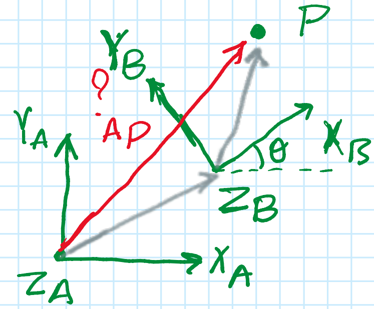
QS. 1

Tuesday, July 21, 2020 9:31 AM

- **Question 1:**
  - A position vector of a point in frame {B} is given by $^{B}P = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$
  - The origin of frame {B} is at [11,-3,9] with respect to frame {A}.
  - Also, frame {B} is rotated by 30 degrees along the z-axis of frame {A}.
  - What is the position vector of the point in frame {A}?
  - Also, write down the homogenous transformation matrix.



$$^{A}P = {}^{A}_{B}R \cdot {}^{B}P + {}^{A}P_{BORG}$$

$$^{A}_{B}R = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & - \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & - \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & - \end{bmatrix} = \begin{bmatrix} \cos 30 & -\sin 30 & 0 \\ \sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{A}_{B}R = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{A}P = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overbrace{\begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}}^{^{B}P} + \begin{bmatrix} 11 \\ -3 \\ 9 \end{bmatrix} =$$

$$^{A}P = \begin{bmatrix} -1.84 \\ 22.32 \\ 30 \end{bmatrix} + \begin{bmatrix} 11 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 9.66 \\ 19.32 \\ 39 \end{bmatrix}$$

$$^{A}_{B}T = \left[\begin{array}{c|c} ^{A}_{B}R & ^{A}P_{BORG} \\ \hline 0\;0\;0 & 1 \end{array}\right] \qquad ^{A}_{B}T = \left[\begin{array}{ccc|c} 0.866 & -0.5 & 0 & 11 \\ 0.5 & 0.866 & 0 & -3 \\ 0 & 0 & 1 & 9 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} ^{A}P \\ \hline 1 \end{bmatrix}}_{4\times1} = \underbrace{^{A}_{B}T}_{4\times4} \cdot \underbrace{\begin{bmatrix} ^{B}P \\ \hline 1 \end{bmatrix}}_{4\times1} = {}^{A}_{B}T \cdot \begin{bmatrix} 10 \\ 20 \\ 30 \\ \hline 1 \end{bmatrix} = \left.\begin{bmatrix} 9.66 \\ 19.32 \\ 39 \\ \hline 1 \end{bmatrix}\right\} {}^{A}P$$

QS. 2

Friday, 24 July 2020 1:25 PM

- **Question 2:**
  - The rotation matrix from A to B is:

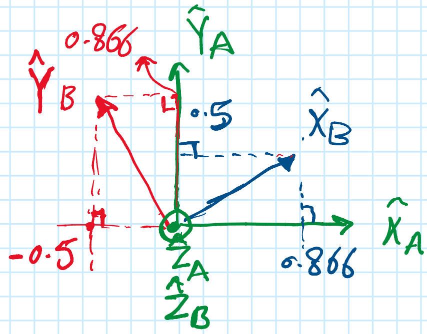
$$\hat{X}_B\cdot\hat{X}_A \qquad \hat{Y}_B\cdot\hat{X}_A \qquad \hat{Y}_B\cdot\hat{Y}_A$$

$${}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{X}_B\cdot\hat{Y}_A \qquad \hat{Z}_B\cdot\hat{Z}_A$$

  - Interpret (by sketch) the meaning of the rotation matrix.
  - What is the rotation matrix from B to A? ${}^{B}_{A}R$?
    - Calculate using matrix inverse.
    - Verify that it is the same as R-transpose.

Rotation Matrix About Z-axis



$${}^{B}_{A}R = \underbrace{{}^{A}_{B}R^{-1}}$$

Properties of Rot. Mat. ${}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T}$

$$1\text{- } \det({}^{A}_{B}R) = 0.866\begin{vmatrix} 0.866 & 0 \\ 0 & 1 \end{vmatrix} - (-0.5)\begin{vmatrix} 0.5 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{vmatrix}$$

$$= 0.866^2 + 0.5^2 = \underline{1}$$

$$2\text{- } \operatorname{adj}({}^{A}_{B}R) = \begin{bmatrix} +\begin{vmatrix} 0.866 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0.5 & 0 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 0.5 & 0.866 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} -0.5 & 0 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 0.866 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0.866 & -0.5 \\ 0 & 0 \end{vmatrix} \\ +\begin{vmatrix} -0.5 & 0 \\ 0.866 & 0 \end{vmatrix} & -\begin{vmatrix} 0.866 & 0 \\ 0.5 & 0 \end{vmatrix} & +\begin{vmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{vmatrix} \end{bmatrix}^{T}$$

$$= \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{T} = \underline{\begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Inverse Using Mathematics

Note $\underline{{}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T}}$

QS. 3

Friday, 24 July 2020 1:27 PM

- **Question 3:**
  - A velocity vector is given by:

$$^{B}V = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

  - Given:

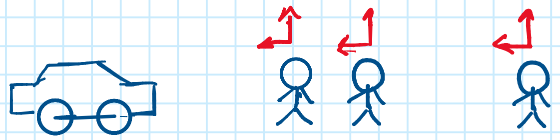
$$^{A}_{B}T = \begin{bmatrix} 0.866 & -0.5 & 0 & 11 \\ 0.5 & 0.866 & 0 & -3 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - Compute:

$$^{A}V$$

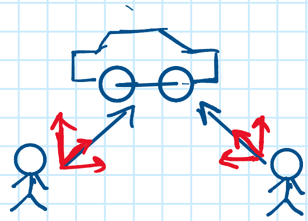
A Velocity Vector is a Free Vector

Meaning that It is only affected by Rotation



100 Km/h

All observers see the same vel.

