MA109 Calculus-I

D4-T6 Tutorial 6

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If $f : D \in \mathbb{R}^n \to \mathbb{R}$ be a function. Then the main difference between a level curve and a contour line is that level curve is a subset of $\mathbb{R}^n$ but contour line is a subset of $\mathbb{R}^{n+1}$.

(i) Given any c from the options, the level curve is the line $x - y = c$ in the XY plane, that is, the set of points $\{(x,\ y) \in \mathbb{R}^2 : x - y = c\}$ in $\mathbb{R}^2$. The contour line for that c is the line in $\mathbb{R}^3$ which consists of the set of points $\{(x,\ y,\ z) \in \mathbb{R}^3 : x - y = c,\ z = c\}$. That is, it is the contour line just shifted parallel-y in the $z-$direction.

(ii) For $c < 0$ the level curve and contour lines are null set. For $c = 0$ the level curve is the singleton set $\{(0,0)\}$ and contour line is the singleton set $\{(0,0,0)\}$. For $c > 0$ the level curve is the circle with center $(0,0)$, radius $\sqrt{c}$ and lies in $\mathbb{R}^2$ and contour line is the circle with center $(0,0,c)$, radius $\sqrt{c}$, parallel to the $x - y$ plane and lies in $\mathbb{R}^3$.

(iii)Here if $c \neq 0$ then the level curve is a hyperbola in $\mathbb{R}^2$ and the contour is a hyperbola which is parallel to the $x - y$ plane in $\mathbb{R}^3$. If $c = 0$ instead of parabola it will be a pair of straight lines.

3(i)

Claim: the function is not continuous at $(0,\ 0)$.
Proof. Consider the following sequence $(x_n,\ y_n) = \left(\frac{1}{n},\ \frac{1}{n^3}\right)$. It is clear that $(x_n,\ y_n) \to (0,\ 0)$.
But $f(x_n,\ y_n) = \frac{1/n^6}{2/n^6} = \frac{1}{2}$. Thus, $f(x_n,\ y_n) \to \frac{1}{2} \neq 0$.
Thus, f is not continuous at $(0,\ 0)$.

3(ii)

Claim: the given function is continuous at $(0,\ 0)$.

Proof. Let $(x_n,\ y_n)$ be any sequence in $\mathbb{R}^2$ such that $(x_n,\ y_n) \to (0,\ 0)$.
Then, $x_n \to 0$ and $y_n \to 0$. (1)

Note that if $(x_n,\ y_n) \neq (0,\ 0)$, then $\left|\dfrac{x^2 - y^2}{x^2 + y^2}\right| \leq 1$.

Thus, $0 \leq |f(x_n,\ y_n)| \leq |x_n y_n|$. (This inequality holds even if $(x_n,\ y_n) = (0,\ 0)$.)

Note that (1) tells us that $x_n y_n \to 0$.

Using Sandwich Theorem we get that $\lim\limits_{n\to\infty} |f(x_n,\ y_n)| = 0$.

6)

(i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the function given.
Then,

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,\ 0) &= \lim_{h \to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h} \\
&= \lim_{h \to 0} \left(h \cdot 0 \cdot \frac{h^2 - 0^2}{h^2 + 0^2} \right) \frac{1}{h} \\
&= 0
\end{aligned}$$

It can be verified that $\frac{\partial f}{\partial y}(0,\ 0)$ also exists and equals 0 in a similar manner

6) continued

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the function given.
Then,

$$\begin{aligned}\frac{\partial f}{\partial x}(0,\ 0) &= \lim_{h \to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h} \\ &= \lim_{h \to 0} \left(\frac{\sin^2(h)}{h|h|} \right)\end{aligned}$$

The right hand limit is 1 and left hand limit is -1. So it doesn't exist. We get the exact same limit for $\frac{\partial f}{\partial y}(0,0)$. So it also doesn't exist.

7)

The continuity of f at $(0,\ 0)$ can be showed using the fact that $|f(x,\ y)| \leq |x^2 + y^2|$. (Use Sandwich Theorem)

It can also be easily verified that $\frac{\partial f}{\partial x}(0,\ 0) = \frac{\partial f}{\partial y}(0,\ 0) = 0$. (Write the expression like the previous questions and arrive at the conclusion.)

