Week 7 – Manipulator Dynamics

Advanced Robotic Systems – MANU2453

Dr Ehsan Asadi, School of Engineering
RMIT University, Victoria, Australia
Email: ehsan.asadi@rmit.edu.au

Lectures

Wk	Date	Lecture (NOTE: video recording)	Maths Difficulty	Hands-on Activity	Related Assessment
1	24/7	• Introduction to the Course • Spatial Descriptions & Transformations			
2	31/7	• Spatial Descriptions & Transformations • Robot Cell Design			Robot Cell Design Assignment
3	7/8	• Forward Kinematics • Inverse Kinematics			
4	14/8	• ABB Robot Programming via Teaching Pendant • ABB RobotStudio Offline Programming		ABB RobotStudio Offline Programming	Offline Programming Assignment
5	21/8	• Jacobians: Velocities and Static Forces			
6	28/8	• Manipulator Dynamics			
7	11/9	• Manipulator Dynamics		MATLAB Simulink Simulation	
8	18/9	• Robotic Vision		MATLAB Simulation	Robotic Vision Assignment
9	25/9	• Robotic Vision		MATLAB Simulation	
10	2/10	• Trajectory Generation			
11	9/10	• Linear & Nonlinear Control		MATLAB Simulink Simulation	
12	16/10	• Introduction to I4.0 • Revision			Final Exam

Content

- Lagrangian Formulation
- Inclusion of Non-Rigid Body Effects
- Explicit Form

Content

- Lagrangian Formulation
- Inclusion of Non-Rigid Body Effects
- Explicit Form

Lagrangian Formulation

- Apart from Newton-Euler's method, there are other approaches to obtain the manipulator's dynamic equation as well.
- The Lagrangian formulation is one such method.
  - It is an "energy-based" approach.
  - The dynamic equations will be derived from the kinetic energy and the potential energy of the manipulator.
- Another approach is the "Explicit Form" method.
  - The V and G vectors can be derived directly from M matrix.

Lagrangian Formulation

- The kinetic energy of each link is:

$$k_i = \frac{1}{2} m_i \upsilon_{c_i}^T \upsilon_{c_i} + \frac{1}{2} \cdot {}^i\omega_i^T \cdot {}^{C_i}I_i \cdot {}^i\omega_i$$

  - and the total kinetic energy of the whole manipulator is: $k = \sum_{i=1}^{n} k_i$

- The potential energy of each link is:

$$u_i = -m_i \cdot {}^0g^T \cdot {}^0P_{C_i} + u_{ref_i}$$

  - where ${}^0g$ is the 3 x 1 gravity vector, ${}^0P_{Ci}$ is the vector representing the position of the centre of the mass of the i^{th} link, and u_{refi} is a constant so that the minimum of u_i is zero.
  - The total potential energy of the manipulator is then: $u = \sum_{i=1}^{n} u_i$

Lagrangian Formulation

- Lagrangian is the difference between the kinetic and potential energy of a mechanical system:

$$L = k - u$$

- The equation of motion for the manipulator is then:

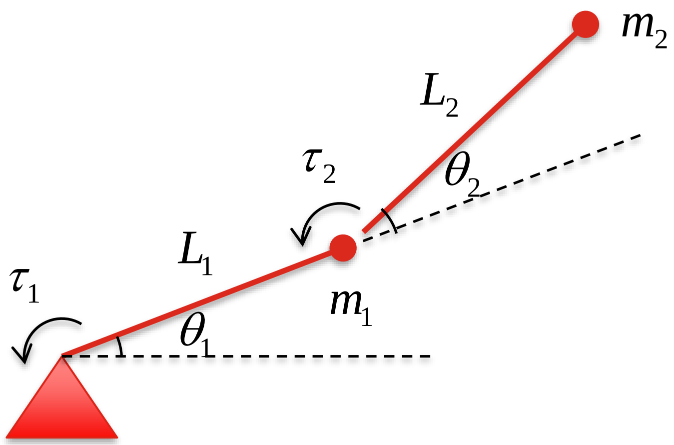
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

- Because the potential energy, u, is independent of velocity, the equation can be written as:

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{q}} - \frac{\partial (k-u)}{\partial q} = \tau$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial k}{\partial \dot{q}} - \frac{\partial k}{\partial q} + \frac{\partial u}{\partial q} = \tau$$

Example



- Let's try this method for the two-link robot from week 6.
- From lecture 5, we had:

- We also need the speed of the origin of frame {3} which is at the tip of manipulator, having the same orientation of frame {2}.
- Using $^{i+1}\upsilon_{i+1} = {}^{i+1}_{i}R \cdot \left({}^{i}\upsilon_i + {}^{i}\omega_i \times {}^{i}P_{i+1} \right)$, we get:

$$^{3}\upsilon_3 = \underbrace{{}^{3}_{2}R}_{I} \cdot \left({}^{2}\upsilon_2 + {}^{2}\omega_2 \times {}^{2}P_3 \right) = \begin{bmatrix} L_1 s_2 \dot{\theta}_1 \\ L_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 s_2 \dot{\theta}_1 \\ L_1 c_2 \dot{\theta}_1 + L_2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right) \\ 0 \end{bmatrix}$$

Example

- Note that the center of mass for 1st link is the origin of frame {2}, and center of mass for 2nd link is the origin of frame {3} / tip of manipulator.
- Therefore:

$$ {}^{2}\upsilon_{C_1} = {}^{2}\upsilon_2 = \begin{bmatrix} L_1 s_2 \dot{\theta}_1 \\ L_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} \qquad {}^{3}\upsilon_{C_2} = {}^{3}\upsilon_3 = \begin{bmatrix} L_1 s_2 \dot{\theta}_1 \\ L_1 c_2 \dot{\theta}_1 + L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ 0 \end{bmatrix} $$

$$ {}^{1}\omega_{C_1} = {}^{1}\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad {}^{2}\omega_{C_2} = {}^{2}\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} $$

- Looking at the kinetic equation again:

$$ k_i = \frac{1}{2} m_i \upsilon_{c_i}^T \upsilon_{c_i} + \frac{1}{2} \cdot {}^{i}\omega_i^T \cdot {}^{C_i}I_i \cdot {}^{i}\omega_i $$

