MATB43 DEFS AND THMS

1.1 The Real Numbers

THM 1.7

$\exists$ an ordered field $\mathbb{R}$ which

(i.) Contains $\mathbb{Q}$ and

(ii.) has the property that any nonempty subset of $\mathbb{R}$ which has an upper bound has a least upper bound.

THM 1.8.

Let $x \in \mathbb{R}$, $x > 0$. Then $\exists\, y \in \mathbb{R}$, $y > 0$ s.t. $y^2 = y \cdot y = x$.

ARCHIMEDEAN PROPERTY

Let a and $b \in \mathbb{R}$, $a, b > 0$. Then $\exists\, n \in \mathbb{N}$ s.t. $na > b$

DENSITY PROPERTY

Let $c, d \in \mathbb{R}$. Then $\exists\, q \in \mathbb{Q}$ with $c < q < d$.

ABSOLUTE VALUE

Let $x \in \mathbb{R}$, $|x| = \begin{cases} x \,; & x > 0 \\ 0 \,; & x = 0 \\ -x \,; & x < 0 \end{cases}$

Pf: $(|x|+|y|)^2 = |x|^2 + 2|xy| + |y|^2$ $\quad (|x|+|y|)^2 \geq |x+y|^2$

$(|x+y|)^2 = x^2 + 2xy + y^2$

TRIANGLE INEQUALITY

Let $x, y \in \mathbb{R}$. $|x+y| \leq |x| + |y|$

APPENDIX Construction Of Real Numbers

FIELD

A set S is a field if it is equipped with a binary operation $(+)$ and a second binary operation $(\cdot)$ such that the following axioms are satisfied:

A1. $x, y \in S \Rightarrow x + y \in S$

A2. $x, y \in S \Rightarrow x + y = y + x$

A3. $x, y, z \in S \Rightarrow x + (y + z) = (x + y) + z$

A4. $\exists\, e_1 \in S$ s.t. $x \in S \Rightarrow x + e_1 = x$ $(e_1 = 0)$

A5. $x \in S \Rightarrow \exists\, -x \in S$ s.t. $x + (-x) = 0$

M1. $x, y \in S \Rightarrow x \cdot y \in S$

M2. $x, y \in S \Rightarrow xy = yx$

M3. $x, y, z \in S \Rightarrow x(yz) = (xy)z$

M4. $\exists\, e_2 \in S$ s.t. $x \in S \Rightarrow x e_2 = x$ $(e_2 = 1)$

M5. if $0 \neq x \in S \Rightarrow \exists\, x^{-1} \in S$ s.t. $x(x^{-1}) = 1$

D1. $x, y, z \in S \Rightarrow x(y + z) = xy + xz$

APPENDIX 1. Elementary Number Systems

PRINCIPAL PROPERTIES OF $\mathbb{N}$

1. 1 is a natural number
2. $n \in \mathbb{N} \Rightarrow \hat{n} \in \mathbb{N}$ where $\hat{n}$ is the successor of n
3. $1 \neq \hat{n}$ $\forall n \in \mathbb{N}$
4. $\hat{m} = \hat{n} \Rightarrow m = n$

THM A2.59

Let S_1, S_2 be countable sets. Let $S = S_1 \cup S_2$. Then S is countable.

THM A2.60

If S and T are each countable sets then so is $S \times T \equiv \{(s,t) : s \in S, t \in T\}$

If $S_1, S_2, \cdots, S_k$ are each countable sets then so is the set $S_1 \times S_2 \times \cdots \times S_k$ $= \{(s_1, \cdots, s_k) \mid s_1 \in S_1, \cdots, s_k \in S_k\}$ consisting of all ordered k-tuples $(s_1, s_2, \cdots, s_k)$ with $s_j \in S_j$.

COR A2.63

The countable union of countable sets is countable

PROP A2.64

The collection P of all polynomials with integer coefficients is countable.

APPENDIX III. Review Of Linear Algebra

LINEAR DEPENDENT

A collection of elements $\vec{u}^1, \vec{u}^2, \ldots, \vec{u}^m \in \mathbb{R}^k$ is said to be linearly dependent if there exists constants $a_1, a_2, \cdots, a_m$, not all zero, such that $\sum_{j=1}^{m} a_j \vec{u}^j = \vec{0}$

2.1 Convergence Of Sequences

SEQUENCE

A sequence of real numbers is a function $\phi : \mathbb{N} \to \mathbb{R}$. We often write the sequence as $\phi(1), \phi(2), \cdots$ or simply as $\phi_1, \phi_2, \cdots$

CONVERGE

A sequence $\{a_j\}$ of real numbers is said to converge to a real number α, if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. if $j > N$ then $|a_j - \alpha| < \varepsilon$.

We call α the limit of the sequence $\{a_j\}$. We write $\lim_{j\to\infty} a_j = \alpha$ $(a_j \to \alpha)$

If a sequence $\{a_j\}$ does not converge then we say it diverges

BOUNDED

A sequence a_j is said to be bounded if $\exists M > 0$ s.t. $|a_j| \leq M \ \forall j \in \mathbb{N}$

PROP 2.5

(a_j) is a convergent sequence $\Rightarrow$ The limit of the sequence is unique and the sequence is bounded.

PROP 2.6

Let $\{a_j\}$ be a sequence of real numbers with limit α and $\{b_j\}$ be a sequence of real numbers with limit β. Then we have:

1. If $c \in \mathbb{R}$ then the sequence $\{a_j \cdot c\}$ converges to $c\alpha$.
2. The sequence $\{a_j + b_j\}$ converges to $\alpha + \beta$
3. The sequence $a_j \cdot b_j$ converges to $\alpha \cdot \beta$.
4. If $b_j \neq 0 \ \forall j$ and $\beta \neq 0$ then the sequence $\{\frac{a_j}{b_j}\}$ converges to $\frac{\alpha}{\beta}$.

