Week 1– Spatial Descriptions & Transformations

Advanced Robotic Systems – MANU2453

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Lectures

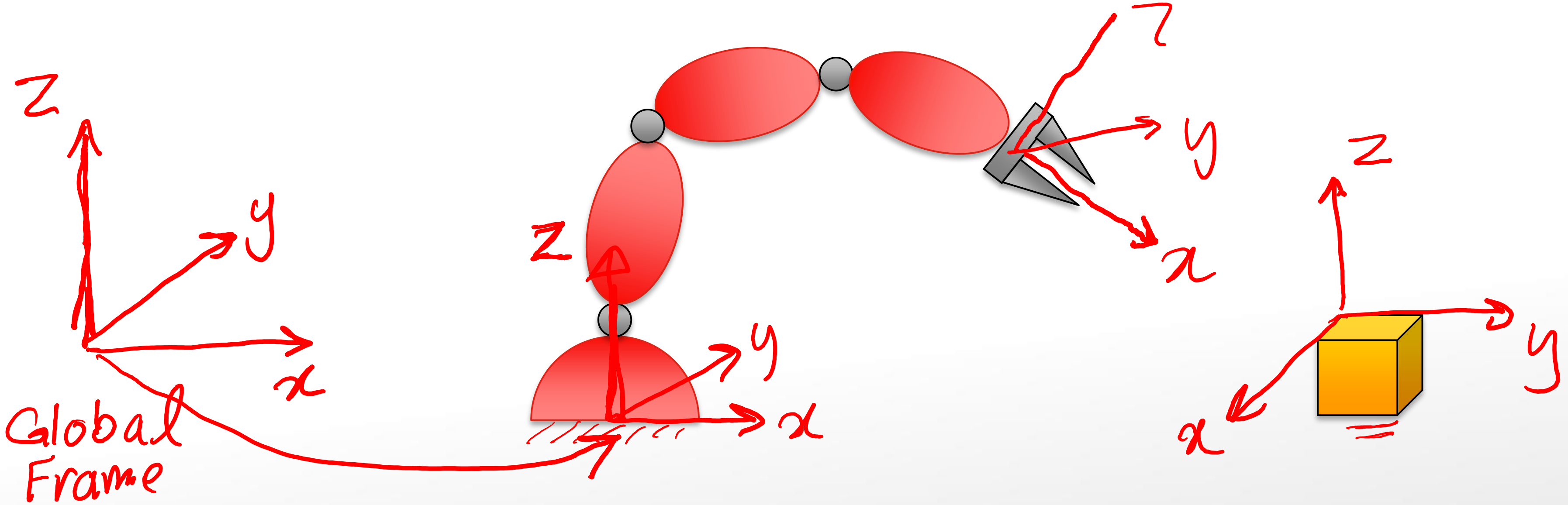
Wk	Date	Lecture (NOTE: video recording)	Maths Difficulty	Hands-on Activity	Related Assessment
1	24/7	• Introduction to the Course • Spatial Descriptions & Transformations			
2	31/7	• Spatial Descriptions & Transformations • Robot Cell Design			Robot Cell Design Assignment
3	7/8	• Forward Kinematics • Inverse Kinematics			
4	14/8	• ABB Robot Programming via Teaching Pendant • ABB RobotStudio Offline Programming		ABB RobotStudio Offline Programming	Offline Programming Assignment
5	21/8	• Jacobians: Velocities and Static Forces			
6	28/8	• Manipulator Dynamics			
7	11/9	• Manipulator Dynamics		MATLAB Simulink Simulation	
8	18/9	• Robotic Vision		MATLAB Simulation	Robotic Vision Assignment
9	25/9	• Robotic Vision		MATLAB Simulation	
10	2/10	• Trajectory Generation			
11	9/10	• Linear & Nonlinear Control		MATLAB Simulink Simulation	
12	16/10	• Introduction to I4.0 • Revision			Final Exam

Content

- Description of Position of a Point
- Description of Position of a Rigid Body
- Description of Orientation of a Rigid Body
- Properties of Rotation Matrix
- Description of Frames.
- Mapping
- Homogeneous Transformation Matrix

Introduction: Description

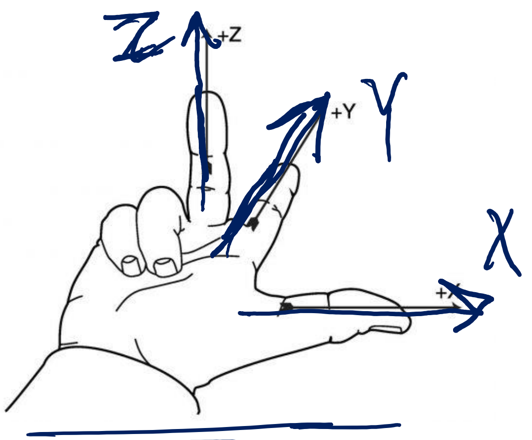
- Robotic manipulation → Parts and tools moved around in space.
- Need to represent position and orientations.
- Define a universe coordinate system to which everything is referenced.



