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INTRO TO LINEAR PROGRAMMING

Mathematical Models, Linear Programming Problems 06/01/2020

Example : Food A has 2 units of p, 1 unit of q, ~~8~~4 units of r
costs $2

Food B has 3 units of p, 3 units of q, 3 units of r
costs $2.5

require 18p, 12q, 24r

want to know : ~~each~~ serving with min cost that meets requirement

	p	q	r	cost
Food A	2	1	4	2
Food B	3	3	3	2.5
want	18	12	24	

min cost = $2x + 2.5y$. where x is # of A, y is # of B

$2x + 3y \geq 18$

$x + 3y \geq 12$

$4x + 3y \geq 24$

above creates a model ~~req~~ ready to be solved

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General LP problem

for values of $x_1, x_2, \ldots, x_n$

maximise or minimise $z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

subject to restrictions

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \,(\leq)(\geq)(=)\, b_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \,(\leq)(\geq)(=)\, b_2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \,(\leq)(\geq)(=)\, b_m$$

A LP problem in standard form

for values of $x_1, x_2, \ldots, x_n$

maximise $z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

subject to restrictions

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m \qquad x_j \geq 0$$

A LP problem in canonical form

for values of $x_1, x_2, \ldots, x_n$

maximise $z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

subject to restrictions

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$
$$\vdots \qquad \vdots \qquad \vdots \qquad x_j \geq 0$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m$$

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Converting GLP to standard LP

1. Minimisation problem as a maximisation problem

min $z = \cdots$ $\rightarrow$ max $z' = -z$

2. Reversing an inequality $k_1x_1 + k_2x_2 + \cdots + k_nx_n \geq b$

$-k_1x_1 - \cdots - k_nx_n \leq -b$

3 Changing an equality to an inequality $= b$

$$\begin{cases} k_1x_1 + \cdots + k_nx_n \leq b \\ k_1x_1 + \cdots + k_nx_n \geq b \end{cases}$$

4. Unconstrained variables (no nonnegative constrain on x_j)

Let $x_j^+, x_j^- \geq 0$ set $x_j = x_j^+ - x_j^-$

5. Changing an inequality to an equality $\leq b$

$k_1x_1 + \cdots k_nx_n + S_1 = b$, $S_1 \geq 0$

slack variable

Converting standard LP to canonical form

Add a different slack variable $S_1, S_2, \cdots, S_n$ to the inequalities and change ~~sig~~ signs to equal.

MATRIX NOTATION AND THEOREM OF LP PROBLEMS 08/01/2020

Standard LP can be expressed in matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

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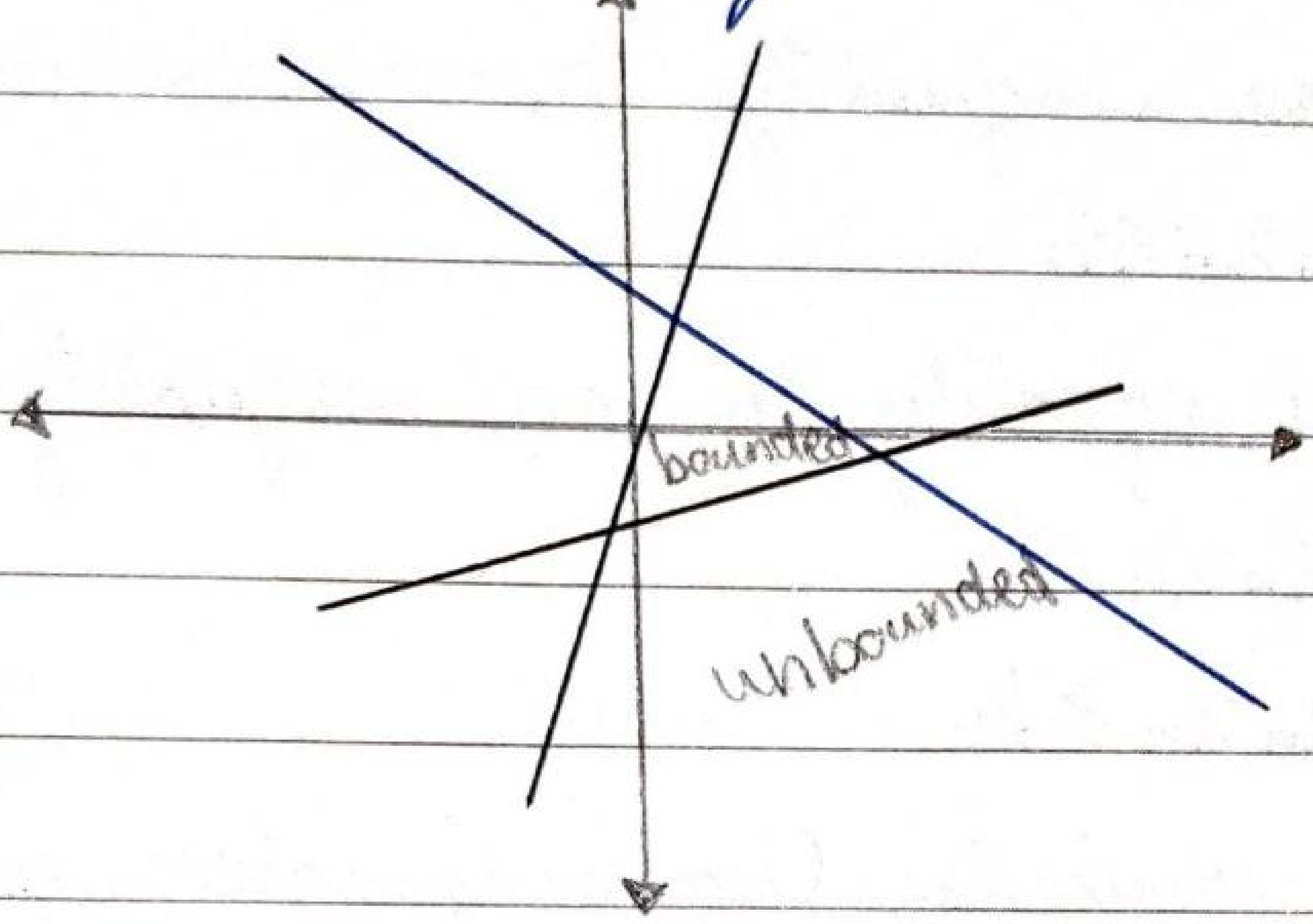
$\Rightarrow$ maximise $\vec{z} = \vec{c}^T \vec{x}$, subject to $A\vec{x} \le \vec{b}$ for $\vec{x} \ge \vec{0}$

