MATC34

Complex Numbers

A complex number is an expression of the form $x+iy$ with $x, y \in \mathbb{R}$ an
i is the "imaginary unit", $i^2=-1$. The set of complex numbers $= \mathbb{C}$.

Modulus

Let $z=x+iy$. The absolute value or modulus of $z=x+iy := |z| = \sqrt{x^2+y^2}$ ~~$x+iy$~~.

Conjugate of Z

Let $z=x+iy$. The conjugate of z is $\bar{z}=x-iy$.

Polar Form

If $z \neq 0$ then $z = re^{i\theta} = r\cos\theta + ir\sin\theta$ with $r>0$ and θ

Fundamental Thm of Algebra

Any polynomial of the form $p(z)=z^n+a_{n-1}z^{n-1}+a_{n-2}z^{n-2}+\cdots+a_1z+a_0$ with $a_i \in \mathbb{C}$
can be factored as $p(z)=(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$ with $\lambda_i \in \mathbb{C}$.
i.e. any polynomial of degree n has n complex roots.

Sequences of Complex Numbers

A sequence of complex numbers is an ordered list $z_1, z_2, \cdots$ of complex numbers.

Convergence

A sequence of complex numbers $\{z_n\}$ converges to z if
$\forall \varepsilon > 0 \ \exists N$ s.t. ~~$\forall n$~~ $n>N \Rightarrow |z_n - z| < \varepsilon$

Sequence Properties

If $\{z_n\}$ and $\{w_n\}$ are sequences of complex numbers s.t. $z_n \to z$
and ~~$z \to z$~~ $w_n \to w$, then

1. $z_n + w_n \to z + w$
2. $z_n w_n \to zw$
3. $\frac{z_n}{w_n} \to \frac{z}{w}$ provided $w \neq 0$ ($w_n \neq 0$) mostly concerned with values $w_n, n>N$

Triangle Inequality

$|z+w| \leq |z| + |w|$

$||z|-|w|| \leq |z-w|$

OPEN DISK

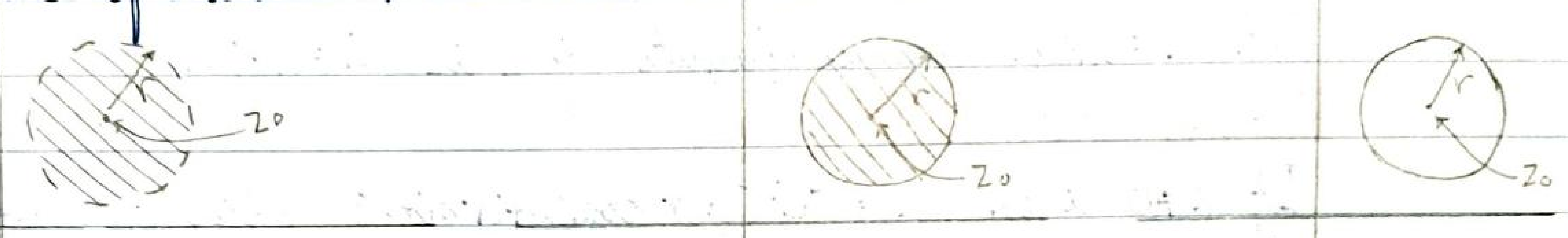
The open disk of radius r centered at $z_0 \in \mathbb{C}$ is $D_r(z_0) = \{z \mid |z-z_0| < r\}$

CLOSED DISK

The closed disk of radius r centered at $z_0 \in \mathbb{C}$ is $\overline{D_r(z_0)} = \{z \mid |z-z_0| \leq r\}$

CIRCLE

The circle of radius r centered at $z_0 \in \mathbb{C}$ is $C_r(z_0) = \{z \in \mathbb{C} \mid |z-z_0| = r\}$



OPEN DISK | CLOSED DISK | CIRCLE

INTERIOR POINT

Let Ω be a set in $\mathbb{C}$. A point $z_0 \in \Omega$ is an interior point of Ω if $\exists\, r > 0$ s.t. $D_r(z_0) \subset \Omega$. ($\forall z$ s.t. $|z-z_0| < r$, $z \in \Omega$)

LIMIT POINT

Let Ω be a set in $\mathbb{C}$. A point $z_0 \in \Omega$ is a limit point of Ω if $\exists$ a sequence $\{z_n\} \subseteq \Omega$ s.t. $z_n \neq z_0$ and $z_n \to z_0$

OPEN SET

A set Ω is open if every point in Ω is an interior point. That is, if whenever $z_0 \in \Omega$, $\exists\, r > 0$ s.t $D_r(z_0) \subset \Omega$.

CLOSED SET

A set Ω is closed if every limit point of Ω is ~~continuous~~ contained in Ω.

CLOSURE

The closure of Ω is the union of Ω with its limit points. $\overline{\Omega}$

INTERIOR

The interior of Ω is the set of all interior points. $\text{int}(\Omega)$ or Ω°

BOUNDARY

The boundary of Ω is $\overline{\Omega} \setminus \text{int}(\Omega)$. Denote as $\partial\Omega$.

THM

A set $\Omega \subset \mathbb{C}$ is closed if and only if $\Omega^c = \mathbb{C} \setminus \Omega$ is open.

BOUNDED

A set $\Omega \subseteq \mathbb{C}$ is bounded if $\exists M > 0$ s.t. $z \in \Omega \Rightarrow |z| \leq M$. The diameter of a bounded set Ω is $\sup |x-y|$, $x, y \in \Omega$.

CAUCHY SEQUENCE

We say a sequence $\{z_n\} \subset \mathbb{C}$ is a Cauchy sequence if $\forall \varepsilon > 0\ \exists N$ s.t. $n, m > N \Rightarrow |z_n - z_m| < \varepsilon$

THM

A sequence converges if and only if it is Cauchy.

COMPACT SET

A set $\Omega \subset \mathbb{C}$ is compact if every covering of Ω by open sets has a finite subcovering

THM

Suppose that $\Omega \subset \mathbb{C}$. Then the following are equivalent:

1. Every covering of Ω by open sets has a finite subcover.
2. Ω is closed and bounded.
3. Every sequence in Ω has a subsequence which converges to a point in Ω.

THM

Suppose $\Omega_1 \supset \Omega_2 \supset \Omega_3 \cdots$ is a sequence of nonempty compact sets such that diameter $(\Omega_j) \to 0$ as $j \to \infty$; then $\bigcap_{j=1}^{\infty} \Omega_j = \{w\}$

CONTINUOUS FUNCTIONS

A function $f : \Omega \to \mathbb{C}$ is continuous at $z_0 \in \Omega$ if

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon \text{ and } z \in \Omega.$$

THM

f is continuous at z_0 iff whenever $\{z_n\}$ is a sequence in Ω s.t. $z_n \to z_0$, $f(z_n) \to f(z_0)$.

CONTINUOUS FUNCTIONS PROPERTIES

Suppose $f, g : \Omega \to \mathbb{C}$ are continuous at z_0. then

1. $f+g$ is continuous at z_0
2. fg is continuous at z_0
3. $|f|$ is continuous at z_0
4. $\max\{f, 0\}$ is continuous at z_0
5. $\frac{f}{g}$ is continuous at z_0 provided $g(z_0) \neq 0$
6. if g is continuous at $f(z_0)$, $g \circ f$ is continuous at z_0.

THM

If f is continuous on the compact set Ω, then f is bounded and ~~it~~ $\exists$ a maximum and minimum value in Ω.

HOLOMORPHIC FUNCTIONS

Let Ω be an open set in $\mathbb{C}$. We say $f:\Omega\to\mathbb{C}$ is holomorphic at z_0 if $\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$ exists. If it exists, then $f'(z_0)=\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$

f is holomorphic in Ω if it is holomorphic at every point in Ω.

THM

Suppose that Ω is an open set in $\mathbb{C}$, and that $f,g:\Omega\to\mathbb{C}$ which are holomorphic at $z_0\in\Omega$. Then:

1. $f+g$ is holomorphic at z_0 and $(f+g)'=f'+g'$
2. fg is holomorphic at z_0 and $(fg)'=f'g+g'f$
3. $\frac{f}{g}$ is holomorphic at z_0 if $g(z_0)\neq 0$ and $\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$

THM

Suppose $f:\mathbb{C}\to\mathbb{C}$ is holomorphic at z_0 and $g:\mathbb{C}\to\mathbb{C}$ is holomorphic at $f(z_0)$. Then $g\circ f$ is holomorphic at z_0 and $(g\circ f)'(z_0)=g'(f(z_0))f'(z_0)$.

CAUCHY-RIEMANN EQUATIONS

Suppose that $f:\Omega\to\mathbb{C}$ is holomorphic at $z_0\in\Omega$. We can write f as a map $f:\mathbb{R}^2\to\mathbb{R}^2 \Rightarrow f(x+iy)=u(x+iy)+iv(x+iy)$. Then $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

THM

Let $f(x+iy)=u(x,y)+iv(x,y)$ be a complex valued function defined in an open set Ω. If u and v are continuously differentiable in Ω and $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ in Ω, then f is holomorphic.