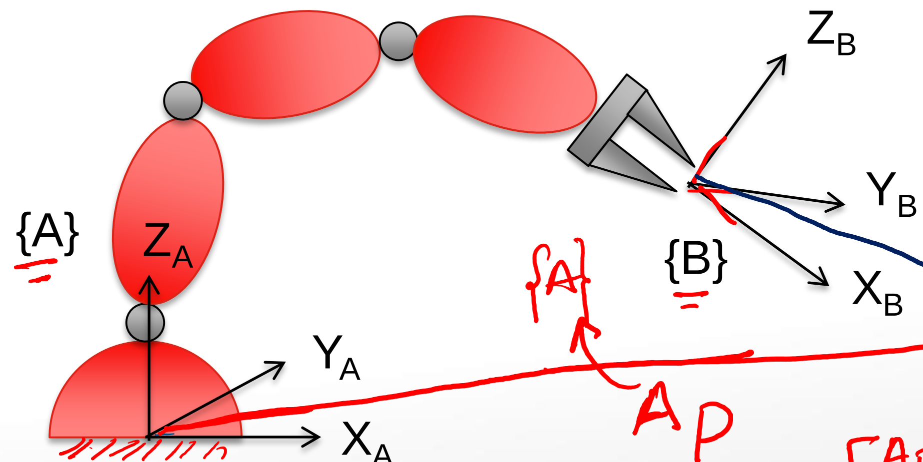
- Also, attach a frame (coordinate system) rigidly to the object, tool etc.
- Then describe the position and orientation of the frame with respect to the reference coordinate system.

Position of a Point

- Any point can be described with respect to any coordinate system using a 3 x 1 position vector.
  - Must specify clearly which coordinate system is being used.



Right hand Rule



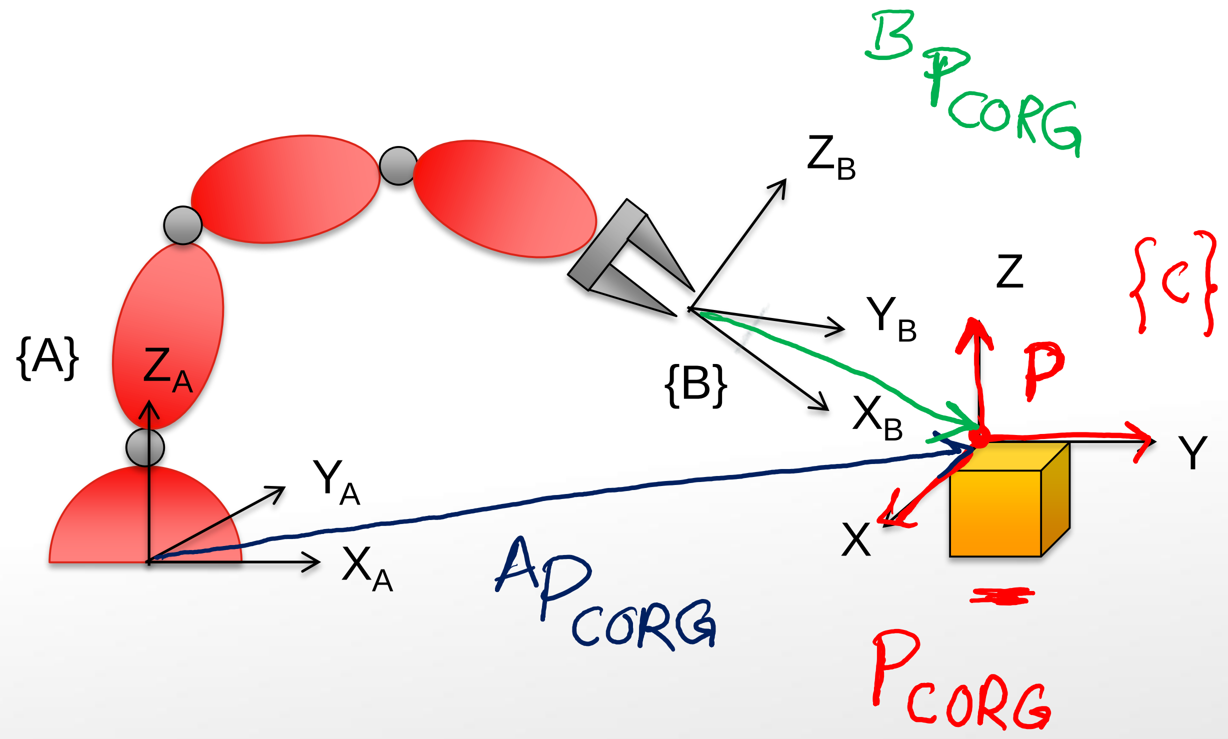
$$^BP = \begin{bmatrix} ^BP_x \\ ^BP_y \\ ^BP_z \end{bmatrix}$$

$$^AP = \begin{bmatrix} ^AP_x \\ ^AP_y \\ ^AP_z \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 6 \end{bmatrix}$$

- In frame {A}, the point P has a position vector ^AP.
- In frame {B}, the point P has a position vector ^BP.

