MATB44

1. Classify ODE

Order of a ODE

Look at the highest "power" of in the whole equation / system

$x' \to 1^{st}$ order $\qquad x'' / \frac{d^2x}{dt^2} \to 2^{nd}$ order

Linear

$$x_i^{(k)} = g_i(t) + \sum_{\ell=1}^{n} \sum_{j=0}^{k-1} f_{i,j,\ell}(t)\, x_\ell^{(j)}$$

x_i a dependent variable differentiated k times wrt t

$g_i(t)$ a function only containing t, e.g. $5, t^2, t\sin t$

$f_{ij\ell}(t)$ a function only containing t

i.e. dependent variables x, x' etc. only on its own or multiplied with independent variables or functions that only contain independent variables.

Not linear if dependent variables have powers / multiplied by eachother

Autonomous

Not directly dependent on independent variable i.e. independent variable not ~~vis~~ visible in equation / system

Homogeneous

If linear and $g_i(t) = 0$ i.e. $x_i^{(k)} = \sum_{\ell=1}^{n} \sum_{j=0}^{k-1} f_{ijk}(t)\, x_\ell^{(j)}$

2. Find an ODE with Given Solution

If n-parameters present, differentiate n times and find a way to put $y^{(n)} / x^{(n)}$ back in the original equation. Simplify.

3. Solve 1^{st} Order ODEs

Autonomous

Move dx / dy / dt and x / y / t on the same side and integrate. Do not forget to simplify and +C!

Separable

$\dot{x}(t,x) = g(t) f(x) \to g(t)dt + f(x)dx = 0$

Rearrange so that all x is with dx and all t is with dt. Do the same thing as in autonomous. Be careful where a certain part is undefined.

LINEAR COEFFICIENTS

$(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0$

Main point here is to make it homogeneous (with ~~c~~ coefficient $=0$) and then solve it with the same ways as homogeneous coefficients

i. Find the intersection point and make it the origin

i.e. solve for x and y using simultaneous equation to get intersection (u,w)

change $x=\bar{x}+u$, $y=\bar{y}+w$

$dx=d\bar{x}$ $dy=d\bar{y}$

after some manipulation it will be look homogeneous

ii. Use the two lines as the new coordinate system

i.e let $u=a_1x+b_1y+c_1$, $v=a_2x+b_2y+c_2$

$du=a_1dx+b_1dy$ $dv=a_2dx+b_2dy$

find dx and dy ~~ante~~ in terms of du and dv

sub everything in and it will be homogeneous

iii. If parallel, only need one substitution

i.e. let $u=a_1x+b_1y+c_1$ $du=a_1dx+b_1dy$

$x=\frac{u-b_1y-c_1}{a_1}$ $dx=\frac{du-b_1dy}{a_1}$

HOMOGENEOUS COEFFICIENTS

If $f(x,y)$ can be written as $f(x,y)=x^n g(u)$ $u=\frac{y}{x}$ or

$f(x,y)=y^n g(u)$ $u=\frac{x}{y}$

let $y=ux \Rightarrow dy=udx+xdu$ and replace y ~~dx~~ and dy. This should make the equation separable

EXACT DIFFERENTIALS

$P(x,y)dx+Q(x,y)dy=0$ where $P(x,y)=\frac{\partial}{\partial x}f(x,y)$, $Q(x,y)=\frac{\partial}{\partial y}f(x,y)$

The differential is exact if $\frac{\partial}{\partial y}P(x,y)=\frac{\partial}{\partial x}Q(x,y)$

$\cancel{f(x,y)=\int_{x_0}^{x}P(\bar{x},y)d\bar{x}+\int_{y_0}^{y}Q(x_0,\bar{y})d\bar{y}}$

1. ~~Integrate~~ Evaluate $\int P(x,y)dx$, the constant c would be $h(y)$
2. Differentiate $\int P(x,y)dx$ ~~in~~ wrt y and compare the ~~ans~~ result with $Q(x,y)$ to get ~~$h(x)$~~ $h'(y)$
3. Evaluate $\int h'(y)dy$
4. Combine $F(x,y)=\int P(x,y)dx+\int h'(y)dy$

INTEGRATING FACTORS

A term when multiplied to the ODE, makes the ODE exact.

if ODE in the form of $\frac{dy}{dx} + P(x)y = Q(x)$ then IF $= e^{\int P(x)dx}$

4. PICARD ITERATES

$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$ where the initial value is given $x(t_0) = x_0$

$\dot{x} = f(t, x)$

5. WORD PROBLEM INTERPRETING ODE

Try to figure out what the question ~~whats~~ wants you to do and if asked graph / $t \to \infty$, plot at least two lines / describe the behaviour.

6. PROOF

Learn the definitions and theorems well.

