FA Assignment 9

Q1.

(a) G is the ordering by the greedy strategy.

$g_i \equiv$ index of last person in car i.

$G \equiv \langle g_1, g_2, \_\_, g_m \rangle$

Similarly,
Z is the ordering by a generic strategy

$Z = \langle z_1, z_2, \_\_, z_k \rangle$

Lemma 1 :

Given a Greedy Strategy ordering $G \equiv \langle g_1, \_\_, g_m \rangle$ and a generic strategy ordering $Z \equiv \langle z_1, \_\_, z_k \rangle$ and the starting position of the first person in the first car to be s for both, we can create a strategy Y such that Y & G agree with the first step of G.
Provided the size of the input for G and Z is equal.

Proof :

Elements in concern, $[w_s, \_\_, w_n]$

Starting position $\equiv s$

Size of input $= n-s+1$

Without loss of generality,

let $g_1 = p$

$\therefore Car_1(G) \equiv [w_s, \_\_\_, w_p]$

$\equiv [s, \_\_\_, p]$

let $z_1 = r$

$\therefore Car_1(Z) \equiv [w_s, \_\_\_, w_r]$

$\equiv [s, \_\_\_, r]$

We show that $r \leq p$,

Suppose r was greater than p.
We know,

$$\sum_{i=s}^{p} w_i \leq C$$

as they have to be in one car.

Also,

$$\sum_{i=s}^{r} w_i \leq C.$$

But for the Greedy strategy, elements from $(p+1)$ to (r) can be incorporated into the current car. Therefore, they will be incorporated into the current car based on the definition of the definition of the Greedy Strategy.

$\therefore$ r cannot be greater than p.

For $r = p$,

$Car_1(Z) = [s, \_\_\_, p]$

$Car_1(G) = [s, \_\_\_, p]$

for Strategy Y, simply taking $Car_1(Z)$,

$\rightarrow Car_1(Y) = [s, \_\_\_, p]$

For some $r < p$.

$$Car_1(2) \equiv [w_s, \_\_, w_r]$$

$$Car_1(G) \equiv [w_s, \_\_, w_p]$$

$$W(Car_1(2)) = w_s + \_\_ + w_r$$

$$W(Car_1(G)) = w_s + \_\_ + w_p$$

As discussed,

$$\sum_{j=s}^{r} w_j \leq \sum_{j=s}^{p} w_j \leq C$$

Now,

$$\sum_{j=s}^{r} w_j + \sum_{j=r+1}^{p} w_j \leq C$$

$$\therefore \sum_{j=r+1}^{p} w_j \leq C - \sum_{j=s}^{r} w_j$$

$\therefore$ Total weight of elements in $Car_1(G)$ but not in $Car_1(2)$

$\leq$ Capacity remaining in $Car_1(2)$ after incorporation of all elements in $Car_1(2)$. α

For strategy 2,

Elements in remaining cars $\equiv [r+1, \_, p,$
$p+1, \_, n]$

For strategy G,

Elements in remaining cars $\equiv$

$$[p+1, \_, n]$$

Following equation α, we can incorporate elements $r+1, \_, p$ into Car 1 for a strategy Y.
Let l be the position of $p+1$ in Z i.e. $(p+1) \in Car_l(Z)$. Elements before $(p+1)$ are now in Car 1 for Y.

For strategy Y,

$$\therefore Car_1(Y) = [s, \_, p]$$

$$Car_1(Z_{old}) = [s, \_, r]$$

$$Car_1(G) = [s, \_, p]$$

$$Z_{old} \equiv \langle z_1, z_2, \_ z_k \rangle$$

$$G \equiv \langle g_1, g_2, \_, g_m \rangle$$

$\therefore$ G and Y agree with respect to the car elements.
Hence our lemma is proved.

$$Size(Car_1(Y)) \geq Size(Car_1(Z))$$

In order to complete the creation of Y from Z, we apply the lemma recursively as

—, we apply the lemma recursively as follows,

$$Z_{new} = \langle z_{\ell}, \_\_, z_k \rangle$$

$$G_{new} = \langle g_2, \_\_, g_m \rangle$$

starting position $= p+1$, for both

Size of Input $= n - (p+1) + 1$

In the final application of the recursion, the last element of G will be the first element of G_{new}. The last element of Y will hence agree with the last element of G.

We hence generate Y.

