Name: SOLUTIONS Date: 11/02/2017

M20550 Calculus III Tutorial
Worksheet 7

1. Using spherical coordinates, compute the volume, $V(R)$ of a sphere of radius R.

Solution: This is equivalent to just computing

$$\iiint_{Sphere} dV$$

(intuitively, we are summing up the volumes of infinitely many infinitesimally small boxes of volume "dV" inside the sphere.) Recall that the standard spherical coordinates are

$$(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$$

for $(\rho, \theta, \phi) \in [0, R] \times [0, 2\pi) \times (0, \pi)$ and the volume element of the sphere with respect to these coordinates is given by $dV = \rho^2 sin\phi d\theta d\phi d\rho$. So,

$$\begin{aligned} V(R) = &\int_0^R \int_0^\pi \int_0^{2\pi} \rho^2 sin\phi d\theta d\phi d\rho \\ =&2\pi \int_0^R \int_0^\pi \rho^2 sin\phi d\phi d\rho \\ =&4\pi \int_0^R \rho^2 d\rho \\ =&\frac{4}{3}\pi R^3 \end{aligned}$$

2. Now compute the surface area, $A(R)$, of a sphere of radius R. Hint: Recall the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x).$$

And recall the common problem from single variable calculus where you have to find the volume of a water tank of height h by integrating the cross sectional area, $A(y)$, over the height.

$$Volume(Tank) = \int_0^h A(y)dy$$

We have a similar formula for the volume of the sphere;

$$V(R) = \int_0^R A(\rho)d\rho.$$

Solution: Let $A(\rho)$ be the surface area of the sphere of radius ρ, we wish to find $A(R)$. Observe

$$\int_0^R A(\rho)d\rho = V(R) = \frac{4}{3}\pi R^3$$

So by the fundamental theorem of calculus, we get

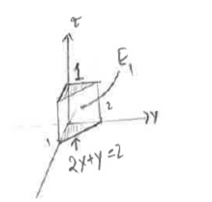
$$A(R) = \frac{d}{dR}\left[\int_0^R A(\rho)d\rho\right] = \frac{dV(R)}{dR} = 4\pi R^2.$$

Another way to solve this problem is to realize through geometric intuition or by reasoning similar to the argument above that

$$A(R) = \int_0^{\pi}\int_0^{2\pi} R^2 sin\phi d\theta d\phi.$$

3. (a) Let E_1 be the solid that lies under the plane $z = 1$ and above the region in the xy-plane bounded by $x = 0$, $y = 0$, and $2x + y = 2$. Write the triple integral $\iiint_{E_1} xz\,dV$ but do not evaluate it.

Solution:



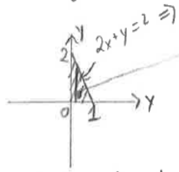
We'll use rectangular coordinates to write $\iiint_{E_1} xz\,dV$.

$$\iiint_{E_1} xz\,dV = \iint_R\left(\int_{z=0}^{z=1} xz\,dz\right)dA$$

(now write $dA = dy\,dx$, and the limits for y and x comes from the picture of R)

in the xy-plane

$2x+y=2 \Rightarrow y = 2-2x$



$$= \boxed{\int_{x=0}^{x=1}\int_{y=0}^{y=2-2x}\int_{z=0}^{z=1} xz\;dz\,dy\,dx}$$

With the order $dy\,dx$:
$0 \le y \le 2-2x$
$0 \le x \le 1$

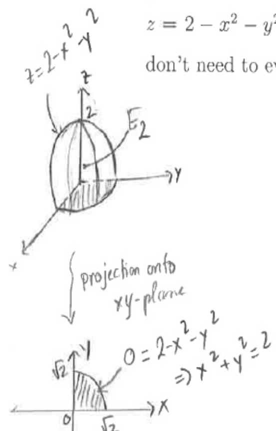
$$\left(\text{Another answer is } \int_{y=0}^{y=2}\int_{x=0}^{x=\frac{2-y}{2}}\int_{z=0}^{z=1} xz\;dz\,dx\,dy\right)$$

(b) Let E_2 be the solid region in the first octant that lies under the paraboloid

$z = 2 - x^2 - y^2$. Write the triple integral $\iiint_{E_2} xz\, dV$ in cylindrical coordinates (you don't need to evaluate it).

