# Week 6 – Manipulator Dynamics

## Advanced Robotic Systems – MANU2453

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# Lectures

| Wk | Date | Lecture (NOTE: video recording) | Maths Difficulty | Hands-on Activity | Related Assessment |
|---|---|---|---|---|---|
| 1 | 24/7 | • Introduction to the Course<br>• Spatial Descriptions & Transformations | | | |
| 2 | 31/7 | • Spatial Descriptions & Transformations<br>• Robot Cell Design | | | Robot Cell Design Assignment |
| 3 | 7/8 | • Forward Kinematics<br>• Inverse Kinematics | | | |
| 4 | 14/8 | • ABB Robot Programming via Teaching Pendant<br>• ABB RobotStudio Offline Programming | | ABB RobotStudio Offline Programming | Offline Programming Assignment |
| 5 | 21/8 | • Jacobians: Velocities and Static Forces | | | |
| 6 | 28/8 | • Manipulator Dynamics | | | |
| 7 | 11/9 | • Manipulator Dynamics | | MATLAB Simulink Simulation | |
| 8 | 18/9 | • Robotic Vision | | MATLAB Simulation | Robotic Vision Assignment |
| 9 | 25/9 | • Robotic Vision | | MATLAB Simulation | |
| 10 | 2/10 | • Trajectory Generation | | | |
| 11 | 9/10 | • Linear & Nonlinear Control | | MATLAB Simulink Simulation | |
| 12 | 16/10 | • Introduction to I4.0<br>• Revision | | | Final Exam |

# Content

- Introduction & Structure of Manipulator’s Dynamic Equations
- Mass Distribution
- Newton-Euler Formulation

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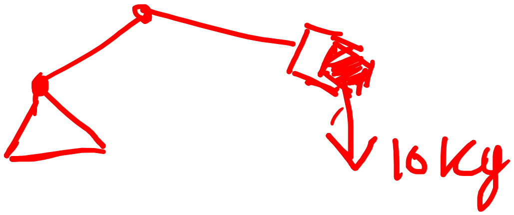
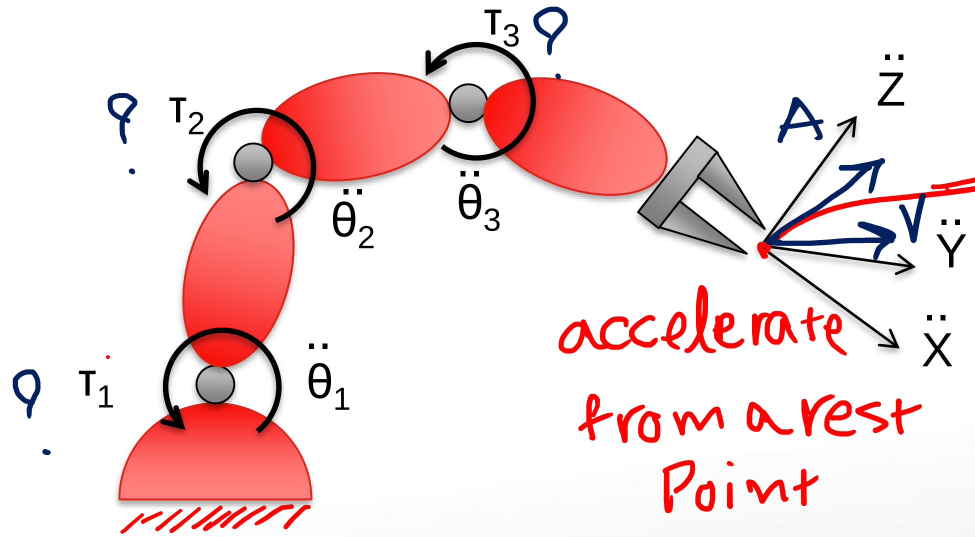
# Introduction

Static Forces
Hold an object




- Manipulator Dynamics:
  - The study of forces which cause motion.