Notice that for any recursive definition of Lemma 1, the size of the car generated for Y will always be greater than the size of the corresponding Z indexed car.
i.e. for $i \in Y\{1, \_\_, m\} \equiv \{Car 1, \_\_ Car\, v\}$, there exists ℓ such that $Size(Car_i(Y)) \geq Size(Car_\ell(Z))$ and $\ell \geq i$. – B
If ℓ was less than i, that would mean elements before ℓ in Z would cover more elements than elements before ℓ in Y, since the association between an element in $Car\ \ell$ for Z and the $Car\ i$ for Y happens when all previous elements in the respective strategies have been covered, we know, from our recursive formulation that any stage of the recursive call, Z does not cover more elements than Y.

$$Cost(Y) = \Sigma\, 1\{Car_i(Y)\} = m$$

$\forall\ i \in \{1, m\}$

For $i = m$, $\exists$ car(l) such that $l \geq m$. – from β.

$\therefore$ Cost $(Y) \geq$ Cost (Z)

$\therefore$ Cost $(G) \geq$ Cost (Z)

– from lemma 1, α, β

Q1.
(b)

$Z \equiv \langle z_1, \_\_, z_n \rangle$

$G \equiv \langle g_1, \_\_\_, g_n \rangle$

$g_i = k$, implies person i is seated in car k.

$F_j(G) = k$ such that person i is seating in car k. $\equiv g_i$

We show that $F_j(G) \leq F_j(Z)$.

Base Case:

For one person,

$G((g_1)) =$ Car$_g$ 1

$G((z_1)) =$ Car$_z$ 1

$F_i(G) \leq F_i(Z)$, # $i = 1$

Consider i people, Assume that $F_i(G) \le F_i(Z)$ for i people.

$$F_i(G) = k = g_i$$

$$F_i(Z) \ge k = z_i$$

$$\begin{array}{ccccc} k-1 & k & & k & \\ g_{j-1} & g_j & \cdots & g_i & g_{i+1} \end{array}$$

$$\begin{array}{ccccc} \ge k-1 & \ge k & & \ge k & \\ z_{j-1} & z_j & \cdots & z_i & z_{i+1} \end{array}$$

Consider g_{i+1}, z_{i+1}.

→ If there is space in Car k in G, $g_{i+1} = k$

→ $z_i \ge k$

$\therefore$ z_{i+1} must be $\ge k$, in $car_Z\, k$ if if space available else in $car_Z (k+1)$

$$\therefore\ g_{i+1} \le z_{i+1}$$

→ If there isn't space in $car_g\, k$.

$$g_{i+1} = k+1$$

$$\begin{array}{ccccc} k-1 & k & & k & \\ g_{j-1} & g_j & \cdots & g_i & g_{i+1} \end{array}$$

$$\begin{array}{ccccc} \geq k-1 & \geq k & & \geq k & \\ z_{j-1} & z_j & ---- & z_i & z_{i+1} \end{array}$$

$$\rightarrow \quad \sum_{s=j-1}^{k} w_s > C$$

If z_{j-1} was $(k-1)$, $z_j \geq k$ based on our inductive assumption, $\therefore$ z_j must start the car k.

Elements $\{j, \_\_, i\}$ constitute the elements in car$_2$ k. Now, our case is that there isn't space in car$_g$ k.

$$\therefore \quad \sum_{s=j}^{i+1} w_s > C$$

$\therefore$ For car$_2$ k, there won't be any space left as well.
$\therefore$ z_{i+1} must be $k+1$. Which satisfies $z_{i+1} \geq g_{i+1}$.

$\rightarrow$ for $z_{j-1} = k$,

We know,
$$\sum_{s=j}^{i+1} w_s > C$$

$$\therefore \quad \sum_{s=j-1}^{i+1} w_s > C$$

$\therefore$ $z_{i+1} \geq k+1$

If z_{j-1} is $k+1$ or greater, the elements ahead of it will have to be in cars $k+1$ or greater, hence validating our inductive assumption.

$$\therefore \; F_j(G) \leq F_j(Z)$$

for $j = n$,

$$F_n(G) \leq F_n(Z)$$

Which completes our proof by GASA.

Q1(d)

→ let the elements in Car 1 from Strategy G_1 be from 1 to i.
→ let the elements in Car 2 from Strategy G_2 be from $i+1$ to j. k is the next element.
→ The elements in Car 1 for G_2 are from 1 to x. Car 2 elements are from $x+1$ to y. v is the next element.

