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Worksheet 6, Math 10560

Times indicate the amount of time that you would be expected to spend on the problem in on an exam. All problems have appeared on old exams for Calculus 2.

1. (4 mins) Evaluate the integral or show that it is divergent

$$\int_{-\infty}^{3} \frac{1}{x^2 - 4x + 5}\, dx$$

Solution: This is an improper integral of type 1(b). We note that the denominator is never zero (the discriminant $b^2-4ac = 16-20 = -4 < 0$, so there are no real solutions to the equation $x^2-4x+5=0$), so $f(x) = \frac{1}{x^2-4x+5}$ is continuous for all real numbers. We complete the square to rewrite $x^2 - 4x + 5 = x^2 - 4x + 2^2 + 5 - 2^2 = (x-2)^2 + 1$, and evaluate the integral as follows:

$$\begin{aligned}\int_{-\infty}^{3} \frac{1}{x^2 - 4x + 5}\, dx &= \lim_{t\to-\infty} \int_{t}^{3} \frac{1}{(x-2)^2+1}\, dx \\ &= \lim_{t\to-\infty} \left[\arctan(x-2)\right]_t^3 \\ &= \lim_{t\to-\infty} \left[\arctan(1) - \arctan(t-2)\right] \\ &= \arctan(1) - \lim_{t\to-\infty} \arctan(t-2) \\ &= \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) \\ &= \frac{3\pi}{4}.\end{aligned}$$

2. (4 mins) Find the arc length of the curve $y = \dfrac{e^{2x} + e^{-2x}}{4}$, $0 \le x \le 2$, using the arc length formula.

Solution: By the arc length formula we have

$$L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Taking derivatives we obtain,

$$\begin{aligned}\frac{dy}{dx} &= \frac{2e^{2x} - 2e^{-2x}}{4} \\ &= \frac{e^{2x} - e^{-2x}}{2}.\end{aligned}$$

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Thus,

$$\begin{aligned}
L &= \int_0^2 \sqrt{1 + \left(\frac{e^{2x} - e^{-2x}}{2}\right)^2}\, dx. \\
&= \int_0^2 \sqrt{1 + \frac{e^{4x} - 2e^{2x}e^{-2x} + e^{-4x}}{4}}\, dx \\
&= \int_0^2 \sqrt{\frac{4 + e^{4x} - 2 + e^{-4x}}{4}}\, dx \\
&= \int_0^2 \sqrt{\frac{e^{4x} + 2 + e^{-4x}}{4}}\, dx \\
&= \int_0^2 \sqrt{\left(\frac{e^{2x} + e^{-2x}}{2}\right)^2}\, dx \\
&= \int_0^2 \frac{e^{2x} + e^{-2x}}{2}\, dx \\
&= \left.\frac{e^{2x} - e^{-2x}}{4}\right|_0^2 \\
&= \frac{e^4 - e^{-4}}{4}.
\end{aligned}$$

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3. (8 mins) Complete the following sentences using the words *converges* and *diverges* :

$\int_1^\infty \frac{1}{x^p} dx$ ___*converges*___ if $p > 1$ and ___*diverges*___ if $p \leq 1$.

$\int_0^1 \frac{1}{x^p} dx$ ___*diverges*___ if $p \geq 1$ and ___*converges*___ if $p < 1$.

Decide whether the following improper integrals converge or diverge by comparing them to a known integral. In each case, state which integral you are comparing the given integral to and state clearly why you can conclude convergence or divergence.

(a) $\int_1^\infty \frac{1}{x^2 + x + 5} \, dx$

> **Solution:** We use the comparison test for integrals. Let $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x^2+x+5}$. Both $f(x)$ and $g(x)$ are continuous on the interval $[1, \infty)$ and we have $f(x) \geq g(x) \geq 0$ for $x \geq 1$. Applying the first statement above with $p = 2$, we know $\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^2} \, dx$. Thus, by the comparison test for integrals, $\int_1^\infty g(x)dx = \int_1^\infty \frac{1}{x^2 + x + 5} \, dx$ converges.