Path

decelerate to a Stop Pos.

a Const vel.

accelerate from a rest Point

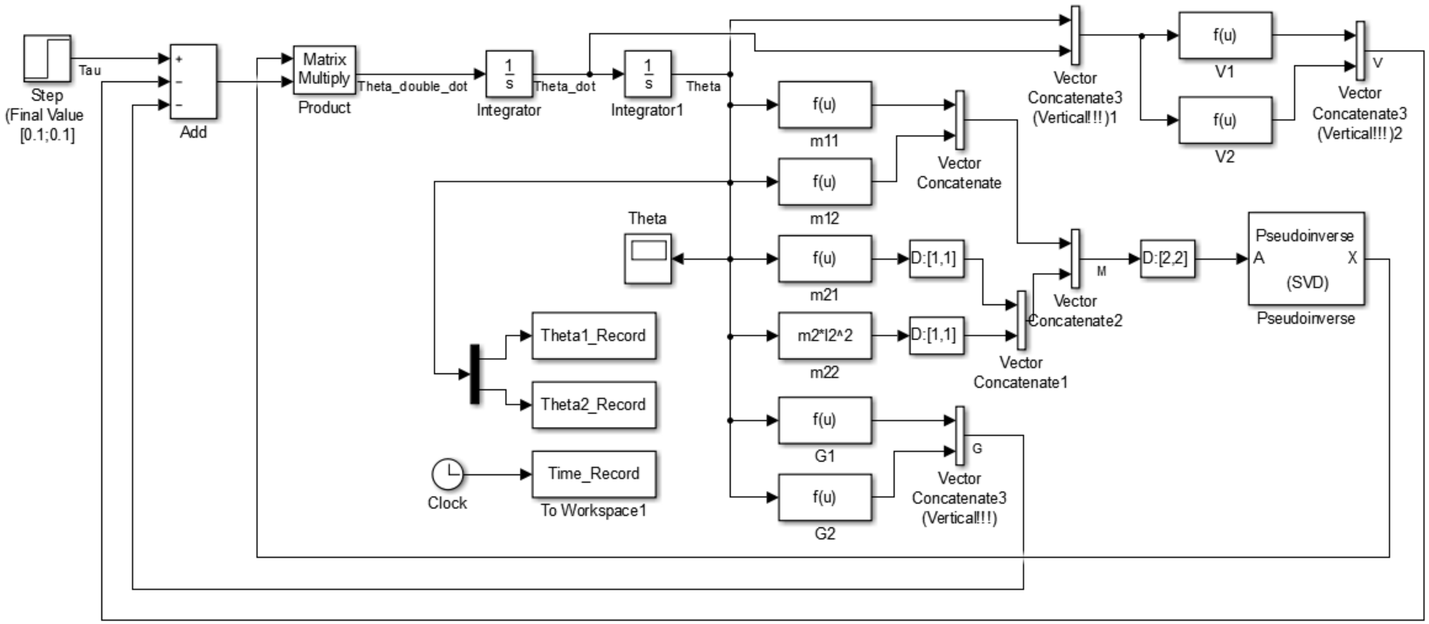
  - How much torque is needed to accelerate the manipulator from rest to constant velocity, and then back to stop?

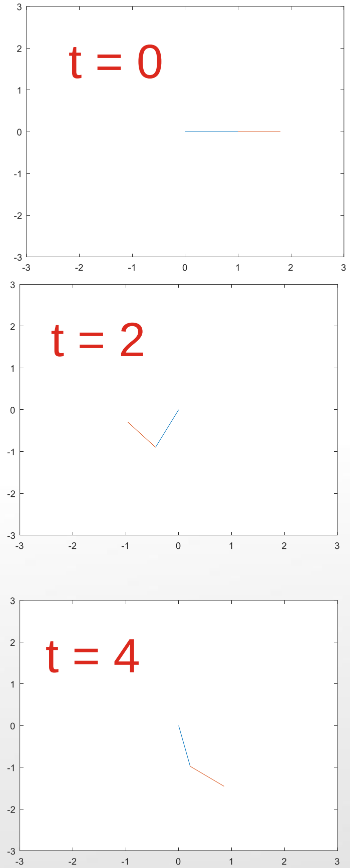
$\tau$ ← The dynamic Eqs of Motion

→ to move along a desire path

# Introduction

- Dynamics also provide us a model (equations of motions) for simulation and control design purpose.




- You will learn how to create a Simulink simulation in week 7.

