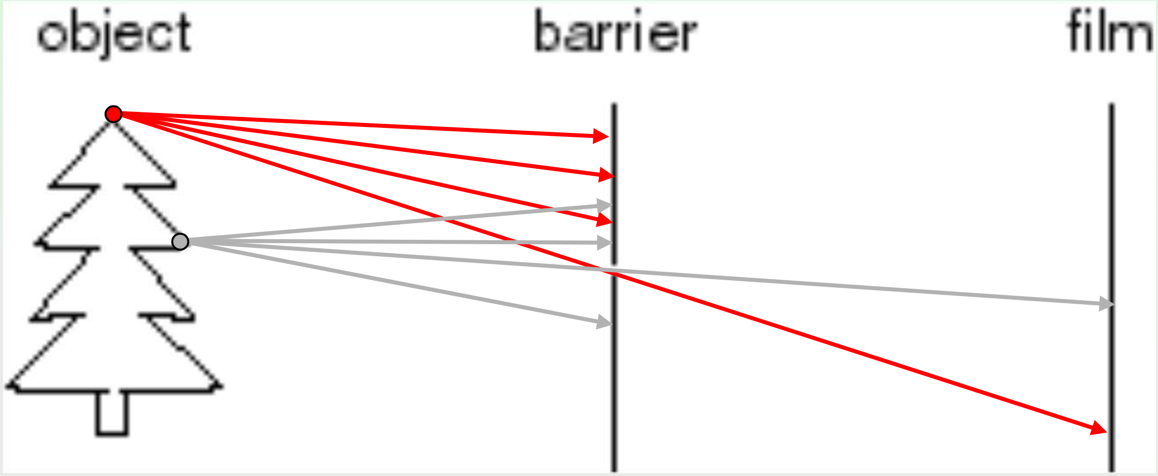
Geometric Camera Models

Lecturer: Sang Hwa Lee

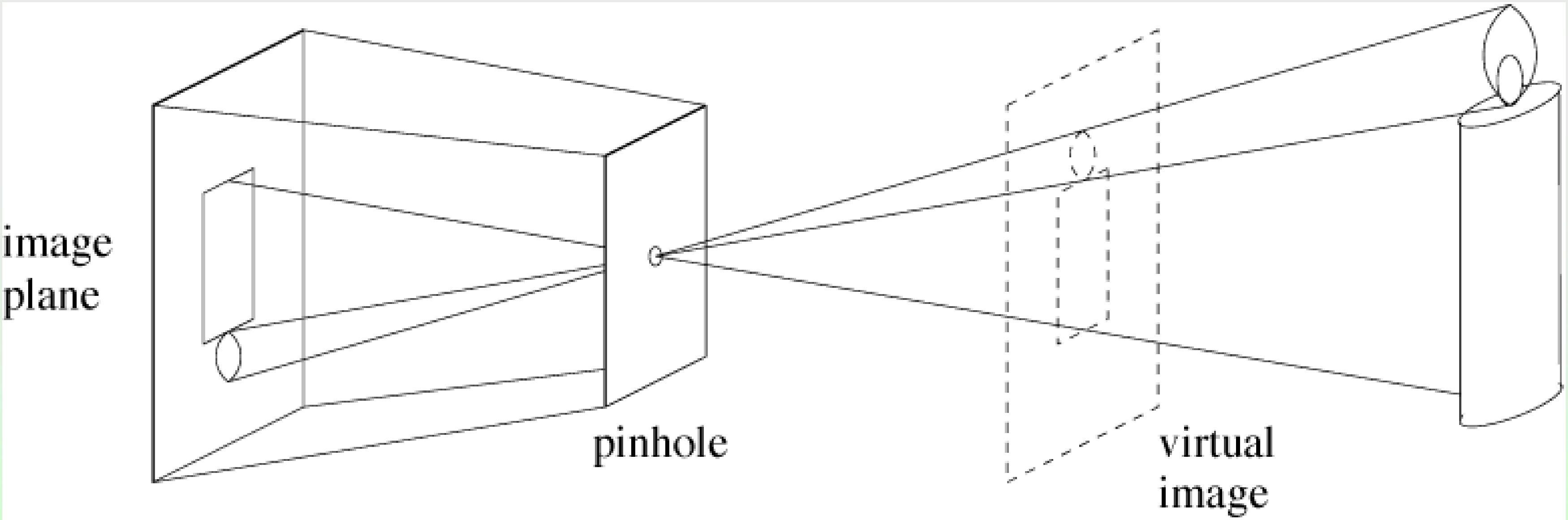
Pinhole Camera (I)



- Add a barrier to block off most of the rays
  - This reduces blurring
  - The opening known as the **aperture**
  - How does this transform the image?
    - it gets inverted

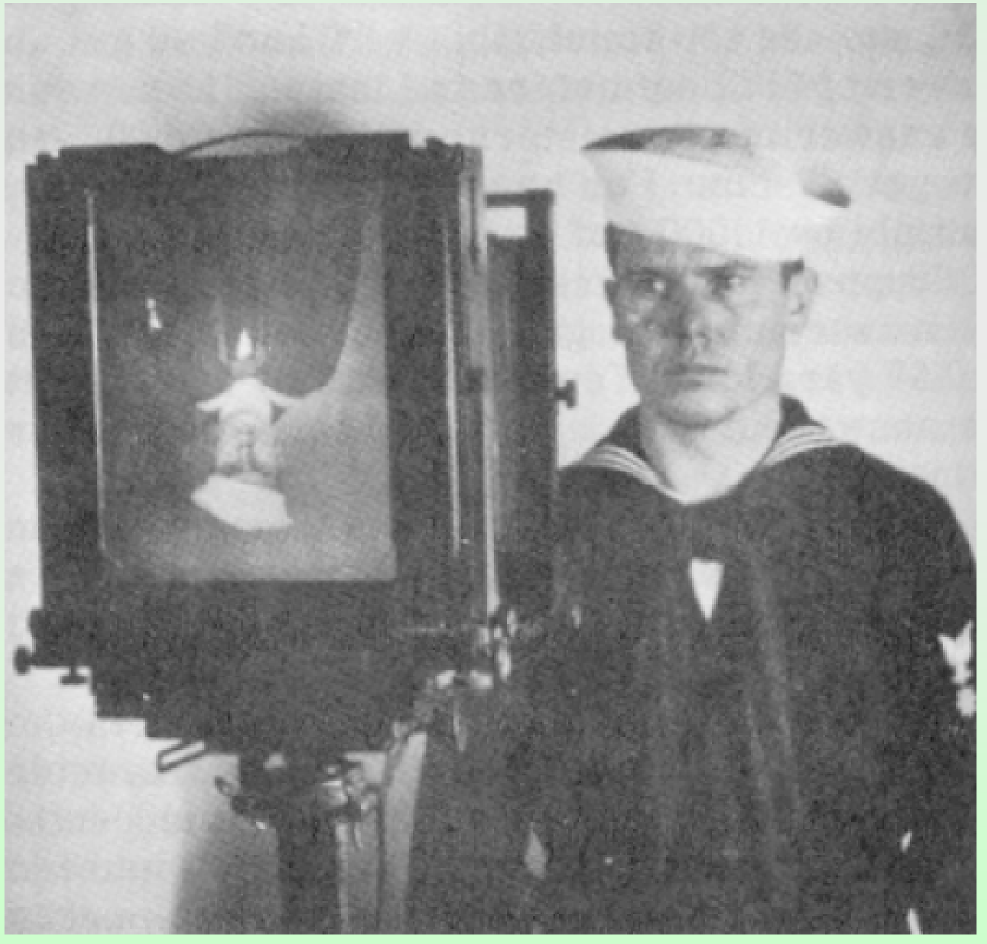
Pinhole Camera (II)

- Abstract camera model: box with a small hole in it
  - By Brunelleschi in 15th
- Pinhole cameras work in practice
  - Acceptable approximation



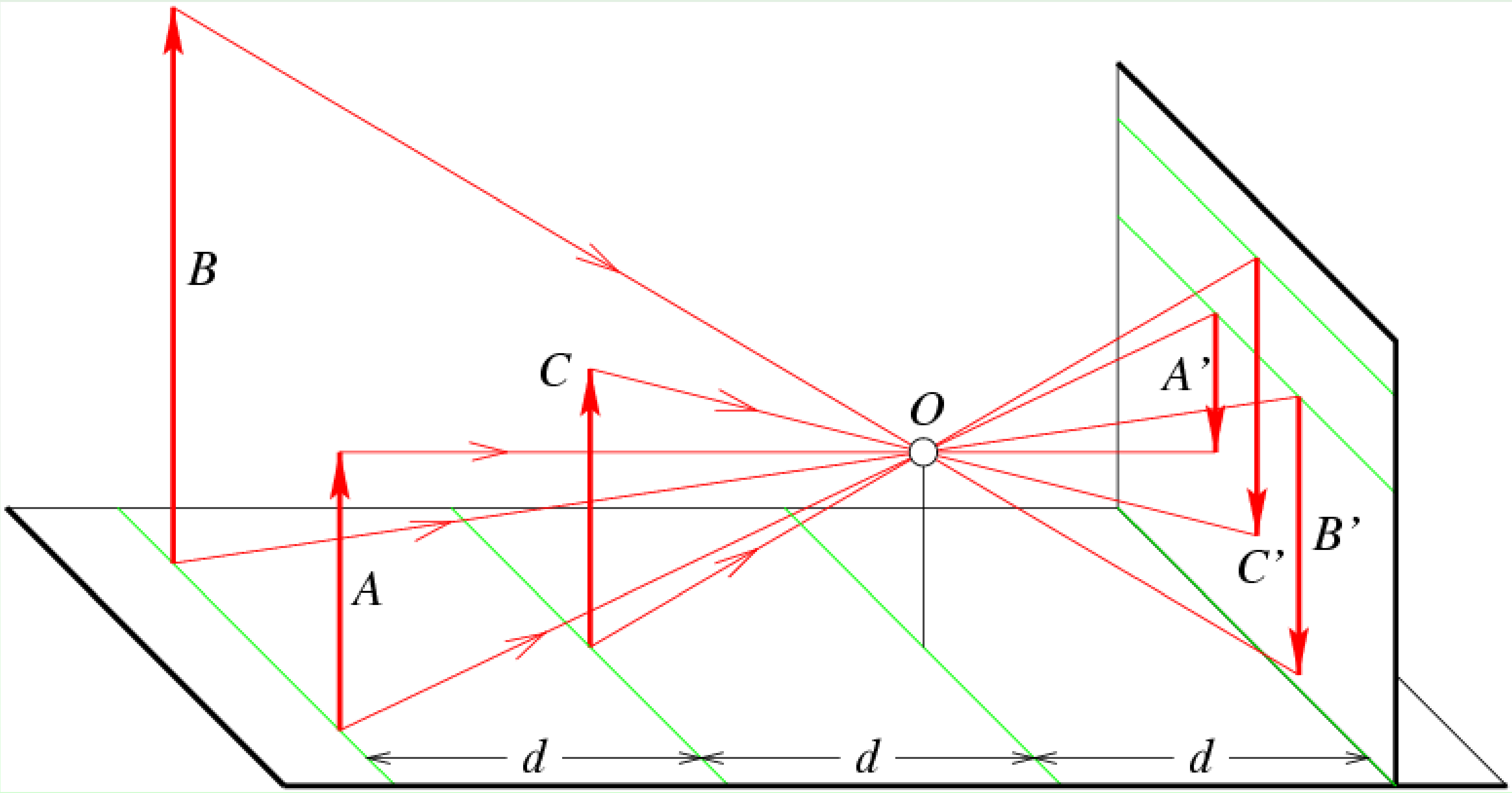
Perspective effects (I)

Inverted image



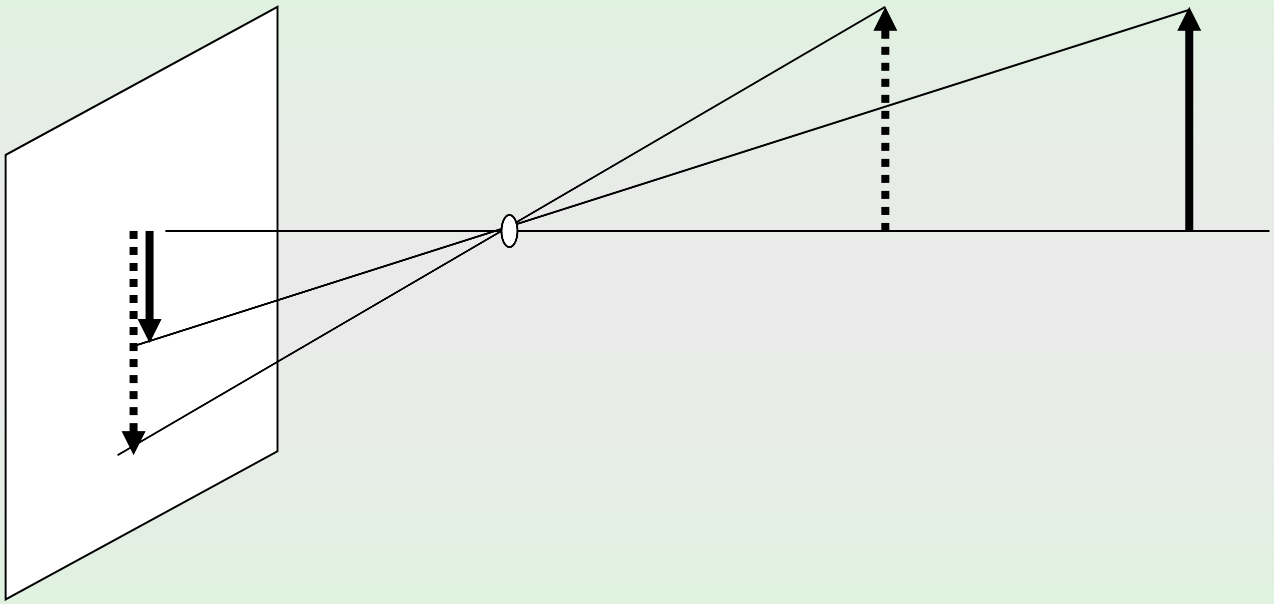
Perspective effects (II)

