MATB41

LINES

vector equation of line: $\vec{x} = \vec{x}_0 + t\vec{v}$ $t\in\mathbb{R}$ ($\vec{x}$: general vector, $\vec{x}_0$: point, $\vec{v}$: direction)

parametric equations of line: $x_i = x_{i0} + tv_i$, $i = 1, 2, \cdots, n$

symmetric equation of line: $\frac{x_1 - x_{10}}{v_1} = \frac{x_2 - x_{20}}{v_2} = \cdots = \frac{x_n - x_{n0}}{v_n}$

PLANES

$ax + by + cz = d$ ← ~~rectangler~~ rectangular description of plane $\in\mathbb{R}^3$

let x_0 be a point on plane, $\vec{n}$ be normal vector, $\vec{x}$ be an arbitrary point on plane $\vec{n}\cdot(x - x_0) = 0$

$\Rightarrow n_1x_1 + n_2x_2 + \cdots + n_nx_n - n_1x_{01} - n_2x_{02} - \cdots - n_nx_{0n} = 0$

$\therefore n_1x_1 + n_2x_2 + \cdots + n_nx_n = d$

$p + s\vec{v} + t\vec{w}$, $s, t \in\mathbb{R}$ ← parametric description ~~on~~ of plane $\in\mathbb{R}^3$

where p is a point on plane, $\vec{v}$ and $\vec{w}$ are vectors on plane

(all points on plane can be described as above equation)

FINDING INTERSECTION OF TWO PLANES

Form an augmented matrix and row-reduce.

FINDING EQN OF PLANE (given point and line in parametric form)

Change line to vector form $\vec{x} = x_0 + t\vec{v}$ where line given is normal

so we have $\vec{n}$, x_0 and one more point

$\vec{n}\cdot(x - x_0) = 0$ is the eqation of plane

d can be found by substituting the other point.

FINDING ANGLE BETWEEN PLANES

$\vec{n}_1\cdot\vec{n}_2 = \|\vec{n}_1\|\,\|\vec{n}_2\|\cos\theta$ where $\vec{n}_1, \vec{n}_2$ are normals to the planes

~~Cu~~ CURVES

The collection C of points $C(t)$ as t varies in $[a,b]$ is called a curve.

path: $c: [a,b] \to \mathbb{R}^n$, $c(t) = (c_1(t), c_2(t), \cdots, c_n(t))$, $t\in[a,b]$

FINDING VELOCITY OF PATH C AT TIME t

$$c'(t) = \lim_{h\to 0} \frac{c(t+h) - c(t)}{h} = (c_1'(t), c_2'(t), \cdots, c_n'(t))$$

(vector tangent to path $c(t)$. $\|c'(t)\|$ is the speed of the path

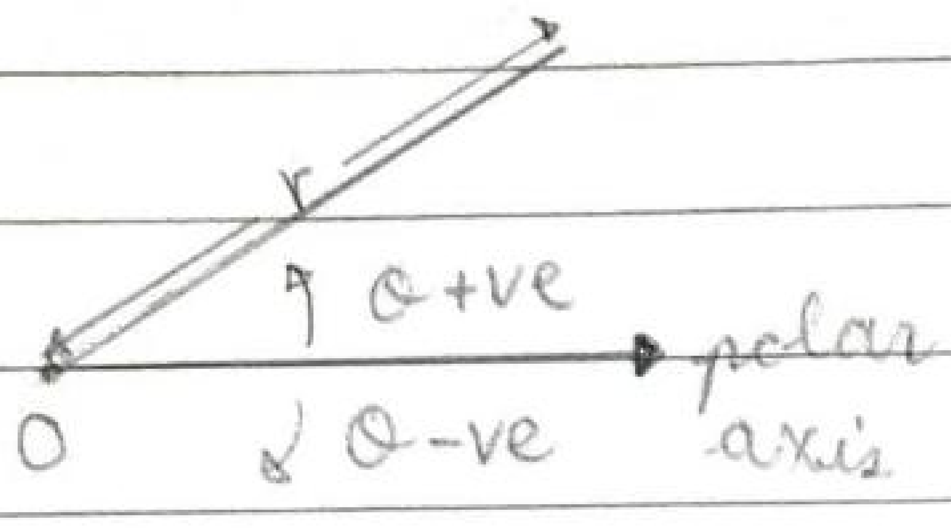
$c(t) = (x(t), y(t), z(t))$

$$c'(t) = \lim_{h\to 0}\left(\frac{x(t+h) - x(t)}{h}, \frac{y(t+h) - y(t)}{h}, \frac{z(t+h) - z(t)}{h}\right) = (x'(t), y'(t), z'(t))$$

TANGENT LINE TO PATH C AT POINT A (Tangent vector of c at a)

$$\vec{x} - \vec{a} = c'(t_0)(t - t_0)$$

POLAR COORDINATES



if $r>0$, (r,θ) lies same quadrant as θ.
if $r<0$, (r,θ) lies opposite quadrant of θ
$(-r,\theta) = (r,\theta+\pi)$

$P(x,y) \rightarrow (r,\theta)$ where r = radius, θ = angle between polar axis and line (anticlockwise)

SKETCH GRAPHS

given (r,θ)

$r=n$ for some $n \in \mathbb{R} \Rightarrow$ a circle
$\theta = \phi$ for some $0 \le \phi \le 2\pi \Rightarrow$ a straight line
$r = \cdots \theta$ then plug in values for θ and sketch