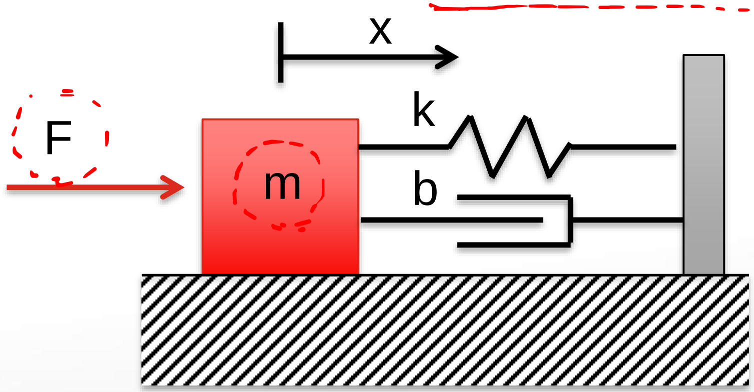
# Manipulator's Dynamic Equations

- Before we go into details of how to derive the manipulator's joint space dynamic equations, let's first have a glimpse of how the equations look like:

$$M(q)\ddot{q}+V(q,\dot{q})+G(q)=\tau$$

$q : \theta, d$

- A comparison with the well-known mass-spring-damper system:



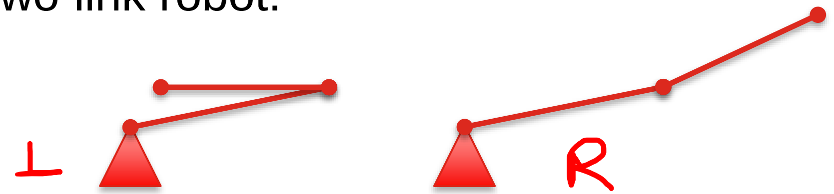

$\sum F = ma \rightarrow$

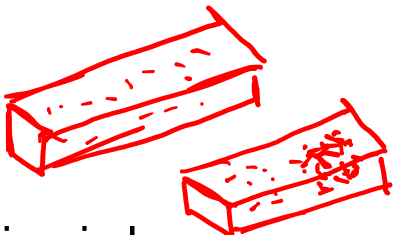
$$m\ddot{x}+b\dot{x}+kx=F$$

- They look somewhat similar.

# Manipulator’s Dynamic Equations

- $M(q)$ is the n x n mass matrix of the manipulator, which depends on the generalized joint coordinates q (angles / displacement).
  - For e.g. two link robot:



  - The “perceived inertia” at joint 1 of the right configuration is larger than that of the left configuration.
  - The “perceived inertia” also depends on the mass distribution and length of the links.
  - $M(q)$ is also called the Kinetic Energy Matrix since Kinetic Energy is

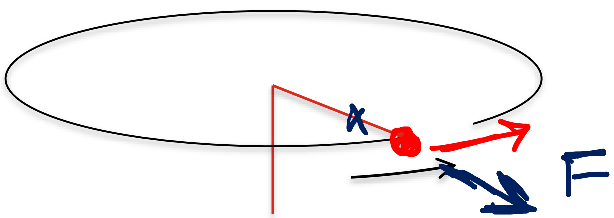
$$K = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

$\dot{m}$ $\frac{1}{2}mv^2$

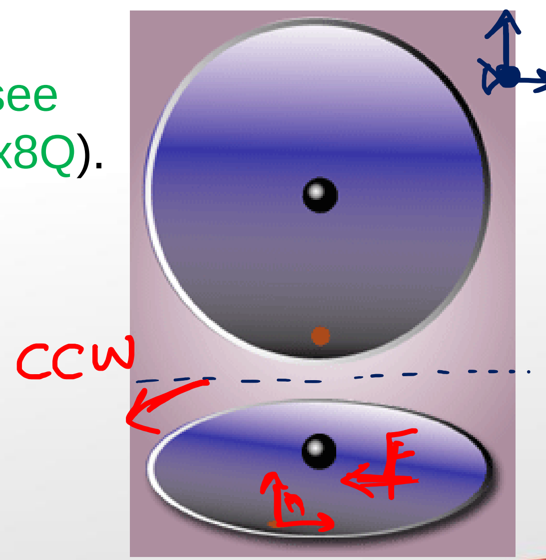
# Manipulator's Dynamic Equations

- $V(q,\dot{q})$ is an n x 1 vector of centrifugal and Coriolis forces.