Far objects become smaller than closer one



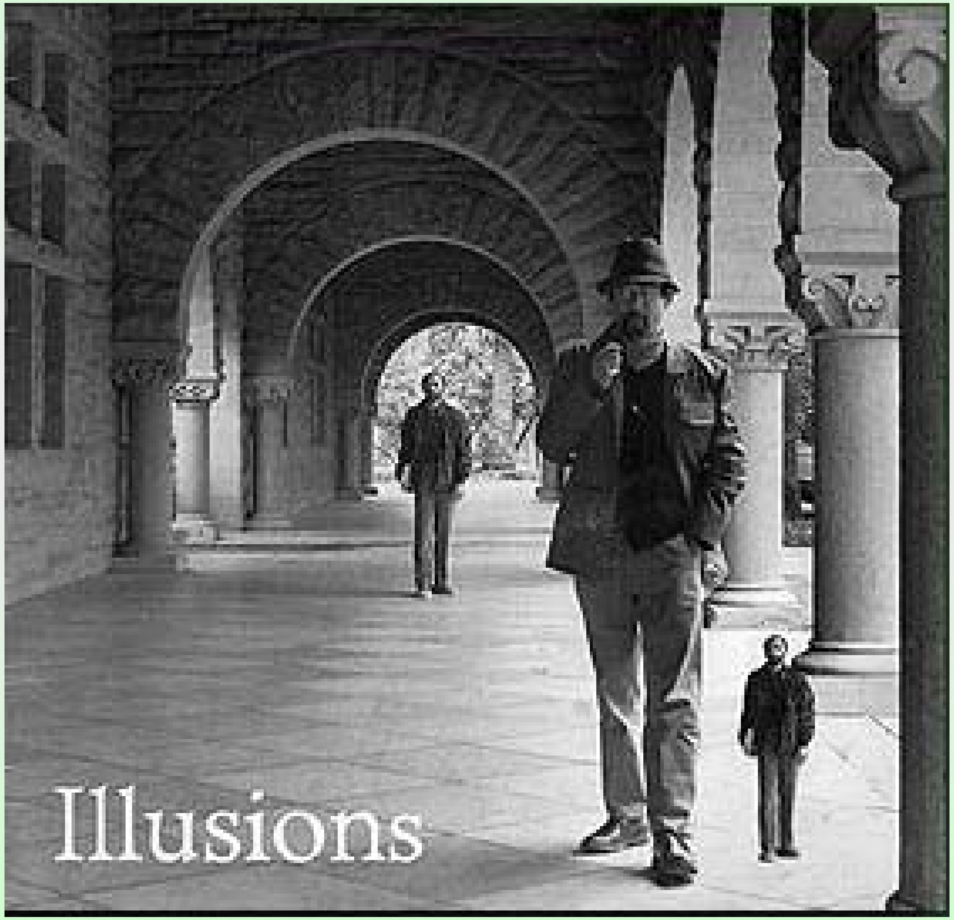
Perspective effects (III)

Measuring distance



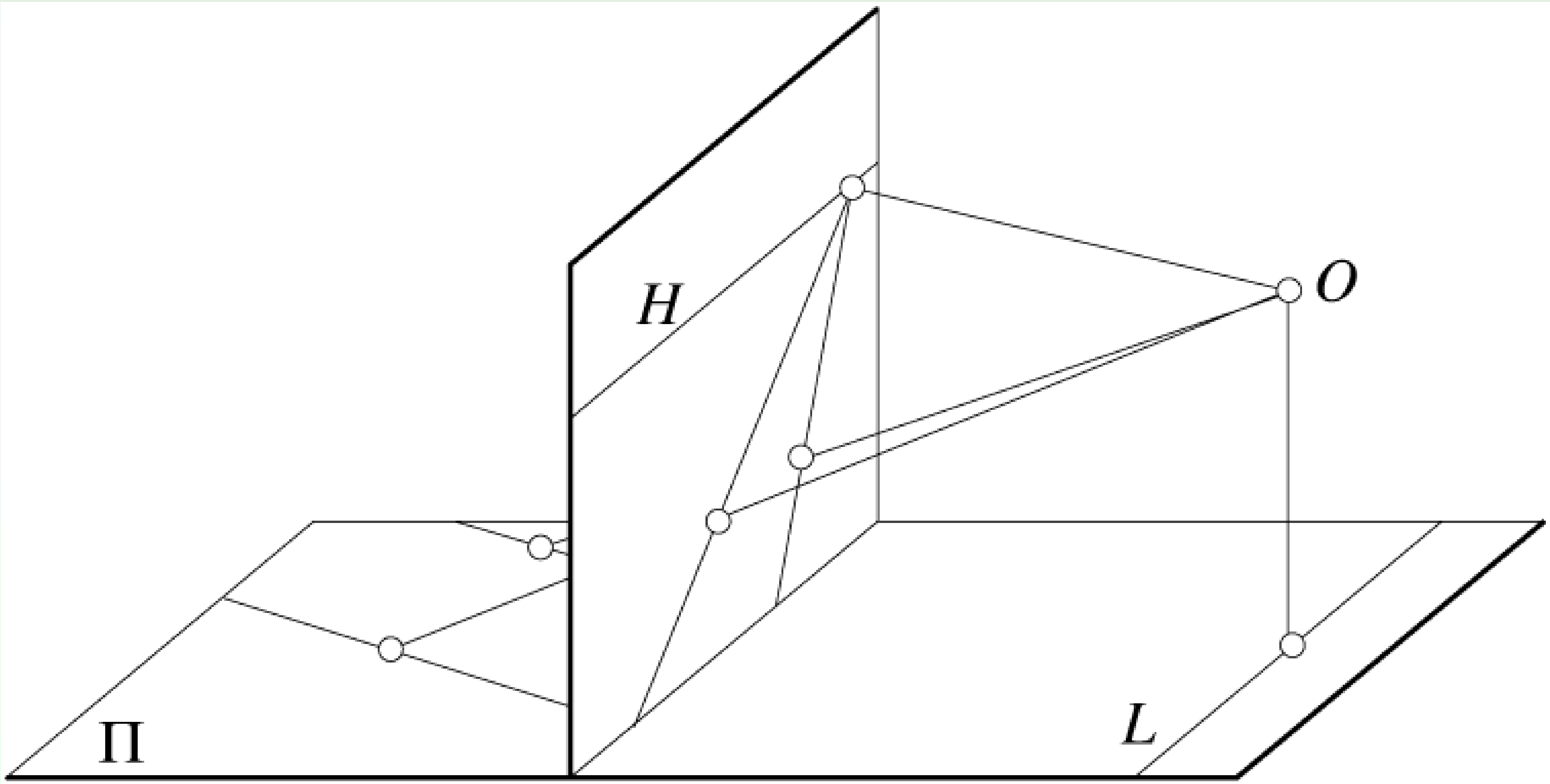
- Object size decreases with distance to the pinhole
- There, given a single projection,
- if we know the size of the object we can know how far it is.
- But for objects of unknown size, the 3D information seems to be lost.

Perspective effects (IV)



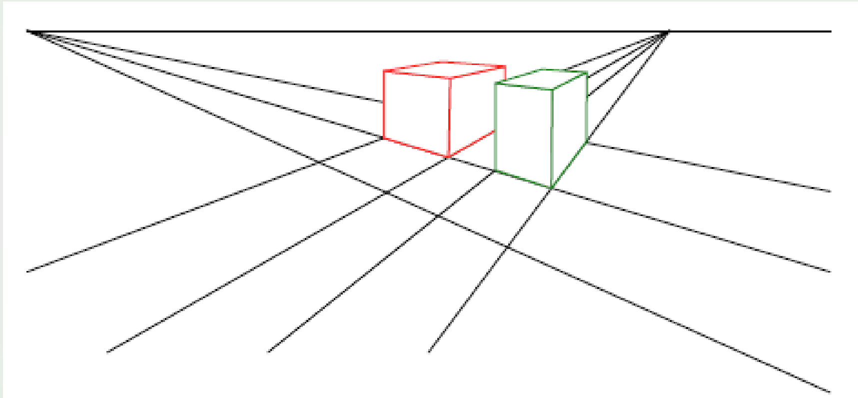
Perspective effects (V)

Two parallel lines intersect at the horizon (vanishing points)

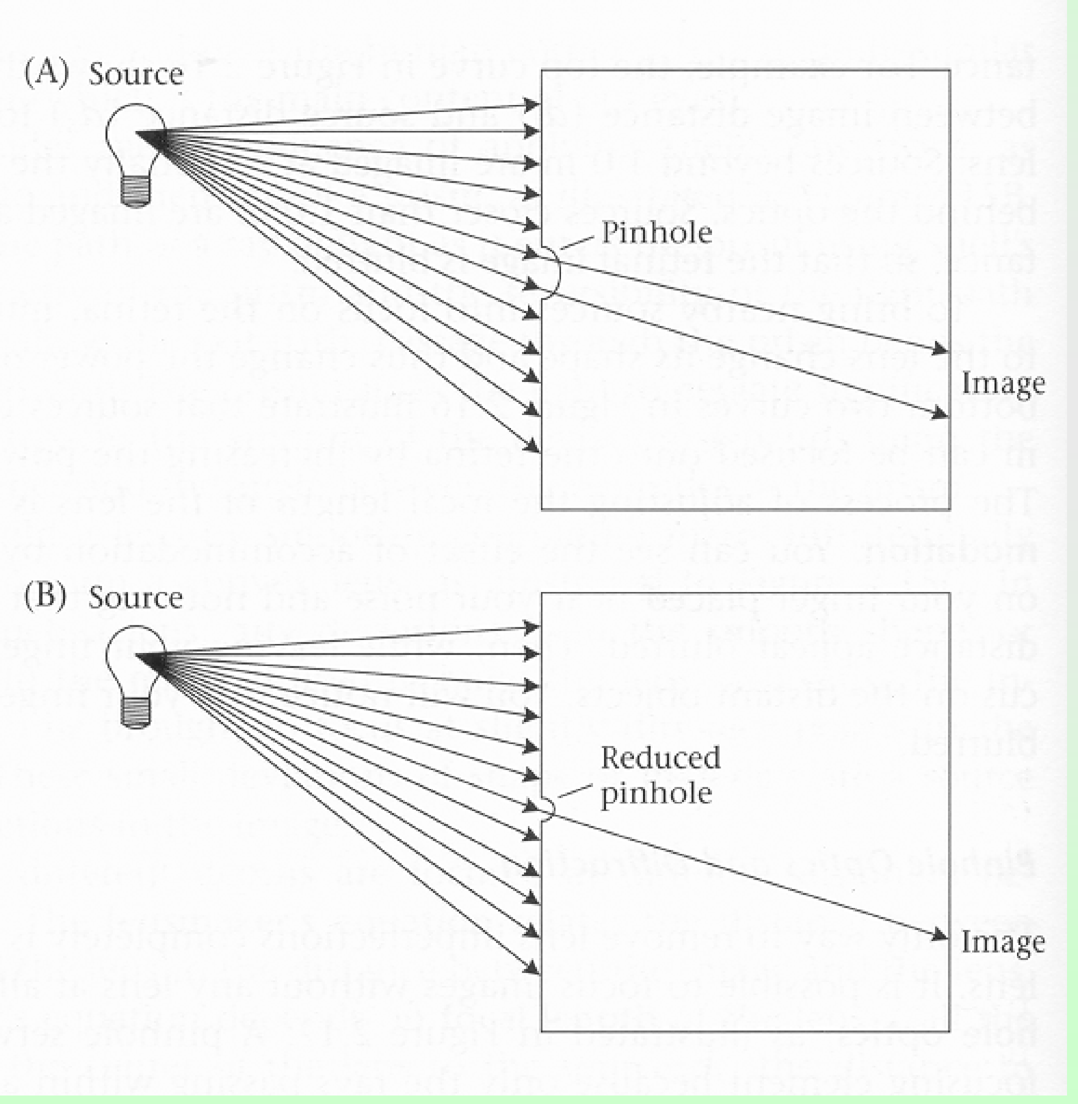


Perspective effects (VI)

- Each set of parallel lines (=direction) meets at a different point
  - The *vanishing point* for this direction
  - Points at infinity
- Sets of parallel lines on the same plane lead to *collinear* vanishing points.
  - The line is called the *horizon* for that plane