BERNOULLI EQUATION

$\frac{dy}{dx} + P(x)y = Q(x)y^n$

$\times (1-n)y^{-n} \Rightarrow (1-n)y^{-n}\frac{dy}{dx} + P(x)(1-n)y^{1-n} = Q(x)(1-n)$

$u = y^{1-n} \quad du = (1-n)y^{-n}dy$

$\Rightarrow \cancel{(1-n)} \frac{du}{dx} + P(x)u = Q(x)(1-n) \Rightarrow$ use IF $= e^{\int (1-n)P(x)dx}$

FIXED POINT

$\forall f: x \to x$, a ~~p~~ fixed point is $f(x) = x$ $\quad K: C \subseteq X \to C \quad x \in C$ s.t. $K(x) = X$

CONTRACTIONS

A mapping that gets everything closer e.g. $f(x): \ln x$ on positive

$K: C \subseteq X \to C \quad \|K(x) - K(y)\| \leq \Theta \|x - y\|, \ x, y \in C$

NORM

Let ~~X~~ be a real vector space. A norm on X is a map $\|\cdot\|: X \to [0, \infty)$ s.t.

i. $\|\vec{0}\| = 0, \ \|\vec{x}\| > 0 \ \forall \vec{x} \neq \vec{0}$

ii. $\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| \ \forall \alpha \in \mathbb{R}$ and $\vec{x} \in X$

iii. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \ \forall \vec{x}, \vec{y} \in X$

CAUCHY SEQUENCE

A sequence $\forall \varepsilon > 0, \exists N > 0$ s.t. $\overset{m,n>N}{\cancel{N>n}} \Rightarrow \|x_m - x_n\| < \varepsilon$ (getting closer)

COMPLETE NORM SPACE / BANACH SPACE

$\Leftrightarrow$ Banach Space where every Cauchy Sequence has a limit

Checking Contraction

$\|K(x)-K(y)\| \leq \theta \|x-y\|$ where $\theta \in [0,1)$

find a θ that works

Consider 1. The mapping puts points in C closer together by θ

2. The map is closed under the interval C

Banach Fixed Point Thm.

Let C be a (nonempty) closed subset of a Banach space X and let $K: C \to C$ be a contraction, then K has a unique fixed point $\bar{x} \in C$

s.t. $\|K^n(x)-\bar{x}\| \leq \frac{\theta^n}{1-\theta}\|K(x)-x\|, x \in C$

Midterm 2

1. Solve Linear ODE of Order n With Constant Coefficients

$$\cancel{a_n}x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 = Q(t) \quad , \quad a_i \in \mathbb{R}$$

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = Q(t) \quad , \quad a_i \in \mathbb{R}$$

$Q(t)$ will have finitely many linearly independent derivatives

The solution would be of the form $x(t) = x_c(t) + x_p(t)$ where $x_c(t)$ is the complementary function (solution to the homogeneous ~~solutio~~ equation, has n-parameters) and $x_p(t)$ is the particular solution to this inhomogeneous equation

Finding $x_c(t)$

1. Form a characteristic polynomial i.e. convert $x^{(i)}$ to m^i and x to 1.
2. Equate the ~~polyn~~ polynomial to 0 and solve for m.

$x_c(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} + \cdots C_n e^{m_n t}$ ✱ if have repeated roots, multiply a t to one of the terms so they remain linearly independent.

~~Finding $x_p(t)$~~ if contains complex roots $\{e^{(\alpha+\beta i)t}, e^{(\alpha-\beta i)t}\}$ span the same space as $\{e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t\}$

Finding $x_p(t)$

1. Assume x_p is a ~~linearly~~ linear combination of $Q, Q', \cdots Q^{(k)}$
2. Form an $x_p(t) = AQ(t) + BQ'(t) + \cdots$ where $Q(t), Q'(t) \cdots$ are linearly independent
3. Differentiate x_p n times so you can substitute it back to the equation we are solving.
4. Equate like terms and solve for $A, B, \cdots$

✱ if one of the linearly independent term is in $x_c(t)$, multiply with t to avoid getting ~~cancele~~ cancelled (not accurately stated)

2. Solve Linear ODE of Order 2

$$f_2(t)\ddot{x}(t) + f_1(t)\dot{x}(t) + f_0(t)x(t) = Q(t)$$

($Q(t)$ may have infinitely many linearly independent derivatives)
(works for non-constant coefficients $u(t)$) VARIATION OF PARAMETERS

Find the solutions to homogeneous equation (from 1.)
- should have 2 in this case (linearly independent + order 2)
  ↳ x_1 , x_2