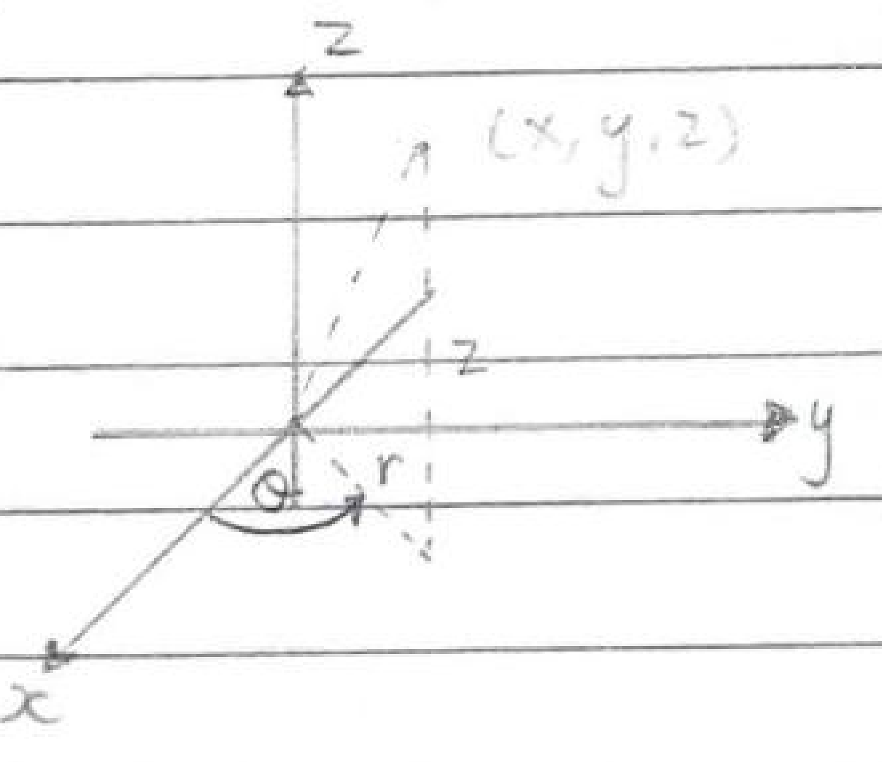
COORDINATE CONVERSION

Polar to Cartesian: $x = r\cos\theta$, $y = r\sin\theta$
Cartesian to Polar: $\theta = \arctan \frac{y}{x}$, $r = \sqrt{x^2+y^2}$

CYLINDRICAL COORDINATES

Basically, changing polar coordinates from $\mathbb{R}^2$ to $\mathbb{R}^3$

$P(x,y,z) \rightarrow (r,\theta,z)$



COORDINATE CONVERSION

Cylindrical to Cartesian: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ $\quad 0 \le r < \infty$, $0 \le \theta \le 2\pi$, $-\infty < z < \infty$
Cartesian to Polar: $\theta = \arctan \frac{y}{x}$, $r = \sqrt{x^2+y^2}$, $z = z$

SKETCH GRAPHS

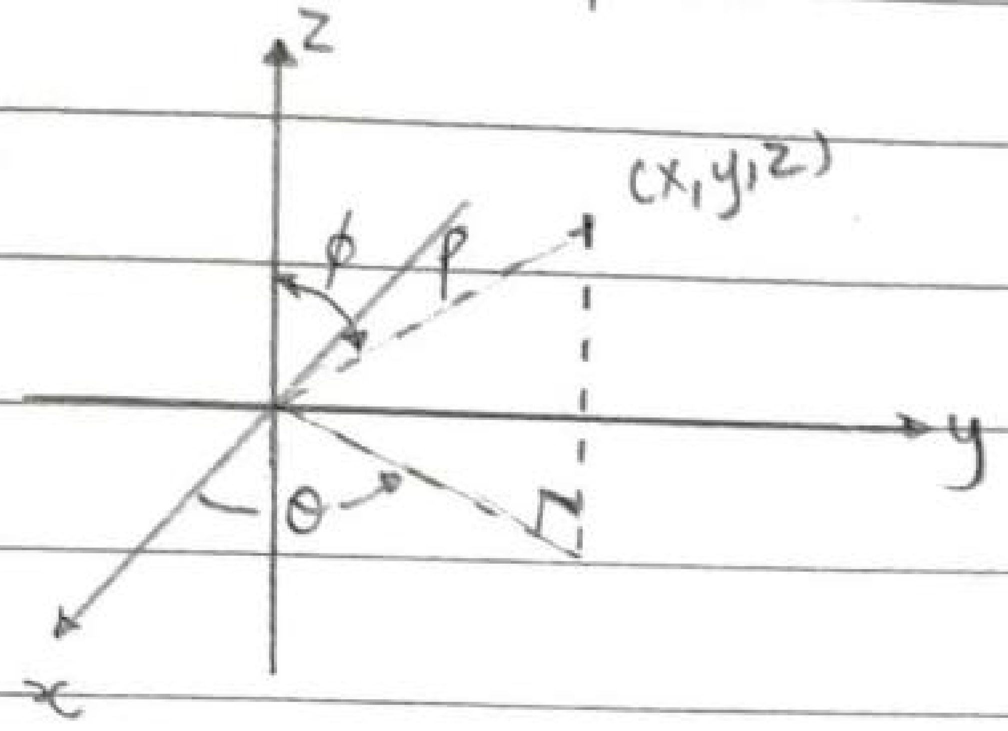
given (r,θ,z)

$r = n$ for some $n \in \mathbb{R} \Rightarrow$ a cylinder $\quad z = n$ $n \in \mathbb{R} \Rightarrow$ a plane?
$\theta = \phi$ for some $0 \le \phi \le 2\pi \Rightarrow$ a plane
$z = nr$ $n \in \mathbb{R} \Rightarrow$ a cone

SPHERICAL COORDINATES

$P(x,y,z) \rightarrow (\rho, \theta, \phi)$

rho theta phi



COORDINATE CONVERSION $0 \leq \rho < \infty$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$

~~Pol~~ Spherical to Cartesian : $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \, \cancel{\cos}\, \sin\theta$, $z = \rho \cos\phi$

Spherical to Cylindrical : $r = \rho \sin\phi$, $\theta = \theta$, $z = \rho\cos\phi$

Cartesian to Spherical : $\rho = \sqrt{x^2+y^2+z^2}$, $\theta = \arctan\left(\frac{y}{x}\right)$, $\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

SKETCH GRAPHS

given (ρ, θ, ϕ)

$\rho = n \quad n \in \mathbb{R} \Rightarrow$ sphere

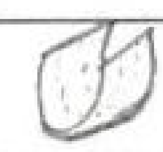
$\theta = \alpha \quad 0 \leq \alpha \leq 2\pi \Rightarrow$ plane

~~VECTP~~ VECTOR FUNCTIONS

A vector-valued function : $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a rule or process that assigns to each input $\vec{x}$ in $\mathbb{R}^n$ corresponding output $\vec{y}$ in $\mathbb{R}^m$ if $m > 1$.

