Ross Problem Set Solutions

Adithya Prabha

March 2023

1 Problem 1

Problem One A

Consider the following sample properties:

(i) If $a^2 = 1$, then $a = \pm 1$.
(ii) If $2x = 0$, then $x = 0$.
(iii) If $c^2 = 0$, then $c = 0$.

Which of the systems above have properties (i), (ii), and/or (iii)?

In this solution, I go through each of the systems individually and examine the three properties. Further, I bring up the contrapositive frequently. The main logic behind it is that if the statement is true, the contrapositive is true and the opposite works as well. Essentially, if there are no solutions to forming the "if" condition of a statement, its always true. If you pass the "if" condition, and the "then" condition fails, the statement is always false.

Starting with $\mathbb{Z}$, we can see that we are dealing with all integers meaning that solving the equations for integers, we see that all 3 properties work.

For $\mathbb{Q}$, we can see that all integers are rational numbers. Thereby, resulting in all three working for $\mathbb{Q}$ as well.

Now moving on to $4\mathbb{Z}$, this is the set of all multiples of 4 that are integers. Looking at property 1, there is no example that we can find that to disprove so taking the contrapositive gives us: If a is not equal to ± 1, a^2 is not equal to 1. Making the property true as the possibility to make it false isn't even there as there is no way to fulfill the "if" condition. Looking at property 2, no multiple of 4 times 2 will equal 0 other than 0 itself meaning that the only value that works is $x = 0$ making it true. Looking at the last property, (similar reasoning to property 2), the only thing that makes $c^2 = 0$ is when $c = 0$ and thus the last property is also true.

For $\mathbb{Z}_3$, the set is simply $\{0, 1, 2\}$. Testing property 1, we see it is true because we could manually test the three and $1^2 \equiv 2^2 \equiv 1 \pmod 3$ where $2^2 \equiv (-1)^2 \pmod 3$. This makes the only two that fulfill the if condition $-1, 1$ and thus setting property 1 as true. For property 2, $2x = 0$ can only happen when $x = 0$ leaving this to be true as well. Now testing property 3, $c^2 = 0$, the only thing that works is $x = 0$ once again which leaves property 3 to be true.

For $\mathbb{Z}_8$, the set is $\{0, 1, 2, 3, 4, 5, 6, 7\}$. Testing property 1, immediately leaves us with false because $3^2 \equiv 1 \pmod 8$ and $3 \neq \pm 1$ (same thing with 5). Now testing property 2, we see it is as false because $x = 4$ works and $4 \neq 0$. We see that $c = 4$ also works for property 3 and thus it is false as well.

For $\mathbb{Z}_9$, the set is $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Testing property 1, we see it is true again by brute force (the only ones that work are 1,8). Testing property 2, we see this is true as well which we can find out by either brute forcing it again or by noticing that for any number x, $2x \neq 9k$ for integers x and k where $x < 9$ because the only time $9k$ can be even are for even k and $k \geq 2$ is too large and thus the only one that works is when $k = 0$ where $x = 0$ implying that this is true. Testing property 3, leads us to a false because $9 = 3^2$ and when $c = 3$, $c^2 = 0$ but $c \neq 0$.

For $4\mathbb{Z}_{12}$, the only numbers in the set are $\{0, 4, 8\}$. Testing property one, we see that no element in the set works for $a^2 = 1$ and thus the answer is automatically true. Now trying property two, we see that this property is true as well by simply plugging it in. Property three is true once again because 0 is the only one that works with respect to $\pmod{12}$.

For $\mathbb{Z})_{13}$, the set is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Testing property one, we see that it works for for only $1, 12 \Rightarrow \pm 1$ meaning that it is true. Testing property two, we see that it is true once again and same for property three.

Generalizations that can be made: First of all one thing that we can notice is that for any even m where $m \neq 2$, for $\mathbb{Z}_m$, the second property is always false. This is because $\frac{m}{2}$ would work which is also $\neq 0$. Further, as we see with $m = 8$, if a number is one less than a perfect square,

Problem One B
Formulate another algebraic property and determine which of those systems have that property. Note: Cardinality is not considered to be an algebraic property. Write down some additional algebraic properties and investigate them.

Property: For all n in set S, there exists k such that $n \cdot k = 1$ What motivated this property is basically the multiplicative modular inverse. This can also be expressed as: $\forall n \in S, \exists k \in S(n \cdot k = 1), S \setminus \{0\}$. Also, for this property wouldn't give the same results if Z_x includes 0, so I try not to include 0 in the system. The reason we don't include 0 is because 0 can never have a modular inverse because its divisible by all numbers.

Starting with $\mathbb{Z}$, we see that it doesn't work. The reason is that k and $\frac{1}{k}$ cannot be both in $\mathbb{Z}$ other than 1 which certainly doesn't account for the whole set.

Now looking at $\mathbb{Q}$, we see that this field does work. This because if k is rational, $\frac{1}{k}$ is also rational. And thus all elements in $\mathbb{Q}$ have its inverse in the set as well.

Now moving on to $4Z$, we see that this doesn't work again. This follows the same reason as $\mathbb{Z}$ in that $4x$ and $\frac{1}{4x}$ cannot be in the same set.

$\mathbb{Z}_3$ we see that the set is $\{0, 1, 2\}$ and after removing 0, it is just $\{1, 2\}$. We see that both of their multiplicative inverse is itself and thus after removing 0 from the set, it works. And thus, when excluding 0, this set is contains the integer and its modular inverse as well.

Now looking at $\mathbb{Z}_8$, we see the set $\{1, 2, 3, 4, 5, 6, 7\}$. In this set, $\{2, 4, 6\}$ do not have a modular inverse with respect to 8. Looking at the rest, 1's modular inverse is itself, 3's modular inverse is also itself, and the same thing for 5 and 7.

