The Multigrid Method

Review of Last Lecture and Outline

- Review Poisson equation
- Overview of Methods for Poisson Equation
- Jacobi's method
- Red-Black SOR method
- Conjugate Gradients

Reduce to sparse-matrix-vector multiply
Need them to understand Multigrid

- Multigrid
- Comparison of methods

Motivation for Multigrid

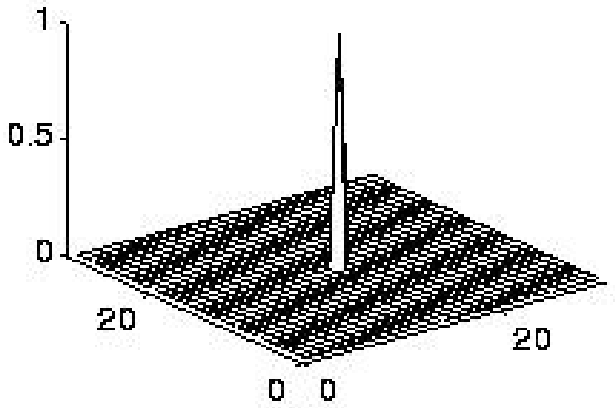
- Recall that Jacobi, SOR, CG, or any other sparse-matrix-vector-multiply-based algorithm can only move information one grid-cell at a time, due to local structure of stencil ➔It takes $O(N^{1/2})$ steps to get information across the grid

- Faster convergence in O(1) steps requires moving information across grid faster than to one neighboring grid cell per step.

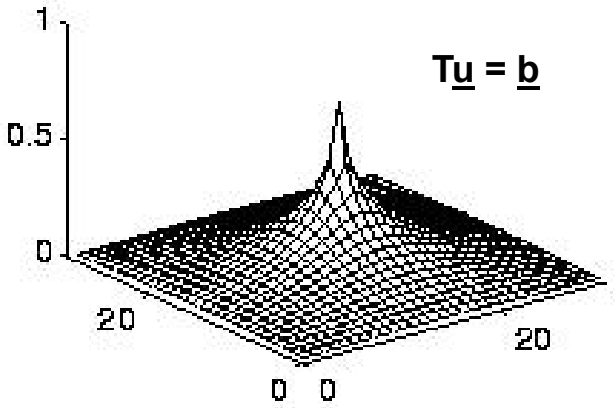
- Can be shown that decreasing error by fixed factor $c<1$ and takes $\Omega(\log n)$ steps → Convergence to a fixed error $\varepsilon < 1$ takes $\Omega(\log n)$ steps

Multigrid Motivation

Right Hand Side

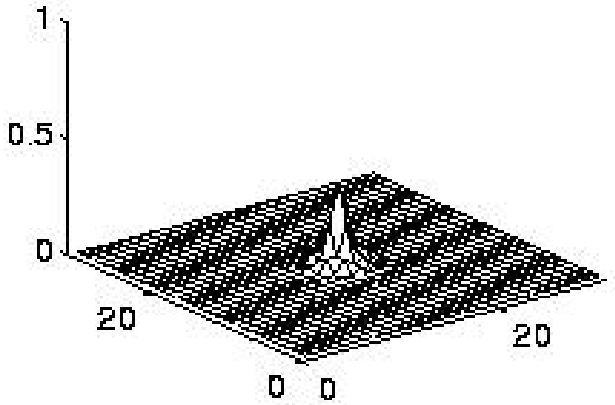


True Solution

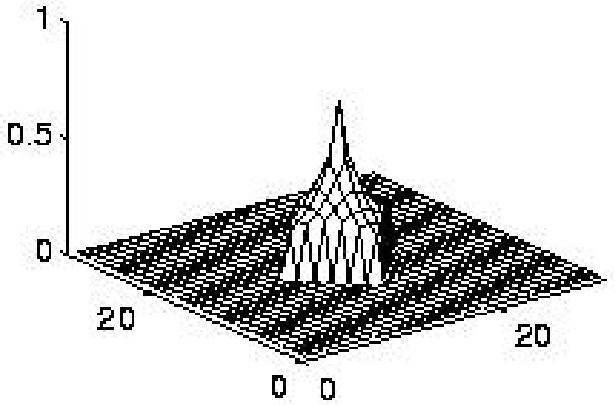


Tu = b

5 steps of Jacobi



Best 5 step solution **(with optimal ω)**



Takes O(n) steps to propagate information across an nxn grid

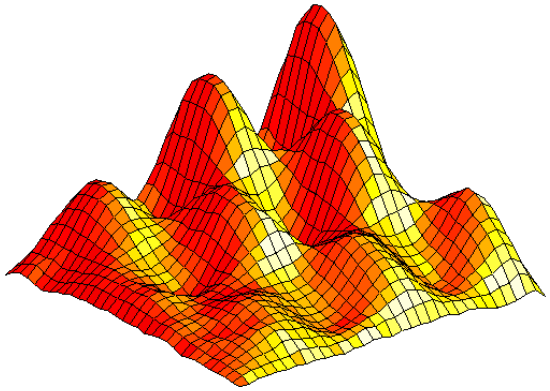
Multigrid Overview

° **Basic Algorithm:**

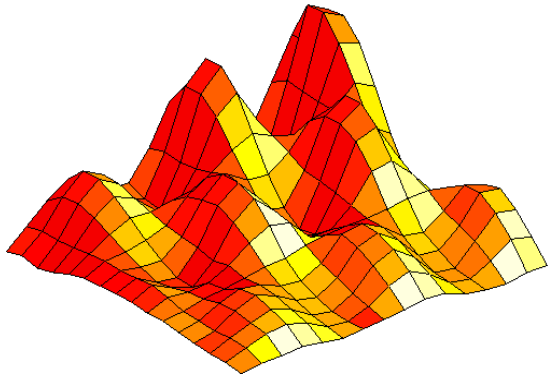
- **Replace problem on fine grid by an approximation on a coarser grid**
- **Solve the coarse grid problem approximately, and use the solution as a starting guess for the fine-grid problem, which is then iteratively updated**
- **Solve the coarse grid problem recursively, i.e. by using a still coarser grid approximation, etc.**

° **Success depends on coarse grid solution being a good approximation to the fine grid**

