[redacted] # MATB24 Definitions & Theorems [redacted]

Set Notation and operations

SET

collection of objects

$a \in S$ - a is an element of the set S

$|S|$ - cardinality of S (size of S)

ϕ - unique set that does not contain elements. $|\phi| = 0$

SUBSET - set x is a subset of S ($x \subseteq S$) $\Leftrightarrow$ $s \in x$, $s \in S$

UNION - $X \cup Y$ $\{a \mid a \in X \text{ or } a \in Y\}$

INTERSECTION - $X \cap Y$ $\{a \mid a \in X \text{ and } a \in Y\}$

~~DIFFENC~~ DIFFERENCE - $X - Y$ or $X \setminus Y$ $\{a \mid a \in X \text{ and } a \notin Y\}$

COMPLEMENT - $X^c = U \setminus X$ $\{a \mid a \notin X\}$ $X \subseteq U$ (universe)

BINARY OPERATION $(S, *)$

A binary operation on a set S is a rule for combining two elements of S to produce a third element of S.

* operation must be closed.

PROPERTIES A BINARY OPERATOR MIGHT HAVE

1. Commutativity $s*t = t*s$ $\forall s \in S, t \in S$
2. Associativity $(s*t)*u = s*(t*u)$ $\forall s,t,u \in S$
3. Identity $\exists e \in S$ s.t. $\forall s \in S$, $e*s = s*e = s$
4. Inverse $\exists e \in S$ s.t. $\forall s, s^{-1} \in S$, $s*s^{-1} = s^{-1}*s = e$

LEMMA

1. The identity element is unique
2. The inverse of $a \in S$ (if it exists) is unique if $(S,*)$ is associative

VECTOR SPACE

A real vector space is a set V of objects called vectors, with a rule for adding vectors in the set and multiplying a ~~s~~ vector by a scalar to produce another vector in V. There must exist a $\vec{0}$, a $-\vec{v}$ $\forall v \in V$ such that A1-A4, S1-S4 holds for all choices of vectors $\vec{u}, \vec{v}, \vec{w} \in V$ and $c_1, c_2 \in \mathbb{R}$.

A1 $\forall v,w \in V$: $v+w = w+v$

A2 $\forall u,v,w \in V$: $u+(v+w) = (u+v)+w$

A3 $\exists 0_v$ s.t. $\forall v \in V$: $0_v + v = v + 0_v = v$

A4 $\forall v \in V$, $\exists \overset{(v^{-1})}{-v} \in V$: $v + \overset{(v^{-1})}{(-v)} = \overset{(v^{-1})}{-v} + v = 0_v$

S1 $\forall c \in \mathbb{R}, u,v \in V$: $c(u+v) = cu + cv$

S2 $\forall c_1, c_2 \in \mathbb{R}, u \in V$: $(c_1 + c_2)u = c_1 u + c_2 u$

S3 $\forall c_1, c_2 \in \mathbb{R}, u \in V$: $(c_1 c_2)u = c_1(c_2 u)$

S4 $\forall v \in V$: $1(v) = v$

* have to be closed under vector addition and scalar multiplication

PROPERTIES OF VECTOR SPACE Let V be vector space and $\vec{v}, \vec{w}, \vec{u} \in V$, $r \in \mathbb{R}$

1. Identity $(\vec{0}_v)$ is unique
2. Inverse $(\vec{v}^{-1})$ is unique
3. If $\vec{u}+\vec{v}=\vec{u}+\vec{w}$, then $\vec{v}=\vec{w}$
4. $0\vec{v}=\vec{0}_v$
5. $r\vec{0}=\vec{0}$
6. $(-r)\vec{v}=-(r\vec{v})=r(-\vec{v})$, $\forall \vec{v} \in V, \forall r \in \mathbb{R}$

~~SR~~ SUBSPACE let V be a vector space

$W \subseteq V$ is said to be a subspace of V if W forms a vector space under the same operations as V. (vector addition and scalar multiplication)

* ~~Ha~~ Has to be non-empty, closed under vector addition and scalar mult. also follows A1-A4, S1-S4

LINEAR COMBINATION let V be a vector space

Let $\{\vec{v}_j\}_{j=1}^{n}$ ~~t~~ $\subseteq V$ and $\{c_j\}_{j=1}^{n} \subseteq \mathbb{R}$, $\vec{w} \in V$ is a linear combination of vectors $\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n$ if $\exists \{c_j\}_{j=1}^{n}$ s.t. $\vec{w}=c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_n\vec{v}_n$