Effect of Aperture (Pinhole) Size



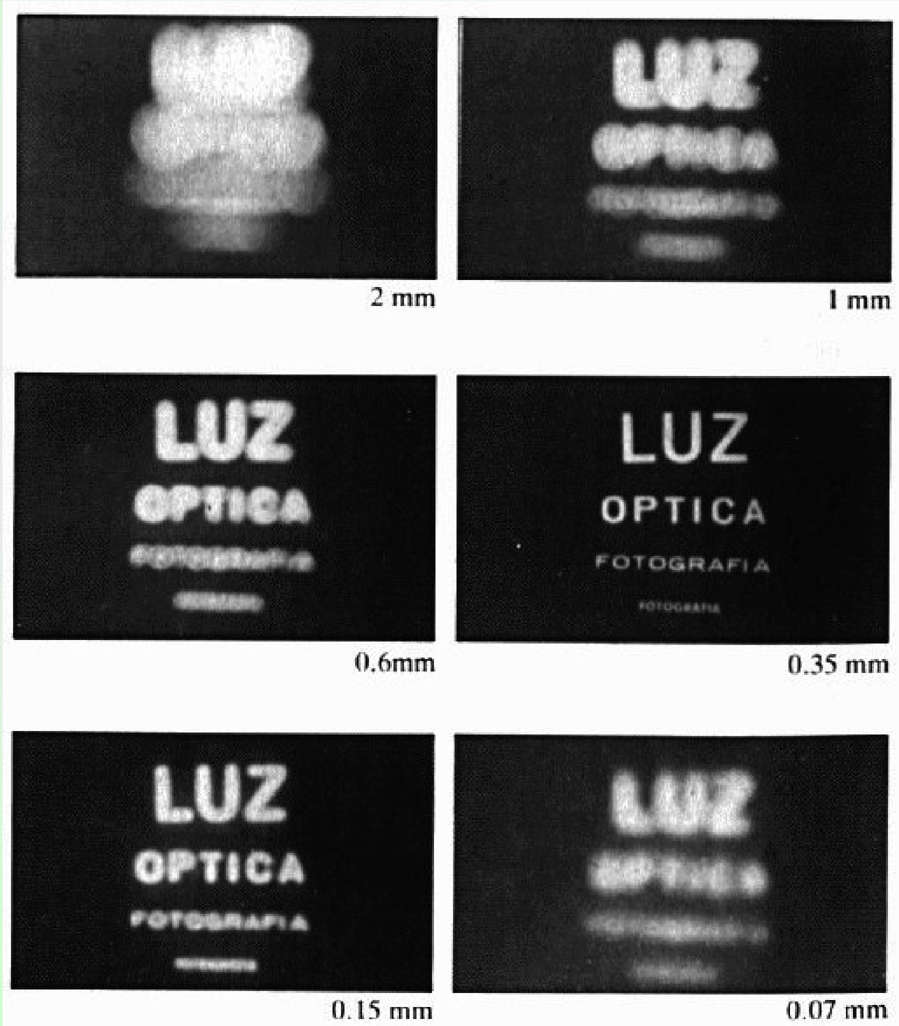
Effect of Aperture (Pinhole) Size

Pinhole too big:

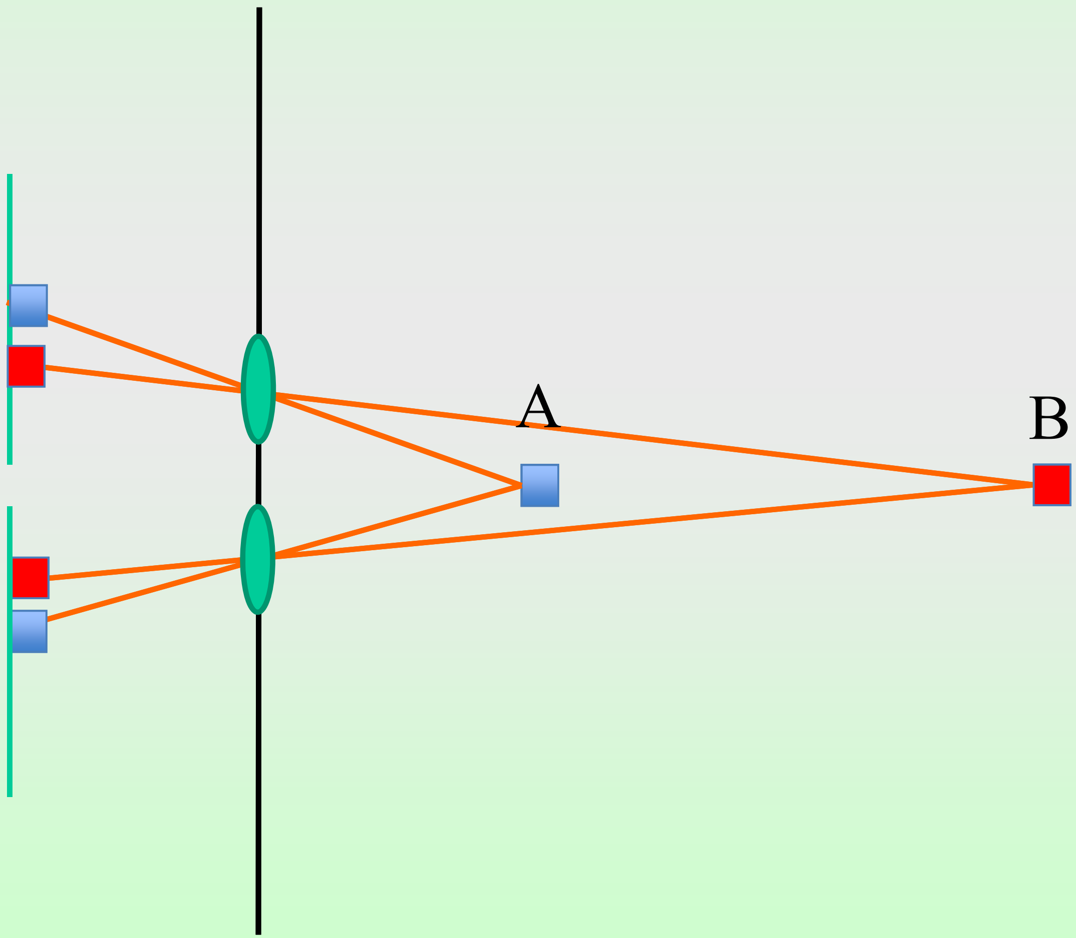
- Many directions are averaged
- Blurring of the image due to out-of-focus

Pinhole too small:

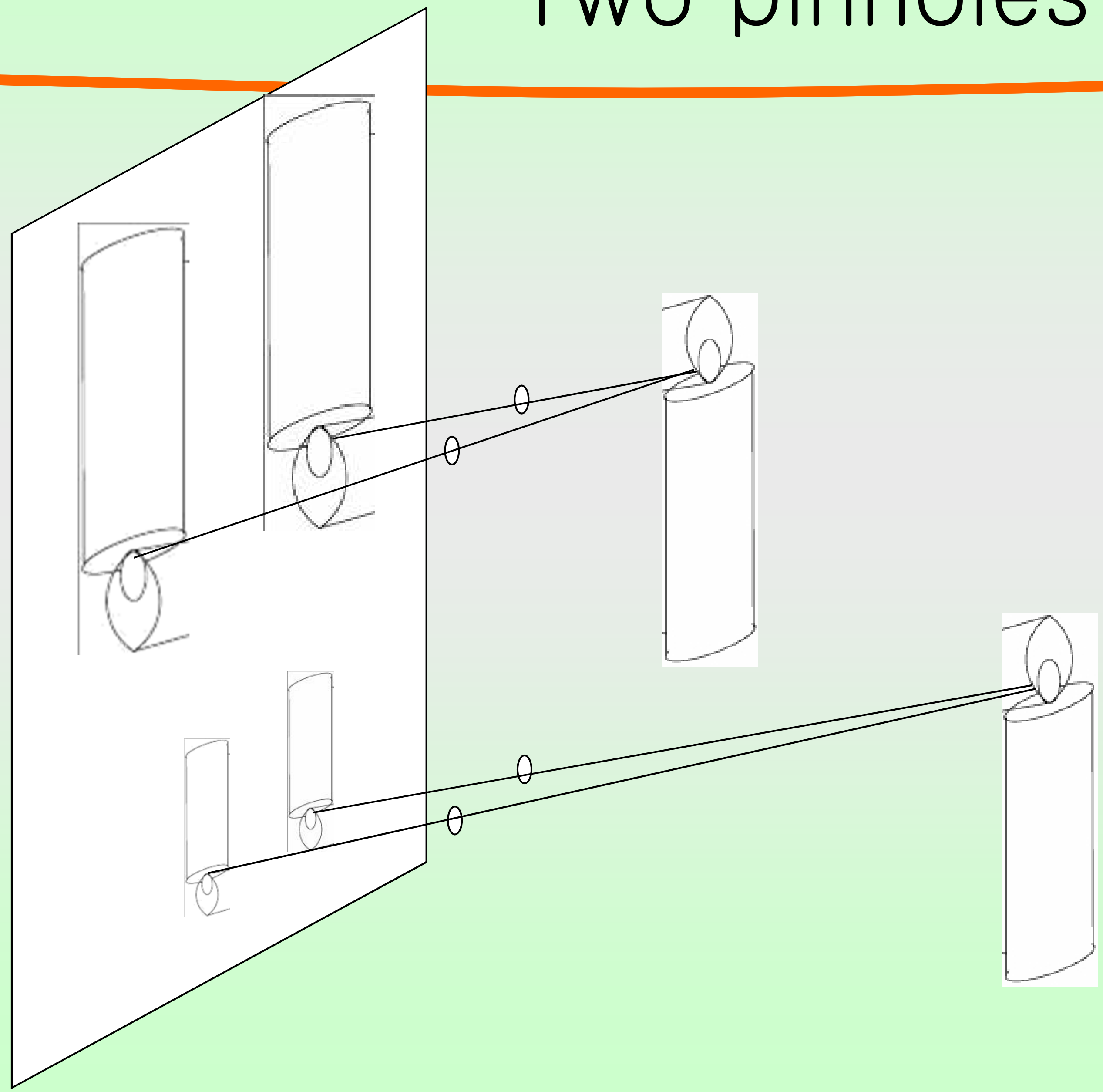
- Diffraction effects blur the image
- Generally, pinhole cameras are *dark*, because a very small set of rays from a particular point hits the screen.



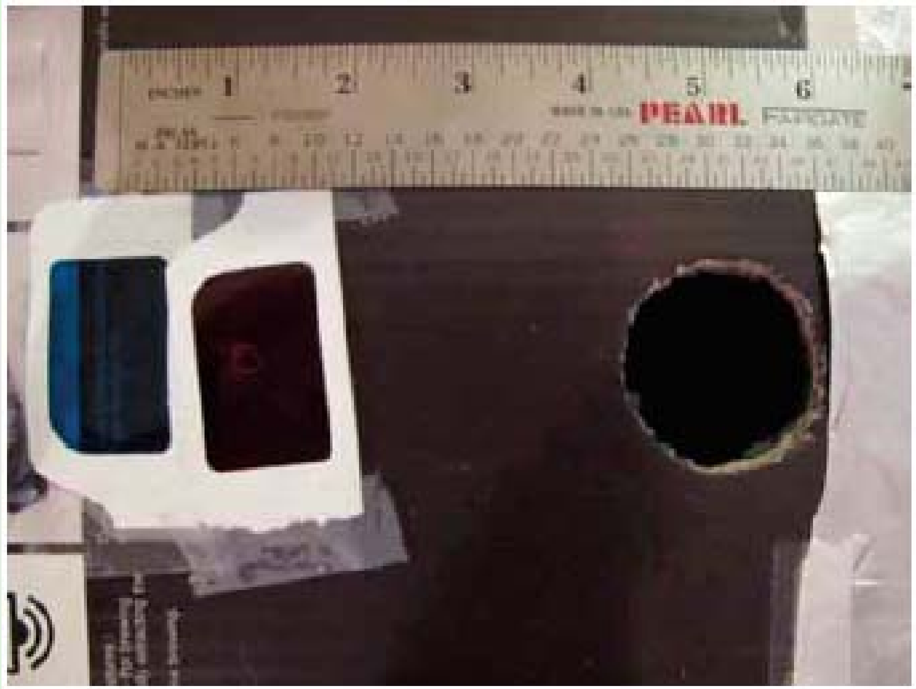
Two Pinholes (I)



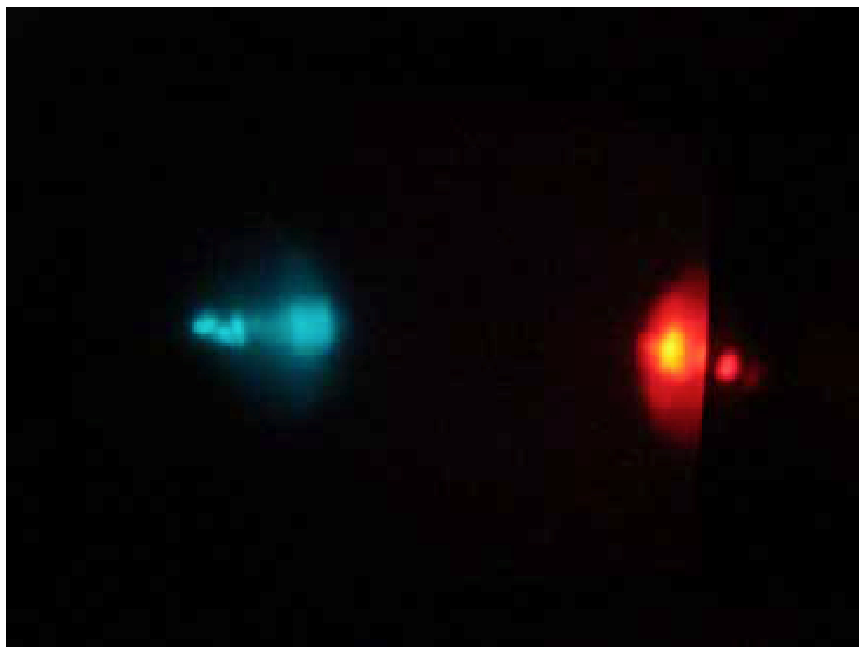
Two pinholes (II)