~~If~~ x_p would be of the form ~~$u_1(x)$~~ $u_1(t)x_1(t) + u_2(t)x_2(t) = x_p(t)$
where $u_i(t)$ are unknown functions
assume ~~$u_1'y_1 + u_2'y_2 = 0$~~ $u_1'x_1 + u_2'x_2 = 0$

$\Rightarrow x_p'(t) = u_1x_1' + u_2x_2'$ and $x_p''(t) = u_1'x_1' + u_1x_1'' + u_2'x_2' + u_2x_2''$

$u_1'x_1 + u_2'x_2 = 0$

$u_1'x_1' + u_2'x_2' = \frac{Q(t)}{a_2}$

if $x''(t)$ has coefficient a_2

solve for u_1' and u_2' using simultaneous equation
integrate u_1' ~~to~~, u_2' and sub back into $x_p = u_1x_1 + u_2x_2$

Reduction of Order
given one solution
use ansatz: $y_2 = x\int u(x)dx$

$y_2' = \int u(x)dx + xu(x)$

$y_2'' = u(x) + u(x) + xu'(x) = 2u(x) + xu'(x)$

~~plg~~ plug the above 3 to your equation (homogeneous) to solve for u.
put u back into ansatz to find y_2.

IVP Cases
1. ~~S~~ Find 2 solutions ~~using~~ of homogeneous equation ~~(either variation of parameter or~~ reduction of order)
2. Find particular solution (variation of parameter)
3. Get a solution $y = c_1y_1 + c_2y_2 + y_p$
4. Plug in initial conditions and solve for c_1, c_2

3. Word Problem

Simple Harmonic Motion

$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$

$\ddot{x} + 2\underset{\text{resistance}}{r}\dot{x} + \underset{\text{natural frequency}}{\omega_0^2} x = f(t)$

RLC Circuits

$L\ddot{I} + R\dot{I} + \frac{1}{C}I = \dot{V}$ $\eta = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

These things will relate to simple harmonic motion

Case 1: $\eta = 0$ / $r = 0$ i.e. no damping

$x_c = C\cos(\omega_0 t - \delta)$ where C is amplitude

since $\ddot{x} + \omega_0^2 x = 0$ δ is a phase shift

Case 2: $\eta > 0$ / $r > 0$

$\ddot{x} + 2\eta\dot{x} + \omega_0^2 x = 0$

$m^2 + 2\eta m + \omega_0^2 = 0$ $\rightarrow$ $\eta > \omega_0$ (over damping)

$\therefore m = -\eta \pm \sqrt{\eta^2 - \omega_0^2}$ 2 distinct real roots, $m_1, m_2 < 0$

$x_c = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

$\downarrow$

$\eta = \omega_0$ (critical damping)

1 distinct real root

$m = -\eta$

$x_c = (c_1 + c_2 t)e^{-\eta t}$

$\downarrow$

$\eta < 0$ (under damping)

2 conjugate complex roots

$m_1, m_2 = -\eta \pm \beta i \rightarrow \beta = \sqrt{\omega_0^2 - \eta^2}$

$x_c = c_1 e^{-\eta t}\cos(\beta t) + c_2 e^{-\eta t}\sin(\beta t)$

$= Ae^{-\eta t}\cos(\beta t - \delta)$

Forcing Function

$\ddot{x} + 2\eta\dot{x} + \omega_0^2 x = f(t)$

Suppose $f(t) = 1$ a constant

$$\left.\begin{aligned} x_p &= c_1 \\ \dot{x}_p &= 0 \\ \ddot{x}_p &= 0 \end{aligned}\right\} c_1\omega_0^2 = 1 \quad c_1 = \frac{1}{\omega_0^2} = x_p$$

$\omega = \omega_0$ damped

$x_p = \frac{F\cos(\omega t - \alpha)}{2\omega_0\eta}$

Suppose $f(t) = \cos(\omega t)$

$\cos(\omega t) = Re(e^{i\omega t})$

$x_p = ke^{i\omega t}$ $\Rightarrow$ $-k\omega^2 e^{i\omega t} + 2\eta k(i\omega e^{i\omega t}) + \omega_0^2 k e^{i\omega t} = e^{i\omega t}$

$\dot{x}_p = ki\omega e^{i\omega t}$ $-k\omega^2 + 2\eta i k + \omega_0^2 k = 1$

$\omega = \omega_0$ undamped $\ddot{x}_p = -k\omega^2 e^{i\omega t}$ $\therefore k = \frac{1}{-\omega^2 + 2\eta i + \omega_0^2} = \frac{1}{\omega_0^2 - \omega^2 + 2\eta i}$

