Anote problem set

Advait Savant

May 2023

1 Q1

Say you are on the N=100th floor of a building. You need to get to floor 1, but there are no stairs so you need to take the broken elevator. The elevator is broken in the sense that every time it stops at a random floor, between floor 1 to floor N-1, with equal probability (in the first case, N = 100, so N - 1 = 99). For example, let's say that the elevator randomly stops at floor 64, for the first try. Then, once you would arrive at floor 64, the counter would += 1, and you would go on the elevator to stop at any random floor between 1 and N-1 = 63. What is the expected value for the number of times you would stop at a random floor (the counter) before you would get to the lobby? Bonus: What is the variance?

For some floor n, let X_n be a random variable which denotes the next floor on which the elevator goes. Since the odds of going to any floor below floor n as equi-probable, We have, $p(X_n = n-1) = 1/(n-1)$ .. $p(X_n = 1) = 1/(n-1)$

For some floor n, let C_n be denoting the number of stops taken to get to floor 1. i.e. the counter variable.

We need to find the expected value $E[C_n]$.

If we start at floor n, and then move to floor k less than n such that $X_n = k$, the number of stops C_n can be written as $1 + C_k$, 1 denoting the current stop and C_k denoting the stops to follow. Given this scenario, based on the linearity of expectation, the expected number of stops we take from floor n is one plus the expected number of stops now starting from floor k. k itself is the instantiation of the random variable X_n. We can write this as,

$$E[C[n]] = 1 + E[C[X_n]]$$

Now, we look at $E[C[X_n]]$,

We have C which is random variable, the domain of which is is dependent on the values of another random variable, X_n.

We can write the expected value of C_n by conditioning it on X_n and then take the expected value of the conditional expected value. This is essentially the law of total expectation.

$$E[C[X_n]] = \sum_{k=1}^{n-1} p(X_n = k) * E[C[k]]$$

This gives us,

$$E[C_n] = 1 + \sum_{k=1}^{n-1} p(X_n = k) * E[C[k]]$$

Therefore,

$$E[C_n] = 1 + \sum_{k=1}^{n-1} \frac{1}{n-1} * E[C[k]]$$

We refer $E[C_n]$ as $E[n]$ for notational simplicity. E[1] is 0, since we are already at floor 1.

$$E[n] = 1 + \sum_{k=1}^{n-1} \frac{1}{n-1} * E[k]$$

$$E[n] = 1 + \frac{1}{n-1} * E[2] + \frac{1}{n-1} * E[3] ... \frac{1}{n-1} * E[n-1]$$

We have, $E[2] = 1$, since when we are at the second floor, there is only one floor which the elevator can go to, and that is the first floor, this is validated by substitution in the recurrence relation.

$$E[2] = 1$$

$$E[3] = 1 + \frac{1}{3-1} * 1 = 1.5$$

Now,

$$E[n] = 1 + \frac{1}{n-1} * E[2] + \frac{1}{n-1} * E[3] ... \frac{1}{n-1} * E[n-1]$$

$$E[n-1] = 1 + \frac{1}{n-2} * E[2] + \frac{1}{n-2} * E[3] ... \frac{1}{n-2} * E[n-2]$$

$$(n-1) * E[n] = (n-1) + E[2] + E[3] ... E[n-1]$$

$$(n-2) * E[n-1] = (n-2) + E[2] + E[3] ... E[n-2]$$

subtracting the two equations, consider the left hand side:

$$(n-1)E[n] - (n-2)E[n-1] = (n-1) * E[n] - (n-1-1) * E[n-1] = (n-1) * (E[n] - E[n-1]) + E[n-1]$$

Consider the right hand side in the subtracting,

$$(n-1) * (E[n] - E[n-1]) + E[n-1] = 1 + E[n-1]$$

$$E[n] - E[n-1] = \frac{1}{n-1}$$

Now this is a telescoping series,
The last equation being,

$$E[2] - E[1] = 1$$

we get the harmonic sum as the value for E[n],
Therefore, $E[n] = H(n-1)$ where H is the Harmonic sum.
Therefore, $E[100] = H(99)$

$$E[100] = 5.18$$

Finding the variance is a more complex problem. I feel the law of total variance will be of utility here.

$$Var[C_n] = Var[E[C_n/X_n]] + E[var[C_n/X_n]]$$

for $Var[C_n/X_n]]$, we can do the following,

$$Var[E[C_n/X_n]] = E[(E[C_n/X_n])^2] - (E[E[C_n/X_n]])^2$$

$$E[var[Y_n/X_n]] = \sum var[Y_k] * P(X_n = k)$$

Computations involve sums of harmonic numbers and their squares, which have closed-form expressions involving the digamma functions, demonstrating the complexity of the problem.

2 Q2

We take turns flipping a fair coin. If the player flips heads, they win. If they flip tails, they pass the coin to the other person. The coin is fair, but the game is not. What is the probability that the player who goes first wins?

Let P_1 and P_2 be the two people who start first and second respectively.

For P_1, if if coin toss 1 lands heads, they win.$p(t_1 = H) = 1/2$ and For P_1, if if coin toss 1 lands heads, they win.$p(t_1 = T) = 1/2$ and

If toss 1 is tails, i.e. $t_1 = T$ materializes, P_2 plays.