CAUCHY SEQUENCE

Let $\{a_j\}$ be a sequence of real numbers. We say that the sequence satisfies the Cauchy criterion / the sequence is Cauchy if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $j, k > N \Rightarrow |a_j - a_k| < \varepsilon$

Every Cauchy sequence is bounded.

THM 2.12

Let $\{a_j\}$ be a sequence of real numbers. The sequence is Cauchy $\Leftrightarrow$ it converges to some limit α.

$\uparrow$, $\downarrow$ AND MONOTONE

Let $\{a_j\}$ be a sequence of real numbers. The sequence is said to be increasing if $a_1 \leq a_2 \leq \cdots$ It is decreasing if $a_1 \geq a_2 \geq \cdots$

A sequence is said to be monotone if it is either increasing or decreasing.

MONOTONE CONVERGENCE THM

If $\{a_j\}$ is an increasing sequence which is bounded above - $a_j \leq M < \infty$ $\forall j \in \mathbb{N} \Rightarrow \{a_j\}$ is convergent.

If $\{b_j\}$ is a decreasing sequence which is bounded below - $b_j \geq K > -\infty$ $\forall j \in \mathbb{N} \Rightarrow \{b_j\}$ is convergent.

COR 2.18

Let S be a nonempty set of real numbers which is bounded above and below. Let β be its supremum and α its infimum. If $\varepsilon > 0$ then there are $s, t \in S$ such that $|s - \beta| < \varepsilon$ and $|t - \alpha| < \varepsilon$.

THE PINCHING PRINCIPLE

Let $\{a_j\}, \{b_j\}, \{c_j\}$ be sequences of real numbers satisfying $a_j \leq b_j \leq c_j$ for every j sufficiently large. If $\lim_{j \to \infty} a_j = \lim_{j \to \infty} c_j = \alpha$ for some real number α, then $\lim_{j \to \infty} b_j = \alpha$.

2.2 Subsequences

SUBSEQUENCES

Let $\{a_j\}$ be a given sequence. If $0 < j_1 < j_2 < \cdots$ are positive integers then the function $k \mapsto a_{j_k}$ is called a subsequence of the given sequence. We usually write the subsequence as $\{a_{j_k}\}_{k=1}^{\infty}$ or $\{a_{j_k}\}$.

PROP 2.23

If $\{a_j\}$ is a convergent sequence with limit α, then every subsequence converges to α.

Conversely, if a sequence $\{b_j\}$ is s.t. each of its subsequences is convergent then $\{b_j\}$ itself is convergent.

BOLZANO - WEIERSTRASS

Let $\{a_j\}$ be a bounded sequence in $\mathbb{R}$. Then there is a subsequence which converges.

DIVERGES

We say a sequence $\{a_j\}$ diverges to $+\infty$ if, $\forall M>0 \ \exists N>0$ s.t. $j>N \Rightarrow a_j > M$
$a_j \to +\infty$

$\{a_j\}$ diverges to $-\infty$ if $\forall K>0 \ \exists N \in \mathbb{N}$ s.t. $j>N \Rightarrow a_j < -K$. $a_j \to -\infty$.

$\Rightarrow$ If S is a set of real numbers which is not bounded above, its supremum is $+\infty$. If T is a set of real numbers which is not bounded below, its infimum is $-\infty$.

2.3 Lim sup AND Lim inf

LIMIT INFIMUM AND LIMIT SUPREMUM

Let $\{a_j\}$ be a real sequence. For each j let $A_j = \inf\{a_j, a_{j+1}, \dots\}$ Then $\{A_j\}$ is an increasing sequence, so it has a limit. Limit infimum of $\{a_j\}$ is $\liminf a_j = \lim_{j\to\infty} A_j \Rightarrow$ smallest possible limit of any subsequence

Let $B_j = \sup\{a_j, a_{j+1}, \dots\}$ Then $\{B_j\}$ is a decreasing sequence, so it has a limit. Limit supremum of $\{a_j\}$ is $\limsup a_j = \lim_{j\to\infty} B_j \Rightarrow$ greatest possible limit of any subsequence.

PROP 2.36

Let $\{a_j\}$ be a sequence of real numbers. Let $\beta = \limsup a_j$ and $\alpha = \liminf a_j$. If $\{a_{j_k}\}$ is any subsequence of the given sequence then $\alpha \le \liminf_{l\to\infty} a_{j_l} \le \limsup_{l\to\infty} a_{j_l} \le \beta$.
Moreover, there is a subsequence $\{a_{j_k}\}$ s.t. $\lim_{k\to\infty} a_{j_k} = \alpha$ and another sequence $\{a_{j_m}\}$ s.t. $\lim_{k\to\infty} a_{j_m} = \beta$.

COR 2.37

If $\{a_j\}$ is a sequence and $\{a_{j_k}\}$ is a convergent subsequence then ~~lim inf a_j~~

$\liminf_{j\to\infty} a_j \le \lim_{k\to\infty} a_{j_k} \le \limsup_{j\to\infty} a_j$

PROP 2.39

Let $\{a_j\}$ be a sequence and let $\limsup a_j = \beta$ and $\liminf a_j = \alpha$. Assume $\alpha, \beta \in \mathbb{R}$. Let $\varepsilon > 0$. Then there are arbitrary large j s.t. $a_j > \beta - \varepsilon$. Also there are arbitrary large k s.t. $a_k < \alpha + \varepsilon$.