$f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called a real-function of several ~~vaal~~ variables

GRAPHS OF FUNCTIONS

$x^2+y^2=r^2 \Rightarrow$ circle $\qquad z = x^2+y^2$  bowl $\qquad z^2 = \sqrt{x^2+y^2}$  $\qquad y = x^2, z \in \mathbb{R}$ 

$x^2+y^2=r^2, z \in \mathbb{R} \Rightarrow$ cylinder $\qquad z = \sqrt{x^2+y^2}$  cone $\qquad x^2+y^2+z^2 = R^2 \Rightarrow$ sphere

LEVEL SETS, CURVES, AND SURFACES

Let $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$, let $k \in \mathbb{R}$. The level set of f at value k is defined to be the set of those point $\vec{x} \in U$ at which $f(\vec{x}) = k$

$$\text{level set of } f = \{(\vec{x}, \vec{k}) \mid \vec{x} \in U\}$$

$n = 2 \Rightarrow$ level curve / level contour

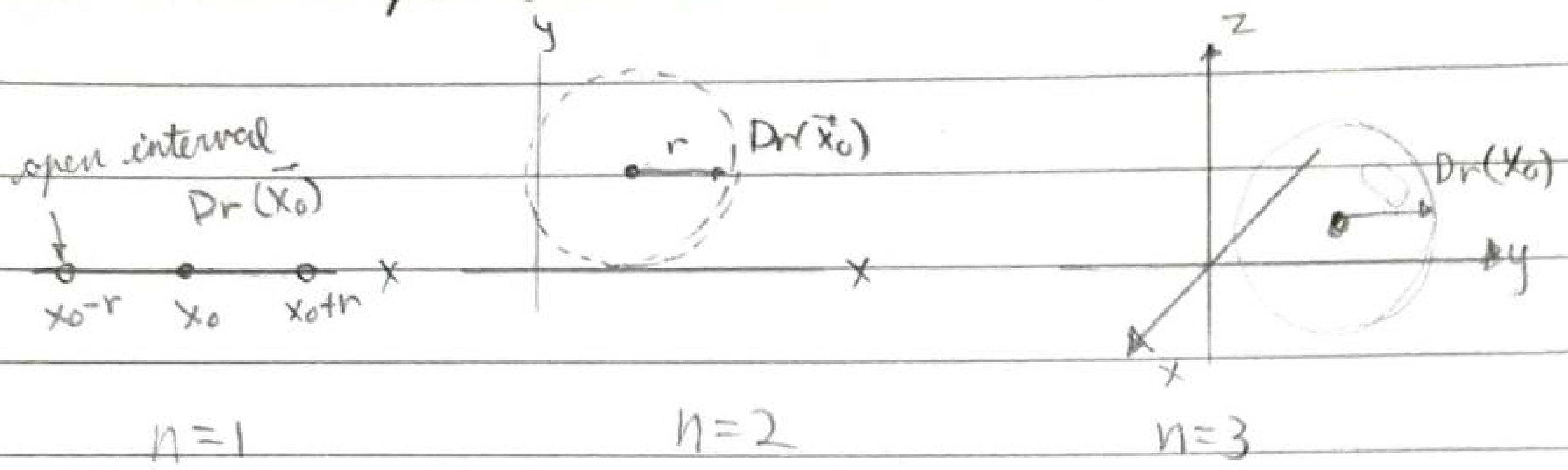
$n = 3 \Rightarrow$ level surface

SKETCHING LEVEL CURVES

Let $f(x,y) = k$ and pick values ~~of~~ for k ~~to~~ in range(f) and sketch the lines (pick 5 values, describe behaviour $\rightarrow \infty$, what is critical point)

LIMITS

Open disk / open ball $D_r(\vec{x}_0) = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x} - \vec{x}_0\| < r\} \subset \mathbb{R}^n$



$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ or $\lim_{(x_1, x_2, \dots, x_n) \to (a_1, a_2, \dots, a_n)} f(x_1, x_2, \dots, x_n) = L$, $f(\vec{x}) \to L$ as $\vec{x} \to \vec{a}$

LIMIT DEFINITION

$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ if

given any $\varepsilon > 0$, there is a

magnitude

absolute value

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $0 < \|\vec{x} - \vec{a}\| < \delta \Rightarrow |f(\vec{x}) - L| < \varepsilon$

USING DEFINITION TO EVALUATE LIMIT

1. WTS $\forall \varepsilon$, find $\delta > 0$ (depending on ε) s.t. $0 < \|\vec{x} - \vec{a}\| < \delta \Rightarrow |f(\vec{x}) - L| < \varepsilon$ (state what you WTS)
2. Manipulate ~~until you~~ $|f(\vec{x}) - L|$ or find things that will relate to $\|\vec{x} - \vec{a}\|$ so you can state $< \delta$ and find an appropriate δ
3. Formally proof using what you found in 2.

※ $0 \le (x \pm y)^2 = x^2 + y^2 \pm 2xy \Rightarrow \mp 2xy \le (x \pm y)^2$

$x^2 \le x^2 + y^2$

WHAT TO DO IF LIMIT DNE

$\vec{x} \to \vec{a}$ along path C, $f(\vec{x}) \to L$, while $\vec{x} \to \vec{a}$ along path D, $f(\vec{x}) \to M$, where $L \neq M$,
$\Rightarrow \lim_{x \to a} f(\vec{x}) \neq L$

$\therefore$ Pick two paths that approaches to two different limits and show limit DNE.

※ if changed to single variable, can use L'Hôpital's Rule.
※ Make sure that path passes through point of interest

TIPS IN EVALUATING LIMITS

a. If deg(numerator) > deg(denominator), limit likely $\to 0$
b. $|x| \le \sqrt{x^2 + y^2}$, $|y| \le \sqrt{x^2 + y^2}$
c. in $\mathbb{R}^2$, change to polar coordinates so $r \to 0^+ \Leftrightarrow (x, y) \to (0, 0)$
d. Squeeze Theorem

~~Limit~~ LIMIT ~~LAM~~ LAWS

$L, M \in \mathbb{R}$. $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ and $\lim_{\vec{x} \to \vec{a}} g(\vec{x}) = M$. Assume c is a constant, $m, n \in \mathbb{Z}$.

a. $\lim_{\vec{x} \to \vec{a}} c = c$

b. $\lim_{\vec{x} \to \vec{a}} c f(\vec{x}) = cL$

c. $\lim_{\vec{x} \to \vec{a}} (f(\vec{x}) \pm g(\vec{x})) = L \pm M$

d. $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) g(\vec{x}) = LM$

e. If $M \neq 0$, $\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{L}{M}$

f. If $n \neq 0$, $\lim_{\vec{x} \to \vec{a}} f(\vec{x})^{\frac{m}{n}} = L^{\frac{m}{n}}$

~~d.~~

Directly plug in numbers if not in indeterminate form.
Simplify until it is not indeterminate otherwise