→ The Elements 1 to i will be in the first car for G_2 as well. The $(i+1)^{th}$ element will be in Car 2 for G_2. At this stage,

$$l_{C_1}^{G_2} = C - \sum_{i=1}^{i} w_i$$

$$l_{C_1}^{G_1} = C - w_{i+1}$$

→ l denotes leftover space in the car. Consider elements $(i+2)$ until j. let these be elements p.

→ For some element in p starting from i+2 to j, if,

$$l_{C_1}^{G_2} - w_p \geq 0$$

$$l_{C_1}^{G_2} := l_{C_1}^{G_2} - w_p$$

w_p is assigned to Car 1 in G_2.

$$\therefore \; N(G_2(\text{Car }1)) \geq N(G_2(\text{Car }2))$$

Meanwhile, that same element is assigned to Car 2 in Strategy G_1.

$$\therefore \; l_{C_2}^{G_1} := l_{C_2}^{G_1} - w_p$$

Also, $l_{C_2}^{G_2}$ remains the same.

$$l_{C_2}^{G_1} \leq l_{C_2}^{G_2}.$$

∴ As we reach j, two things hold,

$$N(G_1(\text{Car }1)) \geq N(G_2(\text{Car }1))$$

$$l_{\text{Car }2}^{G_2} \geq l_{\text{Car }2}^{G_1}$$

→ At person j+1, Car 2 for C_2 is dispatched

→ Since $G_2(\text{Car }2)$ is always having greater capacity than $G_1(\text{Car }2)$ let us say p_1 elements from i+1 to j are assigned to $G_2(\text{car }1)$, Let the rest of the elements be p_2.

Since G_1 (Car 2) supports $(p_1+p_2) \equiv (i+1 \text{ to } j)$ elements. G_2 (Car 2) can definitly support p_2 elements. As the capacity for individual cars is the same.

→ So at person $k = j+1$, there will be capacity in Car 2 for G_2 to accomate person ω_k such that $l^{G_2}_{Car 2} - \omega_k \geq 0$ if

$l^{G_2}_{Car 1} - \omega_k \not< 0$. This is done until

v such that for $\sum_{i \in Car 1} \omega_i + \omega_v > 0$ &

$\sum_{i \in Car 2} \omega_i + \omega_v > 0$ for Strategy G_2.

$\therefore \; v \geq k$,

If Car 1, Car 2 form Pair 1, for elements in Pair 1 for Strategy G_1, atleast as many elements will be present in Pair 1 for for strategy G_2.

$N(\text{Pair 1})$ for $G_2 \equiv \{1, \_, v-1\}$
$\equiv \{1, \_, v'\}$

For elements $s+i$, $i+1$ to j in Car 1 and Car 2 for strategy G_1, elements present in Car 1 and Car 2 for Strategy G_2 will be $s+x$, $x+1$ to v' such that $v' \geq j$.

For the second pair, if elements in Car 3 for G_1 go from k to r, if $v' \leq r$, we can continue the same argument on car capacity to state that for Pair 1, Pair 2 in G_1, the number of elements in Pair 1, Pair 2 in G_2 is atleast as much as the number of elements for G_1.

If pair 2 in G_1 ends at e and Pair 2 in G_2 ends at e'.

$$e \leq e'$$

Q1.(e)

$C = 10$

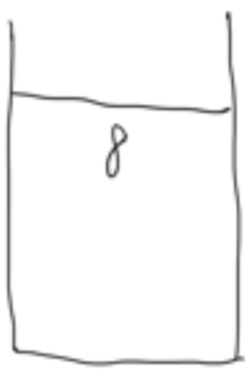
1. G_2 does as good as G_1

$$W = [1, 3, 5, 8]$$

For G_1

Car 1 -

Car 2 -



for G_2 -

G_2

Car 1 = {1, 3, 5}

Car 2 = {8}

$$n_{cars}(G_1) = n_{cars}(G_2) = 2$$

2. G_2 does better than G_1.

$\rightarrow$ $W = [\underset{w_1}{4}, \underset{w_2}{7}, \underset{w_3}{3}, \underset{w_4}{2}]$

for G_1 - car 1

- {4}

car 2

- {7, 3}

Car 3

- {2}

$n_{cars} = 3$

For G_2,

car 1

- {4, 3, 2}

car 2

- {7}

$$n_{cars} = 2$$

$$n_{cars}(G_1) = 3$$

$$n_{cars}(G_2) = 2$$

(c)

$$\sum_{i=1}^{n} w_i > \lfloor (k/2) \rfloor . C$$

$$Cost(Opt(W)) = l$$

l = no. of subsets in the optimal strategy.

