Name: Solutions Date: 11/09/2017

M20580 L.A. and D.E. Tutorial
Worksheet 8
Sections 2.1, 2.2, 2.3

1. Which of the following are the first-order linear differential equations? Check **all** that apply:

   - ☐ $y' = \dfrac{M(x)}{N(y)}$ is separable, not linear
   - ☐ $y'' + P(x)y = Q(x)$ this is second order
   - ☒ $y' + P(x)y = Q(x)$
   - ☐ $P(x)y' + y = Q(x)y^2$ ⟵ non-linear
   - ☒ $P(x)y' + Q(x)y = R(x)$
   - ☒ $y' = P(x) + Q(x)y$ or $y' - Q(x)y = P(x)$

   Write the formula for the integrating factor for each linear equation you found above.

   $$y' + P(x)y = Q(x) \Rightarrow \boxed{\mu(x) = e^{\int P(x)\,dx}}$$

   $$P(x)y' + Q(x)y = R(x) \Leftrightarrow y' + \frac{Q(x)}{P(x)}y = \frac{R(x)}{P(x)} \Rightarrow \boxed{\mu(x) = e^{\int \frac{Q(x)}{P(x)}dx}}$$

   $$y' = P(x) + Q(x)y \Leftrightarrow y' - Q(x)y = P(x) \Rightarrow \boxed{\mu(x) = e^{\int -Q(x)\,dx}}$$

2. Determine whether the following differential equation is first-order linear or separable equation?

   $$\frac{dy}{dx} = \frac{\ln x + y\cos x}{\csc x}$$ ⟵ make it not separable

   $$\Leftrightarrow (\csc x)y' = \ln x + y\cos x \Leftrightarrow (\csc x)y' - (\cos x)y = \ln x$$ is linear eq'n

   If it's a linear equation, find the integrating factor (you don't need to solve it). But, if it's a separable equation, find general solutions to the differential equation

   Rewrite it into $y' - \left(\dfrac{\cos x}{\csc x}\right)y = \dfrac{\ln x}{\csc x} \Leftrightarrow y' - (\cos x\sin x)y = \dfrac{\ln x}{\csc x}$

   $\dfrac{\cos x}{\csc x} = \dfrac{\cos x}{\frac{1}{\sin x}}$

   $$\mu(x) = e^{\int -\cos x\sin x\,dx} = \boxed{e^{-\frac{1}{2}\sin^2 x}}$$

Name: separable

Date: 11/09/2017

3. Let $\phi(x)$ be a solution to $\frac{dy}{dx} = \frac{1+y^2}{x^2}$ that satisfies $\phi(1) = 0$. Find $\phi(2)$.

$$\frac{dy}{dx} = \frac{1+y^2}{x^2} \Leftrightarrow \frac{dy}{1+y^2} = \frac{dx}{x^2} \Leftrightarrow \int \frac{dy}{1+y^2} = \underbrace{\int \frac{dx}{x^2}}_{= \int x^{-2}dx = -x^{-1}}$$

$$\Leftrightarrow \tan^{-1}(y) = -\frac{1}{x} + C$$

So $\tan^{-1}(\phi) = -\frac{1}{x} + C \Rightarrow \phi = \tan\left(-\frac{1}{x} + C\right) \rightarrow$ find C

$$\phi(1) = \tan(C-1) = 0$$

$$\Rightarrow C-1 = \tan^{-1}(0) \Rightarrow C-1 = 0 \Rightarrow C = 1$$

Thus, $\phi(x) = \tan\left(-\frac{1}{x}+1\right)$

$$\phi(2) = \tan\left(-\frac{1}{2}+1\right) = \boxed{\tan\left(\frac{1}{2}\right)}$$

