MA109 Calculus-I

D4-T6 Tutorial 1

Adish Shah

8th December 2021

Definition of convergence of a sequence

Definition

Let (a_n) be a sequence of real numbers. We say that (a_n) is convergent if there is $L \in \mathbb{R}$ such that the following condition holds. For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq n_0$.

This is known as the $\epsilon - n_0$ definition of convergence of a sequence.
In this case, we say that (a_n) converges to L, or that L is the limit of (a_n), and we write

$$\lim_{n \to \infty} a_n = L$$

Some properties of convergent sequences :

- Every convergent sequence is bounded.

Definition of convergence of a sequence

Definition

Let (a_n) be a sequence of real numbers. We say that (a_n) is convergent if there is $L \in \mathbb{R}$ such that the following condition holds. For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq n_0$.

This is known as the $\epsilon - n_0$ definition of convergence of a sequence.
In this case, we say that (a_n) converges to L, or that L is the limit of (a_n), and we write

$$\lim_{n \to \infty} a_n = L$$

Some properties of convergent sequences :

- Every convergent sequence is bounded.
- Every bounded **and** monotonic sequence is convergent.

1 (iii)

To Prove :

$$\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$$

1 (iii)

To Prove :

$$\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$$

Let $\epsilon > 0$ be given. We need to show that $\exists n_0 \in \mathbb{N}$ such that

$$\left| \frac{n^{2/3}\sin(n!)}{n+1} - 0 \right| < \epsilon \quad \forall n \geq n_0$$

1 (iii)

To Prove :

$$\lim_{n \to \infty} \frac{n^{2/3} \sin(n!)}{n+1} = 0$$

Let $\epsilon > 0$ be given. We need to show that $\exists n_0 \in \mathbb{N}$ such that

$$\left| \frac{n^{2/3} \sin(n!)}{n+1} - 0 \right| < \epsilon \quad \forall n \geq n_0$$

Simplifying the LHS : Use the fact that $|sinx| \leq 1 \;\; \forall x$

$$\left| \frac{n^{2/3} \sin(n!)}{n+1} \right| \leq \left| \frac{n^{2/3}}{n+1} \right| < \left| \frac{n^{2/3}}{n} \right| < \epsilon$$

1 (iii)

To Prove :

$$\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$$

Let $\epsilon > 0$ be given. We need to show that $\exists n_0 \in \mathbb{N}$ such that

$$\left| \frac{n^{2/3}\sin(n!)}{n+1} - 0 \right| < \epsilon \quad \forall n \geq n_0$$

Simplifying the LHS : Use the fact that $|sinx| \leq 1 \;\; \forall x$

$$\left| \frac{n^{2/3}\sin(n!)}{n+1} \right| \leq \left| \frac{n^{2/3}}{n+1} \right| < \left| \frac{n^{2/3}}{n} \right| < \epsilon$$

$$\frac{1}{n^{1/3}} < \epsilon \iff \frac{1}{\epsilon^3} < n$$

.

1 (iii)

To Prove :

$$\lim_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$$

Let $\epsilon > 0$ be given. We need to show that $\exists n_0 \in \mathbb{N}$ such that

$$\left| \frac{n^{2/3}\sin(n!)}{n+1} - 0 \right| < \epsilon \quad \forall n \geq n_0$$

Simplifying the LHS : Use the fact that $|sinx| \leq 1 \ \ \forall x$

$$\left| \frac{n^{2/3}\sin(n!)}{n+1} \right| \leq \left| \frac{n^{2/3}}{n+1} \right| < \left| \frac{n^{2/3}}{n} \right| < \epsilon$$

$$\frac{1}{n^{1/3}} < \epsilon \iff \frac{1}{\epsilon^3} < n$$

. Thus, we can choose $n_0 = \lfloor \frac{1}{\epsilon^3} \rfloor + 1$ and the desired inequality holds.

