	Seat No	
	Student Name	Chow Yin
	Student ID	Sample Solution
	Signature	

EXAM COVER SHEET

NOTE: DO NOT REMOVE this exam paper from the exam venue

EXAM DETAILS

Course Code:	**MANU2453**				
Course Description:	**Advanced Robotic Systems**				
Date of exam:	23/10/2017	**Start time of exam:**	5:30 PM	**Duration of exam:**	2hr 15min
Total number of pages (incl. this cover sheet)	5				

ALLOWABLE MATERIALS AND INSTRUCTIONS TO CANDIDATES

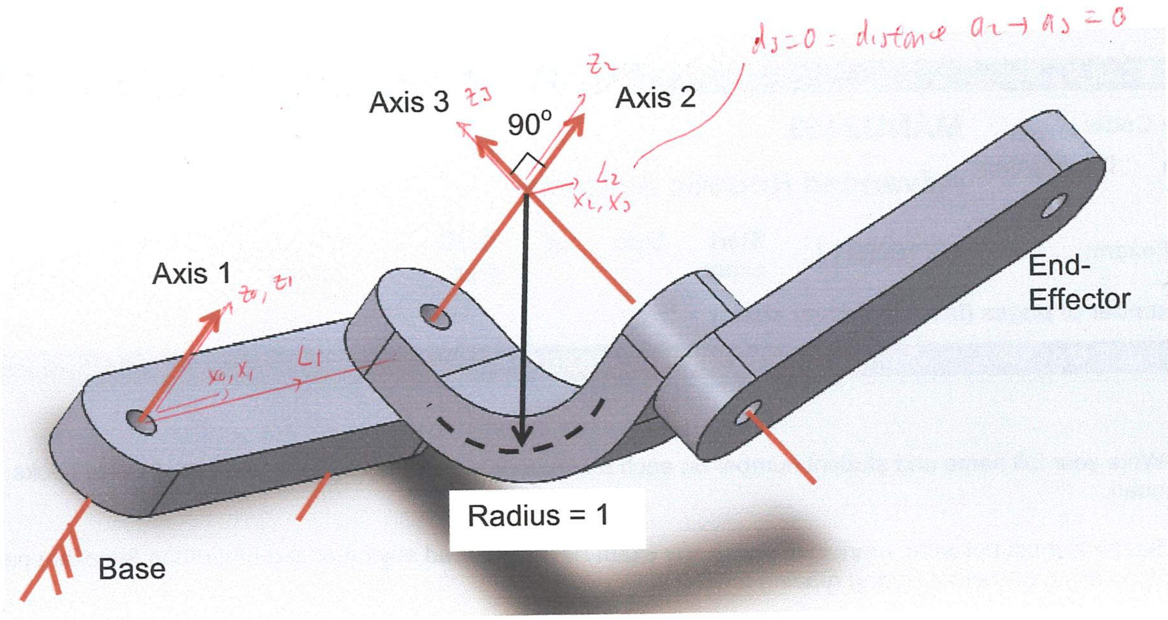
1. Write your full name and student number on each exam booklet together with the number of exam books used.
2. Students must not write, mark in any way any exam materials, read any other text other than the exam paper or do any calculations during reading time.
3. All mobile phones must be switched off and placed under your desk. You are in breach of exam conditions if it is on your person (ie. pocket).
4. This is an **OPEN BOOK** Exam.
5. Commence each question on a new page. Carry out the instructions on the front cover of the exam script book and the front of this exam paper.
6. Non text storing calculators are allowed.

Advanced Robotic Systems – MANU 2453

Final Exam (Semester 2, Year 2017)

Question 1 (11 Marks)

A robot is designed as follows:



(a) Sketch the mutual perpendiculars between the axes directly in the figure above, then label them "L1" and "L2" respectively. (1 Mark)

- Let the sense of direction for these lines be from the left of the page to the right of the page.

(b) Derive the DH-parameters of the manipulator, and tabulate them. You should provide some simple explanations (e.g. convention, definition) of how you get those parameters instead of just writing the final answer (3 Marks)

(c) Calculate the transformation matrices ${}^{0}_{1}T$, ${}^{1}_{2}T$, ${}^{2}_{3}T$, ${}^{0}_{2}T$ and ${}^{0}_{3}T$ (3 Marks).