For $\mathbb{Z}_9$, we see that it is the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. For this, $\{3, 6\}$ do not have a inverse and for the others, 1's and 8's modular inverse is the same. And 2 and 5, 4 and 7 are the inverse pairs.

For $4\mathbb{Z}_{12}$, we see that the set is $\{4, 8\}$ and neither actually work because both neither is relatively prime to 12.

For $\mathbb{Z}_{13}$, we see that all work with $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. First of all numbers in the set are relatively prime to 13. Pairing them up based on what their inverse modulo is, we see 1 and itself, 12 and itself, 2 and 7, 3 and 9, 4 and 10, 5 and 8, 6 and 11.

Observations: First of all, something that is very important to the problem that simplifies all the "bash" is that numbers that are not relatively prime to the modulo cannot have an inverse. This is because, an inverse of k modulo-p is $k \cdot w \equiv 1 \pmod{p}$. This means that, for some b, $k \cdot w = p \cdot b + 1$. If the numbers k and p share some common factor, $p \cdot b + 1$ does not share this same factor because consecutive numbers are always relatively prime. This just shows that a number k can have an inverse modulo p only if $\gcd(k, p) = 1$.

Another observation (which is a little less obvious) is that all $\mathbb{Z}_p$ where p is a prime is a set that works. This is true because for a prime number, no numbers less than it share any factor with it (which is the only thing stopping it from having an inverse). The other numbers just simply pair up.

Problem One C

In your opinion, which of the listed systems are "most similar" to each another?

I think that among the listed systems, the most similar ones are $\mathbb{Z}_3$ and $\mathbb{Z}_{13}$. This is because both have a modulo of a prime number and among the properties given, they share all of them. The reason that I say that they share so many properties is that 1) there are always only 2 factors for any prime number, 2) all theorems such as Fermat's little theorem, Chinese Remainder Theorem, etc are all built upon p being a prime. This may not directly impact the properties a field has, but I think that these theorems would impact the modular arithmetic.

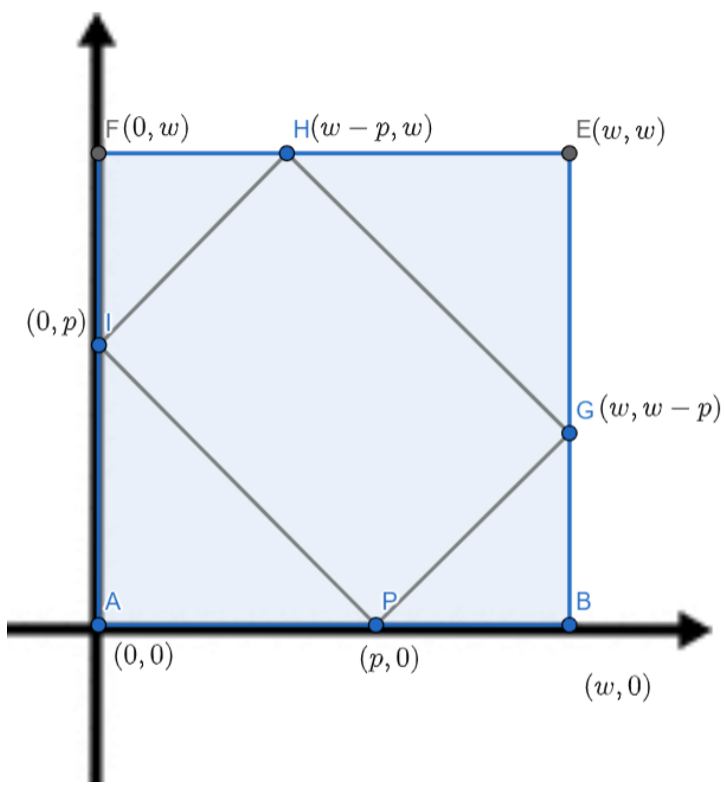
2 Problem 2

I switched the order around a little bit for this problem as I thought that parts b and d were very related and what I derive in part d could easily prove part b, so I answer part b after part d.

Problem Two A

Suppose $s = 1$ so that the path begins at a 45 degree angle. For every starting point P, show: Rossie's path is a stable rectangle. (If P is a corner point, the path degenerates to a line segment traced back and forth.)

We can start by putting this on the coordinate plane. Above, I let the side length of the square



be w and put it on the coordinate plane. Further, I let the point $P(p, 0)$ be on the bottom side with arbitrary coordinates such that it is on AB. First, it is obvious that if the line that goes through $(p, 0)$ has slope 1, it has an equation $y = x - p$ and thus it intersects $x = w$ at $(w, w - p)$. Similarly, the line that goes through $(w, w - p)$ has slope -1 (because it is perpendicular) and thus, $y = -x + c$ goes through $(w, w - p)$ so $w - p = -w + c \Rightarrow c = 2w - p$. Thus, it has equation $y = -x + (2w - p)$. It intersects $y = w$ at $w - p$ by substituting and simplifying. And then the doing the same thing again, it intersects $(0, p)$ and thus intersects $(0, p)$ again. Going through the rectangle with the slopes, we see that we will always form a rectangle (with the parameters specified in the problem that $p < w$ and the slope is 1).

