MA109 Calculus-I

D4-T6 Tutorial 7

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Second Derivative Test

Theorem

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2$$

1. *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum for f*
2. *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum for f*
3. *If $D < 0$, then (x_0, y_0) is asaddle point for f*
4. *If $D = 0$ further examination of the function is necessary*

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(3) We shall assume that z is a "sufficiently smooth" function of x and y. We are given that $\sin(x+y)+\sin(y+z)=1$ and $\cos(y+z)\neq 0$. Differentiating with respect to x while keeping y constant gives us
$$\cos(x+y)+\cos(y+z)\frac{\partial z}{\partial x}=0. \qquad (*)$$

Similarly, differentiating with respect to y while keeping x constant gives us
$$\cos(x+y)+\cos(y+z)\left(1+\frac{\partial z}{\partial y}\right)=0. \qquad (**)$$

Differentiating $(*)$ with respect to y gives us
$$-\sin(x+y)-\sin(y+z)\left(1+\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}+\cos(y+z)\frac{\partial^2 z}{\partial x \partial y}=0.$$ [1]

[1] Note that I have implicitly assumed that $\frac{\partial^2 z}{\partial x \partial y}=\frac{\partial^2 z}{\partial y \partial x}$. However, using a different set of calculations, one can arrive at the same answer without assuming this. I encourage you to try that.

Thus, using $(*)$ and $(**)$, we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{\cos(y+z)} \left[\sin(x+y) + \sin(y+z) \cdot \left(1 + \frac{\partial z}{\partial y}\right) \frac{\partial z}{\partial x} \right]$$

$$= \frac{1}{\cos(y+z)} \left[\sin(x+y) + \sin(y+z) \left(-\frac{\cos(x+y)}{\cos(y+z)}\right) \left(-\frac{\cos(x+y)}{\cos(y+z)}\right) \right]$$

$$= \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z) \frac{\cos^2(x+y)}{\cos^2(y+z)}$$

(4) We have that

$$f_{xy}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}.$$

For $k \neq 0$, we know that

$$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} = -k.$$

We also know that

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$

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Thus, we get that

$$f_{xy}(0,0) = \lim_{k \to 0} \frac{-k - 0}{k} = -1.$$

By similar calculations, we get that $f_{yx}(0,0) = 1$.
Thus, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

For $(x,\ y) \neq (0,\ 0)$, one can calculate the second derivatives and see that they turn out to be discontinuous at $(0,\ 0)$.

$$f_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}, f_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}, f_{yx}(x,y) = \frac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$$

5. (i) $f(x, y) = x^4 + y^4 + 4x - 32y - 7, \quad (x_0, y_0) = (-1, 2)$.
Note that the above function is defined on $D = \mathbb{R}^2$.
Thus, the given point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.
Note that $(\nabla f)(x, y) = (4x^3 + 4, 4y^3 - 32)$. Hence, $(\nabla f)(x_0, y_0) = 0$.
Thus, we can appeal to the determinant test.

$(\Delta f)(x, y) = (12x^2)(12y^2) - (0)^2 = 144x^2y^2$.
Thus, $(\Delta f)(x_0, y_0) > 0$.
Also, $f_{xx}(x_0, y_0) = 12x_0^2 > 0$.

Thus, by the determinant test, we get that f has a local minimum at (x_0, y_0).

5. (ii) $f(x, y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3, \quad (x_0, y_0) = (0, 0)$.
Note that the above function is defined on $D = \mathbb{R}^2$.
Thus, the given point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.
Note that $(\nabla f)(x, y) = (3x^2 + 6x - 2y, -2x + 10y - 12y^2)$. Hence, $(\nabla f)(x_0, y_0) = 0$.
Thus, we can appeal to the determinant test.

$(\Delta f)(x, y) = (6x + 6)(10 - 24y) - (-2)^2$.
Thus, $(\Delta f)(x_0, y_0) = (6)(10) - 4 = 56 > 0$.
Also, $f_{xx}(x_0, y_0) = 6 > 0$.

Thus, by the determinant test, we get that f has a local minimum at (x_0, y_0).

6. (i) $f(x, y) = \left(x^2 - y^2\right) e^{-\left(x^2+y^2\right)/2}$.
Note that the above function is defined on $D = \mathbb{R}^2$.
Thus, every point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere. (How?)
For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.

Note that $f_x(x, y) = xe^{1/2\left(-x^2-y^2\right)} \left(-x^2 + y^2 + 2\right)$.
Also, $f_y(x, y) = ye^{1/2\left(-x^2-y^2\right)} \left(-x^2 + y^2 - 2\right)$.