SPAN let V be a vector space

Let $S \subseteq V$. Span(S) is the set of all linear combinations of vectors $\{v_j\}_{j=1}^{n} \subseteq S$

$$\text{span}(S)=\left\{\sum_{j=1}^{n} c_j\vec{v}_j \mid \vec{v}_j \in S, c_j \in \mathbb{R}, n \in \mathbb{N}\right\}$$

FINITELY GENERATED

If there is a finite number of elements in S and S spans V, then V is ~~finid~~ finitely generated by S

LINEAR DEPENDENCY

$\{\vec{v}_j\}_{j=1}^{n}$ is linearly ~~def~~ dependent if $\exists (c_1, \cdots, c_n) \neq \vec{0}$ s.t. $\sum_{j=1}^{n} c_j\vec{v}_j=\vec{0}$

$(\exists c_i \neq 0$ s.t. $\sum_{j=1}^{n} c_j\vec{v}_j=\vec{0})$

$\{\vec{v}_j\}_{j=1}^{n}$ is linearly independent if $\sum_{j=1}^{n} c_j\vec{v}_j=\vec{0}_v \Rightarrow c_j=0 \ \forall j \in \{1, \cdots, n\}$

* To prove linear independent, differentiate (for sin and cos and e) and pick values of x (do not pick c since WTS $c=0$)

To prove linear dependent, pick c (do not pick x since WTS $\sum_{j=1}^{n} c_j\vec{v}_j=\vec{0}$ for arbitrary x)

LEMMA

Linear dependency $\Rightarrow$ one vector is a linear combination of others

BASIS

Let V be a real vector space. $\{\vec{V}_1, \dots, \vec{V}_n\}$ is a basis $\Leftrightarrow$ a. $\text{span}(\{\vec{V}_1, \dots, \vec{V}_n\}) = V$ and
b. $\{\vec{V}_1, \dots \vec{V}_n\}$ is lin. independent

UNIQUE COMBINATION CRITERION FOR A BASIS

Let V be a vector space. $\{\vec{V}_1, \dots, \vec{V}_n\}$ is a basis $\Leftrightarrow$ each vector in V can be uniquely expressed as a linear combination of $\{\vec{V}_j\}_{j=1}^{n}$

RELATIVE SIZE OF SPANNING AND ~~IM~~ INDEPENDENT SETS

Let V be a vector space. Let ~~s~~ $\text{span}(S) = V$ and $X \subseteq V$ be linearly independent. Then $|X| \leq |S|$

INVARIANCE OF DIMENSION FOR FINITELY GENERATED SPACES

Every vector space has a basis. If we have max # of linearly independent vectors in X and ~~minim~~ min # of vectors in S that spans V, then $|X| = |S|$

DIMENSION

The dimension of a vector space V is given by $n = |S| = |X|$

LEMMA

Suppose $\dim(V) = n$ and $S \subseteq V$

i. if $\text{span}(S) = V$ and $|S| = n$, then S is a basis for V

ii. if S is linearly independent and $|S| = n$, then S is a basis for V.

Any two would work
$|S| = n$
$\text{span}(S) = V$
S is linearly independent

LINEAR TRANSFORMATION

Let V, W be vector spaces. A map $T: V \to W$ is said to be a linear transformation $\Leftrightarrow$

1. $T(\vec{V}_1 + \vec{V}_2) = T(\vec{V}_1) + T(\vec{V}_2)$, $\forall \vec{V}_1, \vec{V}_2 \in V$
2. $T(c\vec{V}_1) = cT(\vec{V}_1)$, $\forall \vec{V}_1 \in V, c \in \mathbb{R}$

ONE-TO-ONE / INJECTIVE

T is 1-1 $\Leftrightarrow$ $T(\vec{V}) = \vec{O}_W \Leftrightarrow \vec{V} = \vec{O}_V$ $\qquad$ $T(\vec{x}) = T(\vec{y}) \Leftrightarrow \vec{x} = \vec{y}$

$\Updownarrow$

$\ker(T) = \{\vec{O}_V\}$

ONTO / SURJECTIVE

$\text{Im}(T) = W \Leftrightarrow T$ is onto $\qquad$ $\forall \vec{w} \in W, \exists \vec{V} \in V$ s.t $T(\vec{V}) = \vec{w}$

TERMS FOR LINEAR TRANSFORMATIONS

IMAGE OF $\vec{V}$ UNDER T — if $\vec{V} \in V$, $T(\vec{V})$ is the image

IMAGE OF U UNDER T — if $U \subseteq V$, $T(U) = \{T(\vec{u}) \mid \vec{u} \in U\}$ is the image $(\text{Im}(T) = T(V))$