A 'fictitious' force acting away from axis of rotation. E.g. whirling a stone on a string



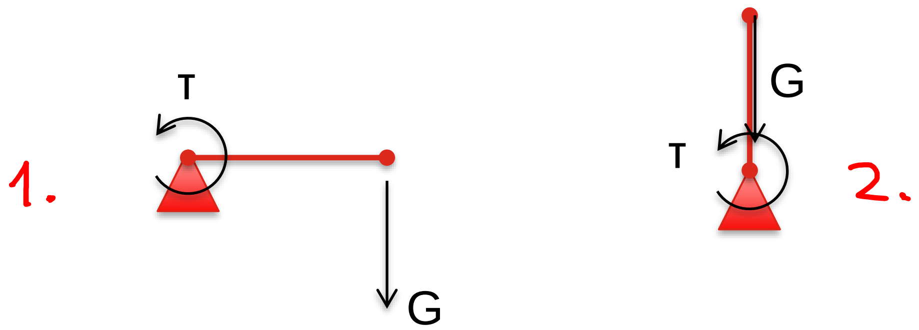

A fictitious force acting on an object that are in motion relative to a rotating reference frame.
In a reference frame with clockwise rotation, the force acts to the left of the motion of the object
(For more details please see https://youtu.be/7TjOy56-x8Q).




- $V(q,\dot{q})$ depends on the generalized joint coordinates q as well as the joint velocities q-dot.
  - It is zero if velocities = 0.
- Also, $V(q,\dot{q})$ can be derived from $M(q)$.
  - It is also zero if M(q) is a constant matrix.

# Manipulator's Dynamic Equations

- $G(q)$ is the n x 1 vector of gravity terms.
  - It is dependent on the joint coordinates / configuration of the robot.




  - In the left figure, the joint torque is nonzero, and in the right figure, the joint torque is zero.

Prismatic Revolute

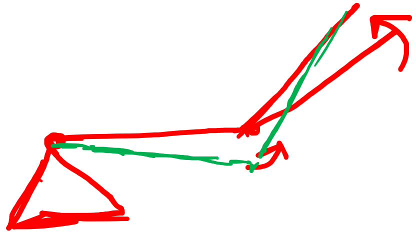
- Finally, $\tau$ is the generalized forces (force or torque) at each joints.

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau$$

$q : \theta , d$

# Manipulator’s Dynamic Equations

- One thing to note is that the dynamic equations show that the links have cross-coupling effects onto one another.
- E.g. 2-link robot: $M(q)\ddot{q}+V(q,\dot{q})+G(q)=\tau$



$$\begin{bmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22}(q) \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}+\begin{bmatrix} V_1(q,\dot{q}) \\ V_2(q,\dot{q}) \end{bmatrix}+\begin{bmatrix} G_1(q) \\ G_2(q) \end{bmatrix}=\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

- Even if $\tau_2 = 0$, there will be an acceleration for $q_2$ because it is affected by $q_1$, which is created by $\tau_1$.
- On the other hand, even if $\tau_1 = 0$, there will be an acceleration for $q_1$ because it is affected by $q_2$, which is created by $\tau_2$.
- These cross coupling are caused by the off-diagonal terms ($m_{12}$, $m_{21}$) in the mass matrix.

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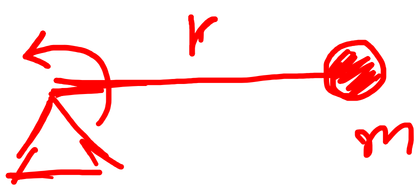
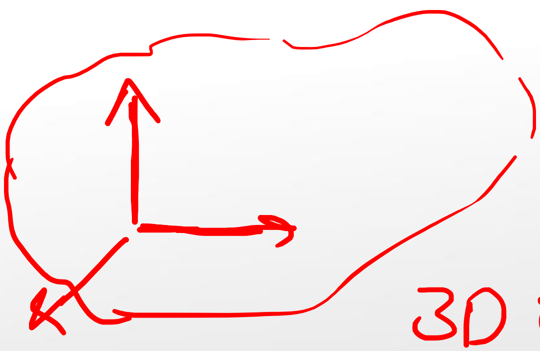
$$F = ma$$