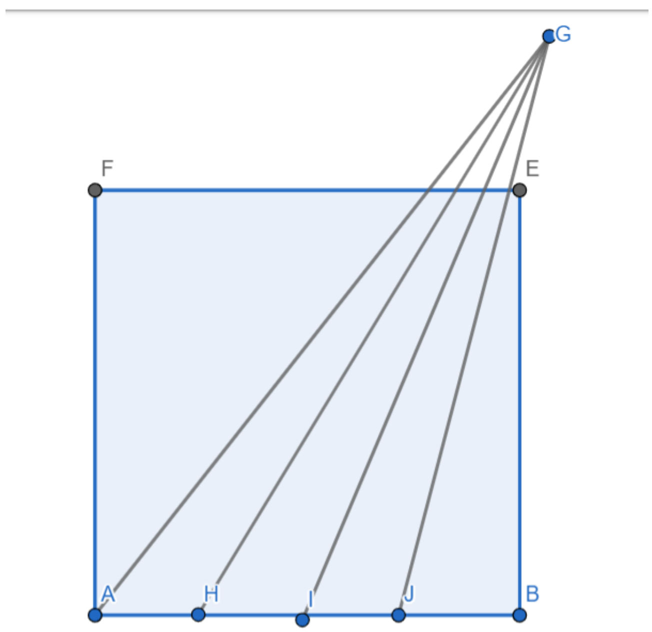
If $(p, 0) = (w, 0)$, the slope will just go back and worth from $(w, 0)$ and $(0, w)$.

A nice observation is that the point H is the reflection of G about $y = x$. Further, $y = x$ is the perpendicular bisector of both HG and PI and thus you just flip the two coordinates to get the other side. This is because the slope of HG and IP is -1 and the slope of $y = x$ is 1. Therefore, they will intersect and thus G is a reflection about $y = x$ of H because it is the perpendicularly bisected. This falls under the latter of the two conditions that work: either $AB + BG = w$ or $AB = BG$. This could be proved by examining the diagonal BH.

Problem Two C

Suppose s is given with $0 < s < 1$. When Rossie starts at point X, let XX' be the first segment of her path. If X is on side AB, must X' be on BC? For P, Q on AB, how is the length $|P'Q'|$ related to $|PQ|$ and s?

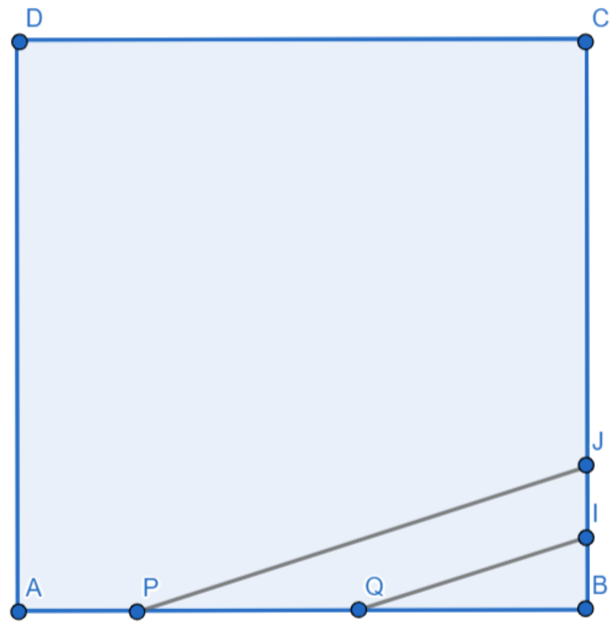
If X is on side AB, X' must be on BC. This is because the slope given to us is positive and less than one. For X' not to be on BC the slope needs to be greater than one for all s. This has to be true because the slope of the diagonal of the square is obviously 1. Further, the closer you get to be on AB, the greater slope required to reach a certain point. We can see this visual in the figure below,



The closer you get to B the greater slope necessary to get to a certain point. This can also be shown algebraically by noticing that the leg on the x-axis gets smaller and the smaller and for any fraction, the smaller the denominator is, the greater the fraction (or in this case the slope) is. For any point, the largest slope such that X' is still in the square is the slope of the line from the X to E and the smallest slope required to get from X to E is when $X = A$ with $s = 1$ (in this case $E = X'$) and thus to leave the square, the slope has to be greater than one ($s > 1$). As we showed above, the slope to any point outside the square is smallest from $X = A$ and if the smallest is $s > 1$, all points closer to B will have greater slopes and thus will have $s > 1$. Meaning that if $s > 1$, we

To leave the square from any point X on AB, the slope, $s > 1$. This results in all $0 < s < 1$. forced to intersect the segment BE at X'.

The second part of the question is interesting as the length $P'Q'$ does not solely depend on the positioning of PQ before hand. The diagram below illustrates this. Clearly $PQ \neq IJ$. The length difference must be influenced by something else. Which we can think about on the coordinate plane once again.



Let $P(p, 0)$ and $Q(q, 0)$. We first notice that $PQ = q - p$. Both lines have slope s meaning that the line through P has equation $y = s \cdot (x - p)$ and the one through Q has equation $y = s \cdot (x - q)$. At $x = w$ (keeping the same notation as the earlier questions where the square has side length w) the y-coordinate of the line through P is $s \cdot (w - p)$ and the latter has $s \cdot (w - q)$. The difference between these two y-coordinates will give us the length IJ. This results in,

$$s \cdot (w - p) - s \cdot (w - q) = s \cdot (q - p)$$

We can see from here that the length $PQ = q - p$ and $JI = s(q - p)$, so $|P'Q'| = s \cdot |PQ|$.

There were many ways to approach this. For example, you could have done it on the coordinate plane like how I did above or you could have used a scale-factor resulting in similar triangles leading us to the same answer.

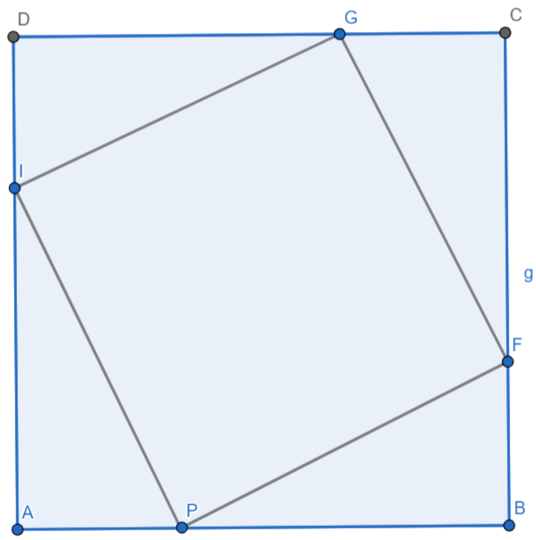
Problem Two D

When $0 < s < 1$, prove or disprove: For any starting point P, Rossie's path converges to a stable rectangle.