PREIMAGE OF $\vec{W}$ UNDER T — if $\vec{w} \in W$, $T^{-1}(\vec{w})$ is the ~~image~~ preimage

PREIMAGE OF X UNDER T — if $X \subseteq W$, $T^{-1}(X) = \{\vec{V} \in V \mid T(\vec{V}) \in X\}$ is the preimage

KERNEL OF T — $T^{-1}(\{O_W\})$ or $\ker(T) = \{\vec{V} \in V \mid T(\vec{V}) = \vec{O}_W\}$

PROPERTIES OF LINEAR MAPS

Let $T: V \to W$ be linear

1. $T(0\vec{v}) = \vec{0}_w$
2. $T(-\vec{v}) = -T(\vec{v}), \forall \vec{v} \in V$

PRESERVATION OF SUBSPACES

Let $T: V \to W$ be a linear transformation and $U \subseteq V$, $X \subseteq W$ where U is a subspace of V and X is a subspace of W

1. $T(U)$ is a subspace of W
2. $T^{-1}(X)$ is a subspace of V

COROLLARY

Let $T: V \to W$ be a linear transformation

1. $\operatorname{Im} T$ is a subspace of W
2. $\operatorname{Ker} T$ is a subspace of V

COMPOSITION OF LINEAR TRANSFORMATION

Let V, W, X be vector spaces

$T_1 : V \to W$ and $T_2 : W \to X \Rightarrow T_2 \circ T_1 : V \to X$ and $(T_2 \circ T_1)(V) = T_2(T_1(V))$

THEOREM

T_1 and T_2 is linear $\Rightarrow T_2 \circ T_1$ is linear

INVERTIBLE LINEAR TRANSFORMATION

$T: V \to W$ is invertible if $\exists$ a linear transformation $T^{-1}: W \to V$ s.t.

1. $T \circ T^{-1} = Id_W$ where $Id_W(\vec{w}) = \vec{w}$ and $Id_V(\vec{v}) = \vec{v}$
2. $T^{-1} \circ T = Id_V$

INVERTIBILITY OF T

T is invertible $\Leftrightarrow$ T is one-to-one and onto

ISOMORPHISM

Let $T: V \to W$ be a invertible linear transformation. Then T is an isomorphism between V and W (denoted as $V \cong W$ if V is isomorphic to W)

COORDINATES

Let V be a finite-dimensional vector space with ordered basis $B = (\vec{b}_1, \vec{b}_2, \dots \vec{b}_n)$. The map $T: V \to \mathbb{R}^n$ defined by $T(\vec{v}) = \vec{v}_B$, the coordinate vector relative to B, is an isomorphism.

THEOREM

$[V]_B$ is a isomorphism between V and $\mathbb{R}^n$

COROLLARY

Every finite dimensional vector space is isomorphic to $\mathbb{R}^n$

LEMMA

$[V]_B$ is invertible

CHANGE OF BASIS

Let B and B' be ordered bases for a finite dimensional vector space V. The change-of-coordinate matrix from B to B' is the unique matrix $C_{B\to B'}$ s.t. $C_{B\to B'}[\vec{V}]_B = [\vec{V}]_{B'}$ for all vectors $\vec{v}$ in V

THEOREM

There exists a linear transformation $T_{B\to A}: \mathbb{R}^n \to \mathbb{R}^n$

$$T_{B\to A}: [\vec{V}]_B \to C_{B\to A}[\vec{V}]_B = [\vec{V}]_A$$

and is given by left multiplication of vector $[\vec{V}]_B \in \mathbb{R}^n$ by change of basis matrix

$$C_{B\to A} = \left([\vec{b_1}]_A \middle| [\vec{b_2}]_A \middle| \cdots \middle| [\vec{b_n}]_A \right) \in M_{n,n}(\mathbb{R})$$

USING BASES TO REPRESENT LINEAR TRANSFORMATIONS

Given an ordered basis in V,

$$[T]_B = \left([T(\vec{b_1})]_B \middle| [T(\vec{b_2})]_B \middle| \cdots \middle| [T(\vec{b_n})]_B \right)$$

transform all bs

$[T(\vec{V})]_B = [T]_B[\vec{V}]_B$ ← make v in terms of B

THM

Let $C_{B\to A}$ be the change of basis matrix. Then: $[T]_B = C_{B\to A}^{-1}[T]_A C_{B\to A}$