$\equiv$ no. of cars deployed.

For each subset s_k,

$$\sum_{j \in s_k} w_j < C$$

Summing for all subsets,

L.H.S :=

$$\sum_{k:=s_1}^{s_l} \sum_{j \in s_k} w_j$$

$$= \sum_{j=1}^{n} w[j]$$

Since each car will lie in one subsets.

R.H.S :=

$$\sum_{k=s_1}^{s_l} C = l \cdot C$$

l subsets

$$\therefore \quad \sum_{j=1}^{n} w_j \leq l \cdot C$$

To prove,

$$\sum_{i=1}^{n} w_i < \lfloor (k/2) \rfloor \cdot C$$

Subsets, $s_1, \ldots, s_k$ for Greedy Strategy.

$$s_1 + s_2 > C$$

Since the greedy strategy will start filling in car 2 after capacity of people under consideration in car 1 (s_1), exceeds C.

$$s_2 + s_3 > C$$

$$s_k + s_{k-1} > C$$

$$2\left(\sum_{i=1}^{k} s_i\right) > k \cdot C$$

We have seen before,

$$\sum_{i=1}^{k} s_i = \sum_{i=1}^{n} w_i$$

$$\therefore \sum_{i=1}^{n} w_i > \frac{k}{2} \cdot C$$

$$\sum_{i=1}^{n} w_i > \lfloor k/2 \rfloor \cdot C$$

Q 2.

(a)

let Coins be a matrix.

$$\text{Coins} \equiv (T+1) \times (1 + p + n + d + q)$$

Create a $(p + n + d + q)$ array for

storing the values of the denominations.

$V \rightarrow [1, \_\_, n']$

where $V[i] = Val(\text{penny}) = 1$ unit
for $i \in \{1, p\}$

$V[i] = val(\text{nickel}) = 5$ unit
for $i \in \{p+1, p+n\}$,

$n' = p + n + d + q$

Coins $[i][j]$ – minumum number of coins to make i dollars such that j coins from V are allowed.

Recersive formulation:

$$\text{Coins}[i][j] = \min \left\{ \begin{array}{l} \text{Coins}[i - V[j]][j-1] + 1 \\ , \\ \text{Coins}[i][j-1] \end{array} \right\}$$

(0)(0)
+1

Base Case:

for j from 1 to n':

$\text{Coins}[0][j] = 0$

for i from 1 to T':

$$Coins[i][0] = \infty$$

$$Coins[0][0] = 0$$

$$Coins[x][1] = 1 \quad \text{if} \quad x == 1 \text{ and } p \geq 1$$

$$Coins[x][1] = \infty \quad \text{else}$$

x goes from 1 to T.

$$Coins[1][y] = 1 \quad \text{if} \quad p \geq 1$$

$$= \infty \quad \text{else}$$

y goes from 1 to n'.

We iterate over the target dollars from 1 to T. For each target dollar i, we fill in for denominations from ledt to right. For each (i,j), the neccessory subproblems required to make the decision will be solved.

As in dynamic programming, we use the ordering of the sub problems in order to fill the Coins table.
There are $O(T \cdot (p+n+d+q))$ entries in the table.
$Coins[T][p+n+d+q]$ will contain the required answer.

$$T(n) = O(T \cdot (p+n+d+q))$$

Q2. (b)

Consider the following scenario,

T	p	n	d	q
30	6	0	4	2

→ The greedy strategy will select the quarter first.

→ For the remaining 5 units, it will select 5 pennies.

→ Coins = 1 {quarter} + 5 {pennies}

→ G (Coins) = 6

Optimal (coins) = 3 {dimes}

10 + 10 + 10

Optimal (Coins) = 3

Greedy (Coins) = 6

∴ Greedy Strategy doesn't find optimal solution.

Q2.

(c)

To prove : Greedy Strategy G is the optimal strategy.

p = 1, n = 5, d = 10, q = 25 - Units

Base Case:

1.1. $T < 5$

G will have to select p's for a feasible solution. The Greedy choice is also the correct choice since p's will consitute a valid optimal solution as all other denominations are greater than p.

