Math 10550 Final Exam Practice Problems

1 Functions

1. Sketch the following curves.
   (a) $y = x$, $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$, $y = x^n$ for even $n \geq 2$, $y = x^n$ for odd $n \geq 3$.
   (b) $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, $y = \frac{1}{x^3}$, $y = \frac{1}{x^4}$, $y = \frac{1}{x^5}$, $y = \frac{1}{x^n}$ for even $n \geq 2$, $y = \frac{1}{x^n}$ for odd $n \geq 1$.
   (c) $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$, $y = \sqrt[5]{x}$, $y = \sqrt[n]{x}$ for even $n \geq 2$, $y = \sqrt[n]{x}$ for odd $n \geq 3$.
   (d) $y = \sin(x)$, $y = \cos(x)$, $y = \tan(x)$, $y = \sec(x)$, $y = \csc(x)$, $y = \cot(x)$.
   (e) $y = \arctan(x)$, $y = \arcsin(x)$, $y = \arccos(x)$.
   (f) $y = e^x$, $y = \ln(x)$.

2. Sketch the following curves.
   (a) $y = 3(x+1)^2 - 1$, $y = -(2x+1)^3 + 1$.
   (b) $y = \frac{1}{(x-1)^2(x+2)}$, $y = \frac{x+2}{(x+1)^3(x-3)^2}$.
   (c) $y = \sqrt[4]{3x^2 - 10}$, $y = 2\sqrt[3]{2x-1} + 3$.
   (d) $y = e^{x+2} - 2$, $y = \ln(x-1) + 1$.

2 Limits and Continuity

3. Compute the following limits.
   (a) $\lim_{x\to -3} \frac{x^3+27}{x+3}$
   (b) $\lim_{x\to 0} \frac{\sqrt{4-x^2}-\sqrt{4+x^2}}{x^2}$
   (c) $\lim_{x\to 0} \frac{\sin(4x)}{\tan(5x)}$
   (d) $\lim_{x\to 0} \frac{e^x - x - 1}{\arctan(x^2)}$
   (e) $\lim_{x\to 0} \frac{\cos(x)-1}{\sin(x^2)}$

4. Find a and b so that the following function is continuous:

$$f(x) = \begin{cases} x^2 - 4 & \text{if } x < -3; \\ ax + b & \text{if } -3 \leq x < 0; \\ e^x + 2 & \text{if } 0 \leq x. \end{cases}$$

5. Give an example of a single function exhibiting all three types of discontinuity: removable, jump, and infinite.

6. Show that the polynomial $f(x) = -5x^4 + 3x^2 - 5x + 8$ has at least one root (you should not try to find this root).

7. Given positive numbers a and b, show that the equation $\frac{a}{x^3+2x^2-1} + \frac{b}{x^3+x-2} = 0$ has at least one solution in the interval $(-1, 1)$.

3 Derivatives

8. Compute the derivatives of the following functions from the definition.
   (a) $f(x) = x^2 - 3x + 1$
   (b) $g(x) = \sqrt{x}$
   (c) $h(x) = \frac{1}{x-3}$
   (d) $k(x) = \frac{1}{\sqrt{x}}$
   (e) $m(x) = x^{5/3}$

9. Find the tangent lines of the following functions at the indicated points.
   (a) $f(x) = x^3 - x - 3$ at $(2, 4)$
   (b) $g(x) = \ln(2x + e) - 1$ at $(0, 0)$
   (c) $h(x) = \frac{x^2-1}{2x-3}$ at $(1, 0)$

10. Find a, b, c, d so that the following function is differentiable:

$$f(x) = \begin{cases} x^2 + x + 1 & \text{if } x < -1; \\ ax^3 + bx^2 + cx + d & \text{if } -1 \leq x < 0; \\ e^x - 1 & \text{if } 0 \leq x. \end{cases}$$

11. Show that there do not exist a, b, c making the following function differentiable:

$$f(x) = \begin{cases} \cos(-2x) & \text{if } x < \frac{-\pi}{4}; \\ ax^2 + bx + c & \text{if } \frac{-\pi}{4} \leq x < 1; \\ \ln(x) & \text{if } 1 \leq x. \end{cases}$$

12. Give an example of a function which fails to be differentiable because of a cusp and one that fails to be differentiable because of a vertical tangent line.

13. Find the derivatives of the following functions.
   (a) $f(x) = \sin(x)\cos(5x)$
   (b) $g(x) = \sqrt{\ln(x-4)}$
   (c) $h(x) = e^{5x^2} \arctan(x)$

14. Use implicit differentiation and that $f(x) = \text{arccot}(x)$ is the inverse of $g(x) = \cot(x)$ to find the derivative of f.