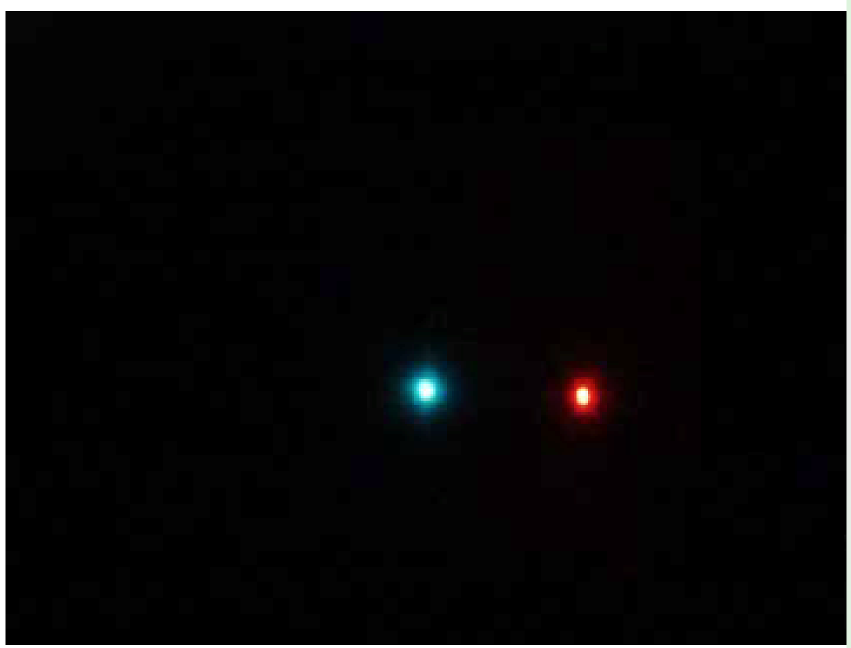


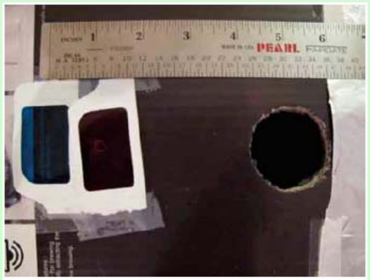
Anaglyph pinhole camera



Anaglyph pinhole camera





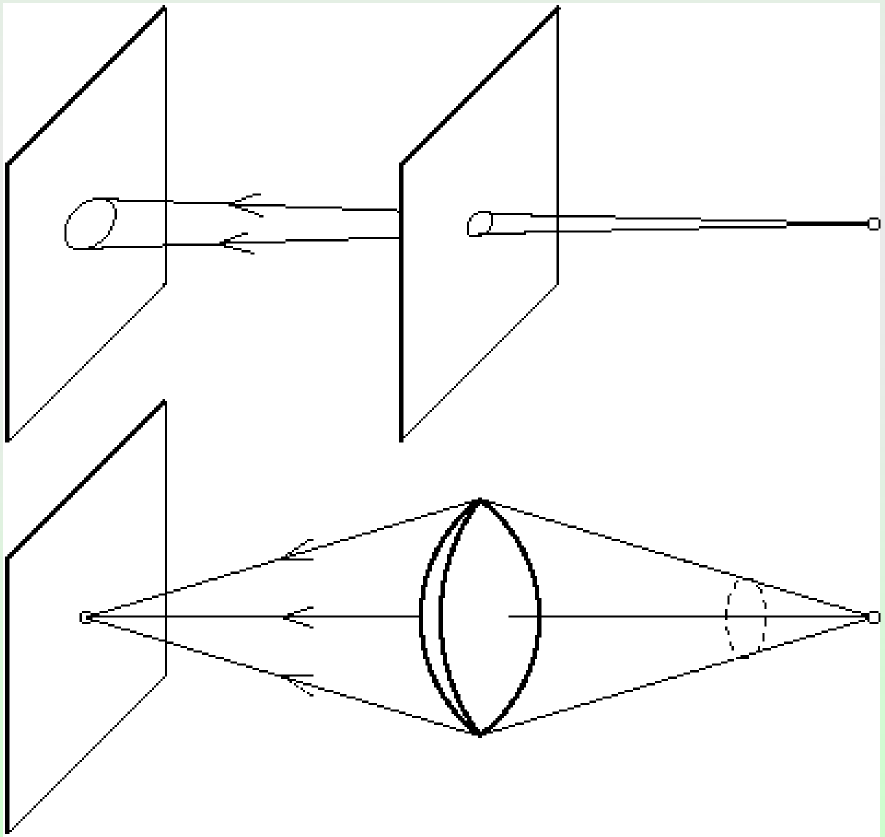


Anaglyph pinhole camera

Anaglyph

The Reason for Lenses

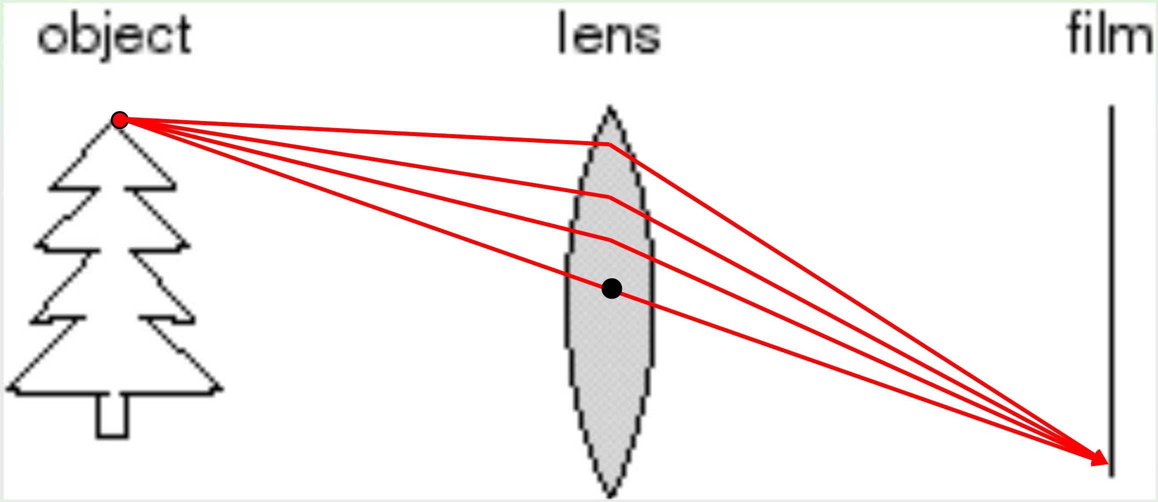
- Gathering light rays of the cone
- Sharp focusing



Without lens

With lens

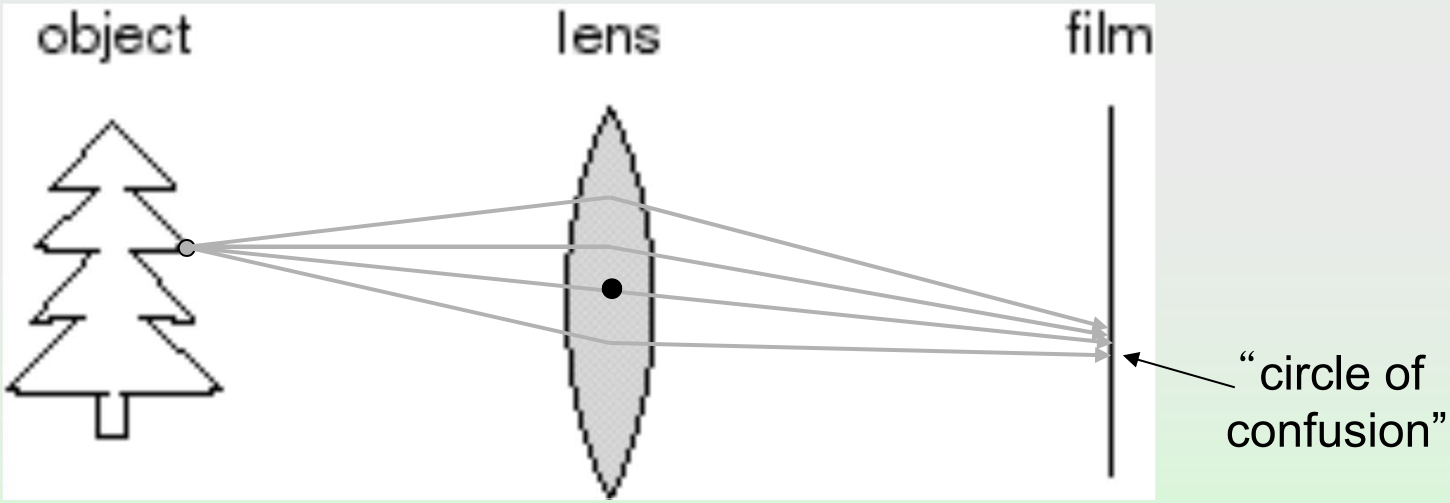
Adding a Lens



- A lens focuses light onto the film
  - There is a specific distance at which objects are "in focus"
    - other points project to a "circle of confusion" in the image
  - Changing the shape of the lens changes this distance

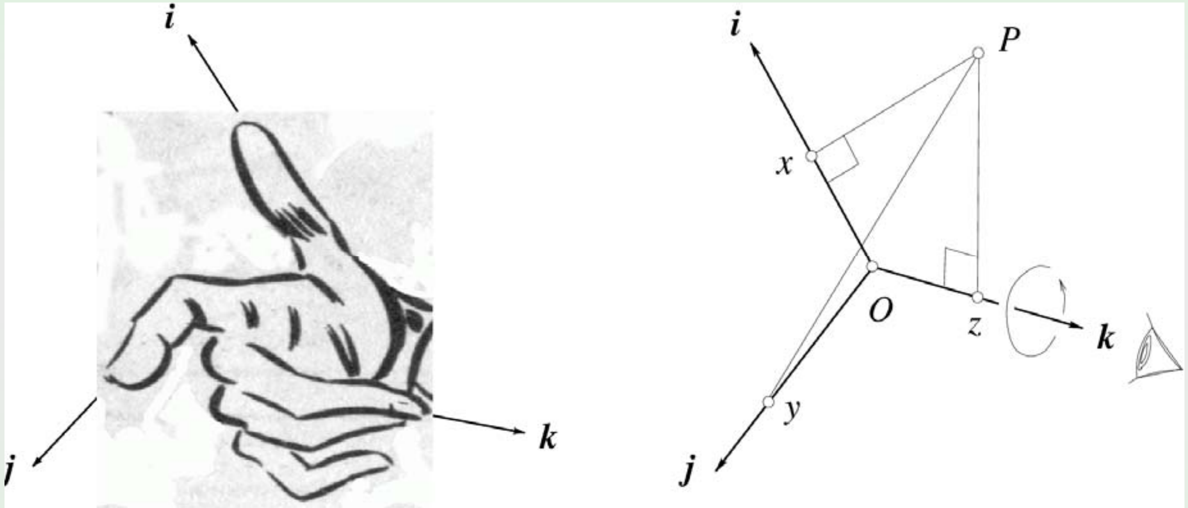
Adding a Lens

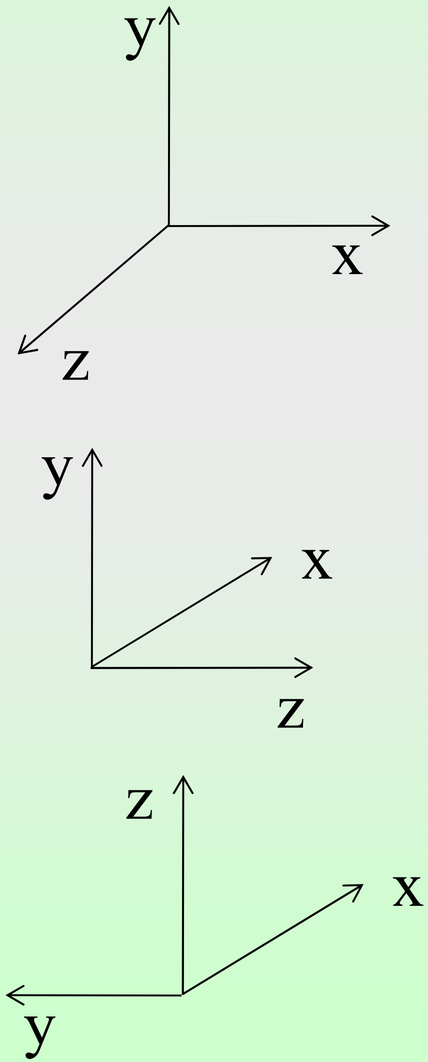
- Out of Focusing according to the distances
  - focus is a cue to perceive 3-D information



Euclidean Coordinates System

Right-handed system





$$\begin{cases} x = \overrightarrow{OP}.\mathbf{i} \\ y = \overrightarrow{OP}.\mathbf{j} \\ z = \overrightarrow{OP}.\mathbf{k} \end{cases} \Leftrightarrow \quad \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \Leftrightarrow \quad \mathbf{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Homogeneous Coordinates

❑ One more coordinates

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D homogeneous coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D homogeneous coordinates

❑ Converting *from* homogeneous coordinates

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Point and Line in 2D HC (I)

- **Homogenous representation**
  - Point: $\mathbf{x} = (x, y)^T \Rightarrow (x, y, 1)^T$
  - Line: $\mathbf{l} = ax+by+c=0 \Rightarrow (a, b, c)^T$
  - Point on line: $\mathbf{l}^T\mathbf{x} = 0$
- **Degrees of freedom (DOF)**
  - Point: two components (x, y-coordinate)
  - Line: two parameters (two independent ratio $\{a:b:c\}$)
- **Intersection point of two lines**
  - For the intersection point, $\mathbf{x}$, $\mathbf{l}_1^T\,\mathbf{x} = \mathbf{l}_2^T\,\mathbf{x} = 0$
  - $\mathbf{l}_1^T(\mathbf{l}_1 \times \mathbf{l}_2) = \mathbf{l}_2^T(\mathbf{l}_1 \times \mathbf{l}_2) = 0$
  - $\mathbf{x} = (\mathbf{l}_1 \times \mathbf{l}_2)$

Point and Line in 2D HC (II)