$\ddot{x} + \omega_0^2 x = Fe^{i\omega_0 t}$

$\therefore x_p = \frac{1}{\omega_0^2 - \omega^2 + 2\eta i} e^{i\omega t}$

$m = \pm i\omega_0$

$x = x_c + x_p \approx x_p$ because $x_c \to 0$ as $t \to \infty$

$\therefore x_c = c_1 e^{-i\omega_0 t} + c_2 e^{i\omega_0 t}$

$x_p = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\omega\eta)^2}} \cdot \cos(\omega t - \alpha)$

$Re(x_p) = \frac{F}{2\omega_0} t \sin(\omega_0 t)$

$\uparrow$

$t \to \infty$ blows up amplitude $= A =$

particular solution periodic, has same frequency as forcing function and amplitude

INAL EXAM

LEMMA FOR DETERMINANTS & LINEAR INDEPENDENCE

The following are equivalent:

$\det(A) \neq 0$

rows of A are linearly independent

columns of A are linearly independent

A has a 0-dimensional nullspace

WRONSKIAN

The wronskian of a set of functions $f_1(x), \cdots, f_n(x) \in C^{n-1}(I)$ is defined to be

$$W(f_1, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

If $f_1, \dots, f_n \in C^{n-1}(I)$ are linearly dependent on I, then $W(f_1, \cdots f_n) \equiv 0$ (determinant is 0)

If not $W(f_1, \cdots, f_n) \equiv 0$ then $f_1, f_2, \cdots, f_n \in C^{n-1}(I)$ are lineakly independent

If $W(y_1, y_2, \cdots, y_n)(x_0) = \vec{0}$, for $x_0 \in I$, $W(y_1, \cdots, y_n) \equiv 0$ where y_i is solution to $f_1, \cdots f_n$

If $W(y_1, \cdots, y_n)(x_0) = 0$ then $y_1, \cdots y_n$ are dependent.

LINEAR EQUATIONS

linear first-order system:

$$\dot{x}_1(t) = A_{11} x_1(t) + A_{12} x_2(t) + \cdots + A_{1n} x_n(t) \qquad x_1(0) = x_{01}$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$\dot{x}_n(t) = A_{n1} x_1(t) + A_{n2} x_2(t) + \cdots + A_{nn} x_n(t) \qquad x_n(0) = x_{0n}$$

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad A \in M_{n\times n}(\mathbb{R}), \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

MATRIX EXPONENTIAL

$$\exp(A) = \sum_{j=0}^{\infty} \frac{1}{j!} A^j, \text{ where } A^0 = I \Rightarrow x(t) = \exp(tA)x_0$$

$\exp(0) = I$

$\exp(X^T) = (\exp X)^T$ T = transpose

$\exp(X^*) = (\exp X)^*$ $*$ = conjugate transpose

$\det(e^A) = e^{\mathrm{tr}(A)}$ trace

if Y is invertible, $\exp(YXY^{-1}) = Y\exp(X)Y^{-1}$

if $XY = YX$, $\exp(X+Y) = \exp(X)\exp(Y)$

$\hookrightarrow \exp((a+b)X) = \exp(aX)\exp(bX) \rightarrow \exp(X)\exp(-X) = I$

$$\exp\left(\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}\right) = \begin{bmatrix} \exp(d_1) & \cdots & & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & \\ 0 & \cdots & 0 & \exp(d_n) \end{bmatrix}$$

given $\dot{x} = Ax$, $x(0) = x_0$, $x = \exp(tA)x_0$

SOLVING 2nd ORDER ODE USING MATRIX

e.g. $\ddot{z} + 3\dot{z} + 2z = 0$, $z(0) = 1$, $\dot{z}(0) = 1$

1. convert to first order system

$$\cancel{\dot{z} = x_1} \qquad z = x_1 \quad \Rightarrow \quad \dot{z} = \dot{x}_1 = x_2$$

$$\cancel{\ddot{z} = \dot{x}_1 = x_2} = \qquad \dot{z} = x_2 \qquad \ddot{z} = \ddot{x}_1 = \dot{x}_2 = -3\dot{z} - 2z = -3x_2 - 2x_1$$

(highest number ~~sub~~ subscript = order, get rid of primes)

2. form a 2×2 matrix so that $A\vec{x} = \dot{\vec{x}}$

$$Z(0) = X_1(0) = 1 \qquad \dot{z}(0) = x_2(0) = 1 \qquad \therefore \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow Ax = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -2x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

we know $X(t) = \exp(tA)X_0$

3. diagonalize A by finding eigenvalues and eigenvectors

$$A = PDP^{-1} \qquad P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}, \ D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \ P^{-1} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

4. calculate $X(t) = \exp(tA)X_0 = \exp(t\,PDP^{-1})X_0 = P\exp(tD)P^{-1}X_0$

$$P\exp(tD)P^{-1}X_0 = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}X_0 = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

notice second row is ~~differentiation~~ derivative of first row.