Position of a Rigid Body

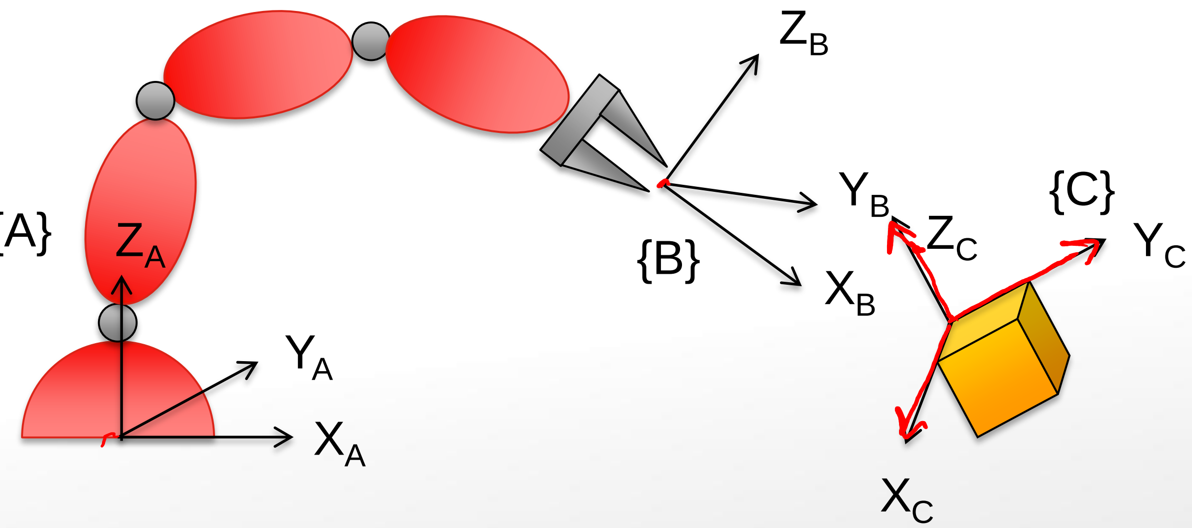
- frames are attached rigidly to each body, the description of the frames with respect to another (e.g. base) is enough to provide Position.



Orientation of a Rigid Body

- Apart from position, we may also need to describe the orientation of arms, tool or object.
- Since frames are attached rigidly to each body, the description of the frames with respect to another (e.g. base) is enough to provide orientation.

Position Vector X Y Z



- E.g. orientation of frame {B} w.r.t. frame {A}.
- E.g. orientation of frame {C} w.r.t. frame {A}.
- E.g. orientation of frame {C} w.r.t. frame {B}.

How → Rotation Matrix

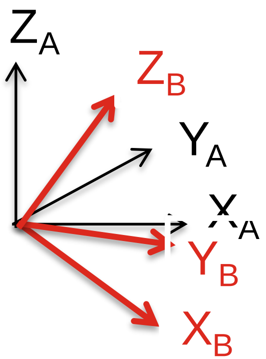
Orientation of a Rigid Body

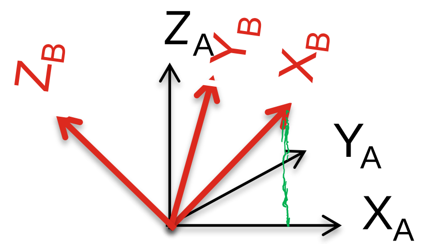
- Let $\hat{X}_B$, $\hat{Y}_B$ and $\hat{Z}_B$ be unit vectors (length = 1) in frame {B}.
- Also, ignore the translation and let the origins coincide.

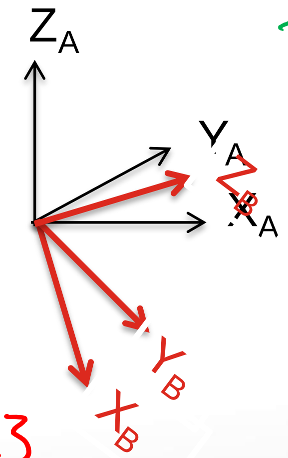
Rotation Matrix From {A} To {B}

Projection

Dot. Product



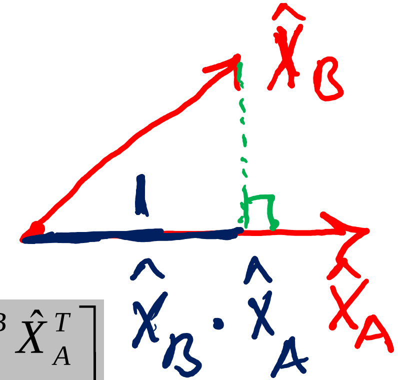




$${}^A_B R = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix}_{3\times 3}$$

$${}^A_B R = \begin{bmatrix} \hat{X}_B \text{ Proj on } \hat{X}_A & \hat{Y}_B \text{ Proj on } \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \text{ Proj on } \hat{Y}_A & \hat{Y}_B \text{ ״ } \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \text{ ״ } \hat{Z}_A & \hat{Y}_B \text{ ״ } \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}$$

Properties of Rotation Matrix



- Look at the rotation matrix again:

$$\rightarrow \quad {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{X}_A^T \\ {}^{B}\hat{Y}_A^T \\ {}^{B}\hat{Z}_A^T \end{bmatrix}$$

