Week 2 – Spatial Descriptions & Transformations

Advanced Robotic Systems – MANU2453

Dr Ehsan Asadi, School of Engineering
RMIT University, Victoria, Australia
Email: ehsan.asadi@rmit.edu.au

Lectures

Wk	Date	Lecture (NOTE: video recording)	Maths Difficulty	Hands-on Activity	Related Assessment
1	24/7	• Introduction to the Course • Spatial Descriptions & Transformations			
2	31/7	• Spatial Descriptions & Transformations • Robot Cell Design			Robot Cell Design Assignment
3	7/8	• Forward Kinematics • Inverse Kinematics			
4	14/8	• ABB Robot Programming via Teaching Pendant • ABB RobotStudio Offline Programming		ABB RobotStudio Offline Programming	Offline Programming Assignment
5	21/8	• Jacobians: Velocities and Static Forces			
6	28/8	• Manipulator Dynamics			
7	11/9	• Manipulator Dynamics		MATLAB Simulink Simulation	
8	18/9	• Robotic Vision		MATLAB Simulation	Robotic Vision Assignment
9	25/9	• Robotic Vision		MATLAB Simulation	
10	2/10	• Trajectory Generation			
11	9/10	• Linear & Nonlinear Control		MATLAB Simulink Simulation	
12	16/10	• Introduction to I4.0 • Revision			Final Exam

Content

- Transformation Arithmetic
- More on Representation of Orientation

Recap From Last Lecture

- Description of Mappings involving both Translation and Rotation.

$$ {}^{A}P = {}^{A}_{B}R \cdot {}^{B}P + {}^{A}P_{BORG} $$

- Description of Mappings using "Homogeneous Transform".

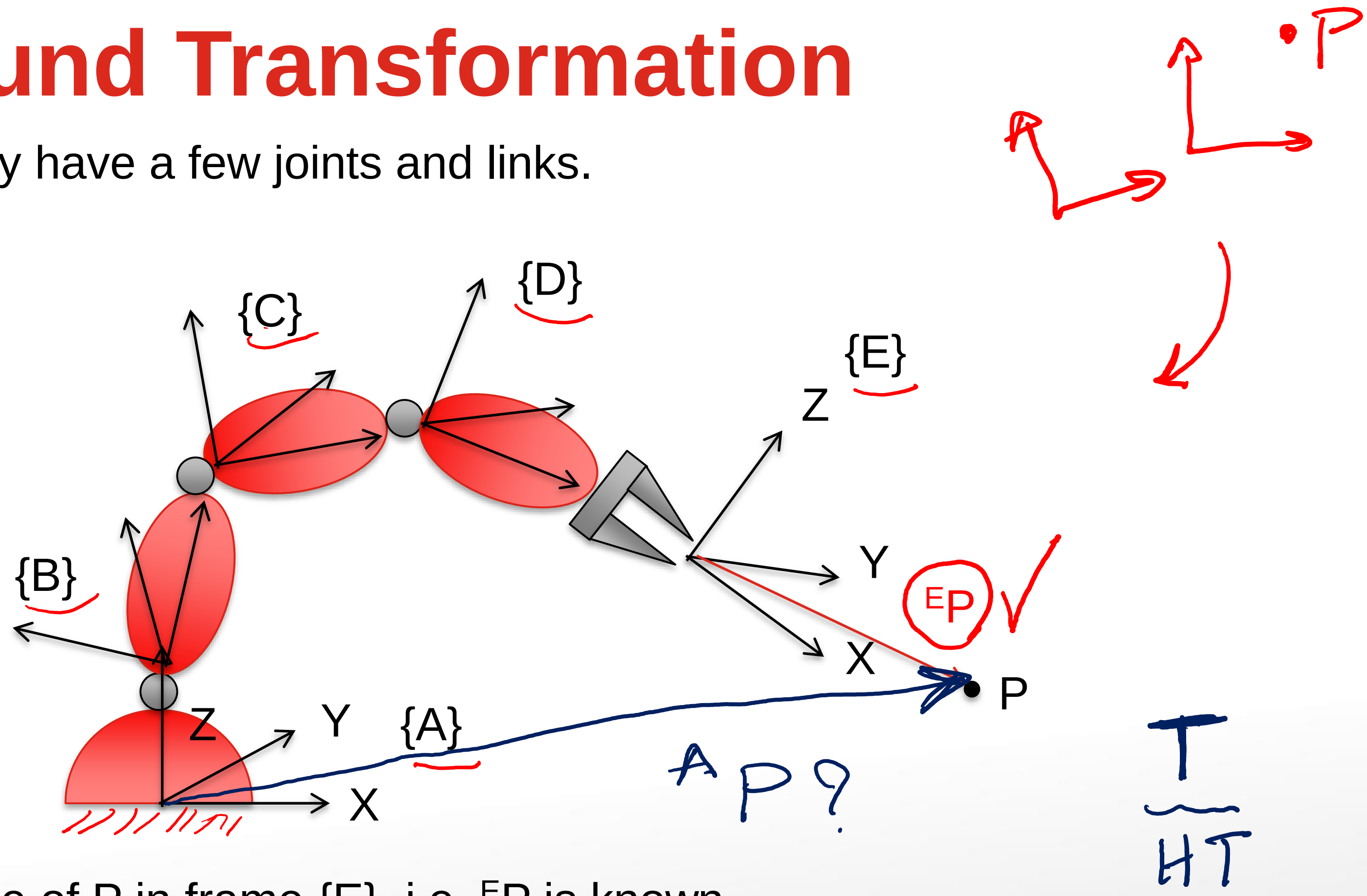
$$ {}^{A}P = {}^{A}_{B}T \cdot {}^{B}P $$

$$ \begin{bmatrix} {}^{A}P \\ \hline 1 \end{bmatrix} = \underbrace{\left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}_{{}^{A}_{B}T} \begin{bmatrix} {}^{B}P \\ \hline 1 \end{bmatrix} $$

4x4

Compound Transformation

- Robots usually have a few joints and links.



- The coordinate of P in frame {E}, i.e. ^{E}P is known.
- {E} relative to {D} is known; {D} relative to {C} is known; {C} relative to {B} is known, and finally {B} relative to {A} is known.
- What is ^{A}P?

