A primer in statistics – Expectation, covariance and the Gaussian distribution

Sensor fusion & nonlinear filtering

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EXPECTED VALUE AND COVARIANCE

- Probability distributions are often characterized by their mean vectors and covariance matrices.

Expected value (mean vector)

- The expected value (mean) of a random vector $\mathbf{x} = [x_1, x_2, \ldots, x_m]^T$ is

μ

$$\mathbb{E}\{\mathbf{x}\} = \int \mathbf{x}\, p(\mathbf{x})\, d\mathbf{x}$$

$\bar{x}$

where $\int d\mathbf{x}$ is shorthand for $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \ldots dx_m$.

Covariance matrix

- The covariance matrix is ($\mathbf{x}$ is a column vector)

$$\mathrm{Cov}\{\mathbf{x}\} = \mathbb{E}\left\{\underbrace{[\mathbf{x} - \mathbb{E}\{\mathbf{x}\}]\,[\mathbf{x} - \mathbb{E}\{\mathbf{x}\}]^T}\right\}$$

m

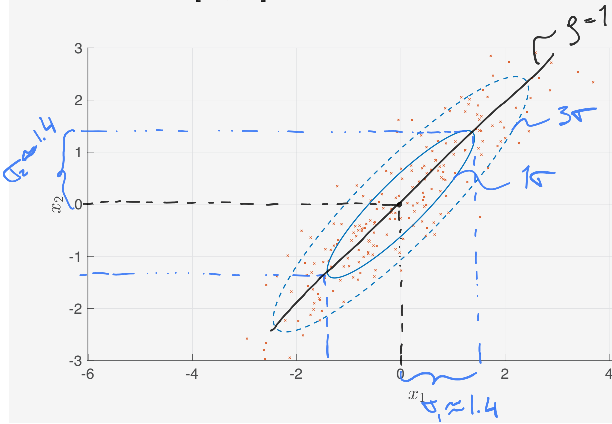
m

- For discrete-valued random variables the above integrals are replaced by the corresponding summations.

GUESS THAT COVARIANCE

Example: Guess that covariance

- Suppose we have independent samples from a zero-mean random vector $\mathbf{x} = [x_1, x_2]^T$. What is the covariance matix of $\mathbf{x}$?



$$\mathrm{Cov}\{x\} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.4^2 & 0.9\,1.4^2 \\ 0.9\,1.4^2 & 1.4^2 \end{bmatrix}$$

$$\approx \begin{bmatrix} 2 & 1.8 \\ 1.8 & 2 \end{bmatrix}$$

LAW OF LARGE NUMBERS

- The law of large numbers states that sample averages converge to expected values.

Law of large numbers

- If $x_1, x_2, \ldots$ are independent and identically distributed random variables distributed according to $p(x)$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i = \mathbb{E}_{p(x)}\{x\}.$$

Example: Throwing a dice many times...

- ...the average face value converges to the expected value

$$\frac{1+2+3+4+5+6}{6} = 3.5$$

GAUSSIAN DISTRIBUTIONS

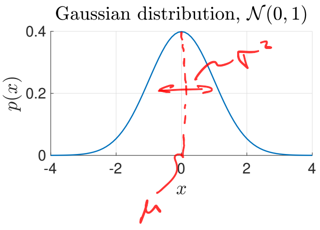
- The most important distribution is the Gaussian distribution (at least in this course).

Gaussian distribution

- We write $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{Q})$ to denote that $\mathbf{x}$ is a Gaussian random variable with mean $\boldsymbol{\mu}$ and covariance $\mathbf{Q}$.
- The pdf of $\mathbf{x}$ is

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}, \mathbf{Q}) = \frac{1}{\sqrt{|2\pi\mathbf{Q}|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{Q}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

where $|\cdot|$ denotes the determinant.



- Note: the pdf of a Gaussian random variable is completely determined by its mean and its covariance matrix.

GAUSSIAN DISTRIBUTIONS

Linear combination of indep. Gaussian random variables

- Let $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}_x, \mathbf{Q}_x)$ and $\mathbf{y} \sim \mathcal{N}(\boldsymbol{\mu}_y, \mathbf{Q}_y)$.
- Then a linear combination of $\mathbf{x}$ and $\mathbf{y}$,

$$\mathbf{z} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y},$$

where $\mathbf{A}$ and $\mathbf{B}$ are deterministic matrices, is also Gaussian with mean

$$\boldsymbol{\mu}_z = \mathbb{E}\left\{\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}\right\} = \mathbf{A}\boldsymbol{\mu}_x + \mathbf{B}\boldsymbol{\mu}_y$$

and covariance

$$\begin{aligned}\mathbf{Q}_z &= \mathrm{Cov}\left\{\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}\right\} = \mathrm{Cov}\left\{\mathbf{A}\mathbf{x}\right\} + \mathrm{Cov}\left\{\mathbf{B}\mathbf{y}\right\} + \underbrace{\mathrm{Cov}\{Ax, By\}}_{=0} + \underbrace{\mathrm{Cov}\{By, Ax\}}_{=0} \\ &= \mathbf{A}\mathbf{Q}_x\mathbf{A}^T + \mathbf{B}\mathbf{Q}_y\mathbf{B}^T.\end{aligned}$$

PROBABILITY THEORY – KEY RESULTS

Conditional distributions: $\begin{cases} p(x,z) = p(z|x)p(x) \\ p(z|x) = \frac{p(x,z)}{p(x)} \end{cases}$

Law of total probability: $\begin{cases} p(z) = \int_x p(x,z)\,dx \\ p(z) = \int_x p(z|x)p(x)\,dx \end{cases}$

1st and 2nd moments: $\begin{cases} \mathbb{E}\{\mathbf{x}\} = \int \mathbf{x}\,p(\mathbf{x})\,d\mathbf{x} \\ \mathrm{Cov}\{\mathbf{x}\} = \mathbb{E}\left\{[\mathbf{x} - \mathbb{E}\{\mathbf{x}\}][\mathbf{x} - \mathbb{E}\{\mathbf{x}\}]^T\right\} \end{cases}$

Gaussian pdf: $\mathcal{N}(\mathbf{x};\boldsymbol{\mu},\mathbf{Q}) = \frac{1}{\sqrt{|2\pi\mathbf{Q}|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\mathbf{Q}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$