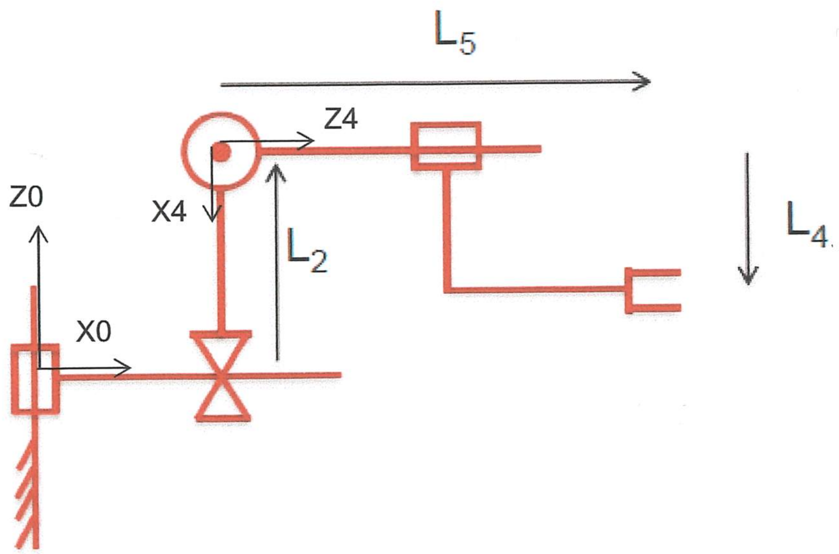
(d) Sketch the frames {0}, {1}, {2} and {3} in the figure above. (1 Mark)

(e) If the end-effector (P) is at a distance of 1m from the axis of rotation of link 3, what is the position vector ${}^{3}P$? (1 Mark)

(f) If L1 = 1, θ1 = 0°, θ2 = 0° and θ3 = 90°, calculate the position of end-effector (P) with respect to the base frame, {0}. (2 Marks)

Question 2 (10 Marks)

Shown in the figure below is a 4-link R-P-R-R robot:



The transformation matrices from frame to frame are:

$$ {}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1_2T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2_3T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & L_2 \\ 0 & 0 & 1 & 0 \\ -s\theta_3 & -c\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

and $$ {}^3_4T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_4 & c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(a) Calculate ${}^0_2T$, ${}^0_3T$ and ${}^0_4T$ (3 marks)

(b) Write down the rotationalJacobian matrix for the robot. Explain how you obtain the answer. (2 marks)

(c) Calculate the translationalJacobian matrix based on your answer in (a). (3 marks)

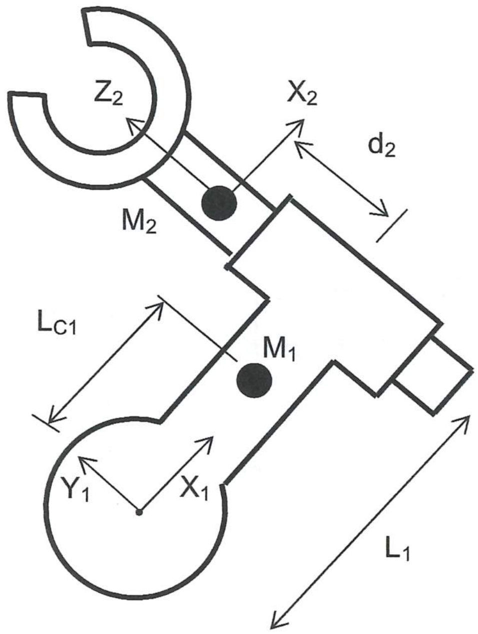
(d) Consider only the x & y portion of your answer in (c), i.e. the upper left 2x2 block of the matrix. Calculate the determinant of this matrix (0.5 marks)

(e) Based on your answer in (d), does the robot have any singularity configuration? If yes, what are the joint angles and link offset at the singularity configuration? (0.5 mark)

(f) Provide a physical interpretation of your result in (e). What happens when the robot is in the singularity configuration? In which direction is the robot not able to move instantaneously? You may use some sketches to explain your answer. (1 Marks)

Question 3 (14 Marks)

A R-P robot is shown in the figure below:



The frames {1} and {2}, and all the geometrical dimensions of the robot are sketched in the diagram. M_1 and M_2 are the masses of each link, and the big dots represent the centres of mass. Also, the inertia tensor of the two links with respect to their centres of mass are:

$$ {}^{C_1}I_1 = \begin{bmatrix} I_{xx_1} & 0 & 0 \\ 0 & I_{yy_1} & 0 \\ 0 & 0 & I_{zz_1} \end{bmatrix}, \quad {}^{C_2}I_2 = \begin{bmatrix} I_{xx_2} & 0 & 0 \\ 0 & I_{yy_2} & 0 \\ 0 & 0 & I_{zz_2} \end{bmatrix} $$

The rotation matrices for the robot are as follows:

$$ {}^{0}_{1}R = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad {}^{1}_{2}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} $$

Also,

${}^{1}P_{2}$: Position vector of frame {2} with respect to {1} = [L1, d2, 0]
${}^{1}P_{C1}$: Position vector of centre of mass of link 1 with respect to {1} = [Lc1, 0, 0]
${}^{2}P_{C2}$: Position vector of centre of mass of link 2 with respect to {2} = [0, 0, 0].