CONTINUITY

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function with domain U. Let $\vec{x}_0 \in U$. f is continuous at $\vec{x}_0 \iff \lim_{\vec{x} \to \vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$

$\lim_{x \to x_0} f(\vec{x}) = f(\vec{x}_0)$ implies

a. $\vec{x}_0 \in U$

b. $\lim_{x \to x_0} f(\vec{x}) = L$ — if one of the three not satisfied, not continuous

c. ~~the~~ $f(\vec{x}_0) = L$

PROPERTIES OF CONTINUOUS FUNCTIONS

a. $f(\vec{x}), g(\vec{x})$ continuous at $\vec{x}_0$, let c be constant. cf, $f \pm g$, fg and $\frac{f}{g}$ ($g(x_0) \neq 0$) are continuous at $\vec{x}_0$

b. $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ with $f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \dots, f_m(\vec{x}))$. f is continuous at $\vec{x}_0 \iff$ $f_1, f_2, \dots f_m$ is continuous at $\vec{x}_0$

c. $g: U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f: U \subset \mathbb{R}^m \to \mathbb{R}^p$. Suppose $g(U_1) \subset U_2$. ~~So~~ so that $f \circ g$ is defined on U_1. g is continuous at $\vec{x}_0$ and f is continuous at $g(\vec{x}_0) \Rightarrow f \circ g$ continuous at $\vec{x}_0$

CONTINUOUS FUNCTION

$f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ with $f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \dots, f_m(\vec{x}))$. f is continuous at $\vec{x}_0$ if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\|f(\vec{x}) - f(\vec{x}_0)\| < \varepsilon \Leftarrow \|x - x_0\| < \delta$ (no $0<$ because $f(x_0)$ has to exist)

~~i.e.~~ continuous on $U \subset \mathbb{R}^n \Rightarrow$ continuous at each point $\vec{x} \in U$.

Polynomials are continuous everywhere.
Rational functions continuous in their domain
Trig functions continuous in their domain

DIFFERENTIATION

DIRECTIONAL DERIVATIVE

Directional derivative of f at $\vec{a}$ in direction $\vec{v}$.

$$D_v(f(\vec{a})) = \lim_{t\to 0} \frac{f(\vec{a}+t\vec{v}) - f(\vec{a})}{t\|\vec{v}\|} \quad \text{for } f: \mathbb{R}^n \to \mathbb{R}$$

$$D_v f(\vec{x}) = \nabla f(\vec{x}) \cdot \frac{\vec{v}}{\|\vec{v}\|} = \left(\frac{\partial f}{\partial x}(\vec{x}), \frac{\partial f}{\partial y}(\vec{x}), \frac{\partial f}{\partial z}(\vec{x})\right) \cdot (v_1, v_2, v_3) \div \|\vec{v}\|$$

$$= \frac{\partial f}{\partial x}(\vec{x}) \cdot v_1 + \frac{\partial f}{\partial y}(\vec{x}) \cdot v_2 + \frac{\partial f}{\partial z}(\vec{x})\, v_3 \div (\sqrt{v_1^2+v_2^2+v_3^2})$$

Best to write down what you need, calculate every part and plug back into the formula.

PARTIAL DERIVATIVES

Taking the derivative with respect to one of the variables and treating others as constants

$$\frac{\partial f}{\partial x_i}(\vec{a}) = f_{x_i}(\vec{a})$$

MATRIX OF PARTIAL DERIVATIVES

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$

$$Df(\vec{a}) = \begin{array}{cc} & \begin{array}{ccc} & & x_n \text{ column} \end{array} \\ & \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \frac{\partial f_1}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{a}) \\ \frac{\partial f_2}{\partial x_1}(\vec{a}) & \frac{\partial f_2}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\vec{a}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & \frac{\partial f_m}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{pmatrix} & \begin{array}{l} \to f_1 \text{ row} \\ \\ \\ \\ \end{array} \end{array}$$

e.g. $f(x,y,z) = (x^2 + y\sin z, xe^y, z\cos x)$

m rows, n columns

m×n dimension

also denoted as $Df(\vec{a}) = \frac{\partial(f_1, f_2, \dots, f_m)}{\partial(x_1, x_2, \dots, x_n)}(\vec{a})$

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$, $Df(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \frac{\partial f}{\partial x_2}(\vec{a}), \dots, \frac{\partial f}{\partial x_n}(\vec{a})\right) = \nabla f(a) = \text{grad}(a)$

we take it as a vector in $\mathbb{R}^n$ and is the gradient of f at $\vec{a}$.

f is differentiable at $\vec{a}$ if

$$\lim_{\vec{x}\to\vec{a}} \frac{\|f(\vec{x}) - f(\vec{a}) - Df(\vec{a})(\vec{x}-\vec{a})\|}{\|\vec{x}-\vec{a}\|} = 0$$

DIFFENTIABILITY OF f AT $\vec{a}$ / CONTINUITY OF f AT $\vec{a}$

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$, suppose all partial derivatives $\frac{\partial f_i}{\partial x_j}$ exists and are continuous on $\vec{a} \in U$. Then f is differentiable at $\vec{a} \in U$

If f is differentiable at $\vec{a}$, then f is continuous at $\vec{a}$ and partial derivatives exist.

PROPERTIES OF DERIVATIVES

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\vec{a} \in U$. let $c \in \mathbb{R}$. Then

a. cf is differentiable at $\vec{a}$, and $D(cf)(\vec{a}) = cDf(\vec{a})$

b. $f \pm g$ is differentiable at $\vec{a}$, and $D(f \pm g)(\vec{a}) = Df(\vec{a}) \pm Dg(\vec{a})$

c. fg is differentiable at $\vec{a}$, and $D(fg)(\vec{a}) = Df(\vec{a})\, g(\vec{a}) + f(\vec{a})\, Dg(\vec{a})$

d. $\frac{f}{g}$ is differentiable at $\vec{a}$, if $g(\vec{a}) \neq 0$, and

$$D\left(\frac{f}{g}\right)(\vec{a}) = \frac{(Df(\vec{a}))g(\vec{a}) - f(\vec{a})Dg(\vec{a})}{(g(\vec{a}))^2}$$