Answer: This is true, and I prove so using some type of limit.

We start by defining some variables. Let w be the side length of the square, let $AP = a$, and we know that the slope is s.

Before finding the answer for convergence, we need to find out for what conditions will there be a stable rectangle formed on the inside on the first go. Keep in mind for the algebra under, I choose $BP = x$ because I found the algebra to be a little bit "better".



We let $BP = x$ and the slope of $PF = s$. We can see that the slope s is also $\tan(\angle FPB)$ and $\angle FPB = \angle GFC = \angle DGI = \angle AIP$ and thus their tangents are all equal. From the tangent, we can see that $a = \frac{BF}{BP} = \frac{x}{BP} \Rightarrow BP = s \cdot x$. Also as the side length is w, $FC = w - s \cdot x$. Doing the same tangent thing again, $s = \frac{GC}{FC} \Rightarrow GC = s \cdot (w - s \cdot x)$. And again, $GD = w - s \cdot (w - s \cdot x)$ and $DI = s \cdot GD = s \cdot (w - s \cdot (w - s \cdot x))$. Instead of continuing with IA, we can note that $AP = w - x$ and thus, $s \cdot \frac{w-x}{AI} \Rightarrow AI = \frac{w-x}{s}$. Since we started with constructing the rectangle on the inside, we need to show that the outside is a square and we can do this by showing that $DI + AI = w$.

$$w = \frac{w-x}{s} + s \cdot (w - s \cdot (w - s \cdot x))$$

Multiplying by s,

$$w \cdot s = (w - x) + s^2 \cdot (w - s \cdot (w - s \cdot x))$$

Now we just simplify until we get all similar terms on one side,

$$w \cdot s = (w - x) + s^2 \cdot (w - s \cdot w + s^2 \cdot x)$$

$$w \cdot s = (w - x) + s^2 \cdot w - s^3 \cdot w + s^4 \cdot x$$

$$-w + w \cdot s - s^2 \cdot w + s^3 \cdot w = x \cdot (s^4 - 1)$$

$$w(-1 + s - s^2 + s^3) = x \cdot (s^4 - 1)$$

Now we can use a clever factorizing trick that $\dfrac{s^4 - 1}{s + 1} = s^3 - s^2 + s - 1$ and substituting,

$$w \cdot \frac{s^4 - 1}{s + 1} = x \cdot (s^4 - 1)$$

$$\frac{w}{s+1} = x$$

However, since we set $PB = x$, we need AP,

$$AP = w - x = w - \frac{w}{s+1} = \frac{w \cdot s}{s+1}$$

This means that to form a stable rectangle with slope s, we simply need $P_nA = \frac{w \cdot s}{s+1}$. Now we change our focus to getting P_nA.

We use the diagram above to solve this problem. We start by doing some algebra. Let $AP = a$. This means that $BP = w - a$ and $FB = s(w-a)$, $CF = w - s(w-a)$, $CG = s(w - s(w-a))$, $GD = w - s(w - s(w-a))$, $DI = s(w - s(w - s(w-a)))$, $IA = w - s(w - s(w - s(w-a)))$,
This results in,

$$P_1A = s(w - s(w - s(w - s(w-a))))$$

An observation that we can see from this is that each time it hits a side of the square, the first length we calculate is in the form $s(w - s(w - s(w \ldots$ when continuing past the point P_1. This seems like its essentially the same function inside a function and so on. Meaning that we can say we can say $P(a) = s(w-a)$ and to find the distance AP_1, it is simply $P(P(P(P(a))))$ and so on.

Geometric Series + Induction:
We first examine the first few $P(P(\ldots(x)\ldots)$.

$P(a) = s(w-a)$
$P(P(a)) = s(w - s(w-a)) = s(w - s \cdot w + s \cdot a) = sw - s^2 \cdot w + s^2 \cdot a$
$P(P(P(a))) = s(w - s(w - s(w-a))) = s(w - (sw - s^2 \cdot w + s^2 \cdot a)) = s \cdot w - s^2 \cdot w + s^3 \cdot w - s^3 \cdot a$
$P(P(P(P(a)))) = s(w - (s \cdot w - s^2 \cdot w + s^3 \cdot w - s^3 \cdot a)) = s \cdot w - s^2 \cdot w + s^3 \cdot w - s^4 \cdot w + s^4 \cdot a$

A clear pattern emerges from evaluating the first four. First, the last term is essentially $(-s)^n$. Also, every other term is essentially forms an infinite geometric sequence, because $0 < s < 1$ meaning the common ratio is in the right range for an infinite geometric series. From this I claim that, Where there are n number of Ps,

$$(\underbrace{P(P(P(\ldots(a)\ldots))}_{n\ Ps}) = w(s - s^2 + s^3 - s^4 + \cdots + (-1)^{n+1} \cdot s^n) + (-s)^n \cdot a$$

To prove this, we use induction. We see that the base case works and now moving on to $n+1$,

$$(\underbrace{P(P(P(\ldots(a)\ldots))}_{n+1\ Ps}) = s(w - \underbrace{w(s - s^2 + s^3 - s^4 + \cdots + (-1)^{n+1} \cdot s^n) + (-s)^n \cdot a)}_{P(a)}$$

$$= s(w - w \cdot s + w \cdot s^2 + \ldots.(-1)^{n+2} \cdot s^{n+1} + (-1)^{n+1} \cdot (s)^n a)$$
$$= s \cdot w - w \cdot s^2 + w \cdot s^3 + \ldots..(-1)^{n+2} \cdot s^{n+2} + (-s)^{n+1} \cdot a$$