Thus, solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us that
$(x_0, y_0) \in \{(0, 0),\ (0, \sqrt{2}),\ (0, -\sqrt{2}),\ (-\sqrt{2}, 0),\ (\sqrt{2}, 0)\}$.
Now, we determine the exact nature using the determinant test.

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Recall that $(\Delta f)(x_0, y_0) := f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.
Hence, in our case,

$$(\Delta f)(x, y) = -e^{-x^2-y^2}\left(x^6 - x^4y^2 - 3x^4 - x^2y^4 + 22x^2y^2 - 8x^2 + y^6 - 3y^4\right.$$

Moreover, $f_{xx}(x, y) = e^{-\left(x^2+y^2\right)/2}(x^4 - x^2y^2 - 5x^2 + y^2 + 2)$
For $(x_0, y_0) = (0, 0)$, it is clear that it is a saddle point for f as discriminant is $-4 < 0$.

Note that if $x = 0$, the discriminant reduces to $-e^{-y^2}(y^6 - 3y^4 - 8y^2 + 4)$. Substituting $y = \pm\sqrt{2}$ gives us that the discriminant is positive with f_{xx} positive and hence, the points are points of local minima.

Similarly, we get that the points $(\pm\sqrt{2}, 0)$ are points of local maxima as they have discriminant positive and f_{xx} negative.

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6. (ii) $f(x, y) = f(x, y) = x^3 - 3xy^2$.
Note that the above function is defined on $D = \mathbb{R}^2$.
Thus, every point is an interior point of D. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere. (How?)
For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.

Note that $f_x(x, y) = 3x^2 - 3y^2$.
Also, $f_y(x, y) = -6xy$.

Thus, solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us that $(x_0, y_0) = (0, 0)$.
Now, we determine the exact nature using the determinant test.

Recall that $(\Delta f)(x_0, y_0) := f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.
Hence, in our case,

$$(\Delta f)(x_0, y_0) = -36(x_0^2 + y_0^2).$$

Thus, for $(x_0, y_0) = (0, 0)$, we get the discriminant is 0.
Hence, we get that the discriminant test is inconclusive!
This means that we must turn to some other analytic methods of determining the nature.

Now, we note that $f(\delta, 0) = \delta^3$ for all $\delta \in \mathbb{R}$.
Thus, given any $\epsilon > 0$, choose $\delta = \pm\epsilon/2$.
This gives us that $(0, 0)$ is saddle point. (How?)

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7. To find: Absolute maxima and minima of
$f(x, y) = \left(x^2 - 4x\right) \cos y$ for $1 \leq x \leq 3, -\pi/4 \leq y \leq \pi/4$.
Note that the domain is a closed and bounded set. As f is continuous on the domain, f does achieve a maximum and a minimum. Note that $f_x(x, y) = (2x - 4) \cos y$ and $f_y(x, y) = -\left(x^2 - 4x\right) \sin y$ for interior points (x, y).
Thus, the only critical point is $p_1 = (2, 0)$.

Now we restrict ourselves to the boundaries to find the local extrema.
"Right boundary:" This is the line segment $x = 3, -\pi/4 \leq y \leq \pi/4$.
The function now reduces to $-3 \cos y$ on this segment.
Using our theory from one-variable calculus, we get that we need to check the points $(3, 0)$, $(3, \pi/4)$, $(3, -\pi/4)$. (How?)

Similar consideration of the "left boundary" gives us the points $(1, 0)$, $(1, \pi/4)$, $(1, -\pi/4)$.

Now, we look at the "top boundary."
The function there reduces to $\frac{x^2-4x}{\sqrt{2}}$.
Once again, using our theory from one-variable calculus, we get that we need to check the points $(1, \pi/4)$, $(2, \pi/4)$, $(3, \pi/4)$.

Similarly, checking the "bottom boundary" gives us the points $(1, -\pi/4)$, $(2, -\pi/4)$, $(3, -\pi/4)$.
We now tabulate our results as follows:

(x_0, y_0)	$(2, 0)$	$(3, 0)$	$(3, \pi/4)$	$(2, \pi/4)$	$(1, \pi/4)$
$f(x_0, y_0)$	-4	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$

(x_0, y_0)	$(1, 0)$	$(1, -\pi/4)$	$(2, -\pi/4)$	$(3, -\pi/4)$	
$f(x_0, y_0)$	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$	

Thus, we get that $f_{\min} = -4$ at $(2, 0)$ and $f_{\max} = -\frac{3}{\sqrt{2}}$ at $(1, \pm\pi/4)$ and $(3, \pm\pi/4)$.