Now, let us evaluate $\frac{\partial f}{\partial x}(x_0,\ y_0)$ for $(x_0,\ y_0) \neq (0,\ 0)$.
It can be easily evaluated using product and chain rules to be:

$$\frac{\partial f}{\partial x}(x_0,\ y_0) = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right).$$

The function $2x\sin\left(\frac{1}{x^2+y^2}\right)$ is bounded in any disc centered at $(0,\ 0)$. (By Sandwich Theorem)

7) continued

However, $\dfrac{2x}{x^2+y^2}\cos\left(\dfrac{1}{x^2+y^2}\right)$ is not bounded in any such disc. To see this, consider any $r > 0$ and any $M \in \mathbb{R}$. One can find an $n \in \mathbb{N}$ such that $\frac{1}{\sqrt{n\pi}} < r$ and $\sqrt{n\pi} > M$. (By Archimedean)
In that case, the point $(x_0,\ y_0) = (1/\sqrt{2n\pi},\ 0)$ will lie in the disc centered at $(0,\ 0)$ with radius r and $f(x_0,\ y_0) > M$.

As the sum of a bounded function and an unbounded function is unbounded, we have proven the result.

8)

The continuity of f is immediate. It is extremely similar to what we've seen many times by now.
Let us show that the partial derivatives don't exist.
The partial derivative of f at $(0,\ 0)$ with respect to the first variable (x) is given by

$$\lim_{h \to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right),$$

which we know does not exist.
Similar considerations apply for the other partial derivative.

9(i)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the function given in the question.
For a unit vector $\mathbf{u} := (u_1, u_2)$ and $t \neq 0$,

$$\lim_{t \to 0} \frac{f(0 + tu_1, 0 + tu_2) - f(0,0)}{t} = u_1 u_2 (u_1^2 - u_2^2) t.$$

Hence, $(\mathrm{D_u} f)(0,0)$ exists and equals 0 for all $\mathbf{u}$. Thus, all directional derivatives exist.

If f is differentiable, then the total derivative *must* be $(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)) = (0,0)$. Let us now see whether this does indeed satisfy the condition for being the total derivative. For that, we must check whether

$$\lim_{(h,k) \to (0,0)} \frac{f(0 + h, 0 + k) - f(0,0) - \frac{\partial f}{\partial x}(0,0)h - \frac{\partial f}{\partial y}(0,0)k}{\sqrt{h^2 + k^2}} = 0.$$

9(i) continued

For $(h, k) \neq (0, 0)$, we have it that

$$\frac{f(0+h, 0+k) - f(0,0) - 0h - 0k}{\sqrt{h^2+k^2}} = hk\frac{(h^2-k^2)}{(h^2+k^2)^{3/2}}.$$

Also, note that

$$\left| hk\frac{(h^2-k^2)}{(h^2+k^2)^{3/2}} \right| \leq \left| h\frac{k}{\sqrt{h^2+k^2}} \right| \leq |h|.$$

Thus, the required limit indeed does exist and equals 0.
Hence, f is differentiable at $(0, 0)$ with (total) derivative equal to $(0, 0)$.

9 (ii)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the function given in the question.
For a unit vector $\mathbf{u} := (u_1, u_2)$ and $t \neq 0$,

$$\lim_{t \to 0} \frac{f(0 + tu_1, 0 + tu_2) - f(0, 0)}{t} = u_1^3.$$

Hence, $(\mathrm{D}_{\mathbf{u}} f)(0, 0)$ exists and equals u_1^3 for all $\mathbf{u}$. Thus, all directional derivatives exist.