} only $U \subset \mathbb{R}^n \to \mathbb{R}$

CHAIN RULE

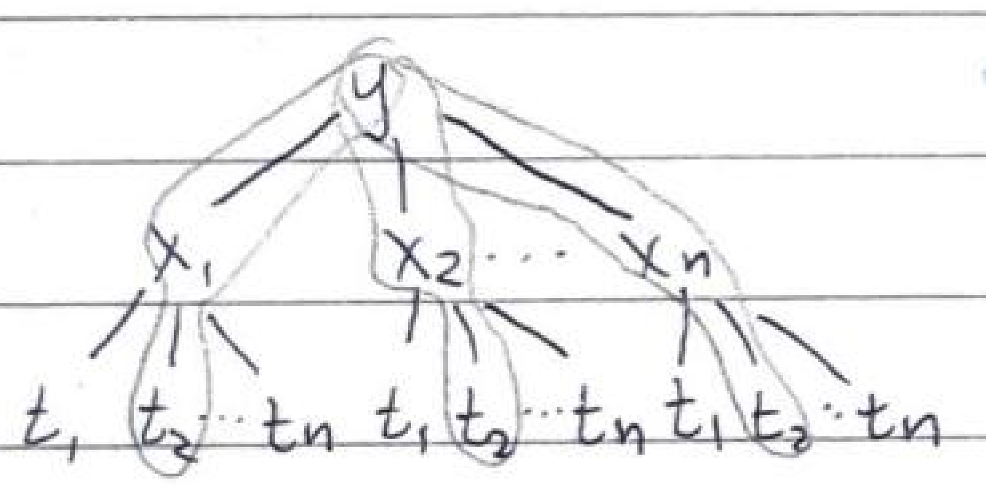
$f: U \subset \mathbb{R}^n \to \mathbb{R}^m$, $g: V \subset \mathbb{R}^m \to \mathbb{R}^k$ so that $g \circ f$ is defined

Let $\vec{a} \in U$ and $\vec{b} = f(\vec{a}) \in V$, if f differentiable at $\vec{a}$ and g differentiable at $\vec{b}$ then $g \circ f$ differentiable at $\vec{a}$ and

$$D(g \circ f)(\vec{a}) = (Dg(\vec{b}))(Df(\vec{a}))$$

computing $g \circ f$ and then $D(g \circ f)$ — compute everything separately and matrix multiplication

$$\cancel{\partial f}\ \frac{\partial y}{\partial t_i} = \cancel{\prod_{j=1}^{n} \partial y}\ \sum_{j=1}^{n} \frac{\partial y}{\partial x_j}\frac{\partial x_j}{\partial t_i}$$



where y = dependent variable

x_j = intermediate variable

t_i = independent variable

always draw diagram to visualize

TANGENT PLANES AND LINEAR ~~APRO~~ APPROXIMATION

$f: \mathbb{R}^3 \to \mathbb{R}$ have continuous partial ~~deri~~ derivatives and let $\vec{x_0} = (x_0, y_0, z_0)$ lie on level surface S - $f(x,y,z) = k$, then $\nabla f(\vec{x_0}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ is normal to level set surface S

TANGENT PLANE

Let S be surface consisting (x,y,z) s.t. $f(x,y,z) = k$, $k \in \mathbb{R}$. Let f be differentiable at $\vec{x}_0 = (x_0, y_0, z_0)$ Tangent plane of S at $\vec{x_0}$ in $\mathbb{R}^3$

$$\nabla f(\vec{x_0}) \cdot (\vec{x} - \vec{x_0}) = 0 \iff \nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

parallel to $(x'(t_0), y'(t_0), z'(t_0)) = c'(t_0)$

$f: \mathbb{R}^2 \to \mathbb{R}$ be differentiable at (x_0, y_0), tangent plane at point $(x_0, y_0, f(x_0, y_0))$

$$z = f(\vec{x_0}) + \nabla f(\vec{x_0}) \cdot (\vec{x} - \vec{x_0}) \qquad \vec{x_0} = (x_0, y_0)$$

LINEAR ~~APR~~ APPROXIMATION

$f: \mathbb{R}^n \to \mathbb{R}^m$ $L(\vec{x}) = f(\vec{x_0}) + Df(\vec{x_0})(\vec{x} - \vec{x_0})$

$f: \mathbb{R}^2 \to \mathbb{R}$ differentiable at $\vec{x_0} = (x_0, y_0)$ — works the same with $\mathbb{R}^3 \to \mathbb{R}$

$$f(\vec{x}) \approx L(\vec{x}) \qquad L(\vec{x}) = f(\vec{x_0}) + \nabla f(\vec{x_0}) \cdot (\vec{x} - \vec{x_0})$$

MEAN VALUE THEOREM

$f(\vec{b}) - f(\vec{a}) = Df(\vec{x})(\vec{b}-\vec{a})$

if $Df(\vec{x}) = \vec{0}$ $\forall x$, f is constant

HIGHER ORDER DERIVATIVES.

CLASS AND PARTIAL DERIVATIVES

If $f(x,y)$ is of class (C^{2+}) (i.e. f is twice continuously differentiable), then ~~the~~

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

~~TAL~~ TAYLOR SERIES AND MACLAURIN SERIES

Assuming f is of class "C^∞" that can be represented by power series near point $x=a$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots \qquad |x-a|<r$$

$f(a) = c_0$

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots$

$f'(a) = c_1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

is called the Taylor series of f at a.

for $a=0$, it is called the Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$
$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

PARTIAL SUM OF THE TAYLOR SERIES

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is a polynomial of degree n called n-th degree Taylor polynomial

* $T_n(x)$ is ~~in~~ of powers of $x-a$, not x

$T_1(x) = f(a) + \frac{f'(a)}{1!}(x-a)$ line of tangent $y=f(x)$ at point $(a, f(a))$

$f''(a)$ = concavity at $(a, f(a))$

$$\left.\begin{aligned} e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} \end{aligned}\right\} \forall x \in \mathbb{R}$$

$$\underbrace{\begin{aligned} (1+x)^a &= \sum_{n=0}^{\infty} \binom{a}{n} x^n = 1 + \frac{a}{1!}x + \frac{a(a-1)}{2!}x^2 \\ \ln(1+x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \\ \tan^{-1}x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \end{aligned}}_{\text{for } |x|<1}$$