Example

- The total kinetic energy of the manipulator is thus:

$$k=\frac{1}{2}m_1\upsilon_{c_1}^T\upsilon_{c_1}+\frac{1}{2}\,{}^1\omega_1^T\cdot\underbrace{{}^{C_1}I_1}_{0}\cdot{}^1\omega_1+\frac{1}{2}m_2\upsilon_{c_2}^T\upsilon_{c_2}+\frac{1}{2}\,{}^2\omega_2^T\cdot\underbrace{{}^{C_2}I_2}_{0}\cdot{}^2\omega_2$$

$$=\frac{1}{2}m_1\begin{bmatrix}L_1s_2\dot{\theta}_1\\L_1c_2\dot{\theta}_1\\0\end{bmatrix}^T\begin{bmatrix}L_1s_2\dot{\theta}_1\\L_1c_2\dot{\theta}_1\\0\end{bmatrix}+\frac{1}{2}m_2\begin{bmatrix}L_1s_2\dot{\theta}_1\\L_1c_2\dot{\theta}_1+L_2\left(\dot{\theta}_1+\dot{\theta}_2\right)\\0\end{bmatrix}^T\begin{bmatrix}L_1s_2\dot{\theta}_1\\L_1c_2\dot{\theta}_1+L_2\left(\dot{\theta}_1+\dot{\theta}_2\right)\\0\end{bmatrix}$$

$$=\frac{1}{2}m_1L_1^2s_2^2\dot{\theta}_1^2+\frac{1}{2}m_1L_1^2c_2^2\dot{\theta}_1^2+\frac{1}{2}m_2L_1^2s_2^2\dot{\theta}_1^2+\frac{1}{2}m_2L_1^2c_2^2\dot{\theta}_1^2+\frac{1}{2}m_2L_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2+m_2L_1L_2c_2\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)$$

$$=\frac{1}{2}(m_1+m_2)L_1^2\dot{\theta}_1^2+\frac{1}{2}m_2L_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2+m_2L_1L_2c_2\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)$$

Example

- The potential energy of the manipulator is:

$$
\begin{aligned}
u &= -m_1 \cdot {}^0 g^T \cdot {}^0 P_{C_1} + u_{ref_1} - m_2 \cdot {}^0 g^T \cdot {}^0 P_{C_2} + u_{ref_2} \\
&= -m_1 \begin{bmatrix} 0 & -g & 0 \end{bmatrix} \begin{bmatrix} L_1 c_1 \\ L_1 s_1 \\ 0 \end{bmatrix} + u_{ref_1} - m_2 \begin{bmatrix} 0 & -g & 0 \end{bmatrix} \begin{bmatrix} L_1 c_1 + L_2 c_{12} \\ L_1 s_1 + L_2 s_{12} \\ 0 \end{bmatrix} + u_{ref_2} \\
&= m_1 g L_1 s_1 + u_{ref_1} + m_2 g \left(L_1 s_1 + L_2 s_{12} \right) + u_{ref_2}
\end{aligned}
$$

- As mentioned, $uref_i$ is chosen such that the minimum of potential energy is zero.
  - For link 1, the minimum of $m_1 g L_1 s_1$ is $-m_1 g L_1$ when $\theta_1 = 270$ deg. Therefore: $u_{ref_1} = m_1 g L_1$
  - Following the same argument, we have: $u_{ref_2} = m_2 g \left(L_1 + L_2 \right)$
- Therefore: $u = m_1 g L_1 s_1 + m_1 g L_1 + m_2 g \left(L_1 s_1 + L_2 s_{12} \right) + m_2 g \left(L_1 + L_2 \right)$

Example

- The kinetic and potential energies are repeated here for convenience sake:

$$k = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + m_2L_1L_2c_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$u = m_1gL_1s_1 + m_1gL_1 + m_2g\left(L_1s_1 + L_2s_{12}\right) + m_2g\left(L_1 + L_2\right)$$

- Now, apply the formula: $\frac{d}{dt}\frac{\partial k}{\partial \dot{\theta}} - \frac{\partial k}{\partial \theta} + \frac{\partial u}{\partial \theta} = \tau$

$$\frac{d}{dt}\begin{bmatrix} (m_1 + m_2)L_1^2\dot{\theta}_1 + m_2L_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right) + 2m_2L_1L_2c_2\dot{\theta}_1 + m_2L_1L_2c_2\dot{\theta}_2 \\ m_2L_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right) + m_2L_1L_2c_2\dot{\theta}_1 \end{bmatrix}$$

$$-\begin{bmatrix} 0 \\ -m_2L_1L_2s_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) \end{bmatrix} + \begin{bmatrix} m_1gL_1c_1 + m_2gL_1c_1 + m_2gL_2c_{12} \\ m_2gL_2c_{12} \end{bmatrix} = \tau$$

Example

- (Continued)

$$\begin{bmatrix} (m_1+m_2)L_1^2\ddot{\theta}_1 + m_2L_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) + 2m_2L_1L_2c_2\ddot{\theta}_1 - 2m_2L_1L_2s_2\dot{\theta}_1\dot{\theta}_2 + m_2L_1L_2c_2\ddot{\theta}_2 - m_2L_1L_2s_2\dot{\theta}_2^2 \\ m_2L_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) + m_2L_1L_2c_2\ddot{\theta}_1 - m_2L_1L_2s_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix}$$
$$-\begin{bmatrix} 0 \\ -m_2L_1L_2s_2\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right) \end{bmatrix} + \begin{bmatrix} m_1gL_1c_1 + m_2gL_1c_1 + m_2gL_2c_{12} \\ m_2gL_2c_{12} \end{bmatrix} = \tau$$

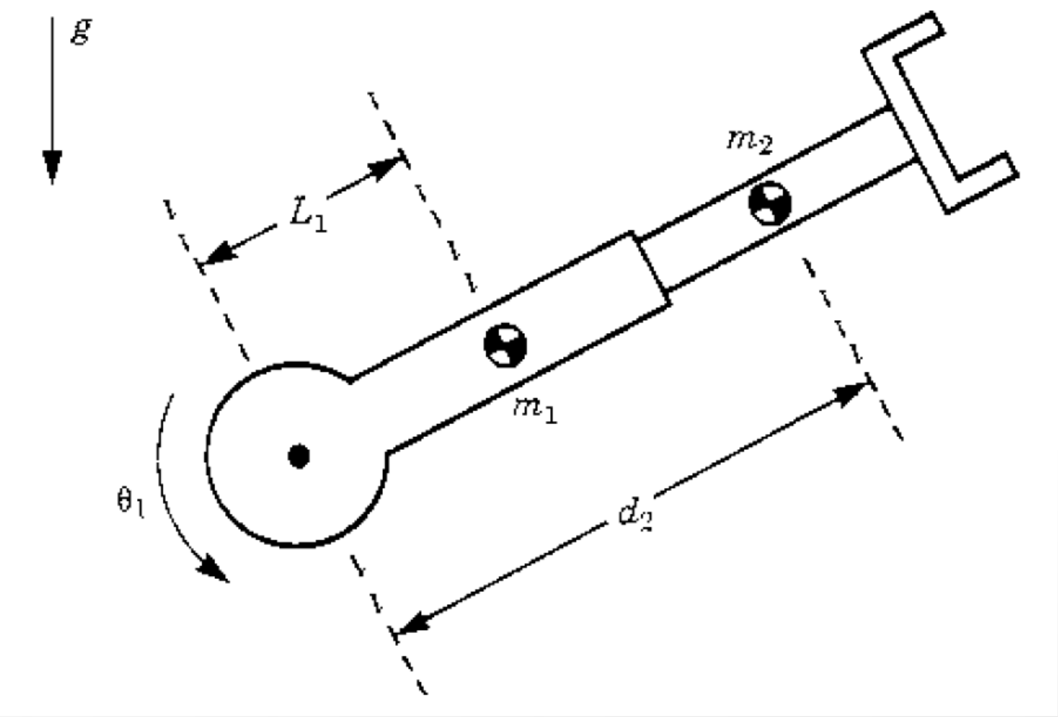
- This gives the following dynamic equation:

$$\begin{aligned} \tau_1 &= m_2L_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) + m_2L_1L_2c_2\left(2\ddot{\theta}_1+\ddot{\theta}_2\right) + (m_1+m_2)L_1^2\ddot{\theta}_1 - m_2L_1L_2s_2\dot{\theta}_2^2 \\ &\quad - 2m_2L_1L_2s_2\dot{\theta}_1\dot{\theta}_2 + (m_1+m_2)gL_1c_1 + m_2gL_2c_{12} \\ \tau_2 &= m_2L_1L_2c_2\ddot{\theta}_1 + m_2L_1L_2s_2\dot{\theta}_1^2 + m_2gL_2c_{12} + m_2L_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) \end{aligned}$$

  - which is exactly the same as the ones derived from Newton-Euler formulation.

Another Example

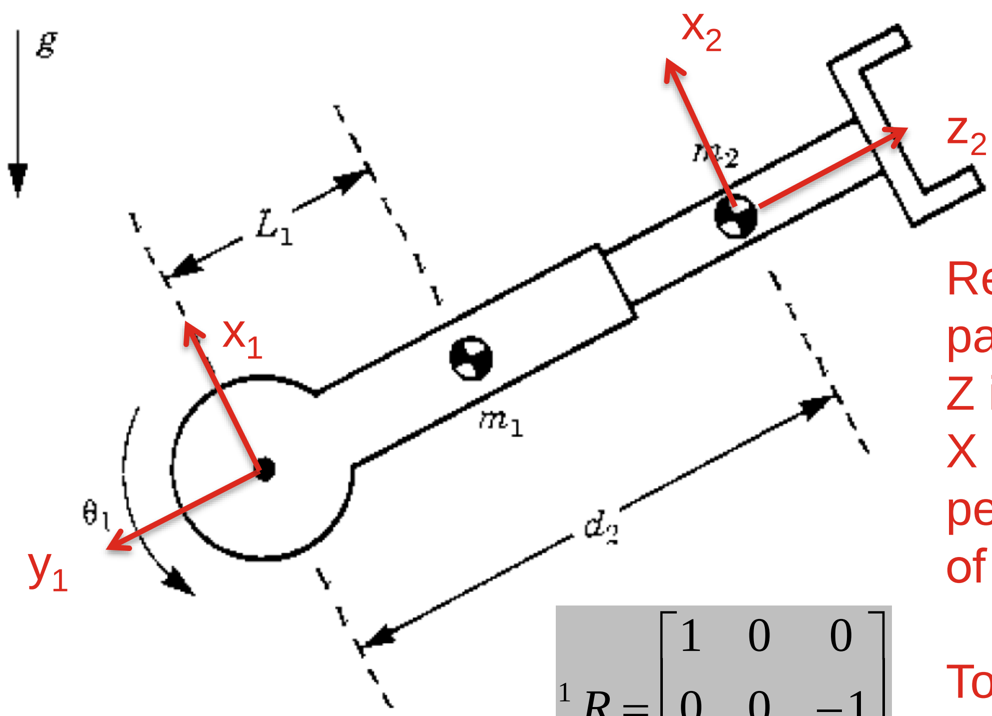
- Consider the following RP manipulator:



- Mass and dimensions are shown in the figure.

Another Example

- The frames are:



$$ {}^{1}_{2}R = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{Rotate 90deg about X}} $$

Remember DH parameters? Z is along axis, X is mutual perpendicular of two Z's.

To get from {1} to {2}, rotate 90deg along x1

Another Example

- Let its inertial tensors be:

$$^{C_1}I_1 = \begin{bmatrix} I_{xx_1} & 0 & 0 \\ 0 & I_{yy_1} & 0 \\ 0 & 0 & I_{zz_1} \end{bmatrix} \qquad ^{C_2}I_2 = \begin{bmatrix} I_{xx_2} & 0 & 0 \\ 0 & I_{yy_2} & 0 \\ 0 & 0 & I_{zz_2} \end{bmatrix}$$

- We calculate the velocities propagation as shown in Lecture 5:

$$^{0}\omega_0 = 0 \qquad ^{0}\upsilon_0 = 0$$

$$^{1}\omega_1 = {}^{1}_{0}R \cdot \underbrace{^{0}\omega_0}_{0} + \dot{\theta}_1 \cdot {}^{1}\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad ^{1}\upsilon_1 = {}^{1}_{0}R \cdot \left(\underbrace{^{0}\upsilon_0}_{0} + \underbrace{^{0}\omega_0}_{0} \times {}^{0}P_1 \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Another Example

- Assume we fix a frame $\{C_1\}$ at center of mass of link 1. Its velocity propagated from frame {1} is then:

$$^{C_1}\upsilon_{C_1} = \underbrace{^{C_1}_{1}R}_{I} \cdot \left(^1\upsilon_1 + {}^1\omega_1 \times {}^1P_{C_1}\right) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix}$$

- For link 2, we have:

$$^2\omega_2 = {}^2_1R \cdot {}^1\omega_1 = {}^1_2R^T \cdot {}^1\omega_1$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}$$

- Assume that the frame of link 2 is located at the center of mass of link 2. Thus:

$$^{C_2}\upsilon_{C_2} = {}^2\upsilon_2 = {}^2_1R \cdot \left(^1\upsilon_1 + {}^1\omega_1 \times {}^1P_{C_2}\right) + \dot{d}_2 \cdot {}^2\hat{Z}_2$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -d_2 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_2 \end{bmatrix} = \begin{bmatrix} d_2\dot{\theta}_1 \\ 0 \\ \dot{d}_2 \end{bmatrix}$$

Another Example

- The total kinetic energy is therefore:

$$k = \frac{1}{2} m_1 \upsilon_{c_1}^T \upsilon_{c_1} + \frac{1}{2} {}^1\omega_1^T \cdot {}^{C_1} I_1 \cdot {}^1\omega_1 + \frac{1}{2} m_2 \upsilon_{c_2}^T \upsilon_{c_2} + \frac{1}{2} {}^2\omega_2^T \cdot {}^{C_2} I_2 \cdot {}^2\omega_2$$

$$= \frac{1}{2} m_1 \begin{bmatrix} l_1\dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} l_1\dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}^T \begin{bmatrix} I_{xx_1} & 0 & 0 \\ 0 & I_{yy_1} & 0 \\ 0 & 0 & I_{zz_1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$+ \frac{1}{2} m_2 \begin{bmatrix} d_2\dot{\theta}_1 \\ 0 \\ \dot{d}_2 \end{bmatrix}^T \begin{bmatrix} d_2\dot{\theta}_1 \\ 0 \\ \dot{d}_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}^T \begin{bmatrix} I_{xx_2} & 0 & 0 \\ 0 & I_{yy_2} & 0 \\ 0 & 0 & I_{zz_2} \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_{zz_1} \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2 \right) + \frac{1}{2} I_{yy_2} \dot{\theta}_1^2$$