$$\tau = I\alpha$$

Inertia




$$I = mr^2$$

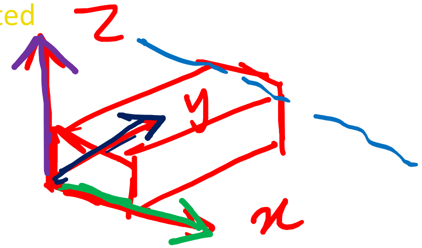


3D space

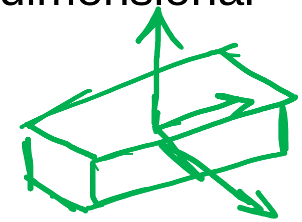
# Mass Distribution

- We are all familiar with Newton’s Law: $F = ma$
  - The acceleration (a) is proportional to force (F) divide by mass (m).
  - If mass is small, then the acceleration is huge.
  - And if the mass is large, then the acceleration is small.
  - The mass presents a “resistance” to the linear motion.
- For the case of rotational motion about a single axis, we have: $\tau = I\alpha$
  - where τ is the torque, I is the moment of inertia, and α is the angular acceleration.
  - The moment of inertia is similar to the mass.
  - It presents a “resistance” to the rotary motion.
- To study the dynamics of the robot, we thus need both the mass/inertia and the acceleration.
  - Let’s start with discussion on mass/inertia first.

# Mass Distribution




- For the case of a rigid body which is free to move in three dimensional space, there are infinitely many possible rotation axis.
- We need a generalization of the moment of inertia.
  - Inertia tensor will be used for this purpose.



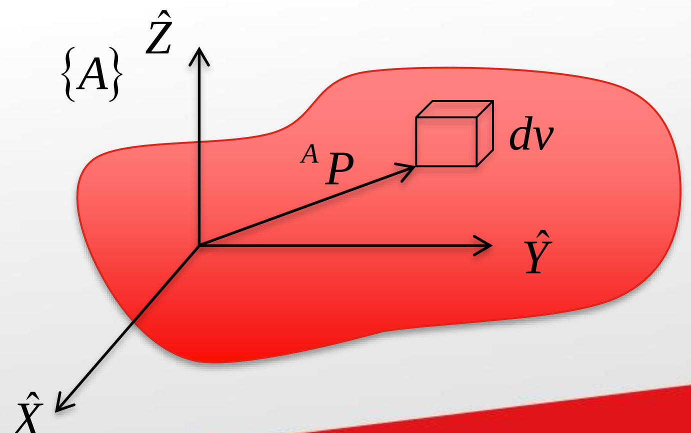
Matrix →

$$
{}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}
$$

3x3

→ shows symmetry

  - It characterizes the mass distribution of a rigid body, wrt to the reference frame (here {A}).




dv is the differential volume element

# Mass Distribution

- The elements of the inertial tensor are:

$$I_{xx} = \iiint_V \left(y^2 + z^2\right)\rho dv$$

$$I_{yy} = \iiint_V \left(x^2 + z^2\right)\rho dv$$

$$I_{zz} = \iiint_V \left(y^2 + y^2\right)\rho dv$$

Mass moment of inertia

$$I_{xy} = \iiint_V xy\rho dv$$

$$I_{xz} = \iiint_V xz\rho dv$$

$$I_{yz} = \iiint_V yz\rho dv$$

Mass products of inertia

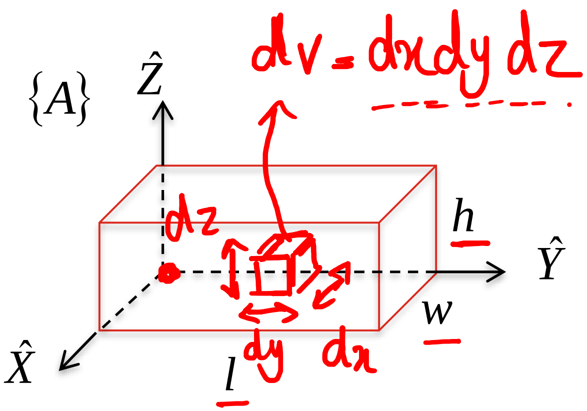
Symmetry

- These elements depend on the position and orientation of the frame.
  - If the frame is at a 'special' orientation, the products of inertia can be zero.
  - In this case, the axes of the frame are called "principal axes", and the moments of inertia are called "principal moments of inertia".