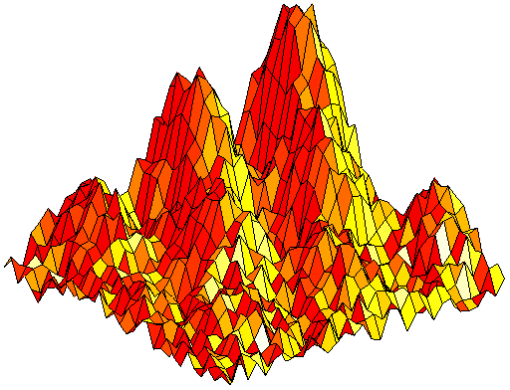
Fine and Coarse Approximations



Fine

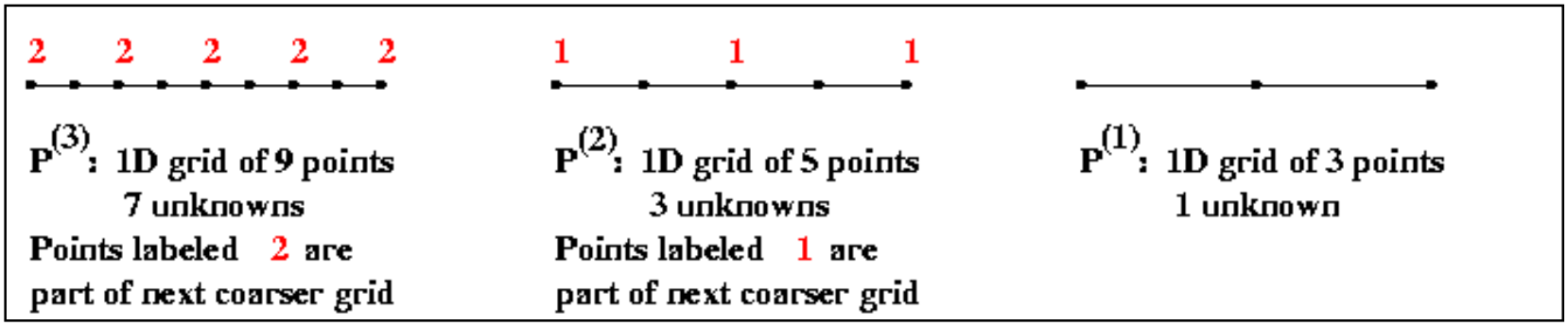


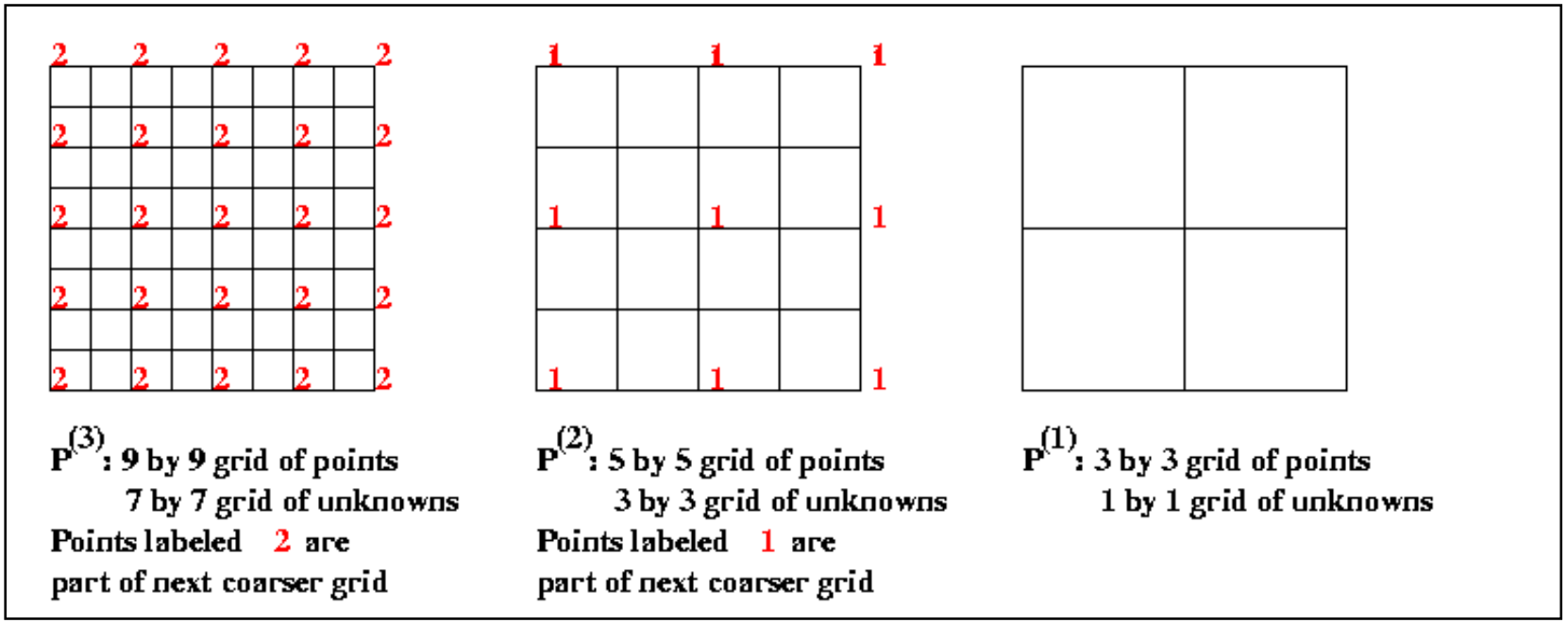
Coarse



Multigrid Sketch, continued

- $P^{(m)}$, $P^{(m-1)}$, … , $P^{(1)}$ is sequence of problems from finest to coarsest





Multigrid Operators (1)

° For problem $P^{(i)}$ solve $T_i x_i = b_i$:
  - b_i is the RHS and
  - x_i is the current estimated solution

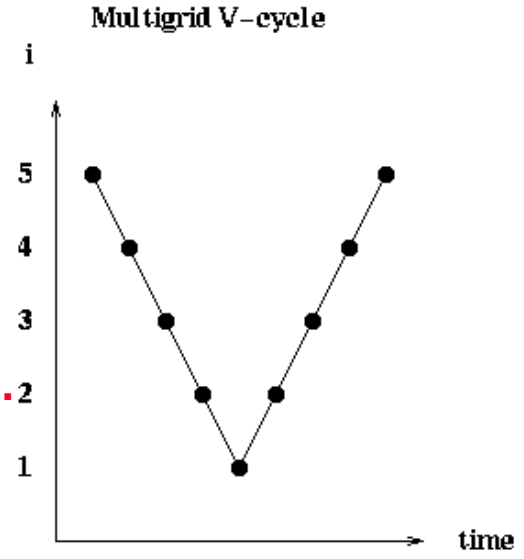
  } both live on grids of size 2^i-1
  - T_i is implicit in the operators below.

° All the following operators (next slide) just average values on neighboring grid points
  - Neighboring grid points on coarse problems are far away in fine problems, so information moves quickly on coarse problems
  - Levels will be constructed explicitly
  - (→ next slide gives an overview of the operators)

Multigrid Operators (2)

- The restriction operator R_i maps $P^{(i)}$ to $P^{(i-1)}$
  - Restricts problem on fine grid $P^{(i)}$ to coarse grid $P^{(i-1)}$ by sampling or averaging
  - $b_{i-1} = R_i (b_i)$
- The interpolation operator In_{i-1} maps an approximate solution x_{i-1} to an x_i
  - Interpolates solution on coarse grid $P^{(i-1)}$ to fine grid $P^{(i)}$
  - $x_i = In_{i-1} (x_{i-1})$
- The solution operator S_i takes $P^{(i)}$ and computes an improved solution x(i) on same grid
  - Uses "weighted" Jacobi or SOR
  - $x_{i, improved} = S_i (b_i , x_i)$
- Details of these operators follow after describing overall algorithm

Multigrid V-Cycle Algorithm (recursive)



Function MGV (b_i , x_i)

… Solve $T_i\ x_i = b_i$ given b_i and an initial guess for x_i

… return an improved x_i

if (i = 1)

compute exact solution x_1 of $P^{(1)}$ only 1 unknown.