Another Example

- As for the potential energy, we have:

$$u_1 = m_1 l_1 g \sin(\theta_1) + m_1 l_1 g$$

$$u_2 = m_2 d_2 g \sin(\theta_1) + m_2 d_{2\max} g$$

  - where d_{2max} is the maximum extension of joint 2.
- The total potential energy is thus:

$$u = m_1 l_1 g \sin(\theta_1) + m_1 l_1 g + m_2 d_2 g \sin(\theta_1) + m_2 d_{2\max} g$$

Another Example

- The kinetic energy and potential energy are repeated here:

$$k = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_{zz_1} \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2 \right) + \frac{1}{2} I_{yy_2} \dot{\theta}_1^2$$

$$u = m_1 l_1 g \sin(\theta_1) + m_1 l_1 g + m_2 d_2 g \sin(\theta_1) + m_2 d_{2\max} g$$

- Applying the Lagrangian formula:

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{q}} - \frac{\partial k}{\partial q} + \frac{\partial u}{\partial q} = \tau$$

$$\frac{d}{dt}\begin{bmatrix} \frac{\partial k}{\partial \dot{q}_1} \\ \frac{\partial k}{\partial \dot{q}_2} \end{bmatrix} - \begin{bmatrix} \frac{\partial k}{\partial q_1} \\ \frac{\partial k}{\partial q_2} \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial q_1} \\ \frac{\partial u}{\partial q_2} \end{bmatrix} = \tau$$

- with $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ d_2 \end{bmatrix}$

- gives (next page)

Another Example

$$\frac{d}{dt}\begin{bmatrix} m_1 l_1^2\dot{\theta}_1 + I_{zz_1}\dot{\theta}_1 + m_2 d_2^2\dot{\theta}_1 + I_{yy_2}\dot{\theta}_1 \\ m_2\dot{d}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ m_2 d_2\dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} m_1 l_1 g c_1 + m_2 d_2 g c_1 \\ m_2 g s_1 \end{bmatrix} = \tau$$

$$\begin{bmatrix} m_1 l_1^2\ddot{\theta}_1 + I_{zz_1}\ddot{\theta}_1 + m_2 d_2^2\ddot{\theta}_1 + 2m_2 d_2\dot{d}_2\dot{\theta}_1 + I_{yy_2}\ddot{\theta}_1 \\ m_2\ddot{d}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ m_2 d_2\dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} m_1 l_1 g c_1 + m_2 d_2 g c_1 \\ m_2 g s_1 \end{bmatrix} = \tau$$

- This gives the structure:

$$\underbrace{\begin{bmatrix} m_1 l_1^2 + I_{zz_1} + m_2 d_2^2 + I_{yy_2} & 0 \\ 0 & m_2 \end{bmatrix}}_{M(q)}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2m_2 d_2\dot{d}_2\dot{\theta}_1 \\ -m_2 d_2\dot{\theta}_1^2 \end{bmatrix}}_{V(q,\dot{q})} + \underbrace{\begin{bmatrix} m_1 l_1 g c_1 + m_2 d_2 g c_1 \\ m_2 g s_1 \end{bmatrix}}_{G(q)} = \tau$$

Content

- Lagrangian Formulation
- Inclusion of Non-Rigid Body Effects
- Explicit Form

Friction

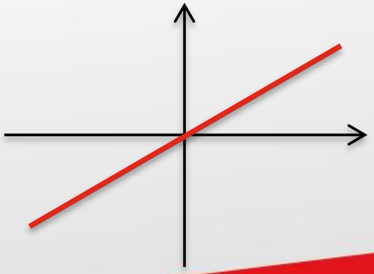
- All mechanisms are affected by friction.
- The manipulator's joint motors need to provide torque to overcome the friction, in addition to all other forces we have seen just now.
- The effect of friction to the manipulator's dynamic can be included in the dynamic equation:

$$M(q)\ddot{q}+V(q,\dot{q})+G(q)=\tau-\tau_{friction}$$

- But how do we model frictional forces?
- One simple model is the viscous friction:

$$\tau_{friction}=k\dot{q}$$

  - k is the viscous-friction constant.

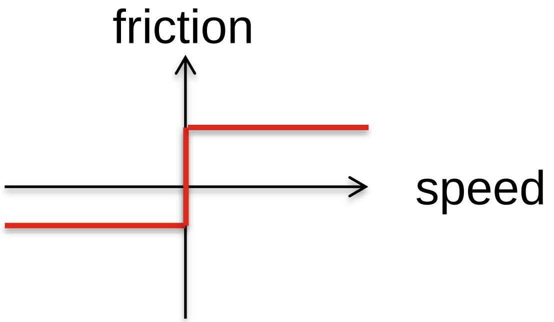


Friction

- Another simple model is the Coulomb-friction:

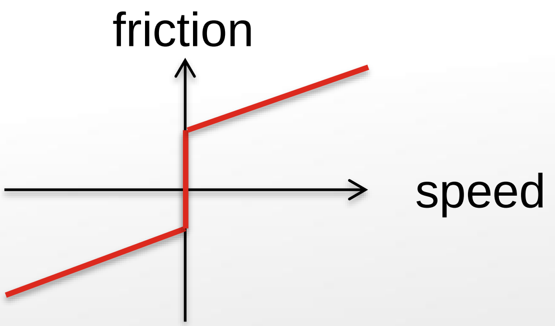
$$\tau_{friction} = c\,\mathrm{sgn}(\dot{q})$$

  - where c is the Coulomb-friction constant.



- A better model would be combination of both viscous and Coulomb friction:

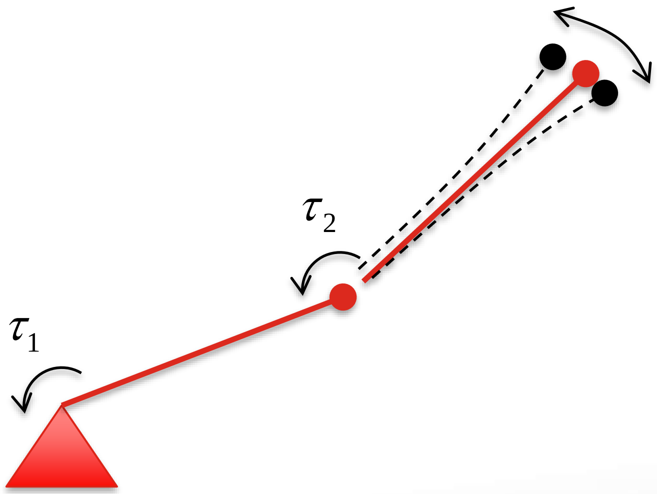
$$\tau_{friction} = c\,\mathrm{sgn}(\dot{q}) + k\dot{q}$$



- There are even more accurate models, for e.g. including Stribeck effect or joint-position-dependent friction.

