MA109 Calculus-I

D4-T6 Tutorial 2

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2)

We are given that $\lim_{x \to \alpha} f(x)$ exists. Let it be $c(\in \mathbb{R})$. Note that it's not necessary that $c = f(\alpha)$.
Let us evaluate $\lim_{h \to 0} f(\alpha + h)$. Let (h_n) be an arbitrary sequence of real numbers such that $h_n \neq 0$ and $h_n \to 0$. We need to find $\lim_{n \to \infty} f(\alpha + h_n)$.
Consider the sequence (x_n) of real numbers defined as $x_n := \alpha + h_n$. Thus, $x_n \neq \alpha$ and $x_n \to \alpha$. By hypothesis, we must have that $\lim_{n \to \infty} f(x_n) = c$.
Thus, by definition of x_n, we must have that $\lim_{n \to \infty} f(\alpha + h_n) = c$. This gives us that $\lim_{h \to 0} f(\alpha + h) = c$.
Similar consideration will give $\lim_{h \to 0} f(\alpha - h_n) = c$ as well.
Using the limit theorems for functions, we have that:

$$\lim_{h \to 0} [f(\alpha + h) - f(\alpha - h)] = \lim_{h \to 0} f(\alpha + h) - \lim_{h \to 0} f(\alpha - h) = c - c = 0.$$

2) Continued

The converse is false.

Let us take a counterexample as follows, consider $\alpha = 0$ and

$$f(x) = \begin{cases} 1 & x \neq 0, \\ \frac{1}{|x|} & x = 0. \end{cases}$$

3 (i)

Claim : The function is continuous everywhere except at $x = 0$
Proof for this is as follows : For $x \neq 0$, f is a composition of continuous functions $\frac{1}{x}$ and $\sin x$. Therefore f is continuous for $x \neq 0$
To see that f is discontinuous at $x = 0$:
Consider the sequence (x_n) where $x_n = \dfrac{2}{(4n+1)\pi}$.
Then, $x_n \to 0$ but $f(x_n) = 1 \quad \forall n \in \mathbb{N}$ and thus, $f(x_n) \to 1 \neq f(0)$.
Thus, f is discontinuous at $x = 0$, by definition.

3 (ii)

Claim : The function is continuous everywhere.
Proof for this is as follows : For $x \neq 0$, f is a product and composition of continuous functions. Therefore f is continuous for $x \neq 0$.
To show continuity at $x = 0$: Let (x_n) be any sequence of real numbers such that $x_n \to 0$. We must show that $f(x_n) \to 0$.
Let $\epsilon > 0$ be given.
Observe that $|f(x_n) - 0| = \left| x_n \sin\left(\frac{1}{x_n}\right) \right| \leq |x_n|$. (as $|\sin x| \leq 1 \quad \forall x$)
Now, we shall use the fact $x_n \to 0$. By this hypothesis, there must exist $n_1 \in \mathbb{N}$ such that $|x_n| = |x_n - 0| < \epsilon \quad \forall n \geq n_1$.
Choosing $n_0 = n_1$, we have it that $|f(x_n) - 0| \leq |x_n| < \epsilon \quad \forall n \geq n_0$.

4)

Given $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Thus, we can let $x = y = 0$. This gives us that:
$f(0+0) = f(0) + f(0) \implies f(0) = 0$.
As f is continuous at 0, we have it that $\lim_{h \to 0} f(h) = f(0) = 0$.

Thus,

$$\lim_{h \to 0} f(c+h) = \lim_{h \to 0} [f(c) + f(h)] = f(c)$$

showing that f is continuous at $x = c$. (As $\lim_{h \to 0} f(c) = f(c)$ (constant sequence))
Optional: First verify the equality for all $k \in \mathbb{Q}$ and then use the continuity of f and density of rationals to establish it for all $k \in \mathbb{R}$.

5)

$f(x) = x^2 sin\dfrac{1}{x}$; if $x \neq 0$ and $f(0) = 0$. As earlier, differentiability of f at $x \neq 0$ follows due to product/composition rules.

$$
\begin{aligned}
f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\
&= \lim_{h \to 0} \frac{h^2 sin(\frac{1}{h})}{h} \\
&= \lim_{h \to 0} hsin(\frac{1}{h}) \\
&= 0
\end{aligned}
$$

Hence, f is differentiable at 0 also, so it is differentiable everywhere.

5) Continued

Now, for $x \neq 0$, we can compute the derivative using product/chain rule.

$$f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Consider the sequence

$$x_n := \frac{1}{2n\pi}, \qquad n \in \mathbb{N}.$$

Clearly, we have that $x_n \to 0$ and $x_n \neq 0$. Thus, we get

$$f'(x_n) = -\cos(2n\pi) = -1.$$

Thus, we see that $f'(x_n) \to -1 \neq f'(0)$.
This shows that f' is not continuous.