If toss 2 is heads, P_1 losses. For the game to continue and P_1 to be in play, we need toss 2 to be tails. $p(t_2 = T) = 1/2$

Now, P_1 can win at toss 3, if they toss an head here.

We see this path as having probability, $p(t_1 = T) * p(t_2 = T) * p(t_3 = H) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2}$

We see a pattern emerging, P_1 can win if toss 1 is heads, or P_1 can win if toss 1 is tails, toss 2 is tails , toss 3 is heads and so on and so forth.

$$p(P_1 = w) = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} ...$$

This is a geometric progression,

$$p(P_1 = w) = \frac{\frac{1}{2}}{1 - \frac{1}{4}}$$

$$p(P_1 = w) = 2/3$$

The probability of player 1 winning is 0.667.

3 Q3

You draw 5 cards from a standard deck of cards (4 suits, 13 cards per suit). The probability of drawing a straight is greater than the probability of drawing a full house (feel free to google these probabilities). Now suppose you start adding more suits to the deck, you increase the size of the deck to N suits - each still with 13 cards. What is the minimum number of suits required such that the probability of drawing a full house is greater than the probability of drawing a straight?

As we increase the number of suits, we are increasing the number of cards in each rank, which increases the chance of drawing a full house more than it does the chance of drawing a straight.

Let the number of suits be N. The total number of cards will be 13N.

The number of ways to draw a 5-card hand from this expanded deck is C(13N, 5).

The probability of drawing a straight doesn't change as it is not dependent on the number of the suits.

For a full house, we have C(N,3) ways to draw three cards of one rank, and C(N,2) ways to draw two cards of another rank. The number of possible full houses is then C(13,1) * C(N,3) * C(12,1) * C(N,2).

We need to find the smallest N such that the probability of drawing a full house is greater than the probability of drawing a straight.

find the smallest N such that,

C(13,1) * C(N,3) * C(12,1) * C(N,2) / C(13N, 5) ¿ P(straight) = 0.0039

We can run a python script for the same.

On doing so, we get the smallest N as 45 The minimum number of suits required is 45.

4 Q4

Flip 3 fair coins and notice the number of heads. Roll 3 fair dice and note the number of 1s or 6s. Finally, draw 3 random cards from a standard deck and note the number of hearts. What's the probability that all these numbers are the same?

For 3 fair coins, the number of heads n will be given by a binomial distribution. k = 0,1,2,3.

$$P(n = k) = C(3, k) * (1/2)^k * (1/2)^(3 - k)$$

For 3 fair dice, for each dice, there is a 1/3 chance that it is a 1 or a 6. The number of 1s or 6s is given by a binomial distribution, k = 0,1,2,3.

$$P(n = k) = C(3, k) * (1/3)^k * (2/3)^(3 - k)$$

For 3 random cards drawn from a standard deck, number of ways of drawing k = 0,1,2,3 heart cards is $C(13, k) * C(39, 3 - k)$. i.e. we chose from the heart cards and then chose from the non heart cards. Total number of ways of drawing 3 cards is C(52, 3). So,

$$P(n = k) = [C(13, k) * C(39, 3 - k)]/C(52, 3)$$

If the three numbers are to be the same, we have, will have the product of probabilities,

$$P(N = k) = p_1(n = k) * p_2(n = k) * p_3(n = k)$$

Here, will take a sum over k from 0 to 3. We can compute the same using a python script. The probability comes out to be 0.0517.

5 Q5

A magic square is a square array of numbers consisting of the distinct positive integers 1, 2, ... n^2 arranged such that the sum of the numbers in any horizontal, vertical or main diagonal is always the same number. What is an n=3, or a 3 by 3 magic square. Prove it.

An example of a 3 by 3 magic square is given by the following array:

$$\begin{array}{ccc} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{array}$$

For this square, we the verify that the sum is 15 by adding the elements along each row, column, and diagonal,

- Rows:
  - $8 + 1 + 6 = 15$
  - $3 + 5 + 7 = 15$
  - $4 + 9 + 2 = 15$
- Columns:
  - $8 + 3 + 4 = 15$
  - $1 + 5 + 9 = 15$
  - $6 + 7 + 2 = 15$
- Diagonals:
  - $8 + 5 + 2 = 15$
  - $6 + 5 + 4 = 15$

As we can see, the sum in each case is 15, confirming that this is a 3 by 3 magic square.

6 Q6

A car travels from New York City to Los Angeles at 60 mph and returns from Los Angeles to New York at 40 mph. What average speed did the car travel at?

Let x be the number of miles between NYC and LA.
From NYC to LA, let the time taken be t_1.

$$t_1 = \frac{x}{60}$$

From LA to NYC, let the time taken be t_2.

$$t_2 = \frac{x}{40}$$

Average speed s is the total distance divided by the total time.

$$s = \frac{2 * x}{\frac{x}{60} + \frac{x}{40}}$$

$$s = \frac{2}{\frac{1}{60} + \frac{1}{40}}$$

$$s = \frac{2 * 60 * 40}{100}$$

$$s = 48$$

Average speed is 48 miles per hour.