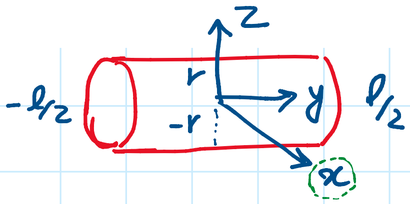
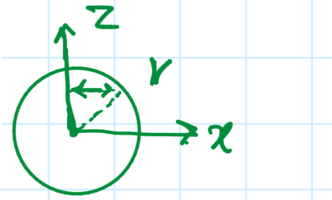
QS. 1

Wednesday, 2 September 2020 7:29 PM

- **Question 1:**
  - Find the inertia tensor of a right cylinder of homogenous density, with respect to a frame with origin at the center of mass of the body.
  - What is its inertia tensor with respect to a frame at one far end of the cylinder?



$$ {}^{c}I = \iiint \cdots \, dx\,dy\,dz $$



How does X vary w.r.t Z

in another word, at each level of z what x do I have?

from $\underbrace{-\sqrt{r^2-z^2}}_{-a} < x < \underbrace{+\sqrt{r^2-z^2}}_{a}$ ← to

$$: I_{xx} = \iiint_V (y^2+z^2)\,\rho\, dv $$

$$ = \int_{-r}^{r}\int_{-\ell/2}^{\ell/2}\int_{-\sqrt{r^2-z^2}}^{+\sqrt{r^2-z^2}} (y^2+z^2)\,\rho\, dx\,dy\,dz = $$

$$ = \int_{-r}^{r}\int_{-\ell/2}^{\ell/2} (y^2+z^2)\, x \Big|_{-\sqrt{r^2-z^2}}^{+\sqrt{r^2-z^2}} \rho\, dy\,dz = $$

$$ = \int_{-r}^{r}\int_{-\ell/2}^{\ell/2} 2(y^2+z^2)\sqrt{r^2-z^2}\,\rho\, dy\,dz = $$

$$ = \int_{-r}^{r} 2\left(\frac{y^3}{3}+z^2 y\right)\sqrt{r^2-z^2}\,\Big|_{-\ell/2}^{\ell/2} \rho\, dz = $$

$$ = \int_{-r}^{r} 2\left(\frac{\ell^3}{12}+z^2\ell\right)\sqrt{r^2-z^2}\,\rho\, dz $$

$$ = 2\int_{-r}^{r}\frac{\ell^3}{12}\sqrt{r^2-z^2}\,\rho\, dz + 2\int_{-r}^{r}\ell z^2\sqrt{r^2-z^2}\,\rho\, dz $$

https://en.wikipedia.org/wiki/Trigonometric_substitution More Info. on Integral of roots

$$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}(x/a)\right]$$

$$\int x^2\sqrt{a^2-x^2}\,dx = -\tfrac{1}{4}x(a^2-x^2)^{3/2} + \tfrac{1}{8}a^2\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)\right]$$

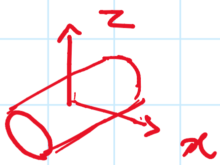
$$= \frac{l^3}{6}\rho\,\frac{1}{2}\left[z\sqrt{r^2-z^2} + r^2\sin^{-1}\left(\frac{z}{r}\right)\right]\Bigg|_{-r}^{r}$$

$$+ 2l\rho\left[-\frac{1}{4}z(r^2-z^2)^{3/2} + \frac{r^2}{8}\left[z\sqrt{r^2-z^2} + r^2\sin^{-1}\left(\frac{z}{r}\right)\right]\right]\Bigg|_{-r}^{r}$$

$$= \frac{l^3}{12}\rho r^2\pi + 2l\rho\frac{r^4}{8}\pi$$

$$I_{xx} = \left(\frac{l^2}{12} + \frac{r^2}{4}\right)\underbrace{\underbrace{\pi r^2 l}_{V}\,\rho}_{M} = \frac{1}{12}Ml^2 + \frac{1}{4}Mr^2$$