7)

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c-h)}{2h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)+f(c)-f(c-h)}{2h}$$

Now, it is given that f is differentiable at c. $\implies \lim\limits_{h\to 0^+} \dfrac{f(c+h)-f(c)}{h}$ exists and is equal to $f'(c)$.

Similarly, the limit $\lim\limits_{h\to 0^+} \dfrac{f(c)-f(c-h)}{h}$ exists and equals $f'(c)$. Now that we know the existence of these limits, we can split the sum above.

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)+f(c)-f(c-h)}{2h}$$

$$= \frac{1}{2}\left(\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} + \lim_{h\to 0^+} \frac{f(c)-f(c-h)}{h}\right)$$

$= \dfrac{1}{2}\left(f'(c)+f'(c)\right) = f'(c)$. Converse isn't true. (Verify by taking $f(x) = |x|$)

9 (i)

Let $f(x) := \cos x$ for $x \in (0, \pi)$. Then f is one-one and continuous.
Consider $c \in (0, \pi)$. Now $f'(c) = -\sin c \neq 0$.
Further, $f((0, \pi)) = (-1, 1)$. If $d \in (-1, 1)$ and $f(c) = \cos c = d$, then

$$(f^{-1})'(d) = \frac{1}{f'(c)} = -\frac{1}{\sin c} = -\frac{1}{\sqrt{1 - \cos^2 c}} = -\frac{1}{\sqrt{1 - d^2}}.$$

(ii) Let $f(x) := \operatorname{cosec} x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$. Then f is one-one and continuous. Consider $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}$. Now
$f'(c) = -\operatorname{cosec} c \cot c = -\operatorname{cosec}^2 c \cos c \neq 0$.
Further, $f\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}\right) = (-\infty, -1) \cup (1, \infty)$. If $|d| > 1$ and
$f(c) = \operatorname{cosec} c = d$, then

$$(f^{-1})'(d) = \frac{1}{f'(c)} = -\frac{1}{\operatorname{cosec}^2 c \cos c} = -\frac{1}{\operatorname{cosec}^2 c \sqrt{1 - \frac{1}{\operatorname{cosec}^2 c}}}$$

$$= -\frac{1}{|d|\sqrt{d^2 - 1}}.$$

10)

Given $y = f\left(\frac{2x-1}{x+1}\right)$ and $f'(x) = \sin x^2$.
Define $g(x) := \dfrac{2x-1}{x+1}$ for $x \in \mathbb{R} \setminus \{1\}$.

Given, $y = (f \circ g)(x)$. As g is differentiable in its domain and so is f, we know that $f \circ g$ is differentiable wherever defined and its derivative is given by:

$$\frac{dy}{dx} = (f \circ g)'(x) = f'(g(x))g'(x) = \sin\left((g(x))^2\right) g'(x).$$

$$g'(x) = \frac{3}{(x+1)^2}$$

$$\therefore \frac{dy}{dx} = \sin\left(\left(\frac{2x-1}{x+1}\right)^2\right) \frac{3}{(x+1)^2}$$

11)

Consider $f(x) := |x| + |1 - x|$ for $x \in \mathbb{R}$. This is continuous everywhere. It is differentiable everywhere except at $x = 0, 1$.

12)

For $c \in \mathbb{R}$, consider a sequence $\{a_n\}$ $n \geq 1$ of rational numbers and a sequence $\{b_n\}$ $n \geq 1$ of irrational numbers, both converging to c.
Then $\{f(a_n)\}$ $n \geq 1$ converges to 1 while $\{f(b_n)\}$ $n \geq 1$ converges to 0, showing that limit of f at c does not exist.

15)

$(i) \Rightarrow (ii)$

Let $\delta > 0$ be such that $(c-\delta, c+\delta) \subseteq (a,b)$. And let $\alpha = f'(c)$, then

$$\epsilon_1(h) = \frac{f(c+h) - f(c) - \alpha h}{h} \text{ if } h \neq 0$$

and $\epsilon_1(0) = 0$. Check that $\lim_{h \to 0} \epsilon_1(h) = f'(c) - \alpha = 0$. So this function satisfies all the properties we need.

$(ii) \Rightarrow (iii)$

$$\begin{aligned} \lim_{h \to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} &= \lim_{h \to 0} |\epsilon_1(h)| \\ &= 0 \end{aligned}$$

Here I used the fact $\lim_{x \to c} f(x) = 0 \Leftrightarrow \lim_{x \to c} |f(x)| = 0$(it is a consequence of $||f(x)| - L| = |f(x) - L|$ for $L = 0$)

15) Continued

$(iii) \Rightarrow (i)$

$$\begin{aligned} \lim_{h\to 0} \frac{|f(c+h)-f(c)-\alpha h|}{|h|} &= 0 \\ \Rightarrow \lim_{h\to 0} \frac{f(c+h)-f(c)-\alpha h}{h} &= 0 \\ \Rightarrow \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} &= \alpha \\ \Rightarrow f'(c) &= \alpha \end{aligned}$$

So $f(x)$ is differentiable at $x = c$. So all (i), (ii) and (iii) are equivalent.