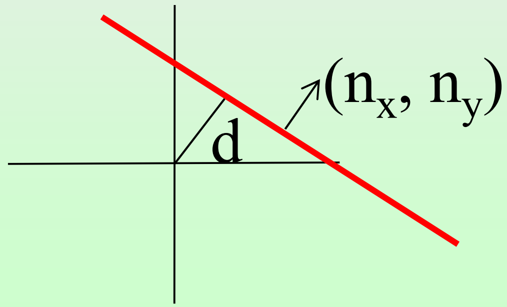
- 2D Points:

$$p = \begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow p' = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad p' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} \longrightarrow p = \begin{bmatrix} x'/w' \\ y'/w' \end{bmatrix}$$

2D Lines: $ax + by + c = 0$

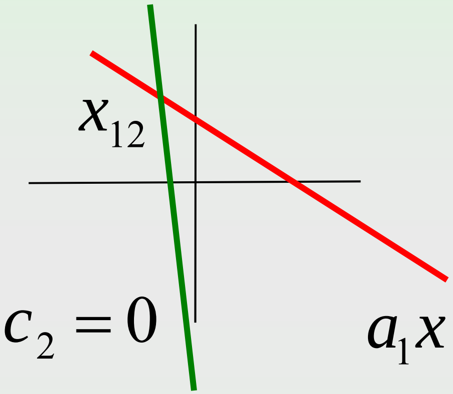
$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$l = \begin{bmatrix} a & b & c \end{bmatrix} \Rightarrow \begin{bmatrix} n_x & n_y & d \end{bmatrix}$$



Point and Line in 2D HC (III)

Intersection between two lines:



$$a_2 x + b_2 y + c_2 = 0 \qquad a_1 x + b_1 y + c_1 = 0$$

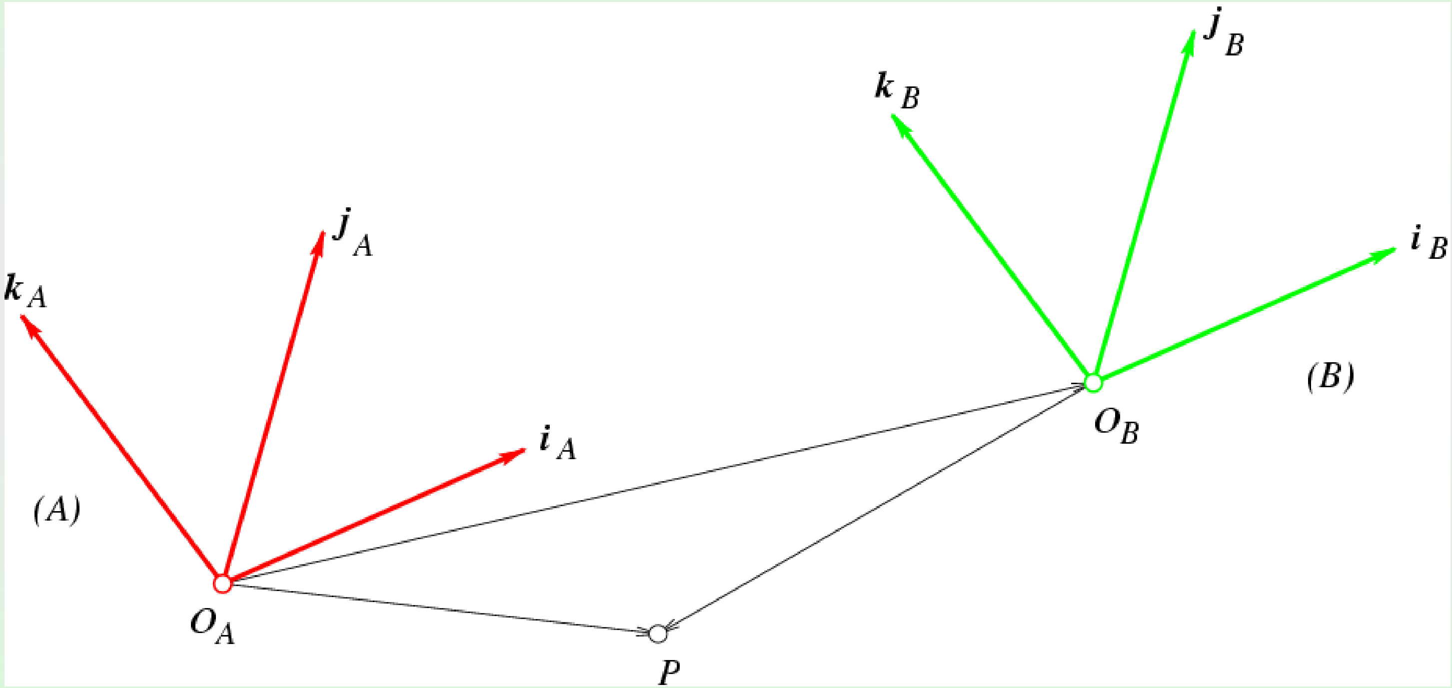
$$\left.\begin{array}{l} l_1 = \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \\[2em] l_2 = \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} \end{array}\right\} x_{12} = l_1 \times l_2$$

Point and Line in 3D HC

- Homogenous representation
  - Point: $\mathbf{x} = (x, y, z)^T \Rightarrow (x, y, z, 1)^T$
  - Plane: $\mathbf{p} = ax+by+cz+d=0 \Rightarrow (a, b, c, d)^T$
  - Point on plane: $\mathbf{p}^T\mathbf{x} = 0$
- Degrees of freedom (DOF)
  - Plane: three parameters (three independent ratio $\{a:b:c:d\}$)
  - Point: three components (x, y, z−coordinate)
  - Line: four component
- Three points define a plane.
- Intersection of three planes define a point.

Coordinates changes (I)

Pure translation

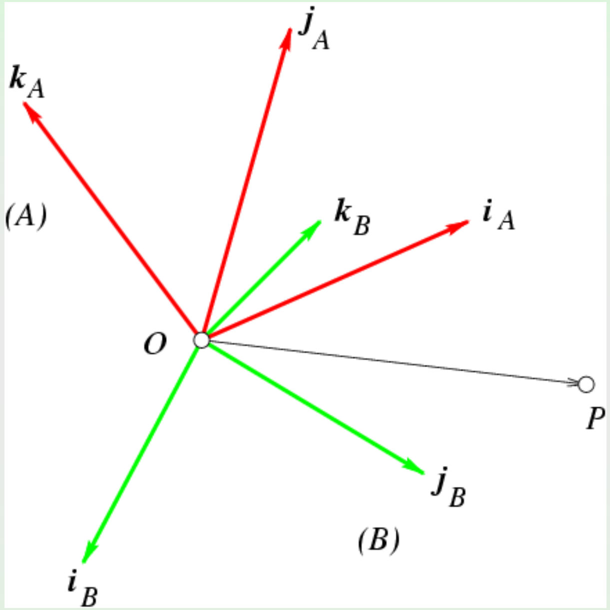


$$\overrightarrow{O_B P} = \overrightarrow{O_B O_A} + \overrightarrow{O_A P} \;, \quad {}^{B}P = {}^{A}P + {}^{B}O_A$$

Coordinates changes (II)

Pure rotation

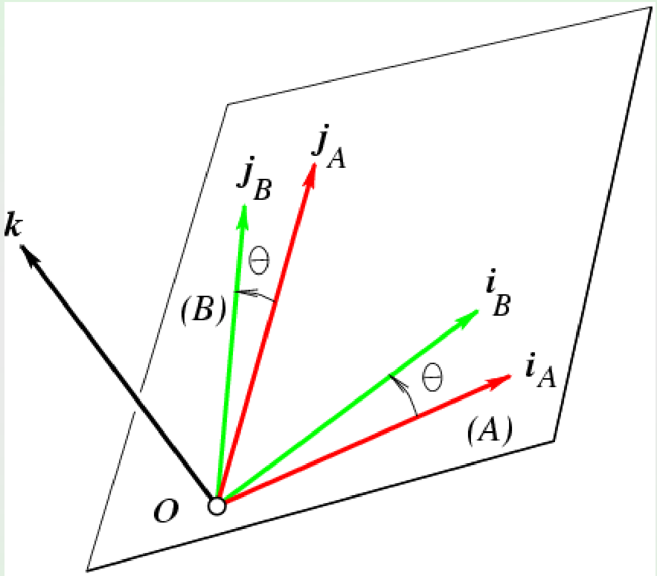
- The inverse is its transpose.
- Determinant is one.
- Associative
- Product of two rotation matrices is also a rotation
- Commutative ???

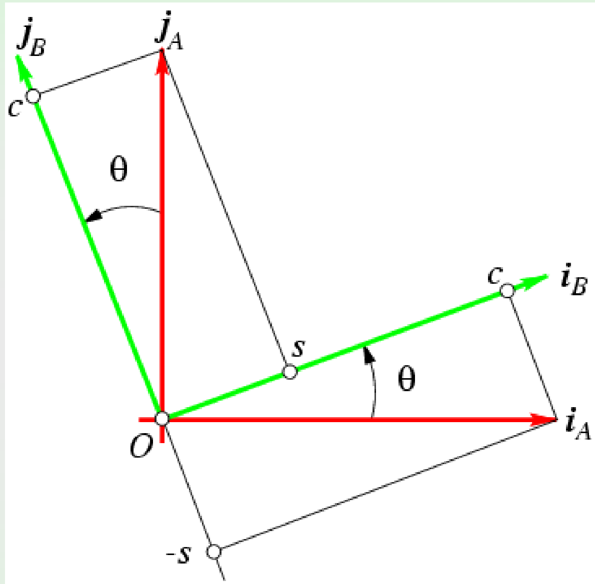


$$
{}^{B}_{A}R = \begin{bmatrix} \mathbf{i}_A.\mathbf{i}_B & \mathbf{j}_A.\mathbf{i}_B & \mathbf{k}_A.\mathbf{i}_B \\ \mathbf{i}_A.\mathbf{j}_B & \mathbf{j}_A.\mathbf{j}_B & \mathbf{k}_A.\mathbf{j}_B \\ \mathbf{i}_A.\mathbf{k}_B & \mathbf{j}_A.\mathbf{k}_B & \mathbf{k}_A.\mathbf{k}_B \end{bmatrix} = \begin{bmatrix} {}^{A}\mathbf{i}_B^T \\ {}^{A}\mathbf{j}_B^T \\ {}^{A}\mathbf{k}_B^T \end{bmatrix} = \begin{bmatrix} {}^{B}\mathbf{i}_A & {}^{B}\mathbf{j}_A & {}^{B}\mathbf{k}_A \end{bmatrix}
$$

Coordinates changes (III)

Rotation about the z axis

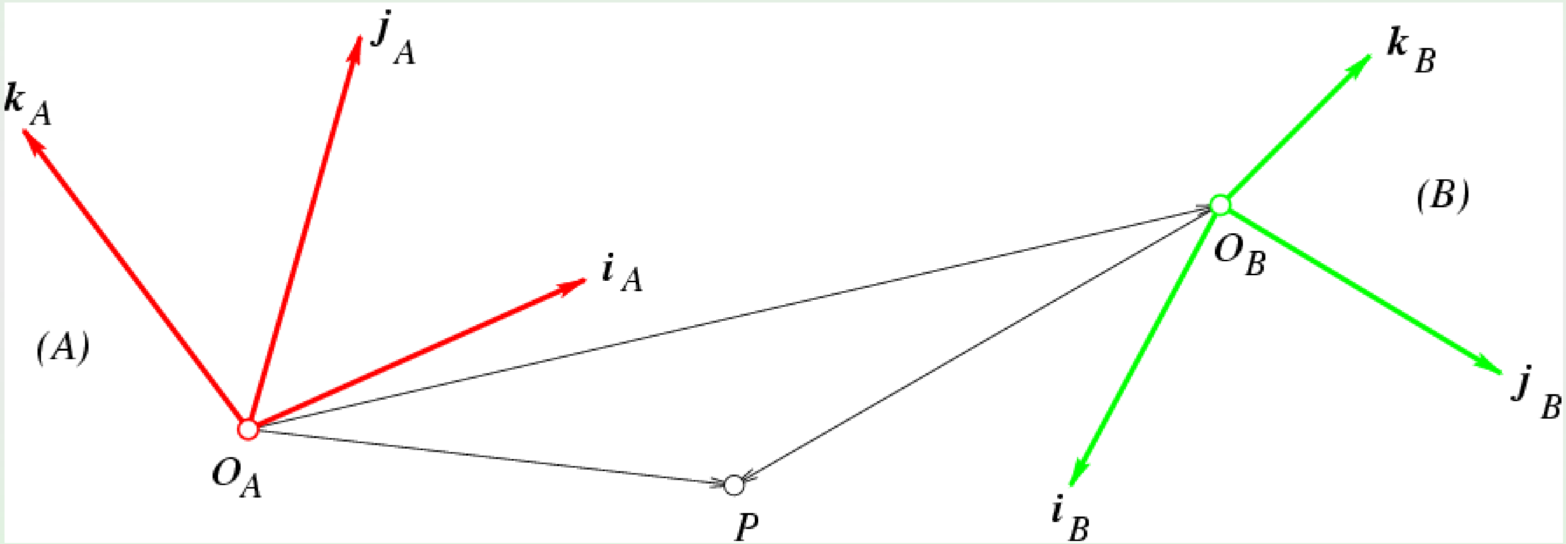




$$
{}^{B}_{A}R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Coordinates changes (IV)