$: I_{zz} = I_{xx}$



$$: I_{yy} = \iiint_V (x^2+z^2)\rho\,dv$$

$$= \int_{-r}^{r}\int_{-l/2}^{l/2}\int_{-\sqrt{r^2-z^2}}^{+\sqrt{r^2-z^2}} (x^2+z^2)\rho\,dx\,dy\,dz =$$

$$= \int_{-r}^{r}\int_{-l/2}^{l/2}\left(\frac{x^3}{3} + z^2x\right)\Bigg|_{-\sqrt{r^2-z^2}}^{+\sqrt{r^2-z^2}}\rho\,dy\,dz =$$

$$= \int_{-r}^{r}\int_{-l/2}^{l/2}\left(2\frac{(\sqrt{r^2-z^2})^3}{3} + 2z^2\sqrt{r^2-z^2}\right)\rho\,dy\,dz =$$

$$= \int_{-r}^{r}\left(2\frac{(\sqrt{r^2-z^2})^3}{3} + 2z^2\sqrt{r^2-z^2}\right)y\Bigg|_{-l/2}^{l/2}\rho\,dz =$$

$$= \int_{-r}^{r} \left(2\frac{(\sqrt{r^2-z^2})^3}{3} + 2z^2\sqrt{r^2-z^2} \right) y \Big|_{-l/2}^{l/2} \rho\, dz =$$

$$= \int_{-r}^{r} \left(2\frac{(\sqrt{r^2-z^2})^3}{3} + 2z^2\sqrt{r^2-z^2} \right) l \rho\, dz =$$

Check Integration Table from Math TextBooks or online Tools

$$= \left\{ \begin{array}{l} 2/3\, r^2 \cdot 1/2 \left[z\sqrt{r^2-z^2} + r^2 \sin^{-1}\left(\frac{z}{r}\right) \right] \\ -4/3 \cdot 1/4 \cdot z\sqrt{r^2-z^2} + 4/24\, r^2 \left[z\sqrt{r^2-z^2} + r^2 \sin^{-1}\left(\frac{z}{r}\right) \right] \end{array} \right\} \Bigg|_{-r}^{r}$$

$$= 1/2\, r^4 \pi l \rho$$

$$I_{yy} = 1/2 \underbrace{\underbrace{\pi r^2 l}_{V} \rho}_{M} r^2 = 1/2\, M r^2$$

: $I_{xy} = \iiint_V xy\, \rho\, dv$

$$= \int_{-r}^{r} \int_{-l/2}^{l/2} \int_{-\sqrt{r^2-z^2}}^{+\sqrt{r^2-z^2}} xy\, \rho\, dx\, dy\, dz =$$

$$= \int_{-r}^{r} \int_{-l/2}^{l/2} \underbrace{1/2\, x^2 \Big|_{-\sqrt{r^2-z^2}}^{+\sqrt{r^2-z^2}}}_{0} y \rho\, dy\, dz = 0$$

Similarly $I_{xz} = 0$ & $I_{yz} = 0$

summary:

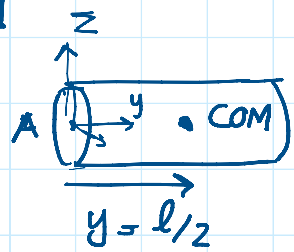
$$^{C}I = \begin{bmatrix} 1/12\, M l^2 + 1/4\, M r^2 & & \\ 0 & 1/2\, M r^2 & \\ 0 & & 1/12\, M l^2 + 1/4\, M r^2 \end{bmatrix}$$

what about the I w.r.t the face of one end ?

$\rightarrow$ use parallel axis theorem

$${}^{A}I = {}^{C}I + M\left[P_C^T P_C \, I - P_C P_C^T \right]$$

in this case $\quad P_C^T = \begin{bmatrix} 0 & l/2 & 0 \end{bmatrix}$



$${}^{A}I = \begin{bmatrix} 1/12\, Ml^2 + 1/4\, Mr^2 & & \\ 0 & 1/2\, Mr^2 & \\ 0 & & 1/12\, Ml^2 + 1/4\, Mr^2 \end{bmatrix}$$

$$+ M \left[\begin{bmatrix} 0 & l/2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ l/2 \\ 0 \end{bmatrix} \cdot I_3 - \begin{bmatrix} 0 \\ l/2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & l/2 & 0 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1/12\, Ml^2 + 1/4\, Mr^2 & & \\ 0 & 1/2\, Mr^2 & \\ 0 & & 1/12\, Ml^2 + 1/4\, Mr^2 \end{bmatrix}$$