3.1 Convergence Of Series

SERIES

$\sum_{j=1}^{\infty} a_j$ is a series, where $a_j \in \mathbb{R}$ (or $\mathbb{C}$) For $N = 1, 2, 3, \cdots$, $S_N = \sum_{j=1}^{N} a_j = a_1 + a_2 + \cdots + a_N$ is called the Nth partial sum of the series. In case $\lim_{N\to\infty} S_N$ exists and is finite then the series converges. The limit of the partial sum is called the sum of the series. If the series does not converge, then we say that the series diverges.

CAUCHY CRITERION FOR SERIES

The series $\sum_{j=1}^{\infty} a_j$ converges if and only if, $\forall \varepsilon > 0$ $\exists N \geq 1 \in \mathbb{N}$ s.t. $n \geq m > N$ $\Rightarrow \left|\sum_{j=m}^{n} a_j\right| < \varepsilon$

THE ZERO TEST

If the series converges then the terms a_j tend to zero as $j \to \infty$

PROP 3.8

A series $\sum_{j=1}^{\infty} a_j$ with all $a_j \geq 0$ is convergent if and only if the sequence of partial sums is bounded.

3.2 THE COMPARISON TEST

Suppose $\sum_{j=1}^{\infty} a_j$ is convergent where $a_j \geq 0$ $\forall j \in \mathbb{N}$. If $\{b_j\}$ are real and $|b_j| \leq a_j$ $\forall j \in \mathbb{N}$ then $\sum_{j=1}^{\infty} b_j$ converges.

If $\sum_{j=1}^{\infty} a_j$ is convergent where $a_j \geq 0$ $\forall j \in \mathbb{N}$ and $0 \leq b_j \leq a_j$ $\forall j \in \mathbb{N}$ then the series $\sum_{j=1}^{\infty} b_j$ converges.

THE CAUCHY CONDENSATION TEST

Assume that $a_1 \geq a_2 \geq \cdots \geq a_j \geq \cdots 0$. The series $\sum_{j=1}^{\infty} a_j$ converges $\Leftrightarrow$ $\sum_{k=1}^{\infty} 2^k \cdot a_{2^k}$ converges.

GEOMETRIC SERIES

$\sum_{j=0}^{N} ar^j = \frac{a(1-r^{n+1})}{r}$

Let α be a complex number. The series $\sum_{j=0}^{\infty} \alpha^j$ is called a geometric series. It converges $\Leftrightarrow |\alpha| < 1$. If $|\alpha| < 1$, then $S_\infty = \frac{1}{1-\alpha}$

COROLLARY 3.17

Let r be a real number. The series $\sum_{j=1}^{\infty} \frac{1}{j^r}$ converges if r exceeds 1 and diverges otherwise

ROOT TEST

Let $\sum_{j=1}^{\infty} a_j$ be a series. If $\limsup_{j\to\infty} |a_j|^{\frac{1}{j}} < 1$ then the series converges.

RATIO TEST

Let $\sum_{j=1}^{\infty} a_j$ be a series. If $\limsup_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| < 1$ then the series converges.

INTEGRAL TEST

Let f be a continuous function on $[0,\infty)$ that is monotonically decreasing. The series $\sum_{j=1}^{\infty} f(j)$ converges $\Leftrightarrow$ the integral $\int_1^{\infty} f(x)\,dx$ converges.

3.3 ADVANCED CONVERGENCE TESTS

ABSOLUTE CONVERGENT

A series of real numbers $\sum_{j=1}^{\infty} a_j$ is said to be absolutely convergent if $\sum_{j=1}^{\infty} |a_j|$ converges.

If the series $\sum_{j=1}^{\infty} a_j$ is absolutely convergent, then it is convergent.

CONDITIONALLY CONVERGENT

A series $\sum_{j=1}^{\infty} a_j$ is said to be conditionally convergent if $\sum_{j=1}^{\infty} a_j$ converges, but it does not converge absolutely.

3.5 OPERATION ON SERIES

PROPOSITION 3.48

Let $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ be convergent series of real or complex numbers; assume that the series sum to limits α and β respectively. Then

a. The series $\sum_{j=1}^{\infty} (a_j + b_j)$ converges to the limit $\alpha+\beta$.

b. If c is a constant then the series $\sum_{j=1} c \cdot a_j$ converges to $c \cdot \alpha$.

THEOREM 3.49

Let $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ be two absolutely ~~conver~~ convergent series which converge to limits α and β respectively. Define the series $\sum_{m=0}^{\infty} c_m$ with summands $c_m = \sum_{j=0}^{m} a_j \cdot b_{m-j}$. Then $\sum_{m=1}^{\infty} c_m$ converges absolutely to $\alpha \cdot \beta$.

4.1 OPEN AND CLOSED SETS

OPEN SET

A subset $U \subseteq \mathbb{R}$ is open (or an open set) $\Leftrightarrow \forall x \in U\ \exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq U$

CLOSED SET

A subset $F \subseteq \mathbb{R}$ is closed (or a closed set) $\Leftrightarrow F^c = \{x \in \mathbb{R} \mid x \notin F\}$ = complement of F is open.

PROPOSITION 4.5

If U_α are open sets, for α in some (possibly uncountable) index set A, then $U = \bigcup_{\alpha \in A} U_\alpha$ is open.

PROPOSITION 4.6

If $U_1, U_2, \dots, U_k$ are open sets then the set $V = \bigcap_{j=1}^{k} U_j$ is also open.

PROPOSITION 4.7

Let $U \subseteq \mathbb{R}$ be a nonempty open set. Then there are either infintely many or countably many pairwise disjoint open intervals I_j such that $U = \bigcup_{j=1}^{\infty} I_j$.

PROPOSITION 4.12

If $\{F_i\}_{i \in I}$ is any collection of closed sets, then the intersection $F = \bigcap_{i \in I} F_i$ is closed.

If $\{E_j\}_{j=1}^{m}$ is a finite collection of closed sets, then $E = \bigcup_{j=1}^{n} E_j$ is closed.

STRUCTURE THM FOR OPEN SETS

Every open set can be expressed as a denumerable (finite or countable) disjoint union of open intervals

NEIGHBOURHOOD NOTATION

Given $x \in \mathbb{R}$ and $r > 0$, write $N(x,r) = (x-r, x+r) = \{t \in \mathbb{R} \mid x-r < t < x+r\}$ which is a symmetric open interval centered at x of "radius" r

$t \in N(x,r) \Leftrightarrow x-r < t < x+r$ or $|t-x| < r$

DELETED NEIGHBOURHOOD OF X OF RADIUS r

$N^*(x,r) = N(x,r) \setminus \{x\} = (x-r, x+r) \setminus \{x\} = \cancel{(x-r) \cup (x} \; (x-r, x) \cup (x, x+r)$

ACCUMLATION POINT / LIMIT PT / CLUSTER POINT

Let $S \subseteq \mathbb{R}$. A number $c \in \mathbb{R}$ is an accumulation point of $S \Leftrightarrow$ $\forall \varepsilon > 0$, $N^*(c, \varepsilon)$ contains infinitely many points of S.

- $\forall \varepsilon > 0$, $N^*(c, \varepsilon) \cap S$ is infinite $S' = \{$accumulation points of $S\}$ = derived set of S

BOUNDARY POINT

Let $S \subseteq \mathbb{R}$. A number $b \in \mathbb{R}$ is a boundary point of $S \Leftrightarrow$ $\forall \delta > 0$, $N(b, \delta) \cap S \neq \emptyset$ and $N(b, \delta) \cap S^c \neq \emptyset$

- b is a boundary point of $S \Leftrightarrow \forall \delta > 0, \exists s \in S$ s.t. $s \in N(b, \delta)$ and $\exists t \notin S$ s.t. $t \in N(b, \delta)$ $\partial(S) = \{$boundary points of $S\}$

INTERIOR POINT

Let $S \subseteq \mathbb{R}$. A point $s \in S$ is an interior point of $S \Leftrightarrow \exists r > 0$ s.t. $N(s, r) \subseteq S$

$S^\circ = \{$interior points of $S\}$

4.2 Further Properties Of Open And Closed Sets

Closure Of S

$\bar{S} = S \cup \partial(S)$ (S union boundary points of S}

Isolated Point Of S

Let $S \subseteq \mathbb{R}$, a point $s \in S$ is an isolated point of $S \Leftrightarrow \exists \varepsilon > 0$ s.t. $N(s, \varepsilon) \cap S = \{s\}$

- The only point in S that is in $N(s, \varepsilon)$ is s itself. iso(S) = { isolated points of S}

Dense

A subset $D \subseteq \mathbb{R}$ is dense $\Leftrightarrow \bar{D} = \mathbb{R}$ ($D \cup \partial(D) = \mathbb{R}$)

Proposition On Dense

A set $D \subseteq \mathbb{R}$ is dense $\Leftrightarrow \forall a, b \in \mathbb{R}$, $a < b$, we have $D \cap (a,b) \neq \emptyset$

- $\forall a, b \in \mathbb{R}$, $a < b$, $\exists d \in D$ s.t. $d \in (a,b)$

Proposition On Derived Set

Let $S \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. $c \in S' \Leftrightarrow \forall \varepsilon > 0$, $N^*(c, \varepsilon) \cap S \neq \emptyset$

Proposition / Definition 4.14

Let $S \subseteq \mathbb{R}$, $c \in \mathbb{R}$. $c \in S' \Leftrightarrow \exists$ an infinite sequence (x_j) of distinct points in S such that $x_j \to c$.

Proposition Closure Is S Union Derived Set

$S \subseteq \mathbb{R}$ $\bar{S} = S \cup S'$

Proposition On Dense (same as above)

A set $D \subseteq \mathbb{R}$ is dense iff $D \cap (a,b) \neq \emptyset$ $\forall a, b \in \mathbb{R}$ $a < b$

Propositions Open Or Closed

Let $S \subseteq \mathbb{R}$.

1. S is open iff $S = S^\circ$
2. S is closed iff $S = \bar{S}$
3. S is closed iff $S' \subseteq S$ (it contains all of its limit points)

Proposition Interior Points / Closure

1. $S^\circ = \bigcup \{V \mid V$ is an open set and $V \subseteq S\} = \{x \in \mathbb{R} \mid \exists$ an open subset $V \subseteq S$ where $x \in V\}$
2. $\bar{S} = \bigcap \{F \mid F$ is a closed set and $S \subseteq F\} = \{t \in \mathbb{R} \mid t \in F$ for all closed sets F s.t. $S \subseteq F\}$

Sequence Characterization Of Closed Sets

S is closed $\Leftrightarrow$ whenever $(x_j)_{j=1}^{\infty}$ is a sequence of points in S that converges, then $\lim_{j \to \infty} x_j \in S$.

Bolzano-Weierstrass Thm

Every bounded infinite set of real numbers has a limit point.

- If S is a bounded infinite set, then $S' \neq \emptyset$.

-SIDE- EQUIVALENCE CLASS

The relation $\sim$ is an equivalence relation if and only if:

1. It is reflexive $\forall a \in \mathcal{U}$, $a \sim a$ (a is equivalent to itself)
2. It is symmetric Let $a, b \in \mathcal{U}$, if $a \sim b$, then $b \sim a$
3. It is transitive Let $a, b, c \in \mathcal{U}$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

Given a set S and an equivalence relation $\sim$ on S, the equivalence class of an element a in S, $[a] = \{x \in S \mid x \sim a\}$.