- Rigid transformation: rotation +translation
- non-commutative between rotation and translation



$$^{B}P = {}^{B}_{A}R\,{}^{A}P + {}^{B}O_{A}$$

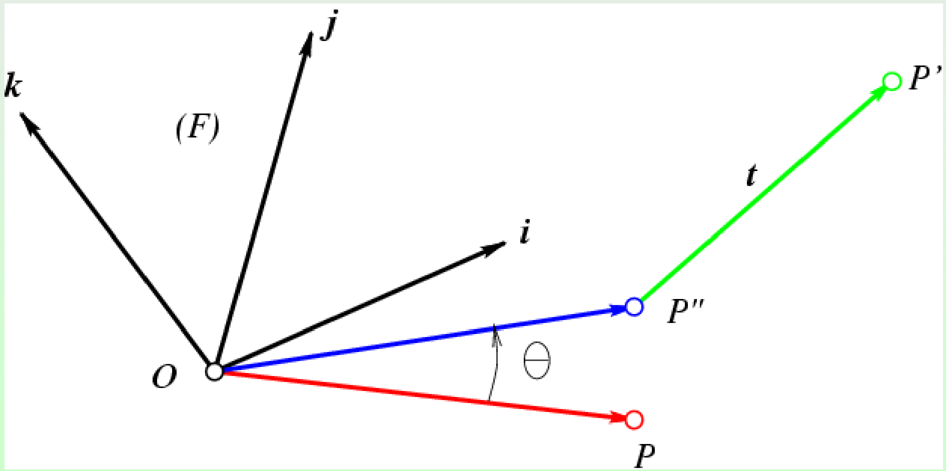
→ B를 좌표원점으로 하고
A를 중심으로 회전한 후,
다시 B를 기준으로 좌표설정

Coordinates changes (V)

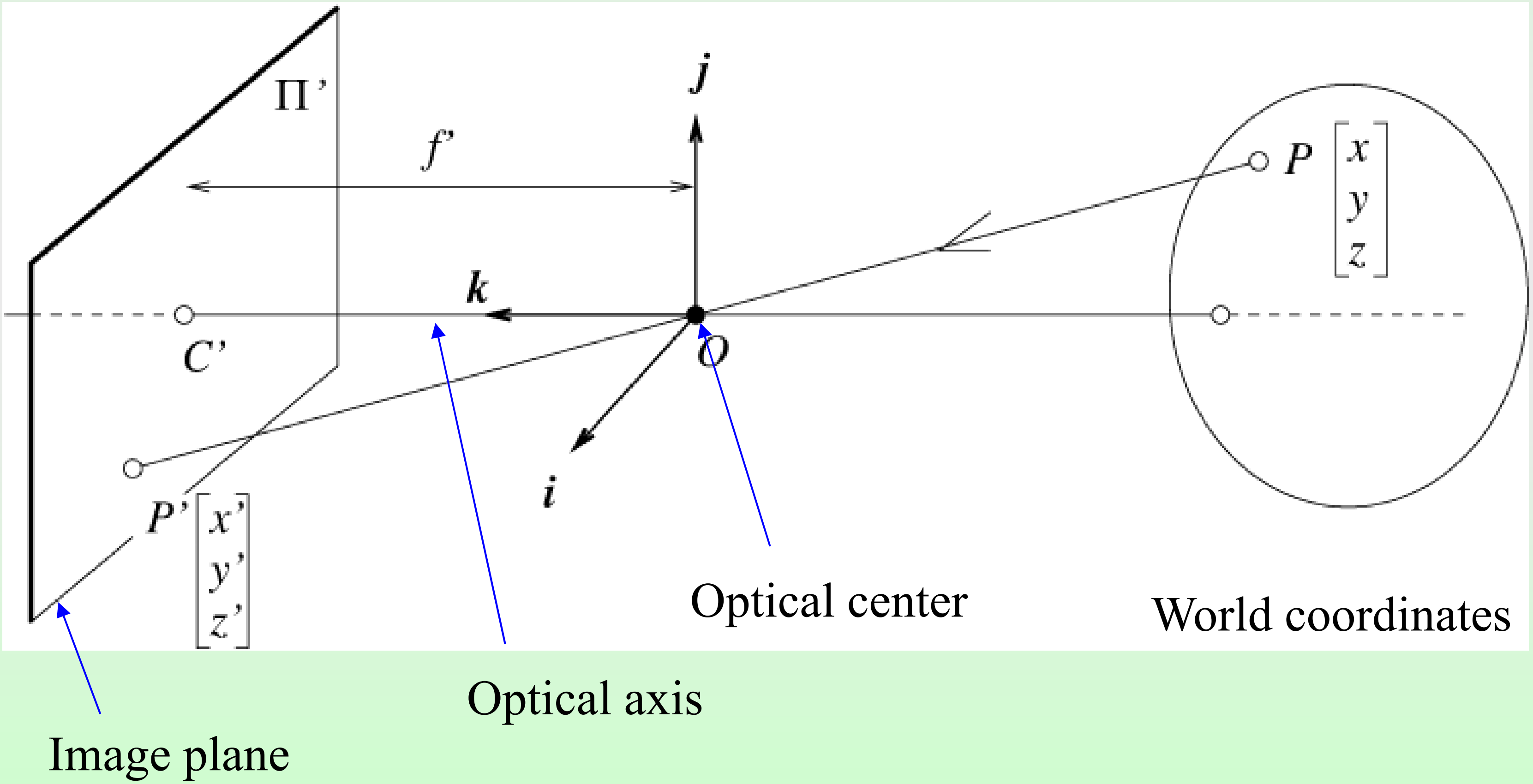
Homogeneous representation of rigid transformations using block matrix

$$\begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R & {}^{B}O_{A} \\ \mathbf{0}^{T} & 1 \end{bmatrix} \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{B}_{A}R\,{}^{A}P + {}^{B}O_{A} \\ 1 \end{bmatrix} = {}^{B}_{A}T \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix}$$

$$ {}^{F}P' = \mathcal{R}\,{}^{F}P + \boldsymbol{t} \Longleftrightarrow \begin{pmatrix} {}^{F}P' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathcal{R} & \boldsymbol{t} \\ \mathbf{0}^{T} & 1 \end{pmatrix} \begin{pmatrix} {}^{F}P \\ 1 \end{pmatrix}$$



Equation of Projection (I)



Equation of Projection (II)

❑ In the Cartesian coordinates:

- We have, by similar triangles, that

  $(x, y, z) \rightarrow (fx/z, fy/z, f)$

- Ignore the third coordinate, and get

$$(x, y, z) \rightarrow (f\frac{x}{z}, f\frac{y}{z})$$

- Not pixel ordering, but the physical distance
- In image coordinates for pixel sites, we have to know the physical distance (radius) between adjacent two pixels.

Equation of Projection (III)

Projection equations

- Compute intersection with PP of ray from (x,y,z) to COP
- Derived using similar triangles

$$(x, y, z) \rightarrow (-d\frac{x}{z},\ -d\frac{y}{z},\ -d)$$

- We get the projection by throwing out the last coordinate:

$$(x, y, z) \rightarrow (-d\frac{x}{z},\ -d\frac{y}{z})$$

Is projection linear ?

- No. Division by z is nonlinear
  - Homogeneity (O), superposition (x)

Equation of Projection (V)

- Homogenous presentation of projective transformation
  - Mapping 3D point in the world coordinates onto 2D point in image coordinates
- Matrix equation

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix}$$

Equation of Projection (VI)

- Projection is the matrix multiplication in the homogeneous coordinates:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ -z/d \end{bmatrix} \Rightarrow (-d\frac{x}{z}, \quad -d\frac{y}{z})$$

- This is known as **perspective projection**
  - The matrix is the projection matrix
  - Can also formulate as a 4x4

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ -z/d \end{bmatrix} \Rightarrow (-d\frac{x}{z}, \quad -d\frac{y}{z})$$

divide by fourth coordinate

Equation of Projection (VII)

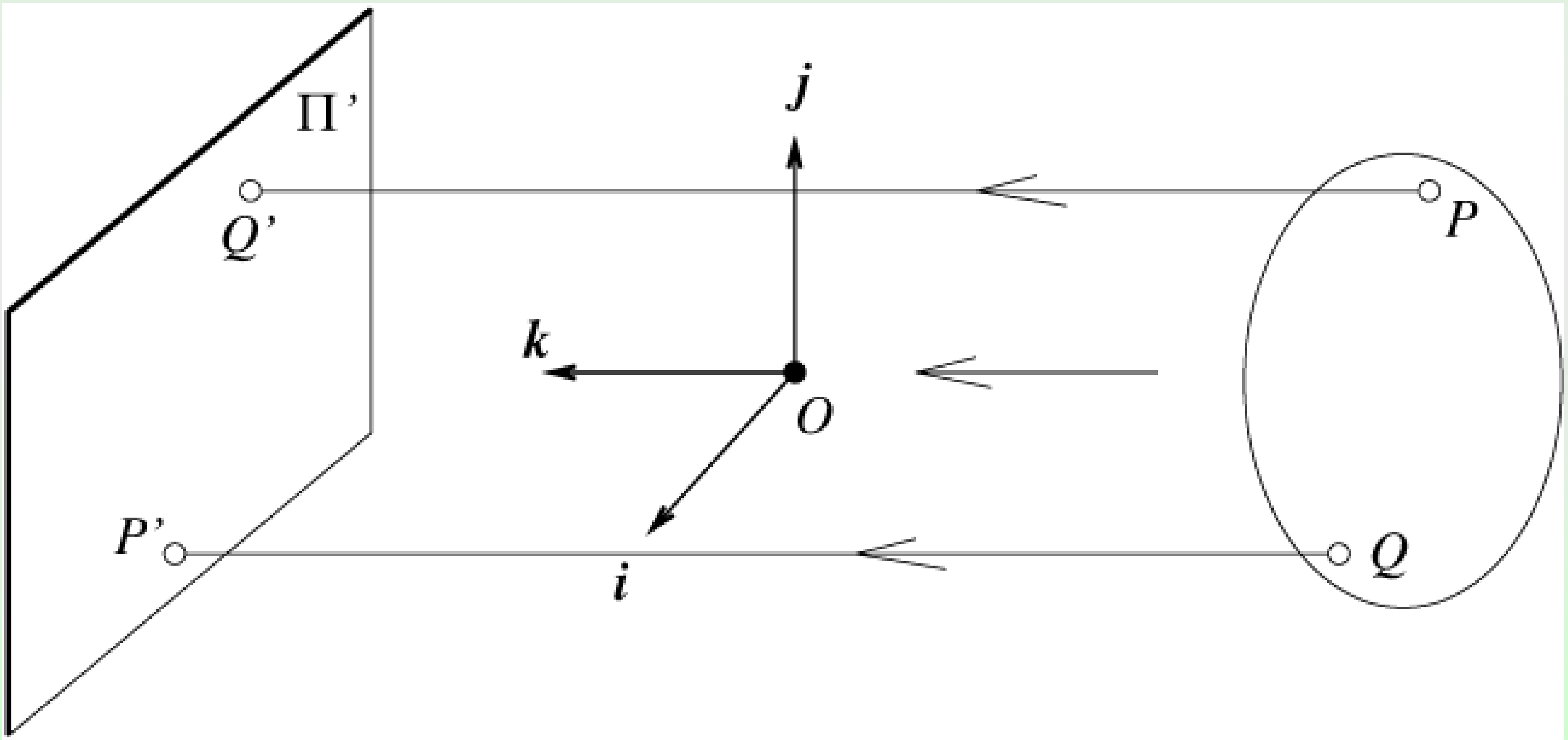
- How does scaling the projection matrix change the transformation?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z/f \end{bmatrix} \Rightarrow \left(f\frac{x}{z}, f\frac{y}{z} \right)$$

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} fx \\ fy \\ z \end{bmatrix} \Rightarrow \left(f\frac{x}{z}, f\frac{y}{z} \right)$$

Orthographic Projection (I)

Perpendicular projection onto the image plane



Orthographic Projection (III)

Depth, Z is ignored.

$$\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{0} & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ \boxed{Z} \\ T \end{pmatrix}$$

Z is ignored

Other Types of Projection

❑ Scaled orthographic

- Also called “weak perspective”

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1/d \end{bmatrix} \Rightarrow (dx, dy)$$

❑ Affine projection

- Also called “para-perspective”

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Three camera projections

3-d point 2-d image position

(1) Perspective: $$(x, y, z) \rightarrow \left(\frac{fx}{z}, \frac{fy}{z} \right)$$

(2) Weak perspective: $$(x, y, z) \rightarrow \left(\frac{fx}{z_0}, \frac{fy}{z_0} \right)$$