DOT PRODUCT

let $\vec{v} \in \mathbb{R}^n$ and $\vec{w} \in \mathbb{R}^n$, dot product $\vec{v}\cdot\vec{w} = \sum_{j=1}^{n} v_j w_j = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$

INNER PRODUCT

An inner product is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies:

1. symmetry / commutativity: $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle \quad \forall \vec{v}, \vec{w} \in V$
2. linearity in first term: $\langle \vec{u} + \lambda\vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \lambda \langle \vec{v}, \vec{w} \rangle$
3. positive definiteness: $\langle \vec{v}, \vec{v} \rangle \geq 0$ ~~iff~~ $\forall \vec{v} \in V$, $\langle \vec{v}, \vec{v} \rangle = 0 \Leftrightarrow \vec{v} = \vec{0}_V$

CAUCHY - SCHWARZ

let V be a vector space $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be a real inner product

$$|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \|\vec{w}\| \quad \forall \vec{v}, \vec{w} \in V$$

MAGNITUDE / NORM

$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$, distance between $\vec{v}$ and $\vec{w}$: $d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$

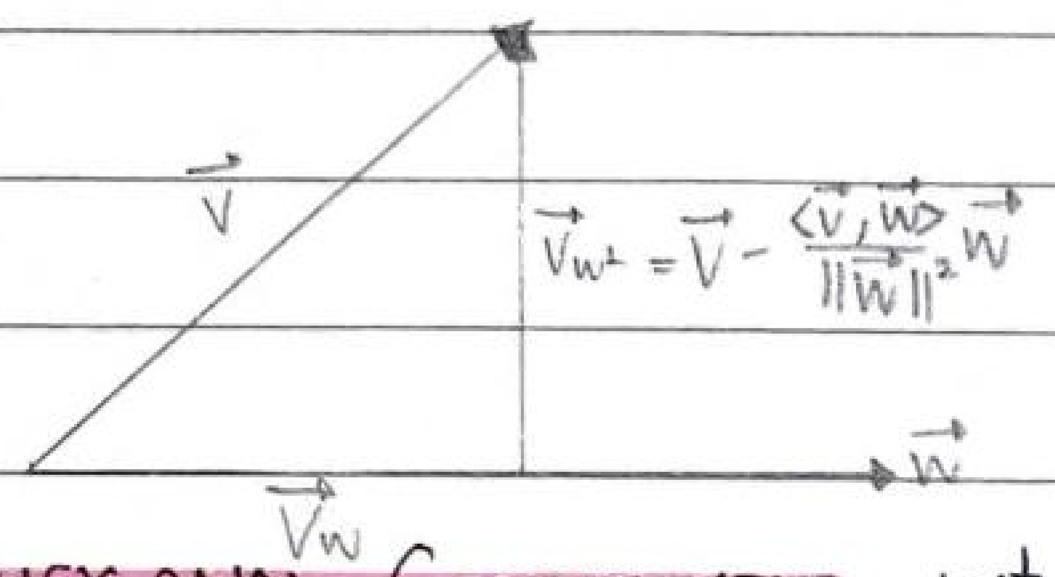
TRIANGLE INEQUALITY

$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

ANGLE BETWEEN VECTORS

$$\theta = \cos^{-1}\left(\frac{\langle\vec{v},\vec{w}\rangle}{\|\vec{v}\|\,\|\vec{w}\|}\right)$$

ORTHOGONAL PROJECTION



$$\vec{v} = \vec{v}_w + \cancel{tt}\,\vec{v}_{w^\perp} = \mathrm{Proj}_{\vec{w}}\vec{v} + \vec{v}_{w^\perp}$$
$$= \left(\frac{\langle\vec{v},\vec{w}\rangle}{\|\vec{w}\|^2}\vec{w}\right) + \left(\vec{v} - \frac{\langle\vec{v},\vec{w}\rangle}{\|\vec{w}\|^2}\vec{w}\right)$$

ORTHOGONAL COMPLEMENT $W^\perp$

Let W be a subspace of $\mathbb{R}^n$. The set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W is the orthogonal complement of W.

$$W^\perp = \{\vec{v} \in V \mid \langle\vec{v},\vec{w}\rangle = 0 \ \forall \vec{w} \in W\}$$

FINDING ORTHOGONAL COMPLEMENT OF A SUBSPACE W OF $\mathbb{R}^n$

1. Find a matrix A having as rows a generating set for W (want row vectors)
2. Row reduce and find the nullspace (solutions to $A\vec{x}=\vec{0}$)