1 (iv)

$$\lim_{n\to\infty}\left(\frac{n}{n+1}-\frac{n+1}{n}\right)=0$$

1 (iv)

$$\lim_{n\to\infty}\left(\frac{n}{n+1}-\frac{n+1}{n}\right)=0$$

Let $\epsilon > 0$ be given. We need to show that $\exists n_0 \in \mathbb{N}$ such that

$$\left|\left(\frac{n}{n+1}-\frac{n+1}{n}\right)-0\right|<\epsilon \quad \forall n \geq n_0$$

1 (iv)

$$\lim_{n\to\infty}\left(\frac{n}{n+1}-\frac{n+1}{n}\right)=0$$

Let $\epsilon > 0$ be given. We need to show that $\exists n_0 \in \mathbb{N}$ such that

$$\left|\left(\frac{n}{n+1}-\frac{n+1}{n}\right)-0\right| < \epsilon \quad \forall n \geq n_0$$

Simplifying the LHS :

$$\left|\frac{n}{n+1}-\frac{n+1}{n}\right| = \left|1-\frac{1}{n+1}-1-\frac{1}{n}\right| = \left|\frac{1}{n+1}+\frac{1}{n}\right| < \frac{2}{n} < \epsilon$$

1 (iv)

$$\lim_{n\to\infty}\left(\frac{n}{n+1}-\frac{n+1}{n}\right)=0$$

Let $\epsilon > 0$ be given. We need to show that $\exists n_0 \in \mathbb{N}$ such that

$$\left|\left(\frac{n}{n+1}-\frac{n+1}{n}\right)-0\right| < \epsilon \quad \forall n \geq n_0$$

Simplifying the LHS :

$$\left|\frac{n}{n+1}-\frac{n+1}{n}\right| = \left|1-\frac{1}{n+1}-1-\frac{1}{n}\right| = \left|\frac{1}{n+1}+\frac{1}{n}\right| < \frac{2}{n} < \epsilon$$

Choose $n_0 = \lfloor \frac{2}{\epsilon} \rfloor + 1$ and the required inequality holds.

2 (i)

Let

$$B_n = \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n} \right)$$

2 (i)

Let

$$B_n = \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n} \right)$$

Consider

$$A_n = \left(\frac{n}{n^2+n} + \frac{n}{n^2+n} + \cdots + \frac{n}{n^2+n} \right)$$

$$C_n = \left(\frac{n}{n^2+1} + \frac{n}{n^2+1} + \cdots + \frac{n}{n^2+1} \right)$$

2 (i)

Let

$$B_n = \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}\right)$$

Consider

$$A_n = \left(\frac{n}{n^2+n} + \frac{n}{n^2+n} + \cdots + \frac{n}{n^2+n}\right)$$

$$C_n = \left(\frac{n}{n^2+1} + \frac{n}{n^2+1} + \cdots + \frac{n}{n^2+1}\right)$$

We can observe that,

2 (i)

Let

$$B_n = \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n} \right)$$

Consider

$$A_n = \left(\frac{n}{n^2+n} + \frac{n}{n^2+n} + \cdots + \frac{n}{n^2+n} \right)$$

$$C_n = \left(\frac{n}{n^2+1} + \frac{n}{n^2+1} + \cdots + \frac{n}{n^2+1} \right)$$

We can observe that,

$$A_n \leq B_n \leq C_n$$

Note that $\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{n^2}{n^2+n} = 1$ and

2 (i)

Let

$$B_n = \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n} \right)$$

Consider

$$A_n = \left(\frac{n}{n^2+n} + \frac{n}{n^2+n} + \cdots + \frac{n}{n^2+n} \right)$$

$$C_n = \left(\frac{n}{n^2+1} + \frac{n}{n^2+1} + \cdots + \frac{n}{n^2+1} \right)$$

We can observe that,

$$A_n \leq B_n \leq C_n$$

Note that $\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{n^2}{n^2+n} = 1$ and $\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$

2 (i)

Let

$$B_n = \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}\right)$$

Consider

$$A_n = \left(\frac{n}{n^2+n} + \frac{n}{n^2+n} + \cdots + \frac{n}{n^2+n}\right)$$

$$C_n = \left(\frac{n}{n^2+1} + \frac{n}{n^2+1} + \cdots + \frac{n}{n^2+1}\right)$$