return x_1

else

$x_i = S_i (b_i , x_i)$ improve solution by damping high frequency error,

$r_i = T_i\ x_i - b_i$ compute residual,

$d_i = In_{i-1}(MGV (R_i(r_i) , 0))$ solve $T_i\ d_i = r_i$ recursively,

$x_i = x_i - d_i$ correct fine grid solution,

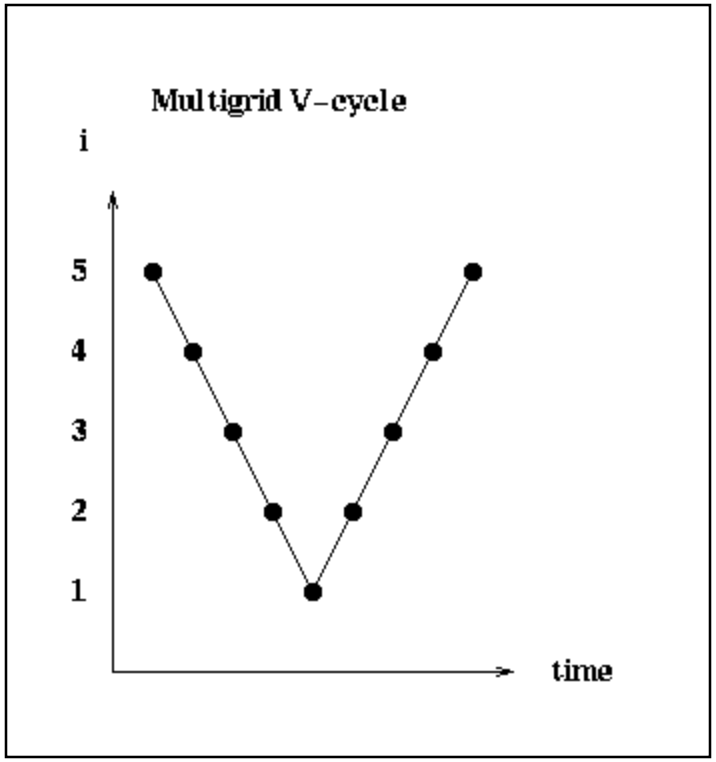
$x_i = S_i(b_i, x_i)$ improve solution again.

return x_i

Why is this called a V-Cycle?

° Just a picture of the call graph

° In time a V-cycle looks like the following



Complexity of a V-Cycle

° On a serial machine

- Work at each "dot" in the V-cycle is *O*(the number of unknowns)
- Cost of level i is $(2^i-1)^2 = O(4^i)$ (for a 2D grid)
- If finest grid level is m, total time is:

$$\sum_{i=1}^{m} O(4^i) = O(4^m) = O(\#\text{ unknowns})$$

° On a parallel machine (PRAM)

- with one processor per grid point and free communication, each step in the V-cycle takes constant time, O(1)
- Total V-cycle time is $O(m)$ = *O*(log #unknowns)

Full Multigrid (FMG)

° **Intuition:**

- improve solution by doing multiple V-cycles
- avoid expensive fine-grid (high frequency) cycles
- analysis of why this works is beyond the scope of this class

Function FMG (b_m , x_m)

… return improved x_m given initial guess

compute the exact solution x_1 of $P^{(1)}$

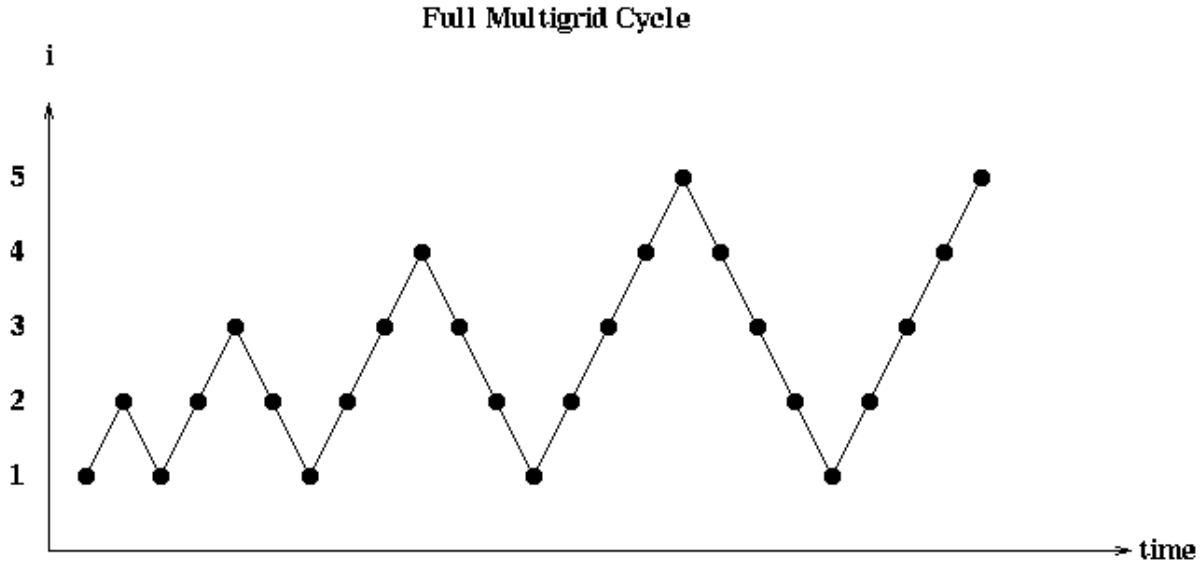
for i=2 to m

x_i = MGV (b_i, In $_{i-1}$ (x_{i-1}))

° **In words:**

- Solve the problem with 1 unknown
- Given a solution to the coarser problem, $P^{(i-1)}$, map it to starting guess for $P^{(i)}$
- Solve the finer problem using the Multigrid V-cycle

Full Multigrid Cost Analysis



° One V for each call to FMG
  - one also use "W"s and other compositions

° Serial time: $\Sigma_{i=1}^{m} O(4^i) = O(4^m) = O(\#\text{ unknowns})$

° PRAM time: $\Sigma_{i=1}^{m} O(i) = O(m^2) = O(\log^2 \#\text{ unknowns})$

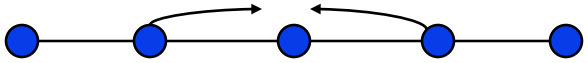
Complexity of Solving Poisson's Equation

- **Theorem: error ε after one FMG call is ≤ c times the error before, where c < 1/2, and independent of # unknowns**
- **x^k = FMG (b, x^{k-1}) ==> $\varepsilon(x^k) < 1/2\ \varepsilon(x^{k-1})$ (i.e., at least 1 bit per FMG iteration. x^k =solution after k^{th} FMG iteration)**
- **Corollary: We can make the error ε < tol, for any fixed tolerance in a fixed number of steps, independent of size of the finest grid**
- **This is the most important convergence property of MG, distinguishing it from other methods, which converge more slowly for large grids**
- **Total complexity is just proportional to the cost of one FMG call**

The Solution Operator S_i - Details

° The solution operator S_i , is a weighted Jacobi op.

° Consider the 1D problem



° At level i, pure Jacobi replaces:

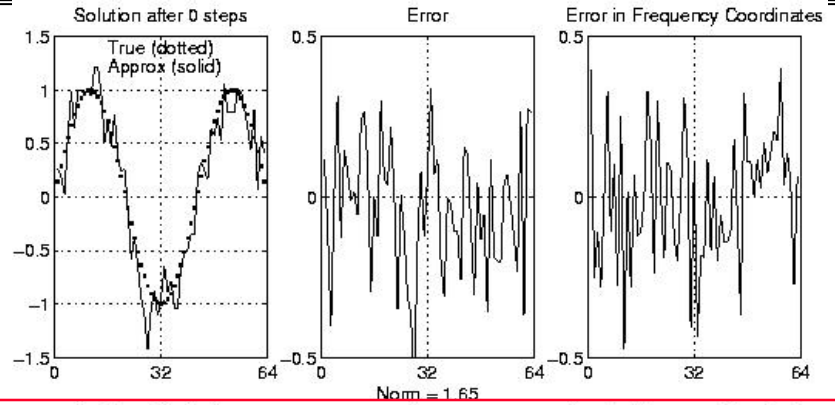
$$x(j) := 1/2\ (x(j-1) + x(j+1) + b(j))$$

° Weighted Jacobi uses:

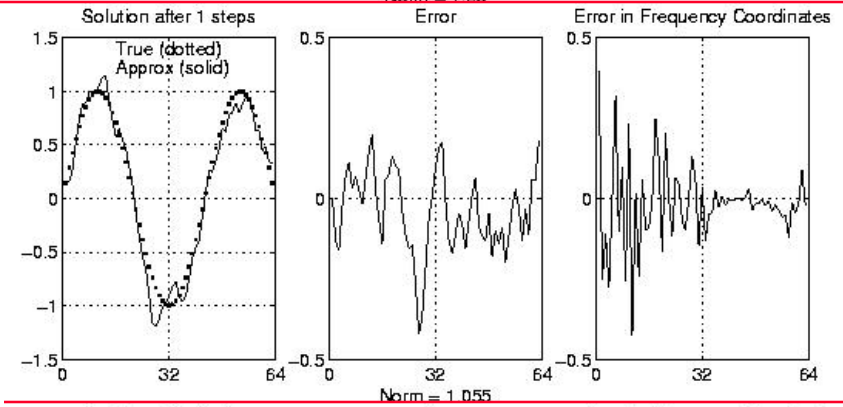
$$x(j) := 1/3\ (x(j-1) + x(j) + x(j+1) + b(j))$$

° In 2D, similar average of nearest neighbors

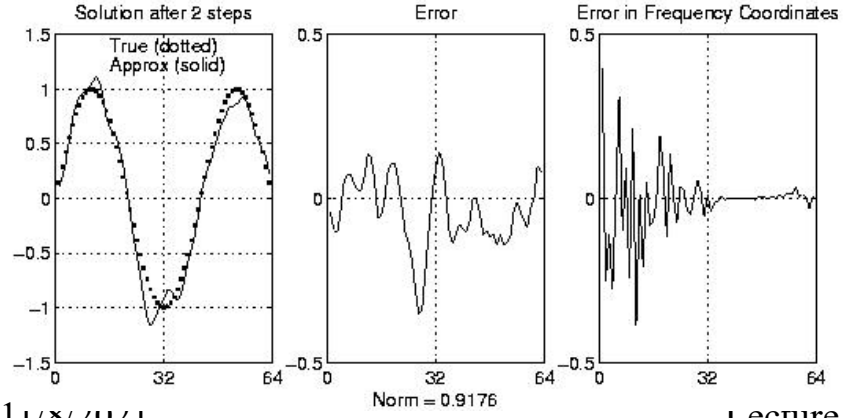
Weighted Jacobi chosen to damp high frequency error



Initial error
- **"Rough"**
- **Lots of high frequency components**
- **Norm = 1.65**



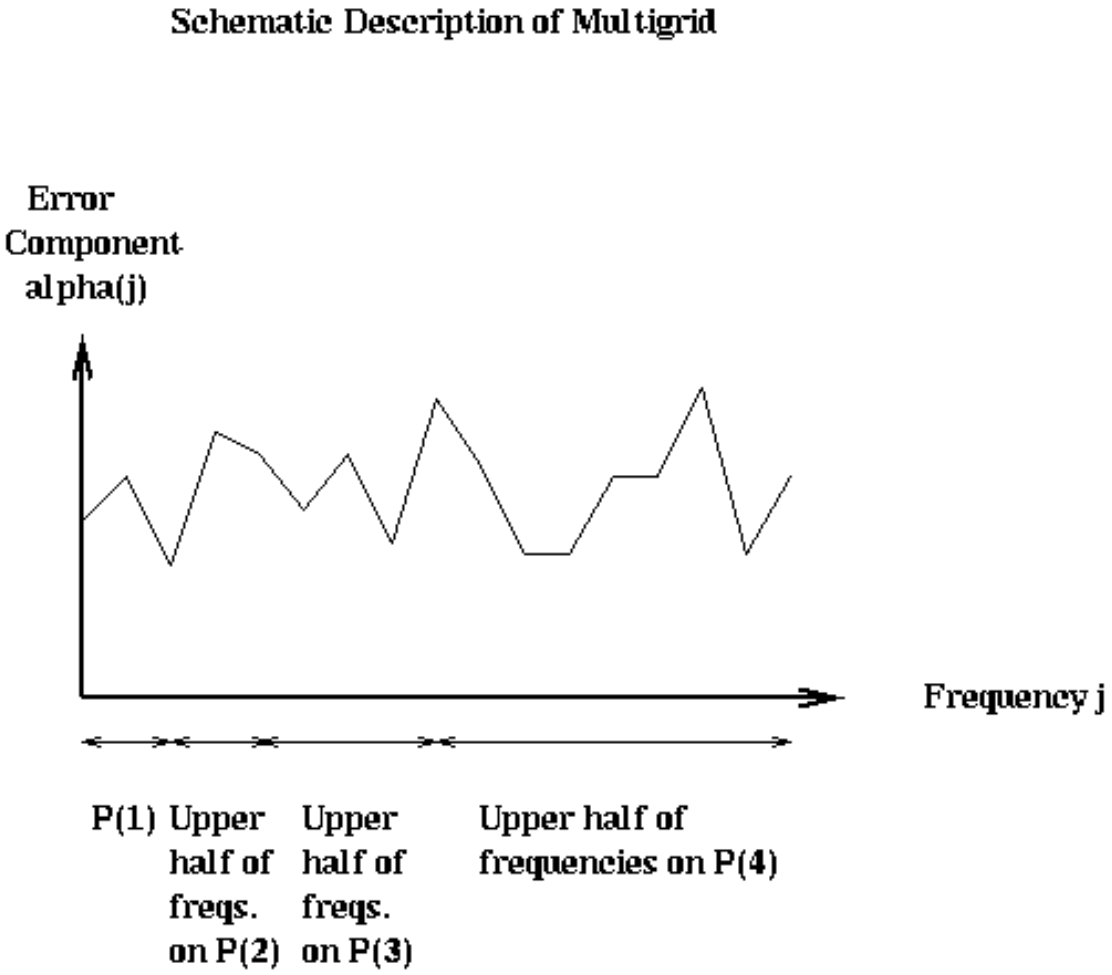
Error after 1 Jacobi step
- **"Smoother"**
- **Less high frequency component**
- **Norm = 1.055**



Error after 2 Jacobi steps
- **"Smooth"**
- **Little high frequency component**
- **Norm = .9176,**
  won't decrease much more

