1 Examples

1.1 Example 1

Prove that $\lim_{x \to 2} x^3 = 8$.

We wish to find $\delta > 0$ in terms of ε such that

$$0 < |x - 2| < \delta \implies \left|x^3 - 8\right| < \varepsilon.$$

We have

$$\begin{aligned}
\left|x^3 - 8\right| &= |x - 2| \left|x^2 + 2x + 4\right| \\
&< \delta \left|(x^2 - 4x + 4) + 6x - 12 + 12\right| \\
&= \delta \left|(x - 2)^2 + 6(x - 2) + 12\right| \\
&< \delta \left|\delta^2 + 6\delta + 12\right|
\end{aligned}$$

Using the triangle inequality, we have

$$\begin{aligned}
\left|x^3 - 8\right| &< \delta \left|\delta^2 + 6\delta + 12\right| \\
&< \delta \left(\left|\delta^2\right| + 6|\delta| + |12|\right) \\
&= \delta \left(\delta^2 + 6\delta + 12\right)
\end{aligned}$$

We have two cases: when $\varepsilon < 19$, and when $\varepsilon \geq 19$.
When $\varepsilon < 19$, we choose $\delta = \frac{\varepsilon}{19} < 1$. Then, we have

$$\begin{aligned} \left|x^3 - 8\right| &< \delta\left(\delta^2 + 6\delta + 12\right) \\ &< \frac{\varepsilon}{19}\left(1^2 + 6 \cdot 1 + 12\right) \\ &= \frac{\varepsilon}{19}(1 + 6 + 12) \\ &= \frac{\varepsilon}{19} \cdot 19 \\ &= \varepsilon \end{aligned}$$

When $\varepsilon \geq 19$, we choose $\delta = 1 \leq \frac{\varepsilon}{19}$. Then, we have

$$\begin{aligned} \left|x^3 - 8\right| &< \delta\left(\delta^2 + 6\delta + 12\right) \\ &= \frac{\varepsilon}{19}\left(1^2 + 6 \cdot 1 + 12\right) \\ &= \varepsilon \end{aligned}$$

So, to make this less than ε, we choose δ such that

$$\delta = \min\left\{\frac{\varepsilon}{19}, 1\right\}$$

Thus, we have proved that, given $\varepsilon < 0$, we can find $\delta > 0$ in terms of ε such that

$$0 < |x - 2| < \delta \implies \left|x^3 - 8\right| < \varepsilon.$$

Therefore, we have proved that $\lim_{x \to 2} x^3 = 8$. $\square$

1.2 Example 2

Let f and g be functions and $a \in \mathbb{R}$. If $\lim\limits_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, find $\lim\limits_{x \to a} (f+g)(x)$.
We can prove that the limit is $L + M$. We need to show that $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$0 < |x - a| < \delta \implies |(f+g)(x) - (L+M)| < \varepsilon.$$

We can also use the fact that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Because $\lim\limits_{x \to a} f(x) = L$, we can choose δ_f such that

$$0 < |x - a| < \delta_f \implies |f(x) - L| < \frac{\varepsilon}{2},$$

and because $\lim\limits_{x \to a} g(x) = M$, we can choose δ_g such that

$$0 < |x - a| < \delta_g \implies |g(x) - M| < \frac{\varepsilon}{2}.$$

If we set $\delta = \min\{\delta_f, \delta_g\}$, the two inequalities on the left are satisfied, so we can write

$$0 < |x - a| < \delta \implies |f(x) - L| < \frac{\varepsilon}{2} \text{ and } |g(x) - M| < \frac{\varepsilon}{2}$$

Adding the two inequalities on the right and applying the triangle inequality, we have

$$\varepsilon > |f(x) - L| + |g(x) - M| \geq |(f+g)(x) - (L+M)|.$$

Therefore, we have proved that $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$0 < |x - a| < \delta \implies |(f + g)(x) - (L + M)| < \varepsilon,$$

so $\lim_{x \to a} (f + g)(x) = L + M$.$\square$

1.3 Example 3

$\lim_{x \to \infty} f(x) = \lim_{z \to 0^+} f\left(\frac{1}{z}\right)$

First, we will show that $\lim_{x \to \infty} f(x) = L$ implies that $\lim_{z \to 0^+} f\left(\frac{1}{z}\right) = L$. From $\varepsilon - N$, we have that since $\lim_{x \to \infty} f(x) = L$, we can find $N > 0$ for every $\varepsilon > 0$ such that

$$x > N \implies |f(x) - L| < \varepsilon.$$

If we let $\delta = \frac{1}{N}$, we can write

$$0 < z < \delta = \frac{1}{N} \implies \frac{1}{z} > N.$$

Setting $x = \frac{1}{z}$, and plugging this into the $\varepsilon - N$ definition for $\lim_{x \to \infty} f(x) = L$, we have

$$\frac{1}{z} > N \implies \left| f\left(\frac{1}{z}\right) - L \right| < \varepsilon.$$