PROPERTIES OF $W^\perp$ ~~let dim(V)=n~~

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. ~~tt~~ $\dim(W^\perp) = n - \dim(W)$
3. $(W^\perp)^\perp = W$
4. $\cancel{b_w}\ \vec{b} = \vec{b}_w + \vec{b}_{w^\perp}\ \forall \vec{b} \in \mathbb{R}^n$, $\vec{b}_w \in W$, $\vec{b}_{w^\perp} \in W^\perp$, where b_w is the projection of $\vec{b}$ on ~~W~~ W
5. $W \cap W^\perp = \{\vec{0}_v\}$

FINDING PROJECTION OF A VECTOR ON A SUBSPACE

1. Find / Given a basis $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ for subspace W.
2. Find a basis $\{\vec{v}_{k+1}, \vec{v}_{k+2}, \cdots, \vec{v}_n\}$ for $W^\perp$.
3. Find $\vec{r}$ s.t. $\vec{b} = r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_n\vec{v}_n$ i.e. Form an Augmented Matrix with basis vectors as columns on the left and a vector $\vec{b}$ on the right.
4. Then $\vec{b}_w = r_1\vec{v}_1 + r_2\vec{v}_2 + \cdots + r_k\vec{v}_k$

RANK-NULLITY THEOREM

Let T be a linear transformation. $\dim V = \dim(\ker T) + \dim(\mathrm{Im}\,T)$

LEMMA ON COMPLEMENT SPACES

$W \oplus W^\perp = V$ $\qquad (\mathrm{Col}(A))^\perp = (\mathrm{Null}(A^T))$

$\mathrm{Col}(A) = (\mathrm{Null}(A^T))^\perp$ $\qquad \mathrm{Proj}_w(\vec{v})$ is vector in W closest to $\vec{v}$

Orthogonal Basis

$\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is an orthogonal basis (for V) iff

i. it is a basis for V

ii. $\langle \vec{v}_j, \vec{v}_k \rangle = 0 \quad \forall j \neq k$ (each basis element is orthogonal to others)

Orthonormal Basis

A basis is orthonormal if it is orthogonal and $\|\vec{v}_j\| = 1 \quad \forall j \in \{1, \cdots, n\}$

Gram-Schmidt Process

To find a basis $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$ that is orthonormal in W.

1. Find a basis for W $\{\vec{v}_1, \cdots \vec{v}_k\}$
2. Let $\vec{w}_1 = \vec{v}_1$
3. Compute $\vec{w}_{2 \cdots k}$ by $\vec{w}_j = \vec{v}_j - \sum_{i=1}^{j} \text{Proj}_{w_i} \vec{v}_j = \vec{v}_j - \sum_{i=1}^{j} \frac{\langle \vec{v}_j, \vec{w}_i \rangle}{\|\vec{w}_i\|^2} \vec{w}_i$

   e.g. $\vec{w}_2 = \cancel{\vec{w}_2}\ \vec{v}_2 - \text{Proj}_{\vec{w}_1} \vec{v}_2$, $\vec{w}_3 = \vec{v}_3 - \text{Proj}_{\vec{w}_1} \vec{v}_3 - \text{Proj}_{\vec{w}_2} \vec{v}_3$
4. Orthonormalize each vector in $\{\vec{w}_1, \cdots, \vec{w}_k\}$ by dividing each with their norm

QR-Factorization

Let A be an $n \times k$ matrix with independent column vectors in $\mathbb{R}^n$. There exists an $n \times k$ matrix Q with orthonormal column vectors and an ~~upper~~ upper-triangular invertible $k \times k$ matrix R such that $A = QR$

$$\underbrace{\begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_k \\ | & | & & | \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} | & | & & | \\ \vec{p}_1 & \vec{p}_2 & \cdots & \vec{p}_k \\ | & | & & | \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} | & | & & | \\ [\vec{a}_1]_p & [\vec{a}_2]_p & \cdots & [\vec{a}_n]_k \\ | & | & & | \end{pmatrix}}_{R}$$

R – upper triangular

$\vec{p}_j = \vec{a}_j - \sum_{i=1}^{j} \frac{\langle a_j, p_i \rangle}{\|p_i\|^2} \vec{p}_i$

$\Rightarrow \vec{a}_j = \vec{p}_j + \sum_{i=1}^{j} \frac{\langle a_j, p_i \rangle}{\|p_i\|^2} \vec{p}_i$ so $\vec{a}_j$ is a linear combination of $\{\vec{p}_1, \vec{p}_2, \cdots, \vec{p}_j\}$.