THM

Let $\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$ where $z=x+iy \Rightarrow x=\frac{z+\bar{z}}{2}$, $y=\frac{z-\bar{z}}{2i}$.

f is holomorphic iff $\frac{\partial f}{\partial \bar{z}}=0 \Rightarrow f'=\frac{\partial f}{\partial z}$.

SERIES

A series is $\sum_{n=0}^{\infty} a_n$ where $\{a_n\}_{n=0}^{\infty}$ is a sequence.
$\sum_{n=0}^{\infty} a_n$ converges if the sequence of partial sums $S_N = \sum_{n=0}^{N} a_n$ converges.

COMPARISON TEST

Suppose $0 \leq a_n \leq b_n$. If $\sum b_n < \infty$, then $\sum a_n < \infty$. (converge)
If $\sum a_n = \infty$, then $\sum b_n = \infty$ (diverge)

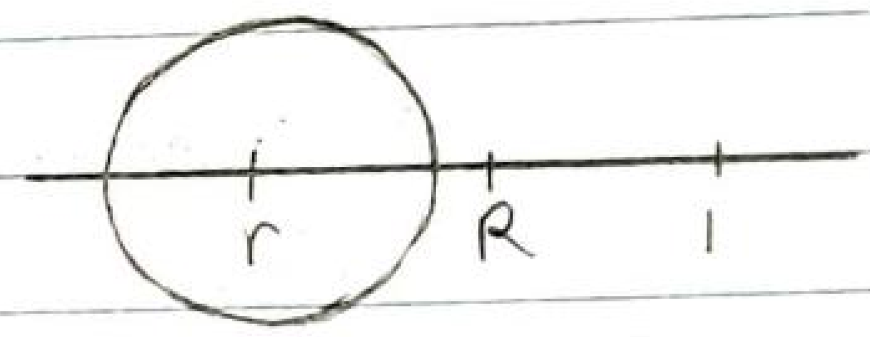
RATIO TEST

Suppose $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r$. If $r < 1$ then $\sum a_n$ converges.
If $r > 1$ then $\sum a_n$ diverges.

No conclusion can be made with $r = 1$.

PROOF

Suppose $r < 1$. Choose R with $r < R < 1$.



Then $\exists N$ s.t. $\left| \frac{a_{n+1}}{a_n} \right| < R \quad \forall n \geq N$.

Then $|a_{N+1}| \leq R \cdot |a_N| \Rightarrow |a_{N+2}| \leq |a_{N+1}| \cdot R \leq |a_{N+1}| R^2$,

$\vdots$

$|a_{N+m}| \leq |a_N| \cdot R^m$

It follows that $\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{N-1} |a_n| + \underbrace{\sum_{n=N}^{\infty} |a_n|}_{= \sum_{m=0}^{\infty} |a_{N+m}| \leq |a_N| R^m}$

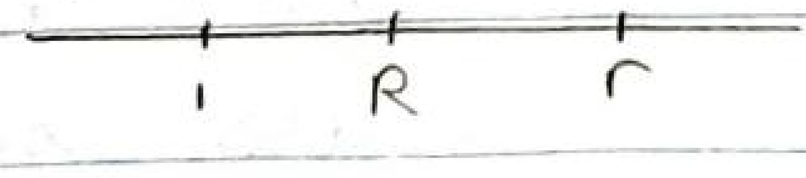
$$\leq \sum_{n=0}^{N-1} |a_n| + |a_N| \sum_{m=0}^{\infty} R^m$$

geometric series converges to $\frac{1}{1-R}$

$$= \sum_{n=0}^{N-1} |a_n| + |a_N| \frac{1}{1-R} < \infty$$

partial converges $\in \mathbb{R}$ $\in \mathbb{R}$

Now suppose $r > 1$. Choose R with $1 < R < r$



Then pick subsequence $\{a_{n_k}\}$ s.t. $\left| \frac{a_{n_{k+1}}}{a_{n_k}} \right| > R$

Then since $R > 1$, $\{|a_{n_k}|\}$ is increasing, so a_{n_k} does not converge to zero.

$\therefore$ The series diverges, as the subsequence diverges.

ROOT TEST

Suppose that $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = r$. If $r<1$ then $\sum a_n$ converges absolutely,
If $r>1$ then $\sum a_n$ diverges.

limsup of $\{a_n\}$

$\limsup_{n\to\infty} a_n = L$ iff $\forall \varepsilon>0\ \exists N$ s.t. $a_n \le L+\varepsilon \quad \forall n \ge N$ and
$\forall \varepsilon>0$ and $\forall N \quad \exists n \ge N$ with $a_n > L-\varepsilon$

e.g. $a_n = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}$ $\quad \lim_{n\to\infty} a_n$ DNE but $\limsup_{n\to\infty} a_n = 1$

PROOF Suppose ~~r→0~~ $r>1$. choose $r<R<1$. Then $\exists N$ s.t. $|a_n|^{\frac{1}{n}} < R$ for all $n \ge N$.

$\Rightarrow |a_n| < R^n$ for $n \ge N$.

$$\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n| \le \sum_{n=0}^{N-1} |a_n| + \sum_{n=N}^{\infty} R^n$$
$$\le \sum_{n=0}^{N-1} |a_n| + \sum_{n=0}^{\infty} R^n \leftarrow \text{geometric}$$
$$= \sum_{n=0}^{N-1} |a| + \frac{1}{1-R} < \infty$$

Now suppose ~~r~~ $r>1$. Choose $1<R<r$. Then $\exists$ a subsequence $\{a_{n_k}\}$ with $|a_{n_k}|^{\frac{1}{n_k}} > R > 1$

$\Rightarrow |a_{n_k}| > 1^{n_k} = 1$ so $a_{n_k} \not\to 0$. By divergence test, the series $\sum a_n$ diverges. ∎

SEQUENCE OF FUNCTIONS CONVERGES POINTWISE

For each $n \in \mathbb{N}$, let f_n be a function defined in Ω. We say that the sequence $\{f_n(z)\}$ converges pointwise to $f(z)$ in Ω if
$$f_n(z) \to f(z) \quad \forall z \in \Omega.$$

CONVERGES UNIFORMLY

We say that $\{f_n(z)\}$ converges uniformly to f in Ω provided $\forall \varepsilon > 0\ \exists N$ s.t. $|f_n(z) - f(z)| < \varepsilon$ whenever $z \in \Omega$ and $n \ge N$.
$$\left(\sup_{z\in\Omega} |f_n(z) - f(z)| < \varepsilon \text{ when } n \ge N \right)$$

UNIFORM CONVERGENCE → CONTINUOUS

If $\{f_n\}$ is a sequence of continuous functions defined on Ω and $f_n \to f$ uniformly in Ω, then f is continuous.

PROOF Choose n s.t. $|f(z) - f_n(z)| < \frac{\varepsilon}{3}$ $\forall z \in \Omega$.

Choose $\delta > 0$ s.t. $|z-w| < \delta \Rightarrow |f_n(z) - f_n(w)| < \frac{\varepsilon}{3}$

$\therefore$ $|f(z) - f(w)| = |f(z) - f_n(z) + f_n(z) - f_n(w) + f_n(w) - f_n(w)|$

$\leq |f(z) - f_n(z)| + |f_n(z) - f_n(w)| + |f_n(w) - f(w)|$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

whenever $|z-w| < \delta$, so f is continuous at z. ∎

CAUCHY SEQUENCE

$\{a_n\}$ converges iff $\forall \varepsilon > 0$ $\exists N$ s.t. $|a_n - a_m| < \varepsilon$ $\forall n, m \geq N$. Then $\{a_n\}$ is a cauchy sequence.

UNIFORMLY CAUCHY

A sequence of functions $\{f_n(z)\}$ is uniformly cauchy on Ω provided $\forall \varepsilon > 0$ $\exists N$ s.t. $m, n \geq N \Rightarrow \sup_{z \in \Omega} |f_n(z) - f_m(z)| < \varepsilon$.

$\{f_n(z)\}$ converges uniformly in Ω iff it is uniformly cauchy.

SERIES OF FUNCTIONS CONVERGENCE

Let $\{f_n(z)\}$ be a sequence of functions defined in Ω.

Let $\underbrace{S_N(z) = \sum_{n=0}^{N} f_n(z)}_{\text{partial sum}}$. Let $\underbrace{\tilde{S}_N(z) = \sum_{n=0}^{N} |f_n(z)|}_{\text{absolute partial sum}}$

PC $\sum_{n=0}^{\infty} f_n(z)$ converges pointwise to $f(z)$ in Ω if $S_N(z)$ converges pointwise to $f(z)$ in Ω

UC $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly to $f(z)$ in Ω if $S_N(z)$ converges uniformly to $f(z)$ in Ω

AC $\sum_{n=0}^{\infty} f_n(z)$ converges absolutely to $f(z)$ in Ω if $\tilde{S}_N(z)$ converges pointwise to $f(z)$ in Ω

AUC $\sum_{n=0}^{\infty} f_n(z)$ converges absolutely uniformly to $f(z)$ in Ω if $\tilde{S}_N(z)$ converges uniformly to $f(z)$ in Ω.