The above thing, is nothing other than $(\underbrace{P(P(P(\ldots(a)\ldots))}_{n+1\ Ps})$ and thus we have finished the induction. Evaluating,

$$(\underbrace{P(P(P(\ldots(a)\ldots))}_{n\ Ps}) = w(s - s^2 + s^3 - s^4 + \cdots + (-1)^{n+1} \cdot s^n) + (-s)^n \cdot = s \cdot w \cdot \frac{1 + (-s)^n}{1+s} + (-s)^n \cdot a$$

Essentially, from here, I need to take the limit as n approaches infinity to show that it eventually converges to the stable rectangle and follows the border of the stable rectangle. This means that taking the limit,

$$\lim_{n\to\infty}\left[s \cdot w \cdot \frac{1+(-s)^n}{1+s} + (-s)^n \cdot a\right]$$

The $(-s)^n$ go to zero because of the interval s is bounded in. And thus it is easy to evaluate the limit now,

$$= \lim_{n\to\infty}\left[s \cdot w \cdot \frac{1}{1+s}\right] = \frac{w \cdot s}{s+1}$$

If we look above to the starting of the problem when we were calculating the required position, it was the exact same thing. This proves the answer to the question to be true.

For any starting point P, Rossie's path converges to a stable rectangle.

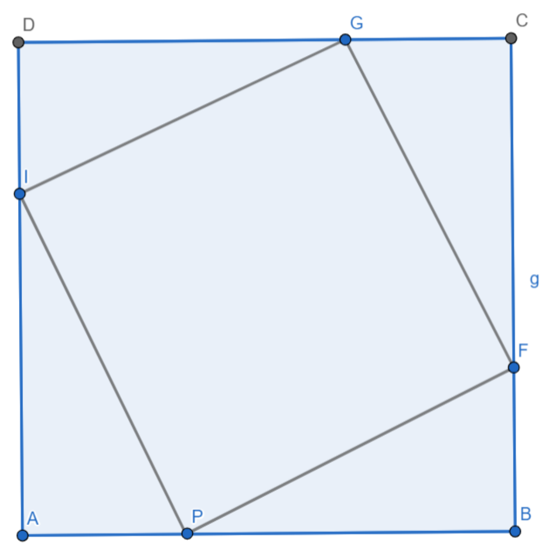
Problem Two B

First consider the case: $0 < s < 1$. For each such s, is there exactly one stable rectangle with slope s? Must that rectangle be a square?

Answer: Yes the rectangle is a square.
From part d, we know that eventually for $0 < s < 1$, a stable rectangle is formed. We can show that the a square is formed by noticing that $P(\frac{sw}{1+s}) = \frac{sw}{1+s}$.

$$P(\frac{sw}{1+s}) = s(w - \frac{sw}{1+s}) = s \cdot \frac{w}{1+s}$$

This essentially means that the shorter distance from the point on the inner rectangle on each side to the vertex are all congruent.



In this situation, this means that $AP = FB = CG = DI$. This actually guarantees us a square on the inside. This is because $FB = FC = GD = IA = w - a$. They are all congruent by SAS and thus have the same hypotenuse length meaning that a square is formed on the inside.

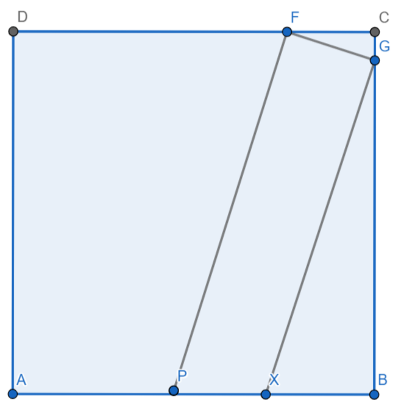
Problem Two E
What is Rossie's behavior when $s > 1$ or when $s < 0$? Does the argument above still apply?

I will first present the algebra I did in the starting which is the one that I did not get too much progress on.

Based on part d, to prove its true/false, we need to show whether or not the position of P after n iterations goes to $\frac{s \cdot w}{s+1}$. An observation that I had when drawing diagrams multiple times is that the core series that we need to use here is

$$1 - \frac{1}{s} + \frac{1}{s^2} \cdots + \frac{(-1)^n}{s^n}$$

The reason I thought this is by first looking at the diagram below,



In this, PF has slope s. Also AP has length a. We can see that the distance $FC = w - \frac{w}{s} - a$. Also, $CG = \frac{FC}{s} = \frac{w}{s} - \frac{w}{s^2} - \frac{a}{s}$, and $BG = w - CG = w - \frac{w}{s} - \frac{w}{s^2} - \frac{a}{s}$. And following, $BX = \frac{BG}{s} = \frac{w}{s} - \frac{w}{s^2} - \frac{w}{s^3} - \frac{a}{s^2}$. This again is similar to the polynomial developed in part d in that it takes 4 cycles to hit AB again.

Here, we see that it is essentially a geometric series on the inside when factoring the w out of the terms (excluding the last one) to see that it models one with ratio $-\frac{1}{s}$.

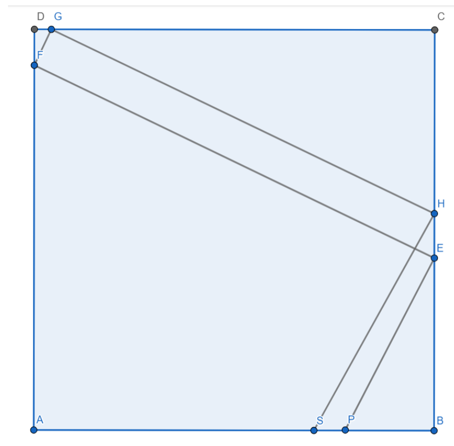
However, it can be a different diagram as well:
In this diagram below, the first thing we see is that $PB = w - a$, $BE = s(w - a)$, $DF = w - s(w - a) - \frac{w}{s}$.