1.2 $5 \leq T < 10$

$$G \equiv \langle g_1, \_\_\_, g_k \rangle$$

$$Z \equiv \langle z_1, \_\_\_, z_m \rangle$$

For Z, we show that

$$Coins(G) \leq Coins(Z)$$

let,

$G \equiv 1n + kp$

$Coins \equiv 1+k$

$k \in \{0, 1, 2, 3, 4\} \equiv$

$t \in \{5, 6, 7, 8, 9\}$

Consider the forms of Z,

$Z = 2n$ is not possible, since $2n >$ target.

if,

$$Z = 0n + mp$$

then,

$$mp = 1n + kp = t$$

$$mp = (5+k)\,p$$

$$m = 5 + k$$

$m = j + k$

$m > k$

$\therefore$ For this form, the Greedy Strategy G is better than Z.

if $Z = 1n + mp$

$$1n + mp = 1n + kp$$
$$m = k$$

$\therefore$ For Case 1.2, comparing the Greedy Strategy with any valid strategy Z, the strategy G is optimal.

1.3. $10 \leq T < 25$

1.3.1 $10 < T < 15$

The Greedy Strategy G will select a dime first. for rest will be pennies.
For any strategy Z, it cannot have 2 dimes.
for a single dime in Z, the rest have to be pennies. This is equivalent to the Greedy Strategy.

If it has no dime,
if it has one nickel, the rest are pennies, the number of coins is more than the greedy solution
if it has two nickels, the rest are pennies, the number of coins is still less than that in the Greedy solution.

Thus, even in this case, $Coins(G) \leq Coins(Z)$

1.3.2.

$15 \leq T < 20$

→ The greedy strategy will choose a dime and a nickel, the rest will be pennies.

→ If the solution 2 doesn't have a dime, it can have upto three nickels, and the rest being pennies, for each such combination, Coins (G) ≤ Coins (2)

→ If the solution 2 has a dime, if it doesn't have nickels, Coins (G) ≤ Coins(2)

→ If it has a dime and a nickel, the rest will be pennies and it will be equivalent to the greedy solution.

→ If it has only pennies, Coins (G) ≤ Coins(2)

1.3.3 $20 \leq T < 25$

→ G will choose two dimes, and then pennies. If 2 has a dime, it can have one or two nickels, the rest being pennies. Again Coins (G) ≤ Coins (2)

→ If 2 has no dimes, similarly we can show considering, the combination of nickels and pennies, Coins (G) ≤ Coins(2)

This is the case analysis for the base case.

Claim 1:

For the base case R, we have shown

$G(R)$ to be optimal. $(R < 25)$

Consider the argument as follows,

let $F_j(G) \equiv$ Cost until target remaining after selection of Coin j in Strategy G.

$$F_j(G) = g_j \equiv t - \sum_{i=1}^{j} c_i$$

selected coin)

Similarly,

$$F_j(z) = z_j$$

Base Case:

1.1 if $1 \leq T < 5$

$c_1^g = 1 \{p\}$ for G

$c_1^z = 1 \{p\}$ for any valid z.

$\therefore g_1 = t - 1$

$z_1 = t - 1$

$g_1 \leq z_1$ is valid.

