Camera Calibration

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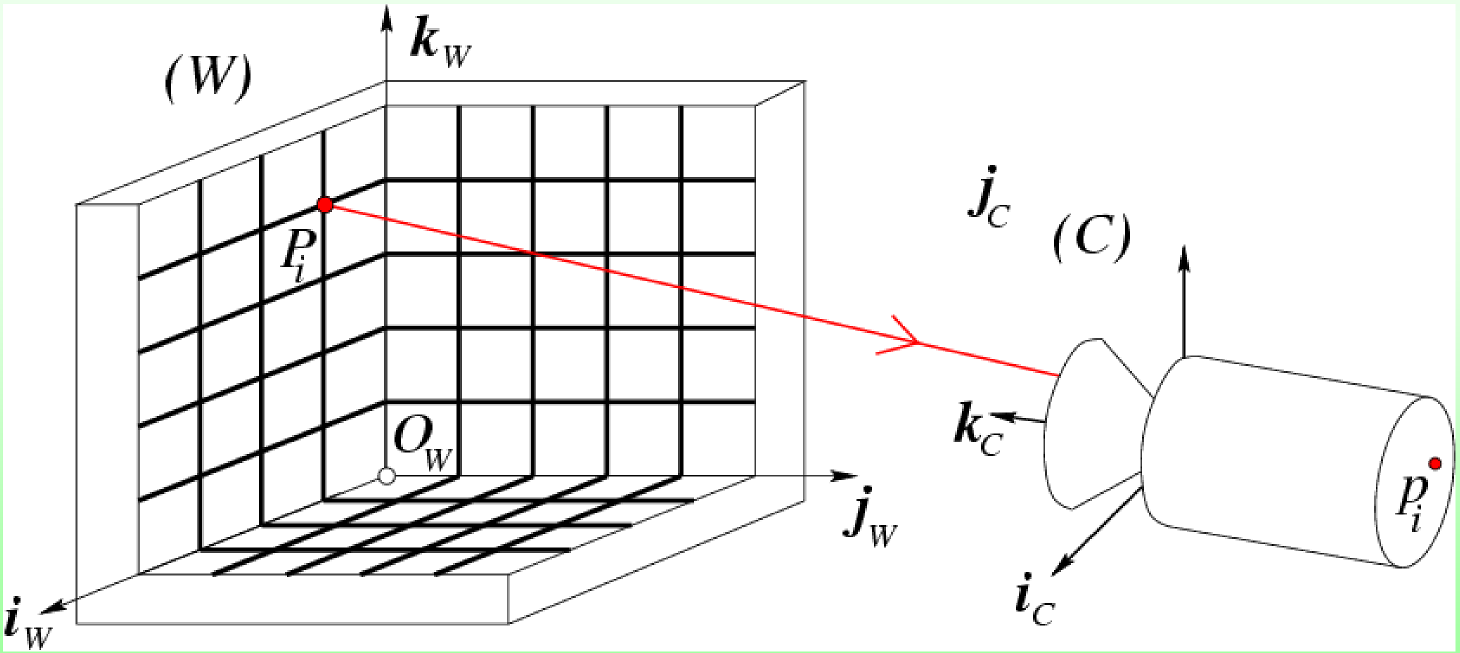
Calibration problem (I)

- Given n points $P_1, P_2, \ldots P_n$ with known positions in 3D space and their images, $p_1, p_2, \ldots, p_n$,
- Find the intrinsic and extrinsic camera parameters



Calibration problem (II)

- Intrinsic parameters
  - What kind of camera is?
  - Focal length, optical center (principal point), skew angle, aspect ratio, radial distortion
- Extrinsic parameters
  - Where is the camera?
  - Rotation and translation of coordinates
- Least squares methods
  - Approximate the coefficients of projection matrix (camera matrix) from multiple points set
  - Newton method, Levenberg-Marquardt method

Calibration procedure (I)

