Week 6 – Manipulator Dynamics

Advanced Robotic Systems – MANU2453

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Lectures

Wk	Date	Lecture (NOTE: video recording)	Maths Difficulty	Hands-on Activity	Related Assessment
1	24/7	• Introduction to the Course • Spatial Descriptions & Transformations			
2	31/7	• Spatial Descriptions & Transformations • Robot Cell Design			Robot Cell Design Assignment
3	7/8	• Forward Kinematics • Inverse Kinematics			
4	14/8	• ABB Robot Programming via Teaching Pendant • ABB RobotStudio Offline Programming		ABB RobotStudio Offline Programming	Offline Programming Assignment
5	21/8	• Jacobians: Velocities and Static Forces			
6	28/8	• Manipulator Dynamics			
7	11/9	• Manipulator Dynamics		MATLAB Simulink Simulation	
8	18/9	• Robotic Vision		MATLAB Simulation	Robotic Vision Assignment
9	25/9	• Robotic Vision		MATLAB Simulation	
10	2/10	• Trajectory Generation			
11	9/10	• Linear & Nonlinear Control		MATLAB Simulink Simulation	
12	16/10	• Introduction to I4.0 • Revision			Final Exam

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Content

- Introduction & Structure of Manipulator's Dynamic Equations
- Mass Distribution
- Newton-Euler Formulation

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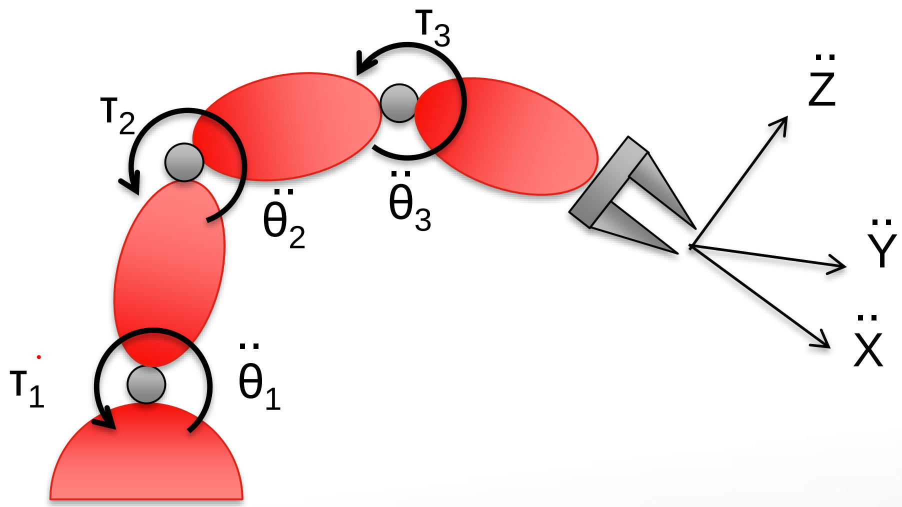
Introduction