- Geometrically and intuitively, if we describe a unit vector of frame {B} in frame {A}, and then describe the "new" vector back into {B}, we should get back the original unit vector.
- The rotation matrix has orthonormal columns. { Normal, orthogonall
  - Each column has unit length. $V = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\sqrt{a^2+b^2+c^2} = 1$
  - Each column is orthogonal (90 degrees) to other columns, i.e. the dot products between two columns are zero. dot product of 2 col. = 0
- Which also means that: Inverse = Transpose

$${}^{B}_{A}R^T = {}^{B}_{A}R^{-1}$$

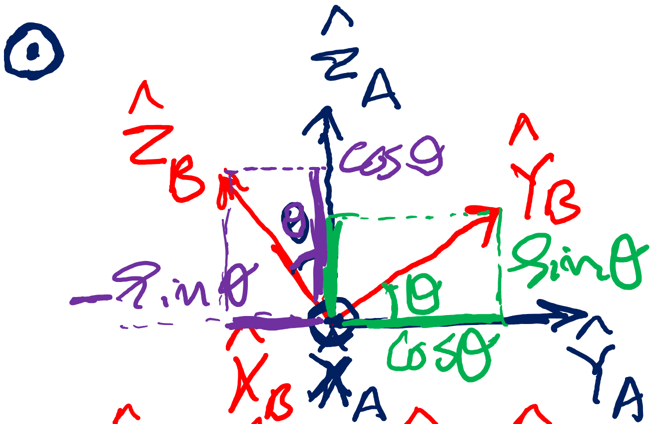
$${}^{B}_{A}R^T = {}^{A}_{B}R$$

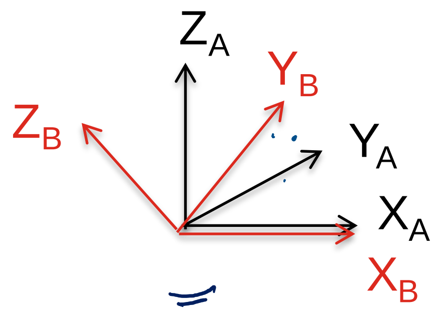
${}^{A}_{B}R \rightarrow {}^{B}_{A}R \qquad {}^{B}_{A}R = {}^{A}_{B}R^{-1} = {}^{A}_{B}R^T$

$\uparrow$ Given

Orientation: Special Case 1

- Example: Rotation θ deg about X-axis only: ${}^{A}_{B}R \xrightarrow{\theta=90} ?$





- Dot product is defined as:

$$P \cdot Q = \|\cancel{P}\|\|\cancel{Q}\|\cos(\theta)$$

$$= \cos\theta$$

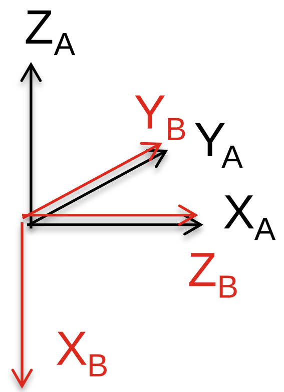
$\hat{X}_B \parallel \hat{X}_A \qquad \hat{X}_B \cdot \hat{X}_A = 1$

$${}^{A}_{B}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ 0 & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Orientation: Special Case 2

- Example: Rotation 90 deg about Y-axis only:

θ



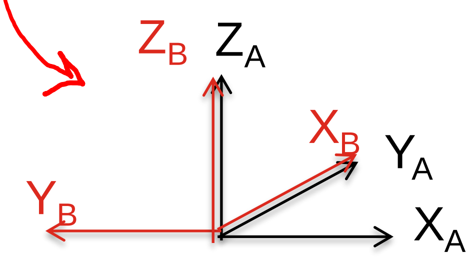
$$
{}^A_B R = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}_{\theta=90^\circ} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}
$$

X_B in {A} $\qquad$ Y_B in {A} $\qquad$ Z_B in {A}

Orientation: Special Case 3

- Example: Rotation 90 deg about Z-axis only:

θ



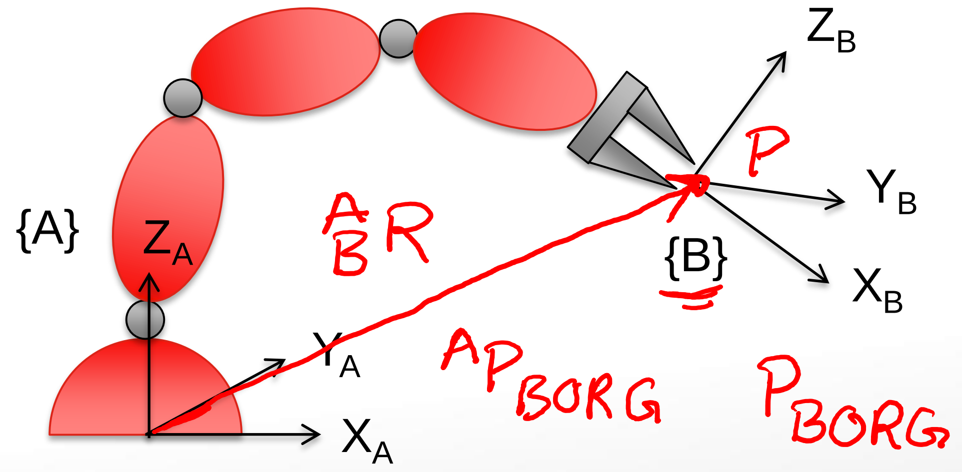
$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\theta=90^\circ} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

X_B in {A} Y_B in {A} Z_B in {A}

Description of a Frame

- So far, we have looked at position and orientation separately.
- To completely specify the whereabouts of the tool / object etc., we need both the position and orientation.

{B} ?

