Three squares touch the circle

Proof

Assumptions : We are assuming three large squares to touch the circle each at a corner. We do not know anything about the fourth square. All squares are in contact with two other squares. The second square is raised by a value 'c'. The distance between the first and third square is taken as 'd'. This arrangement is illustrated in the following diagram. The case a=b is being considered. All big squares have the same side length x_1 .

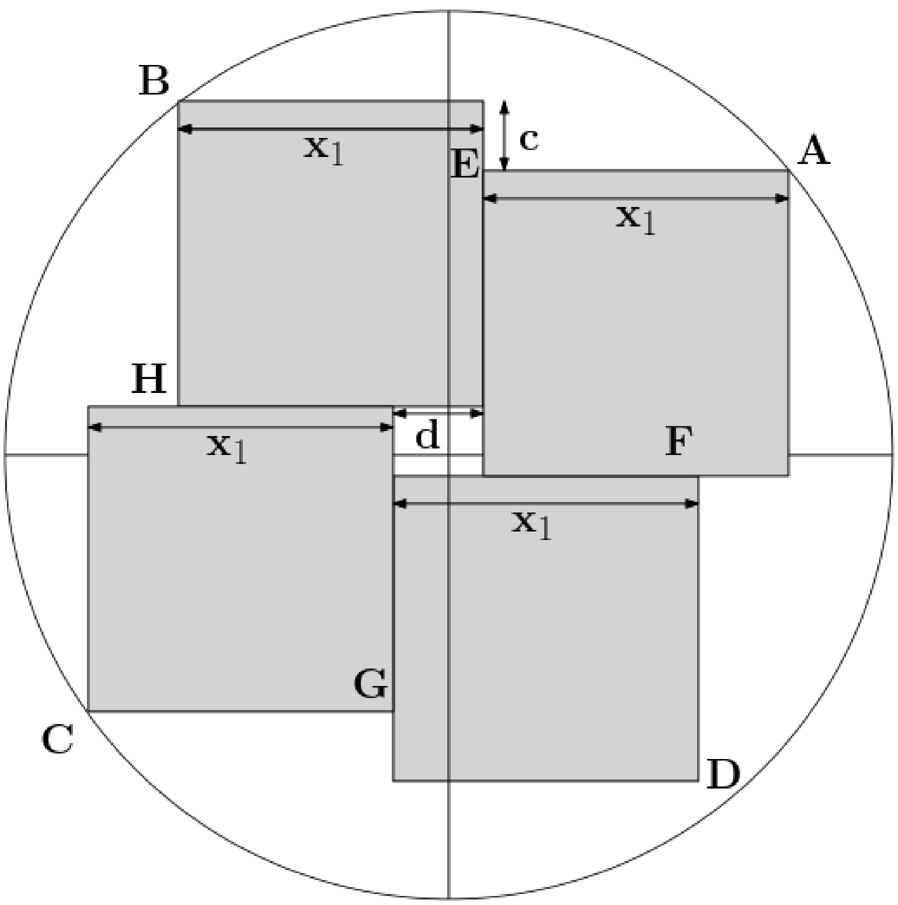

Figure 1

Let A have the coordinates (cos r, sin r). Then we get the following points.

B (cos r - $2x_1$, sin r + c)
C(cos r - $2x_1$ - d, sin r + c - $2x_1$)
D(cos r - d , sin r - $2x_1$)
E(cos r - x_1 , sin r)
F(cos r - d , sin r - x_1)
G(cos r - x_1 - d , sinr + c - $2x_1$)
H(cos r - $2x_1$, sin r + c - x_1)

Since point B lies on the circle we get the following equation.

$(\cos r - 2x_1)^2 + (\sin r + c)^2 = 1$
$\cos^2 r - 4x_1 \cos r + 4{x_1}^2 + \sin^2 r + 2c \sin r + c^2 = 1$
$4x_1^2 + c^2 = 4x_1 \cos r - 2c \sin r$

$$\frac{4x_1^2+c^2}{\sqrt{16x_1^2+4c^2}} = \frac{4x_1\cos r}{\sqrt{16x_1^2+4c^2}} - \frac{2c\sin r}{\sqrt{16x_1^2+4c^2}}$$

let $\sin t = \frac{4x_1}{\sqrt{16x_1^2+4c^2}}, \cos t = \frac{2c}{\sqrt{16x_1^2+4c^2}}$

$$\frac{4x_1^2+c^2}{\sqrt{16x_1^2+4c^2}} = \sin t.\cos r - \cos t.\sin r$$

$$\frac{4x_1^2+c^2}{\sqrt{16x_1^2+4c^2}} = \sin(t-r)$$

$$\sin^{-1}\left(\frac{4x_1^2+c^2}{\sqrt{16x_1^2+4c^2}}\right) = t - r$$

$$r = \sin^{-1}\left(\frac{4x_1}{\sqrt{16x_1^2+4c^2}}\right) - \sin^{-1}\left(\frac{4x_1^2+c^2}{\sqrt{16x_1^2+4c^2}}\right)$$

Now we are going to get d in terms of c, x_1 and r. Since C lies on the circle we get the following equation.

