Derivation of Newton Euler Algorithm

Advanced Robotic Systems – MANU2453

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Content

- Derivation of the Newton-Euler Formulation

Recap: Velocities

- Last week, we learnt about the notation of velocities:
  - Derivative of a position vector, with respect to reference frame {B}: $^{B}V_Q = \frac{d}{dt}\left({}^{B}Q\right)$
  - Linear velocity (differentiation done in {B}), but expressed in frame {A}: $^{A}\left({}^{B}V_Q\right) = {}^{A}_{B}R \cdot {}^{B}V_Q$
  - Linear velocity of the origin of a frame {C}, relative to an understood universal frame {U}. $\upsilon_C = {}^{U}V_{CORG}$ $\quad {}^{A}\upsilon_C = {}^{A}\left({}^{U}V_{CORG}\right)$

  - Rotation of frame {B} relative to {A}: $^{A}\Omega_B$
  - Angular velocity of frame {B} relative to {A}, but expressed in {C}: $^{C}\left({}^{A}\Omega_B\right)$
  - Angular velocity of a frame {C}, relative to an understood universal frame {U}. $\omega_C = {}^{U}\Omega_C$ $\quad {}^{A}\omega_C = {}^{A}\left({}^{U}\Omega_C\right)$

Acceleration of Rigid Body

- Now we can write the definition of acceleration:

$$^{B}\dot{V}_{Q} = \frac{d}{dt}\left(^{B}V_{Q}\right)$$

$$^{A}\dot{\Omega}_{B} = \frac{d}{dt}\left(^{A}\Omega_{B}\right)$$

  - That means, the accelerations are the derivatives of the velocities.

- Similar to the velocities, if the reference frame of differentiation is understood as the universal reference frame {U}, then we write: $\dot{\upsilon}_C = {}^{U}\dot{V}_{CORG}$ $\dot{\omega}_C = {}^{U}\dot{\Omega}_C$
  - e.g. $^{A}\dot{\upsilon}_C = {}^{A}\left(^{U}\dot{V}_{CORG}\right)$
    - Acceleration of origin of frame {C},
    - where differentiation is done in frame {U}.
    - The whole $^{U}\dot{V}_{CORG}$ is then expressed in {A}.

Linear Acceleration

- The velocity of a point can be obtained by differentiating its position

$$ {}^{A}Q = {}^{A}P_{BORG} + {}^{A}_{B}R \cdot {}^{B}Q $$

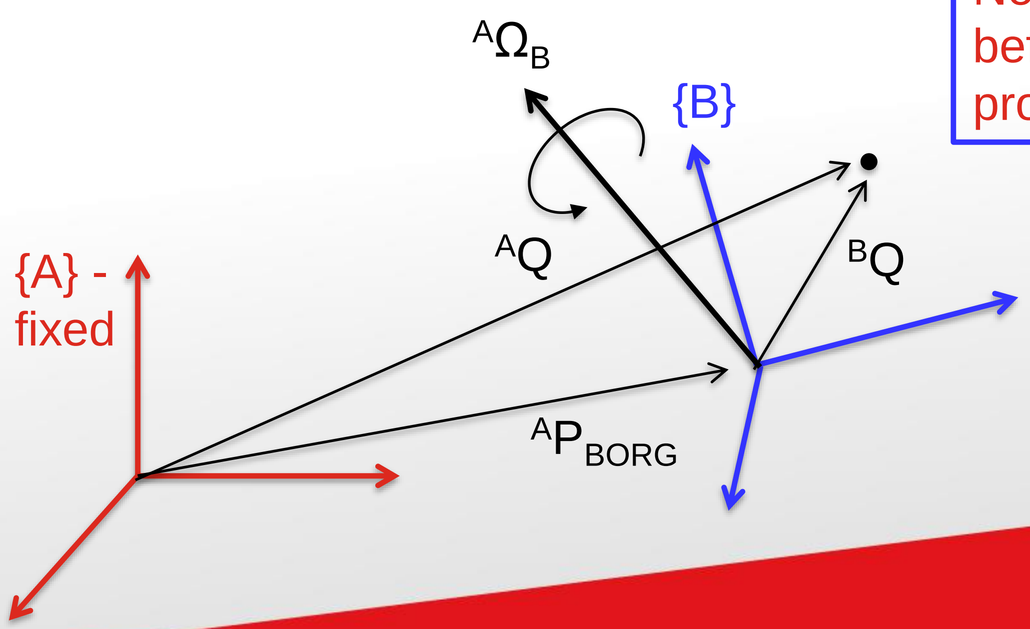
  - which gives:

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R \cdot {}^{B}V_{Q} + {}^{A}_{B}\dot{R} \cdot {}^{B}Q $$

  - It can be shown that the last term is: $ {}^{A}\Omega_{B} \times {}^{A}_{B}R \cdot {}^{B}Q $

Note: the “dot” is not a dot product. Just standard matrix-vector multiplication

Note: Multiplication before cross product



Linear Acceleration

- We now differentiate $^{A}V_Q = ^{A}V_{BORG} + ^{A}_{B}R \cdot ^{B}V_Q + ^{A}\Omega_B \times ^{A}_{B}R \cdot ^{B}Q$ once more.
- This leads to (using chain rule for red and green underlined terms):

$$^{A}\dot{V}_Q = ^{A}\dot{V}_{BORG} + ^{A}_{B}\dot{R} \cdot ^{B}V_Q + ^{A}_{B}R \cdot ^{B}\dot{V}_Q + ^{A}\dot{\Omega}_B \times ^{A}_{B}R \cdot ^{B}Q + ^{A}\Omega_B \times ^{A}_{B}\dot{R} \cdot ^{B}Q + ^{A}\Omega_B \times ^{A}_{B}R \cdot ^{B}V_Q$$