Resonance Modes

- There are also bending effects and resonance in actual robots.



- However, these are very difficult to model and thus will be ignored in this course.

Content

- Lagrangian Formulation
- Inclusion of Non-Rigid Body Effects
- Explicit Form

Explicit Form

- Using Lagrangian formulation, we derived the dynamic equations by differentiating the kinetic and potential energy.
- It turns out that we can even skip calculating the energies!
  - We can write out the dynamic equations just by looking at the structure of the manipulator!
  - → Explicit form.
- Recap: The Lagrangian formulation is given by:

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{q}} - \frac{\partial k}{\partial q} + \frac{\partial u}{\partial q} = \tau$$

- Recap: The structure of manipulator's dynamic equations is:

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau$$

- We will try to find some explicit relationship between the two equations.

Explicit Form – Mass Matrix

- The total kinetic energy of the manipulator, in terms of joint velocities, is:

$$k = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

- Differentiating this in accordance to the Lagrangian formulation gives:

$$\frac{\partial k}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}\dot{q}^T M(q)\dot{q}\right) = M(q)\dot{q}$$

$$\downarrow$$

$$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}}\right) = \frac{d}{dt}\left(M(q)\dot{q}\right) = M(q)\ddot{q} + \dot{M}(q)\dot{q}$$

$$\frac{\partial k}{\partial q} = \frac{\partial}{\partial q}\left(\frac{1}{2}\dot{q}^T M(q)\dot{q}\right) = \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M(q)}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M(q)}{\partial q_n}\dot{q} \end{bmatrix}$$

- Therefore:

$$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}}\right) - \frac{\partial k}{\partial q} = M(q)\ddot{q} + \underbrace{\dot{M}(q)\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M(q)}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M(q)}{\partial q_n}\dot{q} \end{bmatrix}}_{\text{Centrifugal \& Coriolis}}$$

Explicit Form – Mass Matrix

- Thus we see that if we have the mass matrix M(q), we can immediately obtain the inertial forces, centrifugal forces and the Coriolis forces.
- But what is this M(q) matrix?
- We know that the kinetic energy of each link is:

$$k_i = \frac{1}{2} m_i \upsilon_{c_i}^T \upsilon_{c_i} + \frac{1}{2} \cdot {}^i\omega_i^T \cdot {}^{C_i}I_i \cdot {}^i\omega_i$$

  - And the total kinetic energy of the manipulator is:

$$k = \sum_{i=1}^{n} k_i$$

- The kinetic energy of the manipulator, based on the joint velocities (previous slide), and based on the sum of each link (above), must be the same.
- Thus:

$$k = \frac{1}{2} \dot{q}^T M(q) \dot{q} = \frac{1}{2} \sum_{i=1}^{n} \left(m_i \upsilon_{c_i}^T \upsilon_{c_i} + {}^i\omega_i^T \cdot {}^{C_i}I_i \cdot {}^i\omega_i \right)$$

Explicit Form – Mass Matrix

- Now, the link velocities and the joint velocities are related as:

$$\upsilon_{C_i} = J_{V_i}\dot{q}$$
$$^{i}\omega_{C_i} = {}^{i}J_{\omega_i}\dot{q}$$

  - where:

$$J_{V_i} = \begin{bmatrix} \dfrac{\partial P_{C_i}}{\partial q_1} & \cdots & \dfrac{\partial P_{C_i}}{\partial q_i} & 0 & \cdots & 0 \end{bmatrix}$$
$$^{i}J_{\omega_i} = \begin{bmatrix} \bar{\varepsilon}_1 {}^{i}Z_1 & \cdots & \bar{\varepsilon}_i {}^{i}Z_i & 0 & \cdots & 0 \end{bmatrix}$$

  - (Why do we stop at i and have zeros thereafter?)
  - (Because the center of mass of link i is affected by joints 1 to i only!)

Explicit Form – Mass Matrix

- Therefore:

$$
\begin{aligned}
\frac{1}{2}\dot{q}^T M(q)\dot{q} &= \frac{1}{2}\sum_{i=1}^{n}\left(m_i \upsilon_{c_i}^T \upsilon_{c_i} + {}^{i}\omega_i^T \cdot {}^{C_i}I_i \cdot {}^{i}\omega_i\right) \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(m_i \dot{q}_i^T J_{V_i}^T J_{V_i}\dot{q}_i + \dot{q}_i \cdot {}^{i}J_{\omega_i}^T \cdot {}^{C_i}I_i \cdot {}^{i}J_{\omega_i}\dot{q}_i\right) \\
&= \frac{1}{2}\dot{q}^T \underbrace{\left(\sum_{i=1}^{n}\left(m_i J_{V_i}^T J_{V_i} + {}^{i}J_{\omega_i}^T \cdot {}^{C_i}I_i \cdot {}^{i}J_{\omega_i}\right)\right)}_{M(q)}\dot{q}
\end{aligned}
$$

Explicit Form – Mass Matrix

- Summary:
  - Calculate M(q) using Jacobians, mass and inertia tensor of each link:

$$M(q) = \sum_{i=1}^{n} \left(m_i J_{V_i}^T J_{V_i} + {}^{i}J_{\omega_i}^T \cdot {}^{C_i}I_i \cdot {}^{i}J_{\omega_i} \right)$$

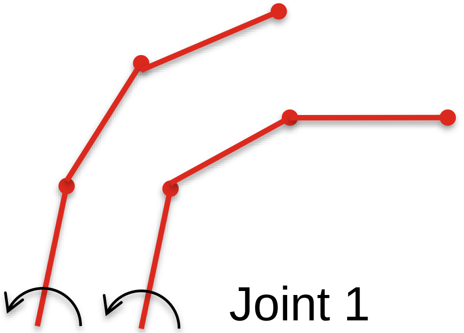
  - The inertial, centrifugal and Coriolis forces can directly be calculated as:

$$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}}\right) - \frac{\partial k}{\partial q} = M(q)\ddot{q} + \dot{M}(q)\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \dfrac{\partial M(q)}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \dfrac{\partial M(q)}{\partial q_n}\dot{q} \end{bmatrix}$$

Side Notes – M(q) Matrix

- Let's try to understand the M(q) matrix more.
- M(q) is an n x n matrix:

$$M(q) = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$



- m_{11} = "perceived inertia" at joint 1, when all other joints are locked. It is a function of q_2 to q_n.
- m_{22} = "perceived inertia" at joint 2, when all other joints are locked. It is a function of q_3 to q_n.
- ……
- $m_{(n-1)(n-1)}$ = "perceived inertia" at joint n-1, when all other joints are locked. It is a function of q_n.
- m_{nn} = "perceived inertia" at joint n, when all other joints are locked. It is a constant!