1.2. if $5 \leq T < 10$

$c_1^g = 5 \{n\}$ for G

For z, possible values of $c_1^z = \{1, 5\}$

In case 1, $z_1 = t - 1$
In case 2 $z_1 = t - 5$

Considering both cases,

$$g_1 \leq z_1$$

1.3. if $10 \leqslant T < 25$

$$c_1^g = 10 \{\text{dime}\}$$

$$g_1 = t - 10$$

$$c_1^z \in \{1, 5, 10\}$$

$$z_1 = t - c_1^z$$

$$\therefore g_1 \leq z_1$$

1.4. if $T \leq 25$

$$c_1^g = 25 \{\text{quarter}\}$$

$$g_1 = t - 25$$

For some valid strategy z,

$$c_1^z \in \{1, 5, 10, 25\}$$

$$z_1 = t - c_1^z$$

$$\therefore g_1 \leq z_1$$

$g_1 = \dots$

Thus, for our Base Case,

$$F_1(G) \leq F_1(Z)$$

Inductive Assumption,

$$\rightarrow \quad F_i(G) \leq F_i(Z)$$

$$F_i(G) = g_i$$

$$F_i(Z) = z_i$$

$$g_i \leq z_i$$

For the $(i+1)^{th}$ stage, G selects,

$$m = \max\{p, n, d, q\} \text{ such that } m \leq g_i$$

$$g_{i+1} = g_i - m$$

$$z_{i+1} = z_i - r \qquad r \in \{p, n, d, q\}$$

$$z_{i+1} - g_{i+1} = z_i - g_i + (m - r)$$

if $m = q$, $\max(r) = q$

$$z_i - g_i \geq 0$$

$$m - r \geq 0$$

$$\therefore \quad z_i - g_i + (m-r) \geq 0$$

$$\therefore \quad z_{i+1} - g_{i+1} \geq 0$$

$$z_{i+1} \geq g_{i+1}$$

Now consider the argument as follows,

Claim 2:

Any number T can be written as $25q + R$ for $T > 25$.

Claim 3:

→ If $T = 25q$, the greedy strategy will select q quarters which will be optimal.

→ Base case, $q=1$, 1 quarter better than any combation of dimes etc.

To show, $q = i$, optimal for $25i$.

let Z be a generic solution $\{n_1, n_2, n_3, n_4\}$

$$n_1 25 + n_2 10 + n_3 5 + n_4 1 = 25i$$

$$n_1 + n_2 \frac{2}{5} + n_3 \frac{1}{5} + n_4 \frac{1}{25} = i$$

$$n_2 \frac{2}{5} + n_3 \frac{1}{5} + n_4 \frac{1}{25} = i - n_1$$

$$n_2 \cdot \frac{2}{5} + n_3 \frac{1}{5} + n_4 \frac{1}{25} = i - n_1$$

$$n_2 \cdot \frac{2}{5} \le n_2$$

$$n_3 \cdot \frac{1}{5} \le n_3$$

$$n_4 \cdot \frac{1}{25} \le n_4$$

$$n_2 + n_3 + n_4 \ge \frac{2}{5} n_2 + \frac{1}{5} n_3 + n_4 \frac{1}{25}$$

$$n_1 + n_2 + n_3 + n_4 \ge n_1 + \frac{2}{5} n_2 + \frac{1}{5} n_3 + \frac{1}{25} n_4$$

$$n_1 + n_2 + n_3 + n_4 \ge i$$

$\therefore$ We have shown that for $T = 25i$, i quarters are optimal.

Claim 4:
If : $O(S_1)$ is an optimal solution for S_1.
If : $O(S_2)$ is an optimal solution for S_2.
For some solution Z for $(S_1 + S_2)$ consisting of coin set C such that $C = \{C_1, C_2\}$ and $Sum(C_1) = S_1$, $Sum(C_2) = S_2$, if Z is the optimal solution and it is producing S_1, S_2 during the process, then C_1 must be $O(S_1)$ and C_2 must be $O(S_2)$.

→ We are claiming on the structure of Z given that Z has picked up coins. If Z consists of C_1, then the subset of coins in Z which represent C_1 must also be optimal. If that were not the case, we would replace the set of coins C_1 with an optimal set of coins for S_1 in order to obtain a better optimal set of coins for $(S_1 + S_2)$ which would no longer make Z the optimal solution.

Claim 5: Any Generic Strategy Z cannot have more quarters than the Greedy Strategy.

Proof by contradiction:

Suppose $G \equiv \langle g_1, \_\_, g_k \rangle$

$Z \equiv (z_1, \_\_, z_r)$

Such that $r < k$

let $G \equiv x_1 q + x_2 d + x_3 n + x_4 p$

, $x_1 + x_2 + x_3 + x_4 = k$

let $Z \equiv y_1 q + y_2 d + y_3 n + y_4 p$

$y_1 + y_2 + y_3 + y_4 = r$

If $x_1 < y_1$:

$\rightarrow t = 25 y_1 + \phi$

$\rightarrow t = 25 (x_1 + \alpha) + \phi$

as Z is a valid solution.

But the Greedy algorithm will then select $x_1 + \alpha = y_1$ quarters and hence x_1 cannot be less than y_1.

If $x_1 = y_1$

$$t = x_1 q + x_2 d + x_3 n + x_4 p$$

$$t = x_1 q + y_2 d + y_3 n + y_4 p$$

For $(x_2 d + x_3 n + x_4 p)$ and $(y_2 d + y_3 n + y_4 p)$, we repeat the same argument on d, n.