AC $\Rightarrow$ PC   UC $\not\Rightarrow$ AC   AUC $\Rightarrow$ UC and AC

UC $\Rightarrow$ PC   AC $\not\Rightarrow$ UC   UC and AC $\not\Rightarrow$ AUC

WEIERSTRAUSS M-TEST

If $|f_n(z)| \leq M_n$ for all n in Ω and $\sum_{n=0}^{\infty} M_n < \infty$, then $\sum_{n=0}^{\infty} f_n(z)$ converges absolutely uniformly in Ω.

PROOF

Let $\tilde{S}_N(z) = \sum_{k=0}^{N} |f_k(z)|$. Then for all $n > m$ we have

$$|\tilde{S}_n(z) - \tilde{S}_m(z)| = \left| \sum_{k=0}^{n} |f_k(z)| - \sum_{k=0}^{m} |f_k(z)| \right| = \left| \sum_{k=m+1}^{n} |f_k(z)| \right|$$

$$= \sum_{k=m+1}^{n} |f_k(z)| \leq \sum_{k=m+1}^{n} M_k$$

for all z in Ω. Since $\sum M_k$ converges, let ~~n>m>L~~ , $L \in \mathbb{N}$

s.t. if $n > m > L$, $\sum_{k=m+1}^{n} M_k < \varepsilon$

$\Rightarrow |\tilde{S}_n(z) - \tilde{S}_m(z)| < \varepsilon$ for all $n > m > L$. $\Rightarrow \tilde{S}_n$ is uniformly cauchy in Ω $\Rightarrow \sum f_n(z)$ converges absolutely uniformly in Ω. ■

POWER SERIES CENTERED AT z_0

A power series centered at z_0 is an expansion of the form $\sum_{n=0}^{\infty} a_n (z-z_0)^n$

when $z_0 = 0$, the expansion becomes $\sum_{n=0}^{\infty} a_n z^n$

THM

If $\sum_{n=0}^{\infty} a_n z^n$ converges when $z = z_0$ with $z_0 \neq 0$, then it converges absolutely uniformly for all $|z| < |z_0|$.

PROOF

Since $\sum_{n=0}^{\infty} a_n z_0^n$ converges, therefore sequence $\{a_n z_0^n\}$ bounded. Let $M > 0$ be s.t. $|a_n z_0^n| < M$.

$$\Rightarrow |a_n z^n| = |a_n z_0^n| \cdot \frac{|z^n|}{|z_0^n|} \leq M \left(\frac{|z|}{|z_0|} \right)^n$$

Let $R = \frac{|z|}{|z_0|}$ $\forall |z| < |z_0|$, $R < 1$, so the geometric series $\sum_{n=0}^{\infty} R^n$ converges

By CT ($\sum |a_n z^n| \leq M \sum R^n < \infty$) $\sum_{n=0}^{\infty} |a_n z^n|$ converges

By weierstrauss M-test $\sum |a_n z^n|$ converges absolutely uniformly in the disk $|z| < |z_0|$ ■

RADIUS OF CONVERGENCE

Let R be the least upper bound (sup) over the set of S s.t. $\sum a_n z^n$ converges absolutely for $|z| < S$. $S = \{ s \mid \forall |z| < s, \sum a_n z^n \text{ converges} \}$. Then R is the radius of convergence of $\sum a_n z^n$. The series converges absolutely for $|z| < R$ and diverges for $|z| > R$. (behaviour unknown when $|z| = R$)

THM

Let R be the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$. Then

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leftarrow \text{always exists}$$

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = s$ exists then $\frac{1}{R} = S$

PROOF

Let $S = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Then the sum converges absolutely if

$$|z| \cdot \frac{1}{R} = \limsup_{n\to\infty} |a_n z^n|^{\frac{1}{n}} < 1 \quad \text{and diverges when}$$

$$|z| \cdot \frac{1}{R} = \limsup_{n\to\infty} |a_n z^n|^{\frac{1}{n}} > 1.$$

So $\sum a_n z^n$ converges absolutely when $|z|<R$ and diverges when $|z|>R$.

$\Rightarrow$ radius of convergence is indeed R. ■

THM

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R>0$, then f defines a holomorphic function in $|z|<R$ and $f'(z) = \sum_{n=1}^{\infty} a_n n z^{n-1}$ for all $|z|<R$.

term-by-term differentiation

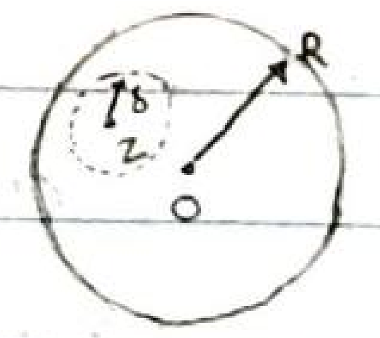
PROOF

since $n^{\frac{1}{n}} \to 1$ as $n\to\infty$ $\Rightarrow \limsup_{n\to\infty} |a_n n|^{\frac{1}{n}} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. $\Rightarrow \sum a_n n z^{n-1}$ has the same radius of convergence as $f(z)$.

BINOMIAL THM (ASIDE)

$f(z+h) = \sum_{n=0}^{\infty} a_n (z+h)^n \Rightarrow (z+h)^n = \sum_{k=0}^{n} z^{n-k} h^{n-k} \binom{n}{k}$ $\quad n \in \mathbb{N}$

$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k \quad r \in \mathbb{R} \quad |x|<1$



Let $z \in \mathbb{C}$ and $\delta>0$ s.t. $|z|+\delta<R$ and $|z|<R$.

For any $|h|<\delta$ we have

$$f(z+h) - f(z) = \sum_{n=0}^{\infty} a_n (z+h)^n - \sum_{n=0}^{\infty} a_n (z^n) = \sum \left(a_n z^n + h z^{n-1} a_n \binom{n}{1} + a_n P_n(z,h)\right) - \sum_{n=0}^{\infty} a_n z^n$$

$$= \underbrace{\sum_{n=0}^{\infty} h z^{n-1} a_n \binom{n}{1}}_{\text{derivative}} + \underbrace{h^2 \sum_{n=0}^{\infty} a_n \frac{P_n(z,h)}{h^2}}_{\text{remainder}} \rightarrow \text{converges absolutely}$$

$$P_n(z,h) = h^2 \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^{k-2}$$

$\Rightarrow \frac{f(z+h)-f(z)}{h} = \sum_{n=0}^{\infty} n a_n z^{n-1} + h \sum a_n \frac{P_n(z,h)}{h}$

Since $\sum |a_n P_n(z,h)| \le \sum |a_n| P_n(|z|,\delta)$ for all $|h|<\delta$,

bounded independent of h

$\Rightarrow h \sum a_n P_n(z,h) \to 0$ as $h \to 0$.

$\Rightarrow \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \sum_{n=0}^{\infty} n a_n z^{n-1}$ ■

COROLLARY

If $f(z)=\sum a_n z^n$ has radius of convergence $R>0$, then f is infinitely ~~differen~~ differentiable in $|z|<R$ and in general, $a_n=\frac{f^{(n)}(0)}{n!}$ for $f(0)$ $z=0$.

ANALYTIC

A function f defined on an open set Ω is <u>analytic at $z_0\in\Omega$</u> if it produce a power series expansion $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ with a radius of converge $R>0$.

We say f is <u>analytic in Ω</u> if it is analytic at each $z_0\in\Omega$

Analytic (admits local power series expansions) $\Rightarrow$ Holomorphic (complex differentiable)

SMOOTH PARAMETERIZATION

A parameterization of the curve $\gamma\subset\mathbb{C}$ is a function $z:\underset{\subseteq\mathbb{R}}{[a,b]}\to\mathbb{C}$ s.t. $z([a,b])=\gamma$. z is a smooth parameterization if z' exists and is continuous and $z'(t)\neq0$ $\forall t\in[a,b]$.

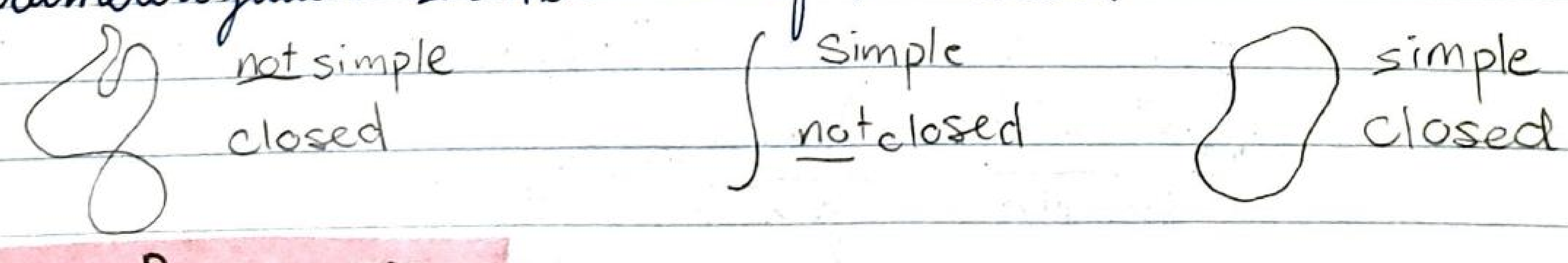
z is a piecewise smooth parameterization of γ provided $\exists$ a partition $a=a_0<a_1<a_2<\cdots<a_n=b$ s.t. z is a smooth parameterization on each subinterval $[a_n,a_{n+1}]$.