- **STEP 1: Find projection matrix M**
  - We need at least 6 points for 3x4 projection matrix
    - 12 parameters in M
  - Given known *n* 3D points and their corresponding pixels
    - Calibration box: Cubic grid

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{m}_1 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \\ \dfrac{\boldsymbol{m}_2 \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3 \cdot \boldsymbol{P}_i} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \boldsymbol{m}_1 - u_i \boldsymbol{m}_3 \\ \boldsymbol{m}_2 - v_i \boldsymbol{m}_3 \end{pmatrix} \boldsymbol{P}_i = 0$$

Homogeneous linear system

Calibration procedure (II)

- Solve linear system equation
  - *n* points →*2n* equations

$$\mathcal{P}\boldsymbol{m} = 0 \text{ with } \mathcal{P} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{P}_1^T & \boldsymbol{0}^T & -u_1\boldsymbol{P}_1^T \\ \boldsymbol{0}^T & \boldsymbol{P}_1^T & -v_1\boldsymbol{P}_1^T \\ \dots & \dots & \dots \\ \boldsymbol{P}_n^T & \boldsymbol{0}^T & -u_n\boldsymbol{P}_n^T \\ \boldsymbol{0}^T & \boldsymbol{P}_n^T & -v_n\boldsymbol{P}_n^T \end{pmatrix} \text{ and } \boldsymbol{m} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \end{pmatrix} = 0$$

- Least squares methods for 12 parameters

$$\sum_{i=1}^{n} \left[\left(u_i - \frac{\boldsymbol{m}_1(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 + \left(v_i - \frac{\boldsymbol{m}_2(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i}{\boldsymbol{m}_3(\boldsymbol{i},\boldsymbol{e}) \cdot \boldsymbol{P}_i} \right)^2 \right] \quad \text{is minimized}$$

Calibration procedure (III)

- STEP 2: Find intrinsic and extrinsic parameters using the obtained projection matrix

$$\mathcal{M} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T & \dfrac{\beta}{\sin\theta} t_y + v_0 t_z \\ \boldsymbol{r}_3^T & t_z \end{pmatrix}$$

$$\rho(\mathcal{A} \quad \boldsymbol{b}) = \mathcal{K}(\mathcal{R} \quad \boldsymbol{t}) \Longleftrightarrow \rho \begin{pmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \boldsymbol{a}_3^T \end{pmatrix} = \begin{pmatrix} \alpha \boldsymbol{r}_1^T - \alpha \cot\theta \boldsymbol{r}_2^T + u_0 \boldsymbol{r}_3^T \\ \dfrac{\beta}{\sin\theta} \boldsymbol{r}_2^T + v_0 \boldsymbol{r}_3^T \\ \boldsymbol{r}_3^T \end{pmatrix},$$

Calibration procedure (IV)

$$\begin{cases} \rho = \varepsilon / |\boldsymbol{a}_3|, \\ \boldsymbol{r}_3 = \rho \boldsymbol{a}_3, \\ u_0 = \rho^2 (\boldsymbol{a}_1 \cdot \boldsymbol{a}_3), \\ v_0 = \rho^2 (\boldsymbol{a}_2 \cdot \boldsymbol{a}_3), \end{cases} \quad \text{where} \quad \varepsilon = \mp 1.$$

$$\begin{cases} \rho^2 (\boldsymbol{a}_1 \times \boldsymbol{a}_3) = -\alpha \boldsymbol{r}_2 - \alpha \cot\theta \boldsymbol{r}_1, \\ \rho^2 (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \dfrac{\beta}{\sin\theta} \boldsymbol{r}_1, \end{cases} \quad \text{and} \quad \begin{cases} \rho^2 |\boldsymbol{a}_1 \times \boldsymbol{a}_3| = \dfrac{|\alpha|}{\sin\theta}, \\ \rho^2 |\boldsymbol{a}_2 \times \boldsymbol{a}_3| = \dfrac{|\beta|}{\sin\theta}. \end{cases}$$