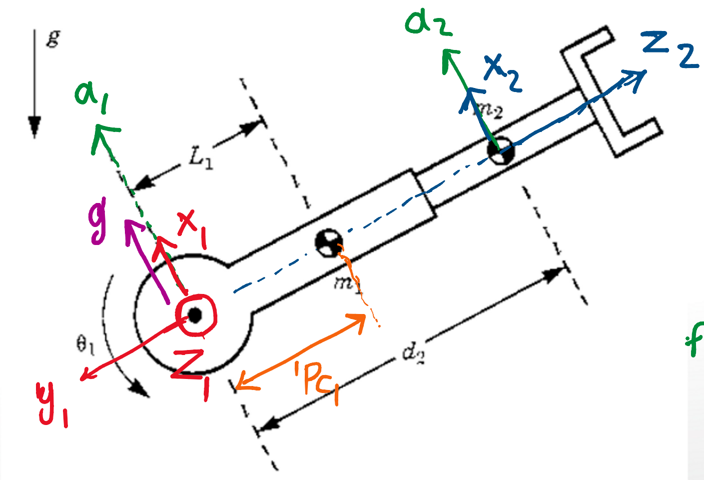
$$+ M \left[\begin{bmatrix} l^2/4 & 0 & 0 \\ 0 & l^2/4 & 0 \\ 0 & 0 & l^2/4 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & l^2/4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]$$

$${}^{A}I = \begin{bmatrix} 1/12\, Ml^2 + 1/4\, Mr^2 + 1/4\, Ml^2 & & \\ 0 & 1/2\, Mr^2 & \\ 0 & & 1/12\, Ml^2 + 1/4\, Mr^2 + 1/4\, Ml^2 \end{bmatrix}$$

QS. 2

Wednesday, 2 September 2020 7:30 PM

- **Question 2:**
  - Consider the following robot with:



$$^{C_1}I_1 = \begin{bmatrix} I_{xx_1} & 0 & 0 \\ 0 & I_{yy_1} & 0 \\ 0 & 0 & I_{zz_1} \end{bmatrix}$$

$$^{C_2}I_2 = \begin{bmatrix} I_{xx_2} & 0 & 0 \\ 0 & I_{yy_2} & 0 \\ 0 & 0 & I_{zz_2} \end{bmatrix}$$

For Last Joint (2)

align a_2 to a_1

$x_2 \parallel x_1$

  - Derive its dynamic equations.

You need to get the frames, then get ${}^0_1R$, ${}^1_2R$, ${}^1_2P$, $^1P_{C_1}$, $^2P_{C_2}$, ...

* Recall from Lecture / week 3 / Forward kinematics

For Last Joint n, align a_n with a_{n-1}

Here $n=2$ $a_2 \parallel a_1 \Rightarrow x_2 \parallel x_1$ when $\theta_1 = 0$

Rotation Matrices:

$${}^0_1R: \text{ Rot. } \underset{\text{variable}}{\theta_1} \text{ about Z-axis}: \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

from {1} to {2}

$${}^1_2R: \text{ Rot. } 90^\circ \text{ about X-axis}: \begin{bmatrix} 1 & 0 & 0 \\ 0 & C90 & -S90 \\ 0 & S90 & C90 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Position Vectors:

$$^1P_2 = \begin{bmatrix} 0 \\ -d_2 \\ 0 \end{bmatrix}, \quad \text{For acc Propagation} \rightarrow \ ^1P_{C_1} = \begin{bmatrix} 0 \\ -L_1 \\ 0 \end{bmatrix} \quad ^2P_{C_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

↑ Preparation

↓ Now start Iterative Newton-Euler Algorithm

– Start with $\{0\}$

$$^0\omega_0 = 0 \quad , \quad ^0\dot{\omega}_0 = 0 \quad , \quad ^0\dot{v}_0 = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix}$$