SIMPLE

A curve is simple if it is nonintersecting.

CLOSED

A curve γ is closed if the beginning point and ending point of any parameterization $z:[a,b]\to\mathbb{C}$ of γ coincide



EQUIVALENT PARAMETERIZATIONS

Let two parameterizations be $z_1:[a,b]\to\mathbb{C}$, $z_2:[c,d]\to\mathbb{C}$. They are equivalent if $\exists$ a continuously differentiable function $f(s)$ which maps $[a,b]$ and $[c,d]$ bijectively and that $z_2(f(s))=z_1(s)$.

INTEGRALS OVER CURVES

Suppose $z:[a,b]\to\mathbb{C}$ is a smooth parameterization of the curve γ. Then for a ~~contin~~ continuous function f given on γ (usually f is defined on a neighbourhood of γ), define

$$\int_\gamma f(z)dz = \int_a^b f(z(t))\, z'(t)dt$$

- equivalent parameterization given the same integral.

THM

If $\{f_n\}$ is a sequence of continuous functions which converge uniformly to f on the open set Ω and γ is a smooth curve in Ω then $\int_\gamma f_n(z)dz \to \int_\gamma f(z)\, dz$ as $n\to\infty$.

PROOF $f_n\to f \Rightarrow \forall\varepsilon>0\ \exists N$ s.t. $\forall n\in\mathbb{N},\ n>N \Rightarrow |f_n(z)-f(z)|<\varepsilon,\ \forall z\in\Omega$

Let $z:[a,b]\to\mathbb{C}$ be a smooth parameterization of γ, let $n>N$ s.t. $|f_n(z)-f(z)|<\varepsilon\ \ \forall z\in\Omega$.

$$\left|\int_\gamma f_n(z)dz-\int_\gamma f(z)dz\right| = \left|\int_\gamma f_n(z)-f(z)dz\right| \leq \text{length}(\gamma)\cdot\sup_{z\in\Omega}|f_n(z)-f(z)|$$
$$< \text{length}(\gamma)\,\varepsilon_1 = \text{length}(\gamma)\frac{\varepsilon}{\text{length}(\gamma)}$$
$$=\varepsilon$$

$\therefore n\to\infty \Rightarrow \left|\int_\gamma f_n(z)dz-\int_\gamma f(z)dz\right|\to 0$ ■

COROLLARY

If $f(z)=\sum_{n=0}^{\infty}a_nz^n$ has a radius of convergence $R>0$ and γ is any smooth closed curve in $\Omega=\{z\,|\,|z|<R\}$ then $\int_\gamma f(z)dz=0$.

INTEGRALS OVER PIECEWISE SMOOTH CURVES

Suppose γ is a piecewise smooth curve, and that $z:[a,b]\to\mathbb{C}$ is a parameterization of γ s.t. z is smooth on each of the subintervals defined by the partition $a=a_1<a_2<\cdots<a_n=b$. Then define

$$\int_\gamma f(z)\,dz=\sum_{j=1}^{n-1}\int_{a_j}^{a_{j+1}} f(z(t))\, z'(t)dt$$

LENGTH OF A SMOOTH CURVE

The length of a smooth curve γ is $\int_a^b|z'(t)|dt$ where $z:[a,b]\to\mathbb{C}$ is a smooth parameterization of γ.

The length of a piecewise smooth curve γ with parametrization $z:[a,b]\to\mathbb{C}$ which is smooth over the intervals defined by the partition $a=a_1<a_2<\cdots<a_n=b$ is $\displaystyle\sum_{j=1}^{n-1}\int_{a_j}^{a_{j+1}}|z'(t)|dt$

THM

If γ is a piecewise smooth curve, f and g are continuous functions $\gamma \to \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$ then

1. $\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$
2. $\left| \int_\gamma f \right| \leq \text{length}(\gamma) \cdot \sup_{z \in \gamma} |f(z)|$
3. If $\tilde{\gamma}$ has the opposite orientation to γ then $\int_{\tilde{\gamma}} f(z)dz = -\int_\gamma f(z)dz$

PRIMITIVES

Suppose that Ω is an open set in $\mathbb{C}$. The holomorphic function $F: \Omega \to \mathbb{C}$ is a primitive for the function $f: \Omega \to \mathbb{C}$ in Ω provided $F'(z) = f(z)$

THM

If a continuous function f has a primitive F in Ω and γ is a piecewise smooth curve in Ω which begins at the point w_1 and ends at the point w_2, then

$$\int_\gamma f(w)dw = F(w_2) - F(w_1)$$

PROOF

Suppose γ is a smooth curve with parameterization $z(t): [a,b] \to \mathbb{C}$.

$\Rightarrow z(a) = w_1$ and $z(b) = w_2$.

$$\therefore \frac{d}{dt}(F(z(t))) = F'(z(t))\, z'(t) = f(z(t)\, z'(t).$$

$$\Rightarrow \int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b \frac{d}{dt}[F(z(t)]\, dt = F(z(b)) - F(z(a))$$
$$= F(w_2) - F(w_1)$$

Now suppose γ is a piecewise smooth curve with parameterization $z(t): [a,b] \to \mathbb{C}$ which is smooth on each of the intervals $[a_j, a_{j+1}]$ defined by the partition $a = a_1 < a_2 < \cdots < a_n = b$

Then $\int_\gamma f(z)dz = \sum_{j=1}^{n-1} \int_{a_j}^{a_{j+1}} f(z(t)\, z'(t)dt = \sum_{j=1}^{n-1} \int_{a_{j-1}}^{a_j} \frac{d}{dt}[F(z(t))]dt$

$$= \sum_{j=1}^{n} (F(z(a_j)) - F(z(a_{j-1}))) = F(a_n) - F(a_1) = F(w_2) - F(w_1) \blacksquare$$

GAUSS GREEN THM (aside)

If Ω is a connected open set in $\mathbb{R}^n$ with smooth boundary then $\int_\Omega \frac{\partial f}{\partial x_i} = \int_{\partial\Omega} f \cdot n_i$ where n_i is the i^{th} component of the outward-pointing unit normal.

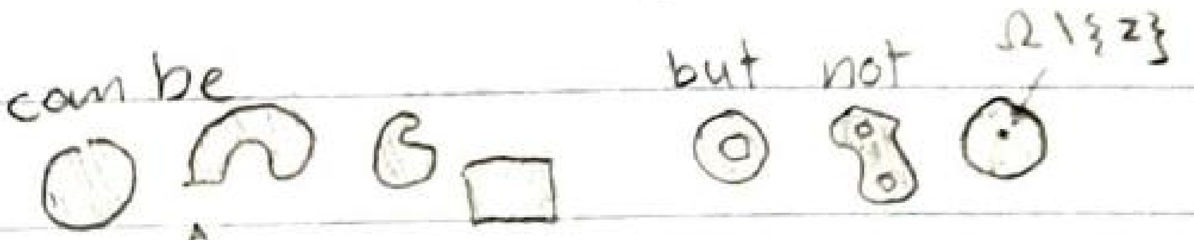
THM

If f is holomorphic on a connected open set Ω and $f'=0$ in Ω, then f is constant on Ω.

PROOF Since f is primitive of f', $f(w_2)-f(w_1)=\int_\gamma f'(z)dz=0$ whenever γ is a curve connecting w_1 to w_2. So $f(w_2)=f(w_1)$ for any point w_1, w_2 in Ω.

COROLLARY

If F and G are primitives of f on a connected open set Ω then $F-G$ is constant on Ω.

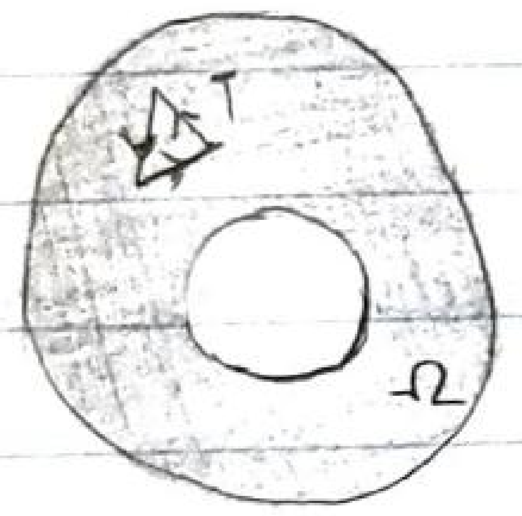


CAUCHY'S THM

If f is holomorphic on an open set Ω and γ is a closed curve in Ω. (domain without ~~at~~ holes => simply connected domain)

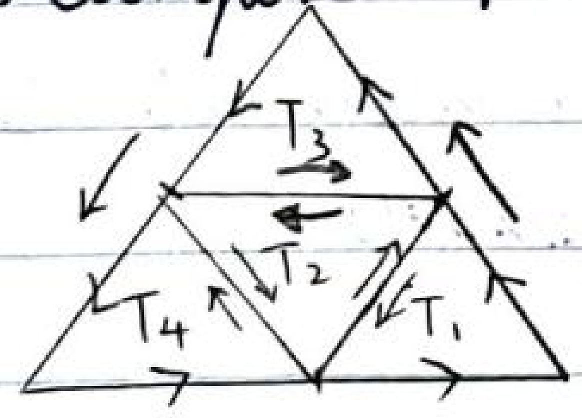
GOURSAT'S THM

If Ω is an open set in $\mathbb{C}$ and $T \subseteq \Omega$ is a triangle where interior is contained in Ω then $\int_T f(z)dz=0$ whenever f is holomorphic on Ω.