N-TH ORDER TAYLOR SERIES $\Leftrightarrow$ n^{th} degree

$$f(x_0+h) = f(x_0) + \sum_{h_1=1}^{n} h_{i_1} \frac{\partial f}{\partial x_{i_1}}(x_0) + \frac{1}{2!}\sum_{h_1,h_2=1}^{n} h_{i_1}h_{i_2} \frac{\partial^2 f}{\partial x_{i_1}\partial x_{i_2}}(x_0) + \cdots$$

e.g. $T_2(x,y) = f(x,y) + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x}\right|_{(x,y)} \cdot (x_i - x_{i0}) + \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} \left.\frac{\partial f}{\partial x_i \partial x_j}\right|_{(x,y)} (x_i - x_{i0})(x_j - x_{j0})$

$$= f(x,y) + \left[x \left.\frac{\partial f}{\partial x}\right|_{(x,y)} + y \left.\frac{\partial f}{\partial y}\right|_{(x,y)} \right] + \frac{1}{2!}\left[x^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x,y)} + 2xy \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x,y)} + y^2 \frac{\partial^2 f}{\partial y^2}(0,0) \right]$$

QUADRATIC FORMS

degree 2 homogeneous polynomial function

$$f(\vec{x}) = f(x_1, x_2, \cdots, x_n) = \sum_{i \le j,\, i,j=1}^{n} u_{ij} x_i x_j$$ where not all u_{ij} are 0.

this can be written as:

$$f(\vec{x}) = \vec{x}^T A \vec{x} = [x_1\ x_2 \cdots x_n] \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

U the upper triangular coefficient matrix

also $= [x_1\ x_2 \cdots x_n] \underbrace{\begin{bmatrix} u_{11} & \frac{1}{2}u_{12} & \cdots & \frac{1}{2}u_{1n} \\ \frac{1}{2}u_{12} & u_{22} & & \vdots \\ \vdots & & \ddots & \\ \frac{1}{2}u_{1n} & \cdots & & u_{nn} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

the ~~symm~~ symmetric coefficient matrix

DIAGONALIZING QUADRATIC FORM

1. Find the symmetric coefficient matrix A
2. Find the eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$ of A
3. Find an orthonormal basis consisting of eigenvectors of A. (orthonormalize eigenvectors of A using Gram-Schmidt)
4. Use the orthonormal basis to form C and substitute $\vec{x} = C\vec{t}$ giving the diagonalizing substitution
5. In diagonal form the quadratic form becomes $\sum_{i=1}^{n} \lambda_i t_i^2 = \vec{t}^T D \vec{t}$

$f(\vec{x}) = \vec{x}^T A \vec{x}$ is said to be , if A is symmetric, then

Positive definite if $f(\vec{x}) > 0$ for $\vec{x} \neq \vec{0}$ / eigenvalues of A are positive

Negative definite if $f(\vec{x}) < 0$ for $\vec{x} \neq \vec{0}$ / eigenvalues of A are negative

Indefinite if $f(\vec{x})$ has both positive and negative values / at ~~les~~ least one positive and one negative

	$f(\vec{x}),\ \vec{x} \neq \vec{0}$	A is symmetric	det of submatrices A (symmetric) $k \times k$
POSITIVE DEFINITE	>0	all $\lambda > 0$	all >0
NEGATIVE DEFINITE	<0	all $\lambda < 0$	$- + - + -$ sign of $\det A_k = (-1)^k$
INDEFINITE	both	has $\lambda > 0$ and $\lambda < 0$	

MAXIMUM AND MINIMUM VALUES

A point x_0 is critical point of f if either f is not differentiable at x_0 or if it is, $Df(x_0) = 0$ $\quad (\nabla f(x_0) = \vec{0})$

First Derivative Test: If $U \subset \mathbb{R}^n$ is open, $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable and x_0 is local extremum, then x_0 is a critical point: $\frac{\partial f}{\partial x_1}(x_0) = 0, \cdots \frac{\partial f}{\partial x_n}(x_0) = 0$

Second Derivative Test: $U \subset \mathbb{R}^n \to \mathbb{R}$ class C^2 $x_0 \in$ open disk $\subset U$ be a critical point

for the Hessian form, positive definite $\Rightarrow (x_0, f(x_0))$ local minimum

negative definite $\Rightarrow (x_0, f(x_0))$ local maximum

else ~~saddle~~ saddle if determint $\neq 0$

~~and neither case~~

else is degenerate

HESSIAN MATRIX

Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ of ~~cat~~ class C^3,

$$H_f(x_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x_0) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x_0) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x_0) & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x_0) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x_0) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n}(x_0) \end{bmatrix}$$

for $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ of class C^2,

$$H_f(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$ critical point satisfies $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$

$D(x,y) = \det(H_f(x,y))$

$D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0 \Rightarrow$ local minimum

$D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0 \Rightarrow$ local maximum

$D(x_0,y_0) < 0 \Rightarrow$ saddle point

$D(x_0,y_0) = 0 \Rightarrow$ test inconclusive

EXTREME VALUE THEOREM

Let D be a compact set in $\mathbb{R}^n$ and let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ be continuous. Then f assumes both global maximum and global minimum on D

FINDING GLOBAL EXTREMA OF CONTINUOUS FUNCTIONS OF A COMPACT SET

1. Find critical points for f on interior of D
2. Find critical points for f restricted to boundary of D
3. Compute f at each of the critical points
4. Compare and pick largest / smallest

FINDING AND CLASSIFYING CRITICAL POINTS

In a region

1. Find all first order partial derivatives and equate to 0
2. Solve for all variables and find critical points
3. Use above classifications to ~~f~~ determine which points are local max / min et

With an equation(s) to ~~sta~~ satisfy

Given $f(\vec{x})$ and $g(\vec{x}) = c$, Define $L(\vec{x}, \lambda) = f(\vec{x}) - \lambda(g(\vec{x}) - c)$

1. Find all first order partial derivatives and equate to 0
2. Solve for all variables (often require to find λ first) and find critical points
3. Classify with same method as above (global extremum?)

MULTIPLE INTEGRALS

Midpoint Rule

Average value of f is $f_{ave} = \frac{1}{A}\iint_R f(x,y)\,dA$, where A is the area of the rectangle R. (Rectangle given by $[a,b]\times[c,d] \Rightarrow$ ~~(a,b), (a,c), (a,d), (b~~ $(a,c)\,(a,d)\,(b,c)\,(b,d)$

Mean Value Theorem for double Integrals

$f(x,y)$ continuous on compact set R (Area = A) in x-y plane, $\exists (x_0,y_0)\in R$ s.t.