If f is differentiable, then the total derivative *must* be $(\frac{\partial f}{\partial x}(0, 0), \frac{\partial f}{\partial y}(0, 0)) = (1, 0)$. Let us now see whether this does indeed satisfy the condition for being the total derivative. For that, we must check whether

$$\lim_{(h,k) \to (0,0)} \frac{f(0 + h, 0 + k) - f(0, 0) - \frac{\partial f}{\partial x}(0, 0)h - \frac{\partial f}{\partial y}(0, 0)k}{\sqrt{h^2 + k^2}} = 0.$$

9 (ii) continued

For $(h, k) \neq (0, 0)$, we have it that

$$\frac{f(0+h, 0+k) - f(0,0) - 1h - 0k}{\sqrt{h^2 + k^2}} = -\frac{hk^2}{(h^2+k^2)^{3/2}}.$$

It can be seen that the limit for the above expression as $(h, k) \to (0, 0)$ does not exist. Indeed, if one approaches $(0, 0)$ along the curve $h = mk$, the limit along that path turns out to be $-m/(1+m^2)^{3/2}$. Thus, taking $m = 1$ and $m = 0$ demonstrates the non-existence of limit.

9 (iii)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the function given in the question.
For a unit vector $\mathbf{u} := (u_1, u_2)$ and $t \neq 0$,

$$\lim_{t \to 0} \frac{f\left(0 + tu_1, 0 + tu_2\right) - f(0,0)}{t} = t \sin\left(\frac{1}{t^2}\right).$$

Hence, $\left(\mathrm{D}_{\mathrm{u}} f\right)(0,0)$ exists and equals 0 for all $\mathbf{u}$. (Sandwich Theorem)
Thus, all directional derivatives exist.

If f is differentiable, then the total derivative *must* be $\left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right) = (0,0)$. Let us now see whether this does indeed satisfy the condition for being the total derivative. For that, we must check whether

$$\lim_{(h,k) \to (0,0)} \frac{f\left(0 + h, 0 + k\right) - f\left(0,0\right) - \frac{\partial f}{\partial x}(0,0)h - \frac{\partial f}{\partial y}(0,0)k}{\sqrt{h^2 + k^2}} = 0.$$

9 (iii) continued

For $(h, k) \neq (0, 0)$, we have it that

$$\frac{f(0 + h, 0 + k) - f(0, 0) - 0h - 0k}{\sqrt{h^2 + k^2}} = \sqrt{h^2 + k^2} \sin\left(\frac{1}{h^2 + k^2}\right).$$

Also, note that

$$\left|\sqrt{h^2 + k^2} \sin\left(\frac{1}{h^2 + k^2}\right)\right| \leq \left|\sqrt{h^2 + k^2}\right|.$$

Thus, the required limit indeed does exist and equals 0.
Hence, f is differentiable at $(0, 0)$ with (total) derivative equal to $(0, 0)$.

10)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \text{ iff } \forall\epsilon > 0 \quad \exists\delta > 0 \text{ such that}$$

$$(x,y) \in D_f, 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon$$

Here for $L = 0$, $\delta = \epsilon$ works $\forall\epsilon > 0$ as $|f(x,y) - L| = \sqrt{x^2 + y^2}$. So it is continuous at $(0,0)$.
For a unit vector $\mathbf{u} := (u_1, u_2)$ and $t \neq 0$,

$$\lim_{t\to 0} \frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} = \begin{cases} 0 & u_2 = 0 \\ \frac{u_2}{|u_2|} & u_2 \neq 0 \end{cases}$$

Hence, $(\mathrm{D_u} f)(0,0)$ exists for all $\mathbf{u}$. Thus, all directional derivatives exist.

10) continued

If f is differentiable, then the total derivative *must* be $(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)) = (0,1)$. Let us now see whether this does indeed satisfy the condition for being the total derivative. For that, we must check whether

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k) - f(0,0) - \frac{\partial f}{\partial x}(0,0)h - \frac{\partial f}{\partial y}(0,0)k}{\sqrt{h^2+k^2}} = 0.$$

For $(h,k) \neq (0,0)$, we have it that

$$\frac{f(0+h,0+k) - f(0,0) - 0h - 1k}{\sqrt{h^2+k^2}} = \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}}.$$

It is clear that the limit of the above expression as $(h,k) \to (0,0)$ does not exist. (Consider the paths $k = mh$.) Hence, f is not differentiable at $(0,0)$.