Side Notes – M(q) Matrix

- M(q) is positive definite:
  - Kinetic energy, $k = \frac{1}{2}\dot{q}^T M(q)\dot{q}$ is always greater or equal to zero.
    - Equals to zero only when velocity is zero.
    - Object cannot have zero mass.
- M(q) is symmetrical.
  - $m_{12} = m_{21}$, and so on.
  - The off-diagonal terms represent couplings between links.

Centrifugal & Coriolis

- We already know:

$$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}}\right)-\frac{\partial k}{\partial q}=\underbrace{M(q)\ddot{q}}_{\text{Inertial forces}}+\underbrace{\dot{M}(q)\dot{q}-\frac{1}{2}\begin{bmatrix}\dot{q}^T\dfrac{\partial M(q)}{\partial q_1}\dot{q}\\ \vdots \\ \dot{q}^T\dfrac{\partial M(q)}{\partial q_n}\dot{q}\end{bmatrix}}_{\text{Centrifugal \& Coriolis}}$$

- Let's look at the Centrifugal and Coriolis forces closer.
- For simplicity, we consider a two-link robot first, and make generalization later.
- M(q) is therefore 2 x 2:

$$M(q)=\begin{bmatrix}m_{11}(q) & m_{12}(q)\\ m_{12}(q) & m_{22}(q)\end{bmatrix}$$

Centrifugal & Coriolis

- Hence:

$$V(q,\dot{q}) = \dot{M}(q)\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \dfrac{\partial M(q)}{\partial q_1}\dot{q} \\ \dot{q}^T \dfrac{\partial M(q)}{\partial q_2}\dot{q} \end{bmatrix}$$

$$m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$$

$$\text{e.g. } m_{121} = \frac{\partial m_{12}}{\partial q_1}$$

$$= \begin{bmatrix} \dot{m}_{11}(q) & \dot{m}_{12}(q) \\ \dot{m}_{12}(q) & \dot{m}_{22}(q) \end{bmatrix}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix}\dot{q} \\ \dot{q}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix}\dot{q} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial m_{11}}{\partial q_1}\dot{q}_1 + \dfrac{\partial m_{11}}{\partial q_2}\dot{q}_2 & \dfrac{\partial m_{12}}{\partial q_1}\dot{q}_1 + \dfrac{\partial m_{12}}{\partial q_2}\dot{q}_2 \\ \dfrac{\partial m_{12}}{\partial q_1}\dot{q}_1 + \dfrac{\partial m_{12}}{\partial q_2}\dot{q}_2 & \dfrac{\partial m_{22}}{\partial q_1}\dot{q}_1 + \dfrac{\partial m_{22}}{\partial q_2}\dot{q}_2 \end{bmatrix}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix}\dot{q} \\ \dot{q}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix}\dot{q} \end{bmatrix}$$

$$= \begin{bmatrix} m_{111}\dot{q}_1 + m_{112}\dot{q}_2 & m_{121}\dot{q}_1 + m_{122}\dot{q}_2 \\ m_{121}\dot{q}_1 + m_{122}\dot{q}_2 & m_{221}\dot{q}_1 + m_{222}\dot{q}_2 \end{bmatrix}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix}\dot{q} \\ \dot{q}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix}\dot{q} \end{bmatrix}$$

Centrifugal & Coriolis

- Expand and simplify gives:

$$V(q,\dot{q}) = \begin{bmatrix} m_{111}\dot{q}_1 + m_{112}\dot{q}_2 & m_{121}\dot{q}_1 + m_{122}\dot{q}_2 \\ m_{121}\dot{q}_1 + m_{122}\dot{q}_2 & m_{221}\dot{q}_1 + m_{222}\dot{q}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \dot{q}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix} \dot{q} \\ \dot{q}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix} \dot{q} \end{bmatrix}$$

$$= \begin{bmatrix} m_{111}\dot{q}_1^2 + m_{112}\dot{q}_1\dot{q}_2 + m_{121}\dot{q}_1\dot{q}_2 + m_{122}\dot{q}_2^2 \\ m_{121}\dot{q}_1^2 + m_{122}\dot{q}_1\dot{q}_2 + m_{221}\dot{q}_1\dot{q}_2 + m_{222}\dot{q}_2^2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} m_{111}\dot{q}_1^2 + m_{121}\dot{q}_1\dot{q}_2 + m_{121}\dot{q}_1\dot{q}_2 + m_{221}\dot{q}_2^2 \\ m_{112}\dot{q}_1^2 + m_{122}\dot{q}_1\dot{q}_2 + m_{122}\dot{q}_1\dot{q}_2 + m_{222}\dot{q}_2^2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} m_{111}\dot{q}_1^2 + (2m_{122} - m_{221})\dot{q}_2^2 \\ (2m_{121} - m_{112})\dot{q}_1^2 + m_{222}\dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112}\dot{q}_1\dot{q}_2 \\ m_{221}\dot{q}_1\dot{q}_2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} m_{111} & 2m_{122} - m_{221} \\ 2m_{121} - m_{112} & m_{222} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112}\dot{q}_1\dot{q}_2 \\ m_{221}\dot{q}_1\dot{q}_2 \end{bmatrix}$$

Centrifugal & Coriolis

- The equation can be written in a special form:

$$V(q,\dot{q}) = \frac{1}{2}\begin{bmatrix} m_{111} & 2m_{122}-m_{221} \\ 2m_{121}-m_{112} & m_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112}\dot{q}_1\dot{q}_2 \\ m_{221}\dot{q}_1\dot{q}_2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} m_{111}+m_{111}-m_{111} & m_{122}+m_{122}-m_{221} \\ m_{121}+m_{121}-m_{112} & m_{222}+m_{222}-m_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112}+m_{121}-m_{121} \\ m_{212}+m_{221}-m_{122} \end{bmatrix}\dot{q}_1\dot{q}_2$$

This will cancel off because $m_{12} = m_{21}$

- Introducing the "Christoffel Symbols":

$$b_{ijk} = \frac{1}{2}\left(m_{ijk} + m_{ikj} - m_{jki}\right)$$

Centrifugal & Coriolis

- Using the Christoffel Symbols, the V matrix can now be written as:

$$V(q,\dot{q}) = \frac{1}{2}\begin{bmatrix} m_{111}+m_{111}-m_{111} & m_{122}+m_{122}-m_{221} \\ m_{121}+m_{121}-m_{112} & m_{222}+m_{222}-m_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112}+m_{121}-m_{121} \\ m_{212}+m_{221}-m_{122} \end{bmatrix}\dot{q}_1\dot{q}_2$$

$$= \underbrace{\begin{bmatrix} b_{111} & b_{122} \\ b_{121} & b_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}}_{\text{C, Centrifugal}} + \underbrace{\begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix}}_{\text{B, Coriolis}}\dot{q}_1\dot{q}_2$$