Orthogonal Linear Transformations

Let $(V, \langle \cdot, \cdot \rangle)$ be inner product space. We say linear transformation $T: V \to V$ is orthogonal if $\langle T(v), T(w) \rangle = \langle v, w \rangle, \forall w, v \in V$ (preservation of innear product)

Preservation of ~~Len~~ Orthogonal Linear Transformation

Let T be orthogonal.

1. $\|T(\vec{v})\| = \|\vec{v}\| \quad \forall \vec{v} \in V$ — preservation of length
2. $\theta_{\vec{v}, \vec{w}} = \theta_{T(\vec{v}), T(\vec{w})}$ — preservation of angle
3. $\|T(\vec{v}) - T(\vec{w})\| = \|\vec{v} - \vec{w}\|$ — preservation of ~~length~~ distance

Orthogonal Matrix

Let $A \in M_{n,n}(\mathbb{R})$, A matrix is orthogonal if $A^T A = I_n$

For $A \in M_{n,n}(\mathbb{R})$ the following are equivalent:

1. A is orthogonal;
2. A^T is orthogonal;
3. col(A) ~~being~~ is an orthonormal basis
4. col(A^T) is an orthonormal basis

THEOREM ON ORTHOGONAL LINEAR TRANSFORMATION AND ORTHOGONAL MATRIX

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal linear transformation, T is orthogonal $\Leftrightarrow$ the standard matrix representation $A = [T]_\varepsilon$ is an orthogonal matrix

Let $B = \{\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n\}$ be an orthogonal basis then $T: \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal linear transformation $\Leftrightarrow [T]_B$ is an orthogonal matrix

PROPERTIES OF THE PROJECTION $\vec{P}$ OF VECTOR $\vec{b}$ ON THE SUBSPACE W

1. The vector $\vec{p}$ must lie in the subspace W
2. The vector $\vec{b} - \vec{p}$ must be perpendicular to every vector in W

PROJECTION b_W OF b ON THE SUBSPACE W

Let $W = sp(\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_k)$ be a k-dimensional subspace of $\mathbb{R}^n$, and A be a matrix with a_i ~~s~~ as columns. The projection of $\vec{b}$ on W is given by

$$\vec{b}_W = A(A^TA)^{-1}A^T\vec{b}$$

where $P = A(A^TA)^{-1}A^T$ is the projection matrix. If $\{\vec{a}_1, \cdots, \vec{a}_k\}$ is orthonormal basis for W, $P = AA^T$

PROPERTIES OF A PROJECTION MATRIX

1. $P^2 = P$ idempotent
2. $P^T = P$ symmetric every $n \times n$ matrix that has ~~this propert~~ these properties is a projection matrix of its column space

LEAST-SQUARES METHOD

Let A be a matrix with independent column vectors. The least-squares solution $\vec{r}$ of $A\vec{r} \approx b$ can be computed by:

1. $\vec{r} = (A^TA)^{-1}A^T\vec{b}$
2. Solve $(A^TA)\vec{r} = A^T\vec{b}$

BILINEAR

Let f be a map $f: V \times V \to \mathbb{R}$, f is bilinear if $f_v: y \to f(v, y)$, $f_w: x \to f(x, w)$ are linear $\forall \vec{v}, \vec{w} \in V$, that is,

1. $f(c\vec{v}_1 + \vec{v}_2, \vec{w}) = cf(\vec{v}_1, \vec{w}) + f(\vec{v}_2, \vec{w})$
2. $f(\vec{v}_1, d\vec{w}_1 + \cancel{d}\vec{w}_2) = df(\vec{v}, \vec{w}_1) + f(\vec{v}, \vec{w}_2)$

MULTILINEAR

Let f be a map $f: V^n \to \mathbb{R}$. f is multilinear if it linear in every factor i.e. $\forall j \in \{1, 2, \cdots, n\}$,

$f(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{j-1}, c\vec{v}_j - \vec{w}_j, \vec{v}_{j+1}, \cdots, \vec{v}_n) = cf(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_j, \cdots, \vec{v}_n) + f(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{j-1}, \vec{w}_j, \cdots, \vec{v}_n)$

ALTERNATING

Let $f: V^n \to \mathbb{R}$, $f(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_{j_1} \cdots \vec{v}_j, \cdots, \vec{v}_n) = 0$;