and canonical form:

Maximise $z = \vec{c}^T \vec{x}$, subject to $A\vec{x} = \vec{b}$ for $\vec{x} \ge \vec{0}$



equation lines split plane into regions

$\{x_1, x_2, \cdots, x_n\}$ is a feasible solution of (1)

$\Leftrightarrow \{X_1, X_2, \cdots, X_{n+m}\}$ is a feasible solution of the canonical form

Feasible Region = {All the feasible solutions}

Bounded

has definite boundaries, else unbounded

Empty

does not contain solution, else nonempty

Objective Function

A function we wish to maximise or minimise under some system of constrains

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Optimum Solution

A solution to the system of constrains that gives the max or min value of the objective function

Linear Programming Theorem

Let f be a linear function. Let U be a nonempty region in $\mathbb{R}^2$ s.t. U is defined by linear inequalities and it includes its boundaries.

a. If U bounded then f has a max and a min on U; values found at corner points of U.

b. If U unbounded and if f has a max and min then this occurs at a corner point of U.

GRAPHIC SOLUTION OF LP PROBLEMS AND GEOMETRY OF LP PROBLEMS

14/01/2020

Given $f(x,y)$ and constraints:

1. Use a point (x_0, y_0) usually $(0,0)$ to determine the feasible region

2a. Bounded: find corner points / intersection of lines

3. Calculate $f(x,y)$ at each point and determine minimum / maximum

2b. Unbounded: let $f(x,y) = p$ and make y as the subject

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⇒ knows the gradient ⇒ "slide the line toward the feasible region, first point the line touches is the minimum (no optimal solution for maximum)

if points close to eachother i.e. cannot determine the minimum by just looking. calculate $f(x,y)$ and compare.

3. If two solutions at corner are equal, then the points on the line connecting the two corners are optimal solutions.

GEOMETRY OF A CONSTRAINT OF A LP PROBLEM:

Closed half space given $\vec{a}^T\vec{x} \leq b_i$ where $\vec{a}^T = [a_{i1}, a_{i2}, \cdots, a_{in}]$

The set of points $\vec{x} = (x_1, x_2, \cdots, x_n)$ that satisfy

Hyperplane

The set of points that ~~sat~~ satisfy $\vec{a}^T\vec{x} = b_i$

It is ~~the~~ a boundary of a closed half space

Set of feasible solutions

Intersection of all closed half-spaces

An objective function let k be a constant

The optimal solution ~~the~~ is the hyperplane that intersects the set of feasible solutions and for ~~which~~ which k is max/min given constraint $a^T x \leq b_i$ and two points x_1, x_2 that satisfy the constraint, any point x in the line segment also satisfy the constraint as $a^T x_1 \leq a^T x \leq a^T x_2 \leq b_i$

Line segment $= \{x \in \mathbb{R}^n \mid x = \lambda x_1 + (1-\lambda)x_2,\ 0 \leq \lambda \leq 1\}$

Convex

A subset K of $\mathbb{R}^n$ for any $x_1, x_2 \in K$.

$$x = \lambda x_1 + (1-\lambda)x_2 \in K \quad , 0 \leq \lambda \leq 1$$

THEOREMS ON CONVEX SET 15/01/2020

The intersection of a finite collection of convex sets is convex

PROOF $\forall x_1, x_2 \in K$. $K = \bigcap_{i=1}^{s} k_i \Rightarrow x_1, x_2 \in k_i$

since k_i is a convex set, therefore $x = \lambda x_1 + (1-\lambda)x_2 \in k_i$

$\Rightarrow x \in k$ $\therefore$ k is a convex $\quad 0 \leq \lambda \leq 1 \quad i = 1, 2, \cdots, s$

CONVEX COMBINATION

A point $x \in \mathbb{R}^n$ is a convex combination of the points $x_1, x_2, \cdots, x_r$ in $\mathbb{R}^n$ if for some real numbers $c_1, c_2, \cdots, c_r$ which satisfy

linear combination with restriction

$$\sum_{i=1}^{r} c_i = 1 \text{ and } c_i \geq 0,\ 1 \leq i \leq r, \text{ we have}$$

$$\vec{x} = \sum_{i=1}^{r} c_i x_i$$

THM. The set of all convex combinations of a finite set of points in $\mathbb{R}^n$ is a convex set.

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EXTREME POINT

A point x in a convex set S that is not an interior point of any line segment in S and not a convex combination of other points in S

BASIC SOLUTIONS 20/01/2020

THM

Let S be the set of feasible solutions to a general LP Problem.