Centrifugal & Coriolis

- We can finally generalize the results to:

$$\underbrace{C(q)}_{n\times n}\underbrace{\left(\dot{q}^2\right)}_{n\times 1}=\begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

$$\underbrace{B(q)}_{n\times\frac{(n-1)n}{2}}\underbrace{\left(\dot{q}\dot{q}\right)}_{\frac{(n-1)n}{2}\times 1}=\begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \ddots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix}\begin{bmatrix} \dot{q}_1\dot{q}_2 \\ \dot{q}_1\dot{q}_3 \\ \vdots \\ \dot{q}_{n-1}\dot{q}_n \end{bmatrix}$$

Centrifugal & Coriolis

- Another summary:
  - Calculate M(q) using Jacobians, mass and inertia tensor of each link:

$$M(q) = \sum_{i=1}^{n} \left(m_i J_{V_i}^T J_{V_i} + {}^i J_{\omega_i}^T \cdot {}^{C_i} I_i \cdot {}^i J_{\omega_i} \right)$$

  - The inertial, centrifugal and Coriolis forces can directly be calculated as:

$$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}}\right) - \frac{\partial k}{\partial q} = M(q)\ddot{q} + \dot{M}(q)\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \dfrac{\partial M(q)}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \dfrac{\partial M(q)}{\partial q_n}\dot{q} \end{bmatrix}$$

  - Use the expressions for B and C on previous slide to get the last two terms.

Explicit Form – Gravity Terms

- So far, we already have the explicit forms for the terms related to kinetic energy.
- What can we say about the potential energy?
- Potential energy of each link was: $u_i = -m_i \cdot {}^0 g^T \cdot {}^0 P_{C_i} + u_{ref_i}$
- The gravity terms in the Lagrangian formulation is obtained by: $\dfrac{\partial u}{\partial q}$
- Hence for each link, the gravity term is:

$$G_j = \frac{\partial u}{\partial q_j} = -\sum_{i=1}^{n} \left(m_i \cdot {}^0 g^T \cdot \frac{\partial {}^0 P_{C_i}}{\partial q_j} \right)$$

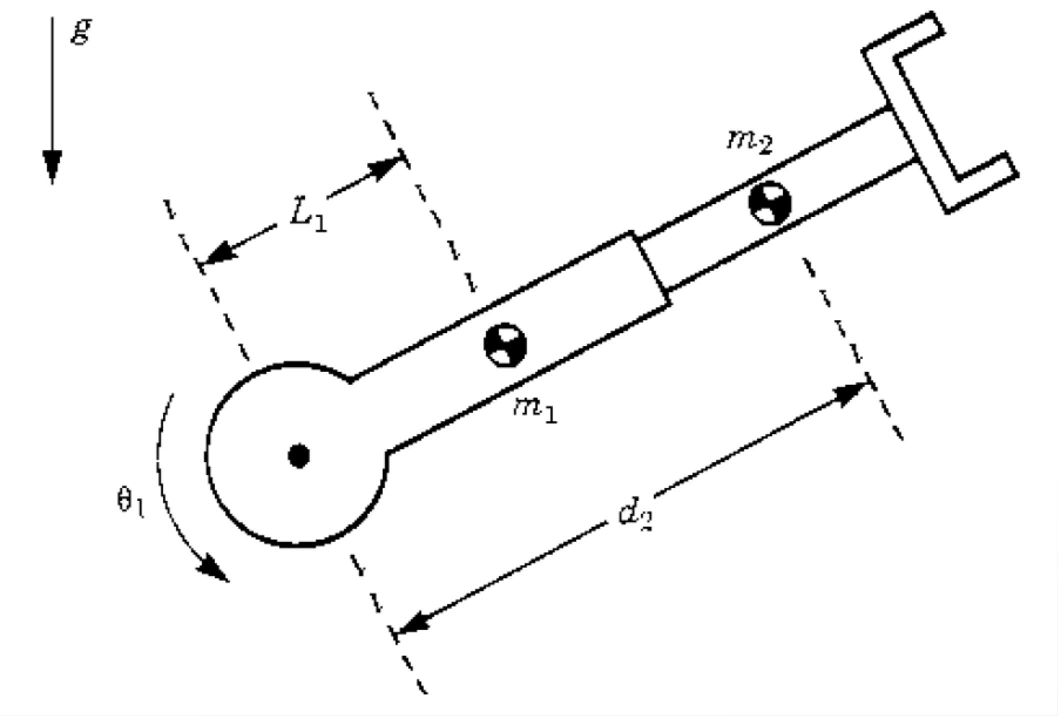
J_{vi}

- Thus we have:

$$G = -\begin{bmatrix} J_{V1}^T & J_{V2}^T & \cdots & J_{Vn}^T \end{bmatrix} \begin{bmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{bmatrix}$$

Example

- Consider the same RP manipulator:



- Mass and dimensions are shown in the figure.

Example

- First, we need to obtain the mass matrix M(q) from:

$$M(q)=\sum_{i=1}^{n}\left(m_i J_{V_i}^T J_{V_i} + {}^i J_{\omega_i}^T \cdot {}^{C_i} I_i \cdot {}^i J_{\omega_i}\right)$$

- To do this, we need the Jacobians.
- The positions of the centres of mass are:

$$ {}^0P_{C_1} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix} \qquad {}^0P_{C_2} = \begin{bmatrix} d_2 c_1 \\ d_2 s_1 \\ 0 \end{bmatrix} $$

- Therefore:

$$ {}^0J_{V_1} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad {}^0J_{V_2} = \begin{bmatrix} -d_2 s_1 & c_1 \\ d_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix} $$

(Frame doesn't matter for linear kinetic energy)

Up to i=1 Add zero column

Example

- This yields:

$$m_1 J_{V_1}^T J_{V_1} = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$m_2 J_{V_2}^T J_{V_2} = \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$

- As for the rotational terms, we have:

$${}^1J_{\omega_1} = \begin{bmatrix} \bar{\varepsilon}_1 {}^1Z_1 & 0 \end{bmatrix} \qquad {}^2J_{\omega_2} = \begin{bmatrix} \bar{\varepsilon}_1 {}^2Z_1 & \bar{\varepsilon}_2 {}^2Z_2 \end{bmatrix}$$

Z_1 in {2}
= y-axis of {2}

- For joint 1, the Jacobian is:

$${}^1J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$ since $${}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$$

- For joint 2, the Jacobian is:

$${}^2J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$ since $${}^2\omega_2 = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$$