We can observe that,

$$A_n \leq B_n \leq C_n$$

Note that $\lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{n^2}{n^2+n} = 1$ and
$\lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$
Using Sandwich Theorem, $\lim_{n\to\infty} B_n = 1$

2 (iv)

$$\lim_{n \to \infty} n^{1/n}$$

2 (iv)

$$\lim_{n\to\infty} n^{1/n}$$

Define $h_n = n^{1/n} - 1$

2 (iv)

$$\lim_{n\to\infty} n^{1/n}$$

Define $h_n = n^{1/n} - 1$

Note that $n^{1/n} \geq 1$, hence $h_n \geq 0 \;\; \forall n \in \mathbb{N}$. For $n > 2$, we have :

2 (iv)

$$\lim_{n\to\infty} n^{1/n}$$

Define $h_n = n^{1/n} - 1$
Note that $n^{1/n} \geq 1$, hence $h_n \geq 0\ \ \forall n \in \mathbb{N}$. For $n > 2$, we have :

$$n = (1 + h_n)^n > 1 + nh_n + n_{C_2} h_n^2 > n_{C_2} h_n^2 > \frac{n(n-1)}{2} h_n^2$$

2 (iv)

$$\lim_{n\to\infty} n^{1/n}$$

Define $h_n = n^{1/n} - 1$
Note that $n^{1/n} \geq 1$, hence $h_n \geq 0 \;\; \forall n \in \mathbb{N}$. For $n > 2$, we have :

$$n = (1 + h_n)^n > 1 + nh_n + n_{C_2} h_n^2 > n_{C_2} h_n^2 > \frac{n(n-1)}{2} h_n^2$$

Thus,

$$0 < h_n < \sqrt{\frac{2}{n-1}} \quad \forall n > 2$$

2 (iv)

$$\lim_{n \to \infty} n^{1/n}$$

Define $h_n = n^{1/n} - 1$
Note that $n^{1/n} \geq 1$, hence $h_n \geq 0 \ \ \forall n \in \mathbb{N}$. For $n > 2$, we have :

$$n = (1 + h_n)^n > 1 + nh_n + n_{C_2} h_n^2 > n_{C_2} h_n^2 > \frac{n(n-1)}{2} h_n^2$$

Thus,

$$0 < h_n < \sqrt{\frac{2}{n-1}} \quad \forall n > 2$$

Hence, by using Sandwich Theorem we get that $lim_{n \to \infty} h_n = 0$

$$\implies \lim_{n \to \infty} n^{1/n} = 1$$

2 (v)

$$\lim_{n\to\infty} \frac{\cos\left(\pi\sqrt{n}\right)}{n^2}$$

2 (v)

$$\lim_{n\to\infty} \frac{\cos\left(\pi\sqrt{n}\right)}{n^2}$$

$$-1 \leq \cos\left(\pi\sqrt{n}\right) \leq 1 \;\; \forall n \in \mathbb{N}$$

2 (v)

$$\lim_{n\to\infty} \frac{\cos\left(\pi\sqrt{n}\right)}{n^2}$$

$$-1 \leq \cos\left(\pi\sqrt{n}\right) \leq 1 \;\; \forall n \in \mathbb{N}$$

Therefore,

$$\frac{-1}{n^2} \leq \frac{\cos\left(\pi\sqrt{n}\right)}{n^2} \leq \frac{1}{n^2}$$

2 (v)

$$\lim_{n \to \infty} \frac{\cos\left(\pi\sqrt{n}\right)}{n^2}$$

$$-1 \leq \cos\left(\pi\sqrt{n}\right) \leq 1 \;\; \forall n \in \mathbb{N}$$

Therefore,

$$\frac{-1}{n^2} \leq \frac{\cos\left(\pi\sqrt{n}\right)}{n^2} \leq \frac{1}{n^2}$$

Hence, using Sandwich Theorem, the limit $= 0$.

