounded annually.

Problem 15. Find the present value of \$20,000 due in 3 years at an interest rate of 12%/year compounded monthly.

Problem 16. Glen invests \$100,000 in an account yielding 6.6% interest compounded monthly. Being unhappy with the return on his investment, he wishes to reinvest the final amount at the end of the first year into a new account where interest is compounded quarterly. What interest rate should he look for if he wishes to obtain \$130,130 at the end of the third year (i.e. after keeping the money for 2 more years in the second account).

Problem 17. The same amount of money is invested in two different accounts. Account A pays simple interest at a rate of 5% per year, while account B pays compound interest at the same rate (compounded annually).

a) Write a formula for the amount of money $A(t)$ in the first account after t years. Do the same for the amount of money $B(t)$ in the second account.
b) How long does it take for the amount in the first account to double in size?
c) Which is bigger: $A(t)$ or $B(t)$? In other words, if the same amount of time passes, which type of interest offers a bigger return?
d) Suppose we have three accounts with each offering the same rate, but one offers simple interest, the second offers compound interest, and the third offers continuously compounded interest. Given the same starting principal amount, which account grows the slowest? Which account grows the fastest?

4. Derivatives of Log and Exponential

Problem 18. Compute dy/dx of the following:

a) $y = \ln\left(\dfrac{x}{x^2+1}\right)$

b) $y = \ln(x\sqrt{x^2-1})$

c) $y = \ln\sqrt{\dfrac{x+1}{x-1}}$ (without logarithmic differentiation)

d) $y = \sqrt{\dfrac{x+1}{x-1}}$ (with logarithmic differentiation)

e) $4xy + \ln(x^2y) = 7$

f) $y = e^{-8x}$

g) $y = x^3e^{x^2}$

h) $y = \ln(1+e^{2x})$

i) $y = \dfrac{e^{2x}}{e^{2x}+1}$

j) $xe^y + ye^x = 1$

k) $y = x^{e^x}$

l) $y = x^{\sqrt{x}}$

m) $y = x^{x^2}$

5. Integration

Problem 19. Compute the following indefinite integrals:

a) $\displaystyle\int \sqrt{2}dx$

b) $\displaystyle\int \frac{1}{x^4}dx$

c) $\displaystyle\int 2x^5 dx$

d) $\displaystyle\int 3r^{-2/3}dr$

e) $\displaystyle\int (x^2 + x + x^{-3})dx$

f) $\displaystyle\int (1 + t + e^t)dt$

g) $\displaystyle\int \frac{x^4-1}{x^2}dx$

h) $\displaystyle\int 5e^{5y}dy$

i) $\displaystyle\int e^{2x-1}dx$

j) $\displaystyle\int x^2e^{2x^3+4}dx$

k) $\displaystyle\int e^x(e^x+5)^2dx$

l) $\displaystyle\int \frac{5-e^x}{e^{2x}}dx$

m) $\displaystyle\int \frac{2x+1}{x^2+x}dx$

n) $\displaystyle\int \frac{2e^{2x}}{1+e^{2x}}dx$

o) $\displaystyle\int xe^{ax^2}dx$ (where a is a positive constant)

p) $\displaystyle\int \frac{-2x}{(x^2+1)^2}dx$

Problem 20. Find the solution (general or particular) to the differential equations:

a) $f'(x) = 6,\ f(0) = 8$

b) $\dfrac{dy}{ds} = 10s - 12s^3,\ y(3) = 2$

c) $y'' = 2,\ y'(0) = 5,\ y(2) = 10$

d) $f''(x) = x^2,\ f'(0) = 8,\ f(0) = 4$

e) $f''(x) = x^{-3/2},\ f'(4) = 2,\ f(0) = 0$

f) $\dfrac{dy}{dx} = 4x + \dfrac{4x}{\sqrt{16-x^2}}$

g) $\dfrac{dy}{dx} = \dfrac{10x^2}{\sqrt{1+x^3}}$

h) $\dfrac{dy}{dx} = \dfrac{x+1}{(x^2+2x-3)^2}$

i) $\dfrac{dy}{dx} = \dfrac{x-1}{\sqrt{x^2-8x+1}}$

Problem 21. Use substitution to compute the following indefinite integrals:

a) $\displaystyle\int x(x^2+1)^2dx$

b) $\displaystyle\int \sqrt{2x-1}dx$

c) $\displaystyle\int x\sqrt{3x-1}dx$

d) $\displaystyle\int 3(3x-1)^4dx$

e) $\displaystyle\int (2x+1)(x^2+x)dx$

f) $\displaystyle\int 3x^2\sqrt{x^3-2}dx$

g) $\displaystyle\int \frac{-4x}{(1-2x)^2}dx$

h) $\displaystyle\int \frac{3x^2}{x^3+1}dx$

i) $\displaystyle\int 2(2x^3+x)e^{x^4+x^2+1}dx$

j) $\int x(x^2+1)^2dx$, compare to $\int (x^2+1)^2dx$

k) $\int \frac{x}{\sqrt{2x-1}}dx$

l) $\int \frac{x^3}{1+x^4}dx$

m) $\int \frac{6x^2}{4x^3-9}dx$

n) $\int \frac{1}{\sqrt{2x}}dx$

o) $\int \left(1+\frac{1}{t}\right)^3\left(\frac{1}{t^2}\right)dt$

p) $\int \frac{1+e^x}{x+e^x}dx$

q) $\int \frac{x}{\sqrt[3]{5x^2}}dx$