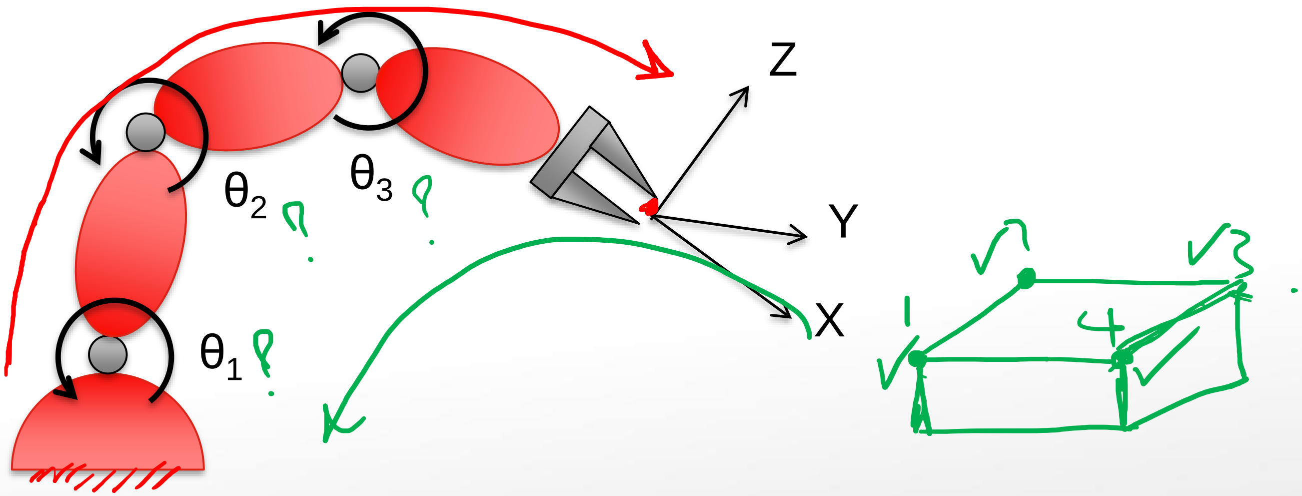
Content

Introduction

- In previous section, we studied the following problem:
  - Given the joint space parameters (angles for revolute joints, or offsets for prismatic joints), as well as the lengths of the links, what is the position and orientation of the end-effector in Cartesian space?



- Now, we will look at the inverse problem (much more difficult!):
  - Given the desired position and orientation of the tool in Cartesian space, what is the set of joint angles that is required to achieve the desired outcome?

Introduction

6 DoF

Match

Fwd Kin

Known

- For example, consider a 6-link robot.
- The homogeneous transform from base to link 6 is:

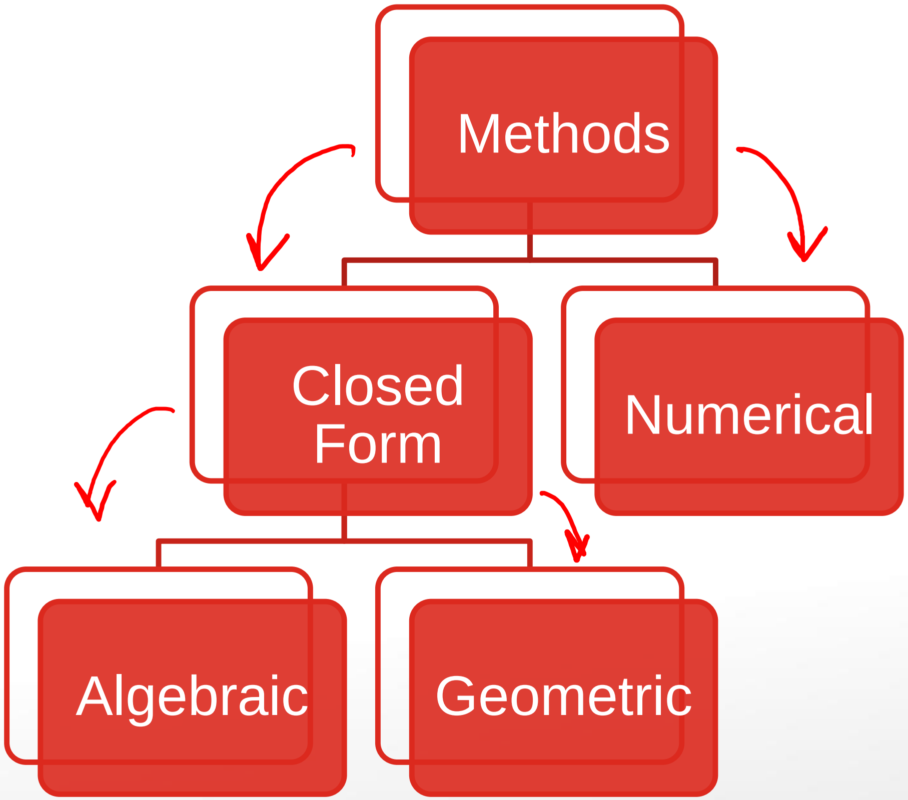
$$
{}^{0}_{6}T = \underbrace{\begin{bmatrix} f_{11}(q_1,\cdots,q_n) & f_{12}(q_1,\cdots,q_n) & f_{13}(q_1,\cdots,q_n) & f_{14}(q_1,\cdots,q_n) \\ f_{21}(q_1,\cdots,q_n) & f_{22}(q_1,\cdots,q_n) & f_{23}(q_1,\cdots,q_n) & f_{24}(q_1,\cdots,q_n) \\ f_{31}(q_1,\cdots,q_n) & f_{32}(q_1,\cdots,q_n) & f_{33}(q_1,\cdots,q_n) & f_{34}(q_1,\cdots,q_n) \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{General}} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Particular / Wanted}}
$$

9 Rot

3 Pos

  - There are 12 non-trivial values in the matrix.
  - From the 9 values related to rotation, only 3 are independent.
  - And we have 3 values related to position.
  - Therefore, there are altogether 6 values / equations.
- From the homogeneous transform, we would like to find the 6 joint angles.
- 6 equations and 6 unknowns ☺
- However, it is not easy to solve…

Methods of Solutions



- We will only look at the closed-form solutions.
- Note: There is no general solution. Every robot has to be analysed in a case-by-case basis.

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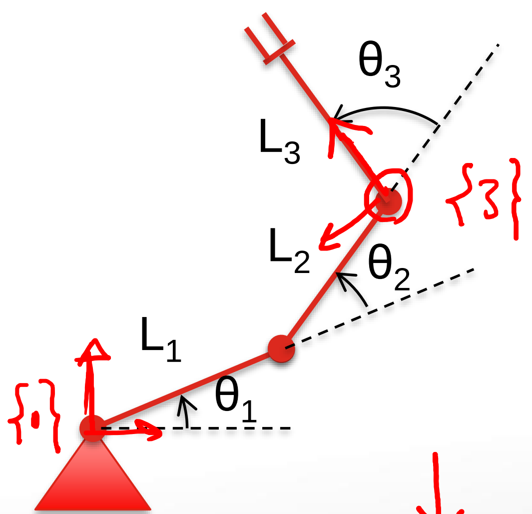
  - Geometric Approach

Algebraic Solutions

- Because there is no general algorithm to solve the inverse kinematic problems, we will only show an example to present the idea.
- E.g. 3-link RRR manipulator.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3



- The general transformation from frame {0} to frame {3} was:

Fwd Kin

$$ {}^0_3T = {}^0_1T \cdot {}^1_2T \cdot {}^2_3T = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & L_2 c\theta_{12} + L_1 c\theta_1 \\ s\theta_{123} & c\theta_{123} & 0 & L_2 s\theta_{12} + L_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Algebraic Solutions

Fwd Kin

$$= \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & L_2 c\theta_{12} + L_1 c\theta_1 \\ s\theta_{123} & c\theta_{123} & 0 & L_2 s\theta_{12} + L_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ 1