For any configuration, we don't know what the nth point on AB will be but we know that it will be in a form like

$$w \cdot (a_1 - \frac{a_1}{s} + \frac{a_1}{s^2} + \dots) + \frac{(-1)^n \cdot a}{s^n}$$
$$= w \cdot a_1 \cdot \frac{s^n + (-1)^{n+1}}{(s+1) \cdot s^n}$$

Taking the limit as n approaches infinity gives us,

$$\frac{w \cdot a_1}{s+1}$$



This is really close to what we are trying to get which is $\frac{w \cdot s}{s+1}$. I didn't have a solid way to continue from here, other than thinking that its most likely that this case converges as well. Then I tried to take a different approach: symmetry.

The main idea I took from this is that after the first line with slope s, the perpendicular line has slope $\frac{-1}{s}$ which is in the range $-1 < \frac{-1}{s} < 0$. In other words, if you reflect this (or picture it as flipping over), $0 < -\frac{1}{s} < 1$. We proved in part d that if the starting slope is between 0 and 1, it converges to a stable rectangle at some point. If a square has a stable rectangle on the inside, if it is rotated or reflected or anything, it would still have a stable rectangle on the inside. Similarly, in this situation, a stable rectangle is formed based on our proof in 3d on the reflected version. Thus, for any starting point P, with $s > 1$, it converges to a stable rectangle based on the prove that was provided in part d.

Now looking at the condition where $s < 0$, we see that it is also essentially a reflection of the condition for $0 < s < 1$, $s = 1$, $s > 1$ as it is essentially flipped over the vertical line that goes through the midpoint of AB.

3 Problem 3

Problem Three A

Find all polynomials f that satisfy the equation: $f(x+2) = f(x)+2$ for every real number x.

Note: n is used frequently and I use it as the degree of the polynomial $f(x)$.

We start by trying random values of $f(x)$ and see which ones work. If $f(x) = x$, $f(x+2) = x+2 = f(x)+2 = x+2 = x+2$. We see that this works.

Now if $f(x) = x^2$, $f(x+2) = (x+2)^2 = x^2+4x+4$ and $f(x)+2 = x^2+2$. Setting them equal, $x^2+4x+4 \neq x^2+2$ for all real x. To show that all real x do not work, take $x = 3$.

$$3^2 + 4 \cdot 3 + 4 \text{ and } 3^2 + 2$$

$$25 \neq 11.$$

Now if we try $f(x) = x^3$, we see that this doesn't work again because

$$(x+3)^2 \text{ and } x^3 + 2$$

$$x^3 + 9x^2 + 27x + 8 \neq x^3 + 2$$

So far, we only have one function that works so maybe we need to try a different approach to finding all functions that work.
We can do so by examining a graphical form of the equation. We choose an arbitrary x and put it on the coordinate plane.

We are trying to find some form of generalization through the graph that will help narrow down what polynomials work for $f(x)$. Notice that the difference the values $x+2$ and x is 2. Similarly, the difference between $f(x+2)$ and $f(x)$ is 2 which we know from the problem itself:

$$f(x+2) = f(x) + 2 - f(x) = 2.$$

Interestingly, we see that the slope of the line that goes through $f(x)$ and $f(x+2)$ is $\dfrac{2}{2} = 1$ for all real x. Thus, this connects all real x meaning that all polynomials need to have a slope or 1. More generally, all polynomials need to have the highest degree as 1 and its coefficient needs to be 1 as well.

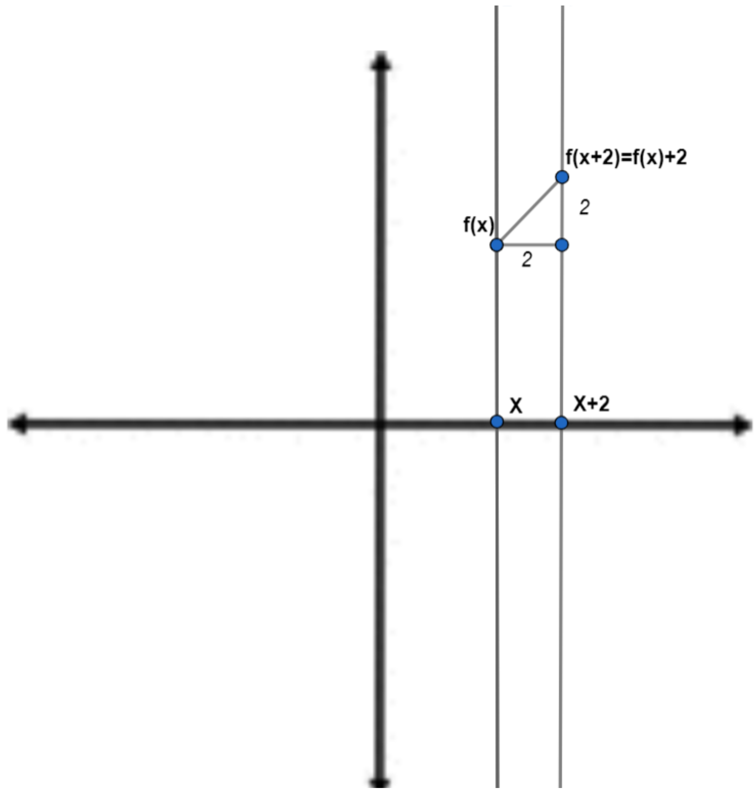
The argument could be made that some higher degree polynomial might work. However, we can disprove this doing some algebra. We say a function $f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + a_{n-2} \cdot x^{n-2} + \cdots + a_1 \cdot x + a_0$ where $a_n \neq 0$ because it is the leading coefficient. We will do this proof by equating the coefficient for x^{n-1}. Notice how I use only the first two terms because the degree for any of the next terms in $f(x)$ do not have a degree of x_{n-1}.