1. If S is nonempty and bounded, then an optimal solution to the problem exists and occurs at an extreme point.
2. If S is nonempty and unbounded, and if an optimal solution to the problem exists, then it occurs at an extreme point.
3. If an optimal solution ~~it~~ to the problem DNE, then either S is empty or S is unbounded

Basic Solutions

canonical form : $\max z = \vec{c}^T \vec{x}$ for $x \in \mathbb{R}^s$

$$\text{s.t. } \underset{n\times s}{A}\,\underset{s\times 1}{\vec{x}} = \underset{n\times 1}{\vec{b}}$$

$$\vec{x} \geq 0,$$

Let S be the convex set of all feasible solutions of $A\vec{x} = \vec{b}$

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Suppose rank(A) = m and last m columns of A are lin. ind.

suppose $x'_1 A'_1 + x'_2 A'_2 + \cdots + x'_m A'_m = b$ (A'_i = i^{th} column of A excluding linearly dependent columns)

where $x'_i \geq 0$ for $i = 1, 2, \ldots, m$. Then

$x = (0, 0, \ldots, 0, x'_1, x'_2, \ldots, x'_m)$ is an extreme point of S.

THM

if $x = (x_1, x_2, \cdots, x_m)$ is an extreme point of S, then columns of A that correspond to positive x_i form a linearly independent set of vectors in $\mathbb{R}^n$

THM

if rank(A) = m, at most m components of any extreme point of S can be +ve, others must be zero.

Basic Solution

~~x = A~~ x where m components are from solving $A\vec{x} = b$ and the rest components are zeros.

Basic Feasible Solution and Extreme Point

A solution is basic feasible $\iff$ it is an extreme point.

and is finite

Solving LP problem in canonical form

1. form an augmented matrix [A|b]
2. find rank of A = m and # of ~~varb~~ variables ξ = s
3. ~~The~~ Compute $\frac{s!}{(s-m)!m!}$ to find the number of basic solutions
4. set s-m variables to zero ~~or~~ i.e. remove their columns from [A|b] and solve
5. Each solution will have m nonzero values and s-m zeros
6. The solutions that do not have negative values are the basic feasible solutions

THE SIMPLEX METHOD

INITIAL FEASIBLE SOLUTION, ADJACENT, INITIAL TABLEAU 22/01/2020

Initial Base Feasible Solution

consider $A\vec{x} = \vec{b}$ and let $\vec{b} \geq \vec{0}$

set all nonslack variables $x_1, \cdots x_n = 0$

$\therefore \vec{x} = [0, 0, \ldots, 0, \underbrace{x_{n+1}}_{=b_1}, \cdots, \underbrace{x_{n+m}}_{=b_m}]$ is initial basic ~~feasile~~ feasible solution

Adjacent Extreme Point

Two distinct extreme points only differ by one basic variable

Initial Tableau

only one position has 1, rest 0

which variable is basic	x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	z	
x_{n+1}	a_{11}	a_{12}	$\cdots$	a_{1n}	1	0	$\cdots$	0	0	b_1
x_{n+2}	a_{21}	a_{22}	$\cdots$	a_{2n}	0	1	$\cdots$	0	0	b_2
$\vdots$					$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
x_{n+m}	a_{m1}	a_{m2}	$\cdots$	a_{mn}	0	0	$\cdots$	1	0	b_m
objective row	$-c_1$	$-c_2$	$\cdots$	$-c_n$	0	0	$\cdots$	0	1	0

value of objective function

let objective function $z = c^T x$ be written as

$$-c_1 x_1 - c_2 x_2 - \cdots - c_n x_n + z = 0$$

each row represents a constrain

FINDING OPTIMAL SOLUTION USING INITIAL TABLEAU

27/01/2020

1. Construct T_1 as above, columns that correspond to variables on leftmost column only has 1 on their row and 0 on every ~~thin~~ where else.
2. Look for the most negative value in objective row. The column will be the entering variable. (pivot column)

3. ~~For~~ Compute θ for that column: ~~$\frac{a_{ij}}{b_i}$~~ $\frac{b_i}{a_{ij}}$ where a_{ij} is positive (ignore the row if a_{ij} not positive)
4. The row that contains the smallest θ will be the departing variable
5. Rewrite the tableau with departing variable replaced by entering variable in leftmost column. Perform row operations so that the column with entering variable ~~ent~~ has 1 one and all zeros.
6. Repeat 1-5 until objective function row has no negative values or all a_{ij} in ~~it~~ column j ~~can~~ is nonpositive

The optimal solution will ~~be~~ have variables in leftmost = value in rightmost, the rest are 0 and z = the bottom right value

(if ~~a_{ij}~~ no θ available, then no ~~feasible~~ optimal solution)

NOTE WHEN DOING SIMPLEX METHOD 29/02/2020

In final tableau there is a zero in objective row for nonbasic variable $\Rightarrow$ multiple optimal solution

Degeneracy

Basic feasible solution where some basic variables are zeros (right most column has zeros)

may lead to cycling

BLAND'S RULE AND TWO PHASES METHOD

03/02/2020

Blands Rule

Used to prevent a cycle when degeneracy occur

Assume each variable has a subscript 1 to m

- pick column with smallest subscript among negative values (entering)
- if $\exists$ two equal θ, pick basic variable with lowest subscript

Two Phases Method

General LP problem

$$\max \quad z = C^T x$$

$$\text{s.t.} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le (\ge)(=)\, b_1$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le (\ge)(=)\, b_m$$

$x_j \ge 0,\ j = 1, 2, \cdots, n$

Rewrite constraints so that its right-hand side is nonnegative

$$\Rightarrow \quad \max\ z = C^T x \quad \text{s.t.} \quad \sum_{j=1}^{s} a'_{ij} x_j = b'_i \qquad i = 1, 2, \cdots m$$