2 (vi)

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n})$$

Rationalize

$$a_n = \sqrt{n}\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

2 (vi)

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n})$$

Rationalize

$$a_n = \sqrt{n}\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n+1}} < a_n < \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n}}$$

2 (vi)

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n})$$

Rationalize

$$a_n = \sqrt{n}\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n+1}} < a_n < \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n}}$$

$$\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n+1}} = \frac{1}{2}\sqrt{\frac{n}{n+1}} = \frac{1}{2}\sqrt{1 - \frac{1}{n+1}} \geq \frac{1}{2}\left(1 - \frac{1}{\sqrt{n+1}}\right)$$

2 (vi)

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n})$$

Rationalize

$$a_n = \sqrt{n}\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n+1}} < a_n < \frac{\sqrt{n}}{\sqrt{n} + \sqrt{n}}$$

$$\frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n+1}} = \frac{1}{2}\sqrt{\frac{n}{n+1}} = \frac{1}{2}\sqrt{1 - \frac{1}{n+1}} \geq \frac{1}{2}\left(1 - \frac{1}{\sqrt{n+1}}\right)$$

We have shown that :

$$\frac{1}{2}\left(1 - \frac{1}{\sqrt{n+1}}\right) < a_n < \frac{1}{2}$$

Using Sandwich Theorem, the limit is $\frac{1}{2}$

3(i)

$$a_n = \frac{n^2}{n+1} \quad \forall n \geq 1$$

3(i)

$$a_n = \frac{n^2}{n+1} \quad \forall n \geq 1$$

$$a_n = \frac{n^2 - 1 + 1}{n+1} = \frac{(n-1)(n+1) + 1}{n+1} = n - 1 + \frac{1}{n+1}$$

3(i)

$$a_n = \frac{n^2}{n+1} \quad \forall n \geq 1$$

$$a_n = \frac{n^2 - 1 + 1}{n+1} = \frac{(n-1)(n+1) + 1}{n+1} = n - 1 + \frac{1}{n+1}$$

Clearly a_n is unbounded and hence it is not a convergent sequence.

4 (i)

$$a_n = \frac{n}{n^2 + 1}$$

4 (i)

$$a_n = \frac{n}{n^2+1}$$

Consider $a_{n+1} - a_n$

$$a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{-n-n^2+1}{((n+1)^2+1)(n^2+1)}$$

4 (i)

$$a_n = \frac{n}{n^2+1}$$

Consider $a_{n+1} - a_n$

$$a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{-n-n^2+1}{((n+1)^2+1)(n^2+1)}$$

$$a_{n+1} - a_n = \frac{-(n^2+n-1)}{((n+1)^2+1)(n^2+1)} < 0$$

4 (i)

$$a_n = \frac{n}{n^2+1}$$

Consider $a_{n+1} - a_n$

$$a_{n+1} - a_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{-n-n^2+1}{((n+1)^2+1)(n^2+1)}$$

$$a_{n+1} - a_n = \frac{-(n^2+n-1)}{((n+1)^2+1)(n^2+1)} < 0$$

Therefore a_n is a decreasing sequence.

4 (iii)

$$a_n = \frac{1-n}{n^2} \quad \forall n \geq 2$$

4 (iii)

$$a_n = \frac{1-n}{n^2} \quad \forall n \geq 2$$

Consider $a_{n+1} - a_n \quad \forall n \geq 2$

$$a_{n+1} - a_n = \frac{1-(n+1)}{(n+1)^2} - \frac{1-n}{n^2} = \frac{n-1}{n^2} - \frac{n}{(n+1)^2}$$

4 (iii)

$$a_n = \frac{1-n}{n^2} \quad \forall n \geq 2$$

Consider $a_{n+1} - a_n \quad \forall n \geq 2$

$$a_{n+1} - a_n = \frac{1-(n+1)}{(n+1)^2} - \frac{1-n}{n^2} = \frac{n-1}{n^2} - \frac{n}{(n+1)^2}$$

$$a_{n+1} - a_n = \frac{(n^2-n-1)}{n^2(n+1)^2} = \frac{(n-2)^2 + 3(n-2) + 1}{n^2(n+1)^2} > 0$$