Solution:

(b) Let E_2 be the solid region in the first octant that lies under the paraboloid $z = 2 - x^2 - y^2$. Write the triple integral $\iiint_{E_2} xz\, dV$ in cylindrical coordinates (you don't need to evaluate it).



$0 \le r \le \sqrt{2}$
$0 \le \theta \le \frac{\pi}{2}$

In cylindrical coordinates, $dV = r\,dz\,dr\,d\theta$

From the picture of E_2, we see that $0 \le z \le 2 - x^2 - y^2 = 2 - r^2$

To get the bounds for r and θ, we look at the projection of the solid E_2 onto the xy-plane. We see that

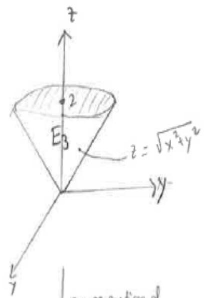
$0 \le r \le \sqrt{2}$ and $0 \le \theta \le \frac{\pi}{2}$.

So, $$\iiint_{E_2} xz\, dV = \int_0^{\pi/2}\int_0^{\sqrt{2}}\int_{z=0}^{z=2-r^2} \underbrace{(r\cos\theta)}_{=\ x \text{ in cylindrical coord.}} z\, r\,dz\,dr\,d\theta$$

$$= \boxed{\int_0^{\pi/2}\int_0^{\sqrt{2}}\int_0^{2-r^2} z\, r^2 \cos\theta\, dz\,dr\,d\theta}$$

(c) Let E_3 be the solid region that lies above the cone $z = \sqrt{x^2+y^2}$ and below the plane $z = 2$. Write the triple integral $\iiint_{E_3} xz\, dV$ in spherical coordinates (you don't need to evaluate it).

Solution:



In spherical coord., $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ and $x = \rho\sin\phi\cos\theta$

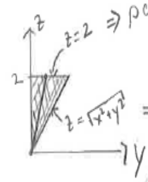
$z = \rho\cos\phi$

We can easily see that $0 \le \theta \le 2\pi$. Now, in order to get the bounds for ρ and ϕ, we can look at the cross section of the solid E_3 with the positive yz-plane (see picture on left). Converting the approriate equations to spherical coord., we get

$$0 \le \rho \le \frac{2}{\cos\phi} \quad \text{and} \quad 0 \le \phi \le \frac{\pi}{4}$$

(recall, ϕ is measured from the positive z-axis)

cross section of E_3 with the positive yz-plane



$z=2 \Rightarrow \rho\cos\phi = 2 \Rightarrow \rho = \frac{2}{\cos\phi}$

$z=\sqrt{x^2+y^2} \Rightarrow \rho\cos\phi = \rho\sin\phi \Rightarrow \cos\phi = \sin\phi \Rightarrow \phi = \frac{\pi}{4}$

$0 \le \rho \le \frac{2}{\cos\phi}$

$0 \le \phi \le \frac{\pi}{4}$

Thus,

$$\iiint_{E_3} xz\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\frac{2}{\cos\phi}} (\rho\sin\phi\cos\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\frac{2}{\cos\phi}} \rho^4\sin^2\phi\cos\phi\cos\theta\,d\rho\,d\phi\,d\theta}$$

4. Write the integral that computes the volume of the part of the solid cylinder $x^2 + y^2 \le 1$ that lies between the planes $z = 0$ and $z = 2 - y$.

Solution: This is best done in cylindrical coordinates,

$$\iiint_R dV = \int_0^1 \int_0^{2\pi} \int_0^{2 - r\sin\theta} r\,dz\,d\theta\,dr.$$

5. Find the mass of the solid between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ whose density is $\delta(x, y, z) = x^2 + y^2 + z^2$.

