Optimal filtering

Sensor fusion & nonlinear filtering

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FILTERING: PROBLEM FORMULATION

- Consider a time-discrete state space model:

$$p(\mathbf{x}_k | \mathbf{x}_{k-1}) \qquad \text{motion model}$$

$$p(\mathbf{y}_k | \mathbf{x}_k) \qquad \text{measurement model,}$$

and suppose that $\mathbf{x}_0 \sim p(\mathbf{x}_0)$ and

$$p(\mathbf{x}_k | \mathbf{x}_{0:k-1}, \mathbf{y}_{1:k-1}) = p(\mathbf{x}_k | \mathbf{x}_{k-1})$$

$$p(\mathbf{y}_k | \mathbf{x}_{0:k}, \mathbf{y}_{1:k-1}) = p(\mathbf{y}_k | \mathbf{x}_k).$$

Objective in filtering

- We seek to compute $p(\mathbf{x}_k | \mathbf{y}_{1:k})$ for $k = 1, 2, 3, \ldots$

A NON-RECURSIVE SOLUTION

- We know Bayesian statistics $\Rightarrow$ we can find $p(\mathbf{x}_k \mid \mathbf{y}_{1:k})$!

Step 1: use Bayes' rule to find

$$p(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k}) = \frac{p(\mathbf{y}_{1:k} \mid \mathbf{x}_{0:k}) p(\mathbf{x}_{0:k})}{p(\mathbf{y}_{1:k})} \propto p(\mathbf{x}_0) \prod_{i=1}^{k} p(\mathbf{y}_i \mid \mathbf{x}_i) p(\mathbf{x}_i \mid \mathbf{x}_{i-1})$$

Step 2: marginalize with respect to $\mathbf{x}_{0:k-1}$

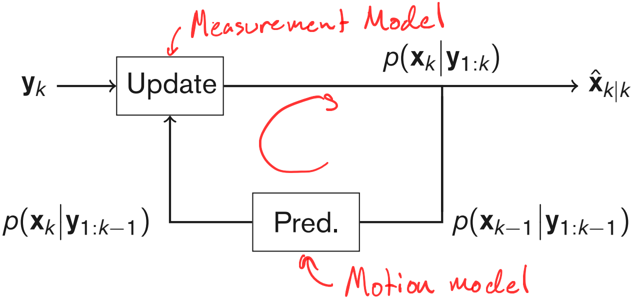
$$p(\mathbf{x}_k \mid \mathbf{y}_{1:k}) = \int p(\mathbf{x}_{0:k} \mid \mathbf{y}_{1:k}) \, d\mathbf{x}_{0:k-1}$$

- Weakness: complexity grows with k.

A RECURSIVE FILTERING SOLUTION

Methodology

- Recursively compute $p(\mathbf{x}_k | \mathbf{y}_{1:k})$ from $p(\mathbf{x}_{k-1} | \mathbf{y}_{1:k-1})$.



- A block diagram illustrating the prediction and update steps that we perform recursively.

THE PREDICTION STEP

Prediction

- Compute $p(\mathbf{x}_k | \mathbf{y}_{1:k-1})$ from $p(\mathbf{x}_{k-1} | \mathbf{y}_{1:k-1})$.

- In this step we use our knowledge regarding $\mathbf{x}_{k-1}$, obtained from $\mathbf{y}_{1:k-1}$, to predict $\mathbf{x}_k$.

$$p(\mathbf{x}_k | \mathbf{y}_{1:k-1}) = \int p(x_k, x_{k-1} | y_{1:k-1}) \, dx_{k-1} = \int p(x_k | x_{k-1}, \cancel{y_{1:k-1}}) \, p(x_{k-1} | y_{1:k-1}) \, dx_{k-1}$$

$$= \int p(x_k | x_{k-1}) \, p(x_{k-1} | y_{1:k-1}) \, dx_{k-1}$$

This is the *Chapman-Kolmogorov* equation.