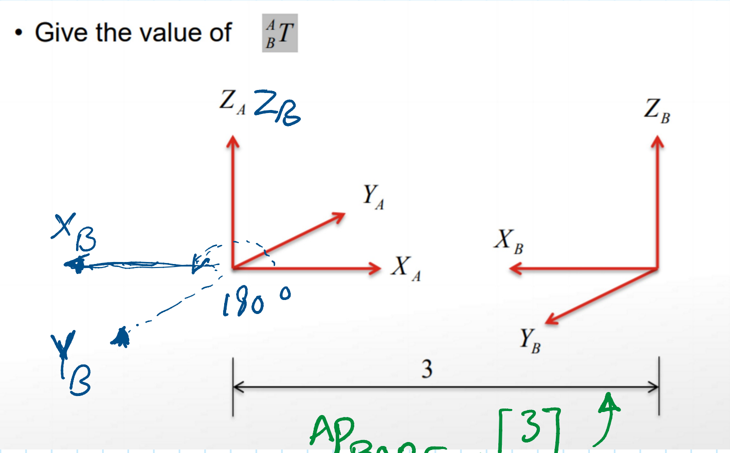


seeing diff. vel. if looking from different Angles

$$^{A}V = {}^{A}_{B}R \cdot {}^{B}V = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} -1.34 \\ 22.32 \\ 30 \end{bmatrix}$$

QS. 4

Friday, 24 July 2020 1:27 PM



$$^AP_{BORG} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$^A_BT = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^A_BT = ?$$

$$^A_BT = \left[\begin{array}{c|c} ^A_BR & ^AP_{BORG} \\ \hline 0 \ 0 \ 0 & 1 \end{array}\right]$$

$^A_BR \rightsquigarrow ?$ 180° about z-axis

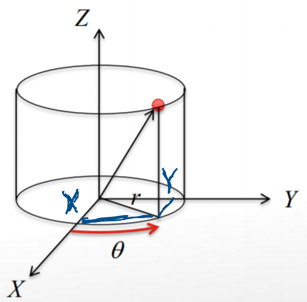
$$^A_BR = \begin{bmatrix} \cos 180 & -\sin 180 & 0 \\ \sin 180 & \cos 180 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^A_BR = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Qs. 5

Friday, 24 July 2020 1:28 PM

- **Question 5:**
  - In the lecture, the positions have been given in Cartesian coordinates.
  - We can also describe the positions in cylindrical coordinates.



  - If the point has Cartesian coordinates $[x, y, z]^T$, what is the position in cylindrical coordinates (function of θ, r, z)?

This is only for your Knowledge
Not to be tested

Cartesian Cylindrical

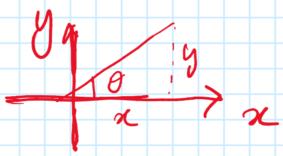
$Z = Z$

$X = r\cos\theta$

$Y = r\sin\theta$

$\rightarrow r, \theta, Z$?

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(\cos^2\theta + \sin^2\theta) = r^2 \rightarrow r = \sqrt{x^2+y^2}$$



$\rightarrow \theta = \arctan(y/x)$

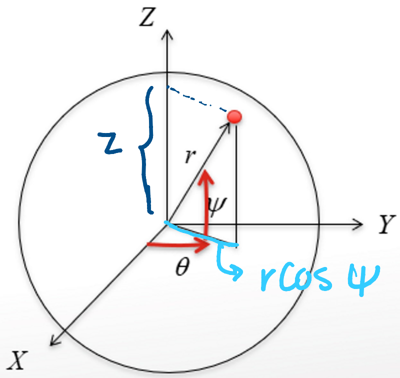
$\theta = \text{arctan2}(y, x)$

Matlab Function

Qs. 6

Thursday, 30 July 2020 3:44 PM

- **Question 6:**
  - Next, we can also describe the positions in spherical coordinates.



  - If the point has Cartesian coordinates $[x, y, z]^T$, what is the position in spherical coordinates (function of θ, r, ψ)?

1- Given θ, ψ, r what are X, Y, Z ?

$z = r \sin \psi$

$x = r \cos \psi \cos \theta$

$y = r \cos \psi \sin \theta$

2. Given X, Y, Z → what are ψ, θ, r

$$x^2 + y^2 + z^2 = r^2 \sin^2 \psi + r^2 \cos^2 \psi \cos^2 \theta + r^2 \cos^2 \psi \sin^2 \theta$$

$$= r^2 \sin^2 \psi + r^2 \cos^2 \psi = r^2$$

$$\Rightarrow \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow \quad \theta = \arctan(y/x)$$

or $\theta = \text{arctan2}(y, x)$ ← Matlab

$$\Rightarrow \quad \sin \psi = z/r$$

$$\psi = \arcsin(z/r)$$

$$= \arcsin\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$