$x_j \ge 0,\ j = 1, 2, \cdots, s$ where $b'_i \ge 0$

have y_i be artificial variables

$$\sum_{j=1}^{s} a'_{ij} x_j + y_i = b'_i \qquad y_i \ge 0,\ i = 1, 2, \cdots, m$$

Phase I

$$\underline{\min} \quad z^* = \sum_{i=1}^{m} y_i$$

$$\text{s.t.} \quad \sum_{j=1}^{s} a'_{ij} x_j + y_i = b_i \qquad x_j \geq 0,\ j=1,2,\cdots,s$$

$$y_i \geq 0,\ i=1,2,\cdots,m$$

$$b'_i \geq 0,\ i=1,2,\cdots,m$$

1. assign -1 and convert to max

let $z = -z^*$ $\quad \max z = -\sum_{i=1}^{m} y_i,\ y_i = b'_i - \sum_{j=1}^{s} a'_{ij} x_i$

2. The auxiliary problem

$$\max \quad z - \sum_{i=1}^{m}\sum_{j=1}^{s} a'_{ij} x_j = -\sum_{i=1}^{m} b'_i$$

s.t.

$$\sum_{j=1}^{s} a'_{ij} x_j + y_i = b'_i$$

$$x_j \geq 0,\ j=1,2,\cdots,s$$

$$y_i \geq 0,\ i=1,2,\cdots,m \quad \text{with } b'_i \geq 0,\ i=1,2,\cdots,m$$

3. Solve auxiliary problem using simplex until one of the three cases:

$\max z < 0$, ≥ 1 artificial variable appears in basic feasible solution at +ve level feasible region empty

$\max z = 0$, ≤ 1 artificial variable appears in basic feasible solution at +ve level phase II

$\max z = 0$, 0 artificial variable appears in basic feasible solution phase II

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2 Phase Method Worked Example

25/02/2020

maximise $z = 5x_1 - 4x_2 + 3x_3$

subject to: $2x_1 + x_2 - 6x_3 = 20$

$6x_1 + 5x_2 + 10x_3 \leq 76$

$8x_1 - 3x_2 + 6x_3 \leq 50$

$x_i \geq 0, \ i = 1,2,3$

Phase I

minimise $z^* = y$

add artificial variable

subject to $2x_1 + x_2 - 6x_3 + y = 20$

$6x_1 + 5x_2 + 10x_3 + S_1 = 76$ ← only need slack

$6x_1 - 3x_2 + 6x_3 + S_2 = 50$ ← variables

$x_i \geq 0 \quad S_j \geq 0 \quad y \geq 0$

$i = 1,2,3 \quad j = 1,2$

let $z^{*'} = -z^* \qquad z^{*'} = -y$

maximise $z^{*'} + y = 0$ don't want y in this eqn

$z^{*'} + (20 - 2x_1 + x_2 - 6x_3) = 0$

The auxiliary LP problem: maximise $z^{*'} - 2x_1 - x_2 + 6x_3 = -20$

subject to $2x_1 + x_2 - 6x_3 + y_3 = 20$

⋮

same as above

basic var

T_1	x_1 ↓	x_2	x_3	S_1	S_2	y		
y	2	1	-6	0	0	1	20	$\frac{20}{2} = 10$
S_1	6	5	10	1	0	0	76	$\frac{76}{6} > 10$
← S_2	6	-3	6	0	1	0	50	$\frac{50}{6} < 10$
	-2	-1	6	0	0	0	-20	

$(0, 0, 0, 76, 50, 20)$
$x_1 \ x_2 \ x_3 \ S_1 \ S_2 \ S_3$

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T_2	x_1	x_2	x_3	S_1	S_2	y		
← y	0	2	-8	0	$\frac{-1}{3}$	1	$\frac{10}{3}$	$\frac{10}{3\times2}=\frac{10}{6}=\frac{5}{3}=1\frac{2}{3}$
S_1	0	8	4	1	-1	0	26	$\frac{26}{8}=\frac{13}{4}=3\frac{1}{4}$
x_1	1	$\frac{-1}{2}$	1	0	$\frac{1}{6}$	0	$\frac{50}{6}$	
	0	-2	8	0	$\frac{1}{3}$	0	$\frac{-10}{3}$	

T_3	x_1	x_2	x_3	S_1	S_2	y	
x_2	0	1	-4	0	$\frac{-1}{6}$	$\frac{1}{2}$	$\frac{5}{3}$
S_1	0	0	36	1	$\frac{1}{3}$	-4	$\frac{38}{3}$
x_1	1	0	-1	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{55}{6}$
	0	0	0	0	0	1	0

$[\frac{55}{6}, \frac{5}{3}, 0, \frac{38}{3}, 0]$

Phase II

is solution to original

get rid of artificial variable y because it is not objective function (no y) a basic variable anymore

notice row for x_1: $1 \; \cancel{4}0 \; -1 \; 0 \; \frac{1}{4} \; \frac{1}{4} \mid \frac{55}{6}$

and for x_2: $0 \; 1 \; -4 \; 0 \; \frac{-1}{6} \; \frac{1}{2} \mid \frac{5}{3}$

original objective function: $z = 5x_1 - 4x_2 + 3x_3 = 0$

$$
\begin{array}{lcccccl}
 & x_1 & x_2 & x_3 & S_1 & S_2 & \\
\text{objective row} & -5 & 4 & -3 & 0 & 0 & \mid 0 \\
 & 5(1 & 0 & -1 & 0 & \frac{1}{4}) & \mid \frac{55}{6}\times 5 \\
 & -4(0 & 1 & -4 & 0 & \frac{-1}{6}) & \mid \frac{5}{3}\times -4 \\
\hline
 & 0 & 0 & 8 & 0 & \frac{3}{2} & \mid \frac{235}{6}
\end{array}
$$