4.3 COMPACT SETS

COMPACT SET

A set $S \subseteq \mathbb{R}$ is called compact $\Leftrightarrow$ if every sequence in S has a subsequence that converges to an element of S. If $\exists$ a sequence (x_n) where $x_n \in S$, $\exists$ a subsequence $(x_{n_j})_{j=1}^{\infty}$ of (x_n) and $x \in K$ s.t. $x_{n_j} \to x$ as $j \to \infty$

HEINE - BOREL THM

A set $S \subseteq \mathbb{R}$ is compact $\Leftrightarrow$ it is closed and bounded.

OPEN COVERING

Let S be a subset of the real numbers. A collection of open sets $\{O_\alpha\}_{\alpha \in A}$ (each O_α is an open set of real numbers) is called an open covering of S if $\bigcup_{\alpha \in A} O_\alpha \supseteq S$

SUBCOVERING

If C, D are open coverings of a set S s.t. $D \subseteq C$ then D is a subcovering of C. D is a finite subcovering if D has just finitely many elements.

THM 4.37 DEF OF COMPACT SETS IN TERMS OF OPEN COVERING

Let $S \subseteq \mathbb{R}$. S is compact $\Leftrightarrow$ every open covering of S has a finite subcovering.

i.e. S is compact $\Leftrightarrow$ given any collection $\mathcal{U} = \{U_i\}_{i=I}$ of open sets s.t. $S \subseteq \bigcup_{i \in I} U_i$, $\exists$ a finite subcollection $T = \{U_{i_j}\}_{j=1}^{N}$ of $\mathcal{U}$ s.t. $S \subseteq \bigcup_{j=1}^{N} U_{i_j}$, $\forall j = 1, 2, \cdots, N$, $U_{i_j} \in \mathcal{U}$

MAX/MIN THM FOR COMPACT SETS

If $K \subset \mathbb{R}$ is compact, then $\min(K)$ and $\max(K)$ exist.

PROP 4.39

Let $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_j \supseteq \cdots$ be non-empty compact sets of real numbers. Set $K = \bigcap_{j=1}^{\infty} K_j$. Then K is compact and $K \neq \emptyset$.

4.4. CANTOR SET

CANTOR $\frac{1}{3}$ SET

Let $S_0 = [0,1]$, $S_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ with the middle $\frac{1}{3}$ of S_0 removed. Let S_{j+1} = The set derived from S_j by extracting middle $\frac{1}{3}$ from each of S_j's subintervals.

$C = \bigcap_{j=1}^{\infty} S_j$, notice C is nonempty, closed and bounded $\Rightarrow$ compact

PROP 4.40

The Cantor set C has zero length, in the sense that the complementary set $[0,1] \setminus C$ has length 1.

PROP 4.41

The Cantor set is uncountable.

THM 4.42

Let C be the Cantor set and define $S = \{x+y \mid x \in C, y \in C\}$. Then $S = [0,2]$.

SET OF MEASURE ZERO

LENGTH OF INTERVAL

Given (a,b) $a,b \in \mathbb{R}$ $a<b$, the length of (a,b) is $\lambda(a,b) = b-a$

SET OF MEASURE 0

A set $A \subset \mathbb{R}$ is a set of measure $0 \Leftrightarrow \forall \varepsilon > 0$ $\exists$ a countable collection (open cover) $\mathcal{C} = \{(a_n, b_n)\}_{n=1}^{\infty}$ of open intervals such that $A \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ $a_n, b_n \in \mathbb{R}$, $a_n < b_n$ and $\sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon$

5.1 DEFINITION AND BASIC PROPERTIES OF THE LIMIT OF A FUNCTION

LIMIT OF A FUNCTION AT A POINT

Let $f: E \to \mathbb{R}$ be a function with domain $E \subseteq \mathbb{R}$. Let $c \in E'$. The limit of f at c exists $\Leftrightarrow$ $\exists\, l \in \mathbb{R}$ st. $\forall \varepsilon > 0$, $\exists \delta > 0$ so that if $x \in E$ and $0 < |x-c| < \delta$ then $|f(x) - L| < \varepsilon$. c need not $\in E$ and $f(c)$ need not be defined.

$l \in \mathbb{R}$ exists $\Rightarrow \lim_{x \to c} f(x) = l$, otherwise lim of f at c DNE.

UNIQUENESS OF LIMIT OF A FUNCTION

Let $f: E \to \mathbb{R}$, $c \in E'$. Suppose $\lim_{x \to c} f(x) = l_1$ and $\lim_{x \to c} f(x) = l_2$, $l_1, l_2 \in \mathbb{R}$

Then $l_1 = l_2$

ELEMENTARY PROPERTIES OF LIMITS OF FUNCTIONS

Let $f: E \to \mathbb{R}$, $g: E \to \mathbb{R}$ and $c \in E'$.

Assume (i.) $\lim_{x \to c} f(x) = l_1$, (ii.) $\lim_{x \to c} g(x) = l_2$

a. $\lim_{x \to c} (f \pm g)(x) = l_1 \pm l_2$

b. $\lim_{x \to c} (f \cdot g)(x) = l_1 \cdot l_2$

c. $\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{l_1}{l_2}$, $l_2 \neq 0$

Sequential Characterization Of Limit Of A Function At Point

Let $f: E \to \mathbb{R}$ be a function with domain $E \subseteq \mathbb{R}$, $c \in E'$.