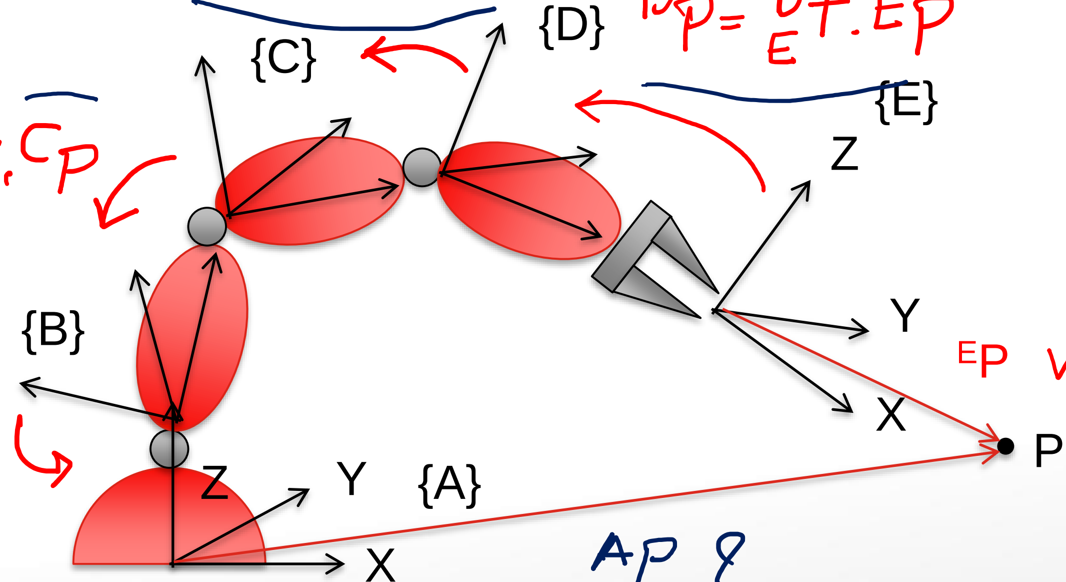
Compound Transformation

- The process is straightforward: Do “mapping” step by step.

${}^{C}P = {}^{C}_{D}T \cdot {}^{D}P$

${}^{D}P = {}^{D}_{E}T \cdot {}^{E}P$

${}^{B}P = {}^{B}_{C}T \cdot {}^{C}P$



${}^{E}P$ ✓

${}^{A}P = {}^{A}_{B}T \cdot {}^{B}P$

${}^{A}P$?

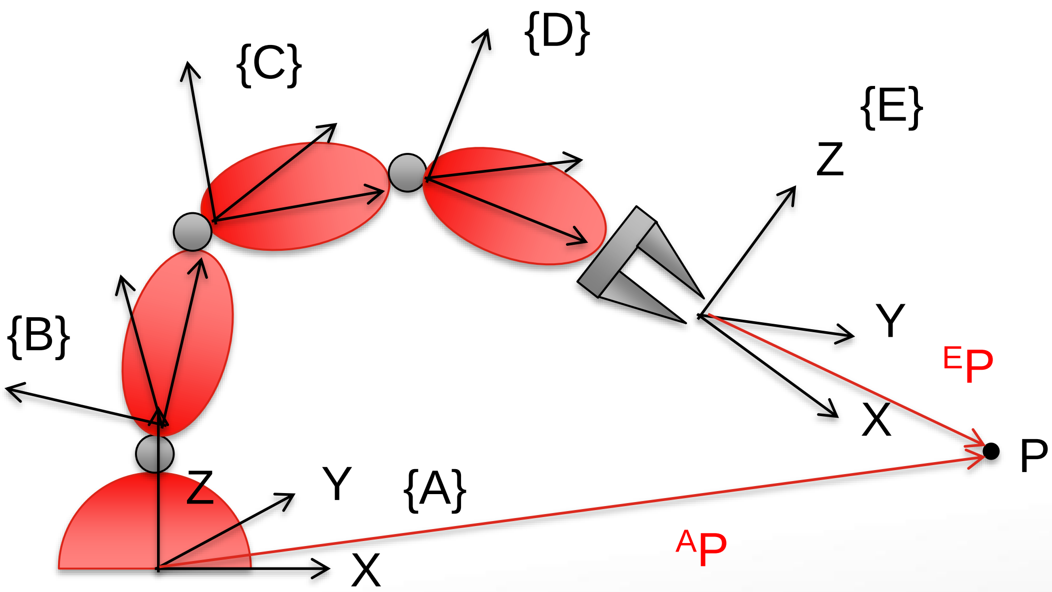
$${}^{A}P = {}^{A}_{B}T \cdot {}^{B}P = {}^{A}_{B}T \cdot {}^{B}_{C}T \cdot {}^{C}P = {}^{A}_{B}T \cdot {}^{B}_{C}T \cdot {}^{C}_{D}T \cdot {}^{D}P = \text{------}$$

$${}^{A}P = {}^{A}_{B}T \cdot {}^{B}_{C}T \cdot {}^{C}_{D}T \cdot {}^{D}_{E}T \cdot {}^{E}P$$

✓

Compound Transformation

- The process is straightforward: Do "mapping" step by step.



- $^{D}P = {}^{D}_{E}T \cdot {}^{E}P$, then $^{C}P = {}^{C}_{D}T \cdot {}^{D}P$, then $^{B}P = {}^{B}_{C}T \cdot {}^{C}P$, and finally $^{A}P = {}^{A}_{B}T \cdot {}^{B}P$.
- Overall, we have:

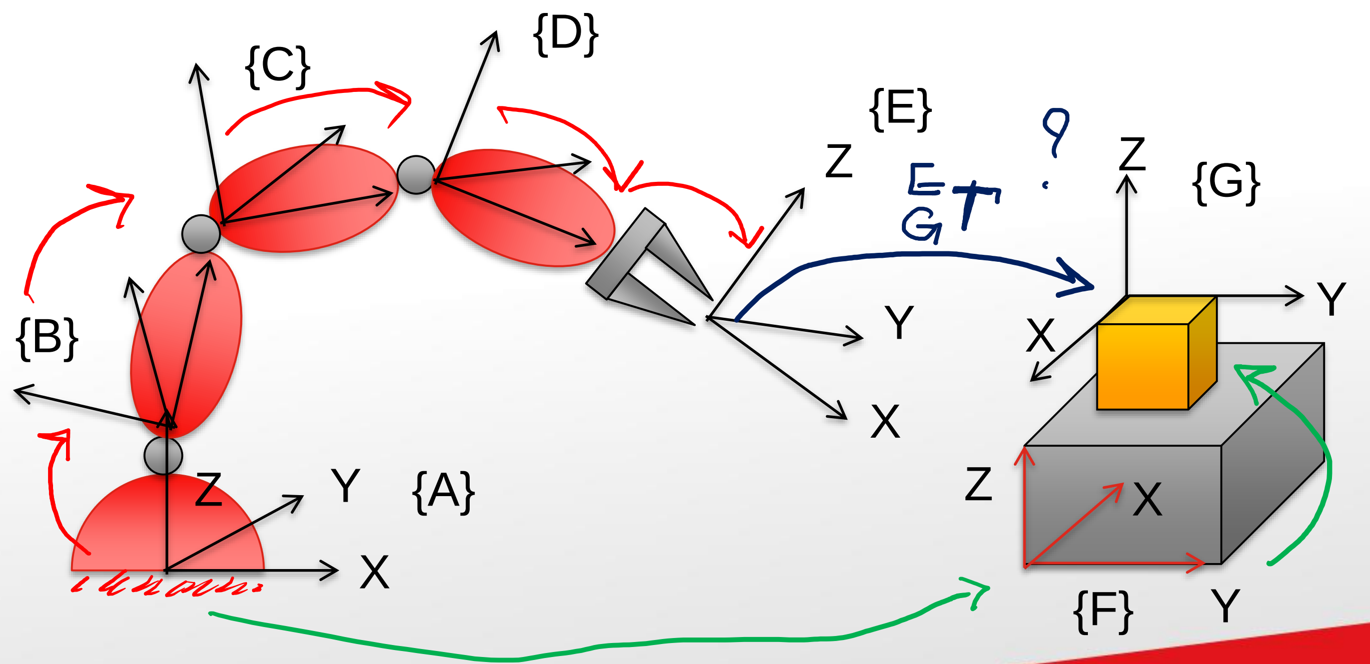
$$^{A}P = {}^{A}_{B}T \cdot {}^{B}P = {}^{A}_{B}T \cdot {}^{B}_{C}T \cdot {}^{C}P = {}^{A}_{B}T \cdot {}^{B}_{C}T \cdot {}^{C}_{D}T \cdot {}^{D}P = {}^{A}_{B}T \cdot {}^{B}_{C}T \cdot {}^{C}_{D}T \cdot {}^{D}_{E}T \cdot {}^{E}P$$

- Or:

$$^{A}P = {}^{A}_{B}T \cdot {}^{B}_{C}T \cdot {}^{C}_{D}T \cdot {}^{D}_{E}T \cdot {}^{E}P$$

Transform Equations

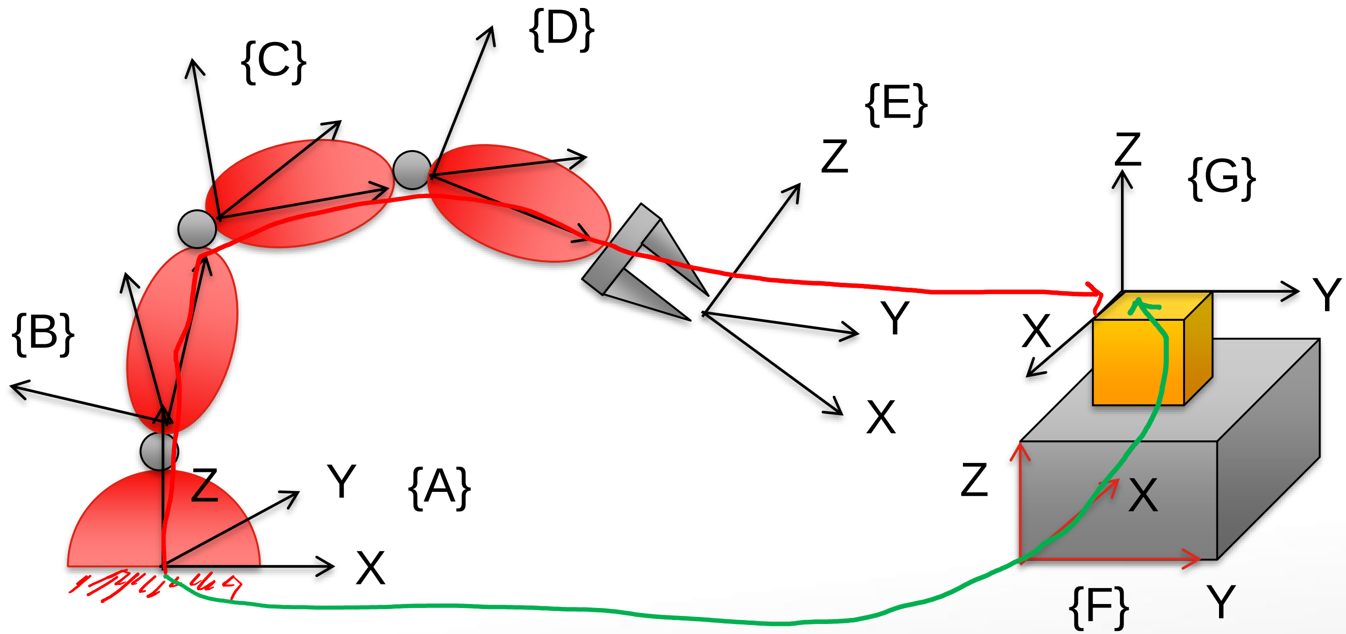
- In the following figure, assume you know the following:
  - All transformations from robot base to end-effector: ${}^{A}_{B}T, {}^{B}_{C}T, {}^{C}_{D}T, {}^{D}_{E}T$
  - Also transformation from robot base to table, and table to object: ${}^{A}_{F}T, {}^{F}_{G}T$
- Then what is the transform from end-effector to object, i.e. ${}^{E}_{G}T$?



