Math 10550 Quiz 1 Solutions

1. The function

$$f(x) = 17 + \ln(x^3 - 7)$$

is a one-to-one function (There is no need to check this). What is $(f^{-1})'(17)$?

Solution: We wish to use the following theorem: If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$ then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

First, observe that $f^{-1}(17) = 2$ since

$$f(2) = 17 + \ln(2^3 - 7) = 17 + \ln(8 - 7) = 17 + \ln(1) = 17 + 0 = 17.$$

Next, compute the derivative

$$f'(x) = \frac{3x^2}{x^3 - 7}.$$

Then, since $f'(f^{-1}(17)) = f'(2) = 12 \neq 0$, we conclude by the theorem

$$(f^{-1})'(17) = \frac{1}{f'(f^{-1}(17))} = \frac{1}{f'(2)} = \frac{1}{12}.$$

2. Differentiate the function

$$f(x) = \frac{(x^4-2)^4 x^2}{(x+3)^5}.$$

Solution: One can directly compute the derivative using the quotient rule, but logarithmic differentiation is more elegant. Taking the logarithm of both sides, we get

$$\ln(f(x)) = \ln\left(\frac{(x^4-2)^4 x^2}{(x+3)^5}\right).$$

Simplifying using log rules,

$$\begin{aligned} \ln(f(x)) &= \ln(x^4-2)^4 + \ln(x^2) - \ln(x+3)^5 \\ &= 4\ln(x^4-2) + 2\ln x - 5\ln(x+3). \end{aligned}$$

Now, differentiating,

$$\begin{aligned} \frac{1}{f(x)} f'(x) &= 4 \cdot \frac{4x^3}{x^3-2} + 2 \cdot \frac{1}{x} - 5 \cdot \frac{1}{x+3} \\ &= \frac{16x^3}{x^3-2} + \frac{2}{x} - \frac{5}{x+3}. \end{aligned}$$

Finally, solving for $f'(x)$,

$$\begin{aligned} f'(x) &= f(x)\left(\frac{16x^3}{x^3-2} + \frac{2}{x} - \frac{5}{x+3}\right) \\ &= \frac{(x^4-2)^4 x^2}{(x+3)^5}\left(\frac{16x^3}{x^3-2} + \frac{2}{x} - \frac{5}{x+3}\right). \end{aligned}$$