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Worksheet 4, Math 10560

Times indicate the amount of time that you would be expected to spend on the problem in on an exam. All problems have appeared on old exams for Calculus 2.

1. (4 mins) Evaluate the integral $\int_1^2 x^3 \ln x \, dx$.

Solution:

We use integration by parts with $u = \ln x$, $dv = x^3\, dx$ so $du = 1/x$, $v = \frac{1}{4}x^4$. So

$$\int_1^2 x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x\Big|_1^2 - \frac{1}{4}\int_1^2 x^4 \frac{1}{x}\, dx = 4\ln 2 - [1 - \frac{1}{16}] = 4\ln 2 - \frac{15}{16}$$

2. (4 mins) Evaluate the integral

$$\int x \sin(3x)\, dx.$$

Solution: We use integration by parts, with $u = x$ and $dv = \sin(3x)\ dx$. Then $du = dx$ and $v = \frac{-\cos(3x)}{3}$. Thus

$$\int x\sin(3x)dx = \frac{-x\cos(3x)}{3} - \int \frac{-\cos(3x)}{3}dx = \frac{-x\cos(3x)}{3} + \frac{\sin(3x)}{9} + C$$

3. (8 mins partial credit) Calculate the integral

$$\int \sqrt{9 - x^2}\ dx\ .$$

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Solution: Use the trigonometric substitution $x = 3\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$; then $\theta = \sin^{-1}\left(\frac{x}{3}\right)$ and $dx = 3\cos\theta d\theta$. Hence

$$\begin{aligned}
\int \sqrt{9-x^2}\, dx &= \int \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\, d\theta \\
&= \int 9\cos\theta\cdot\cos\theta\, d\theta \\
&= 9\int \cos^2\theta\, d\theta && \text{(Use } \cos^2\theta = \tfrac{1+\cos 2\theta}{2}\text{).} \\
&= \frac{9}{2}\int (1+\cos 2\theta) d\theta \\
&= \frac{9}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C && \text{(Use } \sin 2\theta = 2\sin\theta\cos\theta\text{.)} \\
&= \frac{9}{2}\left(\theta + \sin\theta\cos\theta\right) + C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\frac{x}{3}\sqrt{1-\sin^2\theta} + C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C.
\end{aligned}$$

4. (4 mins.) Evaluate the integral

$$\int_0^{\pi/2} \sin^5(x)\cos^3(x) dx\ .$$

Solution: We have odd powers of sin and cos and thus we can either do a $u = \cos x$ substitution or $u = \sin x$. Let's choose $u = \sin x$. So then $du = \cos x dx$ and we will convert the remaining $\cos^2 x$ using $\cos^2(x) = 1 - \sin^2(x)$. Thus we have

$$\begin{aligned}
\int_0^{\pi/2} \sin^5(x)\cos^3(x) dx &= \int_0^{\pi/2} \sin^5(x)\cos^2(x)\cos(x) dx \\
&= \int_0^{\pi/2} \sin^5(x)(1-\sin^2(x))\cos(x) dx \\
&= \int_{\sin 0}^{\sin(\pi/2)} u^5(1-u^2) du \\
&= \int_0^1 (u^5 - u^7) du \\
&= \left[\frac{u^6}{6} - \frac{u^8}{8}\right]_0^1 = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.
\end{aligned}$$

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5. (8 mins partial credit) Calculate the integral

$$\int_0^3 \frac{1}{\sqrt{x^2+9}}\ dx\ .$$

Note: One of the formulas given on the last page of the worksheet may help you with this problem.

Solution: Since the argument of the integral is of the form $\sqrt{x^2+a^2}$, we make the substitution $x = 3\tan(\theta) \Rightarrow dx = 3\sec^2(\theta)d\theta$

$$\int_0^3 \frac{1}{\sqrt{x^2+9}}dx = \int_0^{\pi/4} \frac{3\sec^2(\theta)}{\sqrt{9\tan^2(\theta)+9}}d\theta$$

We factor the 9 and use $1+\tan^2 = \sec^2$ to get

$$\int_0^3 \frac{1}{\sqrt{x^2+9}}dx = \int_0^{\pi/4} \frac{3\sec^2(\theta)}{3\sqrt{\sec^2(\theta)}}d\theta = \int_0^{\pi/4} \sec(\theta)d\theta = \ln|\sec(\theta)+\tan(\theta)|\big|_0^{\pi/4},$$

where in the last equality, we applied the last formula from the 'list of useful trigonometric formulas' at the end of the exam. Evaluating, we get

$$\int_0^3 \frac{1}{\sqrt{x^2+9}}dx = \ln|2/\sqrt{2}+1| - \ln|1+0| = \ln|\sqrt{2}+1|$$

6. (4 mins) Compute $\displaystyle\int_0^{\pi/2} \sin(7x)\ \sin(3x)\ dx$

Solution:

Here we use the product to sum rule for sine

$$\sin A\ \sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

to write

$$\sin(7x)\ \sin(3x) = \frac{1}{2}\left(\cos(4x) - \cos(10x)\right)$$

Then our integral becomes

$$\begin{aligned}\int_0^{\pi/2} \sin(7x)\ \sin(3x)\ dx &= \int_0^{\pi/2} \frac{1}{2}\left(\cos(4x)-\cos(10x)\right)\ dx \\ &= \frac{1}{2}\left(\frac{\sin(4x)}{4} - \frac{\sin(10x)}{10}\right)\Bigg|_0^{\pi/2} \\ &= 0\end{aligned}$$

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The following is the formula sheet included with exam 1:

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x-y) + \sin(x+y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\int \sec\theta = \ln|\sec\theta + \tan\theta| + C$$