MA109 Calculus-I

D4-T6 Tutorial 3

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15th December 2021

Theorem (MVT)

Let $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *be a function such that*
(i) f *is continuous on* $[a, b]$, *and*
(ii) f *is differentiable on* (a, b).

Then there exists $c \in (a, b)$ *such that* $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

1)

$$f(x) = x^3 - 6x + 3$$

Let's find the stationary points of this polynomial.

$$f'(x) = 3x^2 - 6$$

$$\implies x = \pm\sqrt{2}$$

Now, $f(-\sqrt{2}) = 4\sqrt{2} + 3 > 0$ and $f(+\sqrt{2}) = -4\sqrt{2} + 3 < 0$. So by IVT, f has a root in $\left(-\sqrt{2}, \sqrt{2}\right)$

- $f(x) \to -\infty$ as $x \to -\infty$. By IVT, f has a root in $\left(-\infty, \sqrt{2}\right)$
- $f(x) \to +\infty$ as $x \to +\infty$. By IVT, f has a root in $\left(+\sqrt{2}, \infty\right)$

Since f has at most three roots, all its root are real.

4)

Let the 3 distinct roots be $r_1 < r_2 < r_3$
By Rolle's theorem $f'(x)$ has at least two real roots, x_1 and x_2 such that $r_1 < x_1 < r_2$ and $r_2 < x_2 < r_3$.
Since $f'(x) = 3x^2 + p$ this implies that $p < 0$, and $x_1 = -\sqrt{\frac{-p}{3}}$, $x_2 = +\sqrt{\frac{-p}{3}}$
Now, $f''(x_1) = 6x_1 < 0 \implies$ f has a local maximum at $x = x_1$. Similarly, f has a local minimum at $x = x_2$.
This proves parts 1 and 2.

4) continued

Since the quadratic $f'(x)$ is negative between its roots x_1 and x_2 (so that f is decreasing over $[x_1, x_2]$) and f has a root r_2 in (x_1, x_2), we must have $f(x_1) > 0$ and $f(x_2) < 0$.
Further,

$$f(x_1) = q + \sqrt{\frac{-4p^3}{27}}, f(x_2) = q - \sqrt{\frac{-4p^3}{27}}$$

$$f(x_1) \times f(x_2) < 0 \implies \frac{4p^3 + 27q^2}{27} < 0$$

Hence proved.

5)

To prove that $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.
Case 1. $a = b$. We can clearly observe that the condition is satisfied.
Case 2. $a \neq b$. Without loss of generality, we can assume that $a < b$.
As $f := \sin$ is continuous and differentiable on $\mathbb{R}$, there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. (By MVT)

Also, we know that $|f'(c)| = |\cos c| \leq 1$.

Thus, we have it that $\left|\dfrac{f(b) - f(a)}{b - a}\right| \leq 1$.

$$\implies |\sin a - \sin b| \leq |a - b|$$

7)

Assume that $f(0) \neq 0$. Then, there are two possibilities.
Case 1. $f(0) > 0$.
The function f satisfies the hypothesis of MVT, thus there must exist $c \in (-a, 0)$ such that $f'(c) = \dfrac{f(0) - f(-a)}{0 - (-a)} = \dfrac{f(0)}{a} + 1.$
As $f(0) > 0$ and $a > 0$, we get that $f'(c) > 1$ which contradicts the hypothesis.

Case 2. $f(0) < 0$.
The function f satisfies the hypothesis of MVT, thus there must exist $d \in (0, a)$ such that $f'(d) = \dfrac{f(a) - f(0)}{a - 0} = 1 - \dfrac{f(0)}{a}.$
As $f(0) < 0$ and $a > 0$, we get that $f'(d) > 1$ which contradicts the hypothesis.

- (i) Assume that there exists such a function. We are given that f'' exists which implies that f' must be continuous and differentiable everywhere. Since $f'(0) = f'(1)$, by Rolle's Theorem, $f''(c) = 0$ for some $c \in (0,1)$. This contradicts the condition that $f''(x) > 0$ for all $x \in \mathbb{R}$.
- (ii)Such a function exists. Example : $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \frac{x^2}{2} + x$.
- (iv) e^x satisfies all the conditions given. Another possible function is :

$$f(x) = \begin{cases} \frac{1}{1-x} & x \leq 0, \\ 1 + x + x^2 & x > 0. \end{cases}$$

8(iii) Assume that such a function exists. Then, we are given that f'' exists. Thus, f' must be continuous and differentiable everywhere.

As f'' is nonnegative, f' must be increasing everywhere. We are given that $f'(0) = 1$.

Thus, given any $c > 0$, we know that $f'(c) \geq 1$. (1)

Let $x \in (0, \infty)$. By MVT, we know that there exists $c \in (0, x)$ such that $f'(c) = \dfrac{f(x) - f(0)}{x - 0}$.

Thus, by (1), we have it that $f(x) - f(0) \geq x$ for all positive x.

This contradicts that $f(x) \leq 100$ for all positive x.

9)

$$f(x) = \begin{cases} 1 - 12x - 3x^2 & \text{if } x \leq 0 \\ 1 + 12x - 3x^2 & \text{if } x > 0 \end{cases}$$

As $f : [-2, 5] \to \mathbb{R}$ is continuous, we have it that the absolute extrema of f on $[a, b]$ is attained either at a critical point of f or at an end-point of $[a, b]$.
An interior point c of the domain is called a critical point of f if either f is not differentiable at c, or if f is differentiable at c and $f'(c) = 0$.
0 is a critical point as f is not differentiable at 0. Moreover, 0 is the only point at which f is not differentiable.
For $x < 0$, we get the derivative of f as $f'(x) = -12 - 6x = -6(x + 2)$.
Thus, no negative number in the domain is a critical point. (Note that -2 is **not** an interior point of the domain.)

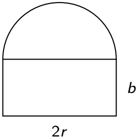
For $x > 0$, we get the derivative as $f'(x) = 12 - 6x = 6(2 - x)$. Thus, 2 is a critical point of f. (Note that 2 **is** an interior point of the domain.)

To summarise,
Critical points of f : $0, 2$. End-points of $[-2, 5]$: $-2, 5$.

x	0	2	-2	5
f(x)	1	13	13	-14

$\therefore$ f attains its global maximum 13 at 2 as well as -2, and its global minimum -14 at 5.

10)



Given perimeter of the window is to be p.

$$p = 2b + 2r + \pi r = 2b + (\pi + 2)r$$

Let $l(r, b)$ denote the amount of light that enters through the window for r and b specified in the diagram.
We know that $l(r, b) = k(2rb) + \left(\frac{k}{2}\right)\left(\frac{\pi r^2}{2}\right)$. Where k is some positive constant.
Rewriting b in terms of p, we get $L(r) = l\left(r, \frac{1}{2}(p - (2+\pi)r)\right) = k\left[r(p - (2+\pi)r) + \frac{\pi r^2}{4}\right] = k\left[pr - 2r^2 - \frac{3\pi r^2}{4}\right]$.

10) continued

As the above is defined for $(0, \infty)$ and is differentiable everywhere, we only need to check where the derivative is zero.

$$L'(r) = k\left[p - 4r - \tfrac{3\pi r}{2}\right] = k\left[p - \left(\tfrac{8+3\pi}{2}\right)r\right].$$
$$\therefore L'(r) = 0 \implies r = \frac{2p}{8+3\pi}.$$

You can verify that for this value of r, $L''(r)$ is negative.

b can now be calculated as we know the relation between b, p and r. It comes out to be $\frac{1}{2}\left(\frac{4+\pi}{8+3\pi}\right)p$.