$$= \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix} \begin{matrix} \leftarrow z(t) \\ \leftarrow \dot{z}(t) \end{matrix}$$

given 2×2 matrix, $\lambda^2 - Tr(A)\lambda + \det(A) = 0$

Or equivalently, use ~~anzat~~ ansatz

$$X(t) = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$$

$$X(0) = \begin{bmatrix} C_1 e^{-t} + C_2 e^{-2t} \\ -C_1 e^{-t} - 2C_2 e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow C_1 = 3, \ C_2 = -2$$

answer equivalent to above

COMPLEX CASE

1. find ~~eig~~ eigenvalues and eigenvectors of the system

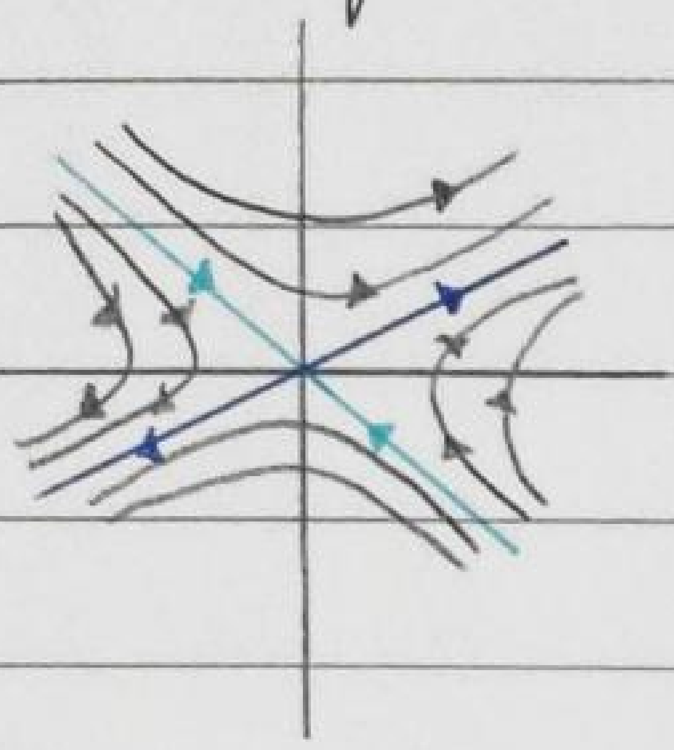
e.g. $A = \begin{bmatrix} -2 & -2 \\ 4 & 2 \end{bmatrix}$ $x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\lambda_1 = 2i$, $v_1 = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$, $\lambda_2 = -2i$, $v_2 = \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$

2. guess ansatz: $\vec{x} = C\vec{V}_1(\cos\beta t + i\sin\beta t)$ and solve for c_1, c_2 where c_1 corresponds to the real parts and c_2 corresponds to the imaginary parts

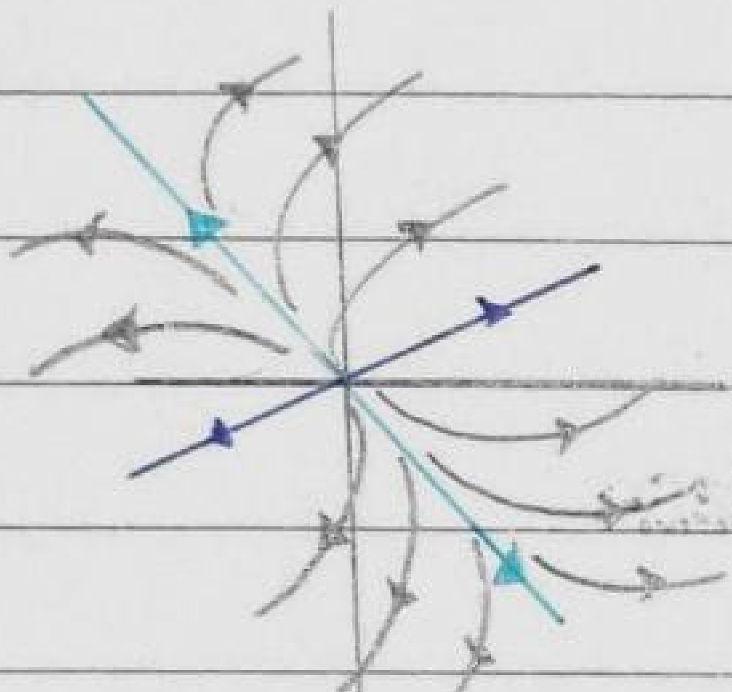
$$\vec{x} = c\begin{pmatrix} -1+i \\ 2 \end{pmatrix}(\cos 2t + i\sin 2t) = C_1\begin{pmatrix} -\cos 2t - \sin 2t \\ 2\cos 2t \end{pmatrix} + C_2\begin{pmatrix} -\sin 2t + \cos 2t \\ 2\sin 2t \end{pmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow c_1 = \frac{1}{2}, \ c_2 = \frac{-1}{2}$$