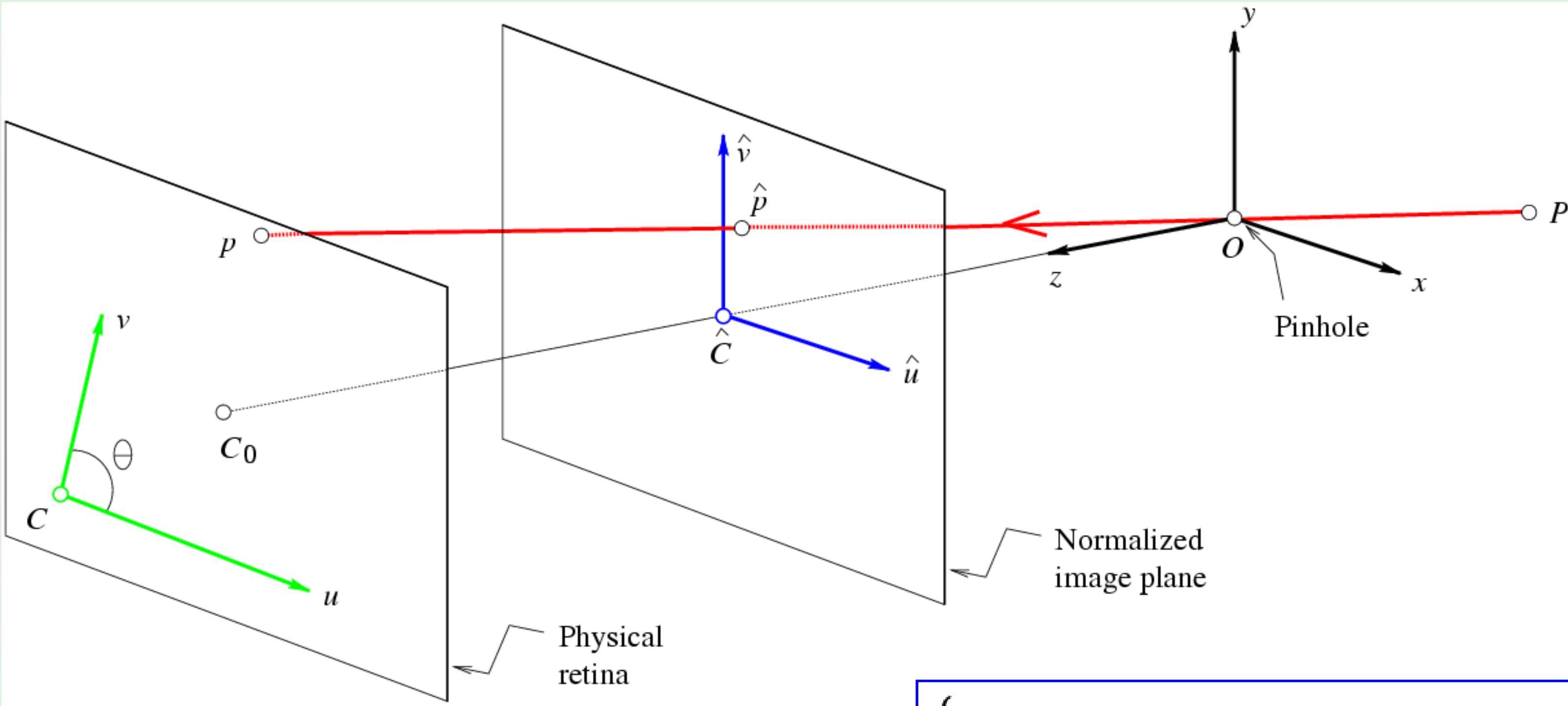
(3) Orthographic: $$(x, y, z) \rightarrow (x, y)$$

Camera parameters

- The world and camera coordinate systems are related by the physical parameters.
- A camera is described by several parameters
  - Translation T of the optical center from the origin of world coordinates
  - Rotation R of the image plane
  - focal length f, principle point (x'_c, y'_c), pixel size (s_x, s_y), skew angle (θ)
  - blue parameters are called "extrinsic," red are "intrinsic"

Intrinsic parameters (I)

Physical and normalized image coordinate systems



Normalized
image coordinate

$$\begin{cases} \hat{u} = \dfrac{x}{z} \\ \hat{v} = \dfrac{y}{z} \end{cases} \iff \hat{\boldsymbol{p}} = \frac{1}{z} \begin{pmatrix} \mathrm{Id} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{P} \\ 1 \end{pmatrix}$$

Intrinsic parameters (II)

Physical image coordinates

$$\begin{cases} u = kf\dfrac{x}{z} \\ v = lf\dfrac{y}{z} \end{cases} \rightarrow \begin{cases} u = \alpha\dfrac{x}{z} + u_0 \\ v = \beta\dfrac{y}{z} + v_0 \end{cases} \rightarrow \begin{cases} u = \alpha\dfrac{x}{z} - \alpha\cot\theta\dfrac{y}{z} + u_0 \\ v = \dfrac{\beta}{\sin\theta}\dfrac{y}{z} + v_0 \end{cases}$$

Units:

k,l : pixel/m

f : m

α,β : pixel

Calibration matrix

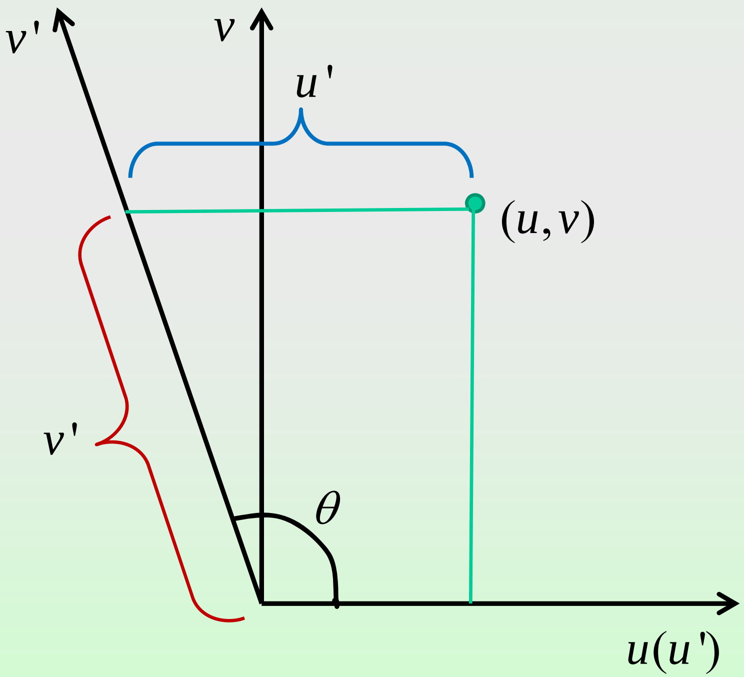
$$\boldsymbol{p} = \mathcal{K}\hat{\boldsymbol{p}}, \quad \text{where} \quad \boldsymbol{p} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{K} \stackrel{\text{def}}{=} \begin{pmatrix} \alpha & -\alpha\cot\theta & u_0 \\ 0 & \dfrac{\beta}{\sin\theta} & v_0 \\ 0 & 0 & 1 \end{pmatrix}$$

Perspective projection equation in homogeneous coordinate

$$\boldsymbol{p} = \frac{1}{z}\mathcal{M}\boldsymbol{P}, \quad \text{where} \quad \mathcal{M} \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{K} & \mathbf{0} \end{pmatrix}$$

Projection Equation (I)

- Skew angle analysis



$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & -\alpha\cot(\theta) & u_0 & 0 \\ 0 & \dfrac{\beta}{\sin(\theta)} & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Projection Equation (I)

- When the camera frame (C) is different from the world frame (W),

$$\begin{pmatrix} {}^{C}P \\ 1 \end{pmatrix} = \begin{pmatrix} {}^{C}_{W}\mathcal{R} & {}^{C}O_{W} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} {}^{W}P \\ 1 \end{pmatrix}.$$

- Thus,

$$\boldsymbol{p} = \frac{1}{z}\mathcal{M}\boldsymbol{P}, \quad \text{where} \quad \begin{cases} \mathcal{M} = \mathcal{K}\begin{pmatrix} \mathcal{R} & \boldsymbol{t} \end{pmatrix}, \\ \mathcal{R} = {}^{C}_{W}\mathcal{R}, \\ \boldsymbol{t} = {}^{C}O_{W}, \\ \boldsymbol{P} = \begin{pmatrix} {}^{W}P \\ 1 \end{pmatrix}. \end{cases}$$

- Note: z is *not* independent of $\mathcal{M}$ and $\boldsymbol{P}$:

$$\mathcal{M} = \begin{pmatrix} \boldsymbol{m}_1^T \\ \boldsymbol{m}_2^T \\ \boldsymbol{m}_3^T \end{pmatrix} \Longrightarrow z = \boldsymbol{m}_3 \cdot \boldsymbol{P}, \quad \text{or} \quad \begin{cases} u = \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}}, \\ v = \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}}{\boldsymbol{m}_3 \cdot \boldsymbol{P}}. \end{cases}$$

Projection Equation (II)

- Projection equation with full camera parameters

$$\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta}\boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

- Decomposition of the projection equation (zero skew)

$$\mathbf{\Pi} = \underbrace{\begin{bmatrix} -fs_x & 0 & x'_c \\ 0 & -fs_y & y'_c \\ 0 & 0 & 1 \end{bmatrix}}_{\boxed{\text{intrinsic}}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\boxed{\text{projection}}} \underbrace{\begin{bmatrix} \mathbf{R}_{3x3} & \mathbf{0}_{3x1} \\ \mathbf{0}_{1x3} & 1 \end{bmatrix}}_{\boxed{\text{rotation}}} \underbrace{\begin{bmatrix} \mathbf{I}_{3x3} & \mathbf{T}_{3x1} \\ \mathbf{0}_{1x3} & 1 \end{bmatrix}}_{\boxed{\text{translation}}}$$

Projection Equation (III)

- 11 parameters
  - 5 intrinsic parameters:
    - two pixel sizes (including focal length), principal point, skew angle,
  - 6 extrinsic parameters:
    - 3 rotation angles, 3 translations for each axis
- A camera with known non-zero skew and non-unit aspect ratio can be transformed into a camera with zero skew and unit aspect ratio by an appropriate change of image coordinates.
- Is an arbitrary 3x4 matrix perspective projection matrix?

Projection Equation (IV)

❑ Theorem by Faugeras (1993)

Let $\mathcal{M} = (\mathcal{A} \quad \boldsymbol{b})$ be a 3×4 matrix and let $\boldsymbol{a}_i^T$ $(i = 1, 2, 3)$ denote the rows of the matrix $\mathcal{A}$ formed by the three leftmost columns of $\mathcal{M}$.

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix is that $\mathrm{Det}(\mathcal{A}) \neq 0$.
- A necessary and sufficient condition for $\mathcal{M}$ to be a zero-skew perspective projection matrix is that $\mathrm{Det}(\mathcal{A}) \neq 0$ and

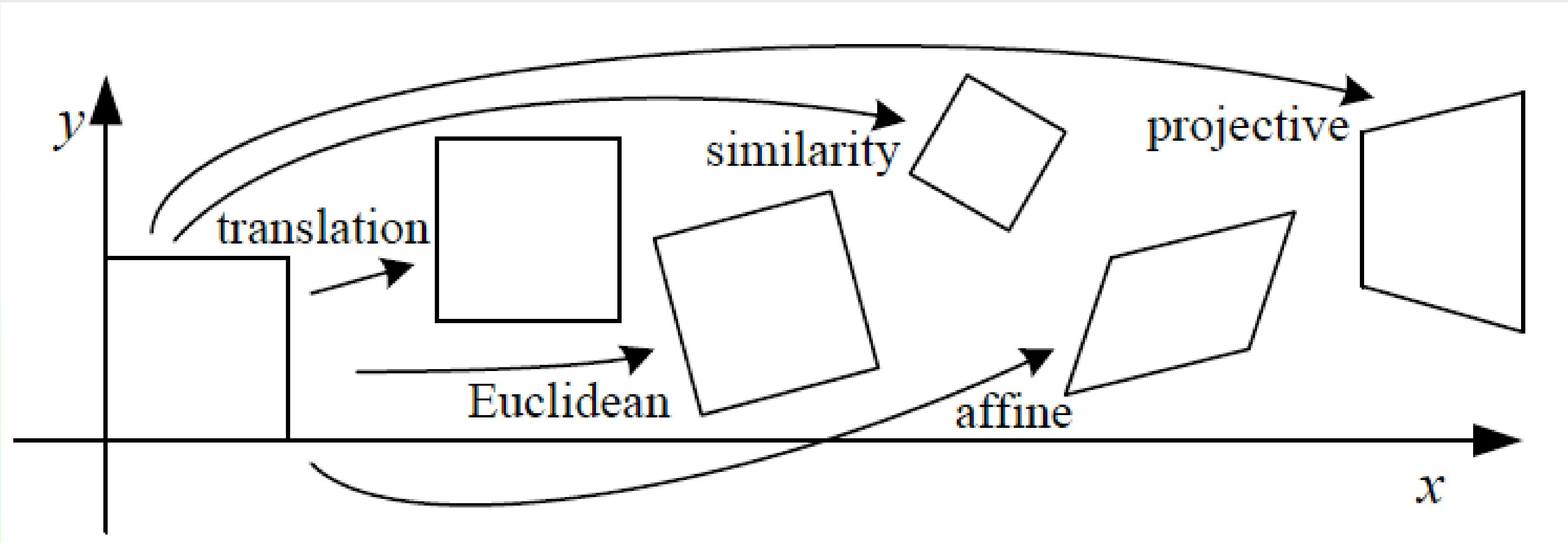
$$(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0.$$

- A necessary and sufficient condition for $\mathcal{M}$ to be a perspective projection matrix with zero skew and unit aspect-ratio is that $\mathrm{Det}(\mathcal{A}) \neq 0$ and

$$\begin{cases} (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = 0, \\ (\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3). \end{cases}$$

2D Projective Transformation (I)

- Projection along rays through a common point (center of projection) defines a mapping from one plane to another

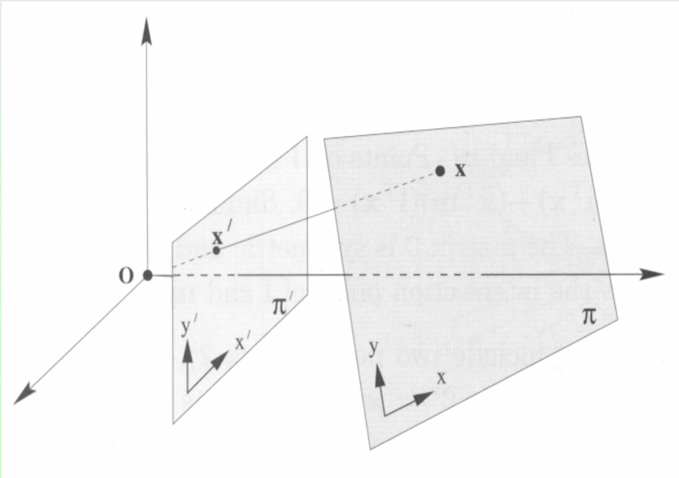


2D Projective Transformation (II)