# Example

$m$

- A rectangular body has uniform density ρ.
- If the frame is attached to one corner as shown, what is the inertia tensor?




small element

- Solution:

$$I_{xx} = \iiint_V (y^2+z^2)\rho \, dv = \int_0^h \int_0^l \int_0^w (y^2+z^2)\rho \cdot dx \cdot dy \cdot dz$$

$$= \int_0^h \int_0^l (y^2+z^2) w\rho \cdot dy \cdot dz = \int_0^h \left(\frac{l^3}{3} + z^2 l\right) w\rho \cdot dz$$

$$= \left(\frac{l^3}{3}h + \frac{h^3}{3}l\right) w\rho = \left(\frac{l^2}{3}hl + \frac{h^2}{3}hl\right) w\rho = \left(\frac{l^2}{3} + \frac{h^2}{3}\right) hlw\rho$$

$$= \left(\frac{l^2}{3} + \frac{h^2}{3}\right) V\rho = \frac{m}{3}\left(l^2 + h^2\right)$$

total mass

$$(y^2+z^2)x\rho \Big|_0^w$$

$$(w^2+z^2)w\rho - 0$$

$$\left(\frac{y^3}{3} + z^2 y\right)w\rho \Big|_0^l$$

$$\left(\frac{l^3}{3}z + \frac{z^3}{3}l\right)w\rho \Big|_0^h$$

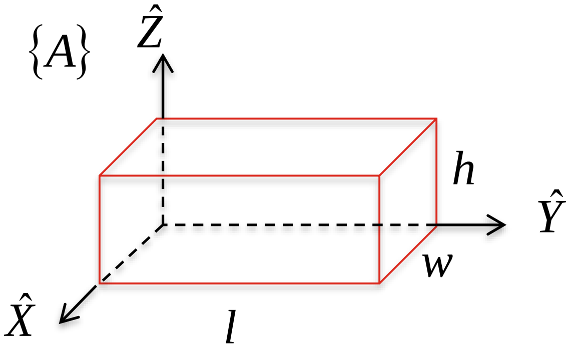
# Example

$I_{xx} = \frac{m}{3}(l_y^2 + h_z^2)$

- Similarly, we can get

$$I_{yy} = \frac{m}{3}\left(w^2 + h^2\right)$$

$$I_{zz} = \frac{m}{3}\left(w^2 + l^2\right)$$




- Next:

$$I_{xy} = \iiint_V xy\rho dv = \left[\int_0^h \left[\int_0^l \left[\int_0^w xy\rho \cdot dx\right] \cdot dy\right] \cdot dz\right]$$

$$= \int_0^h \int_0^l \frac{1}{2} w^2 y\rho \cdot dy \cdot dz = \int_0^h \frac{1}{4} w^2 l^2 \rho \cdot dz$$

$$= \frac{1}{4} w^2 l^2 h\rho = \frac{1}{4} wl \cdot \underbrace{wlh}_{\text{total volume}}\rho$$

$$= \frac{1}{4} wl \cdot \underbrace{V\rho}_{\text{total mass}} = \frac{m}{4} wl$$