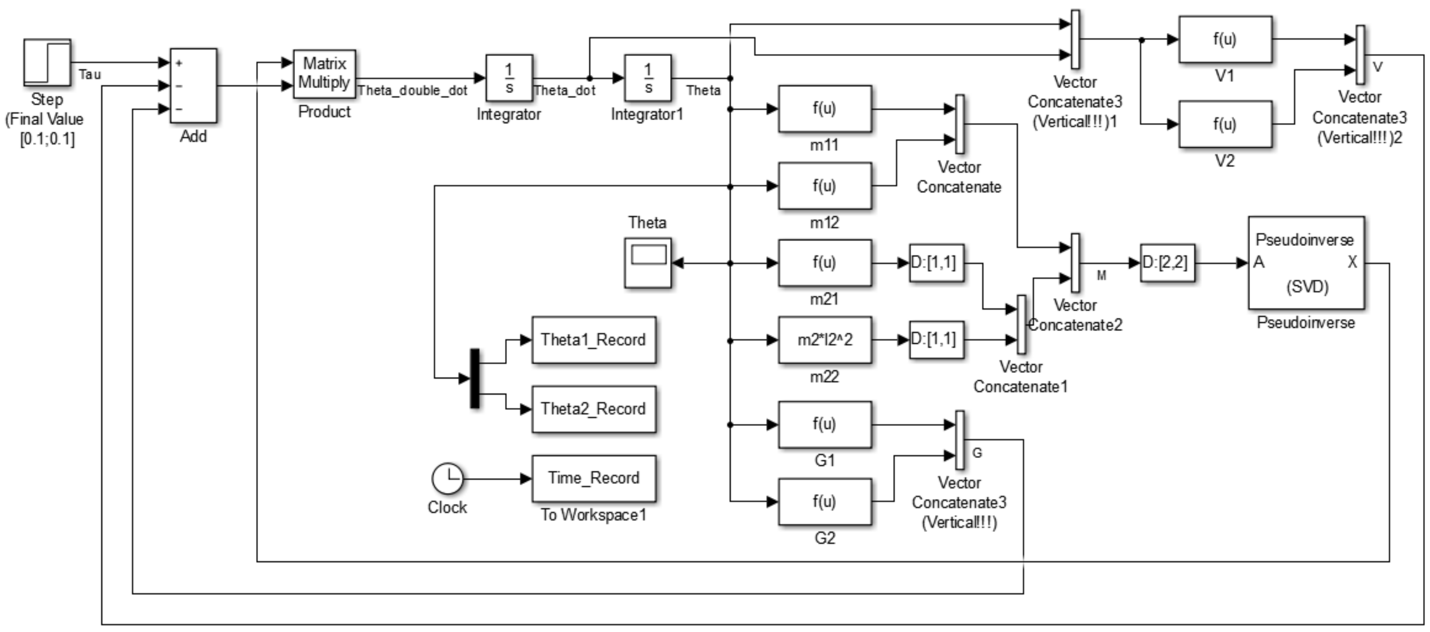
- Manipulator Dynamics:
  - The study of forces which cause motion.

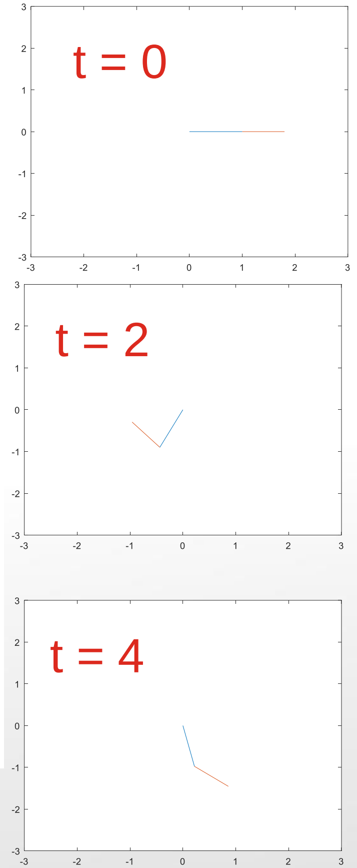


  - How much torque is needed to accelerate the manipulator from rest to constant velocity, and then back to stop?

Introduction

- Dynamics also provide us a model (equations of motions) for simulation and control design purpose.





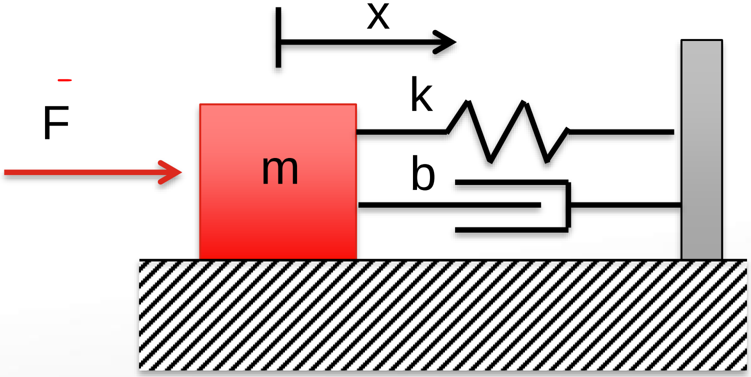
- You will learn how to create a Simulink simulation in week 7.

Manipulator's Dynamic Equations

- Before we go into details of how to derive the manipulator's joint space dynamic equations, let's first have a glimpse of how the equations look like:

$$M(q)\ddot{q}+V(q,\dot{q})+G(q)=\tau$$

- A comparison with the well-known mass-spring-damper system:



$$m\ddot{x}+b\dot{x}+kx=F$$

- They look somewhat similar.