(b) $\int_1^\infty \frac{1}{xe^x} \, dx$

> **Solution:** Again we use the comparison test for integrals. Note that for $x \geq 1$, $e^x \geq x$. So $\frac{1}{x^2} \geq \frac{1}{xe^x} \geq 0$ for $x \geq 1$, and by the comparison test since $\int_1^\infty \frac{1}{x^2} \, dx$ converges, so does $\int_1^\infty \frac{1}{xe^x} \, dx$.

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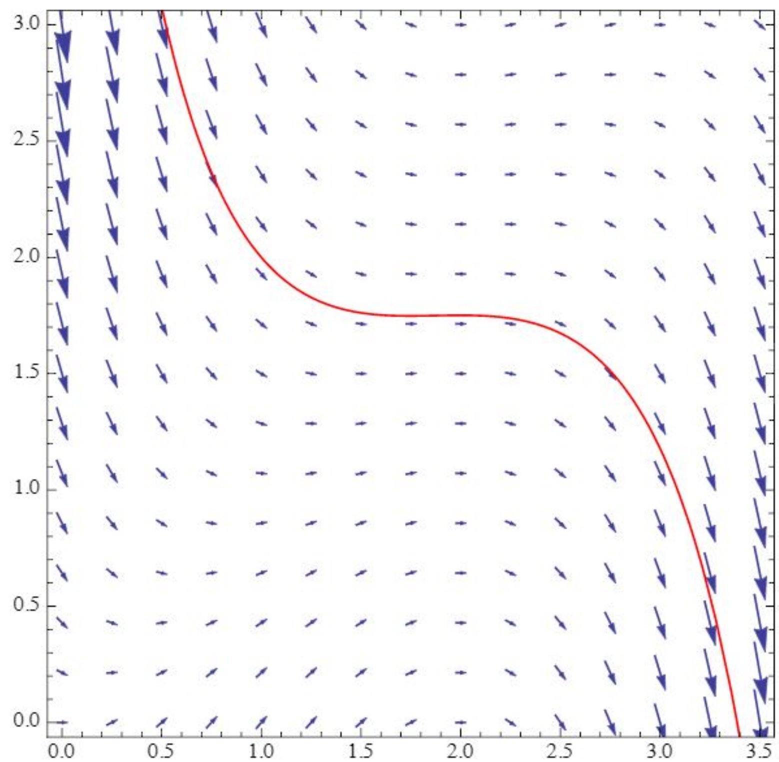
4. Let $y' = (x-2)(y-x)$ with $y(1) = 2$.

   (a) Draw a 3×3 direction field ($x = 1, 2, 3,\ y = 1, 2, 3$), and approximate solution curve.

Solution: We compute $y' = F(x, y) = (x-2)(y-x)$ at the points (x, y) for $x = 1, 2, 3,\ y = 1, 2, 3$.

x	1	1	1	2	2	2	3	3	3
y	1	2	3	1	2	3	1	2	3
y'	0	-1	-2	0	0	0	-2	-1	0

Using this data we can sketch the direction field and an approximate solution to the equation with $y(1) = 2$.



   (b) Use Euler's method with $\Delta x = 1$ to estimate $y(3)$. How close is Euler's method to the solution curve you drew by hand above?

Solution: We use Euler's Method with step size $\Delta x = 1$. Our first point is $(x_0, y_0) = (1, 2)$.

$$x_1 = x_0 + \Delta x = 2 \Rightarrow y_1 = y_0 + F(x_0, y_0) = 2 + (1-2)(2-1) = 1$$

$$x_2 = x_1 + \Delta x = 3 \Rightarrow y_2 = y_1 + F(x_1, y_1) = 1 + (2-2)(1-2) = 1$$

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$\boxed{\text{So } y(3) \approx 1. \text{ The solution drawn in part a gives an estimate of } y(3) \approx 1.2.}$