Example

- Therefore:

$$^{1}J_{\omega_1}^{T} \cdot ^{C_1} I_1 \cdot ^{1} J_{\omega_1} = \begin{bmatrix} I_{zz_1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$^{2}J_{\omega_2}^{T} \cdot ^{C_2} I_2 \cdot ^{2} J_{\omega_2} = \begin{bmatrix} I_{yy_2} & 0 \\ 0 & 0 \end{bmatrix}$$

- Finally, the mass matrix is:

$$M(q) = \sum_{i=1}^{n} \left(m_i J_{V_i}^{T} J_{V_i} + ^{i} J_{\omega_i}^{T} \cdot ^{C_i} I_i \cdot ^{i} J_{\omega_i} \right)$$
$$= \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} I_{zz_1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I_{yy_2} & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} m_1 l_1^2 + I_{zz_1} + m_2 d_2^2 + I_{yy_2} & 0 \\ 0 & m_2 \end{bmatrix}$$

Example

- Next, we will use the Christoffel Symbols to calculate the centrifugal and Coriolis forces:

$$M(q) = \begin{bmatrix} m_1 l_1^2 + I_{zz_1} + m_2 d_2^2 + I_{yy_2} & 0 \\ 0 & m_2 \end{bmatrix}$$

$$b_{ijk} = \frac{1}{2}\left(m_{ijk} + m_{ikj} - m_{jki}\right)$$

$$\underbrace{C(q)}_{n \times n}\underbrace{\left(\dot{q}^2\right)}_{n \times 1} = \begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

$$\underbrace{B(q)}_{n \times \frac{(n-1)n}{2}} \underbrace{\left(\dot{q}\dot{q}\right)}_{\frac{(n-1)n}{2} \times 1} = \begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \ddots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \dot{q}_2 \\ \dot{q}_1 \dot{q}_3 \\ \vdots \\ \dot{q}_{n-1} \dot{q}_n \end{bmatrix}$$

Example

- This gives:

$$\underbrace{C(q)}_{n\times n}\underbrace{(\dot{q}^2)}_{n\times 1}=\begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix} m_{111}+m_{111}-m_{111} & m_{122}+m_{122}-m_{221} \\ m_{211}+m_{211}-m_{112} & m_{222}+m_{222}-m_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

Joint 2 only sees centrifugal force

$$\underbrace{B(q)}_{n\times\frac{(n-1)n}{2}}\ \underbrace{(\dot{q}\dot{q})}_{\frac{(n-1)n}{2}\times 1}=\begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \ddots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix}\begin{bmatrix} \dot{q}_1\dot{q}_2 \\ \dot{q}_1\dot{q}_3 \\ \vdots \\ \dot{q}_{n-1}\dot{q}_n \end{bmatrix}$$

$$=\begin{bmatrix} m_{112}+m_{121}-m_{121} \\ m_{212}+m_{221}-m_{122} \end{bmatrix}\dot{q}_1\dot{q}_2=\begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}\dot{q}_1\dot{q}_2$$

Joint 1 only sees Coriolis force

Example

- Finally, we shall derive the gravity term.
- We use the formula:

$$G = -\begin{bmatrix} J_{V1}^T & J_{V2}^T & \cdots & J_{Vn}^T \end{bmatrix} \begin{bmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{bmatrix} = -J_{V1}^T m_1 g - J_{V2}^T m_2 g$$

- In frame {0}. the gravity vector is: $^0 g = \begin{bmatrix} 0 & -g & 0 \end{bmatrix}^T$

- Therefore:

$$\begin{aligned} G &= -J_{V1}^T m_1 g - J_{V2}^T m_2 g \\ &= -m_1 \begin{bmatrix} -l_1 s_1 & l_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} - m_2 \begin{bmatrix} -d_2 s_1 & d_2 c_1 & 0 \\ c_1 & s_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} m_1 g l_1 c_1 + m_2 g d_2 c_1 \\ m_1 g s_1 \end{bmatrix} \end{aligned}$$

Example

- Combining all the results (for M, C, B and G), we arrive at:

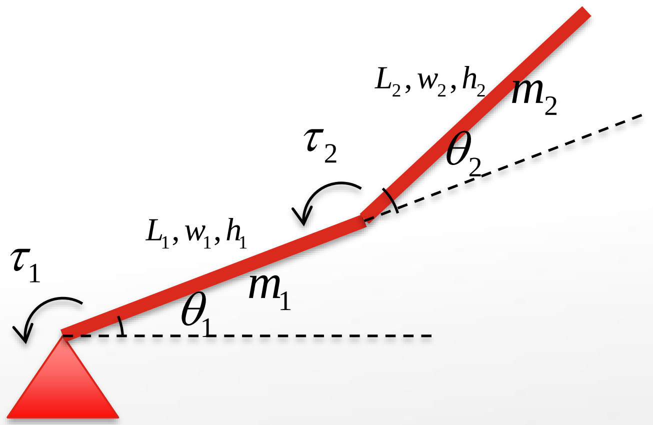
$$\underbrace{\begin{bmatrix} m_1 l_1^2 + I_{zz_1} + m_2 d_2^2 + I_{yy_2} & 0 \\ 0 & m_2 \end{bmatrix}}_{M(q)} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}}_{C(q)\dot{q}^2} + \underbrace{\begin{bmatrix} 2 m_2 d_2 \dot{d}_2 \dot{\theta}_1 \\ 0 \end{bmatrix}}_{B(q)(\dot{q}\dot{q})} + \underbrace{\begin{bmatrix} m_1 l_1 g c_1 + m_2 d_2 g c_1 \\ m_2 g s_1 \end{bmatrix}}_{G(q)} = \tau$$

- which is exactly the same as what we had from differentiation of Lagrangian:

$$\underbrace{\begin{bmatrix} m_1 l_1^2 + I_{zz_1} + m_2 d_2^2 + I_{yy_2} & 0 \\ 0 & m_2 \end{bmatrix}}_{M(q)} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 m_2 d_2 \dot{d}_2 \dot{\theta}_1 \\ -m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}}_{V(q,\dot{q})} + \underbrace{\begin{bmatrix} m_1 l_1 g c_1 + m_2 d_2 g c_1 \\ m_2 g s_1 \end{bmatrix}}_{G(q)} = \tau$$

Tutorial Assignments

- **Question 1:**
  - The following two-link robot has each link as a rectangular solid of homogenous density.
  - Each link has dimension l_i, w_i, h_i, and a total mass of m_i.



  - Derive the dynamic equations using Lagrangian method.

Tutorial Assignments

- **Question 2:**
  - For the same robot in Question 1:
  - (a) Write the dynamic equation, when each joint is subject to viscous and coulomb friction.
  - (b) Calculate the dynamic model in Cartesian space.

Tutorial Assignments

- **Question 3:**
  - For the same robot in Question 1:
  - Derive the dynamic equation using the Explicit method.

Thank you!

Have a good evening.