15. Find $\frac{dy}{dx}$ for each curve below.
   (a) $x^2 + y^2 = 4$
   (b) $xy^2 = 7x^4 + 2x + y$
   (c) $e^{xy} = xy + 1$

16. Find the tangent line to the curve $x^2 + xy + y^2 = x - y + 6$ at the point $(2, 1)$.

17. A particle is traveling along the curve $\frac{x^2}{4} + \frac{y^2}{12} = 1$. As the particle passes through the point $(\sqrt{2}, \sqrt{6})$, its velocity in the x-direction is $\sqrt{3}$ units/second, what is the velocity of the particle in the y-direction at that time?

18. Consider a tank of water obtained by revolving the region bounded by $y = x^2$ and $y = 4$ around the y-axis. If water is added to the tank at a rate of 1 m^3/min, how fast is the depth d of water in the tank increasing when $d = 1$?

19. If the vertex angle θ of an isosceles triangle is decreasing at a rate of 1 radian per second and the length ℓ of the legs is increasing at a rate of 3 centimeter per second, how fast is the length of the base changing when $\theta = \frac{\pi}{3}$ and $\ell = \sqrt{6}$?

20. Use linearizations to approximate $\sqrt{3.9}$, $\cos\left(\frac{\pi}{5}\right)$.

21. Use linearizations at $a = 1$ and $a = \sqrt{3}$ to approximate $\arctan\left(\frac{7}{5}\right)$. Argue using concavity for which gives the better approximation.

4 Applications of Differentiation

22. Find the absolute maximum and absolute minimum values of the functions below on the indicated intervals.

    (a) $f(x) = x^4 - 4x^2 + 4$ on $[-3, 3]$
    (b) $g(x) = 2\sin(x)\cos(x) - 1$ on $[-\pi, \pi]$
    (c) $h(x) = e^{-x}\sin(x)$ on $[0, 2\pi]$

23. Suppose $f(2) = -1$ and $0 \leq f'(x) \leq 3$ for all $x \in [-2, 2]$. What is largest and smallest that $f(-2)$ can be?

24. Show that the equation $\ln(x) = e^{-x}$ has exactly one solution in the interval $(0, \infty)$. Estimate it to within 5 decimal places using Newton's method.

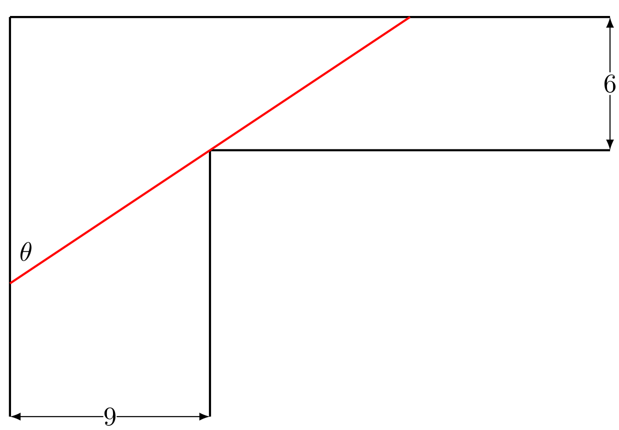
25. Show that the equation $e^x = \sqrt{x + 2}$ has exactly two solutions. Estimate them to within 4 decimal places using Newton's method.

26. Sketch the graphs of the following functions.

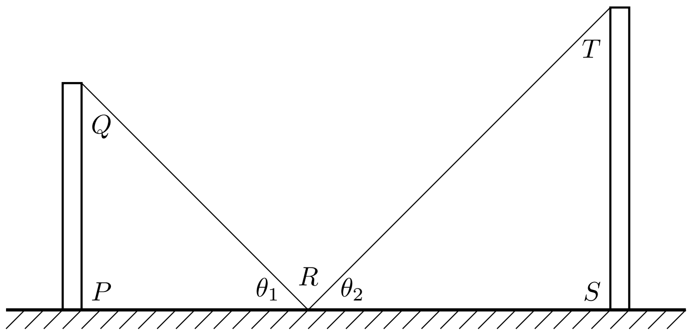
    (a) $f(x) = \cos^2(x) + \cos(x) + 2$
    (b) $g(x) = 5x^{2/3} - 2x^{5/3}$
    (c) $h(x) = \frac{x^4 - 3x^2 - 4}{x^4 - 3x^3 + 4}$

27. A farmer wants to enclose two side-by-side pens next to a river, if the farmer wants each pen to contain and area of $150\,m^2$, what is the minimum amount of fence needed to build the pens? (Note: there is no fence necessary next to the river.)