4 (iii)

$$a_n = \frac{1-n}{n^2} \quad \forall n \geq 2$$

Consider $a_{n+1} - a_n \quad \forall n \geq 2$

$$a_{n+1} - a_n = \frac{1-(n+1)}{(n+1)^2} - \frac{1-n}{n^2} = \frac{n-1}{n^2} - \frac{n}{(n+1)^2}$$

$$a_{n+1} - a_n = \frac{(n^2-n-1)}{n^2(n+1)^2} = \frac{(n-2)^2 + 3(n-2) + 1}{n^2(n+1)^2} > 0$$

Therefore a_n is an increasing sequence.

5 (ii)

$$a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n}\ \ \forall n \geq 1$$

$$a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n}\ \ \forall n \geq 1$$

First we prove that the sequence is bounded.

5 (ii)

$$a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n}\ \ \forall n \geq 1$$

First we prove that the sequence is bounded.
Claim : $\sqrt{2} \leq a_n < 2\ \forall n \geq 1$

5 (ii)

$$a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n}\ \ \forall n \geq 1$$

First we prove that the sequence is bounded.
Claim : $\sqrt{2} \leq a_n < 2\ \forall n \geq 1$
Proof by Induction :
We can see that the claim holds for $n = 1$ as $\sqrt{2} \leq a_1 < 2$.

5 (ii)

$$a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n}\ \ \forall n \geq 1$$

First we prove that the sequence is bounded.
Claim : $\sqrt{2} \leq a_n < 2\ \forall n \geq 1$
Proof by Induction :
We can see that the claim holds for $n = 1$ as $\sqrt{2} \leq a_1 < 2$.
Let us assume that the claim holds for $n = k \implies \sqrt{2} \leq a_k < 2$.

5 (ii)

$$a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n}\ \ \forall n \geq 1$$

First we prove that the sequence is bounded.
Claim : $\sqrt{2} \leq a_n < 2\ \forall n \geq 1$
Proof by Induction :
We can see that the claim holds for $n = 1$ as $\sqrt{2} \leq a_1 < 2$.
Let us assume that the claim holds for $n = k \implies \sqrt{2} \leq a_k < 2$.
Now, $a_{k+1} = \sqrt{2 + a_k} \implies a_{k+1}^2 = 2 + a_k$.

5 (ii)

$$a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n}\ \ \forall n \geq 1$$

First we prove that the sequence is bounded.
Claim : $\sqrt{2} \leq a_n < 2\ \forall n \geq 1$
Proof by Induction :
We can see that the claim holds for $n = 1$ as $\sqrt{2} \leq a_1 < 2$.
Let us assume that the claim holds for $n = k \implies \sqrt{2} \leq a_k < 2$.
Now, $a_{k+1} = \sqrt{2 + a_k} \implies a_{k+1}^2 = 2 + a_k$.
From the induction hypothesis, $a_{k+1}^2 < 2 + 2. \implies a_{k+1} < 2.$ Also

5 (ii)

$$a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n}\ \ \forall n \geq 1$$

First we prove that the sequence is bounded.
Claim : $\sqrt{2} \leq a_n < 2\ \forall n \geq 1$
Proof by Induction :
We can see that the claim holds for $n = 1$ as $\sqrt{2} \leq a_1 < 2$.
Let us assume that the claim holds for $n = k \implies \sqrt{2} \leq a_k < 2$.
Now, $a_{k+1} = \sqrt{2 + a_k} \implies a_{k+1}^2 = 2 + a_k$.
From the induction hypothesis, $a_{k+1}^2 < 2 + 2. \implies a_{k+1} < 2$. Also
$a_{k+1}^2 \geq 2 + \sqrt{2} \implies a_{k+1} \geq \sqrt{2}$.

5 (ii)

Next we show that the sequence is monotonic.