Manipulator's Dynamic Equations

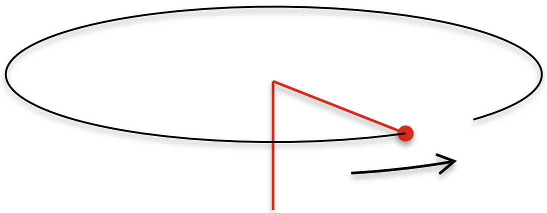
- $M(q)$ is the n x n mass matrix of the manipulator, which depends on the generalized joint coordinates q (angles / displacement).
  - For e.g. two link robot:

  - The "perceived inertia" at joint 1 of the right configuration is larger than that of the left configuration.
  - The "perceived inertia" also depends on the mass distribution and length of the links.
  - $M(q)$ is also called the Kinetic Energy Matrix since Kinetic Energy is
    $$K = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

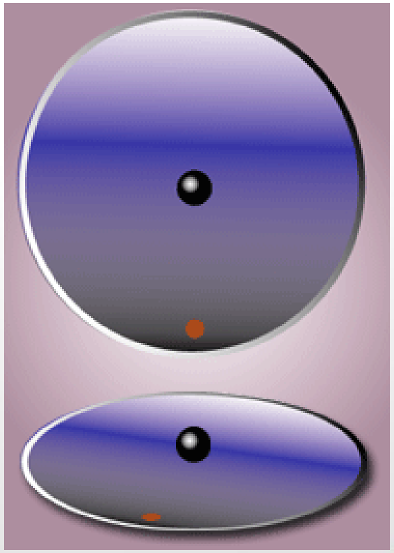
Manipulator's Dynamic Equations

- $V(q,\dot{q})$ is an n x 1 vector of centrifugal and Coriolis forces.

A 'fictitious' force acting away from axis of rotation. E.g. whirling a stone on a string



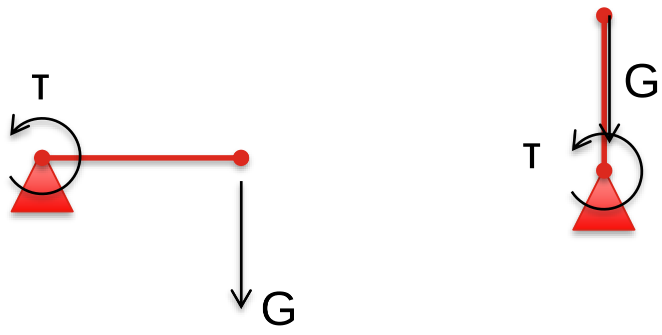
A fictitious force acting on an object that are in motion relative to a rotating reference frame.
In a reference frame with clockwise rotation, the force acts to the left of the motion of the object
(For more details please see https://youtu.be/7TjOy56-x8Q).



- $V(q,\dot{q})$ depends on the generalized joint coordinates q as well as the joint velocities q-dot.
    - It is zero if velocities = 0.
- Also, $V(q,\dot{q})$ can be derived from $M(q)$.
    - It is also zero if M(q) is a constant matrix.

Manipulator’s Dynamic Equations

- $G(q)$ is the n x 1 vector of gravity terms.
  - It is dependent on the joint coordinates / configuration of the robot.



  - In the left figure, the joint torque is nonzero, and in the right figure, the joint torque is zero.

- Finally, τ is the generalized forces (force or torque) at each joints.

Manipulator's Dynamic Equations

- One thing to note is that the dynamic equations show that the links have cross-coupling effects onto one another.
- E.g. 2-link robot: $M(q)\ddot{q}+V(q,\dot{q})+G(q)=\tau$

$$\begin{bmatrix} m_{11}(q) & m_{12}(q) \\ m_{21}(q) & m_{22}(q) \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}+\begin{bmatrix} V_1(q,\dot{q}) \\ V_2(q,\dot{q}) \end{bmatrix}+\begin{bmatrix} G_1(q) \\ G_2(q) \end{bmatrix}=\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

- Even if $\tau_2 = 0$, there will be an acceleration for q_2 because it is affected by q_1, which is created by τ_1.
- On the other hand, even if $\tau_1 = 0$, there will be an acceleration for q_1 because it is affected by q_2, which is created by τ_2.
- These cross coupling are caused by the off-diagonal terms (m_{12}, m_{21}) in the mass matrix.