$$f(x+2) = a_n \cdot (x+2)^n + a_{n-1} \cdot (x+2)^{n-1} + a_{n-2} \cdot (x+2)^{n-2} + \cdots + a_1 \cdot (x+2) + a_0$$

$$= f(x) + 2 = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + a_{n-2} \cdot x^{n-2} + \cdots + a_1 \cdot x + a_0 + 2$$

By the binomial theorem, on the first two terms for $f(x+2)$,

$a_n \cdot (x+2)^n + a_{n-1} \cdot (x+2)^{n-1} = a_n \cdot (x^n + 2 \cdot x^{n-1} + \cdots + 2^n) + a_{n-1} \cdot (x^{n-1} + \ldots 2^{n-1})$.
Similarly, the coefficient of x^{n-1} is a_{n-1}.



The sum results in the coefficient of the x^{n-1} term is $2 \cdot a_n + a_{n-1} = a_{n-1}$ and thus, $a_n = 0$. However, we said at the starting that $a_n \neq 0$ and thus by contradiction from what we said, no functions f with leading term having a degree of greater than one will work. This is because if the leading term has a degree of 1, a_0 is irrelevant.

$f(x) = x + a_0$, $f(x+2) = x + 2 + a_0 = f(x) + 2 = x + a_0 + 2$ which is true. The main idea here stems from the fact that if the degree of x is 0 for the term with degree $n-1$, this term is the constant term in the polynomial and thus doesn't change. Also because the highest degree is 1, $2 = 2$.

Therefore we can say that all polynomials that suit this condition is any polynomial with degree 1 and leading coefficient of 1, which also means that $f(x)$ is in the form $x + a_0$ for any a_0.

Extra:
A nice observation one could make with the functional equation is that $\lfloor x + a_0 \rfloor$ works as well. This can be easily seen by the floor property that $\lfloor x + 2 + a_0 \rfloor = 2 + \lfloor x + a_0 \rfloor = 2 + \lfloor x + a_0 \rfloor$. Which is always true. We could take the 2 out because it doesn't affect the floor whatsoever and thus they are equal.

Problem Three B

Find all polynomials g that satisfy the equation: $g(2x) = 2g(x)$ for every real number x.

We do a similar thing to part a. We say our polynomial is $g(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 \cdot x + a_0$. Once again we equate the coefficients of the same terms.

$$g(2x) = a_n \cdot (2x)^n + a_{n-1} \cdot (2x)^{n-1} + \cdots + a_1 \cdot 2x + a_0$$
$$2g(x) = 2(a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 \cdot x + a_0)$$

Equating leading coefficients:

$$a_n \cdot (2x)^n = 2(a_n \cdot x^n) \Rightarrow (2x)^n = 2x^n \Rightarrow 2^n = 2$$

Looking at this, clearly, $n = 1$ is the only option that works. Once again bringing us to a polynomial with degree $n = 1$.
Trying the next coefficient,

$$a_{n-1} \cdot (2x)^{n-1} = 2a_{n-1} \cdot x^{n-1} \Rightarrow 2^{n-1} = 2 \Rightarrow n = 2$$

This brings some type of question in mind: why doesn't this work? The first set brings $n = 1$ while the other sets $n = 2$. Plugging the respective value of n into the exponent, we see that both of the powers is 1 for x so it is clear that the highest degree is once again 1.

We can also show this in a more "thought-based" way by noticing (instead of doing the algebra) that if the degree is greater than one, the coefficient will be far too large of a power of two. Thus, we know that the degree must be 1. So far, we have that it needs to be in the form: $a_1 \cdot x + c$. Now plugging into the original function, we see that the problem is the constant term is the issue because there is $2c$ on one side and c on the other side. Thus, we need to eliminate the constant term leaving $a_1 \cdot x$. Now this works for all real $a_1 \neq 0$.

Thus all polynomials that satisfy $g(2x) = 2g(x)$ are all polynomials in the form $a_1 \cdot x$ for all a_1.

Problem Three C

Find all polynomials Q that satisfy the equation:

$$\frac{1}{Q(x) - 1} = Q(\frac{1}{x+1})$$

for every real number x such that those denominators are nonzero.

Is this one straightforward to solve?

Before starting the solutions, I just wanted to answer the second question by saying that this was ***not*** *straightforward to solve. This solution took me a decent amount of time but I really enjoyed it.*
We start by multiplying out the denominator of the LHS,

$$1 = (Q(x) - 1) \cdot Q(\frac{1}{x+1})$$

It may not be evident from the problem statement that $(Q(x)-1)$ and $Q(\dfrac{1}{x+1})$ are reciprocals but that is the case. Meaning that $Q(x)-1$ is a regular polynomial and $Q(\dfrac{1}{x+1})$ is the reciprocal (which is not a polynomial because polynomials can't have a negative exponent to x for example $\frac{1}{x}$ is not a polynomial. On the other hand, $\frac{1}{2}$ is because it is a constant still).