Tableau for Phase II: ✓ Might not be final tableau at this point

	x_1	x_2	x_3	S_1	S_2	
x_2	0	1	-4	0	$\frac{-1}{6}$	$\frac{5}{3}$
S_1	0	0	36	1	$\frac{1}{3}$	$\frac{38}{3}$
x_1	1	0	-1	0	$\frac{1}{4}$	$\frac{55}{6}$
	0	0	8	0	$\frac{3}{2}$	$\frac{235}{6}$

optimal solution is $z = \frac{235}{6}$, where $x_1 = \frac{55}{6}$, $x_2 = \frac{5}{3}$

probably easier to keep objective row of phase II along the way

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Phase I

constraint is $\leq$: add slack variable

$=$: add artificial variable

$\geq$: subtract slack variable then add artificial variable

Let $z^* = \sum_{i=1}^{j} y_i$ for j artifitial variables and minimise

$\Leftrightarrow$ maximise $z^{*'} = -y_1 - y_2 - \cdots - y_j$ and write $z^{*'}$ in terms of original variables and slack variables

Set up initial tableau with objective row $z^{*'} + (x_1 \cdots + s_1 \cdots) = c$ where c is the value augmented on the right of the objective row (constant)

(can keep a second row with the original of objective function $z + \cdots = 0$ to make phase II easier)

Perform operations using the initial tableau as usual until artificial variables are not basic

Stop if $z < 0$ by the final tableau (no feasible solution)

Phase II

Remove first objective row (should all be 0s except artificial variable columns) and artificial variable columns

Perform simplex method as usual to find the optimal solution of the original LP problem.

Big M method worked example

maximise $z = x_1 + 2x_2$

subject to $x_1 + x_2 \leq 9$

$x_1 - x_2 \geq 1$ ← need one artificial variable here

$x_i \geq 0,\ i = 1, 2$

$\Rightarrow$ maximise $z = x_1 + 2x_2 - My$ ← M is very large so y has to be 0 (positive)

subject to $x_1 + x_2 + s_1 = 9$

$y = 1 - x_1 + x_2 + s_2$ ← $x_1 - x_2 - s_2 + y = 1$

$x_i \geq 0,\ s_i \geq 0 \quad i = 1, 2 \quad y \geq 0$

substitute $y = 1 - x_1 + x_2 + s_2$ to objective function

maximise $z = x_1 + 2x_2 - M(1 - x_1 + x_2 + s_2)$

$z = x_1 + 2x_2 - M + Mx_1 - Mx_2 - Ms_2$

new objective function $z - (M+1)x_1 + (M-2)x_2 + Ms_2 = -M$

T_1	x_1 ↓	x_2	s_1	s_2	y	
s_1	1	1	1	0	0	9
← y	1	-1	0	-1	1	1
	$-(M+1)$	$(M-2)$	0	M	0	$-M$
	<0	>0		>0		

T_2	x_1	x_2 ↓	s_1	s_2	y	
← s_1	0	2	1	1	-1	8
x_1	1	-1	0	-1	1	1
	0	$(M-2) - (M+1)$	0	$M - (M+1)$	$M+1$	$-M+M+1$
		-3		1	not basic anymore, ignore	1

T_3	x_1	x_2	s_1	s_2	
x_2	0	1	$\frac{1}{2}$	$\frac{1}{2}$	4
x_1	1	0	$\frac{1}{2}$	$\frac{-1}{2}$	5
	0	0	$\frac{3}{2}$	$\frac{5}{2}$	13

~~optimal~~

optimal solution $z = 13$

with $x_1 = 5,\ x_2 = 4$

Big M method

convert LP problem into canonical form by adding slack variables. Add artificial variables as in phase I

Let M be a large nonnegative number and subtract My_i from objective function.

Rewrite objective function as $z + \cdots\cdots = -cM$ where c is a constant and ~~y~~ substitute y in terms of constraints

Set up initial tableau with $-cM$ on the bottom right corner

Perform simplex method and at some point y_i will become nonbasic; get rid of them

if final tableau has $z = -ve$, then there DNE optimal solution

DUALITY

01/03/2020

Primal Problem	Its dual Problem:	
max $z = C^T X$	min $z' = b^T \vec{w}$	STANDARD
s.t. $\underset{m\times n}{A}\underset{n\times 1}{\vec{x}} \le \underset{m\times 1}{\vec{b}}$	s.t. $\underset{n\times m}{A^T}\underset{m\times 1}{\vec{w}} \ge \underset{n\times 1}{\vec{C}}$	FORM
$\vec{x} \ge 0$	$\vec{w} \ge \vec{0}$	

Given a primal problem as in (1), the dual of its dual problem is again the primal problem.

max $z = c^T \vec{x}$	min $z' = b^T \vec{w}$	CANONICAL
s.t. $A\vec{x} = \vec{b}$	s.t. $A^T \vec{w} \geq \vec{c}$	FORM
$\vec{x} \geq 0$	w unrestricted	
Primal Problem	Its Dual Problem	
max $z = c^T \vec{x}$	min $z' = b^T \vec{w}$	
s.t. $A\vec{x} \leq \vec{b}$	s.t. $A^T \vec{w} = c$	
$\vec{x}$ unrestricted	$\vec{w} \geq 0$	