(a) Use the Newton-Euler's iterative formula to calculate the dynamics of the robot. (6 marks)

(b) Use the Lagrangian method to calculate the dynamics of the robot. (6 marks)

(c) Based on your answer in either (a) or (b), write the model of the robot in the familiar "M, V, G" structure. (1 marks)

(d) Design a nonlinear controller for the robot to achieve the desired stiffness Kp and damping Kv throughout the whole workspace. (1 marks)

Question 4 (5 Marks)

A robot joint is required to move from q1 = 30° to q2 = 80° in 5 seconds. Additional requirements include zero velocity and zero acceleration at the start and at the end of the motion.

(a) Calculate the parameters of a quintic polynomial, which would be able to achieve all the requirements. Show your work out, not just the final answer. (2 marks)

(b) If the robot is required to pass through a via point q_{via} = 100 at t = 3 seconds during the motion from q1 to q2, with the velocity at the via point as 2 degrees/second, and accelerations as 0 degrees/second^2, what are the two quintic polynomials for the portion q1→q_{via}, and q_{via}→q2? (3 marks)

1) a) see figure

b) link lengths:

a_1 = distance Axis 1 to Axis 2 = L_1

a_2 = distance Axis 2 to Axis 3 = 0

$a_0 = 0$

link twist:

$\alpha_1 = \measuredangle$ Axis 1 to Axis 2 about $a_1 = 0°$

$\alpha_2 = \measuredangle$ Axis 2 to Axis 3 about $a_2 = -90°$

$\alpha_0 = 0$

link offsets:

d_2 = distance a_1 to a_2 along axis 2 = 1

$\left.\begin{array}{l} d_1 = 0 \\ d_3 = 0 \end{array}\right\}$ convention

Joint angles:

$\theta_1, \theta_2, \theta_3$ variables

Table:

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	1	θ_2
3	-90	0	0	θ_3

(c) $${}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ c\alpha_{i-1} s\theta_i & c\alpha_{i-1} c\theta_i & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\alpha_{i-1} s\theta_i & s\alpha_{i-1} c\theta_i & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ {}^{0}_{1}T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} C_2 & -S_2 & 0 & L_1 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} C_3 & -S_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -S_3 & -C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{2}T = {}^{0}_{1}T \cdot {}^{1}_{2}T $$

$$ = \begin{bmatrix} C_1C_2 - S_1S_2 & -C_1S_2 - S_1C_2 & 0 & L_1C_1 \\ S_1C_2 + C_1S_2 & -S_1S_2 + C_1C_2 & 0 & L_1S_1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} C_{12} & -S_{12} & 0 & L_1C_1 \\ S_{12} & C_{12} & 0 & L_1S_1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{3}T = \begin{bmatrix} C_{12} & -S_{12} & 0 & L_1C_1 \\ S_{12} & C_{12} & 0 & L_1S_1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 & -S_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -S_3 & -C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{12}C_3 & -C_{12}S_3 & -S_{12} & L_1C_1 \\ S_{12}C_3 & -S_{12}S_3 & C_{12} & L_1S_1 \\ -S_3 & -C_3 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

(d) see figure.

(e) $^{3}P = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(f) $^{0}P = {}^{0}_{3}T \cdot {}^{3}P$

$$= \begin{bmatrix} C_{12}C_3 & -C_{12}S_3 & -S_{12} & L_1C_1 \\ S_{12}C_3 & -S_{12}S_3 & C_{12} & L_1S_1 \\ -S_3 & -C_3 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\left.\begin{array}{l} \theta_1 = 0 \Rightarrow C_1 = 1,\ S_1 = 0 \\ \theta_2 = 0 \Rightarrow C_2 = 1,\ S_2 = 0 \end{array}\right\} \begin{array}{l} C_{12} = 1 \\ S_{12} = 0 \end{array}$$

$\theta_3 = 90 \Rightarrow C_3 = 0,\ S_3 = 1$

$$= \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

(max 20mins)