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Content

- Introduction & Structure of Manipulator's Dynamic Equations
- Mass Distribution
- Newton-Euler Formulation

Mass Distribution

- We are all familiar with Newton's Law: $F = ma$
  - The acceleration (a) is proportional to force (F) divide by mass (m).
  - If mass is small, then the acceleration is huge.
  - And if the mass is large, then the acceleration is small.
  - The mass presents a "resistance" to the linear motion.
- For the case of rotational motion about a single axis, we have: $\tau = I\alpha$
  - where τ is the torque, I is the moment of inertia, and α is the angular acceleration.
  - The moment of inertia is similar to the mass.
  - It presents a "resistance" to the rotary motion.
- To study the dynamics of the robot, we thus need both the mass/inertia and the acceleration.
  - Let's start with discussion on mass/inertia first.

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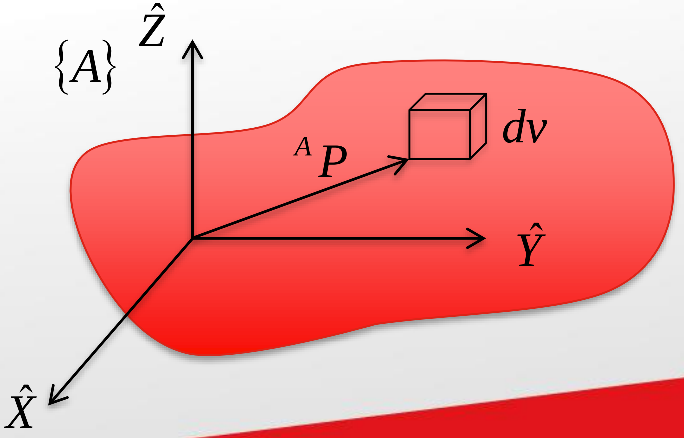
Mass Distribution

- For the case of a rigid body which is free to move in three dimensional space, there are infinitely many possible rotation axis.
- We need a generalization of the moment of inertia.
  - Inertia tensor will be used for this purpose.

$$^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

  - It characterizes the mass distribution of a rigid body, wrt to the reference frame (here {A}).



dv is the differential volume element

Mass Distribution

- The elements of the inertial tensor are:

$$I_{xx} = \iiint_V \left(y^2 + z^2\right)\rho dv$$

$$I_{yy} = \iiint_V \left(x^2 + z^2\right)\rho dv$$

$$I_{zz} = \iiint_V \left(y^2 + y^2\right)\rho dv$$

Mass moment of inertia

$$I_{xy} = \iiint_V xy\rho dv$$

$$I_{xz} = \iiint_V xz\rho dv$$

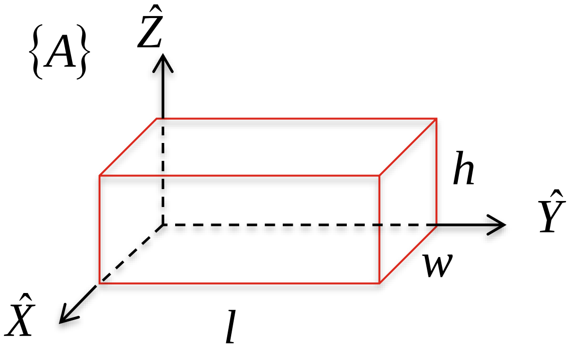
$$I_{yz} = \iiint_V yz\rho dv$$

Mass products of inertia

- These elements depend on the position and orientation of the frame.
  - If the frame is at a ‘special’ orientation, the products of inertia can be zero.
  - In this case, the axes of the frame are called “principal axes”, and the moments of inertia are called “principal moments of inertia”.

Example

- A rectangular body has uniform density ρ.
- If the frame is attached to one corner as shown, what is the inertia tensor?



- Solution:

$$I_{xx} = \iiint_V \left(y^2 + z^2\right)\rho \cdot dv = \int_0^h \int_0^l \int_0^w \left(y^2 + z^2\right)\rho \cdot dx \cdot dy \cdot dz$$

$$= \int_0^h \int_0^l \left(y^2 + z^2\right) w\rho \cdot dy \cdot dz = \int_0^h \left(\frac{l^3}{3} + z^2 l\right) w\rho \cdot dz$$

$$= \left(\frac{l^3}{3}h + \frac{h^3}{3}l\right) w\rho = \left(\frac{l^2}{3}hl + \frac{h^2}{3}hl\right) w\rho = \left(\frac{l^2}{3} + \frac{h^2}{3}\right) hlw\rho$$

$$= \left(\frac{l^2}{3} + \frac{h^2}{3}\right) V\rho = \frac{m}{3}\left(l^2 + h^2\right)$$

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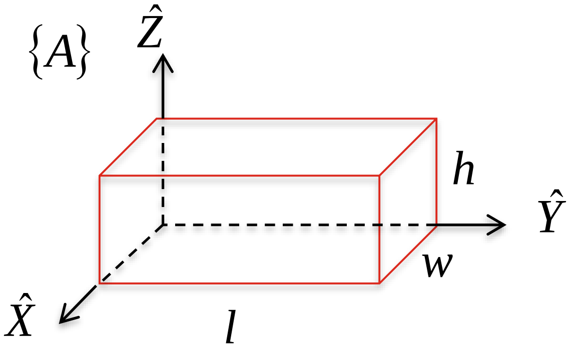
Example

- Similarly, we can get

$$I_{yy} = \frac{m}{3}\left(w^2 + h^2\right)$$

$$I_{zz} = \frac{m}{3}\left(w^2 + l^2\right)$$



- Next:

$$\begin{aligned}
I_{xy} &= \iiint_V xy\rho dv = \int_0^h \int_0^l \int_0^w xy\rho \cdot dx \cdot dy \cdot dz \\
&= \int_0^h \int_0^l \frac{1}{2} w^2 y\rho \cdot dy \cdot dz = \int_0^h \frac{1}{4} w^2 l^2 \rho \cdot dz \\
&= \frac{1}{4} w^2 l^2 h\rho = \frac{1}{4} wl \cdot \underbrace{wlh}_{\text{total volume}}\rho \\
&= \frac{1}{4} wl \cdot \underbrace{V\rho}_{\text{total mass}} = \frac{m}{4} wl
\end{aligned}$$

- Similarly, we have

$$I_{xz} = \frac{m}{4} wh$$

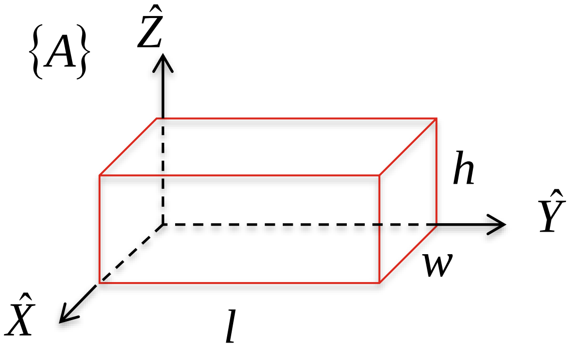
$$I_{yz} = \frac{m}{4} lh$$

Example

- In summary, the inertia tensor is:

$$
{}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}
$$

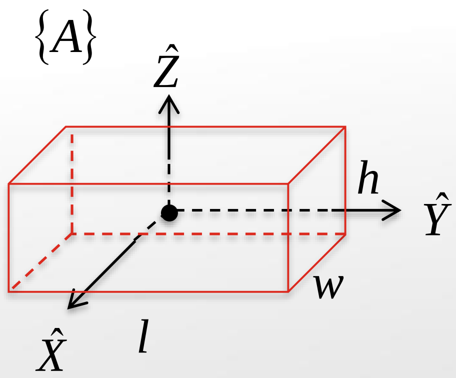
$$
= \begin{bmatrix} \frac{m}{3}\left(l^2 + h^2\right) & -\frac{m}{4}wl & -\frac{m}{4}hw \\ -\frac{m}{4}wl & \frac{m}{3}\left(w^2 + h^2\right) & -\frac{m}{4}hl \\ -\frac{m}{4}hw & -\frac{m}{4}hl & \frac{m}{3}\left(w^2 + l^2\right) \end{bmatrix}
$$



Parallel-Axis Theorem

- In the example just now, the reference frame is placed at one corner of the rectangle.
- We also mentioned that the inertia tensor is dependent on the position and orientation of the frame.
- Since we have already calculated the inertia tensor for one frame, can we get the inertia tensor (of the same object) for another translated frame, without going through the calculation of integration?
- Yes!
  - Parallel-Axis Theorem:

$$^{A}I = {}^{C}I + m\left[P_C^T P_C I_3 - P_C P_C^T\right]$$



  - where “C” means the center of mass.
  - and $P_C = [x_C, y_C, z_C]^T$ is the location of “C” wrt. {A}.

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Parallel-Axis Theorem

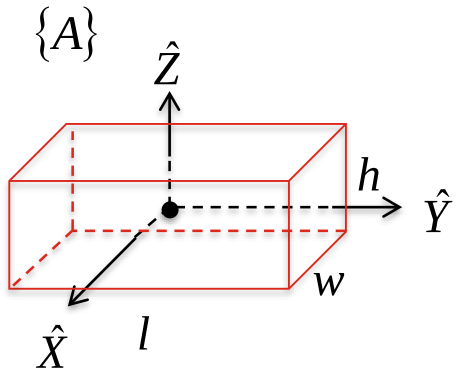
- Using the ${}^{A}I = {}^{C}I + m\left[P_C^T P_C I_3 - P_C P_C^T\right]$ equation, we have:

$$
\begin{aligned}
{}^{A}I &= {}^{C}I + m\left[\begin{bmatrix} x_C & y_C & z_C \end{bmatrix}\begin{bmatrix} x_C \\ y_C \\ z_C \end{bmatrix} I_3 - \begin{bmatrix} x_C \\ y_C \\ z_C \end{bmatrix}\begin{bmatrix} x_C & y_C & z_C \end{bmatrix}\right] \\
&= {}^{C}I + m\left[\begin{bmatrix} x_C^2 + y_C^2 + z_C^2 & 0 & 0 \\ 0 & x_C^2 + y_C^2 + z_C^2 & 0 \\ 0 & 0 & x_C^2 + y_C^2 + z_C^2 \end{bmatrix} - \begin{bmatrix} x_C^2 & x_C y_C & x_C z_C \\ x_C y_C & y_C^2 & y_C z_C \\ x_C z_C & y_C z_C & z_C^2 \end{bmatrix}\right] \\
&= {}^{C}I + m\begin{bmatrix} y_C^2 + z_C^2 & -x_C y_C & -x_C z_C \\ -x_C y_C & x_C^2 + z_C^2 & -y_C z_C \\ -x_C z_C & -y_C z_C & x_C^2 + y_C^2 \end{bmatrix}
\end{aligned}
$$

- or:

$$
{}^{C}I = {}^{A}I - m\begin{bmatrix} y_C^2 + z_C^2 & -x_C y_C & -x_C z_C \\ -x_C y_C & x_C^2 + z_C^2 & -y_C z_C \\ -x_C z_C & -y_C z_C & x_C^2 + y_C^2 \end{bmatrix}
$$

Example



- Consider the same rectangle block as just now.
- The frame for the inertia tensor is now located at the center of mass.
- What is the inertia tensor wrt. to center of mass?
- Answer: We have: $P_C = [x_C, y_C, z_C]^T = \begin{bmatrix} \frac{1}{2}w & \frac{1}{2}l & \frac{1}{2}h \end{bmatrix}^T$
- Applying the parallel-axis formula:

$$ {}^{C}I = {}^{A}I - m \begin{bmatrix} y_C^2 + z_C^2 & -x_C y_C & -x_C z_C \\ -x_C y_C & x_C^2 + z_C^2 & -y_C z_C \\ -x_C z_C & -y_C z_C & x_C^2 + y_C^2 \end{bmatrix} $$

- leads to (next page):

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Example

$$
{}^{C}I = {}^{A}I - m\begin{bmatrix} y_C^2 + z_C^2 & -x_C y_C & -x_C z_C \\ -x_C y_C & x_C^2 + z_C^2 & -y_C z_C \\ -x_C z_C & -y_C z_C & x_C^2 + y_C^2 \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{m}{3}\left(l^2 + h^2\right) & -\frac{m}{4}wl & -\frac{m}{4}hw \\ -\frac{m}{4}wl & \frac{m}{3}\left(w^2 + h^2\right) & -\frac{m}{4}hl \\ -\frac{m}{4}hw & -\frac{m}{4}hl & \frac{m}{3}\left(w^2 + l^2\right) \end{bmatrix} - m\begin{bmatrix} \frac{1}{4}\left(l^2 + h^2\right) & -\frac{1}{4}wl & -\frac{1}{4}wh \\ -\frac{1}{4}wl & \frac{1}{4}\left(w^2 + h^2\right) & -\frac{1}{4}hl \\ -\frac{1}{4}wh & -\frac{1}{4}hl & \frac{1}{4}\left(w^2 + l^2\right) \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{m}{12}\left(l^2 + h^2\right) & 0 & 0 \\ 0 & \frac{m}{12}\left(w^2 + h^2\right) & 0 \\ 0 & 0 & \frac{m}{12}\left(w^2 + l^2\right) \end{bmatrix}
$$