PHASE PORTRAIT

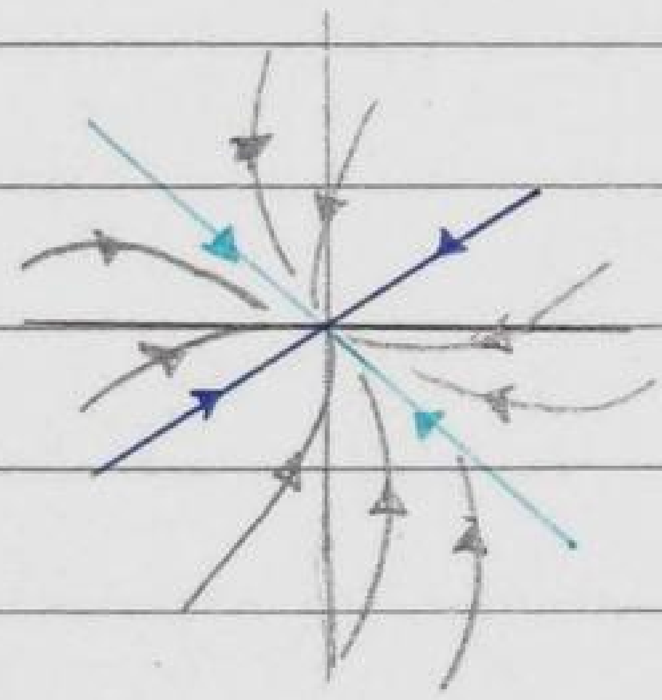
2 real eigenvalues



$\lambda_1 > 0$ Saddle Point
$\lambda_2 < 0$ (1 positive, 1 negative)



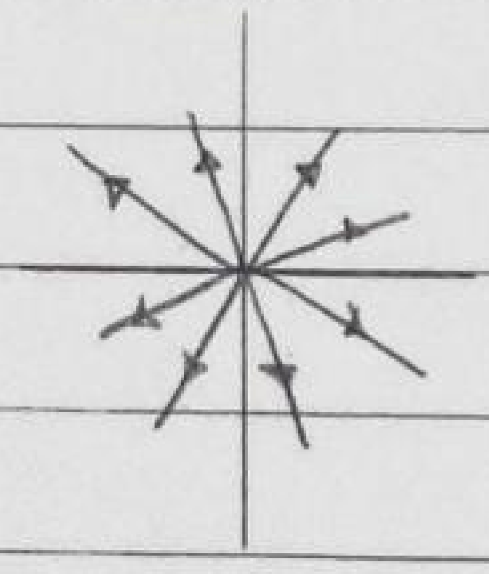
$\lambda_1 > 0$ Source Leaves origin but component along
$\lambda_2 > 0$ v_1 ~~gets~~ leaves quicker
$|\lambda_1| > |\lambda_2|$ .. (both positive)
(lines eventually parallel to more positive one)

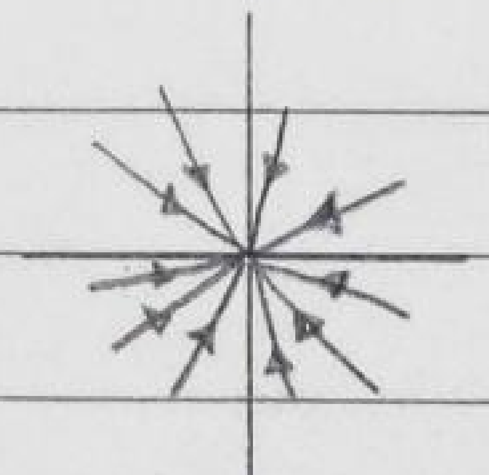


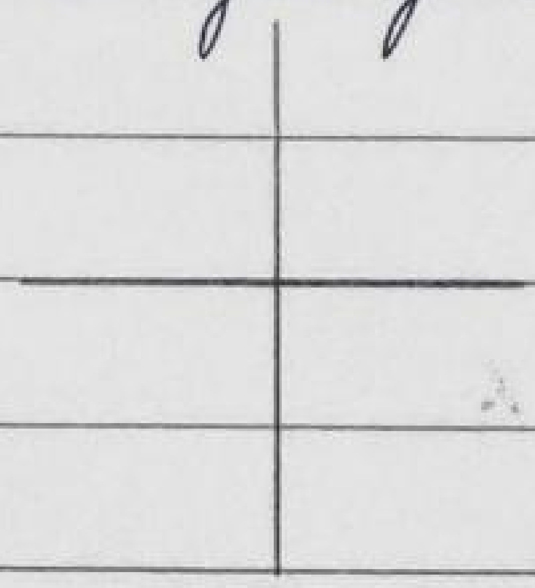
$\lambda_1 < 0$ Sink Goes to zero, but component along
$\lambda_2 < 0$ v_1 decays quicker
$|\lambda_1| > |\lambda_2|$ (both negative)
(lines initially parallel to more negative component)