total volume

total mass

$m$

$\frac{x^2}{2} y\rho \Big|_0^w$

$\frac{w^2}{2} y\rho$

$\frac{1}{2} w^2 \times \frac{y^2}{2}\rho \Big|_0^l$

$\frac{1}{4} w^2 l^2 \rho$

- Similarly, we have

$$I_{xz} = \frac{m}{4} wh$$

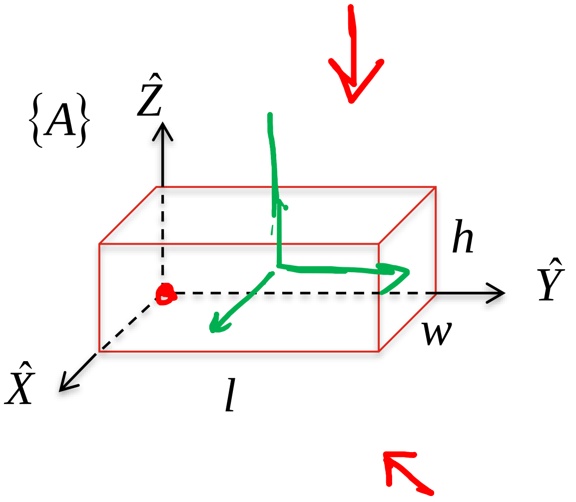
$$I_{yz} = \frac{m}{4} lh$$

# Example

- In summary, the inertia tensor is:

$$
{}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}
= \begin{bmatrix} \frac{m}{3}(l^2+h^2) & -\frac{m}{4}wl & -\frac{m}{4}hw \\ -\frac{m}{4}wl & \frac{m}{3}(w^2+h^2) & -\frac{m}{4}hl \\ -\frac{m}{4}hw & -\frac{m}{4}hl & \frac{m}{3}(w^2+l^2) \end{bmatrix}
$$

$\neq 0$



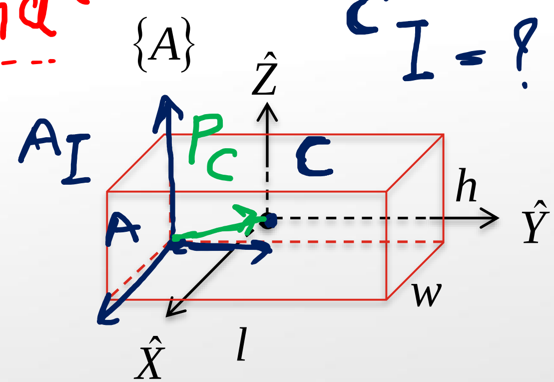
# Parallel-Axis Theorem

$$\int_{-h/2}^{h/2}\int_{-l/2}^{l/2}\int_{-w/2}^{w/2} \text{—}\, dx\,dy\,dz$$

- In the example just now, the reference frame is placed at one corner of the rectangle.
- We also mentioned that the inertia tensor is dependent on the position and orientation of the frame.
- Since we have already calculated the inertia tensor for one frame, can we get the inertia tensor (of the same object) for another translated frame, without going through the calculation of integration?
- Yes!


$^C I = \frac{1}{4} m l^2 \rightarrow {}^A I = {}^C I + m d^2$

$^C I = ?$




  - Parallel-Axis Theorem:

$$^A I = {}^C I + m\left[P_C^T P_C I_3 - P_C P_C^T\right]$$

  - where “C” means the center of mass.
  - and $P_C = [x_C, y_C, z_C]^T$ is the location of “C” wrt. {A}.

$w/2,\ l/2,\ h/2$

# Parallel-Axis Theorem

$P_C = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}_{3\times1} \quad P_C^T = \begin{bmatrix} x_c & \cdots \end{bmatrix}_{1\times3}$

- Using the ${}^{A}I = {}^{C}I + m\left[P_C^T P_C I_3 - P_C P_C^T\right]$ equation, we have:

$$\underbrace{{}^{A}I}_{3\times3} = \underbrace{{}^{C}I}_{3\times3} + m\left[\overbrace{\underbrace{\begin{bmatrix} x_C & y_C & z_C \end{bmatrix}}_{1\times3}\begin{bmatrix} x_C \\ y_C \\ z_C \end{bmatrix}}^{1\times1} I_3 - \overbrace{\begin{bmatrix} x_C \\ y_C \\ z_C \end{bmatrix}\underbrace{\begin{bmatrix} x_C & y_C & z_C \end{bmatrix}}_{1\times3}}^{3\times3}\right]$$