– frame $\{1\}$ / Link 1

$$^1\omega_1 = ({}^1_0R \cdot \underbrace{^0\omega_0}_{0}) + (\dot{\theta}_1 \, {}^1\hat{Z}_1) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$^1\dot{\omega}_1 = ({}^1_0R \cdot \underbrace{^0\dot{\omega}_0}_{0}) + ({}^1_0R \cdot \underbrace{^0\omega_0}_{0} \times \dot{\theta}_1 \, {}^1\hat{Z}_1) + (\ddot{\theta}_1 \, {}^1\hat{Z}_1) = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}$$

$$^1\dot{v}_1 = \underline{({}^1_0R \cdot {}^0\dot{v}_0)} + (2 \cdot {}^1\omega_1 \times \underbrace{\dot{d}_1}_{0} \, {}^1\hat{Z}_1) + (\underbrace{\ddot{d}_1}_{0} \, {}^1\hat{Z}_1)$$

$$+ {}^1_0R({}^0\dot{\omega}_0 \times \underbrace{^0P_1}_{0} + {}^0\omega_0 \times {}^0\omega_0 \times \underbrace{^0P_1}_{0})$$

$$= \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix} = \underline{\begin{bmatrix} c_1 g \\ -s_1 g \\ 0 \end{bmatrix}}$$

COM: $^1\dot{v}_{C_1} = {}^1\dot{v}_1 + ({}^1\dot{\omega}_1 \times {}^1P_{C_1}) + ({}^1\omega_1 \times ({}^1\omega_1 \times {}^1P_{C_1}))$

$$= \begin{bmatrix} c_1 g \\ -s_1 g \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 \\ 0 & -L_1 & 0 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ 0 & -L_1 & 0 \end{vmatrix}$$

$$= \begin{bmatrix} c_1 g \\ -s_1 g \\ 0 \end{bmatrix} + \begin{bmatrix} L_1\ddot{\theta}_1 \\ 0 \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ L_1\dot{\theta}_1 & 0 & 0 \end{vmatrix}$$

$$= \begin{bmatrix} c_1 g + L_1 \ddot{\theta}_1 \\ -s_1 g + L_1 \dot{\theta}_1^2 \\ 0 \end{bmatrix}$$

$F = ma$: $\quad {}^1F_1 = m_1 \, {}^1\dot{v}_{C_1} = \begin{bmatrix} m_1 c_1 g + m_1 L_1 \ddot{\theta}_1 \\ -m_1 s_1 g + m_1 L_1 \dot{\theta}_1^2 \\ 0 \end{bmatrix}$

$${}^1N_1 = {}^{C_1}I_1 \, {}^1\dot{\omega}_1 + {}^1\omega_1 \times {}^{C_1}I_1 \, {}^1\omega_1$$

$$= \underbrace{\begin{bmatrix} I_{xx1} & & \\ & I_{yy1} & \\ & & I_{zz1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} I_{xx1} & & \\ & I_{yy1} & \\ & & I_{zz1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 \\ 0 \\ I_{zz1} \ddot{\theta}_1 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ 0 & 0 & I_{zz1} \dot{\theta}_1 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ I_{zz1} \ddot{\theta}_1 \end{bmatrix}$$

1^{st} Link is done ☺

– frame {2} / Link 2 (Note: we have a Prismatic Joint here)

$${}^2\omega_2 = ({}^2_1R \cdot {}^1\omega_1) + (\dot{\theta}_2 \, {}^2\hat{Z}_2) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}$$

$\dot{\theta}_2 = 0$

$\ddot{\theta}_2 = 0$

$${}^2\dot{\omega}_2 = \underbrace{({}^2_1R \cdot {}^1\dot{\omega}_1)} + \overbrace{({}^2_1R \cdot {}^1\omega_1 \times \underbrace{\dot{\theta}_2}_{0} \, {}^2\hat{Z}_2)}^{0} + (\underbrace{\ddot{\theta}_2}_{0} \, {}^2\hat{Z}_2) = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{bmatrix}$$

Note: we First Calculate this before X Product

$${}^2\dot v_2 = ({}^2_1R \cdot {}^1\dot v_1) + (2\ {}^2\omega_2 \times \dot d_2\ {}^2\hat Z_2) + (\ddot d_2\ {}^2\hat Z_2)$$

$$+\ {}^2_1R\left({}^1\dot\omega_1 \times {}^1P_2 + {}^1\omega_1 \times ({}^1\omega_1 \times {}^1P_2)\right)$$