Transform Equations

Transformation

upper path = lower path



$$^{D}_{E}T^{-1} \cdot {}^{C}_{D}T^{-1} \cdot {}^{B}_{C}T^{-1} \cdot {}^{A}_{B}T^{-1} \cdot \cancel{^{A}_{B}T} \cdot \cancel{^{B}_{C}T} \cdot \cancel{^{C}_{D}T} \cdot \cancel{^{D}_{E}T} \cdot {}^{E}_{G}T = {}^{D}_{E}T^{-1} \cdot {}^{C}_{D}T^{-1} \cdot {}^{B}_{C}T^{-1} \cdot {}^{A}_{B}T^{-1} \cdot {}^{A}_{F}T \cdot {}^{F}_{G}T$$

1

Inverting a Transformation

- Let's think this through: ${}^{A}_{B}T^{-1} \rightarrow [\quad]_{4\times4} \rightarrow$ Not easy
- We know: ${}^{B}_{A}R = {}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T}$

$${}^{A}_{B}T = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$${}^{A}_{B}T^{-1} \neq {}^{A}_{B}T^{T}$$

- What about getting the inverse of ${}^{A}_{B}T$ $\rightarrow$ inverse

${}^{A}_{B}T \rightarrow$ A Trans. from $\{A\}$ to $\{B\}$ ${}^{A}_{B}T^{-1} \rightarrow$ A Tranf. from $\{B\}$ to $\{A\} \rightarrow {}^{B}_{A}T$

$${}^{A}_{B}T^{-1} = {}^{B}_{A}T = \begin{bmatrix} {}^{B}_{A}R & {}^{B}P_{AORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

${}^{B}_{A}R = {}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T} \rightarrow$ known

${}^{B}P_{AORG} \rightarrow$?

Inverting a Transformation

- Let's think this through:
- We know:

$$ {}^{A}_{B}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] $$

- Flipping A & B:

$$ {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{B}_{A}R & & {}^{B}P_{AORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] $$

- We already know ${}^{B}_{A}R = {}^{A}_{B}R^{-1} = {}^{A}_{B}R^{T}$ thus:

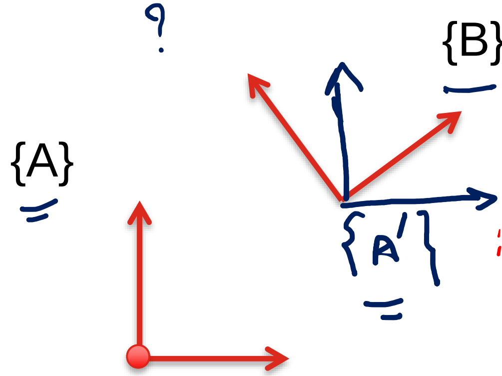
$$ {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & {}^{B}P_{AORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] $$

- What about ${}^{B}P_{AORG}$?

Inverting a Transformation

Pos of BORG in {A} ✓

- Graphical Interpretation: ${}^{B}P_{AORG}$ means position of A-origin in {B}-frame. ?



$\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$${}^{A'}P_{AORG} = -{}^{A}P_{\underline{A'ORG}}$$

BORG

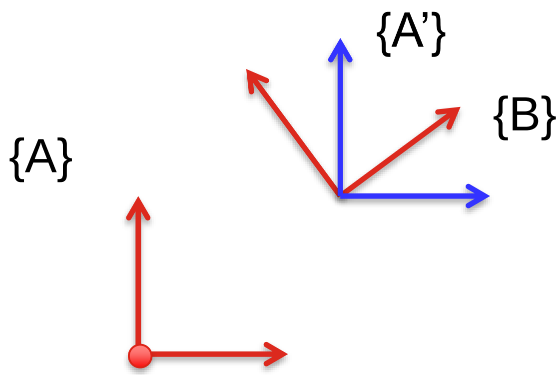
$$= -{}^{A}P_{BORG}$$ ✓

${}^{B}P_{AORG} \longrightarrow$ mapping with only Rot {A'} & {B}

$${}^{B}P_{AORG} = \underbrace{{}^{B}_{A'}R}_{\{A'\} \parallel \{A\},\ 0\ Rot} \cdot \underline{{}^{A'}P_{AORG}} = \underbrace{{}^{B}_{A}R}_{{}^{A}_{B}R^T} \cdot (-{}^{A}P_{BORG}) = -{}^{A}_{B}R^T \cdot {}^{A}P_{BORG}$$

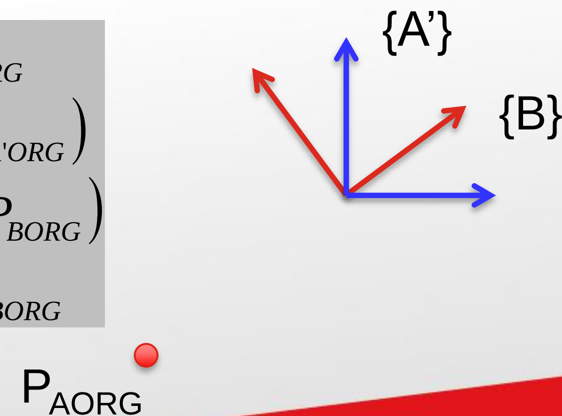
Inverting a Transformation

- It is obvious that $^{A'}P_{AORG} = -^{A}P_{A'ORG}$
  - e.g. if $^{A}P_{A'ORG} = [1,2,0]^T$ then $^{A'}P_{AORG} = -[1,2,0]^T$



- Finally, we use the formula for "mapping involving rotation only"

$$\begin{aligned} ^{B}P_{AORG} &= ^{B}_{A'}R \cdot ^{A'}P_{AORG} \\ &= ^{B}_{A}R \cdot \left(-^{A}P_{A'ORG}\right) \\ &= ^{A}_{B}R^{T} \cdot \left(-^{A}P_{BORG}\right) \\ &= -^{A}_{B}R^{T} \cdot ^{A}P_{BORG} \end{aligned}$$



Inverting a Transformation

- Summary:
- Given:

$$\rightarrow \quad {}^{A}_{B}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \quad \rightarrow \quad {}^{A}_{B}T^{-1}$$

- Then:

$$ {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{B}_{A}R & & {}^{B}P_{AORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T} \cdot {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Inverting a Transformation

- Example: {B} is rotated relative to {A} about z-axis by 30 degrees, and then translated 4 units in X_A and 3 units in Y_A.
- The description of {B} w.r.t. {A} is:

$${}^{A}_{B}R \qquad {}^{A}P_{BORG}$$

$$\rightarrow {}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.5 & 0 & 4 \\ 0.5 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow {}^{A}_{B}T^{-1}$$

$$4 \times 4$$

- And the description of {A} w.r.t. {B} is:

$${}^{A}_{B}R^{T} \qquad -{}^{A}_{B}R^{T} \cdot {}^{A}P_{BORG}$$

$${}^{A}_{B}T^{-1} = {}^{B}_{A}T = \begin{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{T} & -\begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{T} \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$${}^{B}_{A}T = \begin{bmatrix} 0.866 & 0.5 & 0 & -4.964 \\ -0.5 & 0.866 & 0 & -0.598 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Background (represent an orientation)

