Conditional independencies in state space models

Sensor fusion & nonlinear filtering

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STATE SPACE MODELS AND CONDITIONAL INDENDENCIES

- We represent state space models in one of two forms:

$$\begin{cases} \mathbf{x}_k &= f_{k-1}(\mathbf{x}_{k-1}, \mathbf{q}_{k-1}) \\ \mathbf{y}_k &= h_k(\mathbf{x}_k, \mathbf{r}_k) \end{cases} \iff \begin{cases} p(\mathbf{x}_k \,|\, \mathbf{x}_{k-1}) \\ p(\mathbf{y}_k \,|\, \mathbf{x}_k). \end{cases}$$

- For the form on the left hand side we assume:

  Both the motion noise, $\mathbf{q}_{k-1}$, and the measurement noise, $\mathbf{r}_k$, are independent of all other noise vectors.

- The corresponding assumptions for the density representation:

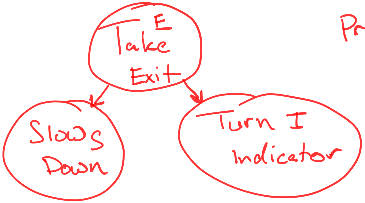
$$p(\mathbf{x}_k \,|\, \mathbf{x}_{0:k-1}, \mathbf{y}_{1:k-1}) = p(\mathbf{x}_k \,|\, \mathbf{x}_{k-1}) \tag{1}$$

$$p(\mathbf{y}_k \,|\, \mathbf{x}_{0:k}, \mathbf{y}_{1:k-1}) = p(\mathbf{y}_k \,|\, \mathbf{x}_k) \tag{2}$$

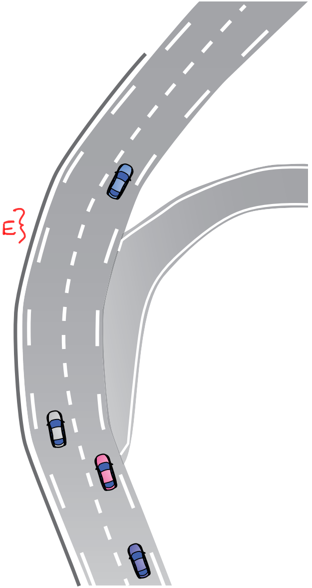
- Note: Both $\mathbf{x}_k$ and $\mathbf{y}_k$ are stochastic processes and the assumption in (1) implies that $\mathbf{x}_k$ is a Markov process.

BAYESIAN NETWORKS AND CONDITIONAL INDEPENDENCIES

- A Bayesian network (also known as belief networks or Bayes net) is a probabilistic graphical model.
- Bayesian networks are directed acyclic graphs that describe how a joint density can be factorized. It also illustrates conditional dependencies.

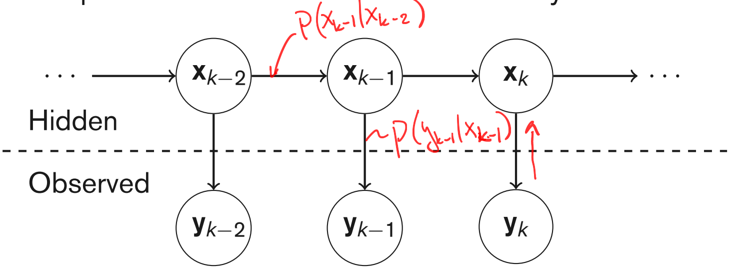


$$\Pr\{S,I,E\} = \Pr\{S|\cancel{I},E\}\Pr\{I|E\}\Pr\{E\}$$
$$= \Pr\{S|E\}\Pr\{I|E\}\Pr\{E\}$$



STATE SPACE MODELS AND BAYESIAN NETWORKS

- A state space model can be described as a Bayesian network:



- The graph illustrates that:

$$
\begin{aligned}
p(\mathbf{x}_{0:k},\mathbf{y}_{1:k}) &= p(\mathbf{y}_{1:k}|\mathbf{x}_{0:k})p(\mathbf{x}_{0:k}) \\
&= p(\mathbf{y}_1|\mathbf{x}_{0:k})p(\mathbf{y}_2|\mathbf{y}_1,\mathbf{x}_{0:k})\dots p(\mathbf{y}_k|\mathbf{y}_{1:k-1},\mathbf{x}_{0:k}) \\
&\quad p(\mathbf{x}_0)p(\mathbf{x}_1|\mathbf{x}_0)p(\mathbf{x}_2|\mathbf{x}_{0:1})\dots p(\mathbf{x}_k|\mathbf{x}_{0:k-1})
\end{aligned}
$$

$$= p(y_1|x_1)p(y_2|x_2)\dots p(y_k|x_k)p(x_0)p(x_1|x_0)p(x_2|x_1)\dots p(x_k|x_{k-1})$$

SELF-ASSESSMENT

Suppose the Bayesian network

describes the joint distribution over variables x_{k-1}, x_k and y_k.

Check all that apply:

- $p(x_k, x_{k-1}, y_k) = p(x_k | x_{k-1}) p(x_{k-1}) p(y_k | x_k)$
- $p(x_k, y_k) = p(y_k | x_k) p(x_k)$
- $p(y_k | x_k, x_{k-1}) = p(y_k | x_k)$