SUMMARY

Primal	Dual
max	min
coefficients of objective func.	right hand side of constraint
coefficients of ith constraint	coefficient of ith variable for each constraint
ith constraint is $\leq$	ith variable is ≥ 0
ith constraint is $=$	ith variable is unrestricted
jth variable unrestricted	jth constraint is $=$
jth variable is ≥ 0	jth constraint is $\geq$
# variables	# constraints

$w_1 + w_2 + 2w_3 + t_1 = 5$

$2w_1 + 3w_2 + 4w_3 + t_2 = 7$

$7w_1 + 6w_2 + 2w_3 + t_3 = 12$

THE DUALITY THEOREM

07/03/2020

Weak Duality Theorem

If $\vec{x}_0$ is a feasible solution to the primal problem

$\max \ z = c^T\vec{x}$

subject to $A\vec{x} \leq \vec{b}$

$\vec{x} \geq \vec{0}$ and if $\vec{w}_0$ is a feasible solution to the dual problem

$\text{minim } z' = b^T\vec{w}$

subject to $A^T\vec{w} \geq \vec{c}$ $\qquad z \leq z'$

$\vec{w} \geq \vec{0}$ then $c^T\vec{x}_0 \leq b^T\vec{w}_0$

That is, the value of the objective function of the dual problem is always greater than or equal to the value of the objective function of the primal problem.

If primal has feasible solutions & but objective function unbounded $\Rightarrow$ dual no feasible solutions

If dual has feasible solutions but objective function unbounded $\Rightarrow$ primal no feasible solutions

primal finite $\Rightarrow$ dual finite

finite infeasible $\Rightarrow$ dual unbounded or infeasible

Duality Theorem

If either the primal or dual problem has a feasible solution with a finite optimal objective value, then the other problem has a feasible solution with the same objective value

If the primal and dual problems has feasible solutions, then

i. primal has optimal solution $\vec{x}_0$

ii. dual has optimal solution $\vec{w}_0$

iii. $c^T\vec{x}_0 = b^T\vec{w}_0$

Complementary Slackness

For any pair of optimal solutions to the primal problem and its dual, we have:

a. For $i = 1, 2, \ldots, m$, the product of the ith slack variable for the primal problem and the ith dual variable is zero. ($x_{n+i} w_i = 0$, $i = 1, 2, \ldots, m$) where x_{n+i} is the ith slack variable for the primal problem

b. For $j = 1, 2, \ldots, n$, the product of the jth slack variable for the dual problem and the jth variable for the primal problem is zero.

* ith slack variable of primal $\neq 0 \Rightarrow$ ith dual variable $=0$
$\therefore$ jth slack variable of dual $\neq 0 \Rightarrow$ jth primal variable $=0$
possible for both slack variable and its corresponding dual variable to be zero.

INTEGER PROGRAMMING

25/03/2020

max $z = C^T\vec{x}$

subject to $A\vec{x} \leq \vec{b}$ $\quad I \subseteq \{1, 2, \cdots, n\}$

$\vec{x} \geq \vec{0}$ $\quad I = \{1, 2, \ldots, n\}$ = pure integer programming problem

x_j = integer if $j \in I$ $\quad I \subset \{1, 2, \cdots, n\}$ = mixed integer programming problem

treating as linear programming problem and rounding afterwards might produce incorrect answers

~~The cutting~~ $[a]$ integer part of a

$[a]$ = largest integer K s.t. $K \leq a$

Fractional part of a

$a = [a] + f_i$, where f is the fractional part

The cutting plane constraint

Suppose problem feasible, finite optimal value exists

assume constraints, objective row have integer entries

the ith constraint in final simplex tableau: $\sum_{j=1}^{n} t_{ij} x_j = x_{B_i}$

x_{Bi} = value of ith basic variable

since $\sum_{j=1}^{n} [t_{ij}] x_j \le \sum_{j=1}^{n} t_{ij} x_j \le [x_{Bi}] \le x_{Bi}$

$\sum_{j=1}^{n} [t_{ij}] x_j + u_i = [x_{Bi}]$ - u_i is slack variable

let $[t_{ij}] + g_{ij} = t_{ij}$

$[x_{Bi}] + f_i = x_{Bi}$

$0 \le g_{ij} < 1$

$0 < f_i < 1$

$$\sum_{j=1}^{n} [t_{ij}] x_j + u_i = [x_{Bi}]$$
$$- \sum_{j=1}^{n} t_{ij} x_j + \cancel{u_i} = x_{Bi}$$
$$\boxed{\sum_{j=1}^{n} -(g_{ij}) x_j + u_i = -f_i}$$

cutting plane constraint to be added to constraints in the problem

1. Solve linear programming problem obtained from given ~~integers~~ integer programming problem by dropping the integer requirements. If solution contains all integers, you're done. Else, step 2.
2. Find a cutting plane constraint of the form $\sum_{j=1}^{n} -(g_{ij}) x_j + u_i = -f_i$. A heuristic rule to use for choosing the constraint in the cutting plane construction is choose the constraint in final tableau that gives <u>largest f_i</u> (largest fraction)
3. Consider the new integer programming problem,

which consists of the same objective function, the cutting plane constraint ~~and the~~ and the same constraints in the original problem. Return to 1.