Multigrid as Divide and Conquer Algorithm

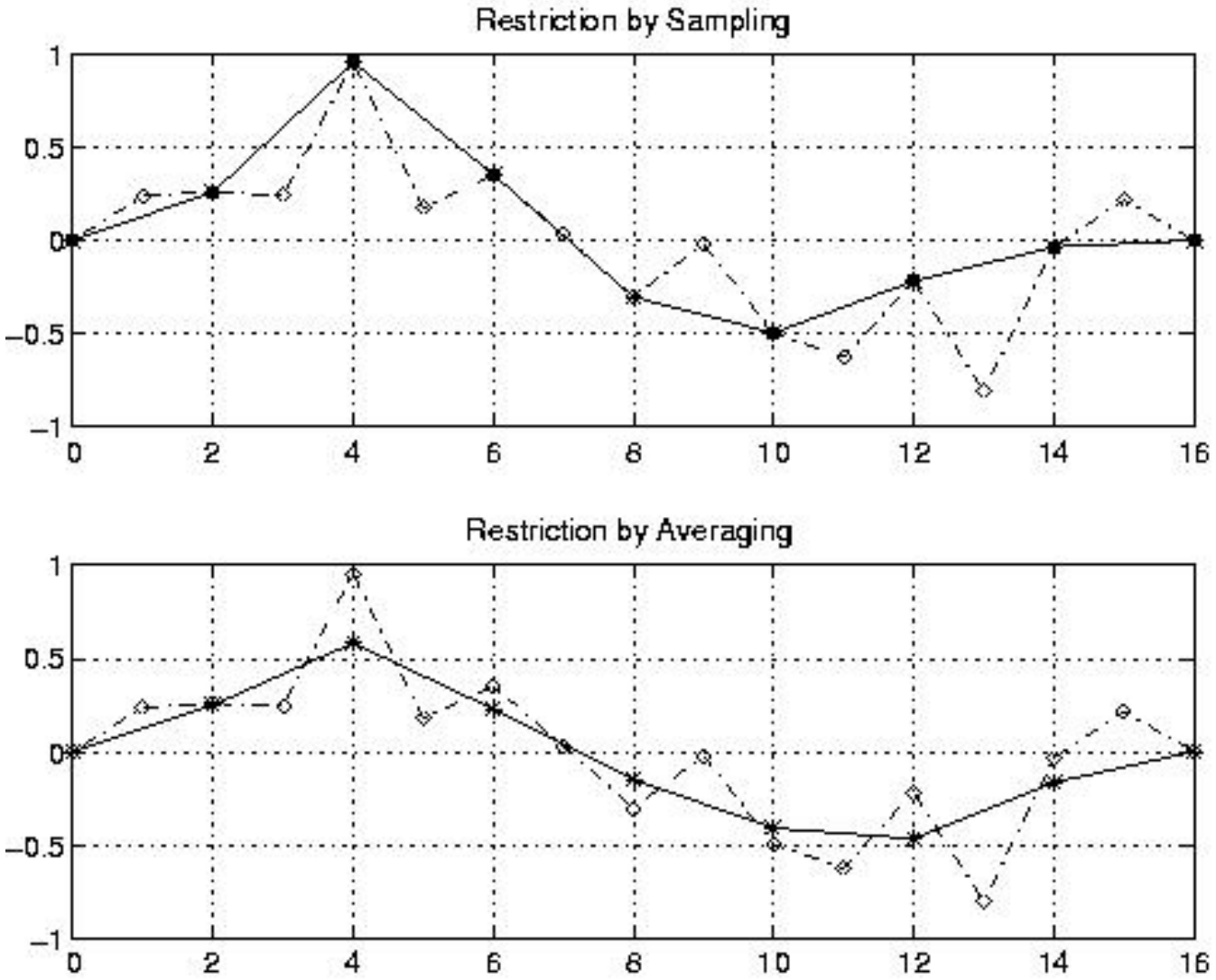
° Each level in a V-Cycle reduces the error in one part of the frequency domain



The Restriction Operator R_i - Details

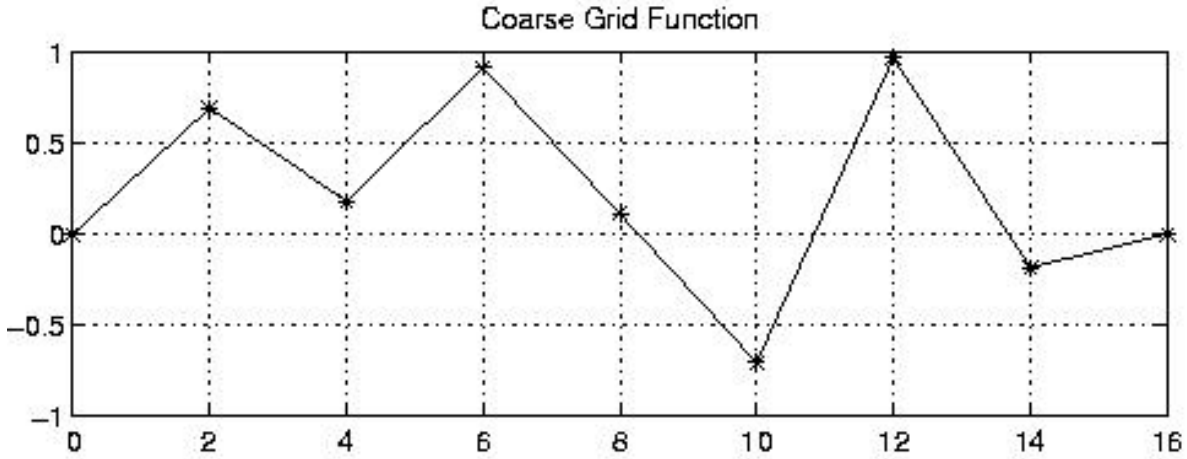
- The restriction operator R_i takes
  - a problem $P^{(i)}$ with RHS b_i and
  - maps it to a coarser problem $P^{(i-1)}$ with RHS b_{i-1}
- In 1D, average values of neighbors
  - $x_{coarse}(j) = 1/4 * x_{fine}(j-1) + 1/2 * x_{fine}(j) + 1/4 * x_{fine}(j+1)$

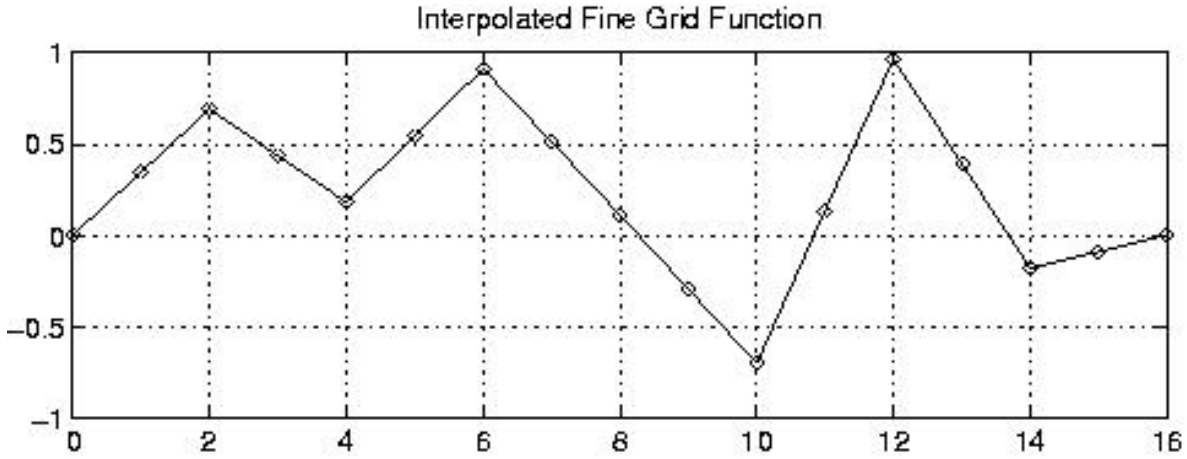
The Restriction Operator R_i - Details



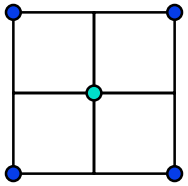
° **In 2D, average with all 8 neighbors (N,S,E,W,NE,NW,SE,SW)**

