# The Kalman filter equations

Sensor fusion & nonlinear filtering

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# KALMAN FILTER

## Kalman filter

- Analytical solution to the filtering equations for linear and Gaussian models.
- The Kalman filter recursively computes

$$\underline{p(\mathbf{x}_k|\mathbf{Y}_{1:k-1})} = \mathcal{N}(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k-1}, \mathbf{P}_{k|k-1}) \leftarrow \text{Prediction step}$$

$$\underline{p(\mathbf{x}_k|\mathbf{Y}_{1:k})} = \mathcal{N}(\mathbf{x}_k; \hat{\mathbf{x}}_{k|k}, \mathbf{P}_{k|k}) \leftarrow \text{Update step}$$

for $k = 1, 2, \ldots$.

Note:

- Only need to compute the moments $\hat{\mathbf{x}}_{k|k-1}$, $\mathbf{P}_{k|k-1}$, $\hat{\mathbf{x}}_{k|k}$ and $\mathbf{P}_{k|k}$ in each recursion.

# KALMAN FILTER: PREDICTION

## Prediction step

$$
\begin{aligned}
\hat{\mathbf{x}}_{k|k-1} &= \mathbf{A}_{k-1}\hat{\mathbf{x}}_{k-1|k-1} \\
\mathbf{P}_{k|k-1} &= \mathbf{A}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{A}_{k-1}^T + \mathbf{Q}_{k-1}
\end{aligned}
$$

$$\left\{\begin{array}{l} x_k = A_{k-1}x_{k-1} + q_{k-1} \\ p(x_{k-1} \mid y_{1:k-1}) = N(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1}) \end{array}\right. \Longrightarrow E\{x_k \mid y_{1:k-1}\} = E\{\underbrace{A_{k-1}x_{k-1}}_{A_{k-1}\cdot\hat{x}_{k-1|k-1}} + \underbrace{q_{k-1}}_{=0} \mid y_{1:k-1}\} = A_{k-1}\hat{x}_{k-1|k-1}$$

Note:

- We assume that process noise $\mathbf{q}_{k-1}$ $\sim N(0, Q_{k-1})$ is zero mean.
- On the right hand side, only $\hat{\mathbf{x}}_{k-1|k-1}$ depend on $\mathbf{y}_{1:k-1}$.

# KALMAN FILTER: UPDATE

## Update step

- The posterior mean $\hat{\mathbf{x}}_{k|k}$ and covariance $\mathbf{P}_{k|k}$ is computed as

Prior

Correction term

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \mathbf{v}_k$$
$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k \mathbf{S}_k \mathbf{K}_k^T$$

where

Kalman Gain $$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1}$$

Innovation $$\mathbf{v}_k = \mathbf{y}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$$

Innovation Covariance $$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k$$

Note:

- The posterior mean $\hat{\mathbf{x}}_{k|k} = \mathbb{E}\{\mathbf{x}_k | \mathbf{y}_{1:k}\}$ is both the MMSE and MAP estimator.