TRANSPORTATION PROBLEM

From ↓ (supply)	To market 1 (100)	market 2 (60)	market 3 (80)	market 4 (120) ← demand
Source 1 (120)	\$5	\$7	\$9	\$6
Source 2 (140)	\$6	\$7	\$10	\$5
Source 3 (100)	\$7	\$6	\$8	\$1

m sources, n dest

↑ cost of transportation from source i to market j

let x_{ij} be amount transported from i to j

LP problem

$$\min \ z = 5x_{11} + 7x_{12} + 9x_{13} + 6x_{14} + 6x_{21} + 7x_{22} + 10x_{23} + 5x_{24} + 7x_{31} + 6x_{32} + 8x_{33} + x_{34}$$

subject to

3 source:

$$\begin{cases} x_{11} + x_{12} + x_{13} + x_{14} \le 120 \\ x_{21} + x_{22} + x_{23} + x_{24} \le 140 \\ x_{31} + x_{32} + x_{33} + x_{34} \le 100 \end{cases}$$

market:

$$\begin{cases} x_{11} + x_{21} + x_{31} \ge 100 \\ x_{12} + x_{22} + x_{32} \ge 60 \\ x_{13} + x_{23} + x_{33} \ge 120 \end{cases}$$

$x_{ij} \ge 0$

$$\text{cost matrix } C = \begin{bmatrix} 5 & 7 & 9 & 6 \\ 6 & 7 & 10 & 5 \\ 7 & 6 & 8 & 1 \end{bmatrix}$$

$$\text{supply vector } S = \begin{bmatrix} 120 \\ 140 \\ 100 \end{bmatrix} \qquad \text{demand vector } d = \begin{bmatrix} 100 \\ 60 \\ 80 \\ 120 \end{bmatrix}$$

$$\sum_{i=1}^{m} S_i \ge \sum_{j=1}^{n} d_j$$

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$\sum_{j=1}^{n} x_{ij} \leq S_i \quad i=1,2,\cdots,m \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \geq \sum_{j=1}^{n} d_j$

$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \leq \sum_{i=1}^{m} S_i$

general transportation problem

minimise $z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij}$

subject to

amount forom a source → $\sum_{j=1}^{n} x_{ij} = S_i \qquad i=1,2,\dots,m$

amount to a destination $\sum_{i=1}^{m} x_{ij} = d_j \qquad j=1,2,\dots,n$

$\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} d_j$

$x_{ij} \geq 0, \; i=1,2,\dots,m \;, \; j=1,2,\dots,n$

This form of the transportation problem has following properties:

1. The problem always has a feasible solution.
2. The set of the feasible solution is bounded.
3. One of $(m+n)$ constraints in $\sum_{j=1}^{n} x_{ij} = S_i$ and $\sum_{i=1}^{m} x_{ij} = d_j$ is redundant.

Each variable appears in exactly two constraints. The column vectors of the constraints are linearly independent.

e.g.

T_1	100	60	80	120	← demand vector
120	5^{100}	7	9	6^{20}	v_1
140	6	7^{40} $+1$	10	5^{100} -1	v_2
100	7	$6^{\boxed{20}}$ -1	8^{80}	1 $+1$	v_3
↑ supply vector	w_1	w_2	w_3	w_4	

cheapest, only need 100

20 remaining, 20 goes here

to balance supply

smallest depart

to balance demand

to balance demand

entering

loop back

↑ initial basic feasible solution: $x_{11}=100$, $C = 2160$
$x_{14}=20$
$x_{22}=40$
$x_{24}=100$
$x_{32}=20$
$x_{33}=80$

Minimum lost

1. Set up a table like the one ~~abt~~ above.
2. For each row, look for lowest cost entry and assign maximum possible amount to fufill demand ~~unit~~ until source is out of supply.
3. Let $v_1, v_2, \cdots v_m$ be dual variables corresponding to supply constraints, let $w_1, w_2, \cdots, w_n$ be the dual variables corresponding to demand constraints.

Dual problem to transportation problem

$$\max \ z' = \sum_{i=1}^{m} s_i v_i + \sum_{j=1}^{n} d_j w_j$$

subject to $v_i + w_j \leq c_{ij}$ $\quad i = 1, 2, \cdots, m \quad j = 1, 2, \cdots, n$

$v_i + w_j + t_{ij} = c_{ij}$ $\quad v_i, w_j$ unrestricted

if x_{ij} basic $\Rightarrow$ dual slack $t_{ij} = 0$

$\Rightarrow v_i + w_j = c_{ij}$

a. Find entering variable

with $6 = m+n-1 = 3+4-1$ basic variables can form 6 equations for 7 dual variables => will have one free variable to set as arbitrary

$x_{11}: v_1 + w_1 = c_{11} = 5$ $x_{32}: v_3 + w_2 = c_{32} = 6$

$x_{14}: v_1 + w_4 = c_{14} = 6$ $x_{33}: v_3 + w_3 = c_{33} = 8$

$x_{22}: v_2 + w_2 = c_{22} = 7$ set one of v_i, w_j to zero

$x_{24}: v_2 + w_4 = c_{24} = 5$ (one with most appearance)

v_1 - 2 times w_1 - 1 time set $v_1 = 0$

v_2 - 2 times w_2 - 2 times $w_1 = 5$ $w_2 = 8$

v_3 - 2 times w_3 - 1 time $w_4 = 6$ $v_3 = -2$

w_4 - 2 times $v_2 = -1$ $w_3 = 10$

Compute $z_{ij} - c_{ij} = v_i + w_j - c_{ij}$ for x_{ij} is nonbasic to get objective row

The entering variable would be x_{ij} where $z_{ij} - c_{ij}$ is largest ~~pos~~ positive value from objective row → if all $z_{ij} - c_{ij}$ are nonpositive, then optimal solution is reached

nonbasic: $z_{ij} - c_{ij} = v_i + w_j - c_{ij}$

$x_{12}: z_{12} - c_{12} = v_1 + w_2 - c_{12} = 0 + 8 - 7 = 1$ $x_{34}: z_{34} - c_{34} = 3$ ↑ entering variable

$x_{13}: z_{13} - c_{13} = 1$ $x_{23}: z_{23} - c_{23} = -1$

$x_{21}: z_{21} - c_{21} = -2$ $x_{31}: z_{31} - c_{31} = -4$

b. Find departing variable

find which basic variable x_{pq} will decrease when x_{ij} increased.