5 (ii)

Next we show that the sequence is monotonic.
Consider $a_{n+1} - a_n$:

$$a_{n+1} - a_n = \sqrt{2 + a_n} - a_n = \frac{(\sqrt{2 + a_n} - a_n)(\sqrt{2 + a_n} + a_n)}{\sqrt{2 + a_n} + a_n}$$

5 (ii)

Next we show that the sequence is monotonic.
Consider $a_{n+1} - a_n$:

$$a_{n+1} - a_n = \sqrt{2 + a_n} - a_n = \frac{(\sqrt{2 + a_n} - a_n)(\sqrt{2 + a_n} + a_n)}{\sqrt{2 + a_n} + a_n}$$

$$\implies a_{n+1} - a_n = \frac{2 + a_n - a_n^2}{\sqrt{2 + a_n} + a_n} = \frac{(2 - a_n)(1 + a_n)}{\sqrt{2 + a_n} + a_n} > 0$$

Therefore a_n is a monotonically increasing sequence.

5 (ii)

Next we show that the sequence is monotonic.
Consider $a_{n+1} - a_n$:

$$a_{n+1} - a_n = \sqrt{2 + a_n} - a_n = \frac{(\sqrt{2 + a_n} - a_n)(\sqrt{2 + a_n} + a_n)}{\sqrt{2 + a_n} + a_n}$$

$$\implies a_{n+1} - a_n = \frac{2 + a_n - a_n^2}{\sqrt{2 + a_n} + a_n} = \frac{(2 - a_n)(1 + a_n)}{\sqrt{2 + a_n} + a_n} > 0$$

Therefore a_n is a monotonically increasing sequence.
We have shown that the sequence is convergent as it is monotonic and bounded.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$$

Take limit on both sides of the equation : $a_{n+1} = \sqrt{2 + a_n}$

$$L = \sqrt{2 + L} \implies L^2 = 2 + L \implies L = 2$$

7)

Given $\lim_{n\to\infty} a_n = L \neq 0$.

7)

Given $\lim_{n\to\infty} a_n = L \neq 0$.
Choose $\epsilon = \frac{|L|}{2}$ (which is indeed > 0).

7)

Given $\lim_{n\to\infty} a_n = L \neq 0$.
Choose $\epsilon = \frac{|L|}{2}$ (which is indeed > 0). By the $\epsilon - n_0$ definition, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon = \frac{|L|}{2}$ for all $n \geq n_0$.

7)

Given $\lim_{n\to\infty} a_n = L \neq 0$.
Choose $\epsilon = \frac{|L|}{2}$ (which is indeed > 0). By the $\epsilon - n_0$ definition, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon = \frac{|L|}{2}$ for all $n \geq n_0$.
Using the triangle inequality,

$$||a_n| - |L|| \leq |a_n - L| < \frac{|L|}{2}$$

7)

Given $\lim_{n\to\infty} a_n = L \neq 0$.
Choose $\epsilon = \frac{|L|}{2}$ (which is indeed > 0). By the $\epsilon - n_0$ definition, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon = \frac{|L|}{2}$ for all $n \geq n_0$.
Using the triangle inequality,

$$||a_n| - |L|| \leq |a_n - L| < \frac{|L|}{2}$$

Thus,

$$-\frac{|L|}{2} < |a_n| - |L| < \frac{|L|}{2}$$

7)

Given $\lim_{n\to\infty} a_n = L \neq 0$.
Choose $\epsilon = \frac{|L|}{2}$ (which is indeed > 0). By the $\epsilon - n_0$ definition, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon = \frac{|L|}{2}$ for all $n \geq n_0$.
Using the triangle inequality,

$$||a_n| - |L|| \leq |a_n - L| < \frac{|L|}{2}$$

Thus,

$$-\frac{|L|}{2} < |a_n| - |L| < \frac{|L|}{2}$$

$$\frac{|L|}{2} < |a_n| < 3\frac{|L|}{2}$$

for all $n \geq n_0$ as required.