28. A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner?



29. Two vertical poles PQ and ST are secured by a rope QRT going from the top of the first pole to a point R on the ground between the poles and then to the top of the second pole as in the figure below. Show that the shortest length of such a rope occurs when $\theta_1 = \theta_2$.
Hint: Start by finding the derivative of θ_1 and θ_2 with respect to changes in PR, then write the total length of QRT as a function of PR, θ_1, θ_2, and PS (which is fixed).



5 Integration

30. Compute the following integrals from the definition. Check each of your answers using the Fundamental Theorem of Calculus.

(a) $\int_1^4 2x^2 + 1\, dx$

(b) $\int_0^2 (x^3 + x^2 + x + 1)\, dx$

(c) $\int_0^1 e^x\, dx$

Hint: Use the summation formula $\sum_{i=1}^{n} a^i = \frac{1-a^{n+1}}{1-a}$ which holds for any real number $a \neq 1$ and the limit $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$.

31. Compute the following integrals (using the Fundamental Theorem of Calculus).

(a) $\int_0^1 \frac{1}{x^2+1}\, dx$

(b) $\int_2^3 \frac{5x^4 - 2x}{x^5 - x^2}\, dx$

(c) $\int_{-1}^2 \sin(x) e^{\cos(x)}\, dx$

(d) $\int_1^5 e^{3x}\sqrt{7 + 3e^{3x}}\, dx$

32. Compute $g'(x)$ for each function below.

(a) $g(x) = \int_5^{x^2} \arctan(t)\, dt$

(b) $g(x) = \int_{\cos(x)}^{-2} (4t^2 + 5)\, dt$

(c) $g(x) = \int_{-x^2}^{3x^4} e^t \cos(t)\, dt$

6 Applications of Integration

33. Find the area bounded by the following curves.

(a) $y = x^4 - 5x^2 + 4$ and $y = -x^2 + 1$

(b) $y = \cos(x)$ and $y = \frac{54}{\pi^2}x^2 - 1$

(c) $y = \cos(\pi x)$ and $y = 4x^2 - 1$

(d) $y = \sec^2(x)$ and $y = 8\cos(x)$ on the interval $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

34. Set up but do not evaluate integrals computing the following volumes.

   (a) The solid whose base is the region bounded by the curves $y = x^3$ and $x = y^2$ and whose cross-sections perpendicular to the x-axis are squares.

   (b) The solid whose base is the region bounded by the curves $y = x^3$ and $x = y^2$ and whose cross-sections perpendicular to the y-axis are semi-circles.

   (c) The solid whose base is the triangle with vertices at $(0,0)$, $(-1,-1)$, and $(1,2)$ and whose cross-sections perpendicular to the x-axis are isosceles right triangles with hypotenuse in the base.

   (d) The solid obtained by revolving the region bounded by $y = -x^2$ and $y = x^4 - 6x^2 + 4$ around the line

      i. $x = 3$ using cylindrical shells.

      ii. $x = -4$ using cylindrical shells.

      iii. $y = 4$ using washers.

      iv. $y = -6$ using washers.

   (e) The solid obtained by revolving the region bounded by $y = -x^2$ and $y = x^4 - 6x^2 + 4$ around the line

      i. $x = 3$ using washers.

      ii. $x = -4$ using washers.

      iii. $y = 4$ using cylindrical shells.

      iv. $y = -6$ using cylindrical shells.

35. A chain lying on the ground is 10 meters long and has a mass of 80 kilograms. How much work is required to raise on end of the chain to a height of 6 meters?

36. Consider a tank of water obtained by revolving the region bounded by $y = x^2$ and $y = 4$ around the y-axis, where units are in meters. Suppose the tank is filled with water to a depth of 3 meters, how much work is required to draw water out of the tank through a spout 2 meters above the top of the tank leaving behind 1 meter of water? Use that the density of water is $1000\,kg/m^3$.

37. A block of ice in the shape of a cube with side length 2 meters has an approximate mass of $7350\,kg$. Such a block of ice is being lifted out of a 120 meter deep ravine at a rate of $2\,m/s$ using a rope with a mass of $180\,kg$. As it is lifted the block of ice melts so that its surface area decreases at a rate of $0.1\,m^2/s$ and the block of ice remains in the shape of a cube. How much work is required to lift the block of ice out of the ravine?
   Hint: Use that you know the surface area as a function of time to find the side length and hence the volume as a function of time. I encourage you to also find the rate of change of the side length and rate of change of the volume using related rates and check that your answers agree.