- The orange term is: $^{A}_{B}\dot{R} \cdot ^{B}Q = ^{A}\Omega_B \times ^{A}_{B}R \cdot ^{B}Q$ (from previous page)
- Analogously, the blue term would be:

$$^{A}_{B}\dot{R} \cdot ^{B}V_Q = ^{A}\Omega_B \times ^{A}_{B}R \cdot ^{B}V_Q$$

- Therefore:

$$^{A}\dot{V}_Q = ^{A}\dot{V}_{BORG} + ^{A}\Omega_B \times ^{A}_{B}R \cdot ^{B}V_Q + ^{A}_{B}R \cdot ^{B}\dot{V}_Q + ^{A}\dot{\Omega}_B \times ^{A}_{B}R \cdot ^{B}Q + ^{A}\Omega_B \times ^{A}\Omega_B \times ^{A}_{B}R \cdot ^{B}Q + ^{A}\Omega_B \times ^{A}_{B}R \cdot ^{B}V_Q$$

- Combining similar terms, we get:

Note: 2nd cross product before the 1st

$$^{A}\dot{V}_Q = ^{A}\dot{V}_{BORG} + 2\,^{A}\Omega_B \times ^{A}_{B}R \cdot ^{B}V_Q + ^{A}_{B}R \cdot ^{B}\dot{V}_Q + ^{A}\dot{\Omega}_B \times ^{A}_{B}R \cdot ^{B}Q + ^{A}\Omega_B \times ^{A}\Omega_B \times ^{A}_{B}R \cdot ^{B}Q$$

- Note: For revolute joint:

$$^{B}V_Q = ^{B}\dot{V}_Q = 0$$

Angular Acceleration

- If {C} rotates relative to {B} with ${}^{B}\Omega_{C}$, and {B} rotates relative to {A} with ${}^{A}\Omega_{B}$, then {C} rotates relative to {A} with:

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R \cdot {}^{B}\Omega_{C} $$

- Differentiating this gives: (using chain rule)

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + {}^{A}_{B}\dot{R} \cdot {}^{B}\Omega_{C} + {}^{A}_{B}R \cdot {}^{B}\dot{\Omega}_{C} $$

- Again, the second term on the RHS can be written using cross product formulation.

$$ {}^{A}_{B}\dot{R} \cdot {}^{B}\Omega_{c} = {}^{A}\Omega_{B} \times {}^{A}_{B}R \cdot {}^{B}\Omega_{C} $$

- Therefore: ➡

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + {}^{A}\Omega_{B} \times {}^{A}_{B}R \cdot {}^{B}\Omega_{C} + {}^{A}_{B}R \cdot {}^{B}\dot{\Omega}_{C} $$

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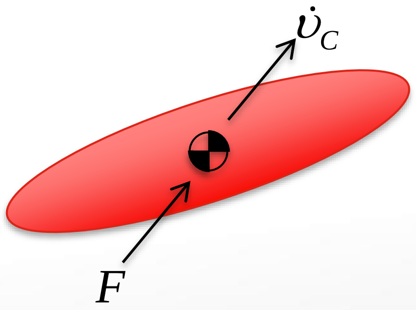
Newton's and Euler's Equations

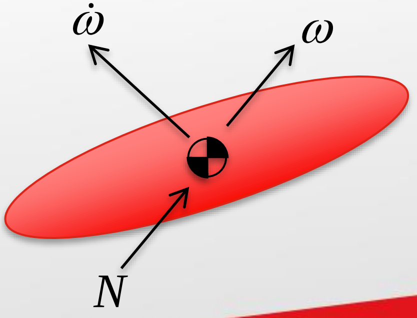
- Consider each link as a rigid body.
- Its mass distribution is characterized by the location of center of mass, and the inertia tensor of the link, both assumed known by now.
- Newtons' law and Euler's equation describe how the links move (accelerate) when given external forces.
  - Newtons' Law: F acting on the center of mass, causes a linear acceleration v-dot.

$$F = m\dot{v}_C$$



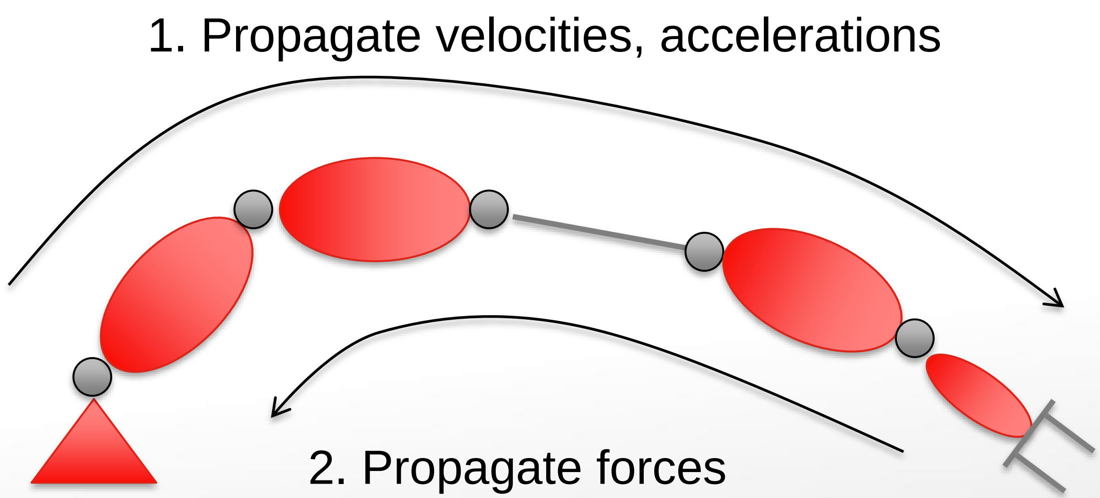
  - Euler's equation: Moment N causes an angular acceleration of the rigid body:

$$N = {}^{C}I\dot{\omega} + \omega \times {}^{C}I\omega$$



Iterative Newton-Euler Formulation

- Now, to calculate the joint torques (for all joints) which are required to move the manipulator along a given trajectory, we can use the iterative Newton-Euler Formulation:



Iterative Newton-Euler Formulation

- Outward iteration (1): Rotational velocities of links:
  - Start with $^{0}\omega_{0}=0$ and $^{0}\dot{\omega}_{0}=0$.
  - Calculate velocity iteratively:

$$^{i+1}\omega_{i+1}=\left({}_{i}^{i+1}R\cdot{}^{i}\omega_{i}\right)+\left(\dot{\theta}_{i+1}\cdot{}^{i+1}\hat{Z}_{i+1}\right)$$

Note: $\dot{\theta}_{i+1}$ is a scalar.
It is multiplied with
$^{i+1}\hat{Z}_{i+1}$ to become vector

- Outward iteration (2): Rotational acceleration of links:
  - Just now, we derived the following equation for angular acceleration:

$$^{A}\dot{\Omega}_{C}={}^{A}\dot{\Omega}_{B}+{}^{A}\Omega_{B}\times{}_{B}^{A}R\cdot{}^{B}\Omega_{C}+{}_{B}^{A}R\cdot{}^{B}\dot{\Omega}_{C}$$

  - Set A = 0, B = i and C = i+1, we have:

$$\begin{aligned}^{0}\dot{\Omega}_{i+1}&=\left({}^{0}\dot{\Omega}_{i}\right)+\left({}^{0}\Omega_{i}\times{}_{i}^{0}R\cdot{}^{i}\Omega_{i+1}\right)+\left({}_{i}^{0}R\cdot{}^{i}\dot{\Omega}_{i+1}\right)\\&=\left({}^{0}\dot{\Omega}_{i}\right)+\left({}^{0}\Omega_{i}\times{}_{i}^{0}R\cdot{}_{i+1}^{i}R\cdot{}^{i+1}\Omega_{i+1}\right)+\left({}_{i}^{0}R\cdot{}_{i+1}^{i}R\cdot{}^{i+1}\dot{\Omega}_{i+1}\right)\end{aligned}$$

Iterative Newton-Euler Formulation

- Last week, we had the following (see section Velocity Propagation):

$$^{i+1}\Omega_{i+1} = \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}$$

- Putting this into the equation from last page gives:

$$\begin{aligned} {}^{0}\dot{\Omega}_{i+1} &= \left({}^{0}\dot{\Omega}_{i}\right) + \left({}^{0}\Omega_{i} \times {}^{0}_{i}R \cdot {}^{i}_{i+1}R \cdot \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left({}^{0}_{i}R \cdot {}^{i}_{i+1}R \cdot \ddot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) \\ &= \left({}^{0}\dot{\Omega}_{i}\right) + \left({}^{0}\Omega_{i} \times {}^{0}_{i+1}R \cdot \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left({}^{0}_{i+1}R \cdot \ddot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) \end{aligned}$$

- Multiply both sides with ${}^{i+1}_{0}R$, and using the identity $R(a \times b) = Ra \times Rb$ where R is a rotation matrix gives:

$$\begin{aligned} {}^{i+1}_{0}R \cdot {}^{0}\dot{\Omega}_{i+1} &= \left({}^{i+1}_{0}R \cdot {}^{0}\dot{\Omega}_{i}\right) + \left({}^{i+1}_{0}R \cdot {}^{0}\Omega_{i} \times {}^{i+1}_{0}R \cdot {}^{0}_{i+1}R \cdot \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left({}^{i+1}_{0}R \cdot {}^{0}_{i+1}R \cdot \ddot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) \\ &= \left({}^{i+1}_{i}R \cdot {}^{i}_{0}R \cdot {}^{0}\dot{\Omega}_{i}\right) + \left({}^{i+1}_{i}R \cdot {}^{i}_{0}R \cdot {}^{0}\Omega_{i} \times \underbrace{{}^{i+1}_{i+1}R}_{I} \cdot \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left(\underbrace{{}^{i+1}_{i+1}R}_{I} \cdot \ddot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) \\ &= \left({}^{i+1}_{i}R \cdot {}^{i}_{0}R \cdot {}^{0}\dot{\Omega}_{i}\right) + \left({}^{i+1}_{i}R \cdot {}^{i}_{0}R \cdot {}^{0}\Omega_{i} \times \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left(\ddot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) \end{aligned}$$

Iterative Newton-Euler Formulation

- Using the notation $\omega_C = {}^U\Omega_C$ and ${}^A\omega_C = {}^A\left({}^U\Omega_C\right)$, which in this case are:

$$\omega_j = {}^0\Omega_j \qquad {}^k\omega_j = {}^k\Omega_j = {}^k_0R \cdot {}^0\Omega_j$$

- gives:

$$\underbrace{{}^{i+1}_{0}R \cdot \underbrace{{}^0\dot{\Omega}_{i+1}}_{\dot{\omega}_{i+1}}}_{{}^{i+1}\dot{\omega}_{i+1}} = \left(\underbrace{{}^{i+1}_{i}R \cdot {}^i_0R \cdot \underbrace{{}^0\dot{\Omega}_i}_{\dot{\omega}_i}}_{{}^i\dot{\omega}_i}\right) + \left(\underbrace{{}^{i+1}_{i}R \cdot {}^i_0R \cdot \underbrace{{}^0\Omega_i}_{\omega_i}}_{{}^i\omega_i} \times \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left(\ddot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right)$$