2) a) ${}^0_2T = {}^0_1T \cdot {}^1_2T$

$$= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & c_1 & -s_1 & -d_2 s_1 \\ 0 & s_1 & c_1 & d_2 c_1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_3T = \begin{bmatrix} 0 & c_1 & -s_1 & -d_2 s_1 \\ 0 & s_1 & c_1 & d_2 c_1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & L_2 \\ 0 & 0 & 1 & 0 \\ -s_3 & -c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_1 s_3 & s_1 c_3 & c_1 & -d_2 s_1 \\ -c_1 s_3 & -c_1 c_3 & s_1 & d_2 c_1 \\ c_3 & -s_3 & 0 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_4T = \begin{bmatrix} s_1 s_3 & s_1 c_3 & c_1 & -d_2 s_1 \\ -c_1 s_3 & -c_1 c_3 & s_1 & d_2 c_1 \\ c_3 & -s_3 & 0 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_1 s_3 c_4 + c_1 s_4 & -s_1 s_3 s_4 + c_1 c_4 & -s_1 c_3 & -d_2 s_1 \\ -c_1 s_3 c_4 + s_1 s_4 & c_1 s_3 s_4 + s_1 c_4 & c_1 c_3 & d_2 c_1 \\ c_3 c_4 & -c_3 s_4 & s_3 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) $J_w = [\,3^{rd}$ col of ${}^0_1T \;\vdots\; 3^{rd}$ col of ${}^0_2T \;\vdots\; 3^{rd}$ col of ${}^0_3T \;\vdots\; 3^{rd}$ col of ${}^0_4T\,]$

0 for prismatic

$$= \begin{bmatrix} 0 & 0 & c_1 & -s_1 c_3 \\ 0 & 0 & s_1 & c_1 c_3 \\ 1 & 0 & 0 & s_3 \end{bmatrix}$$

(c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -d_2 s_1 \\ d_2 c_1 \\ L_2 \end{bmatrix}$

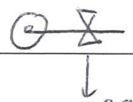
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \frac{dx}{dq_1} & \frac{dx}{dq_2} & \frac{dx}{dq_3} & \frac{dx}{dq_4} \\ \frac{dy}{dq_1} & \frac{dy}{dq_2} & \frac{dy}{dq_3} & \frac{dy}{dq_4} \\ \frac{dz}{dq_1} & \frac{dz}{dq_2} & \frac{dz}{dq_3} & \frac{dz}{dq_4} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -d_2 c_1 & -s_1 & 0 & 0 \\ -d_2 s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{J_v} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}$$

(d) $\begin{vmatrix} -d_2 c_1 & -s_1 \\ -d_2 s_1 & c_1 \end{vmatrix} = -d_2 c_1^2 - d_2 s_1^2 = \underline{-d_2}$

(e) Yes. When $\underline{d_2 = 0}$

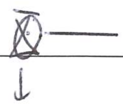
(f) Top view

$d_2 \neq 0$



can move instantaneously

$d = 0$



cannot move instantaneously.

(Max 10 mins)

Write on both sides of the paper

a) → same as 1417

3) b) ${}^0w_0 = 0, \quad {}^0V_0 = 0$

$${}^1w_1 = {}^1_0R \cdot {}^0w_0 + \dot\theta_1 \, {}^1\hat{Z}_1$$

$$= \begin{bmatrix} C_1 & S_1 & 0 \\ -S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix}$$

$${}^1V_1 = {}^1_0R(\underbrace{{}^0V_0}_{0} + \underbrace{{}^0w_0}_{0} \times {}^0P_1) = 0$$

$${}^2w_2 = {}^2_1R \cdot {}^1w_1 + \cancel{\dot\theta_2}^{\,0} \, {}^2\hat{Z}_2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\dot\theta_1 \\ 0 \end{bmatrix}$$

$${}^2V_2 = {}^2_1R({}^1V_1 + {}^1w_1 \times {}^1P_2) + \dot d_2 \, {}^2\hat{Z}_2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{vmatrix} i & j & k \\ 0 & 0 & \dot\theta_1 \\ L_1 & d_2 & 0 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot d_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -d_2\dot\theta_1 \\ L_1\dot\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot d_2 \end{bmatrix}$$

$$= \begin{bmatrix} -d_2\dot\theta_1 \\ 0 \\ L_1\dot\theta_1 + \dot d_2 \end{bmatrix}$$

$$V_{c2} = V_2 = \begin{bmatrix} -d_2\dot{\theta}_1 \\ 0 \\ L_1\dot{\theta}_1 + \dot{d}_2 \end{bmatrix}$$

$$V_{c1} = \underbrace{{}^{c_1}_{1}R}_{I}\,(\underbrace{{}^1V_1}_{0} + {}^1\omega_1 \times {}^1P_{c1})$$