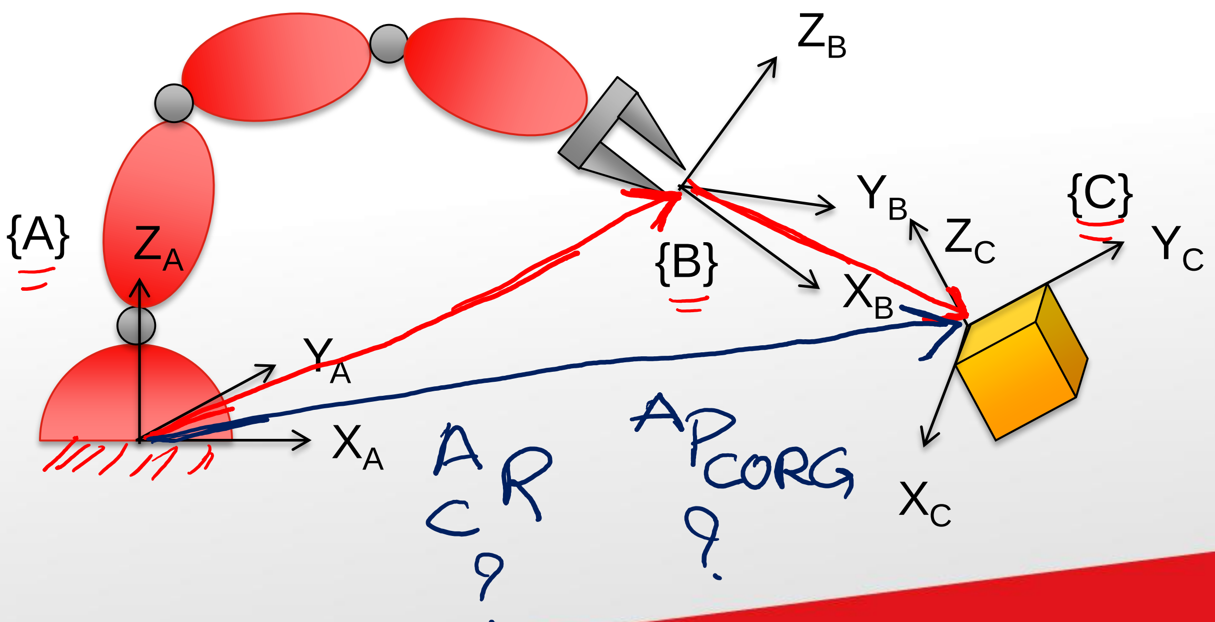


- Use the rotation matrix + the position vector of frame origin to specify:

$$\{B\} = \{ {}^{A}_{B}R, {}^{A}P_{BORG} \}$$

Introduction: Mapping

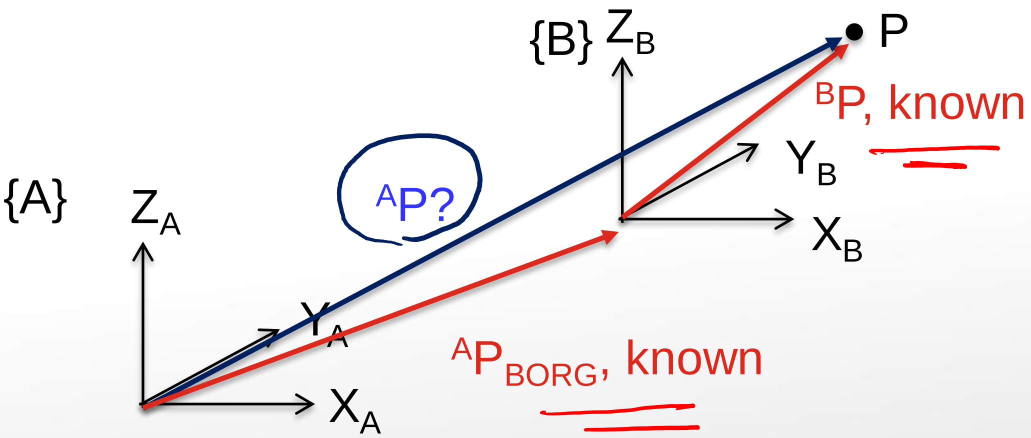
- Example scenario:
- You know the position and orientation of an object {c} described in frame {B}.
- Your know the position and orientation of frame {B} described in frame {A}.
- What is the position and orientation of the object {c} in coordinate system frame {A}?



Mapping involving Translated Frames

- If the frames differ only by a pure translation, then:

$$^{A}P = {}^{B}P + {}^{A}P_{BORG}$$

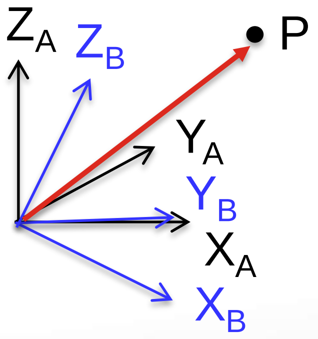


- Caution: Vector addition is allowed only if the orientation of the frames are the same!

Mapping involving Rotated Frames

- If the frames differ only by a pure rotation, then:

$${}^{A}P = {}^{A}_{B}R \cdot {}^{B}P$$



${}^{B}P$, known ←

Rotation matrix of {B} relative to {A} known ${}^{A}_{B}R$ ←

${}^{A}P$?

- Note: The notation is useful to keep track of mappings and frames of reference. E.g. cancellation of both B's in the above equation.

Mapping involving Rotated Frames

- Example:
  - Frames {B} is obtained by rotating {A} along the z-axis by 30deg.
  - In Frame {B}, the point P has a coordinate of $[1, 1, 0]^T$.
  - What is the coordinate of point P in frame {A}?