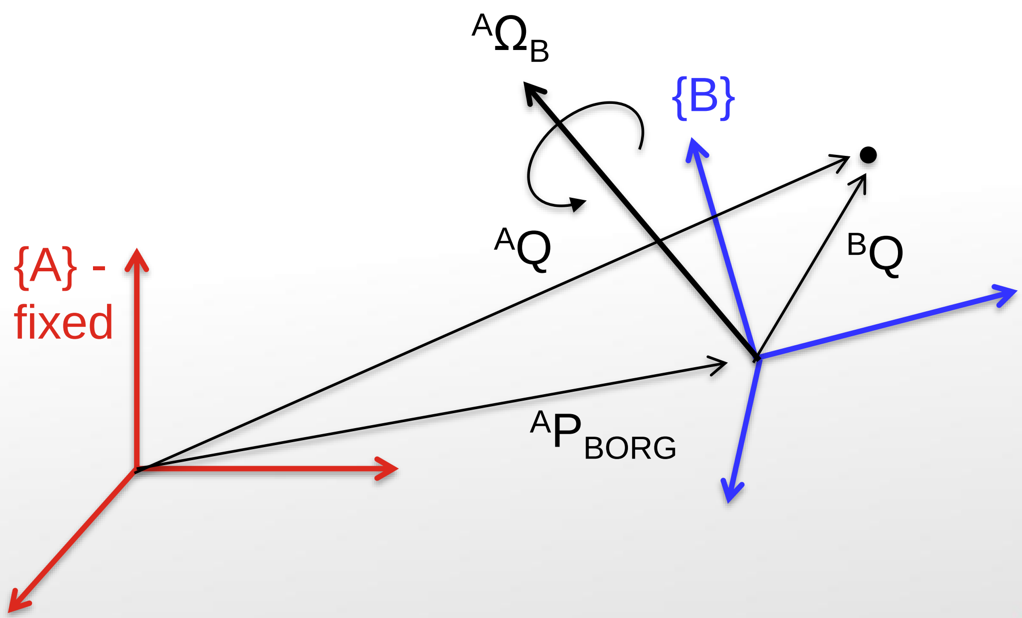
Note: This is allowed because of the property $R(a \times b) = Ra \times Rb$

$$\Rightarrow \quad {}^{i+1}\dot{\omega}_{i+1} = \left({}^{i+1}_{i}R \cdot {}^i\dot{\omega}_i\right) + \left(\underbrace{{}^{i+1}_{i}R \cdot {}^i\omega_i}_{\text{First}} \times \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left(\ddot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right)$$

Iterative Newton-Euler Formulation

- Outward iteration (3): Linear acceleration of link-frame origin
  - Just now, we derived the following equation for angular acceleration:

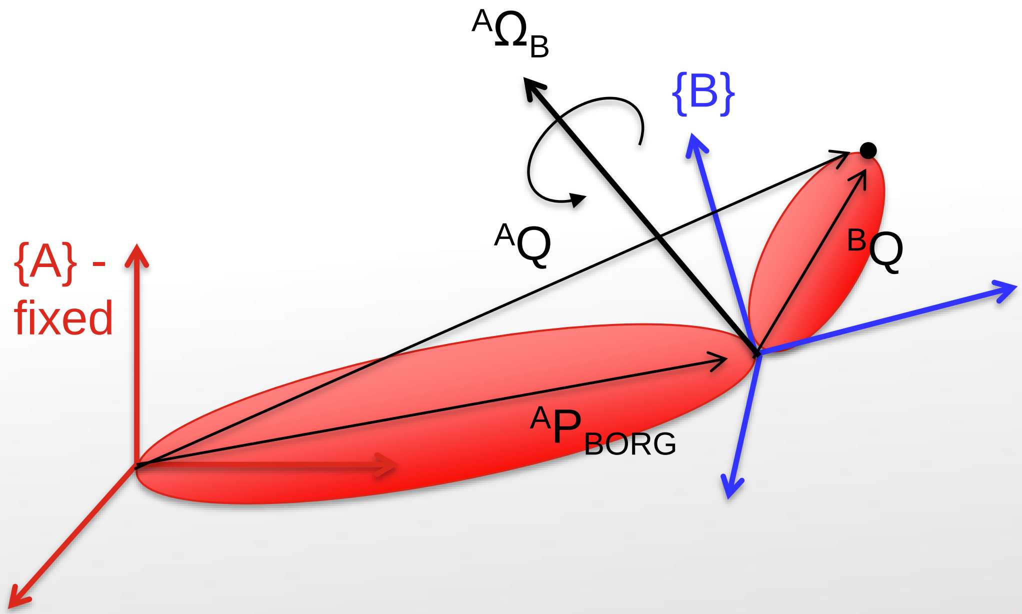
$$^{A}\dot{V}_{Q}=\left(^{A}\dot{V}_{BORG}\right)+\left(2^{A}\Omega_{B}\times_{B}^{A}R\cdot^{B}V_{Q}\right)+\left(_{B}^{A}R\cdot^{B}\dot{V}_{Q}\right)+\left(^{A}\dot{\Omega}_{B}\times_{B}^{A}R\cdot^{B}Q\right)+\left(^{A}\Omega_{B}\times^{A}\Omega_{B}\times_{B}^{A}R\cdot^{B}Q\right)$$



Iterative Newton-Euler Formulation

- Set A = 0, B = i and Q = i+1, we have:

$$^0\dot{V}_{i+1} = \left(^0\dot{V}\right)_i + \left(2\,^0\Omega_i \times {}^0_iR \cdot {}^iV_{i+1}\right) + \left({}^0_iR \cdot {}^i\dot{V}_{i+1}\right) + \left(^0\dot{\Omega}_i \times {}^0_iR \cdot {}^iQ\right) + \left(^0\Omega_i \times {}^0\Omega_i \times {}^0_iR \cdot {}^iQ\right)$$



Iterative Newton-Euler Formulation

- Multiply both sides with ${}^{i+1}_{0}R$, and using the identity $R(a \times b) = Ra \times Rb$ where R is a rotation matrix gives:

$$
\begin{aligned}
{}^{i+1}_{0}R \cdot {}^{0}\dot{V}_{i+1} &= \left({}^{i+1}_{0}R \cdot {}^{0}\dot{V}_{i}\right) + \left(2\,{}^{i+1}_{0}R \cdot {}^{0}\Omega_{i} \times {}^{i+1}_{0}R \cdot {}^{0}_{i}R \cdot {}^{i}V_{i+1}\right) + \left({}^{i+1}_{0}R \cdot {}^{0}_{i}R \cdot {}^{i}\dot{V}_{i+1}\right) \\
&\quad + \left({}^{i+1}_{0}R \cdot {}^{0}\dot{\Omega}_{i} \times {}^{i+1}_{0}R \cdot {}^{0}_{i}R \cdot {}^{i}Q\right) + \left({}^{i+1}_{0}R \cdot {}^{0}\Omega_{i} \times {}^{i+1}_{0}R \cdot {}^{0}\Omega_{i} \times {}^{i+1}_{0}R \cdot {}^{0}_{i}R \cdot {}^{i}Q\right) \\
&= \left({}^{i+1}_{0}R \cdot {}^{0}\dot{V}_{i}\right) + \left(2\,{}^{i+1}_{i}R \cdot {}^{i}_{0}R \cdot {}^{0}\Omega_{i} \times {}^{i+1}_{0}R \cdot {}^{0}_{i}R \cdot {}^{i}V_{i+1}\right) + \left({}^{i+1}_{0}R \cdot {}^{0}_{i}R \cdot {}^{i}\dot{V}_{i+1}\right) \\
&\quad + \left({}^{i+1}_{i}R \cdot {}^{i}_{0}R \cdot {}^{0}\dot{\Omega}_{i} \times {}^{i+1}_{0}R \cdot {}^{0}_{i}R \cdot {}^{i}Q\right) + \left({}^{i+1}_{0}R \cdot {}^{0}\Omega_{i} \times {}^{i+1}_{0}R \cdot {}^{0}\Omega_{i} \times {}^{i+1}_{0}R \cdot {}^{0}_{i}R \cdot {}^{i}Q\right) \\
&= \left({}^{i+1}_{i}R \cdot {}^{i}_{0}R \cdot {}^{0}\dot{V}\right)_{i} + \left(2\,{}^{i+1}_{i}R \cdot {}^{i}_{0}R \cdot {}^{0}\Omega_{i} \times {}^{i+1}V_{i+1}\right) + \left({}^{i+1}\dot{V}_{i+1}\right) \\
&\quad + \left({}^{i+1}_{i}R \cdot {}^{i}_{0}R \cdot {}^{0}\dot{\Omega}_{i} \times {}^{i+1}_{0}R \cdot {}^{0}_{i}R \cdot {}^{i}Q\right) + \left({}^{i+1}_{i}R \cdot {}^{i}_{0}R \cdot {}^{0}\Omega_{i} \times {}^{i+1}_{i}R \cdot {}^{i}_{0}R \cdot {}^{0}\Omega_{i} \times {}^{i+1}_{0}R \cdot {}^{0}_{i}R \cdot {}^{i}Q\right)
\end{aligned}
$$

Iterative Newton-Euler Formulation

- Using the notation $\omega_C = {}^U\Omega_C$, ${}^A\omega_C = {}^A\left({}^U\Omega_C\right)$, $\upsilon_C = {}^U V_{CORG}$ and ${}^A\upsilon_C = {}^A\left({}^U V_{CORG}\right)$ which in this case are:

$$\omega_j = {}^0\Omega_j \qquad {}^k\omega_j = {}^k\Omega_j = {}^k_0R\cdot{}^0\Omega_j$$

$$\upsilon_j = {}^0V_j \qquad {}^k\upsilon_j = {}^k\left({}^0V_j\right) = {}^k_0R\cdot{}^0V_j$$

- gives:

$$\underbrace{{}^{i+1}_0R\cdot{}^0\underbrace{\dot V_{i+1}}_{\dot\upsilon_{i+1}}}_{{}^{i+1}\dot\upsilon_{i+1}} = \left(\underbrace{{}^{i+1}_iR\cdot{}^i_0R\cdot{}^0\underbrace{\dot V_i}_{\dot\upsilon_i}}_{{}^i\dot\upsilon_i}\right) + \left(2\,{}^{i+1}_iR\cdot\underbrace{{}^i_0R\cdot{}^0\underbrace{\Omega_i}_{\omega_i}}_{{}^i\omega_i}\times\underbrace{{}^{i+1}V_{i+1}}_{\dot d_{i+1}\cdot{}^{i+1}\hat Z_{i+1}}\right) + \left(\underbrace{{}^{i+1}\dot V_{i+1}}_{\ddot d_{i+1}\cdot{}^{i+1}\hat Z_{i+1}}\right)$$

$$+\left({}^{i+1}_iR\cdot\underbrace{{}^i_0R\cdot{}^0\underbrace{\dot\Omega_i}_{\dot\omega_i}}_{{}^i\dot\omega_i}\times\underbrace{{}^{i+1}_0R\cdot{}^0_iR}_{{}^{i+1}_iR}\cdot\underbrace{{}^iQ}_{{}^iP_{i+1}}\right) + \left({}^{i+1}_iR\cdot\underbrace{{}^i_0R\cdot{}^0\underbrace{\Omega_i}_{\omega_i}}_{{}^i\omega_i}\times{}^{i+1}_iR\cdot\underbrace{{}^i_0R\cdot{}^0\underbrace{\Omega_i}_{\omega_i}}_{{}^i\omega_i}\times\underbrace{{}^{i+1}_0R\cdot{}^0_iR}_{{}^{i+1}_iR}\cdot\underbrace{{}^iQ}_{{}^iP_{i+1}}\right)$$

Iterative Newton-Euler Formulation

- Finally, we obtain:

$$
\begin{aligned}
{}^{i+1}\dot{\upsilon}_{i+1} = {} & \left({}^{i+1}_{i}R \cdot {}^{i}\dot{\upsilon}_{i}\right) + \left(2\,{}^{i+1}_{i}R \cdot {}^{i}\omega_{i} \times \dot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left(\ddot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) \\
& + \left({}^{i+1}_{i}R \cdot {}^{i}\dot{\omega}_{i} \times {}^{i+1}_{i}R \cdot {}^{i}P_{i+1}\right) + \left({}^{i+1}_{i}R \cdot {}^{i}\omega_{i} \times {}^{i+1}_{i}R \cdot {}^{i}\omega_{i} \times {}^{i+1}_{i}R \cdot {}^{i}P_{i+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
{}^{i+1}\dot{\upsilon}_{i+1} = {} & \left({}^{i+1}_{i}R \cdot {}^{i}\dot{\upsilon}_{i}\right) + \left(2\,{}^{i+1}_{i}R \cdot {}^{i}\omega_{i} \times \dot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left(\ddot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) \\
& + {}^{i+1}_{i}R \cdot \left({}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times {}^{i}\omega_{i} \times {}^{i}P_{i+1}\right)
\end{aligned}
$$