$$= \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ L_{c1} & 0 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ L_{c1}\dot{\theta}_1 \\ 0 \end{bmatrix}$$

$$K = \tfrac{1}{2}m_1 V_{c1}^T V_{c1} + \tfrac{1}{2}\,{}^1\omega_1^T\,{}^{c_1}I_1\,{}^1\omega_1 + \tfrac{1}{2}m_2 V_{c2}^T V_{c2} + \tfrac{1}{2}\,{}^2\omega_2^T\,{}^{c_2}I_2\,{}^2\omega_2$$

$$= \tfrac{1}{2}m_1 L_{c1}^2\dot{\theta}_1^2 + \tfrac{1}{2}I_{zz1}\dot{\theta}_1 + \tfrac{1}{2}m_2(d_2^2\dot{\theta}_1^2 + L_1^2\dot{\theta}_1^2 + 2L_1\dot{\theta}_1\dot{d}_2 + \dot{d}_2^2)$$

$$+ \tfrac{1}{2}I_{yy2}\dot{\theta}_1^2$$

$$U = m_1 g L_{c1}\cos\theta_1 + m_2 g\,(L_1\cos\theta_1 + d_2\sin\theta_1)$$

$$\frac{dk}{d\dot{q}_i} = \begin{bmatrix} \frac{dk}{d\dot{q}_1} \\ \frac{dk}{d\dot{q}_2} \end{bmatrix} = \begin{bmatrix} m_1 L_{c1}^2\dot{\theta}_1 + I_{zz1}\dot{\theta}_1 + m_2 d_2^2\dot{\theta}_1 + m_2 L_1^2\dot{\theta}_1 \\ + m_2 L_1\dot{d}_2 + I_{yy2}\dot{\theta}_1 \\ \hline m_2 L_1\dot{\theta}_1 + m_2\dot{d}_2 \end{bmatrix}$$

$$\frac{d}{dt}\left(\frac{dk}{d\dot{q}}\right) = \begin{bmatrix} m_1 L_{c1}^2\ddot{\theta}_1 + I_{zz1}\ddot{\theta}_1 + m_2 d_2^2\ddot{\theta}_1 + m_2 L_1^2\ddot{\theta}_1 \\ + m_2 L_1\ddot{d}_2 + I_{yy2}\ddot{\theta}_1 \\ \hline m_2 L_1\ddot{\theta}_1 + m_2\ddot{d}_2 \end{bmatrix}$$

$$\frac{dk}{dq} = \begin{bmatrix} \frac{dk}{dq_1} \\ \frac{dk}{dq_2} \end{bmatrix} = \begin{bmatrix} 0 \\ m_2 d_2\dot{\theta}_1^2 \end{bmatrix} \qquad \frac{du}{dq} = \begin{bmatrix} -m_1 g L_{c1} s_1 \\ -m_2 g(L_1 s_1 + d_2 c_1) \\ m_2 g s_1 \end{bmatrix}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dk}{d\dot{q}}\right) - \frac{dk}{dq} + \frac{du}{dq} = \tau$$

c) $$\begin{bmatrix} m_1 L_{c1}^2 + I_{zz1} + m_2 d_2^2 + m_2 L_1^2 + I_{yy2} & m_2 L_1 \\ m_2 L_1 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$+ \begin{bmatrix} -m_1 g L_{c1} s_1 - m_2 g (L_1 s_1 + d_2 c_1) \\ m_2 g s_1 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

d) Robot: $M\ddot{q} + V + G = \tau$

Design $\tau = M\alpha + V + G$

Then $\ddot{q} = \alpha$

Design $\alpha = \ddot{q}_r + k_v(\dot{q}_r - \dot{q}) + k_p(q_r - q)$

Then $\underline{\ddot{e} + k_v \dot{e} + k_p = 0}$

Set any preferred k_p

Finally, set $\underline{\underline{k_v = 2\sqrt{k_p}}}$

(55 mins)

4) a) Quintic polynomial :

$u(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$

$a_0 = u_0 = 30$

$a_1 = 0$

$a_2 = 0$

$a_3 = \frac{20(80) - 20(30)}{2(5^3)} = 4$

$a_4 = \frac{30(30) - 30(80)}{2(5^4)} = -1.2$

$a_5 = \frac{12(80) - 12(30)}{2(5^5)} = 0.096.$

Then: $u(t) = \underline{\underline{30 + 4t^3 - 1.2t^4 + 0.096t^5}}$

~~(max 3 mins)~~

b) 1^{st} potion :