So, this means that

$$0 < z < \delta \implies \left| f\left(\frac{1}{z}\right) - L \right| < \varepsilon,$$

which is the $\delta - \varepsilon$ definition for $\lim_{z \to 0^+} f\left(\frac{1}{z}\right) = L$. Therefore, we have proved that

$$\lim_{x \to \infty} f(x) = L \implies \lim_{z \to 0^+} f\left(\frac{1}{z}\right) = L.$$

Next, we will need to prove the converse of this, which then finishes the proof. From $\delta - \varepsilon$ on $\lim_{z \to 0^+} f\left(\frac{1}{z}\right) = L$, we have that

$$0 < z < \delta \implies \left| f\left(\frac{1}{z}\right) - L \right| < \varepsilon.$$

Let $N = \frac{1}{\delta}$. We then have

$$x > N \implies x > \frac{1}{\delta} \implies 0 < \frac{1}{x} < \delta.$$

Setting $\frac{1}{x} = z$, we can plug this into $\delta - \varepsilon$ on $\lim_{z \to 0^+} f\left(\frac{1}{z}\right) = L$. We have

$$0 < \frac{1}{x} < \delta \implies \left| f\left(\frac{1}{\frac{1}{x}}\right) - L \right| = |f(x) - L| < \varepsilon.$$

This means that $x > N$ implies that $|f(x) - L| < \varepsilon$, which is the $\varepsilon - N$ definition of $\lim_{x \to \infty} f(x) = L$. Therefore, we have proved that

$$\lim_{z \to 0^+} f\left(\frac{1}{z}\right) = L \implies \lim_{x \to \infty} f(x) = L.$$

Therefore, since

$$\lim_{z \to 0^+} f\left(\frac{1}{z}\right) = L \iff \lim_{x \to \infty} f(x) = L,$$

we have proved that $\lim_{x \to \infty} f(x) = \lim_{z \to 0^+} f\left(\frac{1}{z}\right)$. $\square$

1.4 Example 4 (bonus)

$\frac{d}{dx}\int_c^x f(t)\,dt = f(x)$

We define a function g such that

$$g(x) = \int_c^x f(t)\,dt.$$

We will show that $g'(x) = f(x)$ using the limit definition of the derivative and $\delta - \varepsilon$. In order to prove that

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f(x),$$

we will use $\delta - \varepsilon$. We must show that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |h| < \delta \implies \left|\frac{g(x+h)-g(x)}{h} - f(x)\right| < \varepsilon.$$

Since f is continuous, $\lim\limits_{t\to x} f(t) = f(x)$, so using $\delta - \varepsilon$ on this, we can choose δ such that

$$|t-x| < \delta \Rightarrow |f(t)-f(x)| < \varepsilon \Rightarrow f(x)-\varepsilon < f(t) < f(x)+\varepsilon.$$

So, if $|h| < \delta$, we can integrate this from x to $x+h$ to get

$$\int_x^{x+h}(f(x)-\varepsilon)\,dt < \int_x^{x+h} f(t)\,dt < \int_x^{x+h}(f(x)+\varepsilon)\,dt.$$

The integrals on the left and right are both constant, since they are in terms of x, and the integral is evaluated with respect to t. Simplifying this, we have

$$h(f(x) - \varepsilon) < \int_x^{x+h} f(t)\,dt < h(f(x) - \varepsilon).$$

Dividing by h, we have

$$f(x) - \varepsilon < \frac{\displaystyle\int_x^{x+h} f(t)\,dt}{h} < f(x) + \varepsilon.$$

Since

$$\int_x^{x+h} f(t)\,dt = \int_c^{x+h} f(t)\,dt - \int_c^{x} f(t)\,dt = g(x+h) - g(x),$$

we can plug this in to get

$$\begin{aligned} f(x) - \varepsilon < \frac{g(x+h) - g(x)}{h} &< f(x) - \varepsilon \\ -\varepsilon < \frac{g(x+h) - g(x)}{h} - f(x) &< \varepsilon \\ \left| \frac{g(x+h) - g(x)}{h} - f(x) \right| &< \varepsilon. \end{aligned}$$

This means that, from $0 < |h| < \delta$, we can go through all these steps to get

$$\left| \frac{g(x+h) - g(x)}{h} - f(x) \right| < \varepsilon.$$

Therefore, by $\delta - \varepsilon$ we have shown that

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x),$$

so

$$\frac{d}{dx} \int_c^x f(t)\,dt = f(x),$$

as desired. $\square$

Note that this not only does not require the assumption that an antiderivative of f exists, but also, we have proved that, as a result of this, that an antiderivative of f must exist.