$(\cos r - 2x_1 - d)^2 + (\sin r - 2x_1 + c)^2 = 1$
$\cos^2 r + 4x_1^2 + d^2 - 4x_1 \cos r - 2d \cos r + 4x_1 d + \sin^2 r + 4x_1^2 + c^2 - 4x_1 \sin r + 2c \sin r - 4x_1 c = 1$
$d^2 + (4x_1 - 2\cos r)d + 8x_1^2 - 4x_1 \cos r + c^2 - 4x_1 \sin r + 2c \sin r - 4x_1 c = 0$

$$d = \frac{2\cos r - 4x_1 \pm \sqrt{16x_1^2 + 4\cos^2 r - 16x_1\cos r - 32x_1^2 + 16x_1 \cos r - 4c^2 + 16x_1 \sin r - 8c \sin r + 16x_1 c}}{2}$$

$d = \frac{2\cos r - 4x_1 + \sqrt{4\cos^2 r - 16x_1^2 - 4c^2 + 16x_1 \sin r - 8c\sin r + 16x_1 c}}{2}$

$d = \cos r - 2x_1 + \sqrt{1 - \sin^2 r - 4x_1^2 - c^2 + 4x_1 \sin r - 2c\sin r + 4x_1 c}$

$d = \cos r - 2x_1 + \sqrt{1 - (\sin r - 2x_1 + c)^2}$

We have considered the +ve sign as d has to be positive.

Now we are going to place a square having maximum side length such that one corner is at E and the other is on the circle. Let its side length be z_1. For the corner that lies on the circle we get the following equation.
$(z_1 + \cos r - x_1)^2 + (z_1 + \sin r)^2 = 1$

1) For any equation of the following type, we get its roots.

$(z + a)^2 + (z + b)^2 = 1$
$z^2 + 2az + a^2 + y^2 + 2by + b^2 = 1$
$2z^2 + z(2a + 2b) + a^2 + b^2 - 1 = 0$

$z = \frac{-(2a+2b) \pm \sqrt{4a^2+4b^2+8ab-8a^2-8b^2+8}}{4}$

$z = \frac{-(a+b) \pm \sqrt{2-(a-b)^2}}{2}$

Thus for z_1 we get

$z_1 = \frac{-(\cos r - x_1 + \sin r) + \sqrt{2 - (\cos r - x_1 - \sin r)^2}}{2}$
We have taken the +ve sign as z_1 has to be positive.

Now we repeat the procedure at F and get the following equation.

$(\cos r - d + z_2)^2 + (\sin r - x_1 - z_2)^2 = 1$
$(z_2 + \cos r - d)^2 + (z_2 + x_1 - \sin r)^2 = 1$

Using what we derived in 1) we get

$$z_2 = \frac{-(\cos r - d + x_1 - \sin r) + \sqrt{2 - (\cos r - d - x_1 + \sin r)^2}}{2}$$

We have taken the +ve sign as z_2 has to be positive.

For G we have

$(\cos r - x_1 - d - z_3)^2 + (\sin r + c - 2x_1 - z_3)^2 = 1$
$(z_3 + x_1 + d - \cos r)^2 + (z_3 + 2x_1 - \sin r - c)^2 = 1$
Using what we derived in 1) we get

$$z_3 = \frac{-(x_1 + d - \cos r + 2x_1 - \sin r - c) + \sqrt{2 - (x_1 + d - \cos r - 2x_1 + \sin r + c)^2}}{2}$$

$$z_3 = \frac{-(3x_1 + d - \cos r - \sin r - c) + \sqrt{(2 - (d - \cos r - x_1 + \sin r + c)^2}}{2}$$

We have taken the +ve sign as z_3 has to be positive.

For H we have

$(\cos r - 2x_1 - z_4)^2 + (\sin r + c - x_1 + z_4)^2 = 1$
$(z_4 + 2x_1 - \cos r)^2 + (z_4 + \sin r + c - x_1)^2 = 1$

Using what we derived in 1) we get

$$z_4 = \frac{-(2x_1 - \cos r + \sin r + c - x_1) + \sqrt{2 - (2x_1 - \cos r - \sin r - c + x_1)^2}}{2}$$

$$z_4 = \frac{-(x_1 - \cos r + \sin r + c) + \sqrt{2 - (3x_1 - \cos r - \sin r - c)^2}}{2}$$

We have taken the +ve sign as z_4 has to be positive.

z_1, z_2, z_3 and z_4 was plotted for a=0.16. c was taken variable. r and d was calculated by the equations derived above.