Interpolation Operator In_i

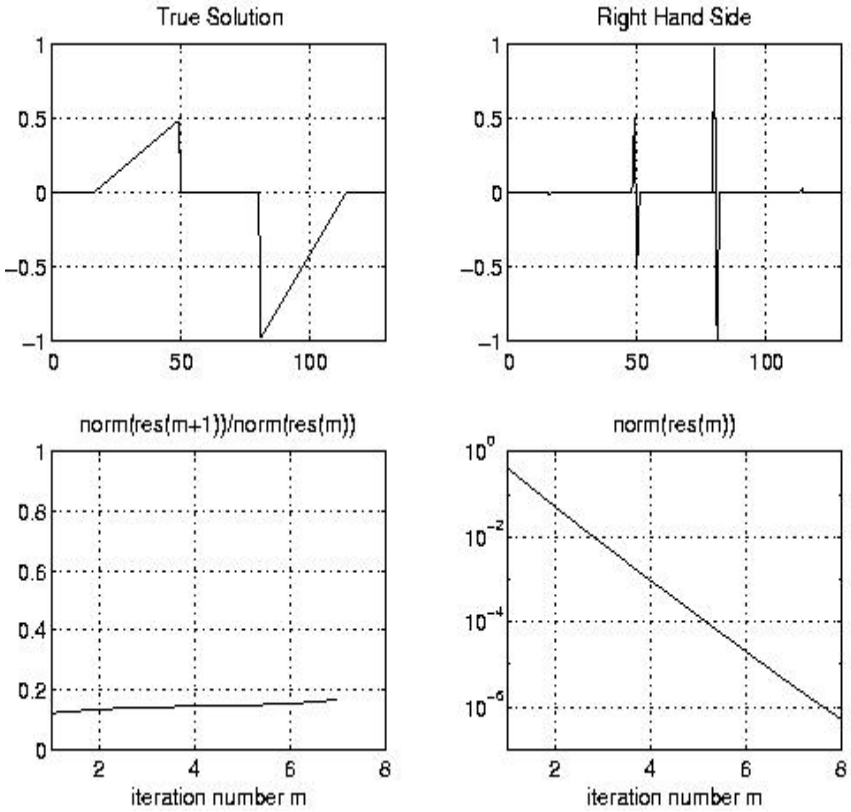


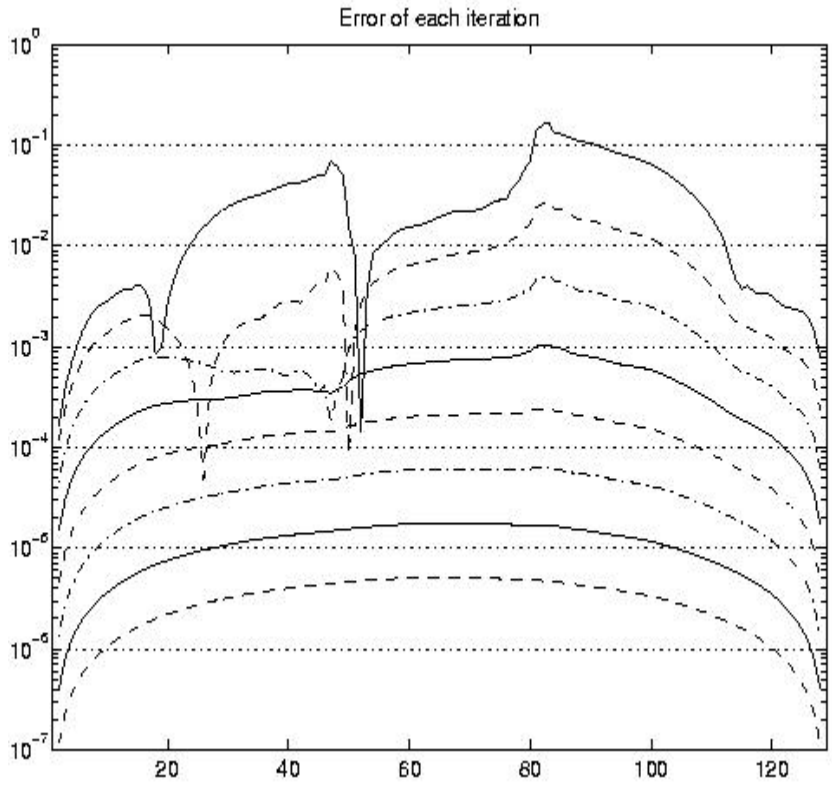


° **In 2D, interpolation requires averaging with 4 nearest neighbors (NW,SW,NE,SE)**



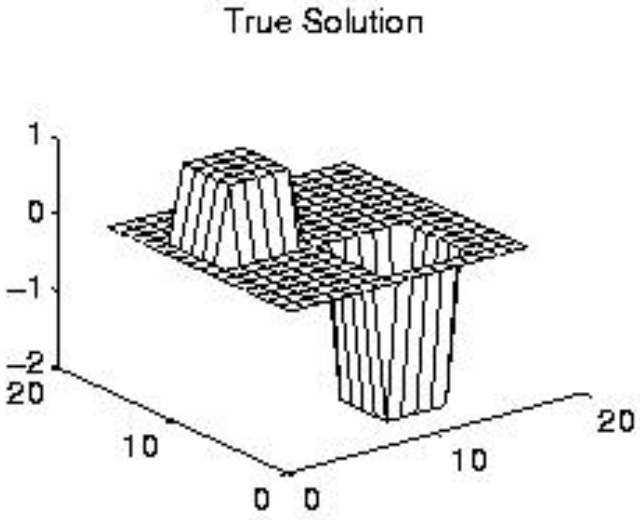
Convergence Picture of Multigrid in 1D

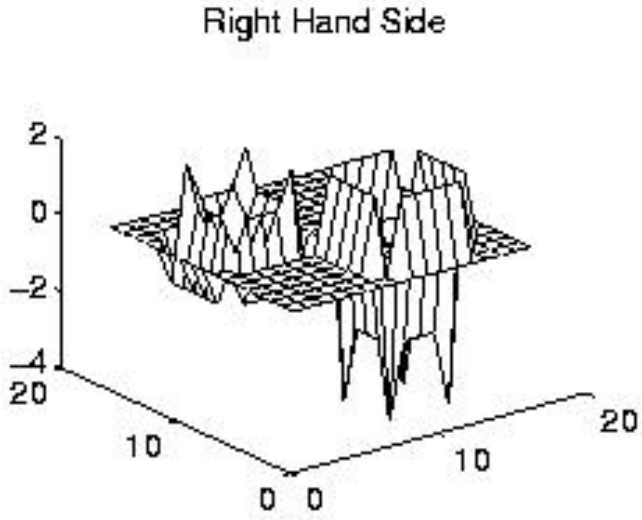


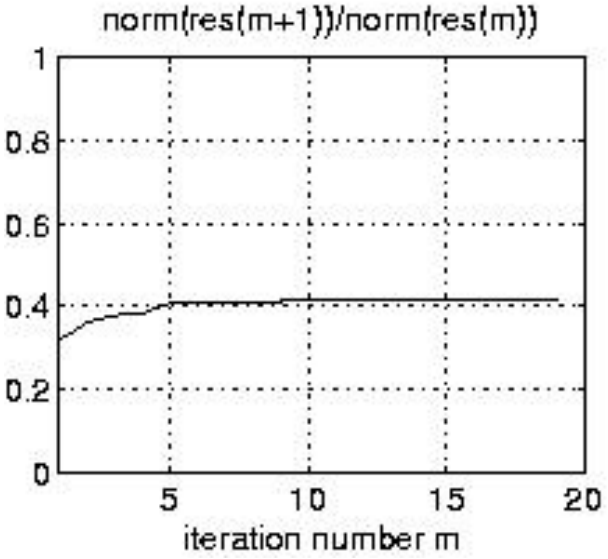


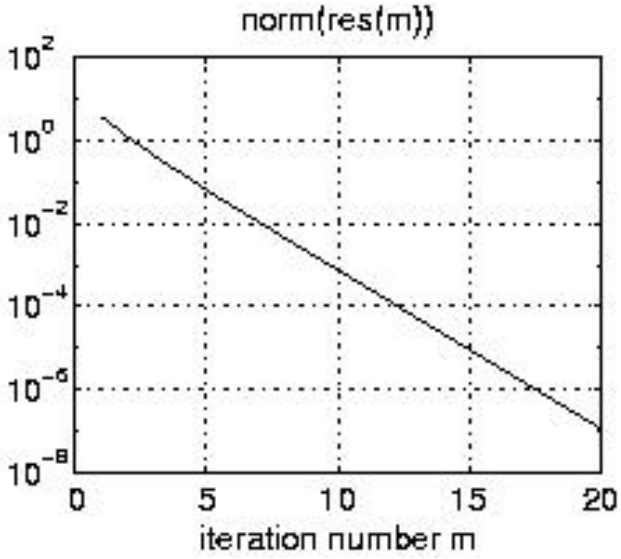
- **Full Multigrid on each iteration**
- **Error decreases by a factor >5 on each iteration**