- The second bracket and the third brackets exist only if joint i+1 is prismatic.
- Since

$$
{}^{i+1}\omega_{i+1} = \left({}^{i+1}_{i}R \cdot {}^{i}\omega_{i}\right) + \underbrace{\left(\dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right)}_{0 \text{ if prismatic}} = \left({}^{i+1}_{i}R \cdot {}^{i}\omega_{i}\right)
$$

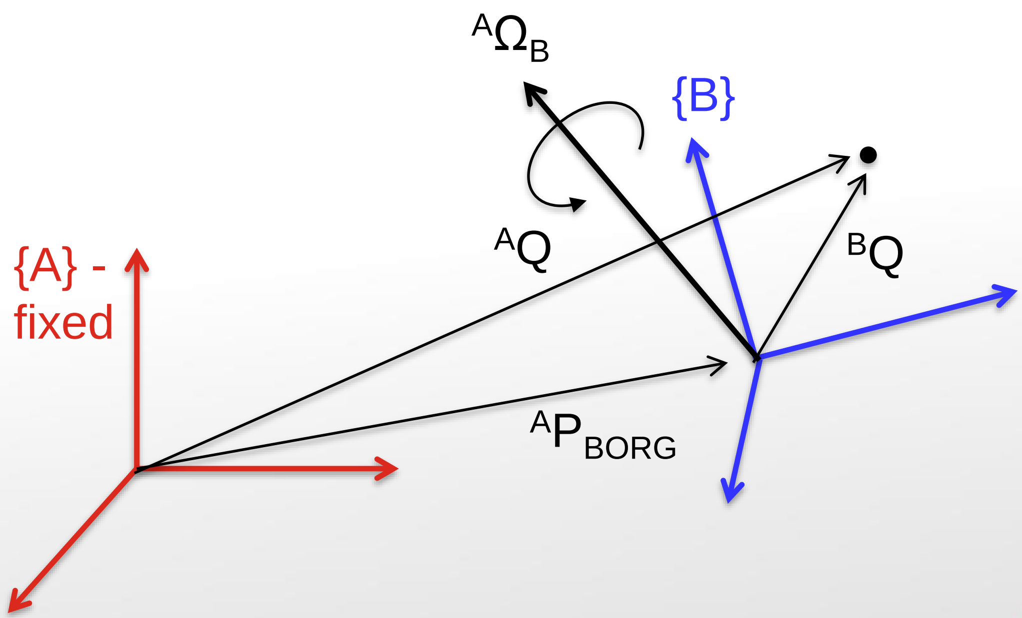
- the linear acceleration equation can be written as:

$$
\begin{aligned}
{}^{i+1}\dot{\upsilon}_{i+1} = {} & \left({}^{i+1}_{i}R \cdot {}^{i}\dot{\upsilon}_{i}\right) + \left(2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left(\ddot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) \\
& + {}^{i+1}_{i}R \cdot \left({}^{i}\dot{\omega}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega_{i} \times {}^{i}\omega_{i} \times {}^{i}P_{i+1}\right)
\end{aligned}
$$

Iterative Newton-Euler Formulation

- Outward iteration (4): Linear acceleration of link's center of mass
  - Just now, we derived the following equation for angular acceleration:

$$^{A}\dot{V}_{Q}=\left(^{A}\dot{V}_{BORG}\right)+\left(2\,^{A}\Omega_{B}\times{}^{A}_{B}R\cdot{}^{B}V_{Q}\right)+\left({}^{A}_{B}R\cdot{}^{B}\dot{V}_{Q}\right)+\left(^{A}\dot{\Omega}_{B}\times{}^{A}_{B}R\cdot{}^{B}Q\right)+\left(^{A}\Omega_{B}\times{}^{A}\Omega_{B}\times{}^{A}_{B}R\cdot{}^{B}Q\right)$$



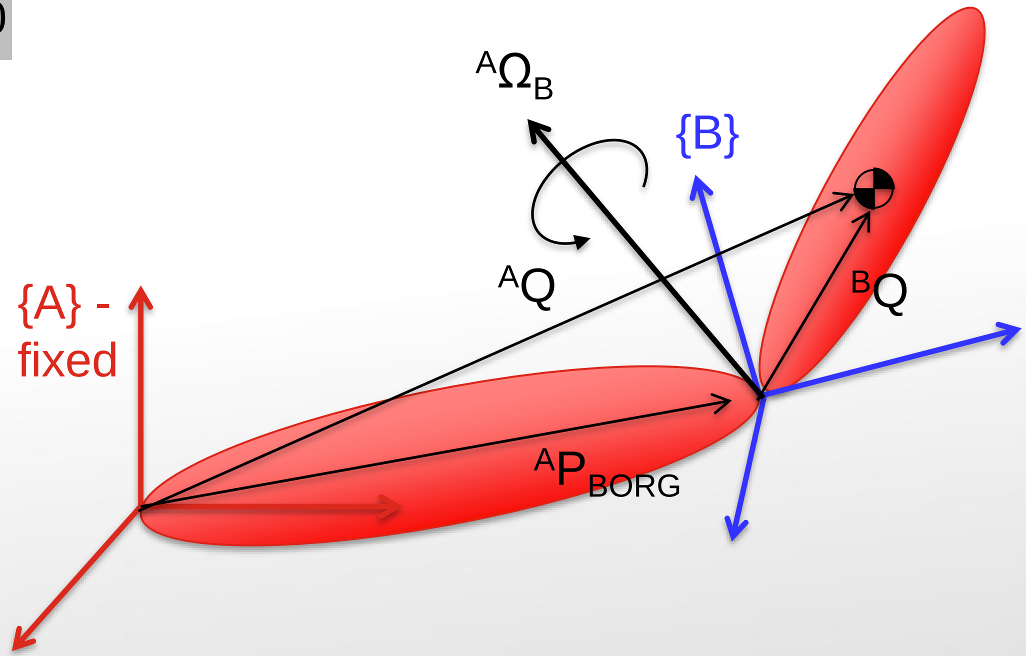
Iterative Newton-Euler Formulation

- Set A = 0, B = i and Q = C_i, we have:

$$ {}^{0}\dot{V}_{C_i} = \left({}^{0}\dot{V}_i\right) + \left(2\,{}^{0}\Omega_i \times {}^{0}_{i}R \cdot {}^{i}V_{C_i}\right) + \left({}^{0}_{i}R \cdot {}^{i}\dot{V}_{C_i}\right) + \left({}^{0}\dot{\Omega}_i \times {}^{0}_{i}R \cdot {}^{i}Q\right) + \left({}^{0}\Omega_i \times {}^{0}\Omega_i \times {}^{0}_{i}R \cdot {}^{i}Q\right) $$

- Now, imagine the frame of C_i has the same orientation as {B}. Also, the center of mass is constant from the ith joint. Therefore:

$$ {}^{i}V_{C_i} = {}^{i}\dot{V}_{C_i} = 0 $$



Iterative Newton-Euler Formulation

- The equation thus simplifies to:

$$ {}^{0}\dot{V}_{C_i} = \left({}^{0}\dot{V}_i\right) + \left({}^{0}\dot{\Omega}_i \times {}^{0}_{i}R \cdot {}^{i}Q\right) + \left({}^{0}\Omega_i \times {}^{0}\Omega_i \times {}^{0}_{i}R \cdot {}^{i}Q\right) $$

- Multiply both sides with ${}^{i}_{0}R$, and using the identity $R(a \times b) = Ra \times Rb$ where R is a rotation matrix gives:

$$ \begin{aligned} {}^{i}_{0}R \cdot {}^{0}\dot{V}_{C_i} &= \left({}^{i}_{0}R \cdot {}^{0}\dot{V}_i\right) + \left({}^{i}_{0}R \cdot {}^{0}\dot{\Omega}_i \times \underbrace{{}^{i}_{0}R \cdot {}^{0}_{i}R}_{I} \cdot {}^{i}Q\right) + \left({}^{i}_{0}R \cdot {}^{0}\Omega_i \times {}^{i}_{0}R \cdot {}^{0}\Omega_i \times \underbrace{{}^{i}_{0}R \cdot {}^{0}_{i}R}_{I} \cdot {}^{i}Q\right) \\ &= \left({}^{i}_{0}R \cdot {}^{0}\dot{V}_i\right) + \left({}^{i}_{0}R \cdot {}^{0}\dot{\Omega}_i \times {}^{i}Q\right) + \left({}^{i}_{0}R \cdot {}^{0}\Omega_i \times {}^{i}_{0}R \cdot {}^{0}\Omega_i \times {}^{i}Q\right) \end{aligned} $$

Iterative Newton-Euler Formulation

- Using the notation $\omega_C = {}^U\Omega_C$, ${}^A\omega_C = {}^A\left({}^U\Omega_C\right)$, $\upsilon_C = {}^U V_{CORG}$ and ${}^A\upsilon_C = {}^A\left({}^U V_{CORG}\right)$ which in this case are:

  $\omega_j = {}^0\Omega_j$ ${}^k\omega_j = {}^k\Omega_j = {}^k_0 R \cdot {}^0\Omega_j$ $\upsilon_j = {}^0 V_j$ ${}^k\upsilon_j = {}^k\left({}^0 V_j\right) = {}^k_0 R \cdot {}^0 V_j$

- gives: $${}^i\dot{\upsilon}_{C_i} = \left({}^i\dot{\upsilon}_i\right) + \left({}^i\dot{\omega}_i \times {}^i P_{C_i}\right) + \left({}^i\omega_i \times {}^i\omega_i \times {}^i P_{C_i}\right)$$

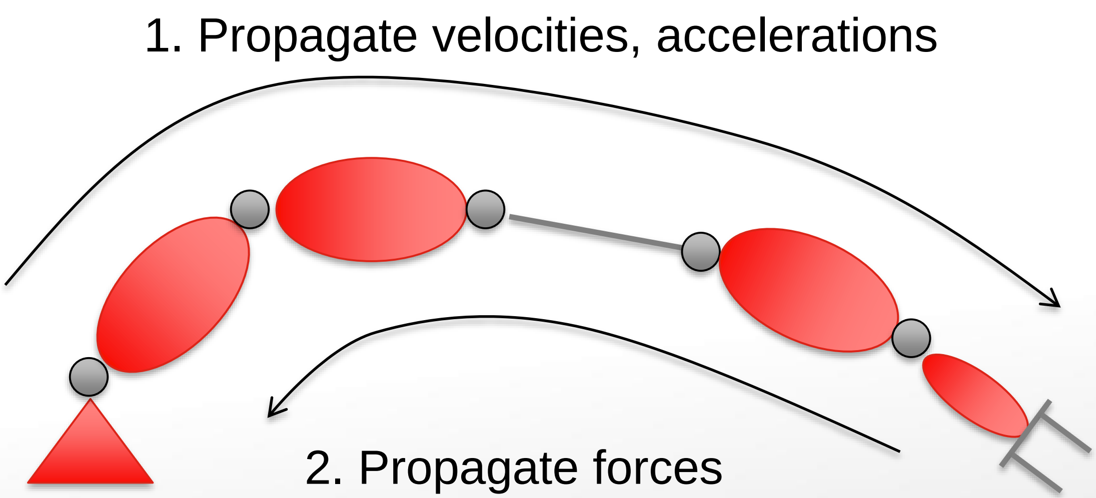
- Outward iteration (5): Force acting on a link
  - Using Newton's law, we have the force acting at the center of mass of each link: $$F_i = m\dot{\upsilon}_{C_i}$$

- Outward iteration (6):Torque acting on a link

$$N_i = {}^{C_i}I\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i$$

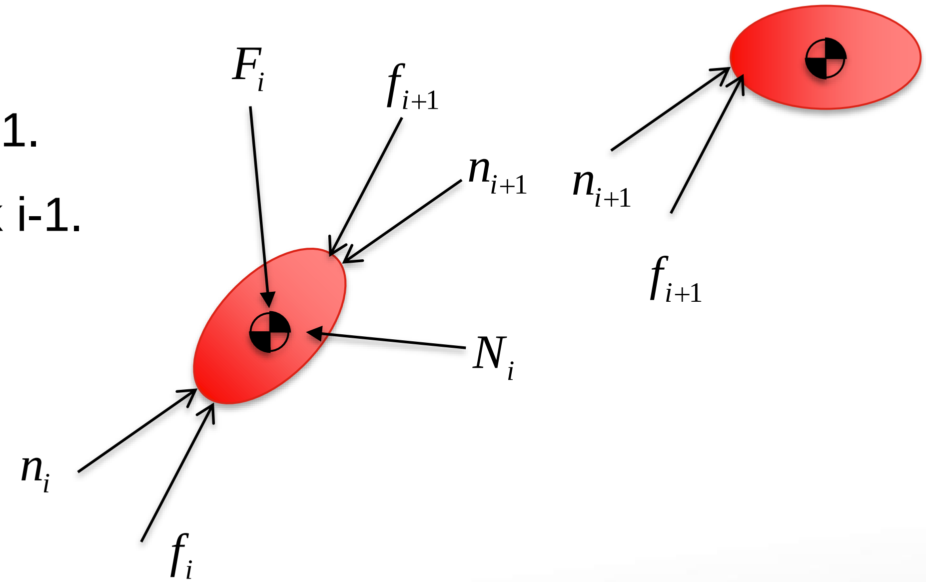
Iterative Newton-Euler Formulation

- So far, we have obtained the equations for outward iterations.
- Now, it's time to do the inward iterations.



Iterative Newton-Euler Formulation

- Define:
  - fi = force exerted on link i by link i-1.
  - ni = torque exerted on link i by link i-1.



- Summing the forces acting on link i gives: $^{i}F_i = {}^{i}f_i - {}^{i}_{i+1}R\, {}^{i+1}f_{i+1}$
- Summing the torques about center of mass gives:

$$^{i}N_i = {}^{i}n_i - {}^{i}n_{i+1} + \left(-{}^{i}P_{C_i}\right)\times {}^{i}f_i - \left({}^{i}P_{i+1} - {}^{i}P_{C_i}\right)\times {}^{i}f_{i+1}$$

Iterative Newton-Euler Formulation

- Substituting ${}^iF_i = {}^if_i - {}^i_{i+1}R^{i+1}f_{i+1}$ into the torque equation gives:

$$\begin{aligned}
{}^iN_i &= {}^in_i - {}^in_{i+1} + \left(-{}^iP_{C_i}\right) \times {}^if_i - \left({}^iP_{i+1} - {}^iP_{C_i}\right) \times {}^if_{i+1} \\
&= {}^in_i - {}^in_{i+1} + \left(-{}^iP_{C_i}\right) \times \left({}^iF_i + {}^i_{i+1}R \cdot {}^{i+1}f_{i+1}\right) - \left({}^iP_{i+1} - {}^iP_{C_i}\right) \times {}^if_{i+1} \\
&= {}^in_i - {}^i_{i+1}R \cdot {}^{i+1}n_{i+1} + \left(-{}^iP_{C_i}\right) \times \left({}^iF_i + {}^i_{i+1}R \cdot {}^{i+1}f_{i+1}\right) - \left({}^iP_{i+1} - {}^iP_{C_i}\right) \times {}^i_{i+1}R \cdot {}^{i+1}f_{i+1}
\end{aligned}$$

- Because cross product is distributive over addition, i.e. $a \times (b + c) = a \times b + a \times c$ we have:

$$\begin{aligned}
{}^iN_i &= {}^in_i - {}^i_{i+1}R \cdot {}^{i+1}n_{i+1} + \left(-{}^iP_{C_i}\right) \times \left({}^iF_i + {}^i_{i+1}R \cdot {}^{i+1}f_{i+1}\right) - \left({}^iP_{i+1} - {}^iP_{C_i}\right) \times {}^i_{i+1}R \cdot {}^{i+1}f_{i+1} \\
&= {}^in_i - {}^i_{i+1}R \cdot {}^{i+1}n_{i+1} - {}^iP_{C_i} \times {}^iF_i - {}^iP_{C_i} \times {}^i_{i+1}R \cdot {}^{i+1}f_{i+1} - {}^iP_{i+1} \times {}^i_{i+1}R \cdot {}^{i+1}f_{i+1} + {}^iP_{C_i} \times {}^i_{i+1}R \cdot {}^{i+1}f_{i+1}
\end{aligned}$$

cancels out

$$= {}^in_i - {}^i_{i+1}R \cdot {}^{i+1}n_{i+1} - {}^iP_{C_i} \times {}^iF_i - {}^iP_{i+1} \times {}^i_{i+1}R \cdot {}^{i+1}f_{i+1}$$

Iterative Newton-Euler Formulation

- Finally, we put those equations into forms suitable for iterations:

$$^{i}f_i = {}^{i}_{i+1}R \, {}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{i}n_i = {}^{i}_{i+1}R \cdot {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R \cdot {}^{i+1}f_{i+1} + {}^{i}N_i$$

- Finally, the torques at each joint are calculated as:
  - Revolute joint: $\tau_i = {}^{i}n_i^T \cdot {}^{i}\hat{Z}_i$
  - Prismatic joint: $\tau_i = {}^{i}f_i^T \cdot {}^{i}\hat{Z}_i$

- Note that these inward iterations started with $^{N+1}f_{N+1} = 0 \;\; \& \;\; {}^{N+1}n_{N+1} = 0$ if the robot is moving in free space, or the actual contact force / torque if the robot is in contact with some objects.

Thank you!

Have a good evening.

RMIT UNIVERSITY