However, there is a solution that lies within the word constant.

$$\frac{1}{c-1} = c$$

Results in the golden ratio meaning that we have found two constant solutions for $Q(x)$.

$$c^2 - c - 1 = 0 \Rightarrow \frac{1 \pm \sqrt{5}}{2}$$

But we need to go further to find a general solution in finding all polynomials that work.
An interesting observation is that if $x = \frac{1-a}{a}$,

$$\frac{1}{x+1} = \frac{1}{\frac{1-a}{a}+1} = \frac{a}{1} = a$$

We can utilize this in finding the general form of all polynomials. We can substitute $x = \frac{1-a}{a}$ into the polynomial making

$$1 = (Q(\tfrac{1-a}{a}) - 1) \cdot Q(a)$$

We want to make this as the product of two polynomials rather than one polynomial and one "not-polynomial" so we make use of the degree by setting $\deg(Q(x)) = t$ We can multiply $(Q(\frac{1-a}{a}) - 1)$ by a^t to make it a polynomial,

$$a^t = \left(a^t \cdot Q(\tfrac{1-a}{a}) - a^t\right) \cdot Q(a)$$

What we now have is very informative about the function $Q(x)$ because of the degree. Note that $\deg(Q(a)) = t$ which leaves us with:

$$\left(a^t \cdot Q(\tfrac{1-a}{a}) - a^t\right)$$

The above cannot have any a in its simplification as it would then increase the degree which we have set to t forcing the above to be a constant term which we can set as c.

$$c = a^t \cdot Q(\tfrac{1-a}{a}) - a^t$$

We plug this back into the original making,

$$a^t = c \cdot Q(a) \Rightarrow Q(a) = \frac{a^t}{c} \tag{1}$$

We can also manipulate the inside of c by dividing by a^t,

$$c = a^t \cdot Q(\tfrac{1-a}{a}) - a^t \Rightarrow 1 + \frac{c}{a^t} = Q(\tfrac{1-a}{a}) \tag{2}$$

Plugging $\frac{1-a}{a}$ in equation (1), $Q(\frac{1-a}{a}) = \dfrac{(\frac{1-a}{a})^t}{c} = \dfrac{(1-a)^t}{c \cdot a^t} = 1 + \frac{c}{a^t}$. Multiplying by a^t, $\dfrac{(1-a)^t}{c} = a^t + c$. For the coefficients of a^t to match, $c = \pm 1$.

First trying $c = 1$, $(1-a)^t = a^t + 1$ and no $t \geq 0$ will work for this. The closest we get is $t = 1$ where the signs for a^t are opposite. Now trying $c = -1$, $(1-a)^t = -a^t + 1$ and $t = 1$ works! Thus we found that the constant $c = -1$ and the degree for $Q(a) = 1$ we can easily build our polynomial that works from where we said earlier on that $Q(a) = \frac{a^t}{c}$ so $Q(x) = -x$.

We can see this works by plugging it back into the problem statement, $\frac{1}{-x-1} = \frac{-1}{x+1}$ which is true.

In hindsight, we didn't need to do the constant solutions at the very starting because we can derive that from the equation: $\dfrac{(1-a)^t}{c} = a^t + c$ by setting $t = 0$, $\frac{1}{c} = 1 + c$ resulting in the golden ratio once again.

Problem Three D

Make your own choice for J and H, formulate the problem, and find a solution. Choose J and H to be non-trivial, but still simple enough to allow you to make good progress toward a solution.

Attempt 1: Originally I thought maybe I should do e^x for both J and H. I got "some" type of progress but it was mostly calculus-based(which involved converting to Taylor and trying to do manipulations based on that) so I got rid of that idea. I honestly don't know what motivated this idea other than the fact that the the derivative of $e^x = e^x$.

Then I thought maybe I should do $\ln(x)$ instead. Then I thought maybe to minimize the function I should make it the reciprocal of $Q(x)$ on the other side. Making it,

$$\ln(Q(x)) = \frac{1}{Q(x)}$$

$$\log_e(Q(x)) = \frac{1}{Q(x)}$$

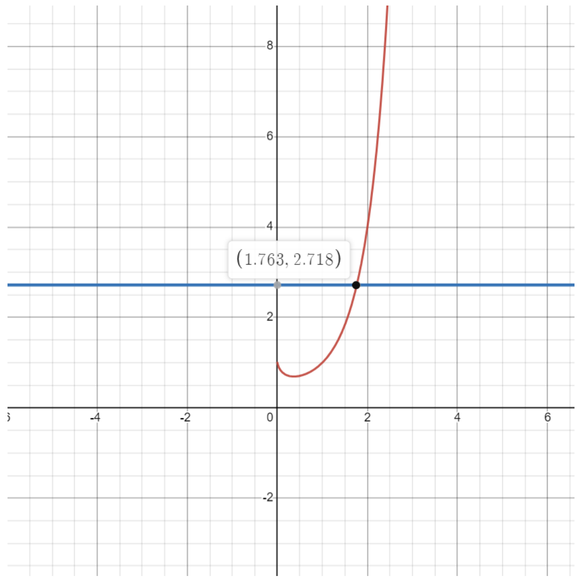
$$e^{\frac{1}{Q(x)}} = Q(x)$$

$$e = [Q(x)]^{Q(x)}$$

It is clear from here that $Q(x)$ is some constant, so let $Q(x) = c$.

$$e = a^a$$

There are two ways to proceed from here. One to graph x^x and see where it intersects $y = e$ and another to use the Lambert W function to find an "explicit" form of the intersection. In the first,



On the graph above, it shows that $x = 1.763$.

Extra: We can also use the Lambert W function to approximate with an explicit form. We can do so by noticing that we could take the natural log of both sides of the above equation.

$$\ln(e) = \ln(a^a)$$

$$1 = a \cdot \ln(a)$$

However, we need to get into the form of $x \cdot e^x$ in order to use the Lambert W function. We can do this by noticing that $a = e^{\ln(a)}$.

$$1 = e^{\ln(a)} \cdot \ln(a)$$

Now taking the "W" of both sides,

$$W(1) = W(e^{\ln(a)} \cdot \ln(a))$$

Because $W(x)$ and $x \cdot e^x$ are inverses, $W(x \cdot e^x) = x$. In this case, the RHS equates to $\ln(a)$.

$$\ln(a) = W(1)$$

Thus,

$$a = Q(x) = e^{W(1)} = \frac{1}{W(1)}.$$

We can get to the last equality case by noticing that

$$W(a)e^{W(a)} = a \text{ so } e^W(a) = \frac{a}{W(a)}$$

Note: To solve this problem, I looked at a paper on the lambert W function by Princeton edu: