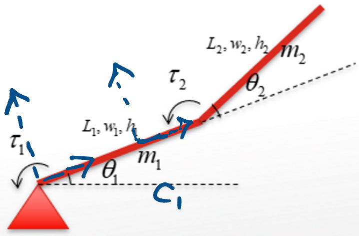
QS. 1

Thursday, 3 September 2020 3:11 pm

Tutorial Assignments

- **Question 1:**
  - The following two-link robot has each link as a rectangular solid of homogenous density.
  - Each link has dimension l_i, w_i, h_i, and a total mass of m_i.



  - Derive the dynamic equations using Lagrangian method.

Recall From week 5. Lecture we have

$$^1v_1 = 0 \qquad ^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} \qquad ^2v_2 = \begin{bmatrix} L_1 S_2 \dot\theta_1 \\ L_1 C_2 \dot\theta_1 \\ 0 \end{bmatrix} \qquad ^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \end{bmatrix}$$

For KE, we need velocities at the COM (center of mass)

Recall: $$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^iv_i + {}^i\omega_i \times {}^iP_{i+1}\right)$$

1: we see "C_1" as "$i+1$" with "1" as "i"

$$^{C_1}v_{C_1} = \underbrace{{}^{C_1}_{1}R}_{I}\left(\underbrace{{}^1v_1}_{0} + {}^1\omega_1 \times {}^1P_{C_1}\right) =$$

$$= \begin{vmatrix} i & j & k \\ 0 & 0 & \dot\theta_1 \\ L_1/2 & 0 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ L_1/2\,\dot\theta_1 \\ 0 \end{bmatrix}$$

2: we see "C_2" as "$i+1$" with "2" as "i"

$$^{C_2}v_{C_2} = \underbrace{{}^{C_2}_{2}R}_{I}\left({}^2v_2 + {}^2\omega_2 \times {}^2P_{C_2}\right) =$$

$$= \begin{bmatrix} L_1 S_2 \dot{\theta}_1 \\ L_1 C_2 \dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 + \dot{\theta}_2 \\ L_2/2 & 0 & 0 \end{vmatrix}$$

$$= \begin{bmatrix} L_1 S_2 \dot{\theta}_1 \\ L_1 C_2 \dot{\theta}_1 + L_2/2 (\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

we also have; $\quad {}^{c_1}\omega_{c_1} = {}^{1}\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad {}^{c_2}\omega_{c_2} = {}^{2}\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$

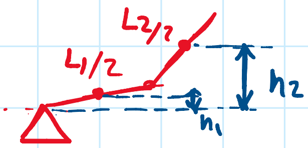
Kinetic Energy: $\quad K = K_1 + K_2$

$$K = \tfrac{1}{2} m_1 v_{c_1}^T v_{c_1} + \tfrac{1}{2} {}^{1}\omega_1^T \, {}^{c_1}I_1 \, {}^{1}\omega_1 + \tfrac{1}{2} m_2 v_{c_2}^T v_{c_2} + \tfrac{1}{2} {}^{2}\omega_2^T \cdot {}^{c_2}I_2 \, {}^{2}\omega_2$$

$$= \tfrac{1}{2} m_1 \frac{L_1^2}{4} \dot{\theta}_1^2 + \tfrac{1}{2} I_{zz1} \dot{\theta}_1^2 + \tfrac{1}{2} m_2 L_1^2 \dot{\theta}_1^2 +$$

$$\tfrac{1}{2} m_2 L_1 L_2 C_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) + \tfrac{1}{8} m_2 L_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + \tfrac{1}{2} I_{zz2} (\dot{\theta}_1 + \dot{\theta}_2)^2$$

Potential Energy:



$$U = m_1 g \underbrace{\frac{L_1}{2} S_1}_{h_1} + m_2 g \underbrace{\left[L_1 S_1 + \tfrac{L_2}{2} S_{12} \right]}_{h_2} + \text{Const.}$$

Lagrangian Formulation

$$\underbrace{\frac{\partial}{\partial t} \Big(\underbrace{\frac{\partial K}{\partial \dot{q}}}_{1} \Big)}_{2} + \underbrace{\frac{\partial K}{\partial q}}_{3} - \underbrace{\frac{\partial U}{\partial q}}_{4} = \tau$$

1) $$\frac{\partial K}{\partial \dot{q}} = \begin{bmatrix} \frac{\partial K}{\partial \dot{\theta}_1} \\ \frac{\partial K}{\partial \dot{\theta}_2} \end{bmatrix} = \begin{bmatrix} m_1 \frac{L_1^2}{4} \dot{\theta}_1 + I_{zz1} \dot{\theta}_1 + m_2 L_1^2 \dot{\theta}_1 + m_2 L_1 L_2 C_2 \dot{\theta}_1 \\ + \tfrac{1}{2} m_2 L_1 L_2 C_2 \dot{\theta}_2 + \tfrac{1}{4} m_2 L_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + I_{zz2} (\dot{\theta}_1 + \dot{\theta}_2) \\ \hline \tfrac{1}{2} m_2 L_1 L_2 C_2 \dot{\theta}_1 + \tfrac{1}{4} m_2 L_2^2 (\dot{\theta}_1 + \dot{\theta}_2) + I_{zz2} (\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix}$$

$$2)\quad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) = \begin{bmatrix} m_1 \frac{L_1^2}{4}\ddot{\theta}_1 + I_{zz1}\ddot{\theta}_1 + m_2 L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 c_2 \ddot{\theta}_1 - m_2 L_1 L_2 S_2 \dot{\theta}_2 \dot{\theta}_1 \\ + \frac{1}{2} m_2 L_1 L_2 (c_2 \ddot{\theta}_2 - s_2 \dot{\theta}_2^2) + \frac{1}{4} m_2 L_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + I_{zz2}(\ddot{\theta}_1 + \ddot{\theta}_2) \\ \hline \frac{1}{2} m_2 L_1 L_2 (c_2 \ddot{\theta}_2 - s_2 \dot{\theta}_1 \dot{\theta}_2) + \frac{1}{4} m_2 L_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + I_{zz2}(\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}$$

$$3)\quad \frac{\partial K}{\partial q} = \begin{bmatrix} \partial K / \partial \theta_1 \\ \partial K / \partial \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} m_2 L_1 L_2 S_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix}$$

$$4)\quad \frac{\partial U}{\partial q} = \begin{bmatrix} \partial U / \partial \theta_1 \\ \partial U / \partial \theta_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} m_1 L_1 g c_1 + m_2 L_1 g c_1 + \frac{1}{2} m_2 L_2 g c_{12} \\ \frac{1}{2} m_2 L_2 g c_{12} \end{bmatrix}$$

Finally $2 - 3 + 4 = \tau$

Put into the general form: $M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau$

$$\begin{bmatrix} m_1 \frac{L_1^2}{4} + I_{zz1} + m_2 L_1^2 + m_2 L_1 L_2 C_2 + \frac{1}{4} m_2 L_2^2 + I_{zz2} & \frac{1}{2} m_2 L_1 L_2 C_2 + \frac{1}{4} m_2 L_2^2 + I_{zz2} \\ \frac{1}{2} m_2 L_1 L_2 C_2 + \frac{1}{4} m_2 L_2^2 + I_{zz2} & \frac{1}{4} m_2 L_2^2 + I_{zz2} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -m_2 L_1 L_2 S_2 \dot{\theta}_1 \dot{\theta}_2 - \frac{1}{2} m_2 L_1 L_2 S_2 \dot{\theta}_2^2 \\ \frac{1}{2} m_2 L_1 L_2 S_2 \dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} m_1 L_1 g C_1 + m_2 L_1 g C_1 + \frac{1}{2} m_2 L_2 g C_{12} \\ \frac{1}{2} m_2 L_2 g C_{12} \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

QS. 2

Monday, 14 September 2020 9:59 PM

- **Question 2:**
  - For the same robot in Question 1:
  - a (a) Write the dynamic equation, when each joint is subject to viscous and coulomb friction.
  - b (b) Calculate the dynamic model in Cartesian space.

a) $M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau - \tau_{friction}$

$$= \begin{bmatrix} \tau_1 - c_1 \operatorname{sign}(\dot{\theta}_1) - k_1 \dot{\theta}_1 \\ \tau_2 - c_2 \operatorname{sign}(\dot{\theta}_2) - k_2 \dot{\theta}_2 \end{bmatrix}$$

Joint Space → Cartesian Space

- Premultiply the joint-space dynamic equation with J^{-T} gives:

$$J^{-T}\tau = J^{-T}M(q)\ddot{q} + J^{-T}V(q,\dot{q}) + J^{-T}G(q) = F$$

- From $\dot{x} = J\dot{q}$, we can obtain through differentiation:

$$\ddot{x} = \dot{J}\dot{q} + J\ddot{q} \quad \text{or} \quad \ddot{q} = J^{-1}\ddot{x} - J^{-1}\dot{J}\dot{q}$$

- Substitute the q-double-dot equation into the first equation above gives:

$$\begin{aligned} F &= J^{-T}M(q)\left(J^{-1}\ddot{x} - J^{-1}\dot{J}\dot{q}\right) + J^{-T}V(q,\dot{q}) + J^{-T}G(q) \\ &= J^{-T}M(q)J^{-1}\ddot{x} - J^{-T}M(q)J^{-1}\dot{J}\dot{q} + J^{-T}V(q,\dot{q}) + J^{-T}G(q) \\ &= \underbrace{J^{-T}M(q)J^{-1}}_{M_x(q)}\ddot{x} + \underbrace{J^{-T}\left(V(q,\dot{q}) - M(q)J^{-1}\dot{J}\dot{q}\right)}_{V_x(q,\dot{q})} + \underbrace{J^{-T}G(q)}_{G_x(q)} \end{aligned}$$

b) We need Jacobians for the tip of robot

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$$J_v = \left[\begin{array}{c|c} -L_1S_1 - L_2S_{12} & -L_2S_{12} \\ \hline L_1C_1 + L_2C_{12} & L_2C_{12} \end{array}\right]$$

From J_v, we calculate J_v^{-1} & $\dot{J}_v$

$$J_v^{-1} = \frac{1}{L_1L_2S_2}\left[\begin{array}{c|c} L_2C_{12} & L_2S_{12} \\ \hline -L_1C_1 - L_2C_{12} & -L_1S_1 - L_2S_{12} \end{array}\right]$$

$$\dot{J}_v = \left[\begin{array}{c|c} -L_1C_1\dot{\theta}_1 - L_2C_{12}(\dot{\theta}_1+\dot{\theta}_2) & -L_2C_{12}(\dot{\theta}_1+\dot{\theta}_2) \\ \hline -L_1S_1\dot{\theta}_1 - L_2S_{12}(\dot{\theta}_1+\dot{\theta}_2) & -L_2S_{12}(\dot{\theta}_1+\dot{\theta}_2) \end{array}\right]$$

Finally:

$$M_x(q) = J_v^{-T} M(q) J_v^{-1} \qquad \text{when } J^{-T} = (J^{-1})^T$$

$$V_x(q,\dot{q}) = J_v^{-T}\left[V - MJ_v^{-1}\dot{J}_v\dot{q}\right]$$

$$G_x(q) = J_v^{-T} G(q)$$

The rest is straight forward

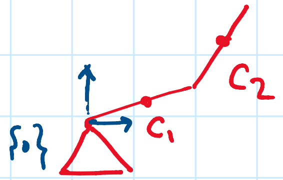
QS. 3

Monday, 14 September 2020 10:00 PM

- **Question 3:**
  - For the same robot in Question 1:
  - Derive the dynamic equation using the Explicit method.

Prepration

speed at COM $\longrightarrow$ Jacobians at COM , Link by Link



$$^0P_{c_1} = \begin{bmatrix} L_1/_2 C_1 \\ L_1/_2 S_1 \\ 0 \end{bmatrix} \qquad ^0P_{c_2} = \begin{bmatrix} L_1C_1 + L_2/_2 C_{12} \\ L_1S_1 + L_2/_2 S_{12} \\ 0 \end{bmatrix}$$

Speed via differentiation

$$^0v_{c_1} = \begin{bmatrix} -\frac{L_1}{2} S_1\dot{\theta}_1 \\ \frac{L_1}{2} C_1\dot{\theta}_1 \\ 0 \end{bmatrix} \qquad ^0v_{c_1} = \begin{bmatrix} -L_1S_1\dot{\theta}_1 - \frac{L_2}{2} S_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ L_1C_1\dot{\theta}_1 + \frac{L_2}{2} C_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

Jacobians

$$^0J_{v_1} = \begin{bmatrix} -\frac{L_1}{2} S_1 & 0 \\ \frac{L_1}{2} C_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad ^0J_{v_2} = \left[\begin{array}{c|c} -L_1S_1 - \frac{L_2}{2} S_{12} & -\frac{L_2}{2} S_{12} \\ L_1C_1 + \frac{L_2}{2} C_{12} & \frac{L_2}{2} C_{12} \\ 0 & 0 \end{array}\right]$$

Also need the angular/Rotational Jacobian

$$^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \quad J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad ^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \quad J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

we have now prepared all the Jacobians :

1) calculate $M(q)$ Matrix

$$M(q) = \Sigma\left(m_i J_{v_i}^T J_{v_i} + {}^iJ_{\omega_i} \, {}^{c_i}I_i \cdot {}^iJ_{\omega_i} \right)$$

$$M(q) = \sum \left(m_i J_{v_i}^T J_{v_i} + {}^{i}J_{\omega_i}^T \cdot {}^{ci}I_i \cdot {}^{i}J_{\omega_i} \right)$$

$$= m_1 \begin{bmatrix} -\frac{L_1}{2} S_1 & \frac{L_1}{2} C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{L_1}{2} S_1 & 0 \\ \frac{L_1}{2} C_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{xx1} & & \\ & I_{yy1} & \\ & & I_{zz1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$+ m_2 \left[\begin{array}{c|c|c} -L_1 S_1 - \frac{L_2}{2} S_{12} & L_1 C_1 + \frac{L_2}{2} C_{12} & 0 \\ -\frac{L_2}{2} S_{12} & \frac{L_2}{2} C_{12} & 0 \end{array}\right] \left[\begin{array}{c|c} -L_1 S_1 - \frac{L_2}{2} S_{12} & -\frac{L_2}{2} S_{12} \\ L_1 C_1 + \frac{L_2}{2} C_{12} & \frac{L_2}{2} C_{12} \\ 0 & 0 \end{array}\right]$$

$$+ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{xx2} & & \\ & I_{yy2} & \\ & & I_{zz2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} & M_{11} \searrow & M_{12} \searrow \\ M(q) = & \left[\begin{array}{c|c} m_1 \frac{L_1^2}{4} + I_{zz1} + m_2 L_1^2 + m_2 L_1 L_2 C_2 + \frac{1}{4} m_2 L_2^2 + I_{zz2} & \frac{1}{2} m_2 L_1 L_2 C_2 + \frac{1}{4} m_2 L_2^2 + I_{zz2} \\ \hline \frac{1}{2} m_2 L_1 L_2 C_2 + \frac{1}{4} m_2 L_2^2 + I_{zz2} & \frac{1}{4} m_2 L_2^2 + I_{zz2} \end{array}\right] \\ & M_{21} \nearrow & M_{22} \nearrow \end{array}$$

2) calculate V : $C(q)\dot{q}^2 + B(q)\dot{q}\dot{q}$

$$C(q)\dot{q} = \frac{1}{2} \left[\begin{array}{c|c} M_{111} + M_{111} - M_{111} & M_{122} + M_{122} - M_{221} \\ M_{211} + M_{211} - M_{112} & M_{222} + M_{222} - M_{222} \end{array}\right] \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

recall - $M_{ijk} = \partial M_{ij} / \partial q_k$

$$= \frac{1}{2} \left[\begin{array}{c|c} 0 + 0 - 0 & -\frac{1}{2} m_2 L_1 L_2 S_2 - \frac{1}{2} m_2 L_1 L_2 S_2 + 0 \\ \hline 0 + 0 + m_2 L_1 L_2 S_2 & 0 + 0 - 0 \end{array}\right] \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\frac{1}{2} m_2 L_1 L_2 S_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{2} m_2 L_1 L_2 S_2 \\ \frac{1}{2} m_2 L_1 L_2 S_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$B(q)\dot{q}\dot{q} = \frac{1}{2} \begin{bmatrix} M_{112} + M_{121} - M_{121} \\ M_{212} + M_{221} - M_{122} \end{bmatrix} \dot{q}_1 \dot{q}_2$$

$$= \begin{bmatrix} -m_2 L_1 L_2 S_2 \\ 0 \end{bmatrix} \dot{q}_1 \dot{q}_2$$

3- calculate $G(q)$ Gravity term

$$G(q) = -\begin{bmatrix} J_{v_1}^T & J_{v_2}^T \end{bmatrix} \begin{bmatrix} m_1 g \\ m_2 g \end{bmatrix} \quad \text{where } {}^0g = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

$$G = -\begin{bmatrix} -\frac{L_1}{2} S_1 & \frac{L_1}{2} C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot m_1 \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

$$- \begin{bmatrix} -L_1 S_1 - \frac{L_2}{2} S_{12} & L_1 C_1 + \frac{L_2}{2} C_{12} & 0 \\ -\frac{L_2}{2} S_{12} & \frac{L_2}{2} C_{12} & 0 \end{bmatrix} \cdot m_2 \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} m_1 \frac{L_1}{2} C_1 g + m_2 (L_1 C_1 + \frac{L_2}{2} C_{12}) g \\ m_2 \frac{L_2}{2} C_{12} g \end{bmatrix}$$

Finally $\rightarrow$ Dynamic Equation:

$$M(q)\ddot{q} + \underbrace{C(q)\dot{q}^2 + B(q)\dot{q}\dot{q}}_{V(q,\dot{q})} + G(q) = \tau$$