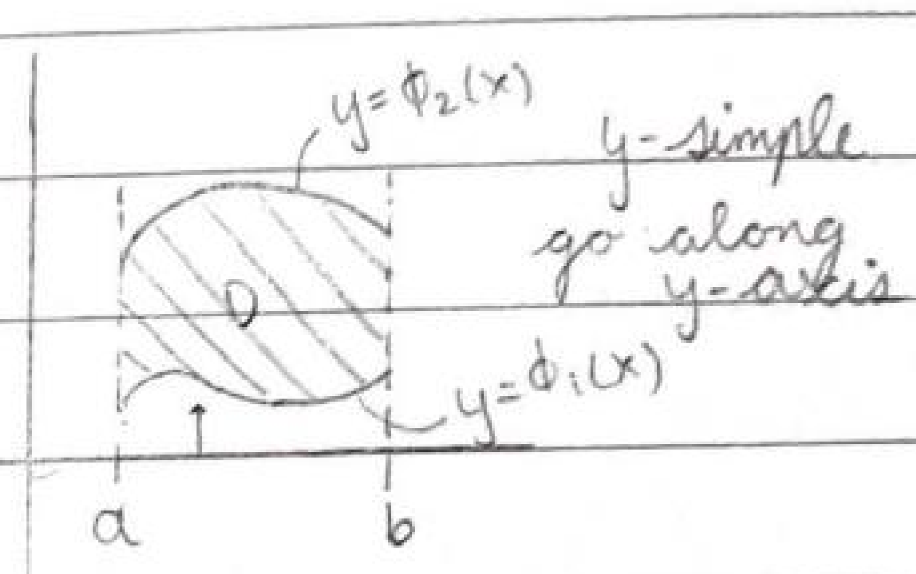
$$\iint_R f(x,y)\,dA = f(x_0,y_0)A$$

DOUBLE INTEGRALS

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy$$ (volume of solid under graph of f above R ($f\geq 0$))

or $$= \int_a^b\int_c^d f(x,y)\,dy\,dx$$ (for RECTANGLES = $a,b,c,d \in \mathbb{R}$)

R = rectangle, D = region

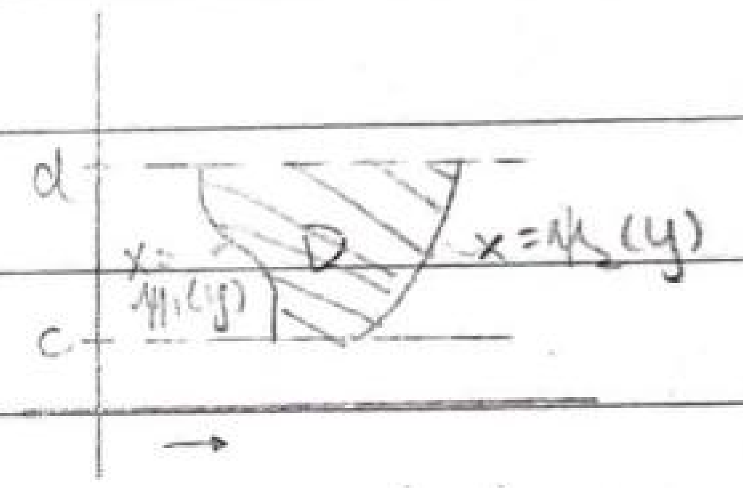


y-simple - integrate y first

$D = \{(x,y) \mid x\in[a,b] \text{ and } \phi_1(x)\leq y\leq \phi_2(x)\}$

bounds (x are numbers while y are functions of x)

$$\iint_D f(x,y)\,dA = \int_a^b\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$



x-simple go along x-axis

x-simple - integrate x first

$D = \{(x,y) \mid y\in[c,d] \text{ and } \psi_1(y)\leq x\leq \psi_2(y)\}$

bounds: x are functions of y while y are numbers

$$\iint_D f(x,y)\,dA = \int_c^d\int_{\psi_1(x)}^{\psi_2(x)} f(x,y)\,\cancel{dy}\,dx\,dy$$

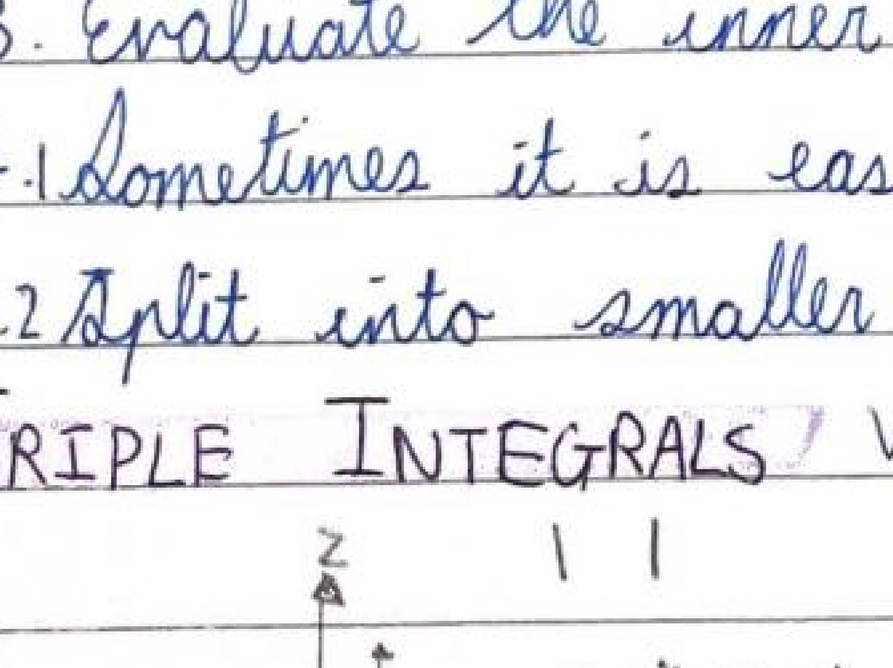
1. Sketch the graph
2. Determine whether to use x-simple / y-simple and define D, form integrals
3. Evaluate the inner integral and treat other variable as constant