$\lim_{x \to c} f(x) = l \iff \forall\ (a_j)_{j=1}^{\infty}$ in $E - \{c\}$ s.t. $a_j \to c$, it follows that $f(a_j) \to l$

5.2 ~~CONT~~ Continuous Functions

Continuous

Let $f: E \to \mathbb{R}$, ~~be a~~ $E \subseteq \mathbb{R}$. Let $c \in E$ s.t. $c \in E'$. f is continuous at $c \iff$ $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$ $\quad \lim_{x \to c} f(x) = f(c)$

f is continuous at $c \iff \forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $x \in N(c, \delta)$ then $f(x) \in N(f(c), \varepsilon)$

f is continuous on $E \iff f$ is continuous at every point $c \in E$

Thm 5.12

Let $f: E \to \mathbb{R}$, $g: E \to \mathbb{R}$, $E \subseteq \mathbb{R}$. Let $c \in E$ s.t. $c \in E'$. If f and g are continuous at c then so are $f \pm g$, $f \cdot g$, and $\frac{f}{g}$ $(g(c) \neq 0)$

Sequential Characterization Of Continuity At A Point

Let $f: E \to \mathbb{R}$, $E \subseteq \mathbb{R}$. Let $c \in E$ s.t. $c \in E'$. f is continuous at $c \iff$ whenever we have a sequence $(a_j)_{j=1}^{\infty}$ in E s.t. $a_j \to c$, then $f(a_j) \to f(c)$ $\quad (a_j \in E, j \geq 1)$

Prop. 5.14

Let $g: D \to E$ and $f: E \to F$ (E is range of g and F is range of f) Let $c \in D$. Suppose $c \in D'$ and $g(c) \in E'$, g is continuous at c and f is continuous at $g(c)$. Then $f \circ g$ is continuous at c.

Inverse Image

Let $f: E \to \mathbb{R}$, $E \subseteq \mathbb{R}$, Let $W \subseteq \mathbb{R}$. $f^{-1}(W) = \{x \in E \mid f(x) \in W\}$. $f^{-1}(W)$ is the inverse image of W under f.

Open Set Characterization Of Continuity ~~At A Point~~

Let $f: E \to \mathbb{R}$, $E \subseteq \mathbb{R}$. f is continuous on $E \iff \forall$ open set G, $\exists$ an open set H s.t. ~~$f^{-1}(G)$~~ $f^{-1}(G) = E \cap H$ / $\forall$ closed set F, $\exists$ a closed set D s.t. $f^{-1}(F) = E \cap D$

Special case $E = \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$ f is continuous on $\mathbb{R} \iff \forall$ open set G, $f^{-1}(G)$ is open

Image

Let $f: E \to \mathbb{R}$ and $L \subseteq E$. $f(L) = \{f(x) \mid x \in L\}$. The set $f(L)$ is called the image of L under f.

Thm 5.21

Let $f: E \to \mathbb{R}$ be continuous. If $K \subseteq E$ is compact then $f(K)$ is compact.

5.3 Topological Properties And Continuity

COROLLARY 5.22

Let f be a continuous, real-valued function with compact domain $K \subseteq \mathbb{R}$. Then there is a number L such that $|f(x)| \leq L$, for all $x \in K$.

EXTREME VALUE THEOREM

Let $f: E \to \mathbb{R}$ be a function on E and let $K \subseteq E$ be compact so f is continuous ~~s~~ on K.

Then f takes its max value and min value on K.

$\exists\, x_1, x_2 \in K$ s.t. $\forall x \in K$, $\underbrace{f(x_1)}_{\text{min value of } f \text{ on } K} \leq f(x) \leq \underbrace{f(x_2)}_{\text{max value of } f \text{ on } K}$

UNIFORMLY CONTINUOUS

Let f be a function with domain $E \subseteq \mathbb{R}$. We say f is uniformly continuous on E if ~~for e~~ $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $s, t \in E$ and $|s-t| < \delta$, then $|f(s) - f(t)| < \varepsilon$

THM 5.27

Let $f: K \to \mathbb{R}$ be continuous, K compact. Then f is uniformly continuous on K.

THM 5.30

Let $f: I \to \mathbb{R}$ be continuous, I be an open interval. Suppose L is a connected subset of I. Then $f(L)$ is connected.

INTERMEDIATE VALUE THEOREM

Let $f: E \to \mathbb{R}$ be a function that is continuous. Let $[a,b] \subseteq E$, $f(a) \leq \gamma \leq f(b)$

Then $\exists\, c \in \mathbb{R}$ ~~s.t.~~, $a \leq c \leq b$ s.t. $f(c) = \gamma$.

LIMITS AT $\pm\infty$

Let $E \subset \mathbb{R}$, E not bounded above. Let $f: E \to \mathbb{R}$. For $l \in \mathbb{R}$, $\lim_{x \to \infty} f(x) = l$
$\Leftrightarrow \forall \varepsilon > 0\ \exists M \in \mathbb{R}$ s.t. $x \in E$ and $x > M \Rightarrow |f(x) - l| < \varepsilon$.

Let $E \subset \mathbb{R}$, E not bounded below. Let $f: E \to \mathbb{R}$, $\lim_{x \to -\infty} f(x) = l \Leftrightarrow \forall \varepsilon > 0\ \exists M \in \mathbb{R}$ s.t. $x \in E$ and $x < M \Rightarrow |f(x) - l| < \varepsilon$.

6.1 THE CONCEPT OF DERIVATIVE

DERIVATIVE OF f AT x

Let $f: I \to \mathbb{R}$, ~~I of~~ $I \in \{(a,b), (-\infty, b), (a, \infty), (-\infty, \infty) \mid a, b \in \mathbb{R}\}$. If $x \in I$ then the limit $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ is the derivative of f at x if it exists. or $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

If the derivative of f at x exists then we say that f is differentiable at x.