8)

Given $a_n \geq 0$ $\lim_{n\to\infty} a_n = 0$. To prove $\lim_{n\to\infty} a_n^{1/2} = 0$

8)

Given $a_n \geq 0$ $\lim_{n\to\infty} a_n = 0$. To prove $\lim_{n\to\infty} a_n^{1/2} = 0$
Let $\epsilon > 0$ be given. $\implies \epsilon^2 > 0$

8)

Given $a_n \geq 0$ $\lim_{n\to\infty} a_n = 0$. To prove $\lim_{n\to\infty} a_n^{1/2} = 0$
Let $\epsilon > 0$ be given. $\implies \epsilon^2 > 0$
By the hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - 0| < \epsilon^2$ for all $n \geq n_0$.

8)

Given $a_n \geq 0$ $\lim_{n\to\infty} a_n = 0$. To prove $\lim_{n\to\infty} a_n^{1/2} = 0$
Let $\epsilon > 0$ be given. $\implies \epsilon^2 > 0$
By the hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - 0| < \epsilon^2$ for all $n \geq n_0$.
Hence,

$$|a_n^{1/2} - 0| = a_n^{1/2} < \epsilon \quad \forall n \geq n_0$$

8)

Given $a_n \geq 0$ $\lim_{n\to\infty} a_n = 0$. To prove $\lim_{n\to\infty} a_n^{1/2} = 0$
Let $\epsilon > 0$ be given. $\implies \epsilon^2 > 0$
By the hypothesis, there exists $n_0 \in \mathbb{N}$ such that $|a_n - 0| < \epsilon^2$ for all $n \geq n_0$.
Hence,

$$|a_n^{1/2} - 0| = a_n^{1/2} < \epsilon \quad \forall n \geq n_0$$

Thus, by the definition of limit we have proved that $\lim_{n\to\infty} a_n^{1/2} = 0$

10)

To prove: a_n is convergent if and only if ($\iff$) a_{2n} and a_{2n+1} are convergent to the same limit.

10)

To prove: a_n is convergent if and only if ($\iff$) a_{2n} and a_{2n+1} are convergent to the same limit.
Proof. ($\implies$) Given $\lim_{n\to\infty} a_n = L$.

10)

To prove: a_n is convergent if and only if ($\iff$) a_{2n} and a_{2n+1} are convergent to the same limit.
Proof. ($\implies$) Given $\lim_{n\to\infty} a_n = L$. Let $\epsilon > 0$ be given.
By the $\epsilon - n_0$ definition of the limit, $\exists n_0 \in \mathbb{N}$ such that
$|a_n - L| < \epsilon \quad \forall n \geq n_0$.

10)

To prove: a_n is convergent if and only if ($\iff$) a_{2n} and a_{2n+1} are convergent to the same limit.
Proof. ($\implies$) Given $\lim_{n\to\infty} a_n = L$. Let $\epsilon > 0$ be given.
By the $\epsilon - n_0$ definition of the limit, $\exists n_0 \in \mathbb{N}$ such that
$|a_n - L| < \epsilon \quad \forall n \geq n_0$.
$2n > n$ and $2n+1 > n$. So $|a_{2n} - L| < \epsilon$ and $|a_{2n+1} - L| < \epsilon \quad \forall n \geq n_0$

10)

To prove: a_n is convergent if and only if ($\iff$) a_{2n} and a_{2n+1} are convergent to the same limit.

Proof. ($\implies$) Given $\lim_{n\to\infty} a_n = L$. Let $\epsilon > 0$ be given.
By the $\epsilon - n_0$ definition of the limit, $\exists n_0 \in \mathbb{N}$ such that
$|a_n - L| < \epsilon \quad \forall n \geq n_0$.
$2n > n$ and $2n + 1 > n$. So $|a_{2n} - L| < \epsilon$ and $|a_{2n+1} - L| < \epsilon \quad \forall n \geq n_0$
Hence, $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$

10) Proof for ($\Longleftarrow$)

Given that $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$, we need to show that $\lim_{n\to\infty} a_n = L$.