- We learnt that to represent an orientation, we could use the 3 x 3 rotation matrix:

$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_{B} & {}^{A}\hat{Y}_{B} & {}^{A}\hat{Z}_{B} \end{bmatrix} = \begin{bmatrix} \hat{X}_{B}\cdot\hat{X}_{A} & \hat{Y}_{B}\cdot\hat{X}_{A} & \hat{Z}_{B}\cdot\hat{X}_{A} \\ \hat{X}_{B}\cdot\hat{Y}_{A} & \hat{Y}_{B}\cdot\hat{Y}_{A} & \hat{Z}_{B}\cdot\hat{Y}_{A} \\ \hat{X}_{B}\cdot\hat{Z}_{A} & \hat{Y}_{B}\cdot\hat{Z}_{A} & \hat{Z}_{B}\cdot\hat{Z}_{A} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

3x3

3DOF

- Do we really need 9 parameters to describe a rotation in 3D-space?
- It turns out that there are 6 constraints for the 9 numbers:

$$ \left|\hat{X}\right| = 1 \quad \left|\hat{Y}\right| = 1 \quad \left|\hat{Z}\right| = 1 \quad \hat{X}\cdot\hat{Y} = 0 \quad \hat{X}\cdot\hat{Z} = 0 \quad \hat{Y}\cdot\hat{Z} = 0 $$

- Therefore, it is possible to describe orientations with just 3 parameters.

Background

- Question 1: How do we write a 3 x 3 rotation matrix by specifying 3 (or 4) parameters?
- Question 2 (inverse): Given a 3 x 3 rotation matrix, how do we visualize the rotation using just 3 (or 4) parameters?

1
- Fixed angles is one of the methods:
  - Rotate about one axis of original frame, then rotate about another axis of original frame, and finally rotate about another axis of original frame.

2
- Euler angles is another method:
  - Rotate about one axis of original frame, then rotate about another axis of rotated frame, and finally rotate about another axis of rotated frame.

3
- Equivalent angle-axis is also a widely-used method:
  - Rotate about an "effective / resultant" axis.

4
- Euler parameters / Quarternion is the final method to be covered.
  - Similar to Equivalent angle-axis but avoids singularity. → ?

X-Y-Z Fixed Angles

- Let's first start with the Fixed Angles representation.
- To describe the orientation of {B}:

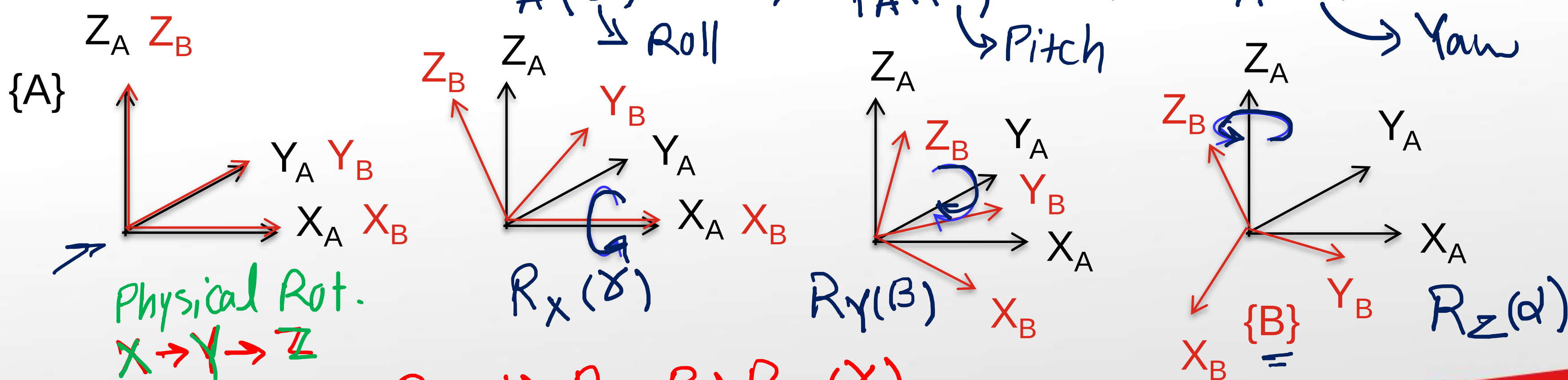
– • Start by aligning {B} with {A}.

1 • First, rotate {B} about X_A by angle γ.

2 • Then, rotate the rotated {B} about Y_A by angle β.

3 • Finally, rotate the rotated {B} about Z_A by angle α.

$X_A(\gamma) \rightarrow Y_A(\beta) \rightarrow Z_A(\alpha)$



$$R_{XYZ}(\gamma, \beta, \alpha) = R_Z(\alpha)\, R_Y(\beta)\, R_X(\gamma)$$

X-Y-Z Fixed Angles

- Question 1: Writing the rotation matrix is straightforward:

3 2 1

$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = R_Z(\alpha)\cdot R_Y(\beta)\cdot R_X(\gamma)
$$

$$
= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}
$$

$$
= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}_{3\times 3}
$$

- Where “c” is the shorthand for cosine, and “s” is the shorthand for sine.

X-Y-Z Fixed Angles

c → Cosine
s → sin

- Question 2: Solve for γ, β, α from the rotation matrix.

$$ {}^A_B R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

$\alpha \rightarrow$
$\beta \rightarrow$
$\gamma \rightarrow$

- We see that $r_{11}^2 + r_{21}^2 = c^2\alpha c^2\beta + s^2\alpha c^2\beta = c^2\beta(\underbrace{s^2\alpha + c^2\alpha}_{1}) = c^2\beta$

$c\beta = \sqrt{r_{11}^2 + r_{21}^2}$

Matlab ↑

$$\tan\beta = \frac{s\beta}{c\beta} = \frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}} \Longrightarrow \beta = \text{arctan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$

$$\tan\alpha = \frac{s\alpha}{c\alpha} = \frac{r_{21}/c\beta}{r_{11}/c\beta} \Longrightarrow \alpha = \text{arctan2}(r_{21}/c\beta, r_{11}/c\beta)$$

$$\tan\gamma = \frac{s\gamma}{c\gamma} = \frac{r_{32}/c\beta}{r_{33}/c\beta} \Longrightarrow \gamma = \text{arctan2}(r_{32}/c\beta, r_{33}/c\beta)$$

$\nearrow \infty$ $\nwarrow 0$

- Can't solve if $c\beta = 0$. ~~α + γ~~

X-Y-Z Fixed Angles

- Question 2: Solve for γ, β, α from the rotation matrix.