4. Solve the differential equation $y' = xy + e^{x^2/2}\sin x$ with $y(0) = 2$

Rewrite the differential equation into $\underbrace{y' - xy = e^{x^2/2}\sin x}$

↑ 1st order linear eq'n

$$\mu(x) = e^{\int -x\,dx} = e^{-\frac{x^2}{2}}$$

So, $\left[ye^{-\frac{x^2}{2}}\right]' = e^{-x^2/2}\left(e^{x^2/2}\sin x\right)$

$$\Rightarrow ye^{-x^2/2} = \int \sin x\,dx \Rightarrow ye^{-x^2/2} = -\cos x + C \Rightarrow y = -e^{+x^2/2}\cos x + Ce^{+x^2/2}$$

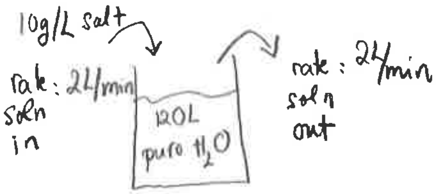
Have $y(0) = -1 + C \overset{\text{given}}{=\!=} 2 \Rightarrow C = 3$

Thus, $y = -e^{x^2/2}\cos x + 3e^{x^2/2}$

Name: Date: 11/09/2017

5. A tank initially contains 120 L of pure water. A mixture containing a concentration of 10 g/L of salt enters the tank at the rate of 2 L/min, and the well-stirred mixture leaves the tank *at the same rate.*

Find an expression for the amount of salt in the tank at any time t. Also find the limit of the amount of salt in the tank as $t \to \infty$.



Let y = the amount of salt in the tank at any time (t)
(gram of salt)

We have $y(0) = 0$ since initially, the tank contains only pure H_2O

We know, $\dfrac{dy}{dt} = \underbrace{(\text{rate in of salt})}_{[\text{unit: g/min}]} - \underbrace{(\text{rate out of salt})}_{[\text{unit: g/min}]}$. Note, 2L/min is the rate in of the salt solution, not of salt alone.

$$= \underbrace{\begin{pmatrix}\text{salt concentration}\\ \text{coming in}\end{pmatrix}\begin{pmatrix}\text{rate in}\\ \text{of}\\ \text{salt solution}\end{pmatrix}}_{(\text{rate in of salt})} - \underbrace{\begin{pmatrix}\text{salt contentration}\\ \text{coming out}\end{pmatrix}\begin{pmatrix}\text{rate out}\\ \text{of}\\ \text{salt solution}\end{pmatrix}}_{(\text{rate out of salt})}$$

$$= \left(10\,\frac{g}{L}\right)\left(\frac{2\,L}{min}\right) - \underbrace{\frac{(y \text{ gram of salt in tank})}{\begin{pmatrix}\text{volume of soln}\\ \text{in the tank}\end{pmatrix}}}_{\text{salt conc. coming out}} \underbrace{\left(\frac{2\,L}{min}\right)}_{\text{rate out of soln.}} \qquad (*)$$

Now, since (rate in of the solution) = (rate out of the solution), the volume of solution inside the tank doesn't change. So, (vol of soln in tank) = 120L

$$\Rightarrow \frac{dy}{dt} = \left(20\,\frac{g}{min}\right) - \frac{y \text{ gram}}{120\,L} \cdot \frac{2\,L}{min}$$

$$\Rightarrow \frac{dy}{dt} = 20\,\frac{g}{min} - \frac{y}{60}\,\frac{g}{min}$$

$$\Rightarrow \frac{dy}{dt} = 20 - \frac{y}{60}$$ (now solve for y. Note, this differential eq'n is both separable and linear equation)

$$\Rightarrow y' + \frac{1}{60}y = 20$$ (treat as linear eq'n)

$\mu(t) = e^{\int 1/60\, dt} = e^{\frac{t}{60}}$. So, $e^{t/60} y = \int 20 e^{\frac{t}{60}} dt \Rightarrow e^{t/60} y = 1200 e^{\frac{t}{60}} + C$

So, $y = 1200 + Ce^{-t/60}$ $\rightarrow$ find C: $y(0) = 1200 + C \overset{\text{given}}{=} 0 \Rightarrow C = -1200$

Thus, $\boxed{y(t) = 1200 - 1200e^{-t/60}}$

$$\lim_{t \to \infty} \left(1200 - 1200e^{-t/60}\right) = \boxed{1200}$$