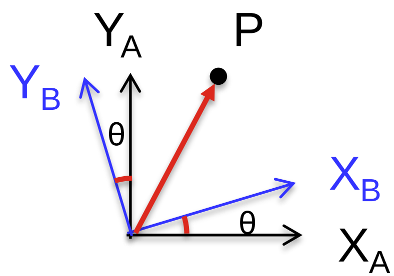
${}^A_B R$

${}^B P$

${}^A P$?

Case 3

Rot Z



30°

- Solution:
  - First, get description of {B} in {A}.

$$
{}^A_B R = \left[{}^A\hat{X}_B \quad {}^A\hat{Y}_B \quad {}^A\hat{Z}_B \right] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

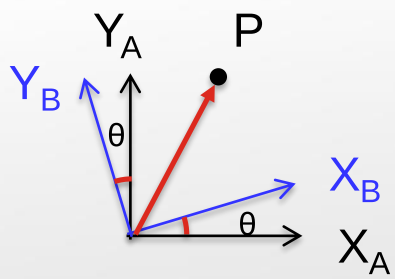
Reminder: X_B as seen in {A}

Mapping involving Rotated Frames

- Secondly, calculate: ${}^{A}P = {}^{A}_{B}R \cdot {}^{B}P$

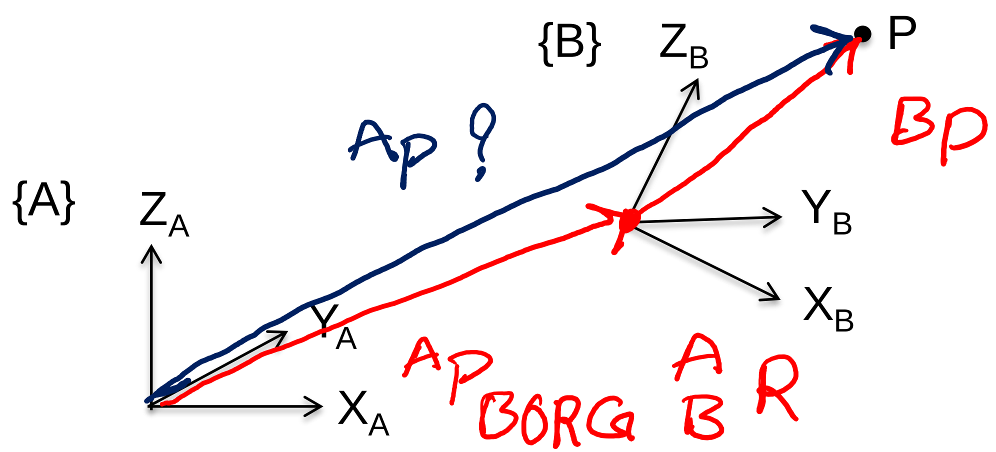
$$= \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.366 \\ 1.366 \\ 0 \end{bmatrix}$$

- The answer looks right from the figure.



Mappings involving both T and R

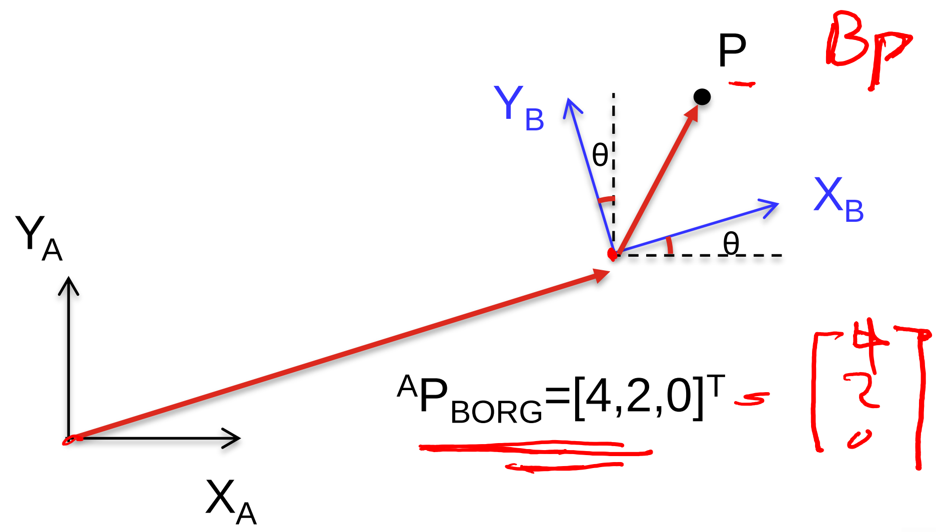
- Next we would like to know about general mappings involving both translation and rotation.



- The formula is: $^{A}P = {}^{A}_{B}R \, {}^{B}P + {}^{A}P_{BORG}$
- Interpretation:
  - 1 First, change the description of ^{B}P into a frame which has the same orientation of {A}.
  - 2 Then perform vector addition for translation. (Remember? Vector addition is allowed only if the frames have same orientation).