$u_0 = 30$

$u_f = u_{via} = 100$

$\dot{u}_0 = 0$

$\dot{u}_{via} = 2$

$\ddot{u}_0 = 0$

$\ddot{u}_{via} = 0$ $\qquad t_f = 3$

Then $a_0 = u_0 = 30$

$a_1 = \dot{u}_0 = 0$ (3 mins)

$a_2 = \ddot{u}_0/2 = 0$

$a_3 = \frac{20(100) - 20(30) - (8\times2 + 12\times0)3 - (3\times0 - 0)3^2}{2(3)^3} =$

$= \frac{1400 - 48}{54} = 25.04$

$$a_4 = \frac{30(70) - 30(100) + 14(2)3 + 0}{2 \cdot 3^4} = -12.44$$

$$a_5 = \frac{12(100) - 12(30) \cancel{+} - 6(2)3 - 0}{2 \cdot 3^5} = \cancel{1.80} \; 1.654$$

$\therefore$ First portion: $\underline{u(t) = 30 + 25.04t^3 - 12.44t^4 + 1.8t^5}$

second portion:

$u_0 = 100$

$u_f = 80$

$\dot{u}_0 = 2$

$\dot{u}_f = 0$

$\ddot{u}_0 = 0$

$\ddot{u}_f = 0$ $\qquad t_f = 2$

$\therefore a_0 = u_0 = 100$

$a_1 = \dot{u}_0 = 2$

$a_2 = \ddot{u}_0/2 = 0$

$$a_3 = \frac{20(80) - 20(100) - (12 \cdot 2)2}{2 \cdot 2^3} = -28$$

$$a_4 = \frac{30(100) - 30(80) + 16 \cdot 2 \cdot 2}{2 \cdot 2^4} = 20.75$$

$$a_5 = \frac{12(80) - 12(100) - (6 \cdot 2)2}{2 \cdot 2^5} = -4.125$$

$\therefore \underline{u(t) = 100 + 2t - 28t^3 + 20.75t^4 - 4.125t^5}$

(max 15 mins)

6) (a) $^1P_2 = [L_1, d_2, 0]$

$^1P_{c1} = [L_{c1}, 0, 0]$

$^2P_{c2} = [0, 0, 0]$

(b) $^0\omega_0 = 0$, $^0\dot{\omega}_0 = 0$, $^0\dot{v}_0 = [0 \quad g \quad 0]$

$$^1\omega_1 = (^1_0R \cdot \underbrace{^0\omega_0}_{=0}) + (\dot{\theta}_1 \, ^1Z_1) = \underline{\underline{\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}}}$$

$$^1\dot{\omega}_1 = (^1_0R \cdot \underbrace{^0\dot{\omega}_0}_{=0}) + (^1_0R \cdot \underbrace{^0\omega_0}_{=0} \times \dot{\theta}_1 \, ^1Z_1) + (\ddot{\theta}_1 \, ^1Z_1) = \underline{\underline{\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}}}$$

$$^1\dot{v}_1 = (^1_0R \cdot {}^0\dot{v}_0) + (2\,^1\omega_1 \times \underbrace{\dot{d}_1}_{=0} \, ^1Z_1) + (\underbrace{\ddot{d}_1}_{=0} \, ^1Z_1)$$

$$+ \, ^1_0R(\underbrace{^0\dot{\omega}_0}_{=0} \times \underbrace{^0P_1}_{=0} + {}^0\omega_0 \times \underbrace{^0\omega_0}_{=0} \times \underbrace{^0P_1}_{=0})$$

$$= \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} s_1 g \\ c_1 g \\ 0 \end{bmatrix}}}$$

$$^1\dot{v}_{c1} = (^1\dot{v}_1) + (^1\dot{\omega}_1 \times {}^1P_{c1}) + (^1\omega_1 \times (^1\omega_1 \times {}^1P_{c1}))$$

$$= \begin{bmatrix} s_1 g \\ c_1 g \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 \\ L_{c1} & 0 & 0 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ L_{c1} & 0 & 0 \end{vmatrix}$$

$$= \begin{bmatrix} s_1 g \\ c_1 g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ L_{c1}\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ 0 & L_{c1}\dot{\theta}_1 & 0 \end{vmatrix} = \underline{\underline{\begin{bmatrix} s_1 g - L_{c1}\dot{\theta}_1^2 \\ c_1 g + L_{c1}\ddot{\theta}_1 \\ 0 \end{bmatrix}}}$$

$$ {}^1F_1 = m_1\, {}^1\dot{V}_{c1} = \begin{bmatrix} m_1 s_1 g - m_1 L_{c1} \dot{\theta}_1^2 \\ m_1 c_1 g + m_1 L_{c1} \ddot{\theta}_1 \\ 0 \end{bmatrix} $$