Claim 6:
For some $T = 25q + R$, $G(T) \equiv \{q + G(R)\}$ is the optimal strategy.

Base Case:
$q = 0$, shown to be optimal for R.

Inductive Assumption:
Suppose the claim is valid for all $q < i$

$$T = 25i + R \quad , \quad R < 25$$

Following from our arguments,
Suppose for some generic strategy Z,
quarters $\equiv i - \alpha$
Suppose Z was optimal,

$$T \equiv 25(i-\alpha) + R_Z$$

$$25(i-\alpha) + R_Z = 25i + R$$

$$R_Z = R + 25\alpha\{Z\}$$ (25α in terms of dimes, nickels & pennies.)

Z produces,

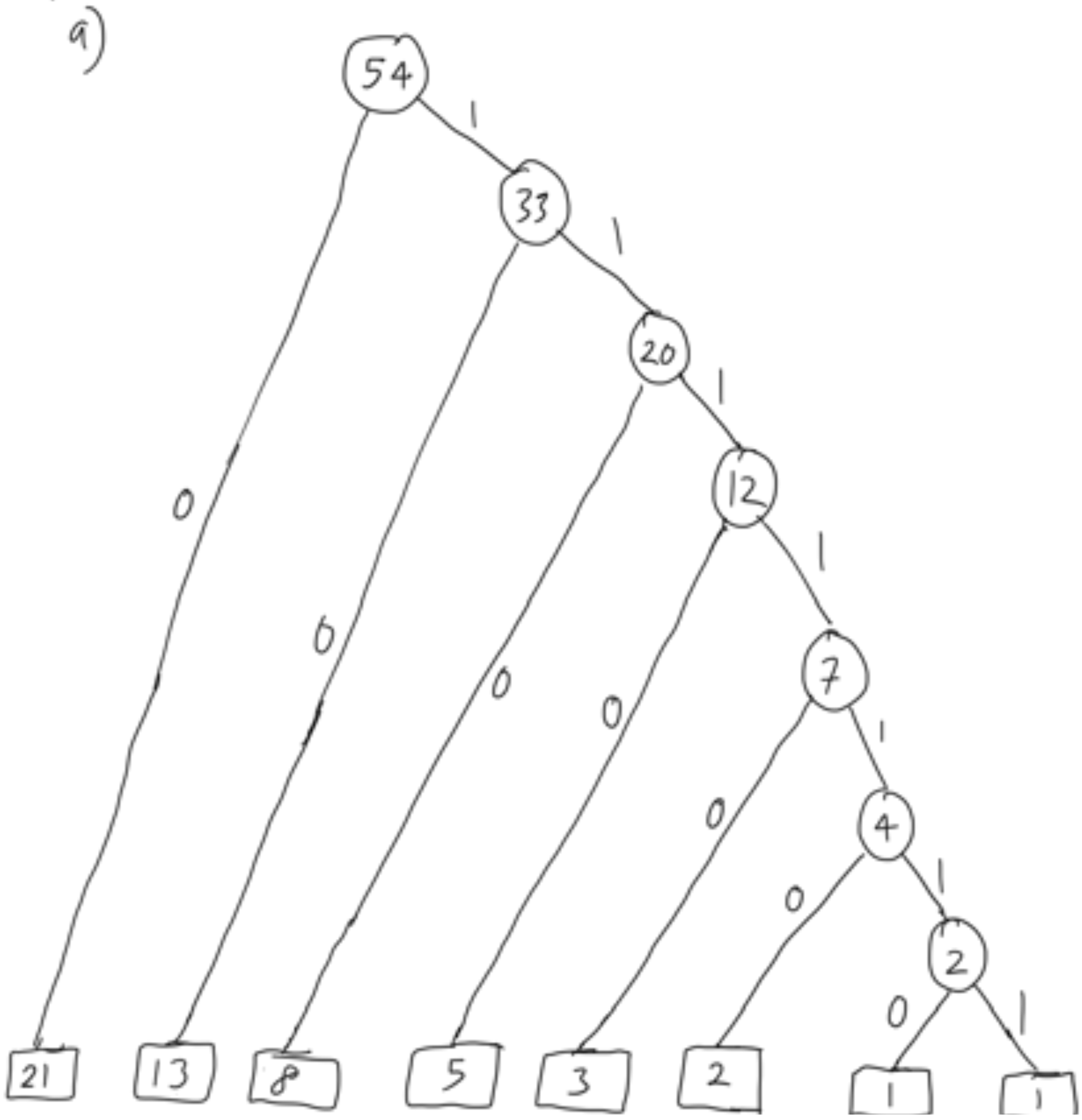
$$T = \underbrace{25(i-\alpha) + R}_{\text{optimal according to our inductive assumption}} + \underbrace{25\alpha\{2\}}_{\text{sub optimal}}$$

For 25α, α quarters is the optimal strategy, any strategy consisting of dimes, nickels and pennies we have shown to be suboptimal.