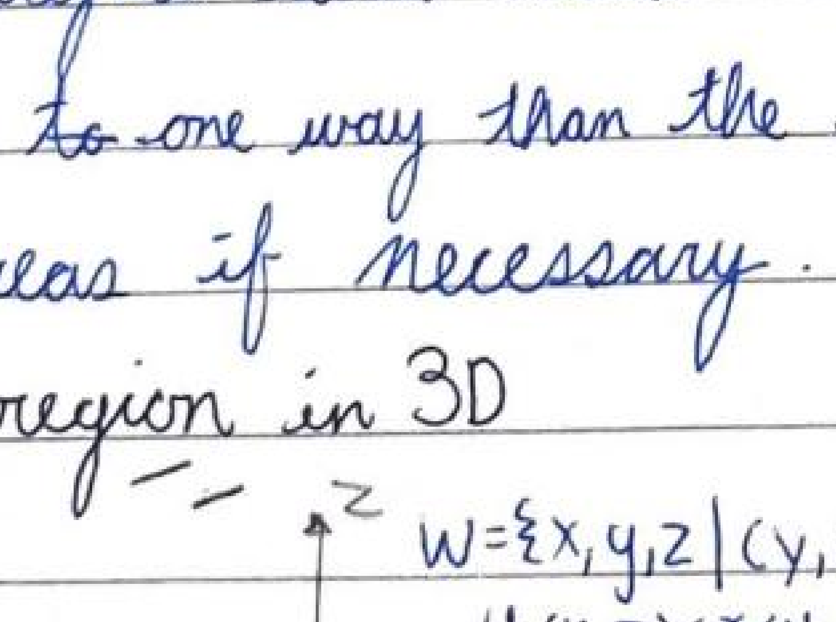
4.1 Sometimes it is easier ~~to~~ one way than the other

4.2 Split into smaller areas if necessary.

TRIPLE INTEGRALS

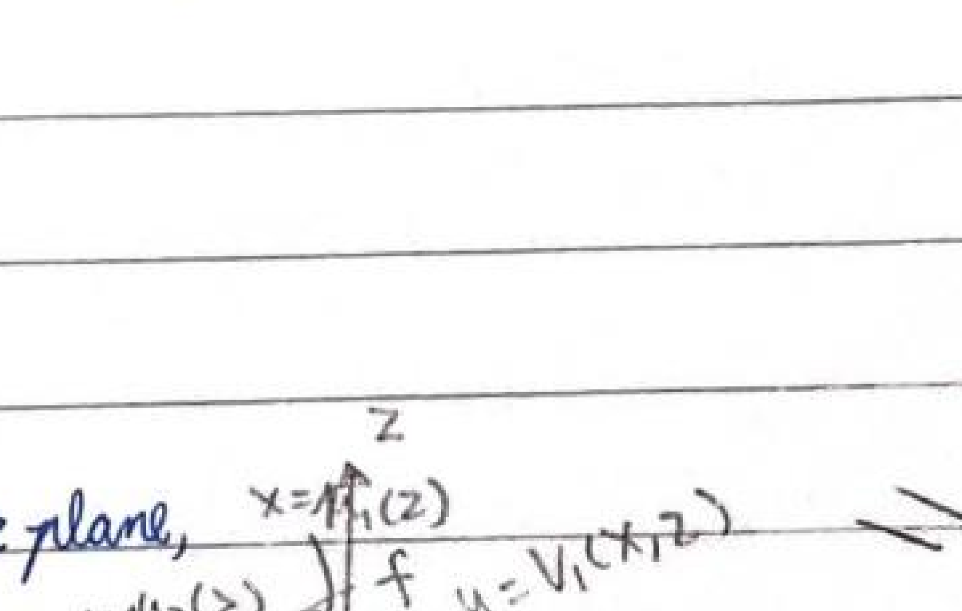
W = region in 3D





$W = \{x,y,z \mid (y,z)\in D$ in yz plane, $u_1(y,z)\leq x\leq u_2(y,z)$ both continuous$\}$

$$\int_e^f\int_{\phi_1(z)}^{\phi_2(z)}\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\,dy\,dz$$



$W = \{(x,y,z) \mid (x,y)\in D$ in x-simple/ y-simple, $\gamma_1(x,y)\leq z\leq \gamma_2(x,y)$, both continuous$\}$

$$\iiint_W f(x,y,z)\,dV = \iint_D \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz\,dA$$

$$= \int_a^b\int_{\phi_1(x)}^{\phi_2(x)}\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

$W = \{(x,y,z) \mid (x,z)\in D$ in xz plane, $V_1(x,z)\leq y\leq V_2(x,z)\}$

$$\int_e^f\int_{\psi_1(z)}^{\psi_2(z)}\int_{V_1(x,z)}^{V_2(x,z)} f(x,y,z)\,dy\,dx\,dz$$

* first octant ⇒ $x \geq 0, y \geq 0, z \geq 0$ ⇒ they are usually lower bounds
sketch the graph on 2D plane by setting third variable constant/zero
e.g. if x,y plane ⇒ $dx\,dy$ will be outside

THE CHANGE OF VARIABLES IN MULTIPLE INTEGRALS

Jacobian determinant

Let $T: D \subset \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 transformation given by $T(u,v) = (x(u,v), y(u,v))$. $\frac{\partial(x,y)}{\partial(u,v)}$ is determinant of the derivative matrix $DT(u,v)$ of T:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

CHANGE OF VARIABLES

Let $T: D^* \subset \mathbb{R}^2 \to D \subset \mathbb{R}^2$ be a C^1 transformation given by $T(u,v) = (x(u,v), y(u,v))$.

$$A(D^*) = \iint_{D^*} dA^*$$

absolute value

$$A(D) = \iint_D dA = \iint_{D^*} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA^*$$

1. Given 4 equations of x, y, form ~~tw~~ two with u = constant, v = constant e.g. $u=a$, $u=b$, $v=c$, $v=d$
   → u and v in terms of x and y so that they can be bounded by constants
2. Find x and y in terms of u and v
3. Find $\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$
4. Evaluate $\int_c^d \int_a^b \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv = \iint_D dA$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

DOUBLE INTEGRALS OVER POLAR & RECTANGULAR REGIONS

f continuous on x-y plane $D^* = \{(r,\theta) \mid 0 \leq h_1(\theta) \leq r \leq h_2(\theta), \alpha \leq \theta \leq \beta\}$, $0 < \beta - \alpha \leq 2\pi$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(r,\theta), y(r,\theta)) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr\,d\theta = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(x(r,\theta), y(r,\theta)) \boxed{|r|}\, dr\,d\theta$$

Same as above * remember to make f in terms of new variables

TRIPLE INTEGRALS

Let W and W^* be elementary regions in $\mathbb{R}^3$, $T: W^* \to W$, $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ $W = T(W^*)$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

Cylindrical Coordinates:

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z) \boxed{|r|}\, dr\,d\theta\,dz$$

Spherical Coordinates

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(\rho \sin\phi \cos\theta, \rho\, \overset{\sin\phi \sin\theta}{\cancel{\cos\phi \cos\theta}}, \rho\cos\phi) \boxed{\rho^2 \sin\phi}\, d\rho\, d\theta\, d\phi$$