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

- We see that $r_{11}^2 + r_{21}^2 = c^2\beta \Rightarrow c\beta = \sqrt{r_{11}^2 + r_{21}^2}$

$$ \rightarrow \quad \tan\beta = \frac{s\beta}{c\beta} = \frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}} \Rightarrow \beta = \arctan 2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right) $$

$$ \rightarrow \quad \tan\alpha = \frac{s\alpha}{c\alpha} = \frac{r_{21}/c\beta \rightarrow \infty}{r_{11}/c\beta \rightarrow 0} \Rightarrow \alpha = \arctan 2(r_{21}/c\beta, r_{11}/c\beta) $$

$$ \rightarrow \quad \tan\gamma = \frac{s\gamma}{c\gamma} = \frac{r_{32}/c\beta \rightarrow \infty}{r_{33}/c\beta \rightarrow 0} \Rightarrow \gamma = \arctan 2(r_{32}/c\beta, r_{33}/c\beta) $$

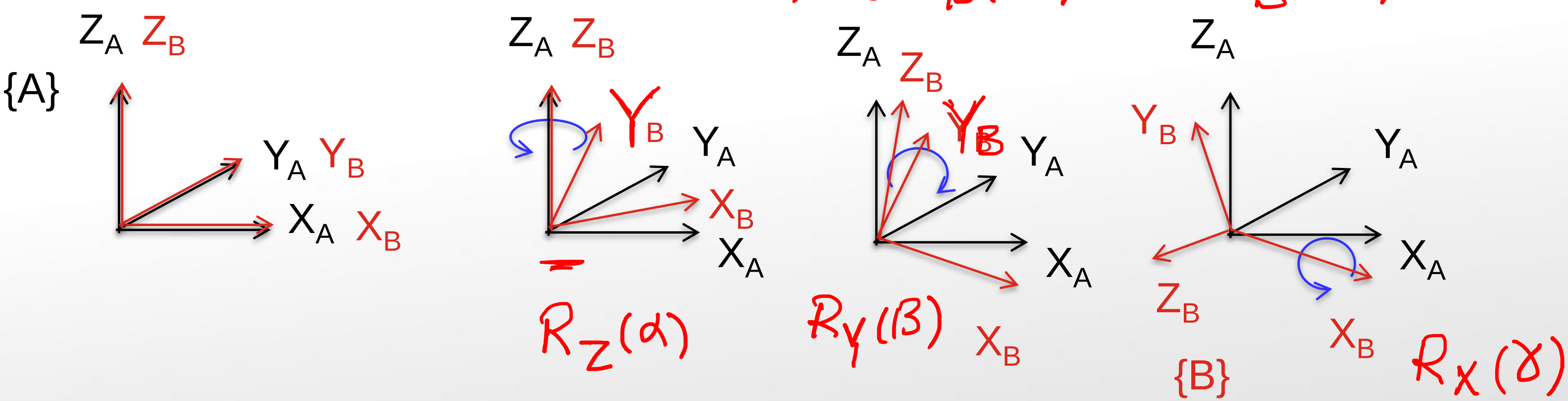
- Can't solve if cβ = 0. $\rightarrow \alpha, \gamma \rightarrow \beta = \pm 90^\circ \rightarrow$ Singularity

Z-Y-X Euler Angles

- To describe the orientation of {B}:
  - – Start by aligning {B} with {A}.
  - 1 First, rotate {B} about Z_B by angle α.
  - 2 Then, rotate {B} about the new Y_B by angle β.
  - 3 Finally, rotate {B} about X_B by angle γ.

~~$Y_A(\beta)$~~

$Z_B(\alpha) \rightarrow$ New $Y_B(\beta) \rightarrow X_B(\gamma)$



$$R_{ZYX} = R_Z(\alpha) R_Y(\beta) R_X(\gamma) \quad *$$

Z-Y-X Euler Angles

- Question 1: Writing the rotation matrix is done as follows:
- Using the rotation of the intermediate frames, we have:

$$
\begin{aligned}
{}^{A}_{B}R_{Z'Y'X'}(\alpha,\beta,\gamma) &= R_Z(\alpha)\cdot R_Y(\beta)\cdot R_X(\gamma) \\
&= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} \\
&= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
\end{aligned}
$$

- The result is the same as having three rotations taken in the opposite order about fixed axis!!
- Question 2: Same as X-Y-Z Fixed Angles.

Fixed Angles & Euler Angles

Rot

- We learnt about X-Y-Z fixed angles and Z-Y-X Euler angles.
- In fact, we can have 24 combinations of them:

? Yes

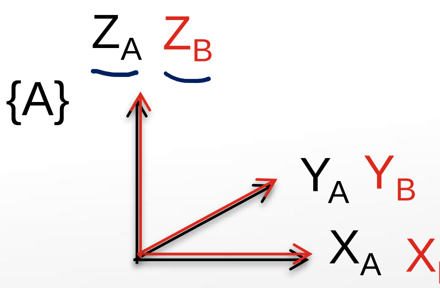
order
Rot

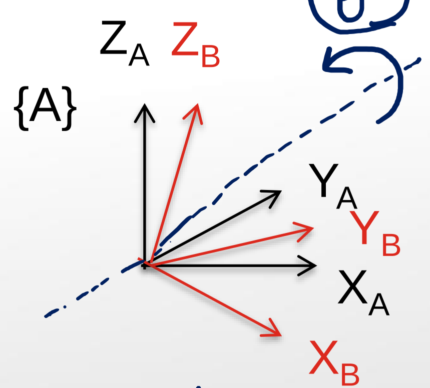
Fixed Angles 12		Euler Angles 12	
X-Y-Z *	X-Y-X	Z'-Y'-X' *	X'-Y'-X'
X-Z-Y	X-Z-X	Y'-Z'-X'	X'-Z'-X'
Y-X-Z	Y-X-Y	Z'-X'-Y'	Y'-X'-Y'
Y-Z-X	(40) Y-Z-Y (30)	X'-Z'-Y'	(40) Y'-Z'-Y' (30)
Z-X-Y	Z-X-Z	Y'-X'-Z'	Z'-X'-Z'
Z-Y-X	Z-Y-Z	X'-Y'-Z'	Z'-Y'-Z'

- However, remember the equivalence of fixed angles and Euler angles (opposite orders). Therefore there are actually only 12 different sets.