1 real eigenvalue if matrix is diagonalizable

$\lambda > 0$	$\lambda < 0$	$\lambda = 0$
source	sink	everything stays constant







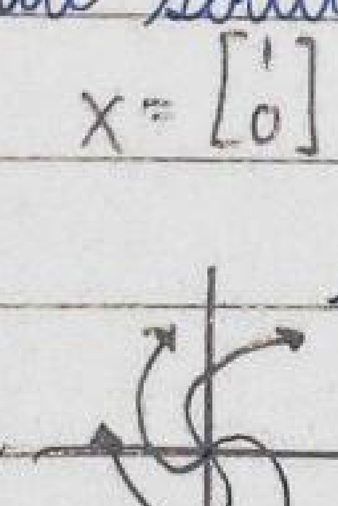
2 complex conjugate eigenvalues

$\vec{u}$ = real part of eigenvector $\vec{w}$ = imaginary part

general solution: $x(t) = e^{\alpha t}[c_1(\vec{u}\cos\beta t - \vec{w}\sin\beta t) + c_2(\vec{u}\sin\beta t + \vec{w}\cos\beta t)]$

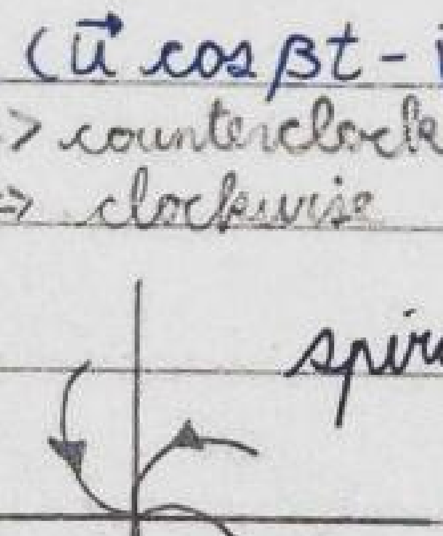
at $x = \begin{bmatrix}1\\0\end{bmatrix}$: $y' = c$ if $c > 0 \Rightarrow$ counterclockwise, $c < 0 \Rightarrow$ clockwise

spiral source



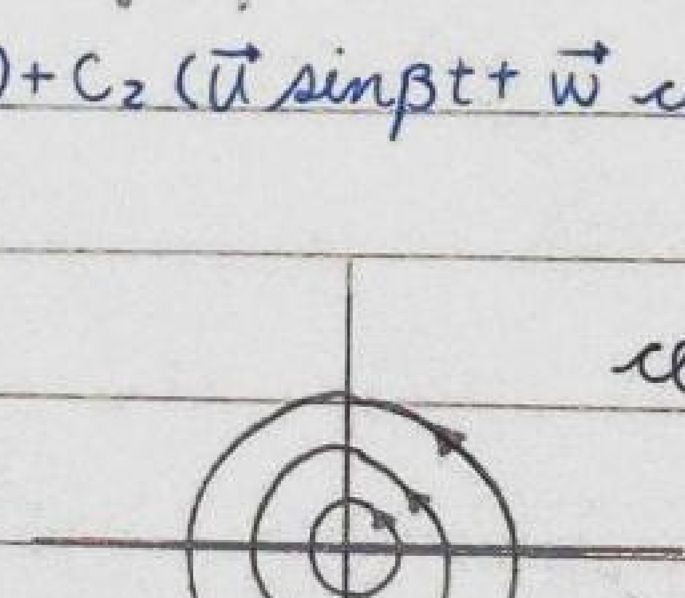
$\alpha > 0$

spiral sink



$\alpha < 0$

center



$\alpha = 0$

INHOMOGENEOUS SYSTEMS

Given $\dot{x}(t) = Ax(t) + g(t)$, $x(0) = x_0$

$$x(t) = \exp(tA)x_0 + \int_0^t \exp((t-s)A)g(s)ds$$

given $x^{(n)} + c_{n-1}x^{(n-1)} + c_{n-2}x^{(n-2)} + \dots c_1 x' + c_0 x = g(t)$

$$x(t) = x_h(t) + \int_0^t u(t-s)g(s)ds \quad \text{always works.}$$

COMPLEX CASE PART 2

1. find eigenvalues and eigenvectors as before but guess ansatz

$$x(t) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} e^{\alpha t}\cos\beta t + \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} e^{\alpha t}\sin\beta t$$