2 Exercises

2.1 Exercise 1.1

Find, with proof, $\lim_{x \to -1} (1 - 2x)$

2.2 Exercise 1.2

Prove that $\lim_{x \to c} \left(\frac{1}{x} \right) = \frac{1}{c}$ where $c > 0$.

2.3 Exercise 1.3

Show that $\lim_{x \to 3} (4x - 5) \neq 10$.

2.4 Exercise 1.4

Prove that

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \iff \lim_{x \to a} f(x) = L.$$

3 Exercise solutions

3.1 Exercise 1.1

Find, with proof, $\lim_{x \to -1}(1-2x)$ We can show that the limit is 3. We need to show that $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$0 < |x-(-1)| = |x+1| < \delta \implies |(1-2x)-3| = |-2x-2| < \varepsilon.$$

The second inequality simplifies down into

$$|-2x-2| = |(-2)(x+1)| = 2|x+1| < 2\delta < \varepsilon.$$

We then choose $\delta = \dfrac{\varepsilon}{2}$, which satisfies the conditions. $\square$

3.2 Exercise 1.2

Prove that $\lim_{x \to c}\left(\dfrac{1}{x}\right) = \dfrac{1}{c}$ where $c > 0$. We need to show that $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$0 < |x-c| < \delta \implies \left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon.$$

The right inequality simplifies into

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c-x}{xc}\right| = \frac{|x-c|}{c|x|} < \frac{\delta}{c|x|}.$$

Since we want x near c, we can restrict x to an interval around c, for example, $|x-c| < \frac{c}{2}$. So, we get

$$
\begin{aligned}
-\frac{c}{2} < x - c < \frac{c}{2} \\
0 < \frac{c}{2} < x < \frac{3c}{2} \\
\frac{1}{x} < \frac{2}{c}.
\end{aligned}
$$

Then, we have

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \frac{\delta}{c|x|} < \frac{2\delta}{c^2}.$$

We choose $\delta = \frac{c^2}{2}\varepsilon$, so we have

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \frac{2\delta}{c^2} = \varepsilon,$$

which is what we are looking for.
From the two restrictions we have, $|x-c| < \frac{c}{2}$ and $\delta = \frac{c^2}{2}\varepsilon$, we have that $\delta = \min\left\{\frac{c}{2}, \frac{c^2}{2}\varepsilon\right\}$. Thus, we have proved that $\lim\limits_{x\to c}\left(\frac{1}{x}\right) = \frac{1}{c}$ where $c > 0$. $\square$

3.3 Exercise 1.3

Show that $\lim\limits_{x\to 3}(4x-5) \neq 10$.
We will use proof by contradiction. We can assume that $\lim\limits_{x\to 3}(4x-5) =$

10, and we can choose a value of ε, say 1. Then, $\exists\delta > 0$ such that

$$0 < |x - 3| < \delta \implies |(4x - 5) - 10| < 1.$$

Simplifying the inequality on the right, we have

$$\begin{aligned} |(4x - 5) - 10| = |4x - 15| &< 1 \\ -1 < 4x - 15 &< 1 \\ 14 < 4x &< 16 \\ 3.5 < x &< 4. \end{aligned}$$

No matter what we choose δ to be, if we choose a value of x less than 3, but still within δ of 3, the above inequality cannot be true, which is a contradiction. Thus, $\lim_{x\to 3}(4x - 5) \neq 10$. $\square$

3.4 Exercise 1.4

Prove that

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L \iff \lim_{x\to a} f(x) = L.$$

We can first prove that

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L \implies \lim_{x\to a} f(x) = L.$$

Then, we have that $\forall\varepsilon > 0, \exists\delta_-$ such that

$$0 < a - x < \delta_- \implies |f(x)_L| < \varepsilon,$$

and $\exists \delta_+ > 0$ such that

$$0 < x - a < \delta_+ \implies |f(x)_L| < \varepsilon.$$

If we let $\delta = \min\{\delta_-, \delta_+\}$, then the top two statements are both satisfied, and we have

$$0 < |x - a| < \delta \implies |f(x)_L| < \varepsilon.$$

Thus, we have proved that

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \implies \lim_{x \to a} f(x) = L.$$

Next, we prove the converse of this. We know that $\lim_{x \to a} f(x) = L$, so $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x)_L| < \varepsilon.$$

We need to find $\delta_- > 0$ such that

$$0 < a - x < \delta \implies |f(x)_L| < \varepsilon.$$

If we let $\delta_- = \delta$, this is satisfied, so we have $\lim_{x \to a^-} f(x) = L$. The same logic applies to δ_+ and $\lim_{x \to a^+} f(x) = L$, so we have proved that that

$$\lim_{x \to a} f(x) = L \implies \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Therefore, we have proved that

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \iff \lim_{x \to a} f(x) = L.$$

$\square$

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Figure 1: how you should feel now