PROOF Denote the original triangle by $T^{(0)}$.

Decompose $T^{(0)}$ as follows:



Connect the midpoints to make 4 subtriangles T_1, T_2, T_3, T_4

$$\Rightarrow \int_{T^{(0)}} f(z)dz = \sum_{j=1}^{4}\int_{T_j} f(z)dz$$

longest side

$\Rightarrow$ diameter(T_j) = diameter$(T^{(0)})\cdot\frac{1}{2}$

perimeter(T_j) = perimeter$(T^{(0)})\cdot\frac{1}{2}$

Let $T^{(1)}$ be one of T_1, T_2, T_3, T_4 with the property that

$$\left|\int_{T^{(1)}} f(z)dz\right| \geq \frac{1}{4}\left|\int_{T^{(0)}} f(z)dz\right|$$

Continuing in this fashion, sequence $T^{(0)} \supset T^{(1)} \supset T^{(2)} \supset \cdots$ s.t.

diameter$(T^{(j)}) = 2^{-j}$ diameter$(T^{(0)})$

perimeter$(T^{(j)}) = 2^{-j}$ ~~d~~ perimeter$(T^{(0)})$

$\left|\int_{T^{(j)}} f(z)dz\right| \geq 4^{-j}\left|\int_{T^{(0)}} f(z)dz\right|$

Let $S^{(j)}$ be the closure of the interior of $T^{(j)}$. Then $S^{(0)} \supset S^{(1)} \supset S^{(2)} \supset \cdots$ is a sequence of decreasing compact sets and $\exists$ a point z_0 in all of the $S^{(j)}$.

$$\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)$$

Since f is holomorphic ~~on~~ at z_0, write $f(z) = f(z_0) + f'(z_0)(z-z_0) + \psi(z)(z-z_0)$ (error) where $\psi(z) \to 0$ as $z \to z_0$

$\psi(z)(z-z_0) = f(z) - f(z_0) - f'(z_0)(z-z_0)$ ← from $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$

rearrange: $f(z) = f(z_0) + f'(z_0)(z-z_0) + \psi(z)(z-z_0)$ $\quad \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) = \psi(z)$ difference

$$\Rightarrow \int_{T^{(n)}} f(z)\,dz = \int_{T^{(n)}} f(z_0)\,dz + \int_{T^{(n)}} f'(z_0)(z-z_0)\,dz + \int_{T^{(n)}} \psi(z)(z-z_0)\,dz$$

$=0$ because have primitives $=0$

$$= f(z_0)\int_{T^{(n)}} 1\,dz + f'(z_0)\int_{T^{(n)}} (z-z_0)\,dz + \int_{T^{(n)}} \psi(z)(z-z_0)\,dz$$

$$= \int_{T^{(n)}} \psi(z)(z-z_0)\,dz$$

length of curve

Now $\left|\int_{T^{(n)}} f(z)\,dz\right| = \left|\int_{T^{(n)}} \psi(z)(z-z_0)\,dz\right| \le \sup_{z \in T^{(n)}} |z-z_0| \sup_{z \in T^{(n)}} |\psi(z)|\, \text{perimeter}(T^{(n)})$

$$\le \text{diameter}(T^{(n)})\,\text{perimeter}(T^{(n)}) \sup_{z \in T^{(n)}} |\psi(z)|$$

$$= 4^{-n} \sup_{z \in T^{(n)}} |\psi(z)|\, \text{diameter}(T^{(0)}).$$

$$\Rightarrow \left|\int_{T^{(0)}} f(z)\,dz\right| \le 4^n \left|\int_{T^{(n)}} f(z)\,dz\right| \le 4^n 4^{-n} \sup_{z \in T^{(n)}} |\psi(z)|\, \text{diameter}(T^{(0)})$$

$$= \text{diam}(T^{(0)}) \underbrace{\sup_{z \in T^{(n)}} |\psi(z)|}_{\to 0 \text{ as } n \to \infty}$$

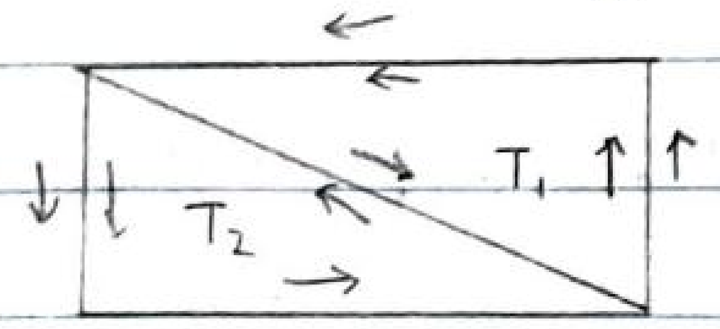
$$\Rightarrow \int_T f(z)\,dz = 0$$

■

COROLLARY

If f is holomorphic on an open set Ω which has the rectangle R in its interior, then $\int_R f(z)\,dz = 0$

PROOF



$$\int_R f(z)\,dz = \int_{T_1} f(z)\,dz + \int_{T_2} f(z)\,dz = 0 + 0 = 0$$

■

PRIMITIVES ON A DISK

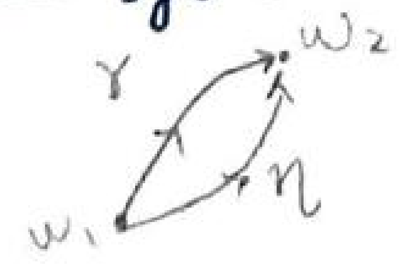
A holomorphic function f on an open disk D admits a primitive.

CAUCHY'S THM FOR A DISK

If f is holomorphic on a disc D and γ is a closed curve in D then $\int_\gamma f(w)\,dw = 0$

HOMOTOPY FORM OF CAUCHY'S THM

If γ is "closed to" η in Ω and f is holomorphic in Ω, then $\int_\gamma f(w)\,dw = \int_\eta f(w)\,dw$.



unit normal.

Simply Connected Domain

A simply-connected domain is one for which any closed curve can be continuously deformed to a point.

CAUCHY'S THM FOR SIMPLY-CONNECTED DOMAIN

If f is holomorphic on the simply-connected domain Ω and γ is a closed curve in Ω then $\int_\gamma f(w)dw = 0$.

COROLLARY

If f is holomorphic on a simply-connected open set Ω then the value of the line integral $\int_\gamma f(w)dw$ depends only on the starting and ending point of γ and not on the path used to connect them.

COROLLARY

A holomorphic function in a simply-connected set Ω admits a primitive.

THM

Let Ω be an open set, let γ and η be curves which are homotopic in Ω. Then $\int_\gamma f(z)dz = \int_\eta f(z)dz$ whenever f is holomorphic in Ω. (same endpoints, can continuously deform γ to become η, all deformed curves stay in Ω.

THM

Let Ω be a simply connected open set, and that f is holomorphic on $\Omega \setminus \{z_0\}$ with z_0 in Ω. If $f(z)$ admits an expansion of the form $f(z) = \sum_{n=-k}^{\infty} a_k (z-z_0)^n$ near z_0 then

$\int_\gamma f(z)dz = 2\pi i\, a_{-1}$ $\qquad = a_{-k}(z-z_0)^{-k} + \dots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \dots$

whenever γ is a simple closed curve in $\Omega \setminus \{z_0\}$.

Residue

The residue of $f(z) = \sum_{n=-k}^{\infty} a_n (z-z_0)^n$ at z_0 is a_{-1}. $\text{Res}_{z_0}(f) = a_{-1}$

THM CAUCHY INTEGRAL FORMULA

Let Ω be a simply-connected open set, and f is holomorphic in Ω. If γ is a simple closed curve with counter-clockwise orientation then $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dz$

PROOF

Since $\frac{f(w)}{w-z}$ is holomorphic on region between γ and circle $C_\varepsilon(z)$:

$$\int_\gamma \frac{f(w)}{w-z}\,dw = \int_{C_\varepsilon(z)} \frac{f(w)}{w-z}\,dw \quad \text{(homotopic)}$$

Since f holomorphic at z,

$$\frac{f(w)-f(z)}{w-z} = f'(z) + \delta(w) \quad \text{and } \delta(w) \to 0 \text{ as } w \to z.$$

$\Rightarrow \frac{f(w)-f(z)}{w-z}$ bounded for w near z $\Rightarrow \lim_{\varepsilon\to 0} \int_{C_\varepsilon(z)} \frac{f(w)-f(z)}{w-z}\,dw = 0$

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon(z)} f'(z) + \delta(w)\,dw = 0$$

constant

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon(z)} \frac{f(w)}{w-z}\,dw = \lim_{\varepsilon\to 0} \int_{C_\varepsilon(z)} \frac{f(w)-f(z)}{w-z}\,dw + f(z)\int_{C_\varepsilon(z)} \frac{1}{w-z}\,dw$$

$$= f(z)\left(\lim_{\varepsilon\to 0} \int_{C_\varepsilon(z)} \frac{1}{z-w}\,dw\right) = 2\pi i\, f(z)$$

$$\Rightarrow \quad \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = f(z) \quad \blacksquare$$

THM

Let f be holomorphic on the open set Ω, and $D_r(z_0) \subset \Omega$. Then f admits a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ which converges in $D_r(z_0)$.

PROOF By Cauchy's formula, $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{\partial D_r(z_0)} \frac{f(w)}{w-z}\,dw$

boundary inside & circle

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\,\frac{1}{1-\boxed{\frac{z-z_0}{w-z_0}}} \quad <1$$

& since $w - z_0 > z - z_0$

$$= \frac{1}{w-z_0}\left(1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0}\right)^2 + \cdots\right) \leftarrow \text{geometric series converges uniformly}$$

for $|z-z_0| < s < 1$ since $|w-z_0| = 1$ & choose $r = 1$?

$$\therefore f(z) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{w-z_0}\left(1 + \frac{z-z_0}{w-z_0} + \left(\frac{z-z_0}{w-z_0}\right)^2 + \cdots\right)dw$$

$$= \sum_{n=0}^{\infty} (z-z_0)^n \underbrace{\left(\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{w-z_0}\,dw\right)}_{a_n} \quad \blacksquare$$

COROLLARY (CAUCHY'S FORMULA)

$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\,dw$ If f is holomorphic on the open set Ω and $\overline{D_r(z_0)} \subset \Omega$. Then $\uparrow$ $\forall z_0 \in D_r(z_0)$.

THM LIOUVILLE

If f is bounded and entire (holomorphic in $\mathbb{C}$), then f is constant.

PROOF

Let $z_0 \in \mathbb{C}$, for all $r>0$, $f'(z_0) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^2} dw$ by Cauchy's Formula.

Since f is bounded, $\exists C>0$ s.t. $|f(z)| \leq C$ $\forall z$.

$$|f'(z_0)| \leq \frac{1}{2\pi i} \text{ length}(C_r(z_0)) \max\left(\frac{f(w)}{(w-z_0)^2}\right)$$

$$\leq \frac{1}{2\pi i} \cdot 2\pi r \cdot \frac{C}{r^2} = \frac{C}{r} \quad \text{for } r \to \infty, \ |f'(z_0)| \to 0$$

$\Rightarrow f$ is constant.

FUNDAMENTAL THM OF ALGEBRA

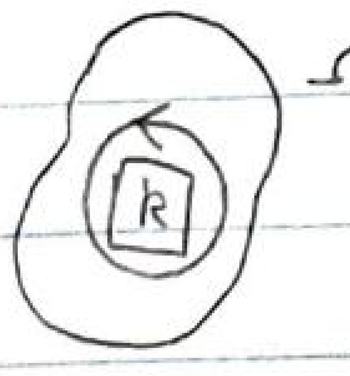
If P is a polynomial of degree $n \geq 1$, then $\exists z_1, \dots, z_n$ and $a_n \neq 0$ s.t. $p(z) = a_n(z-z_1)\cdots(z-z_n)$.

THM

Let $\{f_n\}$ be a sequence of holomorphic functions on an open set Ω, and $f_n \to f$ converges uniformly on compact subsets of Ω. Then f is holomorphic in Ω and $f_n^{(k)} \to f^{(k)}$ converges uniformly on compact subsets of Ω.

PROOF

CLAIM: f is holomorphic. Let k be a compact subset of Ω. Choose a simple closed curve in Ω which encloses k.



$\forall z \in k$, $f_n(z) = \frac{1}{2\pi i}\int \frac{f_n(w)}{w-z} dw$. Since $f_n \to f$ uniformly, as $n \to \infty$, $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z} dz$ for any rectangle R in Ω,

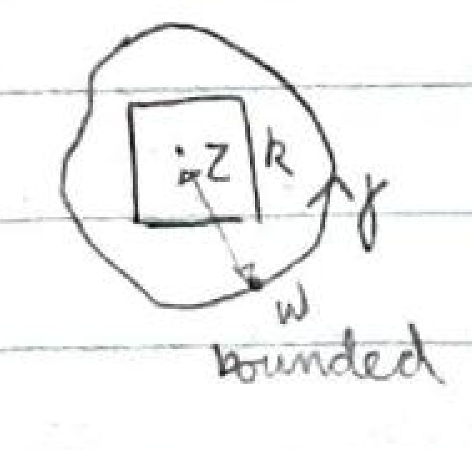
$\int_k f_n(w) dw = 0 \Rightarrow \int_k f(w) dw = \lim_{n\to\infty} \int_R f_n(w) dw = 0$

$$\therefore f^{(k)}(z) = \frac{k!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{k+1}} dw = \lim_{n\to\infty} \frac{k!}{2\pi i}\int_\gamma \frac{f_n(w)}{(w-z)^{k+1}} dw = \lim_{n\to\infty} f_n^{(k)}(z)$$

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{k!}{2\pi}\left|\int_\gamma \frac{f(w)-f_n(w)}{(w-z)^{k+1}} dw\right| \leq \frac{k!}{2\pi} \text{length}(\gamma) \cdot \sup_{w\in\gamma}|f(w)-f_n(w)| \cdot \frac{1}{\delta^{k+1}}$$

$\to 0$

for some $\delta \leq |z-w|$



THM

If f is continuous on the simple closed curve γ then $g(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$ is holomorphic on the interior of γ.

THM

If f is continuous in an open set Ω and $\int_K f(z)dz = 0$ for a rectangle K in Ω, then f is holomorphic.

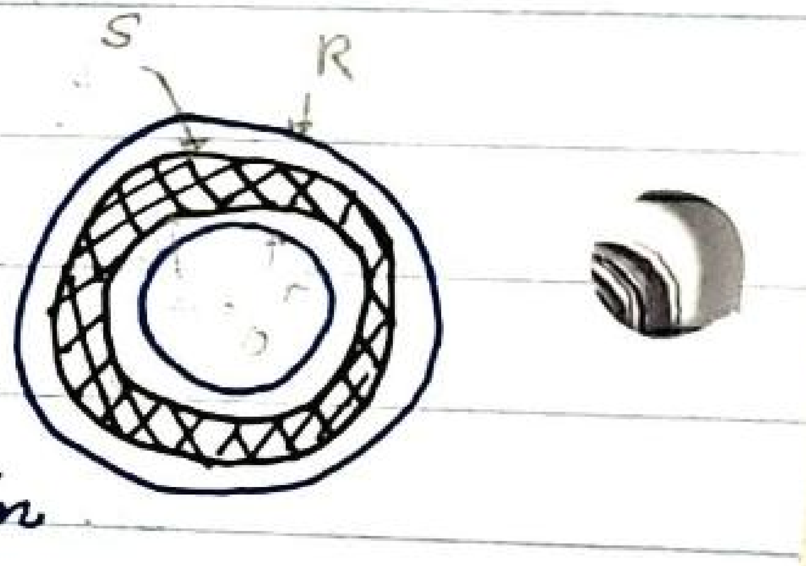
SINGULAR ~~FUNCTION~~

A function f is singular at a point z_0 if it $f(z_0)$ is not defined and f is not holomorphic at z_0.

ISOLATED SINGULARITY

A singularity z_0 of a function f is isolated if $f(z)$ is holomorphic on a deleted neighbourhood of z_0 $(D_r(z_0) \setminus \{z_0\})$

THM



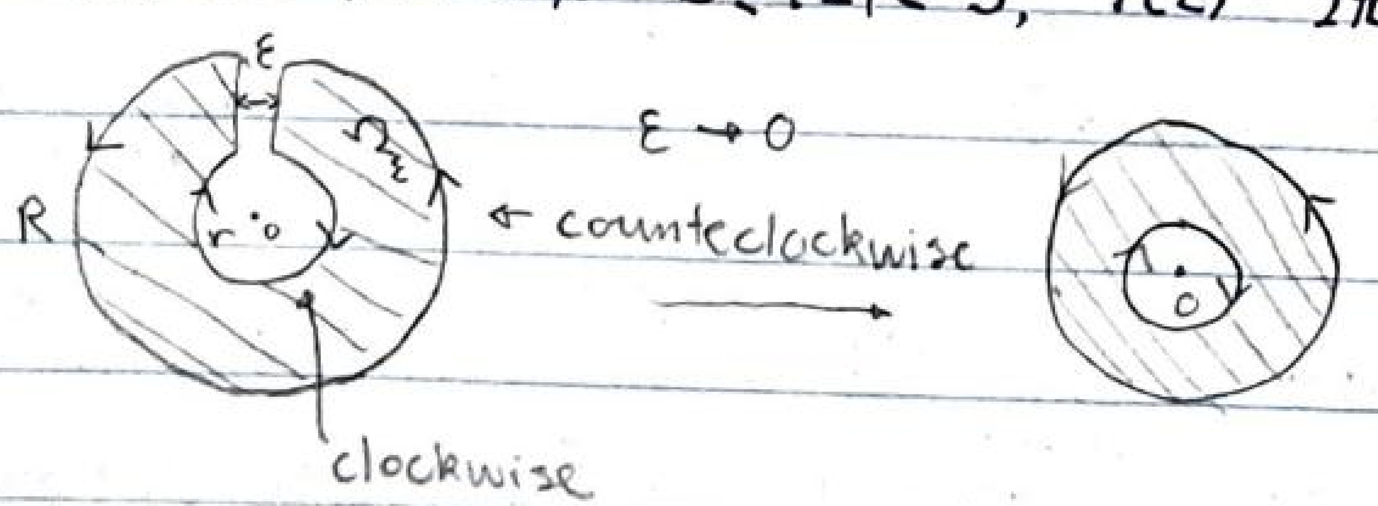
Let f be holomorphic on the annulus $r < |z| < R$.
If $r < s < S < R$ then f admits an expansion of the form $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ which is absolutely uniformly convergent in $s < |z| < S$.