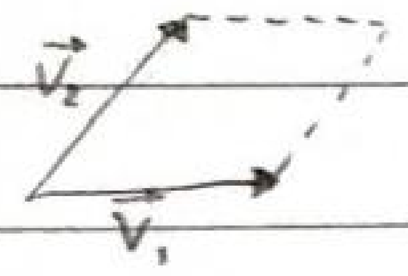
SKEW-SYMMETRIC

Let $f: V^n \to \mathbb{R}$, $f(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_j, \cdots, \vec{v}_k, \cdots, \vec{v}_n) = -f(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k, \cdots, \vec{v}_j, \cdots, \vec{v}_n)$

THEOREM: ALTERNATING IFF SKEW SYMMETRIC

Let $f: V^n \to \mathbb{R}$ be multilinear. Then f is alternating $\Leftrightarrow$ skew symmetric

GEOMETRIC INTERPRETATION OF DETERMINANT (IN $\mathbb{R}^2$)



$\det(\vec{v}_1, \vec{v}_2) =$ area of parallelogram

N-BOX

The N-box P_n of $\{\vec{v}_1, \cdots \vec{v}_n\}$ satisfies $S = \{\sum_{j=1}^{n} t_j \vec{v}_j \mid t_j \in [0,1]\}$

VOLUME OF AN N-BOX

$n = 1$: $Vol(P_1) = \|\vec{v}_1\|$

$n > 1$: $Vol(P_n) = \|\vec{b}\|(Vol(P_{n-1}))$ where $\vec{b} = \vec{v}_1 - Proj_{sp(\vec{v}_2, \cdots, \vec{v}_n)}(\vec{v}_1)$

$(\vec{b} \in sp(\vec{v}_2, \cdots, \vec{v}_n)^{\perp} \Rightarrow \langle \vec{b}, \vec{v}_j \rangle = 0 \ \forall j \in \{2, 3, \cdots, n\})$

The volume of n-box of P_n in $\mathbb{R}^m$ is given by $Vol(P_n) = \sqrt{\det(A^T A)}$

where $A = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \in M_{m,n}\mathbb{R}$ $(n \leq m)$

If $m = n$, $Vol(P_n) = |\det(A)|$

VOLUME-CHANGE FACTOR FOR $T: \mathbb{R}^n \to \mathbb{R}^n$

Let G be a region in $\mathbb{R}^n$ of volume V, and let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of rank n with standard matrix representation A. Then the volume of the image of G under T is $|\det(A)| \cdot V$

DIAGONALIZABILITY

Let T be a linear transformation $T: V \to V$ is diagonalizable if $\exists$ a basis B for V s.t. $[T]_B$ is diagonal.

FIELD

F is a field if:

1. $(+, \times)$ commutative
2. $(+, \times)$ associative
3. $(+, \times)$ have identity elements
4. Every element has an additive inverse
5. $\forall x \in \mathbb{F} \setminus \{0_F\}$, x has a multiplicative inverse
6. Distributive under $(+, \times)$

FUNDAMENTAL THEOREM OF ALGEBRA

A polynomial of degree n $P_n(x)$ over $\mathbb{C}$ has n (possibly non distinctive) roots

Hermitian Inner Product

Let V be a complex vector space. $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is a Hermitian inner product if

1. $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}$ for all $\vec{v}, \vec{w} \in V$
2. $\langle \vec{v}_1 + r\vec{v}_2, \vec{w} \rangle = \langle \vec{v}_1, \vec{w} \rangle + \bar{r} \langle \vec{v}_2, \vec{w} \rangle \quad \forall \vec{v}_1, \vec{v}_2, \vec{w} \in V,\ r \in \mathbb{C}$
3. $\langle \vec{v}, \vec{w}_1 + r\vec{w}_2 \rangle = \langle \vec{v}, \vec{w}_1 \rangle + r \langle \vec{v}, \vec{w}_2 \rangle \quad \forall \vec{v}, \vec{w}_1, \vec{w}_2 \in V,\ r \in \mathbb{C}$
4. $\langle \vec{v}, \vec{v} \rangle \geq 0$ and $\langle \vec{v}, \vec{v} \rangle = 0 \iff \vec{v} = \vec{0}_v$

Euclidean Inner Product

The Euclidean Inner Product on $\mathbb{C}^n$ is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ defined by

$$\left\langle \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \begin{bmatrix} z_1' \\ z_2' \\ \vdots \\ z_n' \end{bmatrix} \right\rangle = \bar{z}_1 z_1' + \bar{z}_2 z_2' + \dots + \bar{z}_n z_n'$$

Conjugate And Conjugate Transpose

Let $A \in M_{m,n}(\mathbb{C})$

The conjugate of $A = \bar{A} = (\overline{a_{ij}}) = (\overline{a_{ij}})$

The conjugate transpose of $A = A^* = (\bar{A})^T = (\overline{a_{ij}})^T = (\overline{a_{ji}})$