Convergence Picture of Multigrid in 2D



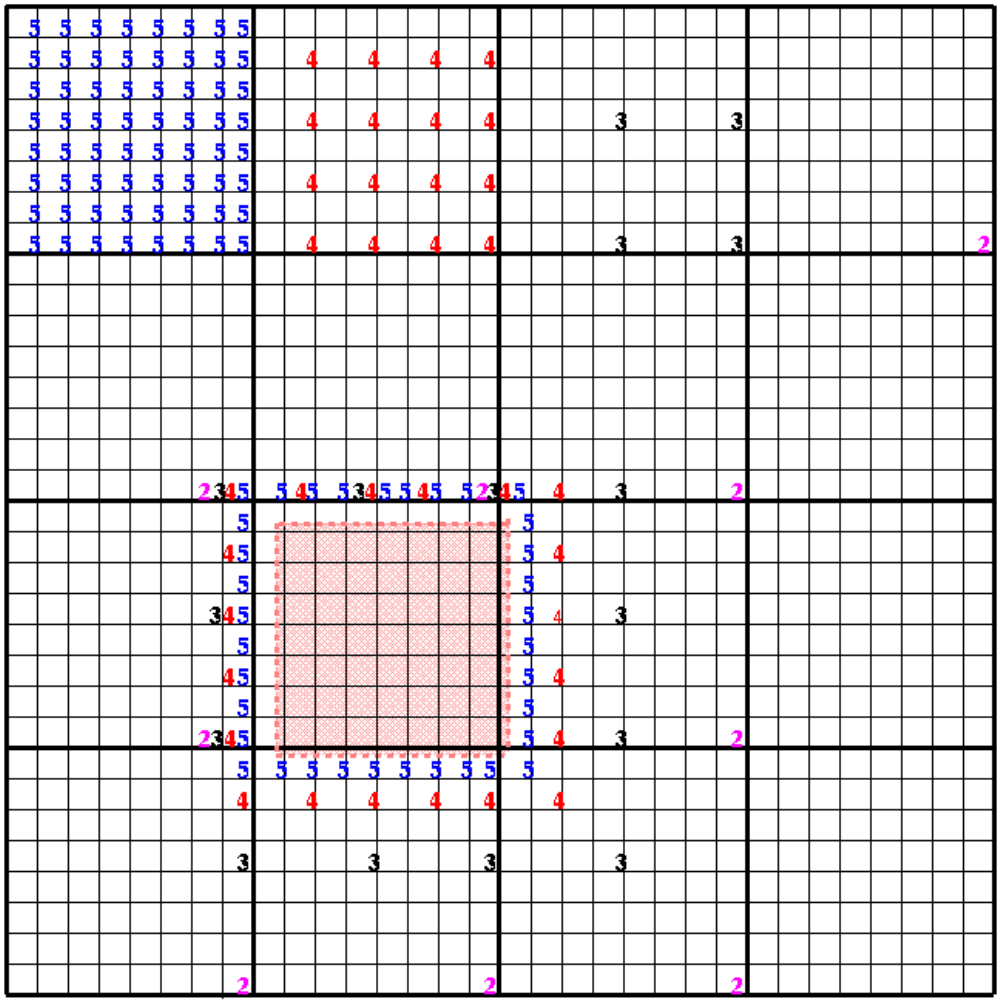




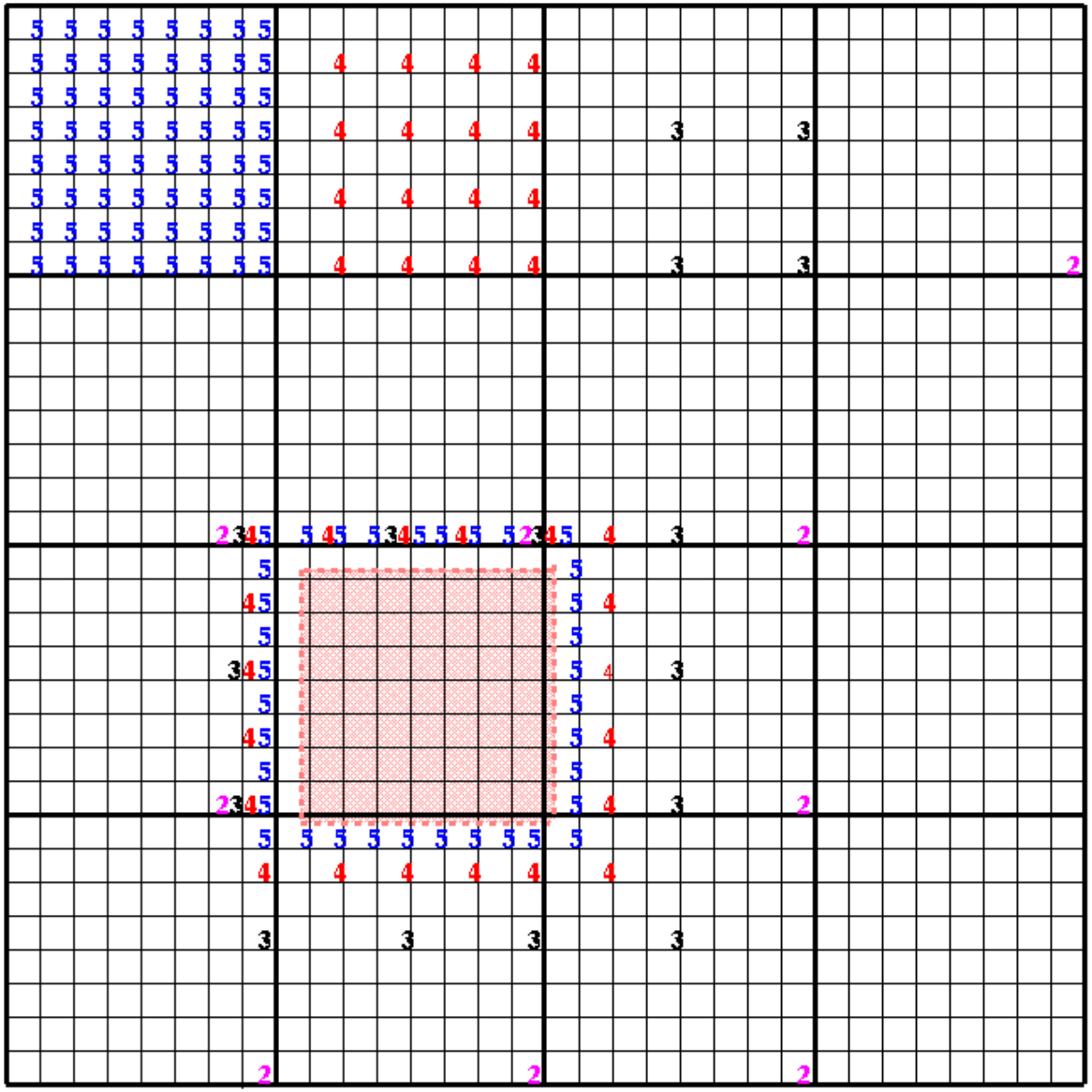


Parallel 2D Multigrid

- Multigrid on 2D requires nearest neighbor (up to 8) computation at each level of the grid
- Start with $n=2^m+1$ by 2^m+1 grid (here m=5)
- Use an s by s processor grid (here s=4)