Mappings involving both T and R

- Example: Same rotation as in previous example, but now with translation of the frames:



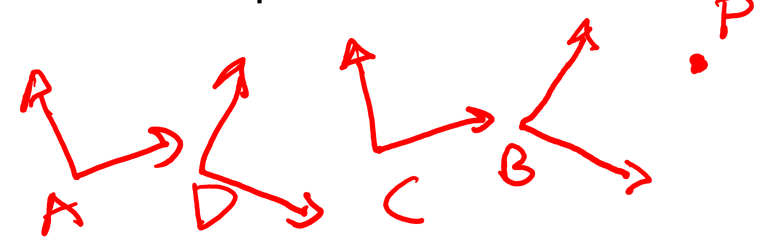
$$^{A}P = {}^{A}_{B}R \cdot {}^{B}P + {}^{A}P_{BORG} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.366 \\ 1.366 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.366 \\ 3.366 \\ 0 \end{bmatrix}$$

Ap

Homogeneous Transformation Matrix

- The equation ${}^{A}P = {}^{A}_{B}R \cdot {}^{B}P + {}^{A}P_{BORG}$ is not very appealing.
- Can we get an equation which is just a simple matrix multiplication, i.e.

$${}^{A}P = {}^{A}_{B}T \cdot {}^{B}P$$



Yes.

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} \underset{3\times3}{{}^{A}_{B}R} & \underset{3\times1}{{}^{A}P_{BORG}} \\ \hline 0 \quad 0 \quad 0 & 1 \end{array}\right]}_{4\times4} \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

$${}^{A}P = \underbrace{{}^{A}_{D}T\,{}^{D}_{C}T\,{}^{C}_{B}T}\,{}^{B}P$$

- The 4 x 4 matrix operator is called “Homogeneous Transform”.

Homogeneous Transformation Matrix

- The equation $^{A}P = {}^{A}_{B}R \cdot {}^{B}P + {}^{A}P_{BORG}$ is not very appealing.
- Can we get an equation which is just a simple matrix multiplication, i.e.

$$^{A}P = {}^{A}_{B}T \cdot {}^{B}P$$

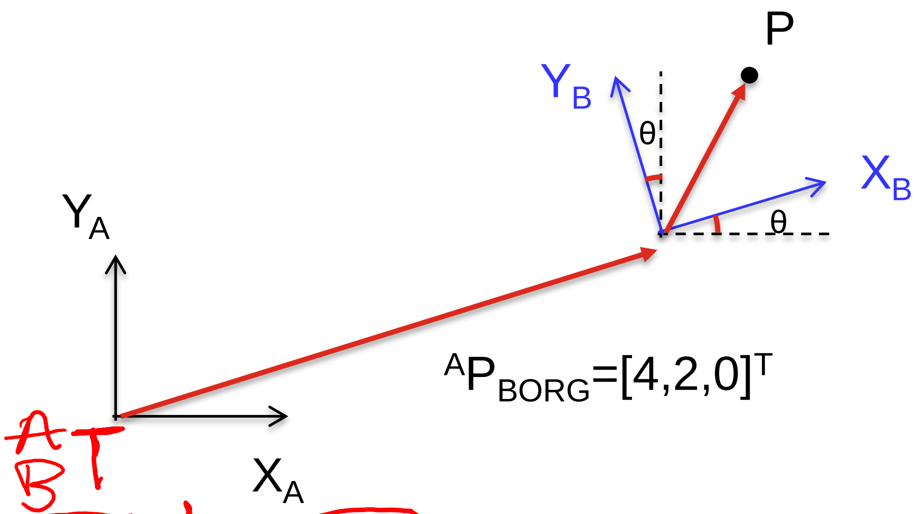
- Yes! By a clever trick:

$$\begin{bmatrix} ^{A}P \\ \hline 1 \end{bmatrix} = \underbrace{\left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}_{{}^{A}_{B}T} \begin{bmatrix} ^{B}P \\ \hline 1 \end{bmatrix}$$

- The 4 x 4 matrix operator is called "Homogeneous Transform".

Homogeneous Transformation Matrix

- Same example as just now:



$$\begin{bmatrix} {}^{A}P \\ \hdashline 1 \end{bmatrix} = \overbrace{\left[\begin{array}{c:c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hdashline \boxed{0 \quad 0 \quad 0} & \boxed{1} \end{array}\right]}^{{}^{A}_{B}T} \begin{bmatrix} {}^{B}P \\ \hdashline 1 \end{bmatrix} = \left[\begin{array}{ccc:c} 0.866 & -0.5 & 0 & 4 \\ 0.5 & 0.866 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 1 \\ 1 \\ 0 \\ \hdashline 1 \end{bmatrix} = \left.\begin{bmatrix} 4.366 \\ 3.366 \\ 0 \\ \hdashline 1 \end{bmatrix}\right\} {}^{A}P$$

- Just ignore the last row from the final result.

Summary of Interpretations

- The homogeneous transform, a 4 x 4 matrix containing orientation and position information, can be interpreted in three different ways:
  - Description of a frame.
    - ${}^{A}_{B}T$ describes the frame {B} relative to frame {A}.
  - Mapping.
    - ${}^{A}_{B}T$ maps ${}^{B}P$ in ${}^{A}P$.
    - Coordinate of point P in {B} → coordinate of point P in {A}.
  - Operator.
    - T operates on ${}^{A}P_1$ to create ${}^{A}P_2$.

Tutorial Assignments

- **Question 1:**
  - A position vector of a point in frame {B} is given by $$ {}^{B}P = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} $$
  - The origin of frame {B} is at [11,-3,9] with respect to frame {A}.
  - Also, frame {B} is rotated by 30 degrees along the z-axis of frame {A}.
  - What is the position vector of the point in frame {A}?
  - Also, write down the homogenous transformation matrix.

Tutorial Assignments

- **Question 2:**
  - The rotation matrix from A to B is:

$$ {}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

  - Interpret (by sketch) the meaning of the rotation matrix.
  - What is the rotation matrix from B to A?
    - Calculate using matrix inverse.
    - Verify that it is the same as R-transpose.