compact subset of annulus

$$a_n = \begin{cases} \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(w)}{w^{n+1}}dw & n \geq 0 \\ \frac{1}{2\pi i}\int_{C_r(0)} \frac{f(w)}{w^{n+1}}dw & n \leq -1 \end{cases}$$

PROOF: choose $r < s < S < R$

CLAIM. $\forall z$ s.t. $s \leq |z| \leq S$, $f(z) = \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C_r(0)} \frac{f(w)}{w-z}dw$



By Cauchy's formula, $f(z) = \frac{1}{2\pi i}\int_{\partial\Omega_\varepsilon} \frac{f(w)}{w-z}dw$

as $\varepsilon \to 0$, $f(z) = \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C_r(0)} \frac{f(w)}{w-z}dw$

$|w| > |z|$ $\qquad$ $|w| < |z|$

$w \in C_R(0)$: $\frac{1}{w-z} = \frac{1}{w} \cdot \underbrace{\frac{1}{1-\frac{z}{w}}}_{\text{geometric series}} = \frac{1}{w}\left(1 + \frac{z}{w} + \left(\frac{z}{w}\right)^2 + \cdots\right)$ since $\overset{R}{|w|} > |z|$, $\frac{|z|}{|w|} < 1$

this converges uniformly for $s \leq |z| \leq S < R$

$f_n \rightrightarrows f \Rightarrow \int f_n \to \int f$ from above

$w \in C_r(0)$: $\frac{1}{w-z} = \frac{-1}{z} \cdot \frac{1}{1-\frac{w}{z}} = \frac{-1}{z}\left(1 + \frac{w}{z} + \left(\frac{w}{z}\right)^2 + \cdots\right)$ converges uniformly as well.

$\therefore$ for $s \leq |z| \leq S$, $f(z) = \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C_r(0)} \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{C_R(0)} \frac{f(w)}{w}\cdot\left(1 + \frac{z}{w} + \cdots\right)dw + \frac{1}{2\pi i}\int_{C_r(0)} \frac{f(w)}{z}\left(1 + \frac{w}{z} + \cdots\right)dw$

$$= \sum_{n=0}^{\infty} z^n \left(\frac{1}{2\pi i}\int_{C_R(0)} \frac{f(w)}{w^{n+1}}dw\right) + \sum_{n=-\infty}^{-1} z^n \left(\frac{1}{2\pi i}\int_{C_r(0)} \frac{f(w)}{w^{n+1}}dw\right)$$

■

LAURENT SERIES

A Laurent series is an expansion of the form $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.
This series converges absolutely / uniformly / uniformly absolutely in an annulus A provided $f^+(z) = \sum_{n=0}^{\infty} a_n z^n$ and $f^-(z) = \sum_{n=-\infty}^{-1} a_n z^n$ converges absolutely / uniformly / uniformly absolutely in A. Then, we regard

$$f(z) = f^+(z) + f^-(z).$$

THM

If f has an isolated singularity at z_0 and f is bounded on a deleted neighbourhood of z_0, then $\exists$ a unique value of $f(z_0)$ which makes f holomorphic in a neighbourhood of z_0.

REMOVABLE SINGULARITY

f has a removable ~~sign~~ singularity at z_0 if f has an isolated singularity at z_0 and there is a unique value of $f(z_0)$ which makes f holomorphic in a neighbourhood of z_0.

e.g. $f(z) = \frac{z^2-1}{z-1} = \frac{(z-1)(z+1)}{z-1} = z+1$ if set $f(1)=2$, then f is holomorphic

f has a removable singularity at z_0 iff f is bounded and holomorphic in a deleted neighbourhood of z_0.

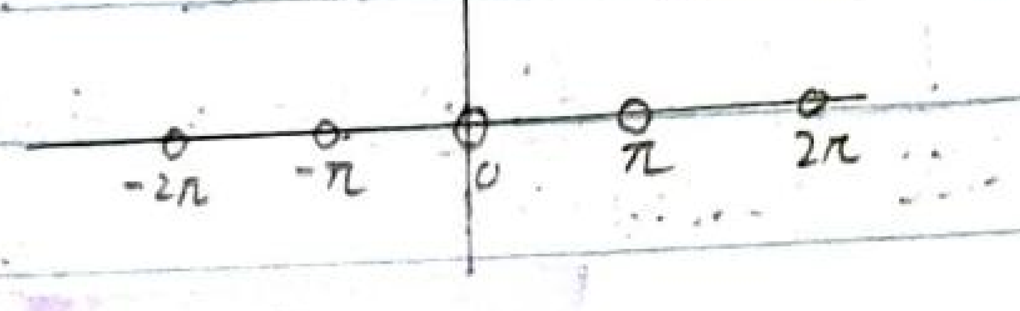
POLE OF ORDER m

An isolated singularity z_0 of f is a pole of order m if the Laurent expansion of f around z_0 is of the form $f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n$ with $a_{-m} \neq 0$. (finite negative terms)

MEROMORPHIC

A function f is meromorphic on an open set Ω if f is holomorphic on $\Omega \setminus \{z_n\}$ where $\{z_n\}$ is a discrete set. (sequence with no limit point)

e.g. $\frac{1}{\sin(z)}$ meromorphic in $\mathbb{C}$



SINGULARITIES IN TERMS OF LAURENT SERIES

Removable Singularity : $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ ← no negative terms

Pole : $f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n$ ← finite negative terms

Essential Singularity : $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ ← infinite negative terms

THM

If f has an essential singularity at z_0 and it is holomorphic on $D_r(z_0)\setminus\{z_0\}$ then $f(D_r(z_0)\setminus\{z_0\})$ is dense in $\mathbb{C}$

DENSE

A set A is dense in set B if $\forall y \in B$ and $\forall \varepsilon > 0$ $\exists x \in A$ s.t. $|x-y| < \varepsilon$ ($\forall w \in \mathbb{C}$ $\forall \varepsilon > 0$ $\exists z$ s.t. $|f(z)-w| < \varepsilon$)

PROOF Suppose $f(D_r(z_0)\setminus\{z_0\})$ is not dense in $\mathbb{C}$.

Then $\exists w \in \mathbb{C}$ and $\delta > 0$ s.t. $|f(z)-w| \geq \delta$ $\forall z \in D_r(z_0)\setminus\{z_0\}$.

Let $g(z) = \frac{1}{f(z)-w}$ $\Rightarrow |g(z)| \leq \frac{1}{\delta}$. g is holomorphic and bounded on $D_r(z_0)\setminus\{z_0\}$

So g has a removable singularity at z_0 so we can regard g as holomorphic on $D_r(z_0)$.

$$g(z) = \frac{1}{f(z)-w} \Rightarrow \frac{1}{g(z)} = f(z)-w \quad \therefore \quad f(z) = \frac{1}{g(z)} + w$$

If $g(z_0) \neq 0$ then f is holomorphic at z_0 CONTRADICTION

$\Rightarrow g(z_0) = 0$ and admits a power series expansion

$$g(z) = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \cdots \text{ with } a_m \neq 0 \text{ and } m \geq 1.$$

$$g(z) = a_m(z-z_0)^m(1 + b_1(z-z_0) + b_2(z-z_0)^2 + \cdots)$$

$$g(z) = a_m(z-z_0)^m(1-p(z)) \qquad p(z) \to 0 \text{ as } z \to 0$$

$$\Rightarrow \frac{1}{g(z)} = \frac{1}{a_m(z-z_0)^m}\frac{1}{1-p(z)} = \frac{1}{a_m(z-z_0)^m}\underbrace{(1 + p(z) + p(z)^2 + \cdots)}_{\text{holomorphic}}$$

$\Rightarrow \frac{1}{g(z)}$ has a pole at z_0 – CONTRADICTION. ■

THM

Let f be meromorphic in the open set Ω, and $D_r(z_0) \subset \Omega$. If f has no zeros or poles on $C_r(z_0)$ then

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'(w)}{f(w)}\,dw = \#\text{ of zeros of } f \text{ in } D_r(z_0) - \#\text{ of poles of } f \text{ in } D_r(z_0)$$

counting multiplicities

ROUCHÉ'S THM

Let Ω be an open set and f, g are holomorphic in Ω. If $D_r(z_0) \subset \Omega$ and $|g(z)| < |f(z)|$ on $C_r(z_0)$ then f and $f+g$ have the same numbers of zeros in $D_r(z_0)$, counting multiplicities.