Communication pattern for Multigrid on 33 by 33 mesh with 4 by 4 processor grid
In top processor row, grid points labeled m are updated in problem P(m) of multigrid
Pink processor owns grid points inside pink box
In lower half of graph, grid points labeled m need to be communicated to pink processor in problem P(m) of multigrid



Communication pattern for Multigrid on 33 by 33 mesh with 4 by 4 processor grid
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Pink processor owns grid points inside pink box
In lower half of graph, grid points labeled m need to be communicated to pink processor
in problem P(m) of multigrid

Performance Model of parallel 2D Multigrid

- Assume 2^m+1 by 2^m+1 grid of unknowns, $n= 2^m+1$, $N=n^2$
- Assume $p = 4^k$ processors, arranged in 2^k by 2^k grid
  - Each processor starts with 2^{m-k} by 2^{m-k} subgrid of unknowns
- Consider V-cycle starting at level m
  - At levels m through k of V-cycle, each processor does some work
  - At levels k-1 through 1, some processors are idle, because a 2^{k-1} by 2^{k-1} grid of unknowns cannot occupy each processor

Performance Model of parallel 2D Multigrid (2)

° **Cost of one level (j) or $P^{(j)}$ in V-cycle**

- **If level j >= k, then cost =**

  $O(4^{j-k})$ …. Flops, proportional to the number of grid points/processor

  $+ O(1)\ \alpha$ …. Send a constant # messages to neighbors

  $+ O(2^{j-k})\ \beta$ …. Number of words sent

- **If level j < k, then cost =**

  $O(1)$ …. Flops, proportional to the number of grid points/processor

  $+ O(1)\ \alpha$ …. Send a constant # messages to neighbors

  $+ O(1)\ \beta$ …. Number of words sent

° **Sum over all levels in all V-cycles in FMG to get complexity**

Comparison of Methods (using p processors)

	# Flops	# Messages	# Words sent
MG	N/p + log p * log N	$(\log N)^2$	$(N/p)^{1/2}$ + log p * log N
FFT	N log N / p	$p^{1/2}$	N/p
SOR	$N^{3/2}/p$	$N^{1/2}$	N/p

- SOR is slower than others on all counts
- Flops for MG and FFT depends on accuracy of MG
- MG communicates less total data (bandwidth)
- Total messages (latency) depends …
  - This coarse analysis can't say whether MG or FFT is better when $\alpha >> \beta$

Practicalities

- In practice, we don't go all the way to $P^{(1)}$
- In sequential code, the coarsest grids are negligibly cheap, but on a parallel machine they are not.
  - Consider 1000 points per processor
  - In 2D, the surface to communicate is 4*sqrt(1000) ~= 128, or 13%
  - In 3D, the surface is $1000\text{-}8^3$ ~= 500, or 50%
  - Data locality ratio α (large α is preferred):

$$\alpha = \frac{\text{computation time between two communication steps}}{\text{communication time}}$$

- Apply communication avoiding Jacobi iterations!
- Dealing with coarse meshes efficiently
  - Should we switch to using fewer processors on coarse meshes?
  - Should we switch to another solver on coarse meshes?

Quiz 4

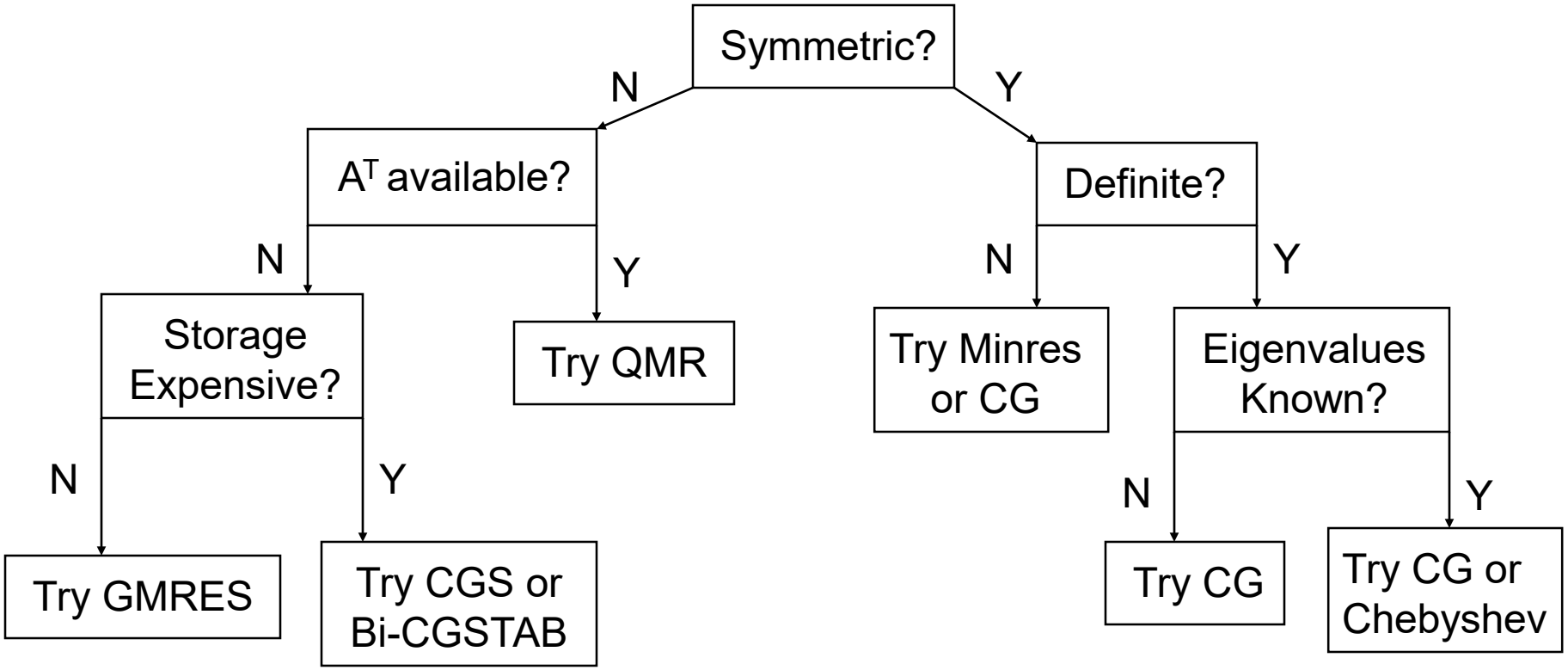
Deadline: Tuesday, November 16, 8:45AM

Extra Slides

(Not a part of discussions of the lecture)

Templates for choosing an iterative algorithm for Ax=b

- **Use only matrix-vector-multiplication, dot, saxpy, …**
- **www.netlib.org/templates**



Summary of Jacobi, SOR and CG

- Jacobi, SOR, and CG all perform sparse-matrix-vector multiply
- For Poisson, this means nearest neighbor communication on an n-by-n grid
- It takes $n = N^{1/2}$ steps for information to travel across an n-by-n grid
- Since solution on one side of grid depends on data on other side of grid faster methods require faster ways to move information
  - FFT
  - Multigrid

Multigrid Methods

- We studied several iterative methods
  - Jacobi, SOR, Guass-Seidel, Red-Black variations, Conjugate Gradients (CG)
  - All use sparse matrix-vector multiply (nearest neighbor communication on grid)
- Key problem with iterative methods is that:
  - detail (short wavelength) is correct
  - convergence controlled by coarse (long wavelength) structure
- In simple methods one needs of order N^2 iterations to get good results
  - Ironically, one goes to large N (fine mesh) to get detail
  - If all you wanted was coarse structure, a smaller mesh would be fine
- Basic idea in multigrid is key in many areas of science
  - Solve a problem at multiple scales
- We get coarse structure from small N and fine detail from large N
  - Good qualitative idea but how do we implement?