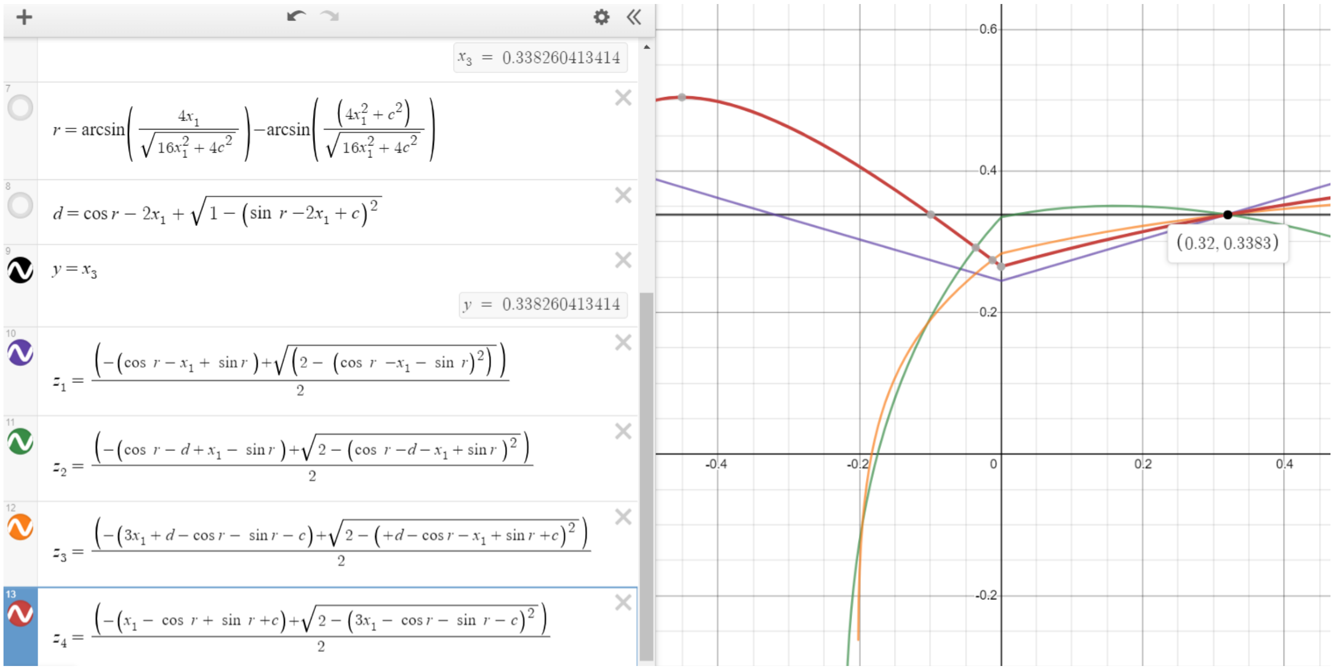

Figure 2

It is observed that that all four functions z_1, z_2, z_3 and z_4 are intersecting each other at only one point. The c value is 0.32 and each z_1, z_2, z_3 and z_4 have the same value at this point which is 0.3383. This value is exactly the value for the squares of side x_3 as well as x_4 (as a=b).

$$x_1 = \frac{(b-a)+\sqrt{2-(a+b)^2}}{2}$$

$$x_3 = \frac{-(a+x_1-b)+\sqrt{2-(x_1-b-a)^2)}}{2}$$

For a=0.16 we have $x_1 = 0.6888$ and $x_3 = 0.3383$

Some conclusions and observations

Only for one value of c we are getting all the four smaller squares to fit in their respective position.

When c was fixed as 0.32 , the value of d we got by the formula that was derived before is also 0.32. d=c=2a at the point where all the four functions are intersecting.

We also checked if point D lies on the circle for c=0.32. This can be done as follows

t $= (\cos r - d)^2 + (\sin r - 2x_1)^2$

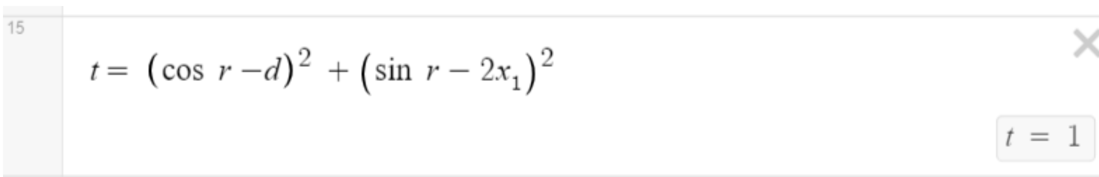

Figure 3

t is found to be 1 which confirms that D lies on the circle when c=0.32.