THM

$f:[0,1]\to\mathbb{R}$ is continuous iff $f^{-1}(O)=\{x\in[0,1]: f(x)\in O\}$ is open whenever O is open

OPEN MAP

A map f is open if it maps open sets to open sets.

THM

If $f:U\to V$ is a map between open sets in $\mathbb{C}$, then f is continuous iff $f^{-1}(O)=\{x: f(x)\in O\}$ is open whenever O is open. i.e. The inverse image of an open set is open.

COROLLARY

If $f:U\to V$ is a bijective continuous open mapping between open sets in $\mathbb{C}$, then $f^{-1}:V\to U$ is continuous.

THM OPEN MAPPING THM

If $f: U\to V$ is a holomorphic function between open sets in $\mathbb{C}$, then f is an open mapping.

COROLLARY

If $f:U\to V$ is a bijective holomorphic function between open sets in $\mathbb{C}$, then $f^{-1}:V\to U$ is holomorphic.

THM

If $f:U\to V$ is a nonconstant holomorphic map between open sets, then f is an open mapping.

PROOF Suppose $f(z_0)=w_0$. We want to show that $\forall\, w$ close to w_0 $\exists\, z$ s.t. $f(z)=w$

Let $g(z)=f(z)-w = f(z)-w_0+w_0-w$

$= F(z) + G(z) \leftarrow$ constant $\forall z$

Since f is nonconstant, there is a deleted neighbourhood of z_0 in which $f\neq w_0$

Choose $\delta>0$ s.t. $D_\delta(z_0)\subset U$ and $f(z)\neq w_0$ for $0<|z-z_0|\leq\delta$

Choose $\varepsilon>0$ s.t. $|f(z)-w_0|\geq\varepsilon$ for $|z-z_0|=\delta$

If $|w-w_0|<\varepsilon$ then $|G(z)|<\varepsilon<|f(z)-w_0| = |F(z)|$

By Rouché's Thm, $F(z)$ and $G(z)+F(z)=f(z)-w$ has same # of zeros in $D_\delta(z_0)$.

Since F has at least one zero in $D_\delta(z_0)$, $f(z)-w = g(z)$ has at least 1 zero in $D_\delta(z)$. $\Rightarrow$ $\exists\, |z-z_0|<\delta$ s.t. $f(z)=w$ $\wedge$ $|w-w_0|<\varepsilon \Rightarrow$

COROLLARY

Suppose f is holomorphic on an open set Ω and $f'(z) \neq 0$ for some z_0 in Ω, then $\exists$ an open set $U \supset \{z_0\}$ s.t. f is injective on U.

THM

Let f be holomorphic on an open set Ω containing the point z_0. Then $\exists$ an open set U in Ω containing z_0 on which f is injective iff $f'(z_0) \neq 0$.

(surjective not as important; once injective $f: U \to f(U)$ is bijective

~~THM~~

~~If $f: U \to V$ is a bijective holomorphic map between~~

ANALYTIC ISOMORPHISM

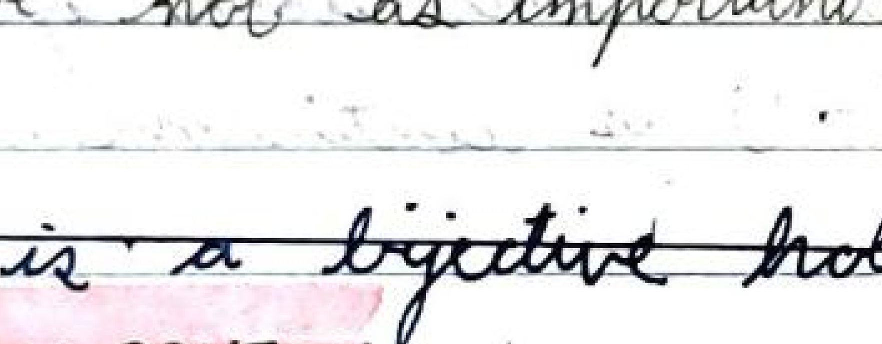
A mapping $f: U \to V$ between open sets in $\mathbb{C}$ is an analytic isomorphism if:

1. f is holomorphic
2. f is bijective
3. f^{-1} is holomorphic (1 and 2 implies 3)

LOCAL ANALYTIC ISOMORPHISM

A holomorphic map $f: U \to V$ between open sets in $\mathbb{C}$ is a local analytic isomorphism at z_0 if $\exists$ a set $W \supset \{z_0\}$ s.t. $f: W \to f(W)$ is an analytic isomorphism

e.g.



f is injective and invertible locally

$f: U \to V$ not isomorphic (cannot map simply connected set to non simply connected set bijectively)

COROLLARY

Let $f: U \to V$ be a holomorphic map between open sets in $\mathbb{C}$. Then f is a local analytic isomorphism at $z_0 \in U$ iff $f'(z_0) \neq 0$.

result can be applied to other z_0

PROOF "$\Leftarrow$" Suppose $f'(z_0) \neq 0$. WLOG assume $z_0 = 0$, $f(z_0) = 0$, $f'(z_0) = 1$

$f(0) = a_0$ — just a constant, so set $a_0 = 0$

$f'(z) = (\sum a_n z^n)'$ so $f'(z_0) = a_1$, divide $f(z)$ by a_1

So at $z_0 = 0$, $f(z) = z + \underbrace{a_2 z^2 + a_3 z^3 + \dots}_{= p(z)}$

$\lim_{z \to 0} \frac{p(z)}{z} = 0$ so $\exists r > 0$ s.t.: $|z| \le r \Rightarrow \left|\frac{p(z)}{z}\right| < \frac{1}{2}$

$\therefore$ when $|z| \le r$, $|p(z)| < \frac{1}{2}|z| \le \frac{1}{2} r$

Choose $|\alpha| < \frac{r}{2}$, $f(z) - \alpha = \underbrace{f(z) - z}_{p(z)} + \underbrace{z - \alpha}_{:= q(z)}$

Since $|p(z)| < \frac{r}{2} \le |q(z)|$ for $|z| = r$, by Rouché's Thm, $q(z)$ and $f(z)$ have the same number of zeros in $|z| < r$.

For each $|\alpha| < \frac{r}{2}$, ~~f(z) =~~ $q(z) = z - \alpha$ has exactly one solution in the disk $|z| < r$. Let W be the inverse image of f for $D_{\frac{r}{2}}(0)$

i.e. $W = f^{-1}(D_{\frac{r}{2}}(0))$ then f is injective on W. $\Rightarrow$ local analytic isomorphism (holomorphic given)

surjective

"$\Rightarrow$" Suppose f is injective on an open neighbourhood W of z_0. WLOG assume $z_0 = 0$, $f(z_0) = 0$. Suppose by contradiction $f'(z_0) = 0$.

$\Rightarrow$ The power series expansion of f near $z_0 = 0$ is

$$f(z) = a_m z^m (1 + p(z)) \text{ with } m > 1,\ a_m \neq 0 \text{ and } p(z) \to 0$$

as $z \to 0$.

For a sufficiently small r, $|a_m z^m (p(z))| < |a_m z^m|$ when $z = r$

by Rouché's Thm, $a_m z^m$ and $f(z) = a_m z^m (1 + p(z))$ has the same number of zeros, i.e. $= m > 1$ zeros in every disk of radius $\le r$ ~~&~~ centered at zero.

$\Rightarrow f$ is not injective — contradiction $\blacksquare$

THM

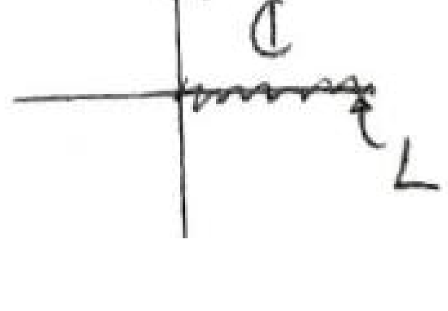
Let Ω be a simply-connected domain which does not contain 0. Then there is a holomorphic function $f: \Omega \to \mathbb{C}$ s.t. $e^{f(z)} = z$ for all z in Ω. Moreover, if g is any other such function then $f - g = 2\pi i k$ with $k \in \mathbb{Z}$.

LOGARITHMIC FUNCTIONS

$\log(re^{i\theta}) = \log(r) + i\theta + 2\pi i k$ $\quad k \in \mathbb{Z}$, $-\pi < \theta < \pi$

real part / rotate / does not affect $e^{i\theta}$

Defined on cut-phase $\mathbb{C} \setminus L$



Given a simply-connected domain Ω and a determination of log, we can define log in ~~C~~ Ω.
$z^\alpha = e^{\alpha \log z}$

THM

Let $f: U \to V$ and $g: V \to U$ are holomorphic functions acting on connected open sets. If $g(f(z)) = z \quad \forall z$ in U then
$f(g(w)) = w \quad \forall w$ in V

PROOF Since $g(f(z)) = z \quad \forall z$

$$f(g(f(z))) = f(z) \quad \forall z, \text{ let } w = f(z)$$

$$\Rightarrow f(g(w)) = w \quad \forall w \in f(U)$$

Now $f(U)$ is an open set in the connected set V,
so $f(g(w)) = w \quad \forall w \in V$.