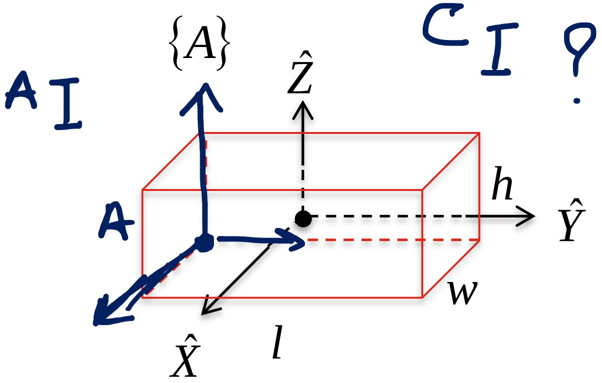
$$= {}^{C}I + m\left[\begin{bmatrix} x_C^2+y_C^2+z_C^2 & 0 & 0 \\ 0 & x_C^2+y_C^2+z_C^2 & 0 \\ 0 & 0 & x_C^2+y_C^2+z_C^2 \end{bmatrix} - \begin{bmatrix} x_C^2 & x_C y_C & x_C z_C \\ x_C y_C & y_C^2 & y_C z_C \\ x_C z_C & y_C z_C & z_C^2 \end{bmatrix}\right]$$

$$A\ I = {}^{C}I + m\underbrace{\begin{bmatrix} y_C^2+z_C^2 & -x_C y_C & -x_C z_C \\ -x_C y_C & x_C^2+z_C^2 & -y_C z_C \\ -x_C z_C & -y_C z_C & x_C^2+y_C^2 \end{bmatrix}}_{3\times3}$$

- or:

$${}^{C}I = {}^{A}I - m\begin{bmatrix} y_C^2+z_C^2 & -x_C y_C & -x_C z_C \\ -x_C y_C & x_C^2+z_C^2 & -y_C z_C \\ -x_C z_C & -y_C z_C & x_C^2+y_C^2 \end{bmatrix}$$

# Example




- Consider the same rectangle block as just now.
- The frame for the inertia tensor is now located at the center of mass.
- What is the inertia tensor wrt. to center of mass?
- Answer: We have: $P_C = [x_C, y_C, z_C]^T = \begin{bmatrix} \frac{1}{2}w & \frac{1}{2}l & \frac{1}{2}h \end{bmatrix}^T$
- Applying the parallel-axis formula:

$$\Rightarrow \quad {}^{C}I = {}^{A}I - m\begin{bmatrix} y_C^2 + z_C^2 & -x_C y_C & -x_C z_C \\ -x_C y_C & x_C^2 + z_C^2 & -y_C z_C \\ -x_C z_C & -y_C z_C & x_C^2 + y_C^2 \end{bmatrix}$$

- leads to (next page):

# Example

$$
{}^{C}I = {}^{A}I - m\begin{bmatrix} y_C^2 + z_C^2 & -x_C y_C & -x_C z_C \\ -x_C y_C & x_C^2 + z_C^2 & -y_C z_C \\ -x_C z_C & -y_C z_C & x_C^2 + y_C^2 \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{m}{3}\left(l^2 + h^2\right) & -\frac{m}{4}wl & -\frac{m}{4}hw \\ -\frac{m}{4}wl & \frac{m}{3}\left(w^2 + h^2\right) & -\frac{m}{4}hl \\ -\frac{m}{4}hw & -\frac{m}{4}hl & \frac{m}{3}\left(w^2 + l^2\right) \end{bmatrix} - m\begin{bmatrix} \frac{1}{4}\left(l^2 + h^2\right) & -\frac{1}{4}wl & -\frac{1}{4}wh \\ -\frac{1}{4}wl & \frac{1}{4}\left(w^2 + h^2\right) & -\frac{1}{4}hl \\ -\frac{1}{4}wh & -\frac{1}{4}hl & \frac{1}{4}\left(w^2 + l^2\right) \end{bmatrix}
$$

$$
{}^{C}I = \begin{bmatrix} \frac{m}{12}\left(l^2 + h^2\right) & 0 & 0 \\ 0 & \frac{m}{12}\left(w^2 + h^2\right) & 0 \\ 0 & 0 & \frac{m}{12}\left(w^2 + l^2\right) \end{bmatrix}
$$

Note: {C} must be the principal axes of the body, since the products of inertia are zero