Equivalent Angle-Axis

- Equivalent Angle-Axis representation is another way of looking at the rotation matrix.
- Any rotation may be obtained through one proper axis and angle. 3
- To describe the orientation of {B}:
  - Start by aligning {B} with {A}.
  - Rotate {B} about a vector ${}^{A}\hat{K}$ by angle θ.





3rd θ

$\hat{K}$

$$\hat{K} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$$

$$k_x^2 + k_y^2 + k_z^2 = 1$$

2 parameters

Equivalent Angle-Axis

$$K = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

- Question 1: The equivalent rotation matrix is given by:

$$\hat{k}, \theta \rightarrow R_K(\theta) = I + (\sin\theta) K + (1-\cos\theta) K^2$$

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

- Where “c” is the shorthand for cosine, “s” is the shorthand for sine, and:

$$v\theta = 1 - \cos\theta$$

$c\theta \rightarrow \text{Cosine}\ \theta$

$s\theta \rightarrow \sin\theta$

Equivalent Angle-Axis

- Question 2: To obtain the parameters ${}^{A}\hat{K} = \begin{bmatrix} k_x & k_y & k_z \end{bmatrix}^T$ and θ from a rotation matrix:

$$R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

$\longrightarrow \theta, \hat{K}$

- The solution is:

$$\underbrace{(k_x^2 + k_y^2 + k_z^2)}_{1} v\theta + 3c\theta = r_{11} + r_{22} + r_{33}$$

$$1 - c\theta + 3c\theta = r_{11} + r_{22} + r_{33}$$

$$\theta = \arccos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\hat{K} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \frac{1}{2\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Equivalent Angle-Axis

- Question 2: To obtain the parameters $^{A}\hat{K} = \begin{bmatrix} k_x & k_y & k_z \end{bmatrix}^T$ and θ from a rotation matrix:
- The solution is:

$$\theta = \arccos\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) \qquad \hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

∞

0

0, 180

- Note: This solution always gives θ between 0 and 180 degrees. $0 < \theta < 180$
- For any ($^{A}\hat{K}$, θ) there is another pair, ($-^{A}\hat{K}$, -θ) which gives the same rotation.
- Also note: The solution fails if θ = 0 or θ = 180 degrees.

$\theta = 0$

$\theta = 180$

Singularity

Euler Parameters / Quaternions

3-methods (3 Param.) → Singularity

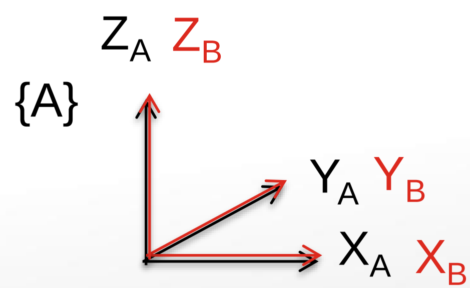
- We saw that in all previous methods with 3 parameters, there will be cases where no solutions to Question 2 can be found.
- To tackle this issue, we need to increase the number of parameters to 4.
- Euler parameters (also called Quaternions) is one such representation:
  - From the equivalent angle-axis representation, calculate the following Euler parameters:

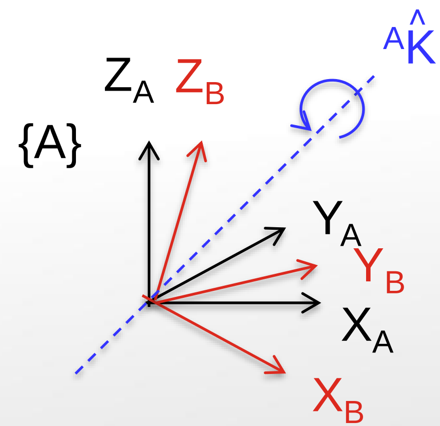
$\xi_1 = K_x \sin(\theta/2)$

$\xi_2 = K_y \sin(\theta/2)$

$\xi_3 = K_z \sin(\theta/2)$

$\xi_4 = \cos(\theta/2)$





  - The parameters are not independent, and satisfy:

$$\rightarrow \quad \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 1$$

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Euler Parameters / Quaternions

- Question 1: The rotation matrix R, consisting of Euler parameters, is:

$$R_\varepsilon = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix}$$

Euler Parameters / Quaternions

- Question 2: To calculate the Euler parameters from a given rotation matrix R, note that:

$$R_\varepsilon = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix}$$

$\rightarrow \xi_1, \xi_2, \xi_3, \xi_4$?

$$r_{11}+r_{22}+r_{33} = 3-4\underbrace{(\xi_1^2+\xi_2^2+\xi_3^2)}$$

$$\xi_1^2+\xi_2^2+\xi_3^2 = 1-\xi_4^2$$

$$\Rightarrow r_{11}+r_{22}+r_{33} = 3-4(1-\xi_4^2) = -1+4\xi_4^2$$

$$\xi_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}$$

R_ξ

$$\xi_1 = \frac{r_{32}-r_{23}}{4\xi_4}, \quad \xi_2 = \frac{r_{13}-r_{31}}{4\xi_4}, \quad \xi_3 = \frac{r_{21}-r_{12}}{4\xi_4}$$

Euler Parameters / Quaternions

- Question 2: To calculate the Euler parameters from a given rotation matrix R, note that:

$$R_\varepsilon = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1-2\varepsilon_2^2-2\varepsilon_3^2 & 2(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4) & 2(\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_3^2 & 2(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4) \\ 2(\varepsilon_1\varepsilon_3-\varepsilon_2\varepsilon_4) & 2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) & 1-2\varepsilon_1^2-2\varepsilon_2^2 \end{bmatrix}$$

