Sets

A **set** is a collection of objects. The objects are called **elements** of the set.

A set can be described as a list, for example $D = \{5, 6, 7\}$ or with words (often in many different ways):

$$\begin{aligned} D &= \{\text{All whole numbers between 5 and 7 inclusive}\} \\ &= \{\text{All integers bigger than 4 and less than 8}\} \end{aligned}$$

Repetitions in the list or changes in the order of presentation do not change the set. For example, the following lists describe the same set:

$$\{5, 5, 6, 7\} = \{5, 6, 7\} = \{6, 5, 7\}.$$

Sets

When describing the elements of a set, we should be careful that there is **no ambiguity in our description** and the set is well-defined.

For example, to talk about "the set of the 15 sexiest men of the twenty-first century" does not make sense, since the description is subject to personal opinion and different people may produce different sets from this description.

On the other hand, the **set** of "men named 'sexiest man alive' by *People* magazine between 2001 and 2017 (inclusive)" is unambiguous.

Notation

We read the notation "$5 \in D$" as "*5 is an element of D*".
We read the notation "$2 \notin D$" as "*2 is not an element of D.*"

Equal Sets: We say two sets are **equal** if they consist of exactly the same elements. For example consider the sets:

$A = \{\text{odd integers between 2 and 8}\}$,
$B = \{\text{prime numbers bigger than 2 and less than 10}\}$.

These two sets are equal and have three elements. We have $A = B = \{3, 5, 7\}$.

Infinite Sets and dot notation

A set can have infinitely many elements, so we can't list all of them. For example let
$E = \{\text{all even integers greater than or equal to } 1\}$.

We write this as $E = \{2, 4, 6, 8, \dots\}$, where "$\dots$" should be read as "*et cetera*".

When we place an element after the dots, as in
$K = \{2, 4, 6, 8, \dots, 100\}$, this indicates that we are talking about the finite set of even numbers greater than 0 and less than or equal to 100 (the last element on the list is 100).

Note: This notation requires us to write enough terms for the pattern to become clear. For example, $\{2, 4, \dots\}$ is unclear as it may describe even integers, but may also describe powers of 2.

Set-Builder Notation

A description of the set $D = \{5, 6, 7\}$ from before may also be written using *set-builder notation*:

$$D = \{x \mid x \text{ is an integer between 5 and 7 inclusive}\}$$

or

$$D = \{x \mid x \text{ is an integer and } \; 5 \leq x \leq 7\}$$

Here the symbol $|$ is read as "such that" and the upper mathematical sentence above reads as "D is the set of all x such that x is an integer between 5 and 7 inclusive".

The Empty Set

The **empty set** is the set with no elements, i.e. the list of its elements is a blank list. It is denoted by the symbol $\emptyset$. One can think of the empty set as an empty list: $\{\ \}$.

This set can have many verbal descriptions, for example:

$\{$all students in this class which are math majors$\} = \emptyset$.

$\{$all even prime numbers bigger than 10$\} = \emptyset$.

$\{$years in which Adrian didn't have a Dracula accent$\} = \emptyset$.

Subsets

A **subset of a set** A is a collection of elements of A. We have B is a subset of A (written as $B \subseteq A$) if every element of B is also an element of A. We say that B is a **proper subset** of A if $B \subseteq A$, but $B \neq A$.

Example : Suppose $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, $C = \{3, 4, 5, 6\}$ and $D = \{2, 6\}$.

Is $D \subseteq A$? No: $6 \in D$ but $6 \notin A$.

Is $D \subseteq B$? Yes. $2 \in B$ and $6 \in B$, so every element in D is also in B.

Is $D \subseteq C$? No: $2 \in D$ but $2 \notin C$.

Is D a proper subset of B? Yes: Notice $8 \in B$ but $8 \notin D$, so the two sets are not equal.

Subsets

Any set is a subset of itself, because it complies with the requirement in the definition of subsets. However, it is not a proper subset.

The empty set is a subset of any other set: $\emptyset \subseteq A$ for every set A. It is always proper (unless $A = \emptyset$).

Two sets A and B are **equal** if and only if they are subsets of each other, i.e. $A \subseteq B$ and $B \subseteq A$.

The collection of all the subsets of a given set is called the **power set**. For example, the power set of $\{a, b, c\}$ has eight elements: $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ and $\{a, b, c\}$

Universal Sets

Sometimes we wish to restrict our attention to a particular set, called a **universal set** and usually denoted by U.

For example, in doing a voter polling survey you might want to restrict attention to likely voters. In this case the universal set would be likely voters (however you define it). Or you might want to survey likely Republican voters which would give you a different universal set.

Universal Sets

Example: If we do a survey on music preferences in our class, we would use $U = \{\text{all students in our class}\}$. Then

$R = \{\text{students in the class who like Rap music}\}$,

$C = \{\text{students in the class who like Classical music}\}$,

$E = \{\text{students in the class who like 80's music}\}$,

are all subsets of our universal set U.