- Assume we want to put the end-effector at $[x, y, 0]^T$ with orientation Φ.
- Thus the specific transformation is:

known

$${}^0_3T = \begin{bmatrix} c_\phi & -s_\phi & 0 & x \\ s_\phi & c_\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ 2

- By comparing the general transform and the specific transform, we see that there are four equations and three unknowns:
  - Equations

$$\begin{aligned} c_\phi &= c_{123} \\ s_\phi &= s_{123} \\ x &= L_1 c_1 + L_2 c_{12} \\ y &= L_1 s_1 + L_2 s_{12} \end{aligned}$$

Unknowns:

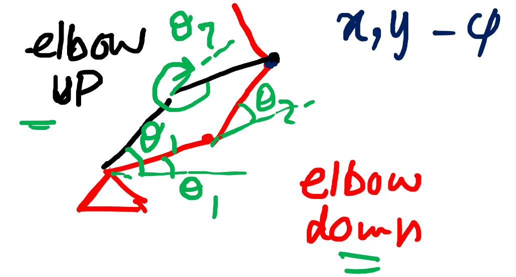
$$\theta_1 \quad \theta_2 \quad \theta_3$$

?

Algebraic Solutions

θ_1, θ_2

$$
\begin{cases}
c_\phi = c_{123} \\
s_\phi = s_{123} \\
x = L_1 c_1 + L_2 c_{12} \\
y = L_1 s_1 + L_2 s_{12}
\end{cases}
$$



- First, square both x and y equations and add them:

tricks;
- square & add
- trigonometry sub.

$\cos(a-b) = c_a c_b - s_a s_b$

$$x^2 = L_1^2 c_1^2 + 2L_1L_2 c_1 c_{12} + L_2^2 c_{12}^2$$

$$y^2 = L_1^2 s_1^2 + 2L_1L_2 s_1 s_{12} + L_2^2 s_{12}^2$$

add

$$x^2 + y^2 = L_1^2 \times 1 + 2L_1L_2(c_1 c_{12} + s_1 s_{12}) + L_2^2 \times 1$$

$$= L_1^2 + 2L_1L_2 \cos(\theta_1 - (\theta_1 + \theta_2)) + L_2^2$$

$$= L_1^2 + 2L_1L_2 c_2 + L_2^2$$

$\cos(\theta_1 - (\theta_1 + \theta_2)) = \cos(-\theta_2) = \cos(\theta_2)$

Note:
cos(- θ) = cos(θ)

- Thus: $$c_2 = \frac{x^2 + y^2 - L_1^2 - L_2^2}{2L_1L_2}$$

  - This value must be between -1 and 1 for a solution to exist.

- We also write: $s_2 = \pm\sqrt{1 - c_2^2}$ where c_2 is a value calculated above.

$s_2^2 + c_2^2 = 1$

Algebraic Solutions

- With these, we can compute θ$_2$ using:

$$\theta_2 = \arctan 2(s_2, c_2)$$

- Note: The choice of sign in $s_2 = \pm\sqrt{1-c_2^2}$ corresponds to the "elbow-up" or "elbow-down" solutions.
  - This is an example of multiple solutions.

θ$_2$ ✓

- Next, we shall try to solve for θ$_1$.

known known

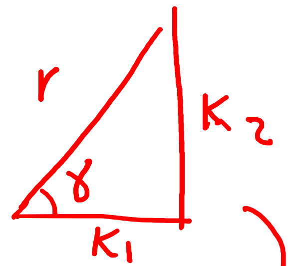
$$\begin{cases} x = L_1c_1 + L_2c_{12} = L_1c_1 + L_2c_1c_2 - L_2s_1s_2 = (L_1 + L_2c_2)c_1 - (L_2s_2)s_1 = K_1c_1 - K_2s_1 \\ y = L_1s_1 + L_2s_{12} = L_1s_1 + L_2s_1c_2 + L_2c_1s_2 = (L_1 + L_2c_2)s_1 + (L_2s_2)c_1 = K_1s_1 + K_2c_1 \end{cases}$$