If f is differentiable at every $x \in I$ then f is differentiable on I.

LEMMA 6.2

If f is differentiable at a point x then f is continuous at x. In particular, $\lim_{t \to x} f(t) = f(x)$

THM 6.3

Let $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ and differentiable at $x \in I$. Then $f \pm g$, $f \cdot g$, and $\frac{f}{g}$ are differentiable at $x \in I$. ~~Then $f \pm g, f$~~ ($g(x) \neq 0$ for $\frac{f(x)}{g(x)}$).

a. ~~$f(+$~~ $(f \pm g)'(x) = f'(x) \pm g'(x)$

b. $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

c. $\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g^2(x)}$

THM 6.5

Define $\psi: \mathbb{R} \to \mathbb{R}$ by $\psi(x) = \begin{cases} x-n & \text{if } n \leq x < n+1 \text{ and } n \text{ is even} \\ n+1-x & \text{if } n \leq x < n+1 \text{ and } n \text{ is odd} \end{cases}$ for $n \in \mathbb{Z}$. Then
$f(x) = \sum_{j=1}^{\infty} \left(\frac{3}{4}\right)^j \psi(4^j x)$ is continuous at every real x and differentiable at no $x \in \mathbb{R}$.

CHAIN RULE

Let g be a differentiable function on an open interval I and let f be a ~~d~~ differentiable function on an open interval that contains the range of g. Then $f \circ g$ is differentiable on I and $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$ for each $x \in I$.

6.2 THE MEAN VALUE THEOREM AND APPLICATIONS

LOCAL EXTREMA

Let $f: (a,b) \to \mathbb{R}$. $x \in (a,b)$ is a local maximum for f if $\exists \delta > 0$ s.t. $f(t) \leq f(x)$ $\forall$ $t \in (x-\delta, x+\delta)$. $x \in (a,b)$ is a local minimum for f if $\exists \delta > 0$ s.t. $f(t) \geq f(x)$ $\forall$ $t \in (x-\delta, x+\delta)$.

Local minima and local maxima are referred to collectively as local extrema.

FERMAT'S THM FOR EXTREMA (INTERIOR EXTREMA THM)

Let $f: (a,b) \to \mathbb{R}$; $a, b \in \mathbb{R}$, $a < b$

If f has a local extrema at $c \in (a,b)$ and if $f'(c)$ exists, then $f'(c) = 0$

DARBOUX'S THEOREM

Let f be differentiable on an open interval I. Pick points $s < t$ in I and suppose $f'(s) < p < f'(t)$. Then there is a point u between s and t such that $f'(u) = p$.

ROLLE'S THEOREM

Let f be a continuous function on the closed interval $[a,b]$ which is differentiable on (a,b). If $f(a) = f(b) = 0$ then there is a point $\xi \in (a,b)$ s.t. $f'(\xi) = 0$.

MEAN VALUE THEOREM

Let f be a continuous function, ~~$f: [a,b] \to \mathbb{R}$~~ on $[a,b]$ and differentiable on (a,b). There exists a point $\xi \in (a,b)$ s.t. $\frac{f(b) - f(a)}{b-a} = f'(\xi)$

COROLLARY 6.13

If f is a differentiable function on the open interval I and if $f'(x)=0$ for all $x \in I$ then f is a constant function

COROLLARY 6.14

If f is differentiable on an open interval I and $f'(x) \geq 0$ for all $x \in I$, then f is increasing on I; that is, if $s<t \in I$, then $f(s) \leq f(t)$.

If f is differentiable on an open interval I and $f'(x) \leq 0$ for all $x \in I$, then f is decreasing on I; that is, if $s<t \in I$, then $f(s) \geq f(t)$

CAUCHY'S MEAN VALUE THM

Let f and g be continuous functions on the interval $[a,b]$ which are both differentiable on the interval (a,b). Assume that $g' \neq 0$ on the interval. Then there is a point $\xi \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$

8.1 PARTIAL SUMS AND POINTWISE CONVERGENCE

SEQUENCE OF FUNCTIONS

A sequence of functions is usually written $f_1, f_2, \cdots$ or $\{f_j\}_{j=1}^{\infty}$
$f_j : S \to \mathbb{R}$ $\forall j \in \mathbb{N}$. S is the "common domain"

CONVERGE POINTWISE $f_j \to f_{pw}$ Choice of N depend on c and ε

A sequence of functions $\{f_j\}_{j=1}^{\infty}$ with domain $S \subseteq \mathbb{R}$ converges pointwise $\Leftrightarrow$ $\forall c \in S$, the real sequence $(f_j(c))_{j=1}^{\infty}$ converges $\forall c \in S, \forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $j > N$ $\Rightarrow |f_j(c) - f(c)| < \varepsilon$

i.e. $(f_j)_{j=1}^{\infty}$ converges pointwise $\Leftrightarrow$ $\exists$ a function $f: S \to \mathbb{R}$ s.t. $f(c) = \lim_{j \to \infty} f_j(c)$, $\forall c \in S$

CONVERGE UNIFORMLY $f_j \to f_u$ Choice of N only depend on ε and would work $\forall c \in S$

Let f_j be a sequence of function on domain S. The functions f_j converge uniformly to f on S $\Leftrightarrow$ $\exists$ a function $f: S \to \mathbb{R}$ s.t. $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall j \geq N$, $|f_j(c) - f(c)| < \varepsilon$ $\forall c \in S$

UNIFORM LIMIT THM FOR CONTINUITY

Let $f_j : E \to \mathbb{R}$ be continuous on E $\forall j \in \mathbb{N}$. Assume f_j converges uniformly to f on E. Then f is continuous on E.