$$ {}^1N_1 = {}^{c_1}I_1\, {}^1\dot{\omega}_1 + {}^1\omega_1 \times {}^{c_1}I_1\, {}^1\omega_1 $$

$$ = \begin{bmatrix} I_{xx1} & & \\ & I_{yy1} & \\ & & I_{zz1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} I_{xx1} & & \\ & I_{yy1} & \\ & & I_{zz1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} $$

$$ = \begin{bmatrix} 0 \\ 0 \\ I_{zz1} \ddot{\theta}_1 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ 0 & 0 & I_{zz1}\dot{\theta}_1 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_{zz1} \ddot{\theta}_1 \end{bmatrix} $$

Link 2:

$$ {}^2\omega_2 = ({}^2_1R \cdot {}^1\omega_1) + \underbrace{\dot{\theta}_2}_{=0}\, {}^2Z_2 $$

$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \underbrace{\dot{\theta}_2}_{=0} \end{bmatrix} = \begin{bmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{bmatrix} $$

$$ {}^2\dot{\omega}_2 = ({}^2_1R \cdot {}^1\dot{\omega}_1) + ({}^2_1R \cdot {}^1\omega_1 \times \overbrace{\dot{\theta}_2}^{=0}\, {}^2Z_2) + (\overbrace{\ddot{\theta}_2}^{=0}\, {}^2Z_2) $$

$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} + \cancel{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ -\ddot{\theta}_1 \\ 0 \end{bmatrix} $$

$$ {}^2\dot{V}_2 = ({}^2_1R \cdot {}^1\dot{V}_1) + (2\, {}^2\omega_2 \times \dot{d}_2\, {}^2Z_2) + (\ddot{d}_2\, {}^2Z_2) $$
$$ + ({}^2_1R\, ({}^1\dot{\omega}_1 \times {}^1P_2 + {}^1\omega_1 \times {}^1\omega_1 \times {}^1P_2) $$

$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s_1 g \\ c_1 g \\ 0 \end{bmatrix} + 2 \begin{vmatrix} i & j & k \\ 0 & -\dot{\theta}_1 & 0 \\ 0 & 0 & \dot{d}_2 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{d}_2 \end{bmatrix} $$

$$ + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \left\{ \begin{vmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 \\ L_1 & d_2 & 0 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ L_1 & d_2 & 0 \end{vmatrix} \right\} $$

$$= \begin{bmatrix} S_1 g \\ 0 \\ C_1 g \end{bmatrix} + 2 \begin{bmatrix} -\dot{\theta}_1 \dot{d}_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{d}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} -d_2 \ddot{\theta}_1 \\ L_1 \ddot{\theta}_1 \\ 0 \end{bmatrix} + \cancel{\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}} \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ -d_2 \dot{\theta}_1 & L_1 \dot{\theta}_1 & 0 \end{vmatrix} \right\}$$

$$= \begin{bmatrix} S_1 g - 2\dot{\theta}_1 \dot{d}_2 \\ 0 \\ C_1 g + \ddot{d}_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -d_2 \ddot{\theta}_1 - L_1 \dot{\theta}_1^2 \\ L_1 \ddot{\theta}_1 - d_2 \dot{\theta}_1^2 \\ 0 \quad 0 \end{bmatrix}$$

$$^2\dot{V}_2 = \begin{bmatrix} S_1 g - 2\dot{\theta}_1 \dot{d}_2 - d_2 \ddot{\theta}_1 - L_1 \dot{\theta}_1^2 \\ 0 \\ C_1 g + \ddot{d}_2 + L_1 \ddot{\theta}_1 - d_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$^2\dot{V}_{C_2} = (^2\dot{V}_2) + (\underbrace{^2\dot{W}_2 \times {}^2P_{C_2}}_{=0}) + (\underbrace{^2W_2 \times {}^2W_2 \times {}^2P_{C_2}}_{=0})$$

$$= \begin{bmatrix} S_1 g - 2\dot{\theta}_1 \dot{d}_2 - d_2 \ddot{\theta}_1 - L_1 \dot{\theta}_1^2 \\ 0 \\ C_1 g + \ddot{d}_2 + L_1 \ddot{\theta}_1 - d_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$^2F_2 = m_2 \, {}^2\dot{V}_{C_2} = \begin{bmatrix} m_2 S_1 g - 2 m_2 \dot{\theta}_1 \dot{d}_2 - m_2 d_2 \ddot{\theta}_1 - m_2 L_1 \dot{\theta}_1^2 \\ 0 \\ m_2 C_1 g + m_2 \ddot{d}_2 + m_2 L_1 \ddot{\theta}_1 - m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}$$