❑ Projective transformation

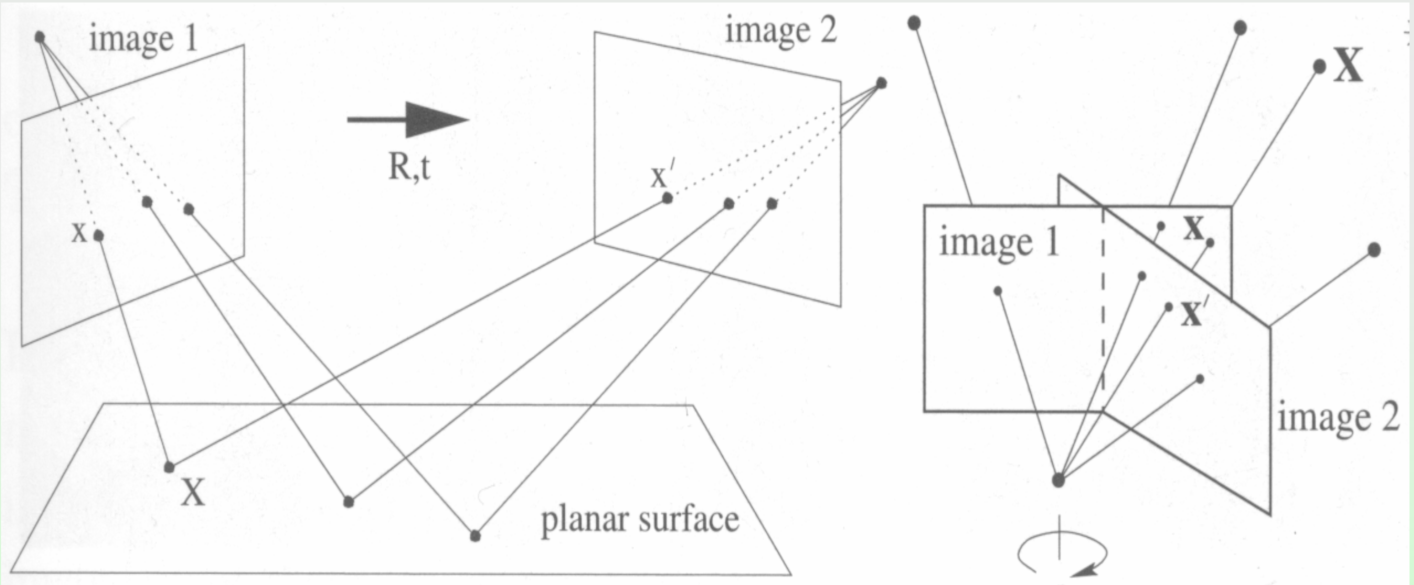
- Invertible mapping from projective plane $\mathbf{P}^2$ to $\mathbf{P}^2$
  - ➢ non-singular 3x3 matrix $\mathbf{H}$
- Three points on the same line are transformed onto a line
- Collineation, homography
- Point: $\mathbf{x}' = \mathbf{H}\mathbf{x}$, Line: $\mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



2D Projective Transformation (III)

- Perspective images from projective transformation



Hierarchy of Transformations (I)

- Isometry transformation
  - Orientation preserving mapping
  - Rotation and translation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \mathbf{x}$$

- 3 DOF
  - Invariant: area, length, shape…

Hierarchy of Transformations (II)

- Similarity transformation
  - Orientation-preserving
  - Scaling

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\ \sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \mathbf{x}$$

- 4 DOF
  - Invariant: ratio of lengths, angles, shape…

Hierarchy of Transformations (III)

Affine transformation

- Concatination of scaling and rotation with respect to x and y coordinate
- Non-isotropic scaling

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \mathbf{x}$$

- 6 DOF
- Invariant: parallelism, ratio of areas and collinear parallel lines, linear combinations of vectors (centroid) line at infinity…

Hierarchy of Transformations (IV)

Projective transformation

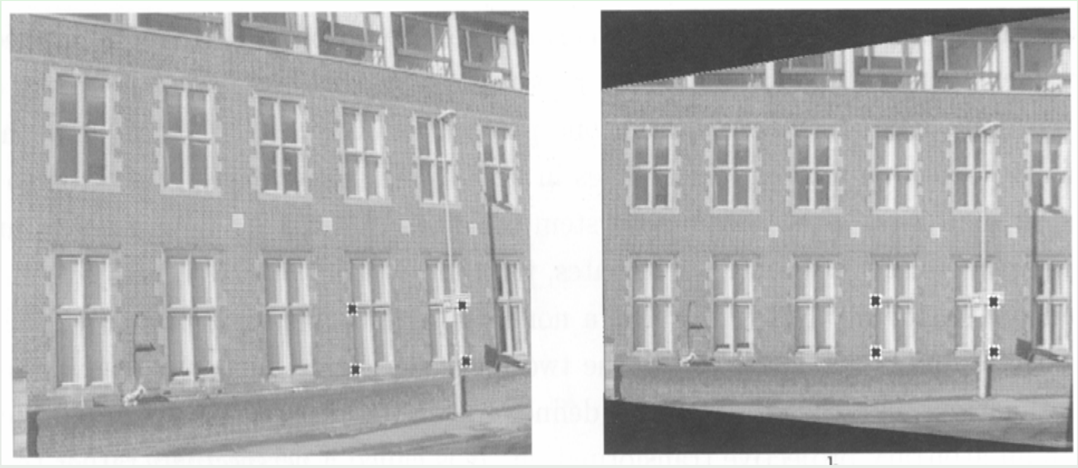
- Generalization of non-singular transformations in homogeneous representation

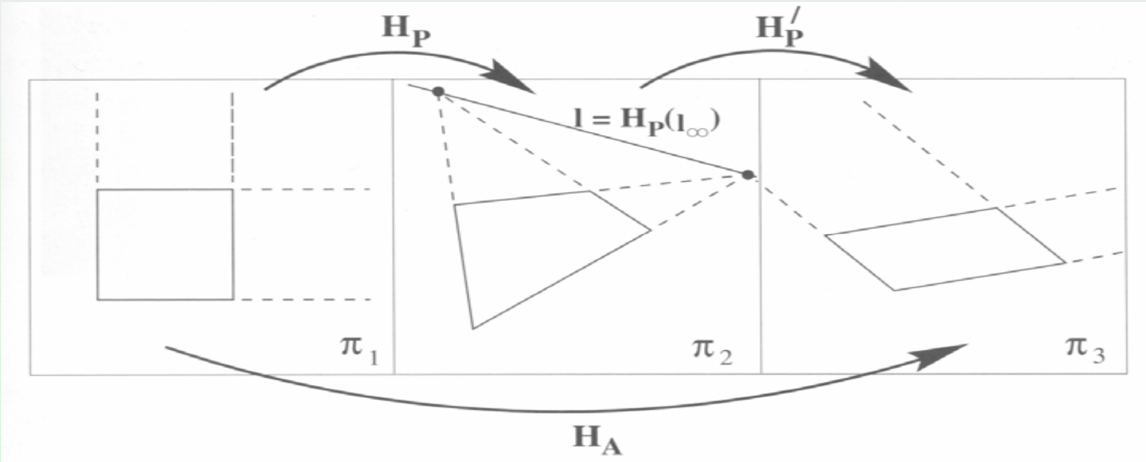
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^{\mathrm{T}} & 1 \end{bmatrix} \mathbf{x}$$

- 8 DOF
- Invariant: collinearity, cross ratio …
- A projective transformation between two planes can be computed from four point correspondences without collinearity of any three points

Conversion of Transformations

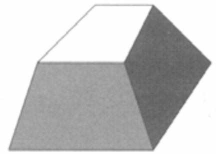
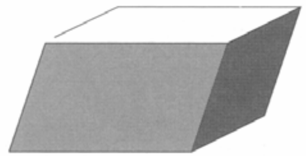
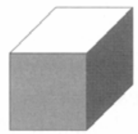
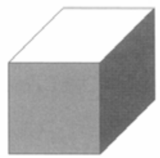
- An example: Removing projective distortion





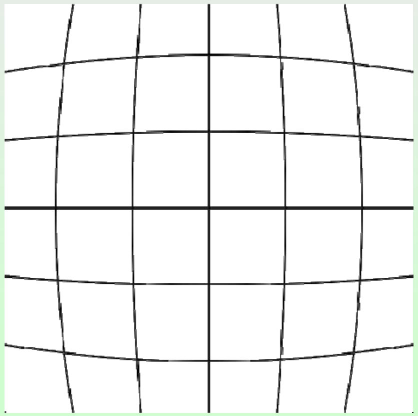
3D Projective Transformation

- Projective transformation of plane: **$\mathbf{p}' = H^{-T}\mathbf{p}$**
- H: **4x4 non-singular matrix**
- Same hierarchy as 2D

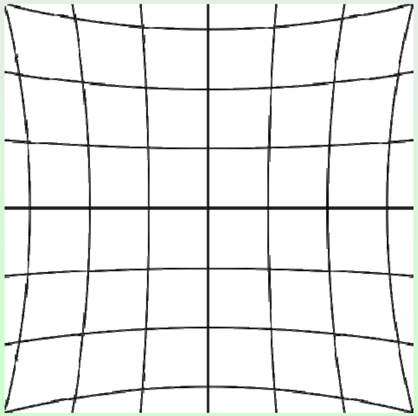
Group	Matrix	Distortion
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^{\top} & v \end{bmatrix}$	
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^{\top} & 1 \end{bmatrix}$	
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\top} & 1 \end{bmatrix}$	
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^{\top} & 1 \end{bmatrix}$	

Radial Distortion (I)

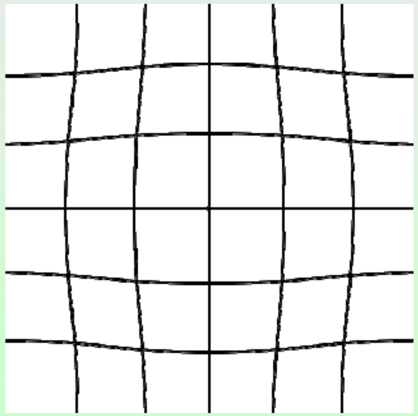
- Non-linear geometric distortion by lens



Barrel distortion



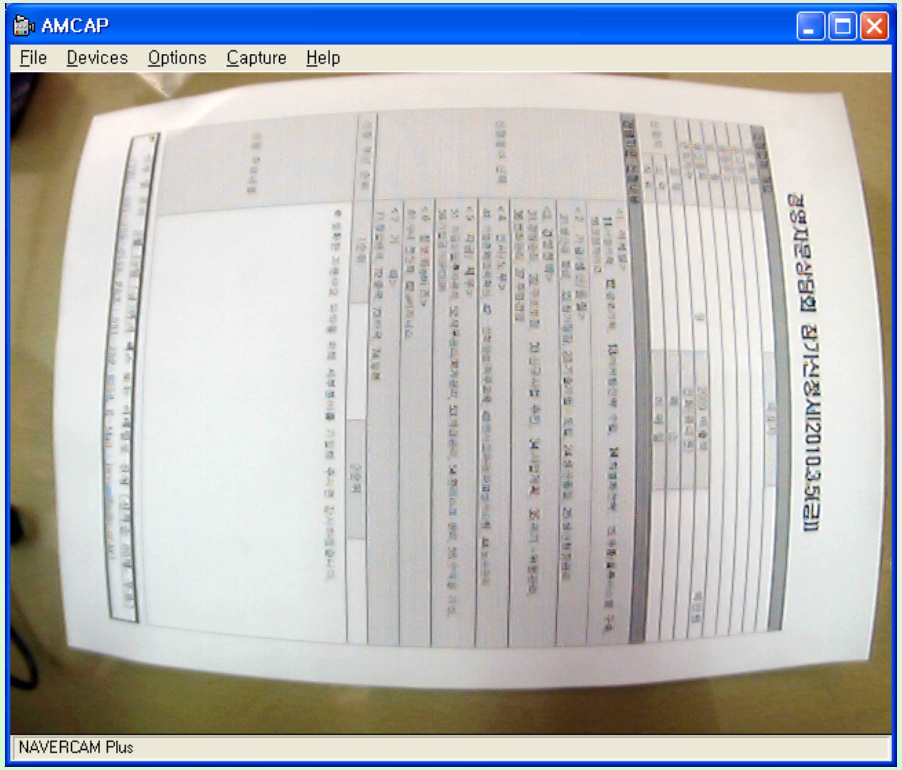
Pincushion distortion



Mustache distortion

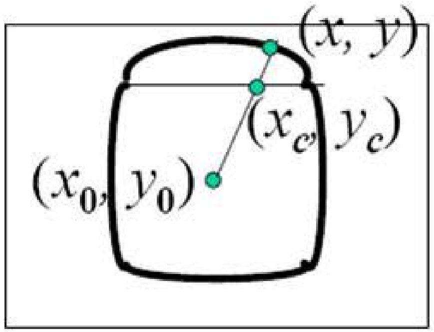
Radial Distortion (II)

- Example: Barrel distortion of usual lens



Radial Distortion (III)

- Modeling by even powers of radial distance



$$x_c - x = L(r)(x - x_0)$$
$$y_c - y = L(r)(y - y_0)$$

with

$$r^2 = (x - x_0)^2 + (y - y_0)^2$$
$$L(r) = 1 + \kappa_1 r + \kappa_2 r^2 + \ldots$$

$$\boxed{\begin{aligned} \delta x &= x(\kappa_1 r^2 + \kappa_2 r^4 + \ldots) \\ \delta y &= y(\kappa_1 r^2 + \kappa_2 r^4 + \ldots) \end{aligned}}$$

For principal point (0,0)

Barrel distortion: all $k_i > 0$
Pincushion distortion: some $k_i < 0$

Radial Distortion (IV)

- Minimize the errors using multiple grid points

$$f(\kappa_1, \kappa_2) = \sum (x'_i - x_{ci})^2 + (y'_i - y_{ci})^2$$

- Correction of distortion