The record of these changes forms a loop - a sequence of entries that satisfies

1. The sequence consists of alternating horizontal and vertical segments
2. Each segment joins exactly two cells
3. First cell is also last cell with no other cells used twice

↳ at least 4 different cells, 2 consecutive cells in same row/column

no three consecutive cells lie in same row/column

first and last cell in same row/col

The departing variable is the one from the loop with smallest value

$x_{22} \leftarrow x_{24}$

$\downarrow \qquad\quad \uparrow$

departing $\rightarrow x_{32} \longrightarrow x_{34} \leftarrow$ entering

$x_{32} = 20$

c. Form a new tableau

departing variable = 0

entering variable = prev departing variable

adjust basic ~~vari~~ variable values to make supply and constraints hold.

T_2	100	60	80	120
120	5 100	7	9	6 20
140	6	7 60	10	5 80
100	7	6 0	8 80	1 20

Repeat a-c

$C = 500 + 120 + 420 + 400 + 640 + 20$

$= 2100 < 2160$

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Degeneracy

e.g. has 6 basic variables and 8 dual variables

$m=4, n=4$ but $m+n-1 \neq 6$

- set 1 variable to 0 (basic) when solving for dual variables i.e. set $v_i / w_j = 0$ and solve until more info needed so that you can evaluate all dual variables

More Procedures for generating an initial BFS

Northwest Corner

start on upper left corner. assign a value so that either a demand or supply will be exhausted

1. assign min (row supply, column demand). Reduce column/row by that value.
2. if column exhausted, move right
   else ~~if~~ move down. but if both exhausted, place a 0 on right/down then move down/right
3. repeat 1, 2 until initial basic feasible solution obtained

Vogel's Method

analyses ~~cheaps~~ differences between cheapest and next cheapest in each row/column.

1. for each row/column

   row or column penalty = difference between two cheapest cells in same row/~~column~~ column.

2. for row/column with largest penalty, ~~assins~~ assign the cheapest cell ~~to~~ the min(row supply, column demand)

   if only one cell remaining in col/row, satisfy that col/row first. Reduce column/row by assigned value.

3. if row/column exhausted, remove and recalculate col/row penalty. if both zero, eliminate one of them and recalculate penalities. (next step will be assigning 0 units

4. Repeat 2 and 3 until initial basic feasible solution is obtained.

Assignment Problem

n task performed by n individuals. Cost of person i to perform task j is c_{ij}. Want to assign task to minimise cost to complete task.

let $x_{ij} = \begin{cases} 1 & \text{if person } i \text{ does task } j \\ 0 & \text{otherwise} \end{cases}$ $\quad i,j = 1,2,\cdots n$

LP problem: minimise $C = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$

s.t. $\sum_{j=1}^{n} x_{ij} = 1$ for $i = 1,2,\cdots,n$

$\sum_{i=1}^{n} x_{ij} = 1$ for $1,2,\cdots,n$

$x_{ij} = 0$ or 1 for $i,j = 1,2,\cdots,n$

jobs

person	1	2	3	4	5
1	2	4	5	1	4
2	4	7	6	11	7
3	3	9	8	10	5
4	1	3	5	1	4
5	7	1	2	1	2

permutation matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

only 1 entry in each row/column =1, rest are 0.

Theorem:

If cost matrix for an assignment problem has nonnegative entries ~~at~~ and at least n zeros, then an optimal solution to the problem exists if n of the zeros lie in the positions of ones of some n×n permutation matrix P. P represents an optimal solution

Theorem 2:

1.
```
for row in matrix:
    num = min(entries in row)
    for entry in row:
        entry = entry - num
for col in matrix:
    perform same operations
```

2. Using as few lines as possible, cross out rows/columns containing zeros. If # lines used =n, an optimal solution is available, stop. Else, step 3

3. Form new cost matrix by finding smallest not crossed entry and subtract all noncrossed ~~entry~~ entries with it. Add it to all entries with 2 lines through them. Go to step 2. (with lines removed)

Then assign the first zero entry in each row that does not lie in a previously assigned column

Maximise case

	1	2	3
A	6	8	12
B	3	4	10
C	5	3	8

Form LP Problem

$$\max \quad P = \sum_{i=1}^{3} \sum_{j=1}^{3} p_{ij} x_{ij}$$

subject to $\sum_{j=1}^{3} x_{ij} = 1$ for $i = 1, 2, 3$ $\quad x_{ij} \geq 0$

$\sum_{i=1}^{3} x_{ij} = 1$ for $j = 1, 2, 3$ $\quad$ for $i, j = 1, 2, 3$

convert to minimization problem

~~max~~ minimise $P' = \sum_{i=1}^{3} \sum_{j=1}^{3} (-p_{ij}) x_{ij}$

subject to $\sum_{j=1}^{3} x_{ij} = 1$ for $i = 1, 2, 3$

$\sum_{i=1}^{3} x_{ij} = 1$ for $j = 1, 2, 3$

$x_{ij} \geq 0$, for $i, j = 1, 2, 3$

The smallest entry in $[-p_{ij}]$ is -12. So we add 12 to each entry in $[p_{ij}]$