- where $\begin{cases} K_1 = L_1 + L_2c_2 \\ K_2 = L_2s_2 \end{cases}$

Algebraic Solutions

- Introduce:

$$r = +\sqrt{K_1^2 + K_2^2}$$
$$\gamma = \arctan 2(K_2, K_1)$$



- Then K_1 and K_2 can be written as:

$$K_1 = r\cos\gamma$$
$$K_2 = r\sin\gamma$$

- With these, x and y can be written as:

K1 K2

$$x = (r\cos\gamma)c_1 - (r\sin\gamma)s_1$$
$$y = (r\cos\gamma)s_1 + (r\sin\gamma)c_1$$

K1 K2

cos(a+b)

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1 = \cos(\gamma+\theta_1)$$
$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1 = \sin(\gamma+\theta_1)$$

sin(a+b)

- Finally:

$$\gamma + \theta_1 = \arctan 2\left(\frac{y}{r}, \frac{x}{r}\right)$$
$$\theta_1 = \arctan 2\left(\frac{y}{r}, \frac{x}{r}\right) - \arctan 2(K_2, K_1)$$

Algebraic Solutions

θ_1, θ_2

- Finally, we can solve for θ_3 easily:

$$\left\{\begin{aligned} c_\phi &= c_{123} \\ s_\phi &= s_{123} \end{aligned}\right. \rightarrow \tan(\underbrace{\theta_1+\theta_2+\theta_3}) = \frac{s_\phi}{c_\phi}$$

Known $\leftarrow$

$$\theta_1+\theta_2+\theta_3 = \arctan 2(s_\phi, c_\phi)$$

$$\theta_3 = \arctan 2(s_\phi, c_\phi) - \theta_1 - \theta_2$$

Note:
θ_1 and θ_2 known

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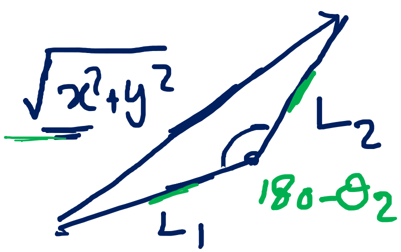
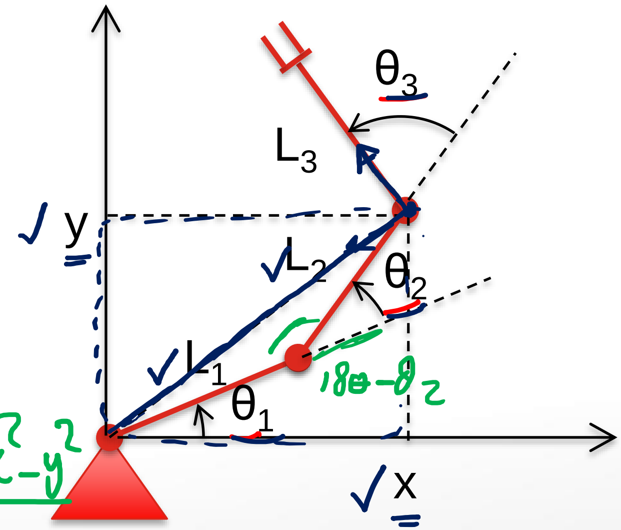
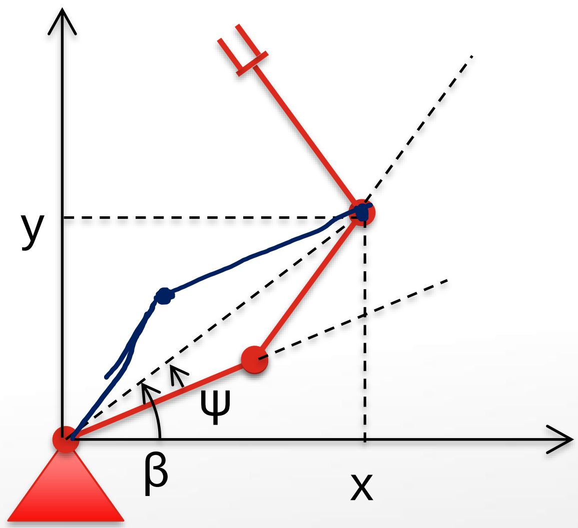
Geometric Solutions