SELF-ASSESSMENT ON THE PREDICTION STEP

Suppose $x_k = x_{k-1} + q_k$ where $q_k \sim \mathcal{N}(0, 1)$. The uncertainties in $p(x_k | y_{1:k-1})$ are then normally [select a suitable word below] than the uncertainties in $p(x_{k-1} | y_{1:k-1})$.

- smaller
- larger
- neither larger nor smaller

Only one answer applies.

THE MEASUREMENT UPDATE STEP

Measurement update

- Compute $p(\mathbf{x}_k \middle| \mathbf{y}_{1:k})$ from $p(\mathbf{x}_k \middle| \mathbf{y}_{1:k-1})$.

- In this step, we update our knowledge about $\mathbf{x}_k$ using the new measurement $\mathbf{y}_k$.

$$p(\mathbf{x}_k \middle| \mathbf{y}_{1:k}) = p(x_k | y_k, y_{1:k-1}) = \frac{p(y_k | x_k, \cancel{y_{1:k-1}})\, p(x_k | y_{1:k-1})}{p(y_k | y_{1:k-1})}$$

$$\propto p(y_k | x_k)\, p(x_k | y_{1:k-1})$$

- Note: the prediction and update equations are general.
  They provide a recursive solution to any filtering problem!

SELF-ASSESSMENT ON THE UPDATE STEP

Suppose $y_k = x_k + r_k$ where $r_k \sim \mathcal{N}(0, 1)$. The uncertainties in $p(x_k | y_{1:k})$ are then normally [select a suitable word below] than the uncertainties in $p(x_k | y_{1:k-1})$.

- smaller
- larger
- neither larger nor smaller

Only one answer applies.

OPTIMAL FILTER EXAMPLE

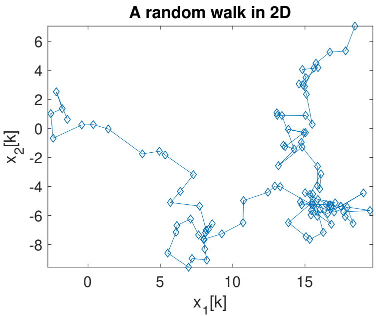
2D random walk with position observations

- Let us consider a 2D state vector, $\mathbf{x}_k = [x_1,\ x_2]^T$, with the following system model

$$\mathbf{x}_k = \mathbf{x}_{k-1} + \mathbf{q}_{k-1} \qquad \mathbf{q}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$$

$$\mathbf{y}_k = \mathbf{x}_k + \mathbf{r}_k \qquad \mathbf{r}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$$

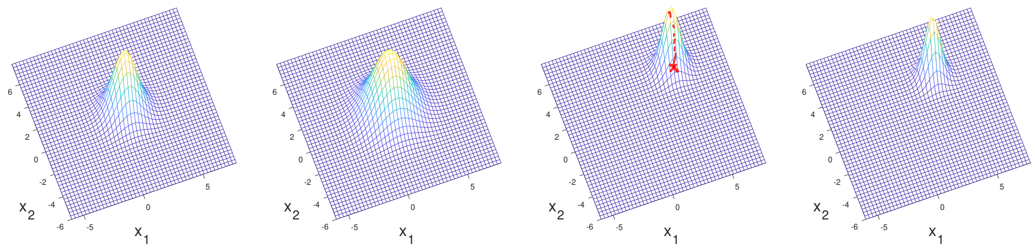
and $\mathbf{x}_0 \sim \mathcal{N}(\mathbf{0}, \mathbf{P}_0)$



PREDICTION AND UPDATE ILLUSTRATIONS

- Optimal filter recursion:

Prior | Predicted density | Likelihood | Updated density



$p(\mathbf{x}_{k-1}|\mathbf{y}_{1:k-1})$ $\quad$ $p(\mathbf{x}_k|\mathbf{y}_{1:k-1})$ $\times$ $p(\mathbf{y}_k|\mathbf{x}_k)$ $\propto$ $p(\mathbf{x}_k|\mathbf{y}_{1:k})$

- Note 1: uncertainties increase during prediction step.
- Note 2: posterior $\propto$ prior (predicted density) x likelihood.