- Therefore:

$$\varepsilon_4 = \frac{1}{2}\sqrt{1+r_{11}+r_{22}+r_{33}}$$

- Substituting this back into R_ε, we can obtain:

$$\varepsilon_1 = \frac{r_{32}-r_{23}}{4\varepsilon_4} \quad \varepsilon_2 = \frac{r_{13}-r_{31}}{4\varepsilon_4} \quad \varepsilon_3 = \frac{r_{21}-r_{12}}{4\varepsilon_4}$$

$\varepsilon_4 = 0$

Euler Parameters / Quaternions

- It seems like there is still singularity when $\varepsilon_4 = 0$, or equivalently when θ = 180 degrees!!
- However: All the expressions remain finite due to their definition, which is:

$$\varepsilon_1 = k_x \sin(\theta/2) \quad \varepsilon_2 = k_y \sin(\theta/2) \quad \varepsilon_3 = k_z \sin(\theta/2) \quad \varepsilon_4 = \cos(\theta/2)$$

- There is also another way out of this problem: If $\varepsilon_4 = 0$, we can always look for another ε_i to be the denominators.
  - E.g. If we want to use ε_1 as denominator, then:

$\xi_1 \rightarrow$

$$r_{11} - r_{22} - r_{33} = -1 + 4\varepsilon_1^2 \Rightarrow \varepsilon_1 = \frac{1}{2}\sqrt{1 + r_{11} - r_{22} - r_{33}} \rightarrow \xi_2 \; \xi_3 \; \xi_4$$

  - Or ε_2:

$$-r_{11} + r_{22} - r_{33} = -1 + 4\varepsilon_2^2 \Rightarrow \varepsilon_2 = \frac{1}{2}\sqrt{1 - r_{11} + r_{22} - r_{33}} \rightarrow \xi_1 \; \xi_3 \; \xi_4$$

  - Or ε_3:

$$-r_{11} - r_{22} + r_{33} = -1 + 4\varepsilon_3^2 \Rightarrow \varepsilon_3 = \frac{1}{2}\sqrt{1 - r_{11} - r_{22} + r_{33}} \rightarrow$$

  - Substitute one of these and calculate the other ε's.

Equivalent Angle-Axis

- Example: A frame {B} which is initially coincident with {A} is rotated about the vector $^{A}K = [0.707\ 0.707\ 0]^T$ (passing through the origin) by 30 degrees. What is the frame description of {B}?

- Answer:

$$
\begin{aligned}
v\theta &= 1-\cos\theta \\
&= 1-0.866 \\
&= 0.134
\end{aligned}
$$

$$
R_K(\theta)=\begin{bmatrix} k_xk_xv\theta+c\theta & k_xk_yv\theta-k_zs\theta & k_xk_zv\theta+k_ys\theta \\ k_xk_yv\theta+k_zs\theta & k_yk_yv\theta+c\theta & k_yk_zv\theta-k_xs\theta \\ k_xk_zv\theta-k_ys\theta & k_yk_zv\theta+k_xs\theta & k_zk_zv\theta+c\theta \end{bmatrix}
$$

$$
=\begin{bmatrix} 0.707^2\times 0.134+0.866 & 0.707^2\times 0.134 & 0.707\times 0.5 \\ 0.707^2\times 0.134 & 0.707^2\times 0.134+0.866 & -0.707\times 0.5 \\ -0.707\times 0.5 & 0.707\times 0.5 & 0.866 \end{bmatrix}
$$

$$
=\begin{bmatrix} 0.933 & 0.067 & 0.354 \\ 0.067 & 0.933 & -0.354 \\ -0.354 & 0.354 & 0.866 \end{bmatrix}
$$

- The frame is:

$$
{}^{A}_{B}T=\begin{bmatrix} 0.933 & 0.067 & 0.354 & 0 \\ 0.067 & 0.933 & -0.354 & 0 \\ -0.354 & 0.354 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Tutorial Assignments

- **Question 1:**
  - A vector ^{A}P is rotated about Z_A by θ degrees.
  - It is subsequently rotated about X_A by Φ degrees.
  - Give the rotation matrix that accomplishes these rotations in the specified order.

Tutorial Assignments

- **Question 2:**
  - A vector ^{A}P is rotated about Y_A by 30 degrees.
  - It is subsequently rotated about X_A by 45 degrees.
  - Give the rotation matrix that accomplishes these rotations in the specified order.

Tutorial Assignments

- **Question 3:**
  - A frame {B} is originally coincident with frame {A}.
  - We first rotate {B} about Z_B by θ degrees.
  - Then we rotate the resulting frame about X_B by Φ degrees.
  - Give the rotation matrix that will change the descriptions of vectors from ^{B}P to ^{A}P.

Tutorial Assignments

- **Question 4:**
  - The axis of a particular rotation is K = [2, 1, 2]. (NOTE: this is not yet a unit vector).
  - This axis passes through the origin.
  - The angle of rotation is 45 degrees.
  - Derive the rotation matrix based on equivalent angle-axis representation.

- **Question 5**:
  - For the same rotations as in Question 4, derive the rotation matrix based on Euler parameters.

Tutorial Assignments

- **Question 6:**
  - A rotation matrix is given by:

$$R = \begin{bmatrix} 0.8373 & -0.4063 & 0.3659 \\ 0.5365 & 0.7397 & -0.4063 \\ -0.1055 & 0.5365 & 0.8373 \end{bmatrix}$$

  - Find out all the parameters in terms of:
    - X-Y-Z fixed angles
    - Z-Y-X Euler angles
    - Equivalent Angle-Axis representation
    - Euler parameters

Tutorial Assignments

- **Question 7:**
  - The following frame definitions are given as known.

$$ {}^{U}_{A}T = \begin{bmatrix} 0.866 & -0.5 & 0 & 11 \\ 0.5 & 0.866 & 0 & -1 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{B}_{A}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 10 \\ 0 & 0.5 & 0.866 & -20 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{C}_{U}T = \begin{bmatrix} 0.866 & -0.5 & 0 & -3 \\ 0.433 & 0.75 & -0.5 & -3 \\ 0.25 & 0.433 & 0.866 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

  - Solve for: ${}^{B}_{C}T$

Tutorial Assignments

- **Question 8:**
  - For sufficiently small rotations so that the approximations $\sin\theta = \theta$

    $\cos\theta = 1$ $\theta^2 = 0$
  - Derive the rotation matrix equivalent to a rotation of θ about a general axis, $\hat{K}$
  - Show that two infinitesimal rotations commute (i.e. the order in which the rotations are performed is not important).

Thank you!

Have a good evening.