MA109 Calculus-I

D4-T6 Tutorial 5

Adish Shah

4th January 2022

1) (i)

Writing y in terms of x gives us $y = 1 - 2\sqrt{x} + x$.
The desired area is

$$\int_0^1 y dx = \int_0^1 (1-2\sqrt{x}+x) dx = \left(x - 2 \cdot \frac{2}{3} x^{3/2} + \frac{1}{2} x^2 \right) \Big|_0^1 = 1 - \frac{4}{3} + \frac{1}{2} = \frac{1}{6}.$$

2)

It is easy to see that the curves $y = f(x)$ and $y = g(x)$ intersect at $(0, 0)$ and $(1 - a, a - a^2)$.
Let us assume that $a < 1$ and find the area A.

$$A = \int_0^{1-a} (x - x^2 - ax)dx = \frac{(1-a)^3}{6}.$$

As $A = 4.5$, we get that $(1 - a)^3 = 27$. Thus, we have it that $a = -2$.
Now, if we assume that $a > 1$, we get that $A = \frac{(a-1)^3}{6} = 4.5$ which gives us that $a = 4$.

3)

Solving for the intersection point of the two curves gives us

$$6a\cos\theta = 2a(1+\cos\theta) \text{ or } \theta = \pm\frac{\pi}{3}.$$

It is for $\theta \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ that the circle is outside the cardioid. Thus, the desired area is

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (6a\cos\theta)^2 - (2a(1+\cos\theta))^2 d\theta$$

$$= 2a^2\int_{-\pi/3}^{\pi/3} (8\cos^2\theta - 1 - 2\cos\theta)d\theta$$

$$= 2a^2\int_{-\pi/3}^{\pi/3} (4\cos(2\theta) - 2\cos\theta + 3)d\theta = 4\pi a^2$$

Arc length :

$$L = \int_1^3 \sqrt{1 + y'(x)^2} dx$$

$$= \int_1^3 \sqrt{1 + (x^2 - \frac{1}{4x^2})^2} dx$$

$$= \int_1^3 \sqrt{1 + x^4 + \frac{1}{16x^4} - \frac{1}{2}} dx = \int_1^3 \sqrt{(x^2 + \frac{1}{4x^2})^2} dx$$

$$= \left(\frac{x^3}{3} - \frac{1}{4x} \right)_1^3 = \frac{26}{3} + \frac{1}{4}(1 - \frac{1}{3}) = \frac{53}{6}$$

5) continued

Area of surface of revolution :
Curve C is parametrised as $(t, \frac{t^3}{3} + \frac{1}{4t}),\ \ 1 \leq t \leq 3$
$\rho(t)$ is the distance of C from the line L : y = -1
$\rho(t) = \frac{t^3}{3} + \frac{1}{4t} - (-1)$

$$Area = 2\pi \int_1^3 (\frac{t^3}{3} + \frac{1}{4t} + 1)\sqrt{1 + (t^2 - \frac{1}{4t^2})^2}dt$$

$$= 2\pi \int_1^3 (\frac{t^3}{3} + \frac{1}{4t} + 1)(t^2 + \frac{1}{4t^2})dt$$

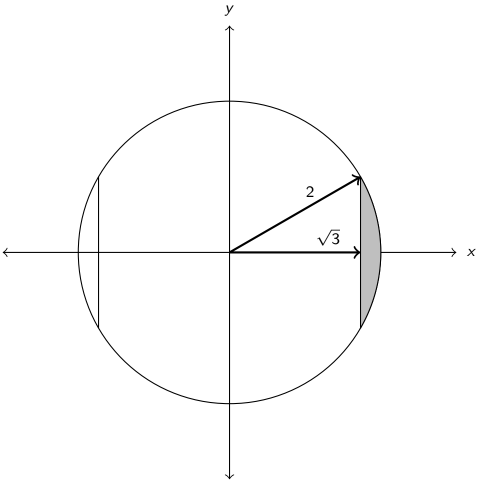
$$= 2\pi \int_1^3 (\frac{t^5}{3} + \frac{t}{12} + \frac{t}{4} + \frac{1}{16t^3} + t^2 + \frac{1}{4t^2})dt$$

$$= \frac{1823\pi}{18}$$

8)

8. We can fix the line to be along $z-$axis, $0 \leq z \leq h$.
For any fixed z, the area of the cross-section of the solid is r^2.
Thus, the required volume is $\int_0^h r^2 dz = hr^2$.

10)



10. We can compute the volume of the portion cut out by finding the volume of revolution (about the $y-$axis) of the shaded part. This can be done easily using washer method.

$$V' = \pi \int_{-1}^{1} \left(\sqrt{4-y^2}\right)^2 - \left(\sqrt{3}\right)^2 dy = \frac{4\pi}{3}.$$

The total volume of the ball is $V = \frac{4\pi}{3}(2)^2$.

Thus, the volume cut out is $V - V' = \frac{28\pi}{3}$.