Hermitian Matrix (similar to symmetric)

A matrix $A \in M_{m,m}(\mathbb{C})$ is Hermitian if $A = A^*$

Unitary Matrix (similar to orthogonal)

A matrix $U \in M_{n,n}(\mathbb{C})$ is unitary if $UU^* = I_n$ or $U^*U = I_n$

Spectral Theorem For Hermitian Matrices

Let $A \in M_{n,n}(\mathbb{C})$ be Hermitian. Then A is unitarily diagonalizable and all eigenvalues are real, all eigenvectors are orthogonal.

Lemma

If A is Hermitian and unitarily diagonalizable, then the eigenvalues are real

Schur's Lemma

Let $A \in M_{n,n}(\mathbb{C})$. There is a unitary matrix U s.t. $U^{-1}AU$ is upper triangular

Properties Of Conjugate Transpose

Let A and B be $m \times n$ matrices. Then

1. $(A^*)^* = A$
2. $(A+B)^* = A^* + B^*$
3. $(zA)^* = \bar{z}A^*$ for $z \in \mathbb{C}$
4. If A and B are square matrices, $(AB)^* = B^*A^*$

Unitary Equivalence $A \underset{u.e}{\sim} B$

Let $A, B \in M_n(\mathbb{C})$. A, B are unitarily equivalent if $\exists$ a unitary matrix $U \in M_n(\mathbb{C})$ s.t. $A = UBU^*$

PROPERTIES OF UNITARY EQUIVALENCE

1. $A \underset{u.E}{\sim} A$ — reflexitivity
2. $A \underset{u.E}{\sim} B \Rightarrow B \underset{u.E}{\sim} A$ — symmetry
3. $A \underset{u.E}{\sim} B$ and $B \underset{u.E}{\sim} C \Rightarrow A \underset{u.E}{\sim} C$ — transitivity

EQUIVALENCE CLASS

Let $A, B \in M_{n,n}(\mathbb{C})$. A is in equivalence class E_B if $\exists$ a unitary matrix U s.t.
$A = UBU^*$ $\Rightarrow E_B = \{A \in M_n(\mathbb{C}) \mid A \underset{u.E}{\sim} B\}$

1. Every matrix belongs to only one equivalence class
2. $A \in E_B \Rightarrow B \in E_A$
3. $A \underset{u.E}{\sim} B \Rightarrow E_A = E_B$
4. $C \in E_A \cap E_B \Rightarrow E_A = E_B$

NORMAL

A matrix $A \in M_{n,n}(\mathbb{C})$ is normal if $AA^* = A^*A$
If $A \in M_n(\mathbb{C})$ is normal, then $\forall B \in E_A$, B is normal

THM
Let $A \in M_n(\mathbb{C})$. A is unitarily diagonalizable $\Leftrightarrow$ A is normal.

JORDAN DECOMPOSITION

A matrix A can be expressed in the form
$A = UJU^*$, where J is a matrix of Jordan Block

JORDAN BLOCK

An mxm matrix is a Jordan Block if it is structured as:
1. All diagonal entries are equal.
2. Each entry immediately above a diagonal entry is 1.
3. All other entries are zero.

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

ORTHOGONAL PROJECTION OF VECTOR $\vec{x} \in \mathbb{R}^n$ ONTO SUBSPACE W

The unique vector $\vec{y} \in V$ such that for every $\vec{w} \in W$, $\|\vec{x}-\vec{y}\| \leq \|\vec{x}-\vec{w}\|$

The unique vector $\vec{y} \in V$ such that $\vec{x}-\vec{y} \in V^{\perp}$

Let $(u_1, u_2, \cdots, u_k)$ be an orthonormal basis for V, then the orthogonal projection of $\vec{x}$ onto V is the vector

$$(\vec{x}\cdot\vec{u}_1)\vec{u}_1 + \cdots + (\vec{x}\cdot\vec{u}_k)\vec{u}_k$$

or let A be the matrix with $(u_1, u_2, \cdots, u_k)$ as columns, then the orthogonal projection of $\vec{x}$ onto V is the vector $A(A^TA)^{-1}A^T\vec{x}$.

JORDAN CANONICAL FORM

An $n \times n$ matrix J is a Jordan canonical form if it consists of Jordan blocks, placed corner-to-corner along the main diagonal~~, as in matrix (4)~~, with only zero ~~et~~ entries outside these Jordan blocks.