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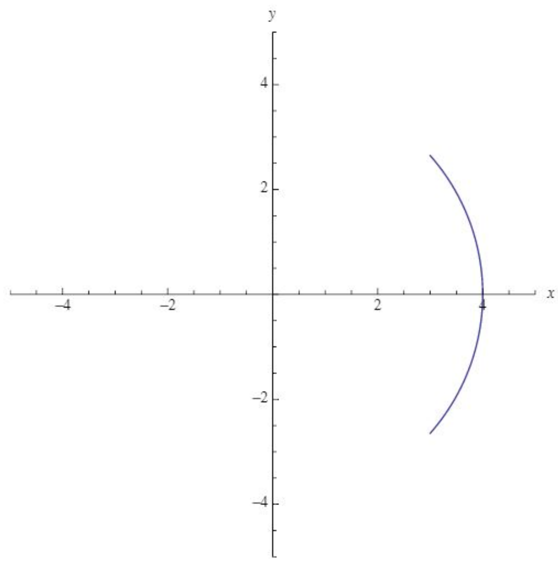
5. Find the arc length of the circle $y^2 + x^2 = 16$ between the point $(3, \sqrt{7})$ and $(3, -\sqrt{7})$ (using the shorter arc between them) using the arc length formula.

Solution: The portion of the circle we are interested in is drawn in the graph below.



We can solve the equation for x to obtain $x = \sqrt{16 - y^2}$ (we take the positive square root because x is positive on the portion of the circle we are interested in). Then by the formula for arc length

$$
\begin{aligned}
L &= \int_{-\sqrt{7}}^{\sqrt{7}} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= 2 \int_{0}^{\sqrt{7}} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy
\end{aligned}
$$

where the last equality comes from the symmetry of the problem. Taking the derivative we obtain:

$$
\begin{aligned}
\frac{dx}{dy} &= \frac{1}{2}(16 - y^2)^{-1/2}(-2y) \\
&= \frac{-y}{\sqrt{16 - y^2}}.
\end{aligned}
$$

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Now,

$$
\begin{aligned}
L &= 2\int_0^{\sqrt{7}} \sqrt{1+\left(\frac{-y}{\sqrt{16-y^2}}\right)^2}\,dy \\
&= 2\int_0^{\sqrt{7}} \sqrt{1+\frac{y^2}{16-y^2}}\,dy \\
&= 2\int_0^{\sqrt{7}} \sqrt{\frac{16}{16-y^2}}\,dy \\
&= 8\int_0^{\sqrt{7}} \frac{dy}{\sqrt{16-y^2}} \\
&= 8\arcsin\left(\frac{y}{4}\right)\Big|_0^{\sqrt{7}} \\
&= 8(\arcsin\left(\frac{\sqrt{7}}{4}\right)-\arcsin(0)) \\
&= 8\arcsin\left(\frac{\sqrt{7}}{4}\right) \\
&\approx 5.78
\end{aligned}
$$

Alternative Solution: We can also solve for y to obtain $y=\pm\sqrt{16-x^2}$. The $\frac{dx}{dx}=\frac{\pm x}{\sqrt{16-x^2}}$. Our formula for arc length in this case becomes:

$$
\begin{aligned}
L &= \int_3^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx - \int_4^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\
&= 2\int_3^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \\
&= 2\int_3^4 \sqrt{1+\left(\frac{\pm x}{\sqrt{16-x^2}}\right)^2}dx \\
&= 8\int_3^4 \frac{dx}{\sqrt{16-x^2}} \\
&= 8\arcsin\left(\frac{x}{4}\right)\Big|_3^4 \\
&= 8\left(\arcsin(1)-\arcsin\left(\frac{3}{4}\right)\right) \\
&\approx 5.78
\end{aligned}
$$

Remark: A good way to check our work is to make sure the formula we have works for an arc length we know. We know the arc length from $(0,-4)$ to $(0,4)$ is

$\frac{2\pi r}{2} = \pi r = 4\pi$. According to our computations above,

$$\begin{aligned} L &= 2\int_0^4 \sqrt{1 + \left(\frac{-y}{\sqrt{16-y^2}}\right)^2}\, dy \\ &= 8 \arcsin\left(\frac{y}{4}\right)\bigg|_0^4 \\ &= 8(\arcsin(1) - \arcsin(0)) \\ &= 8 \cdot \frac{\pi}{2} \\ &= 4\pi. \end{aligned}$$