MA109 Calculus-I

D4-T6 Tutorial 4

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Fundamental Theorem of Calculus : Part 1

Theorem (Fundamental Theorem of Calculus : Part 1)

Let f be integrable on $[a, b]$. For $x \in [a, b]$,define

$$F(x) := \int_a^b f(t)dt$$

Then F is continuous on $[a, b]$. Morover, if f is continuous at $c \in [a, b]$, then F is differentiable at c, and $F'(c) = f(c)$

Fundamental Theorem of Calculus : Part 2

Theorem (Fundamental Theorem of Calculus : Part 2)

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable on $[a, b]$,. Then

$$\int_a^b f'(x)dx = f(b) - f(a)$$

5)

The given function is reimann integrable as it is a monotonically increasing function.
Let P_n be the partition of [0,2] into 2×2^n parts.
Then $U(P_n, f) = 3$ and

$$L(P_n) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = 1 + 1 \times \frac{1}{2^n} + 2 \times \frac{2^n - 1}{2^n} \to 3$$

as $n \to \infty$.
Thus, $\int_0^2 f(x)dx = 3$

6)

(a)We know that $L(P) \leq \int_a^b f(x)dx \leq U(P)$,

$$L(P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$
$$\Rightarrow L(P) \geq 0$$
$$\Rightarrow \int_a^b f(x)dx \geq 0$$

Since $m_i \geq 0 \ \forall i$. Further, if f is continuous let $F(x)$ be defined by $F(x) = \int_a^x f(t)dt$, then from FTC

$$F'(x) = f(x) \geq 0 \forall x \in [a, b]$$

Now we know that $F'(x) \geq 0$, $F(b) = F(a) = 0 \Rightarrow$ $F(x) = 0 \forall x \in [a, b] \Rightarrow f(x) = 0 \forall x \in [a, b]$

6) continued

(b) Take $f(x) = 0$ if $x \neq \frac{a+b}{2}$, $f(\frac{a+b}{2}) = 1$. Then this function is Riemann integrable and

$$\int_a^b f(x)dx = 0$$

7)

i) Note that

$$S_n = \frac{1}{n^{5/2}} \sum_{i=1}^{n} i^{3/2} = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{3/2} \left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := \frac{2}{5}x^{5/2}$. Then, we have that $f'(x) = x^{3/2}$. As f' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $t_i := i/n$ for $i = 1, 2, \ldots, n$. Then, $S_n = R(f', P_n, t)$. Since $||(P_n)|| = \frac{1}{n} \to 0$, it follows that

$$R(f', P_n, t) \to \int_0^1 x^{3/2} dx = \int_0^1 f'(x) dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x) dx = f(1) - f(0) = \frac{2}{5}.$$

7)

(iii) Note that

$$S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{in + n^2}} = \sum_{i=1}^{n} \frac{1}{\sqrt{\left(\frac{i}{n}\right) + 1}} \left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := 2\sqrt{x+1}$. Then, we have that $f'(x) = \frac{1}{\sqrt{x+1}}$.
As f' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $t_i := i/n$ for $i = 1, 2, \ldots, n$.
Then, $S_n = R(f', P_n, t)$. Since $||(P_n)|| = \frac{1}{n} \to 0$, it follows that

$$R(f', P_n, t) \to \int_0^1 \frac{1}{\sqrt{x+1}} dx = \int_0^1 f'(x) dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n \to \infty} S_n = \int_0^1 f'(x) dx = f(1) - f(0) = 2\sqrt{2} - 2.$$

7)

(iv) Note that

$$S_n = \frac{1}{n}\sum_{i=1}^{n}\cos\left(\frac{i\pi}{n}\right) = \sum_{i=1}^{n}\cos\left(\frac{i\pi}{n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := \frac{1}{\pi}\sin(\pi x)$. Then, we have that $f'(x) = \cos(\pi x)$.
As f' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $t_i := i/n$ for $i = 1, 2, \ldots, n$.
Then, $S_n = R(f', P_n, t)$. Since $||(P_n)|| = \frac{1}{n} \to 0$, it follows that

$$R(f', P_n, t) \to \int_0^1 \cos(\pi x)dx = \int_0^1 f'(x)dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)dx = f(1) - f(0) = 0.$$