- Sometimes (for planar robot), the inverse kinematic problem can be solved easier using geometric approach.
- Again, this is done on a case-by-case basis.
- Same example:

θ_2







$\cos(180-\theta_2) = \dfrac{L_1^2+L_2^2-x^2-y^2}{2L_1L_2}$

$= -\cos\theta_2$

- Using cosine rule:

$$x^2+y^2 = L_1^2+L_2^2-2L_1L_2\cos(180-\theta_2)$$
$$= L_1^2+L_2^2+2L_1L_2\cos\theta_2$$

$$c_2 = \frac{x^2+y^2-L_1^2-L_2^2}{2L_1L_2}$$

- Using symmetry for “Elbow-up” case: $\theta_2' = -\theta_2$

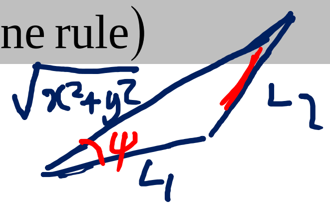
Geometric Solutions

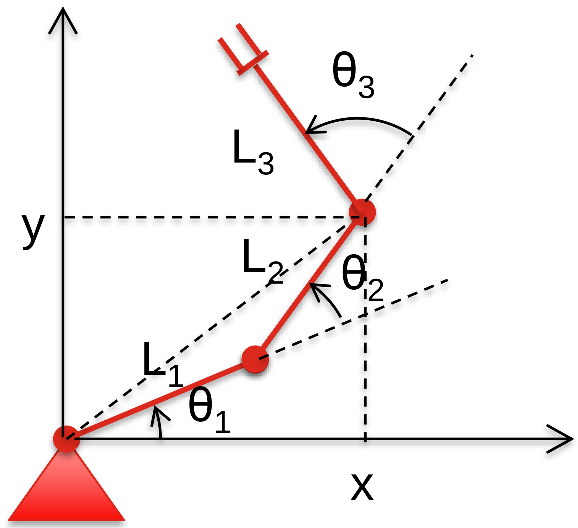
- To solve for θ_1, note that:

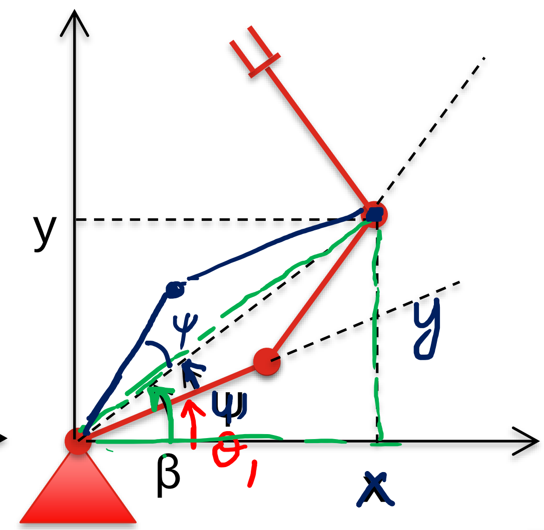
$$\beta = \arctan 2(y, x)$$

$$\cos\psi = \frac{L_1^2 + x^2 + y^2 - L_2^2}{2L_1\sqrt{x^2+y^2}}$$

(cosine rule)







$\theta_1 = \beta - \psi$

- The arc-cosine must be solved so that $0 \le \psi \le 180^\circ$
- Finally: $\theta_1 = \beta - \psi$ and using symmetry for "elbow up" case: $\theta_1 = \beta + \psi$

- θ_3 can be solved easily, because the sum of joint angles = final orientation.

$$\theta_1 + \theta_2 + \theta_3 = \phi \quad \Rightarrow \quad \theta_3 = \phi - \theta_1 - \theta_2$$

known

Thank you!

Have a good evening.