First Calc. This

$$= \begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}\begin{bmatrix}c_1 g\\ -s_1 g\\ 0\end{bmatrix} + 2\begin{vmatrix} i & j & k\\ 0 & \dot\theta_1 & 0 \\ 0 & 0 & \dot d_2\end{vmatrix} + \begin{bmatrix}0\\0\\ \ddot d_2\end{bmatrix}$$

$$+ \begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}\left\{\begin{vmatrix} i & j & k\\ 0 & 0 & \ddot\theta_1 \\ 0 & -d_2 & 0\end{vmatrix} + \begin{bmatrix}0\\0\\ \dot\theta_1\end{bmatrix} \times \begin{vmatrix} i & j & k\\ 0 & 0 & \dot\theta_1 \\ 0 & -d_2 & 0\end{vmatrix}\right\}$$

$$= \begin{bmatrix}c_1 g\\ 0\\ s_1 g\end{bmatrix} + 2\begin{bmatrix}\dot\theta_1 \dot d_2\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0\\0\\ \ddot d_2\end{bmatrix}$$

$$+ \begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}\left\{\begin{bmatrix}d_2\ddot\theta_1\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0\\ d_2\dot\theta_1\\ 0\end{bmatrix}\right\}$$

$$= \begin{bmatrix}c_1 g + 2\dot\theta_1\dot d_2 + d_2\ddot\theta_1\\ 0\\ s_1 g + \ddot d_2 - d_2\dot\theta_1^2\end{bmatrix}$$

COM: $${}^2\dot v_{C_2} = {}^2\dot v_2 + ({}^2\dot\omega_2 \times \overbrace{{}^2P_{C_2}}^{0}) + ({}^2\omega_2 \times ({}^2\omega_2 \times \overbrace{{}^2P_{C_2}}^{0}))$$

$$= \begin{bmatrix}c_1 g + 2\dot\theta_1\dot d_2 + d_2\ddot\theta_1\\ 0\\ s_1 g + \ddot d_2 - d_2\dot\theta_1^2\end{bmatrix}$$

F=ma : $${}^2F_2 = m_2\,{}^2\dot{v}_{C_2} = \begin{bmatrix} m_2 c_1 g + 2m_2\dot{\theta}_1\dot{d}_2 + m_2 d_2\ddot{\theta}_1 \\ 0 \\ m_2 s_1 g + m_2\ddot{d}_2 - m_2 d_2\dot{\theta}_1^2 \end{bmatrix}$$

$${}^2N_2 = {}^{C_2}I_2\,{}^2\dot{\omega}_2 + {}^2\omega_2 \times {}^{C_2}I_2\,{}^2\omega_2$$

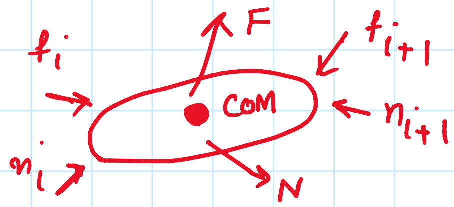
$$= \underbrace{\begin{bmatrix} I_{xx2} & & \\ & I_{yy2} & \\ & & I_{zz2} \end{bmatrix}\begin{bmatrix} 0 \\ \ddot{\theta}_1 \\ 0 \end{bmatrix}} + \underbrace{\begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} I_{xx2} & & \\ & I_{yy2} & \\ & & I_{zz2} \end{bmatrix}\begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 0 \\ I_{yy2}\ddot{\theta}_1 \\ 0 \end{bmatrix} + \underbrace{\begin{vmatrix} i & j & k \\ 0 & \dot{\theta}_1 & 0 \\ 0 & I_{yy2}\dot{\theta}_1 & 0 \end{vmatrix}}_{0} = \begin{bmatrix} 0 \\ I_{yy2}\ddot{\theta}_1 \\ 0 \end{bmatrix}$$

2nd Link is done

outward iteration done !! ☺

↓ Now start Inward Iteration



- Start with End of Robot

assumed not touching environment : ${}^3f_3 = 0$ ${}^3n_3 = 0$

- Joint 2

$${}^2f_2 = \overbrace{{}^2_3R \cdot \underbrace{{}^3f_3}_{0}}^{0} + {}^2F_2 = \begin{bmatrix} {}^2F_{2x} \\ {}^2F_{2y} \\ {}^2F_{2z} \end{bmatrix}$$

$${}^2n_2 = {}^2_3R \cdot \underbrace{{}^3n_3}_{0} + \underbrace{{}^2P_{C_2}}_{0} \times {}^2F_2 + {}^2P_3 \times {}^2_3R \cdot \underbrace{{}^3f_3}_{0} + {}^2N_2 = \begin{bmatrix} {}^2N_{2x} \\ {}^2N_{2y} \\ {}^2N_{2z} \end{bmatrix}$$