From Claims 5, 4, using 2, G, following from the previous equation, Z cannot be optimal. Hence $q = i$, our Greedy Strategy follows to be optimal for $T = 25q + R$.

Q3.

a)



Huffman codes:

21 - 0
13 - 10
8 - 110
5 - 1110
3 - 11110
2 - 111110
1 - 1111110
1 - 1111111

Q3. (b)

Base Case:

$S_1 = f_0 + f_1$

$= 2$

$f_3 - 1 = 3 - 1 = 2$

$S_1 = f_3 - 1$

For the inductive Assumption,
Suppose, $S_j = f_{j+2} + 1 \quad \forall j$ less than or equal to $i-1$.

\# $S_{i-1} = f_{i+1} + 1 \;, j = i-1$

For step i,

$$S_i = \sum_{j=0}^{i} f_j$$

$$= \sum_{j=0}^{i-1} f_j + f_i$$

$j=0$

$$= S_{i-1} + f_i$$

$$= f_{i+1} + 1 + f_i$$

$$= f_{i+1} + f_i + 1$$

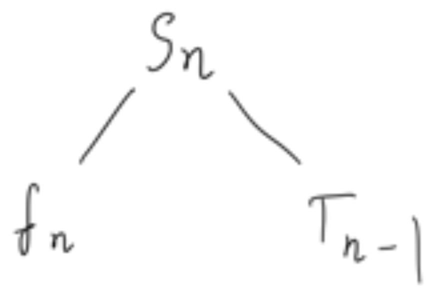
$$S_i = f_{i+2} + 1$$

Hence our proof by induction is valid.

Q3. (c)

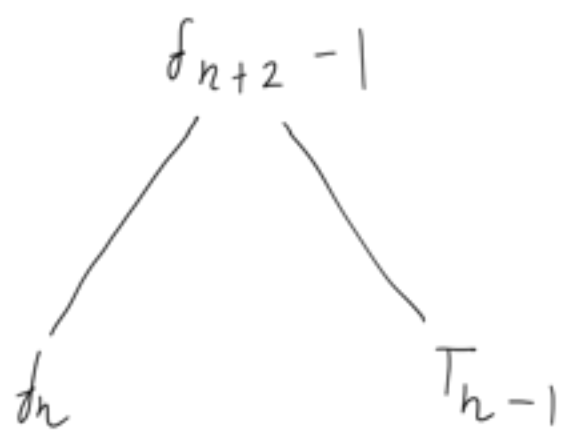
For the Huffman Tree with $(n+1)$ fibonacci numbers, the optimal solution will be as follows,

$T_n \equiv$



T_n - $(n+1)$ fibonacci numbers
T_{n-1} - n fibonacci numbers

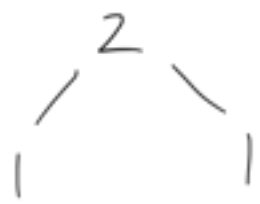
$\therefore T_n \equiv$



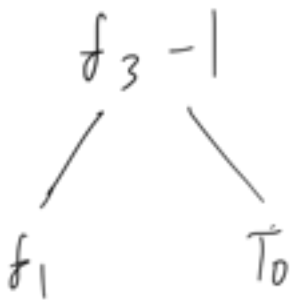
Base Case

Base case

$T_1 \equiv$



$\equiv$



Assume that the structure is true for T_n.

For T_{n+1}, since f_{n+1} the largest frequency, it will be the shortest code based prefix corresponding to the left of the root. The right sub tree will contain the rest of the frequencies will be correct based on our inductive assumption. The root will thus be $f_{n+1} + S_n \equiv S_{n+1} \equiv f_{n+3} + 1$

$\therefore$ Structure of Tree is valid.