(v) Note that

$$S_n = \frac{1}{n}\left\{\sum_{i=1}^{n}\left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n}\left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n}\left(\frac{i}{n}\right)^{2}\right\}.$$

We shall find $\lim_{n\to\infty} S_n$ by finding the limits of the individual sums and showing that they all exist.

$$S_n \to \int_0^1 x dx + \int_1^2 x^{3/2} dx + \int_2^3 x^2 dx = \frac{1}{2} + \frac{2}{5}(4\sqrt{2} - 1) + \frac{19}{3}$$

8 b)

Let u and v be differentiable functions defined on appropriate domains. Let g be a continuous function. Define $G(x) := \int_a^x g(t)dt$. Then $G'(x) = g(x)$, by Fundamental Theorem of Calculus (Part 1). Note that

$$\int_{u(x)}^{v(x)} g(t)dt = \int_a^{v(x)} g(t)dt - \int_a^{u(x)} g(t)dt = G(v(x)) - G(u(x)).$$

Thus, by the Chain Rule, one has

$$\frac{d}{dx}\int_{u(x)}^{v(x)} g(t)dt = G'(v(x))v'(x) - G'(u(x))u'(x) = g(v(x))v'(x) - g(u(x))u'$$

We can now easily solve the question.

8b) continued

(i)

Given, $F(x) = \int_1^{2x} \cos\left(t^2\right) dt$

$$\begin{aligned} \therefore \frac{dF}{dx} &= \cos\left((2x)^2\right)(2x)' - \cos(1)(1)' \\ &= 2\cos\left(4x^2\right). \end{aligned}$$

8b) continued

(ii)

Given, $F(x) = \int_0^{x^2} \cos(t)dt$

$$\begin{aligned}\therefore \frac{dF}{dx} &= \cos\left(x^2\right)(x^2)' - \cos(0)(0)' \\ &= 2x\cos\left(x^2\right).\end{aligned}$$

9)

Define $F : \mathbb{R} \to \mathbb{R}$ as

$$F(a) := \int_a^{a+p} f(t)dt.$$

If we show that F is constant, then we are done.
As f is a continuous, Fundamental Theorem of Calculus (Part 1) tells us that F is differentiable everywhere. Using the result we had shown earlier, we have it that $F'(a) = f(a+p) \cdot 1 - f(a) \cdot 1 = 0$.
As F is defined on an interval ($\mathbb{R}$), we have it that F is constant. $\blacksquare$

10)

$$
\begin{aligned}
g(x) &= \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)dt \\
&= \frac{1}{\lambda}\int_0^x f(t)\left(\sin\lambda x\cos\lambda t - \cos\lambda x\sin\lambda t\right)dt \\
&= \frac{1}{\lambda}\sin\lambda x\int_0^x f(t)\cos\lambda t dt - \frac{1}{\lambda}\cos\lambda x\int_0^x f(t)\sin\lambda t dt
\end{aligned}
$$

Now, we can differentiate g using product rule and Fundamental Theorem of Calculus (Part 1).

$$
\therefore g'(x) = \cos\lambda x\int_0^x f(t)\cos\lambda t dt + \sin\lambda x\int_0^x f(t)\sin\lambda t dt
$$

10) continued

It is easy to verify that both $g(0)$ and $g'(0)$ are 0.
We can differentiate g' in a similar way and get,

$$
\begin{aligned}
g''(x) &= -\lambda \sin \lambda x \int_0^x f(t) \cos \lambda t dt + f(x) \cos^2 \lambda x + \lambda \cos \lambda x \int_0^x f(t) \sin \lambda t d \\
&\quad + f(x) \sin^2 \lambda x \\
&= f(x) - \lambda^2 \left(\frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \cos \lambda x \sin \lambda t \right) dt \right) \\
&= f(x) - \lambda^2 g(x) \\
\implies g''(x) + \lambda^2 g(x) &= f(x)
\end{aligned}
$$