Tutorial Assignments

- **Question 3:**
  - A velocity vector is given by:

$$^{B}V = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

  - Given:

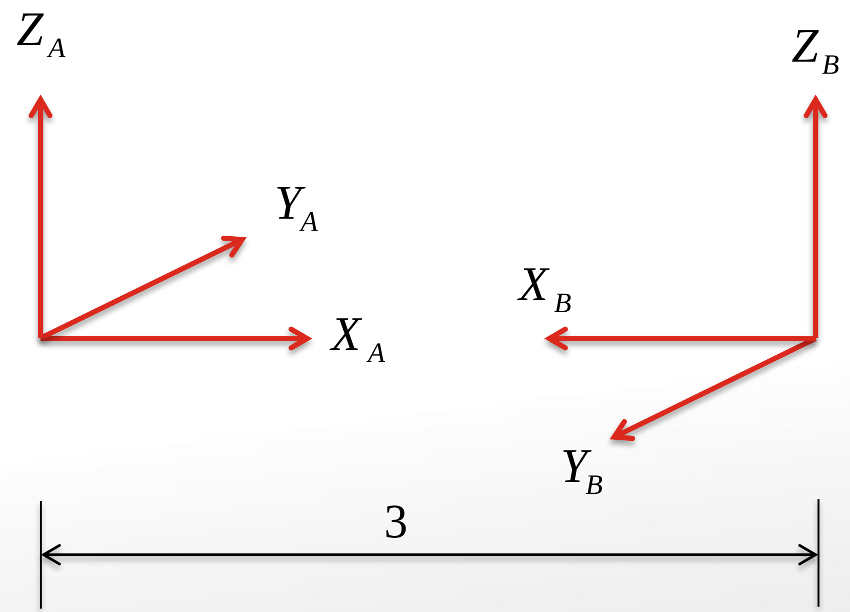
$$^{A}_{B}T = \begin{bmatrix} 0.866 & -0.5 & 0 & 11 \\ 0.5 & 0.866 & 0 & -3 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - Compute:

$$^{A}V$$

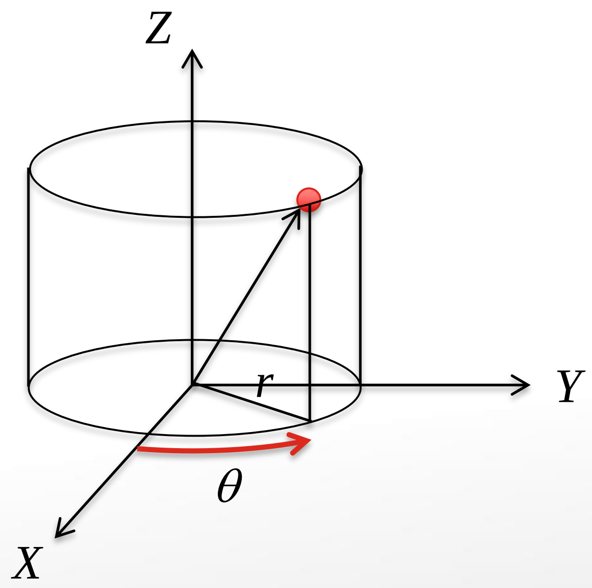
Tutorial Assignments

- **Question 4:**
  - Give the value of $^{A}_{B}T$



Tutorial Assignments

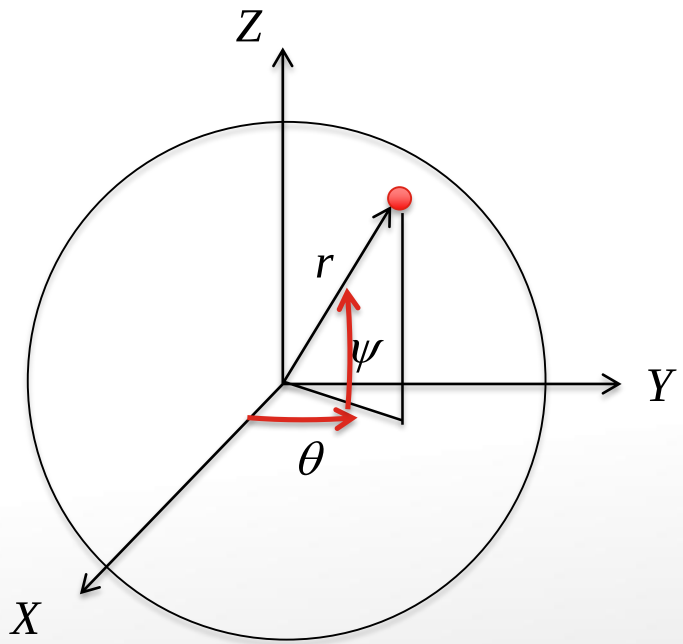
- **Question 5:**
  - In the lecture, the positions have been given in Cartesian coordinates.
  - We can also describe the positions in cylindrical coordinates.



  - If the point has Cartesian coordinates $[x, y, z]^T$, what is the position in cylindrical coordinates (function of θ, r, z)?

Tutorial Assignments

- **Question 6:**
  - Next, we can also describe the positions in spherical coordinates.



  - If the point has Cartesian coordinates $[x, y, z]^T$, what is the position in spherical coordinates (function of θ, r, ψ)?

Thank you!

Have a good evening.