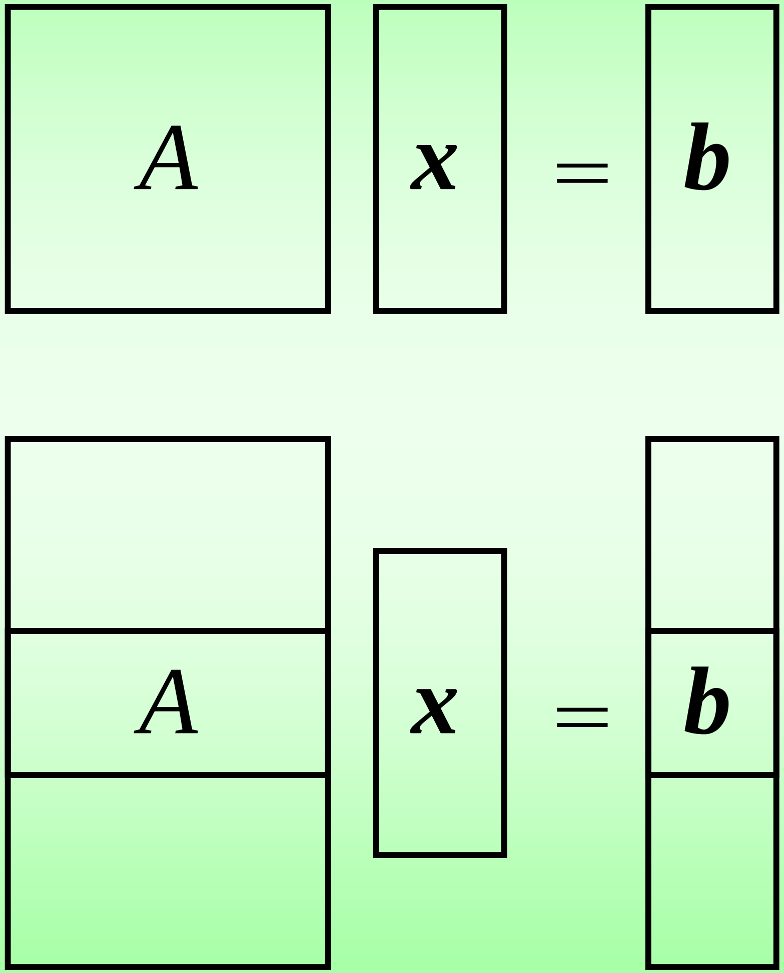
$$\begin{cases} \cos\theta = -\dfrac{(\boldsymbol{a}_1 \times \boldsymbol{a}_3) \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)}{|\boldsymbol{a}_1 \times \boldsymbol{a}_3||\boldsymbol{a}_2 \times \boldsymbol{a}_3|}, \\ \alpha = \rho^2 |\boldsymbol{a}_1 \times \boldsymbol{a}_3| \sin\theta, \\ \beta = \rho^2 |\boldsymbol{a}_2 \times \boldsymbol{a}_3| \sin\theta, \end{cases}$$

$$\begin{cases} \boldsymbol{r}_1 = \dfrac{\rho^2 \sin\theta}{\beta} (\boldsymbol{a}_2 \times \boldsymbol{a}_3) = \dfrac{1}{|\boldsymbol{a}_2 \times \boldsymbol{a}_3|} (\boldsymbol{a}_2 \times \boldsymbol{a}_3), \\ \boldsymbol{r}_2 = \boldsymbol{r}_3 \times \boldsymbol{r}_1. \end{cases}$$

Calibration toolbox

- Matlab toolbox for camera calibration
  - http://www.vision.caltech.edu/bouguetj/calib_doc/
- Procedure
  - Images loading
  - Extraction of 4 grid corners
  - Main calibration
    - Iterative distortion correction
  - Results
    - Camera parameters
    - Reprojection errors

Least square problem (I)



Square linear system:

- unique solution
- Gaussian elimination

Rectangular system ??