Solution: Let E be the solid in consideration. Now, to find the mass, we simply integrate the density function over the entire solid to get;

$$\begin{aligned}\int_1^2\int_0^\pi\int_0^{2\pi}\delta(\rho)\rho^2 sin\phi d\theta d\phi d\rho =&\int_1^2 A(\rho)\delta(\rho)d\rho\\ =&\int_1^2 4\pi\rho^2\rho^2 d\rho\\ =&4\pi\frac{\rho^5}{5}\bigg|_1^2\\ =&4\pi(\frac{32}{5}-\frac{1}{5})\\ =&\frac{124\pi}{5}.\end{aligned}$$

Note: The fact that the density only depended on ρ simplified our work here.

6. Find the center of mass of the solid S bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$ if S has constant density 1 and total mass $\frac{\pi}{2}$. (Hint: $\overline{x}$ and $\overline{y}$ can be found by symmetry of the solid being considered).

Solution: Since the density is constantly 1, we just need to compute the average values of x, y and z inside this solid. Because the solid is rotationally symmetric about the z-axis, we get $\overline{x} = \overline{y} = 0$. Now we compute

$$\begin{aligned}\overline{z} =&\frac{2}{\pi}\int_0^1\int_0^{2\pi}\int_0^{\sqrt{z}} zrdrd\theta dz\\ =&\frac{2}{\pi}\int_0^1\int_0^{2\pi} z\frac{r^2}{2}\bigg|_0^{\sqrt{z}} d\theta dz\\ =&\frac{2}{\pi}\int_0^1\int_0^{2\pi}\frac{z^2}{2}d\theta dz\\ =&2\int_0^1 z^2 dz\\ =&\frac{2}{3},\end{aligned}$$

so the center of mass is given by $(0, 0, \frac{2}{3})$.

7. Consider the solid cylinder of height 1, radius 2, and whose density at any point is equal to that points distance to the central axis. Find the moment of inertia of this cylinder around its central axis.

Solution: Recall that the moment of inertia of a point mass about an axis is equal to the square of its distance the the axis times its mass, and to compute the moment of inertia of a solid about an axis, we just sum up all of the moments of inertia of the points that make up the solid (in other words we integrate the function $r^2\delta$ over the solid, where r is the distance to the axis with respect to which we are computing the moment of inertia and δ is the density!). So we must compute the following integral;

$$\iiint_{Cylinder} r^2 \delta dV$$

We do this in cylindrical coordinates and get,

$$\begin{aligned}\int_0^1 \int_0^{2\pi} \int_0^2 r^4 dr d\theta dz &= \int_0^1 \int_0^{2\pi} \frac{32}{5} d\theta dz \\ &= \frac{64\pi}{5}.\end{aligned}$$

8. In this problem, we are going to calculate the same integral in two different ways by changing coordinates. Compute the following integral;

$$\int_0^1 \int_0^1 x^3 y dx dy$$

first, by making the coordinate change $u = x^2, v = xy$, and then as you normally would. (Don't forget to multiply by the Jacobian!)

Solution:

We first compute the Jacobian;

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{u}} & 0 \\ \frac{-v}{u^{\frac{3}{2}}} & \frac{1}{\sqrt{u}} \end{vmatrix} = \frac{1}{2u}$$

(note: u is always positive so we don't need to take the absolute value) now, we know by the change of variables formula that

$$\int_0^1 \int_0^1 x^3 y dx dy = \int_0^1 \int_0^{\sqrt{u}} uv \frac{1}{2u} dv du = \int_0^1 \frac{v^2}{4}\Big|_{v=0}^{v=\sqrt{u}} du = \int_0^1 \frac{u}{4} du = \frac{1}{8}.$$

If we compute this integral in the usual way, we get;

$$\int_0^1 \int_0^1 x^3 y dx dy = \int_0^1 \frac{y}{4} dy = \frac{1}{8}.$$