- Joint 1

$${}^1f_1 = {}^1_2R \cdot {}^2f_2 + {}^1F_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} {}^2F_{2x} \\ {}^2F_{2y} \\ {}^2F_{2z} \end{bmatrix} + \begin{bmatrix} {}^1F_{1x} \\ {}^1F_{1y} \\ {}^1F_{1z} \end{bmatrix} = \begin{bmatrix} {}^2F_{2x} + {}^1F_{1x} \\ -{}^2F_{2y} + {}^1F_{1y} \\ {}^2F_{2z} + {}^1F_{1z} \end{bmatrix} \leftarrow \text{3rd row}$$

$$ {}^1n_1 = \underbrace{{}^1_2R \cdot {}^2n_2}_{} + \underbrace{{}^1P_{c_1} \times {}^1F_1}_{} + \underline{{}^1P_2 \times {}^1_2R} \cdot {}^2f_2 + \underline{\underline{{}^1N_1}} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} {}^2N_{2x} \\ {}^2N_{2y} \\ {}^2N_{2z} \end{bmatrix} + \begin{vmatrix} i & j & K \\ 0 & -L_1 & 0 \\ {}^1F_{1x} & {}^1F_{1y} & {}^1F_{1z} \end{vmatrix}$$

$$+ \begin{bmatrix} 0 \\ -d_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} {}^2F_{2x} \\ {}^2F_{2y} \\ {}^2F_{2z} \end{bmatrix} + \begin{bmatrix} {}^1N_{1x} \\ {}^1N_{1y} \\ {}^1N_{1z} \end{bmatrix}$$

$$\text{3rd row} \rightarrow = \begin{bmatrix} * \\ * \\ {}^2N_{2y} \end{bmatrix} + \begin{bmatrix} * \\ * \\ L_1 \cdot {}^1F_{1x} \end{bmatrix} + \begin{vmatrix} i & j & K \\ 0 & -d_2 & 0 \\ {}^2F_{2x} & {}^2F_{2y} & {}^2F_{2z} \end{vmatrix} + \begin{bmatrix} * \\ * \\ {}^1N_{1z} \end{bmatrix}$$

$$= \begin{bmatrix} * \\ * \\ I_{yy2}\ddot{\theta}_1 + m_1L_1c_1g + m_1L_1^2\ddot{\theta}_1 + m_2c_1d_2g + m_2d_2^2\ddot{\theta}_1 \\ + 2m_2d_2\dot{\theta}_1\dot{d}_2 + I_{zz1}\ddot{\theta}_1 \end{bmatrix}$$

Finally since this is a R-P robot

$\tau_1 =$ 3rd row of ${}^1n_1$ (Revolute)

$\tau_2 =$ 3rd row of ${}^1f_1$ (Prismatic)

$$\tau_1 = I_{yy2}\ddot{\theta}_1 + m_1L_1c_1g + m_1L_1^2\ddot{\theta}_1 + m_2c_1d_2g + m_2d_2^2\ddot{\theta}_1 + 2m_2d_2\dot{\theta}_1\dot{d}_2 + I_{zz1}\ddot{\theta}_1$$

$$\tau_2 = {}^2F_{2z} + {}^1F_{1z} =$$

$$\tau_2 = {}^2F_{2z} + {}^1F_{1z} =$$
$$= m_2 s_1 g - m_2 d_2 \dot{\theta}_1^2 + m_2 \ddot{d}_2$$

Last Step: Put into $M\ddot{q} + V + G = \tau$

$$\begin{bmatrix} I_{yy2} + m_1 L_1^2 + m_2 d_2^2 + I_{zz1} & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix}$$
$$+ \underbrace{\begin{bmatrix} 0 \\ -m_2 d_2 \dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 \\ 0 \end{bmatrix}}_{V} + \underbrace{\begin{bmatrix} m_1 L_1 c_1 g + m_2 c_1 d_2 g \\ m_2 s_1 g \end{bmatrix}}_{G} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Thats all ☺