2. equate $\dot{x}(t)$ with $Ax(t)$ by differentiating above equation ~~(x(t) is~~
3. Solve for all the k_is by equating initial condition with $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$

THINGS TO NOTE FOR SYSTEM OF RABBITS AND SHEEPS

Say # of rabbits $= x(t)$, growth would be $\dot{x}(t) = 3x$ and $x(t) = x_0 e^{3t}$

With limited resources, logistic equation with fixed carrying capacity

$\dot{x} = x(3-x)$ $\qquad \frac{A}{x} + \frac{B}{3-x} = 1$

$\frac{dx}{dt} = x(3-x)$ $\qquad A(3-x) + Bx = 1$

$\frac{dx}{x(3-x)} = dt$ $\qquad$ if $x=0$, if $x=3$

$\qquad A = \frac{1}{3}$ $\quad B = \frac{1}{3}$

$\frac{1}{3x} + \frac{1}{3(3-x)}dx = dt$

$\frac{1}{3}\ln|x| - \frac{1}{3}\ln(3-x) = t + c$ $\qquad$ since absolute value, split to

$\ln\left(\frac{|x^{\frac{1}{3}}|}{|3-x|^{\frac{1}{3}}}\right) = t + c$ $\qquad$ cases $0 < x < 3$, $x < 0$ or $x > 3$

$\left|\left(\frac{x}{3-x}\right)\right|^{\frac{1}{3}} = Ae^{t} \rightarrow \left|\frac{x}{3-x}\right| = Ae^{3t}$ $\qquad$ and $x = 0$, $x = 3$

in case where $\dot{x}$, $\dot{y}$ have x and y terms, equate $\dot{x}$ and $\dot{y}$ to 0 and find critical points

JACOBIAN MATRIX

if $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ $\qquad \dot{z}(x,y) = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x}(x_0,y_0) & \frac{\partial \dot{x}}{\partial y}(x_0,y_0) \\ \frac{\partial \dot{y}}{\partial x}(x_0,y_0) & \frac{\partial \dot{y}}{\partial y}(x_0,y_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$

Use Jacobian Matrix to find eigenvalues and eigenvectors to determine the phase portrait

STABILITY

Fixed point attracting if all trajectories starting near it approaches it as $t \to \infty$

Liapunov stable if all trajectories start sufficiently close remain close $t > 0$ $t \in \mathbb{R}$

~~Asy~~ asymptotically stable if both conditions hold neutrally stable if Liapunov stable but not attracting

DIFFERENCE EQUATIONS

First order difference equation

$y_{n+1} = f(n, y_n)$, $n = 0, 1, 2, \dots$ $y_n \in \mathbb{R}$ it is linear if the function f is linear. We can give initial conditions $y_0 = \alpha$.

Questions with interest

y_0 = starting amount r = rate (compound monthly or yearly) p = payment or gain

n = months/years passed $y_{n+1} = y_n \cdot r \pm p$ (+ if gain/deposit, − if payment)

1. Form an equation of above form/similar
2. Find y_n in terms of y_0 $\qquad \sum_{i=1}^{n-1} r^i = \frac{1-r^n}{1-r}$
3. Calculate the missing information

Formal Power Series

$f(x) = a_0 + a_1x + a_2x^2 + \cdots$

$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$

$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n$

Some possible ~~mau~~ manipulation given $f(x) = \sum_{i=0}^{\infty} a_i x^i$ $g(x) = \sum_{i=0}^{\infty} b_i x^i$

$f(\lambda x) = \sum_{i=0}^{\infty} \lambda^i a_i x^i$

$f(x) + g(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$

$x f(x) = \sum_{i=0}^{\infty} (a_i) x^{i+1} = \sum_{i=1}^{\infty} (a_{i-1}) x^i = \sum_{i=0} (i+1) a_{i+1} x^i$

$f'(x) = \sum_{i=0}^{\infty} i a_i x^{i-1} = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i$

if $f(x)$ is a generating function of a_n and $g(x)$ is for b_n, then

$f(x) \cdot g(x)$ is the generating function for $c_n = \sum_{i=0}^{n} a_i b_{n-i}$

Solving difference equations / recurrence relations given $a_n =$ ~~f(an~~ $g(a_{n-1})$ and a_0

1. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$
2. Change index, add/subtract terms to get $f(x)$.
3. Solve for $f(x)$ and put it back to a summation formula with ~~fe~~ x^n the remaining portion in the summation is a_n (likely be using partial fraction decomposition