$${}^2N_2 = \left({}^{C_2}I_2\, {}^2\dot{\omega}_2\right) + \left({}^2\omega_2 \times {}^{C_2}I_2 \cdot {}^2\omega_2\right)$$

$$= \begin{bmatrix} I_{xx_2} & & \\ & I_{yy_2} & \\ & & I_{zz_2} \end{bmatrix} \begin{bmatrix} 0 \\ -\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & -\dot{\theta}_1 & 0 \\ 0 & -I_{yy_2}\dot{\theta}_1 & 0 \end{vmatrix} = \underline{\underline{\begin{bmatrix} 0 \\ -I_{yy_2}\ddot{\theta}_1 \\ 0 \end{bmatrix}}}$$

<u>inwards:</u>

$${}^3f_3 = 0 \quad , \quad {}^3n_3 = 0$$

$${}^2f_2 = \left({}^2_3R \cdot \underbrace{{}^3f_3}_{=0}\right) + {}^2F_2 = \begin{bmatrix} m_2 s_1 g - 2m_2\dot{\theta}_1\dot{d}_2 - m_2 d_2\ddot{\theta}_1 - m_2 L_1\dot{\theta}_1^2 \\ 0 \\ m_2 c_1 g + m_2\ddot{d}_2 + m_2 L_1\ddot{\theta}_1 - m_2 d_2\dot{\theta}_1^2 \end{bmatrix}$$

$${}^2n_2 = \left({}^2_3R \cdot \underbrace{{}^3n_3}_{=0}\right) + \left(\underbrace{{}^2P_{C_2}}_{=0} \times {}^2F_2\right) + \left({}^2P_3 \times {}^2_3R \cdot \underbrace{{}^3f_3}_{=0}\right) + \left({}^2N_2\right) = \begin{bmatrix} 0 \\ -I_{yy_2}\ddot{\theta}_1 \\ 0 \end{bmatrix}$$

${}^1f_1$ = not important (revolute joint)

$${}^1n_1 = \left({}^1_2R \cdot {}^2n_2\right) + \left({}^1P_{C_1} \times {}^1F_1\right) + \left({}^1P_2 \times {}^1_2R \cdot {}^2f_2\right) + \left({}^1N_1\right)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -I_{yy_2}\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ L_{C_1} & 0 & 0 \\ {}^1F_{1x} & {}^1F_{1y} & {}^1F_{1z} \end{vmatrix} + \begin{bmatrix} L_1 \\ d_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} {}^2f_{2x} \\ {}^2f_{2y} \\ {}^2f_{2z} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ I_{zz_1}\ddot{\theta}_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ I_{yy_2}\ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ -L_{C_1}\,{}^1F_{1z} \\ L_{C_1}\,{}^1F_{1y} \end{bmatrix} + \begin{vmatrix} i & j & k \\ L_1 & d_2 & 0 \\ {}^2f_{2x} & {}^2f_{2z} & -{}^2f_{2y} \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ I_{zz_1}\ddot{\theta}_1 \end{bmatrix}$$

$$= \begin{bmatrix} * \\ * \\ I_{yy_2}\ddot{\theta}_1 + L_{C_1}\,{}^1F_{1y} + L_1\,{}^2f_{2z} - d_2\,{}^2f_{2x} + I_{zz_1}\ddot{\theta}_1 \end{bmatrix}$$

$$
{}^{1}n_1 = \begin{bmatrix} * \\ * \\ I_{yy2}\ddot{\theta}_1 + L_{c1}(m_1 c_1 g + m_1 L_{c1}\ddot{\theta}_1) \\ + L_1(m_2 c_1 g + m_2 \ddot{d}_2 + m_2 L_1 \ddot{\theta}_1 - m_2 d_2 \dot{\theta}_1^2) \\ - d_2(m_2 s_1 g - 2m_2 \dot{\theta}_1 \dot{d}_2 - m_2 d_2 \ddot{\theta}_1 - m_2 L_1 \dot{\theta}_1^2) \\ + I_{zz1}\ddot{\theta}_1 \end{bmatrix}
$$

$\therefore$ $\boxed{\begin{array}{l} \tau_1 = 3^{rd} \text{ row of } {}^{1}n_1 \\ \tau_2 = 3^{rd} \text{ row of } {}^{2}f_2 \end{array}}$

(max 30 mins)

$\Rightarrow$ I solved the whole paper in 1.5 hours