- underconstrained:
  infinity of solutions
  regularization

- overconstrained:
  no solution

Minimize $||Ax-b||^2$

Least square problem (II)

How to solve overconstrained linear equations ??

- Define $E = |\boldsymbol{e}|^2 = \boldsymbol{e} \cdot \boldsymbol{e}$ with

$$\begin{aligned} \boldsymbol{e} &= A\boldsymbol{x} - \boldsymbol{b} = \left[\begin{array}{c|c|c|c} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \dots & \boldsymbol{c}_n \end{array} \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \boldsymbol{b} \\ &= x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \cdots x_n\boldsymbol{c}_n - \boldsymbol{b} \end{aligned}$$

- At a minimum,

$$\begin{aligned} \frac{\partial E}{\partial x_i} &= \frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e} + \boldsymbol{e} \cdot \frac{\partial \boldsymbol{e}}{\partial x_i} = 2\frac{\partial \boldsymbol{e}}{\partial x_i} \cdot \boldsymbol{e} \\ &= 2\frac{\partial}{\partial x_i}(x_1\boldsymbol{c}_1 + \cdots + x_n\boldsymbol{c}_n - \boldsymbol{b}) \cdot \boldsymbol{e} = 2\boldsymbol{c}_i \cdot \boldsymbol{e} \\ &= 2\boldsymbol{c}_i^T(A\boldsymbol{x} - \boldsymbol{b}) = 0 \end{aligned}$$

- or

$$0 = \begin{bmatrix} \boldsymbol{c}_i^T \\ \vdots \\ \boldsymbol{c}_n^T \end{bmatrix} (A\boldsymbol{x} - \boldsymbol{b}) = A^T(A\boldsymbol{x} - \boldsymbol{b}) \Rightarrow A^TA\boldsymbol{x} = A^T\boldsymbol{b},$$

where $\boxed{\boldsymbol{x} = A^\dagger \boldsymbol{b} \text{ and } A^\dagger = (A^TA)^{-1}A^T \text{ is the } \textit{pseudoinverse} \text{ of } A \text{ !}}$

Least square problem (III)

$$A \, \boldsymbol{x} = \boldsymbol{0}$$

Homogenous square system

- unique solution: 0
- unless Det(A)=0

$$A \, \boldsymbol{x} = \boldsymbol{0}$$

Rectangular system ??

- 0 is always a solution

Minimize $||Ax||^2$
under constraint $||x||^2 = 1$
due to many solutions
with scalars and **0**

Least square problem (IV)

How do you solve overconstrained homogeneous linear equations ??

$$E = |\mathcal{U}\boldsymbol{x}|^2 = \boldsymbol{x}^T(\mathcal{U}^T\mathcal{U})\boldsymbol{x}$$

- Orthonormal basis of eigenvectors: $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q$.
- Associated eigenvalues: $0 \leq \lambda_1 \leq \ldots \leq \lambda_q$.
- Any vector can be written as

$$\boldsymbol{x} = \mu_1\boldsymbol{e}_1 + \ldots + \mu_q\boldsymbol{e}_q$$

for some μ_i $(i = 1, \ldots, q)$ such that $\mu_1^2 + \ldots + \mu_q^2 = 1$.

$$\begin{aligned} E(\boldsymbol{x}) - E(\boldsymbol{e}_1) &= \boldsymbol{x}^T(\mathcal{U}^T\mathcal{U})\boldsymbol{x} - \boldsymbol{e}_1^T(\mathcal{U}^T\mathcal{U})\boldsymbol{e}_1 \\ &= \lambda_1^2\mu_1^2 + \ldots + \lambda_q^2\mu_q^2 - \lambda_1^2 \\ &\geq \lambda_1^2(\mu_1^2 + \ldots + \mu_q^2 - 1) = 0 \end{aligned}$$

$\boxed{\text{The solution is } \boldsymbol{e_1}\text{ .}}$

$\boxed{\text{Minimum error is } \lambda_1\text{ .}}$