THM 8.8

Let (f_j) be integrable functions on a nontrivial bounded interval $[a,b]$ and suppose that the functions f_j converge uniformly to the limit function f. Then f is integrable on $[a,b]$ and $\lim_{j \to \infty} \int_a^b f_j(x)\,dx = \int_a^b f(x)\,dx$

THM

Let $E \subseteq \mathbb{R}$, $E \neq \emptyset$ and $\forall j \in \mathbb{N}$, let $f_j : E \to \mathbb{R}$ be uniformly continuous on E. If $(f_j)_{j=1}^{\infty}$ converges uniformly to $f: E \to \mathbb{R}$, then f is uniformly continuous on E.

8.2 More On Uniform Convergence

Thm 8.10

Fix a set S and a point $s \in S$. Suppose the functions f_j converge uniformly on the domain $S \setminus \{s\}$ to a limit function f. Suppose that each function $f_j(x)$ has a limit as $x \to s$. Then f itself has a limit as $x \to s$ and

$\lim_{x \to s} f(x) = \lim_{j \to \infty} \lim_{x \to s} f_j(x) = \lim_{x \to s} \lim_{j \to \infty} f_j(x)$.

Uniformly Cauchy Sequence

A sequence of functions f_j on a domain S is called a uniformly Cauchy sequence, if, for each $\varepsilon > 0$, there is an $N > 0$ s.t. $j, k > N \Rightarrow |f_j(x) - f_k(x)| < \varepsilon \ \forall x \in S$

- $\forall \varepsilon > 0 \ \exists N > 0$ s.t. $j, k > N \Rightarrow |f_j(x) - f_k(x)| < \varepsilon \ \forall x \in S$

* the choice of N does not depend on x.

~~Uniform Continuity Of $f_j \xrightarrow{uc} f$~~ **Prop 8.13**

A sequence of functions f_j is uniformly Cauchy on a domain S if and only if the sequence converges uniformly to a limit function f on S.

UConverge Implies PW Converges

If (f_j) converges uniformly to f, then it converges pointwise to f.

Thm 8.15

Let $I = (a,b)$ and $\forall j \in \mathbb{N}$, let $f_j : I \to \mathbb{R}$ be differentiable on I. Suppose $(f_j)_{j=1}^{\infty}$ converges pointwise to $f: I \to \mathbb{R}$ and suppose the derivative sequence $(f_j')_{j=1}^{\infty}$ converges uniformly to $g: I \to \mathbb{R}$. Then f is differentiable on I and $f' = g$ on I.

8.3 Series Of Functions

Series Of Functions

Let $E \subseteq \mathbb{R}$, $E \neq \emptyset$, $f_j : E \to \mathbb{R}$ $\forall j \in \mathbb{N}$. $\sum_{j=1}^{N} f_j(x)$ is called a series of functions.

For $N = 1, 2, 3, \cdots$. $S_N(x) = \sum_{j=1}^{N} f_j(x) = f_1(x) + f_2(x) + \cdots + f_N(x)$ is called the Nth partial sum for the series.

$\lim_{N \to \infty} S_N(x)$ exists and is finite $\Rightarrow$ series converges at x. Otherwise the series diverges at x.

Series Converges Pointwise choice of k depend on ε and x

The infinite series of functions $\sum_{j=1}^{\infty} f_j$ converges pointwise to $f: E \to \mathbb{R} \Leftrightarrow$ the sequence of partial sums functions $(S_N)_{N=1}^{\infty}$ converges pointwise to f. $\forall x \in E, \forall \varepsilon > 0 \ \exists k \in \mathbb{N}$ s.t. $N \geq k$ then $|S_N(x) - f(x)| < \varepsilon$

Series Converges Uniformly choice of k only depend on ε

The infinite series of functions $\sum_{j=1}^{\infty} f_j$ converges uniformly on $E \Leftrightarrow \exists$ a function $f: E \to \mathbb{R}$ s.t. $(S_N)_{N=1}^{\infty}$ converges uniformly to f. $\forall \varepsilon > 0 \ \exists k \in \mathbb{N}$ s.t. $N \geq k \Rightarrow |S_N(x) - f(x)| < \varepsilon$, $\forall x \in E$

THM: SERIES CONTINUOUS

If f_j is continuous on E and if $\sum_{j=1}^{\infty} f_j \xrightarrow{uc} f$ then f is continuous on E.

THM: SERIES INTEGRABILITY

Let $f_j : [a,b] \to \mathbb{R}$ and integrable on $[a,b]$. If $\sum_{j=1}^{\infty} f_j \xrightarrow{uc} f$ then f is integrable on $[a,b]$ and $\int_a^b f(x)dx = \sum_{j=1}^{\infty} \int_a^b f_j(x)dx = \int_a^b \sum_{j=1}^{\infty} f_j(x)dx$

WEIERSTRASS M-TEST

Let $f_j : E \to \mathbb{R}$. Assume $\forall j \in \mathbb{N}$, $\exists M_j \geq 0$ s.t.

1. $|f_j(x)| \leq M_j \quad \forall x \in E$
2. $\sum_{j=1}^{\infty} M_j$ converges $\left(\sum_{j=1}^{\infty} M_j < \infty \right)$

Then $\sum_{j=1}^{\infty} f_j \xrightarrow{uc} f$ on E, $E \subseteq \mathbb{R}$, $E \neq \emptyset$, also, $\sum_{j=1}^{\infty} M_j$ converges $\Rightarrow \lim_{j\to\infty} M_j = 0$