Note: To avoid ambiguity in the definition of such sets, it is common in surveys to restrict answers to the given questions to "yes" and "no":

Do you like Rap music?   Yes   No   (circle one)

The resulting sets are well defined, but we have ignored the tastes of those who like some rap music but not all.

Set Unions

Given two sets, A and B, we define their **union**, denoted $A \cup B$ (read "A union B"), to be the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Important note: when we say "or" we **always** mean it in the inclusive sense. That is, $A \cup B$ consists of the elements that are in A or B, or both A and B.

Properties:

- $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
- $A \cup B = B \cup A$.
- $A \cup \emptyset = \emptyset \cup A = A$.

Set Unions

Example: If $A = \{1, 2, 6, 7, 8\}$ and $B = \{-1, 3, 6, 8\}$, what is $A \cup B$? $A \cup B = \{1, 2, 6, 7, 8, -1, 3\}$.

Example: If $X = \{-1, 0, 1, 2\}$ and $Y = \{-2, 2\}$, what is $X \cup Y$? $X \cup Y = \{-2, -1, 0, 1, 2\}$.

Example: If $D = \{1, 3, 5, 7, 9\}$, what is $D \cup \emptyset$?
$D \cup \emptyset = \{1, 3, 5, 7, 9\} = D$.

Set Unions

Union of 3 sets If A and B and C are sets, their union $A \cup B \cup C$ is the set whose elements are those objects which appear in at least one of A **or** B **or** C.

Example: If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$, list the elements of the set $A \cup B \cup C$.

$A \cup B = \{1, 2, 3, 4, 6, 8\}$, $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 8\}$.

$B \cup C = \{2, 3, 4, 5, 6, 8\}$, $A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6, 8\}$.

$A \cup C = \{1, 2, 3, 4, 5, 6\}$, $B \cup (A \cup C) = \{1, 2, 3, 4, 5, 6, 8\}$.

Note: Set union combines all the elements together.

Set Intersections

For sets A and B, their **intersection** $A \cap B$ (read "A intersect B") is the set of those elements which are in A **and** in B (i.e. the common elements of A and B).

Example: If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, list the elements of the set $A \cap B$. $\{2, 4\}$

If A and B and C are sets, their intersection $A \cap B \cap C$ is the set whose elements are those objects which appear in A **and** B **and** C (i.e. those elements which are common to all three sets, or belong to all three sets simultaneously).

Example If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{1, 3, 4, 5, 6\}$, find $A \cap B \cap C$. $\{4\}$.

Universal set and complements

Given a subset A of the universal set U, the **complement** of A, denoted by A', consists of all the elements of U which are **not** in A.

Example: If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$ are subsets of the universal set $U = \{1, 2, \ldots, 10\}$, find $A' \cup B \cup C$.

$A' = \{5, 6, 7, 8, 9, 10\}$
$B \cup C = \{2, 3, 4, 5, 6, 8\}$
$A' \cup B \cup C) = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Universal set and complements

Example: Give a verbal description of the set $R \cup C \cup E$ from our class survey example. Are you in this set?
The people in the class who liked rap, classical, or 80's music.

Example: Give a verbal description of the set $(R \cup E \cup C)'$. Are you in this set?
The people in the class who liked none of rap, classical or 80's music.

Properties of the empty set and complements

Properties of the empty set: For any set A,

$$\emptyset \cup A = A, \qquad \emptyset \cap A = \emptyset, \quad \text{and} \quad \emptyset \subset A.$$

Properties of the complement:

$$A \cap A' = \emptyset, \qquad (A')' = A, \quad \text{and} \quad A \cup A' = U$$

DeMorgan laws for sets:

$$(A \cup B)' = A' \cap B' \quad \text{and} \quad (A \cap B)' = A' \cup B'$$

Set Difference

For sets A and B, the **set difference** $A \setminus B$ (read "A minus B") is the set of those elements which are in A **but not in** B (i.e. the common elements of A and B).

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\} = A \cap B'$$

$A \setminus B$ is **the complement of** B **relative to** A. In a sense, we are restricting our universal set to A and take the complement there.

Example: If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, list the elements of the set $A \setminus B$. $\{1, 3\}$

What about $B \setminus A$? $\{6, 8\}$.

Disjoint sets

Two sets are **disjoint** if they have no elements in common (i.e. their intersection is empty).

Example: Consider the sets
$A = \{\text{all odd integers}\}$
$B = \{\text{all integers which are divisible by } 2\}$
Then A and B are disjoint, since there are no odd integers which are divisible by 2 (i.e. $A \cap B = \emptyset$).

Example: Is there a disjoint pair of sets from among $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, and $C = \{3, 5, 7, 9\}$?

$B \cap C = \emptyset$, so B and C are disjoint.
A and B have elements in common, so they are not disjoint. Also, A and C are not disjoint.