10) Proof for ($\Longleftarrow$)

Given that $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$, we need to show that $\lim_{n\to\infty} a_n = L$.
Let $\epsilon > 0$ be given. By the $\epsilon - n_0$ definition of the limit, $\exists n_1, n_2 \in \mathbb{N}$ such that $|a_{2n} - L| < \epsilon \quad \forall n \geq n_1$ and $|a_{2n+1} - L| < \epsilon \quad \forall n \geq n_2$

10) Proof for ($\Longleftarrow$)

Given that $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$, we need to show that $\lim_{n\to\infty} a_n = L$.
Let $\epsilon > 0$ be given. By the $\epsilon - n_0$ definition of the limit, $\exists n_1, n_2 \in \mathbb{N}$ such that $|a_{2n} - L| < \epsilon \quad \forall n \geq n_1$ and $|a_{2n+1} - L| < \epsilon \quad \forall n \geq n_2$
Choose $n_0 = max(2n_1, 2n_2 + 1)$. Assume $n > n_0$ so that $n > 2n_1$ and $n > 2n_2 + 1$.

10) Proof for ($\Longleftarrow$)

Given that $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$, we need to show that $\lim_{n\to\infty} a_n = L$.

Let $\epsilon > 0$ be given. By the $\epsilon - n_0$ definition of the limit, $\exists n_1, n_2 \in \mathbb{N}$ such that $|a_{2n} - L| < \epsilon \quad \forall n \geq n_1$ and $|a_{2n+1} - L| < \epsilon \quad \forall n \geq n_2$

Choose $n_0 = max(2n_1, 2n_2 + 1)$. Assume $n > n_0$ so that $n > 2n_1$ and $n > 2n_2 + 1$.

If n is even, then $n = 2m$ for some $m \in \mathbb{N}$, and $2m > 2n_1 \implies m > n_1$, hence

$$|a_n - L| = |a_{2m} - L| < \epsilon$$

10) Proof for ($\Longleftarrow$)

Given that $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$, we need to show that $\lim_{n\to\infty} a_n = L$.

Let $\epsilon > 0$ be given. By the $\epsilon - n_0$ definition of the limit, $\exists n_1, n_2 \in \mathbb{N}$ such that $|a_{2n} - L| < \epsilon \quad \forall n \geq n_1$ and $|a_{2n+1} - L| < \epsilon \quad \forall n \geq n_2$

Choose $n_0 = max(2n_1, 2n_2 + 1)$. Assume $n > n_0$ so that $n > 2n_1$ and $n > 2n_2 + 1$.

If n is even, then $n = 2m$ for some $m \in \mathbb{N}$, and $2m > 2n_1 \implies m > n_1$, hence

$$|a_n - L| = |a_{2m} - L| < \epsilon$$

If n is odd, then $n = 2m + 1$ for some m, and $2m + 1 > 2n_2 + 1 \implies m > n_2$, hence

$$|a_n - L| = |a_{2m+1} - L| < \epsilon$$

10) Proof for ($\Longleftarrow$)

Given that $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n+1} = L$, we need to show that $\lim_{n\to\infty} a_n = L$.

Let $\epsilon > 0$ be given. By the $\epsilon - n_0$ definition of the limit, $\exists n_1, n_2 \in \mathbb{N}$ such that $|a_{2n} - L| < \epsilon \;\; \forall n \geq n_1$ and $|a_{2n+1} - L| < \epsilon \;\; \forall n \geq n_2$

Choose $n_0 = max(2n_1, 2n_2 + 1)$. Assume $n > n_0$ so that $n > 2n_1$ and $n > 2n_2 + 1$.

If n is even, then $n = 2m$ for some $m \in \mathbb{N}$, and $2m > 2n_1 \implies m > n_1$, hence

$$|a_n - L| = |a_{2m} - L| < \epsilon$$

If n is odd, then $n = 2m + 1$ for some m, and $2m + 1 > 2n_2 + 1 \implies m > n_2$, hence

$$|a_n - L| = |a_{2m+1} - L| < \epsilon$$

In either case, $|a_n - L| < \epsilon \;\; \forall n > n_0$. Hence $\lim_{n\to\infty} a_n = L$