Note: {C} must be the principal axes of the body, since the products of inertia are zero

Content

- Introduction & Structure of Manipulator’s Dynamic Equations
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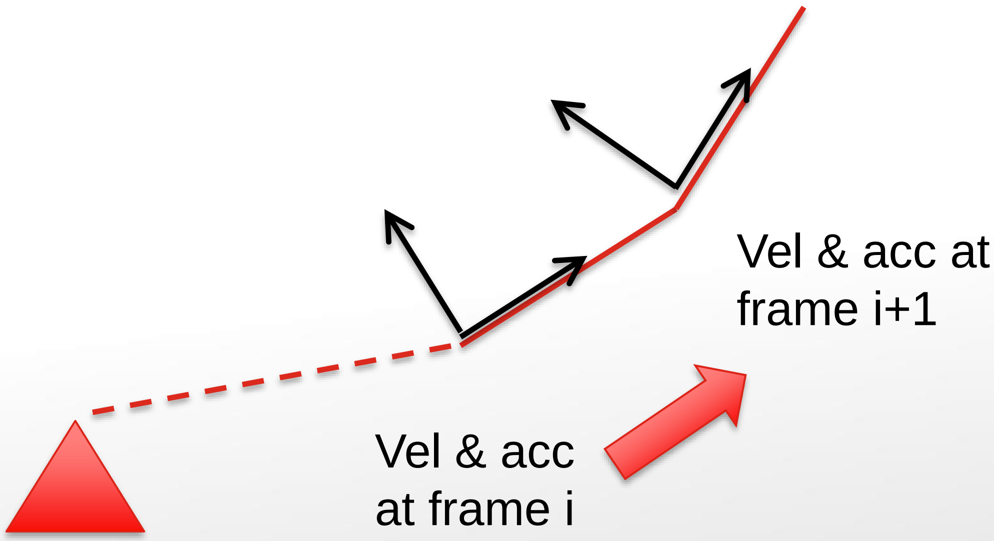
- Newton-Euler Formulation

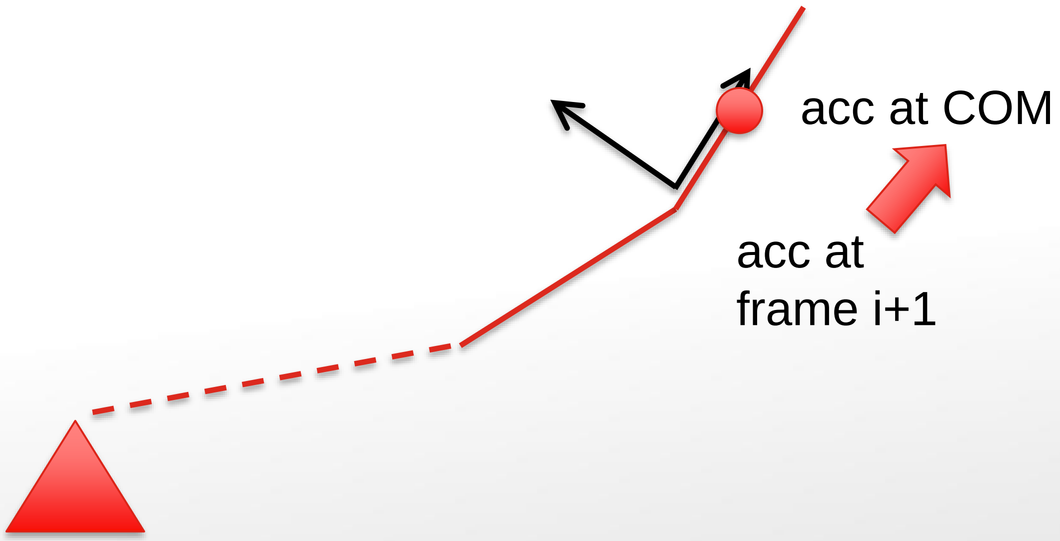
Iterative Newton-Euler Formulation

- Basic idea:
- Firstly, similar to velocity propagation which you learnt last week, acceleration can also be propagated from lower frame to upper frame.



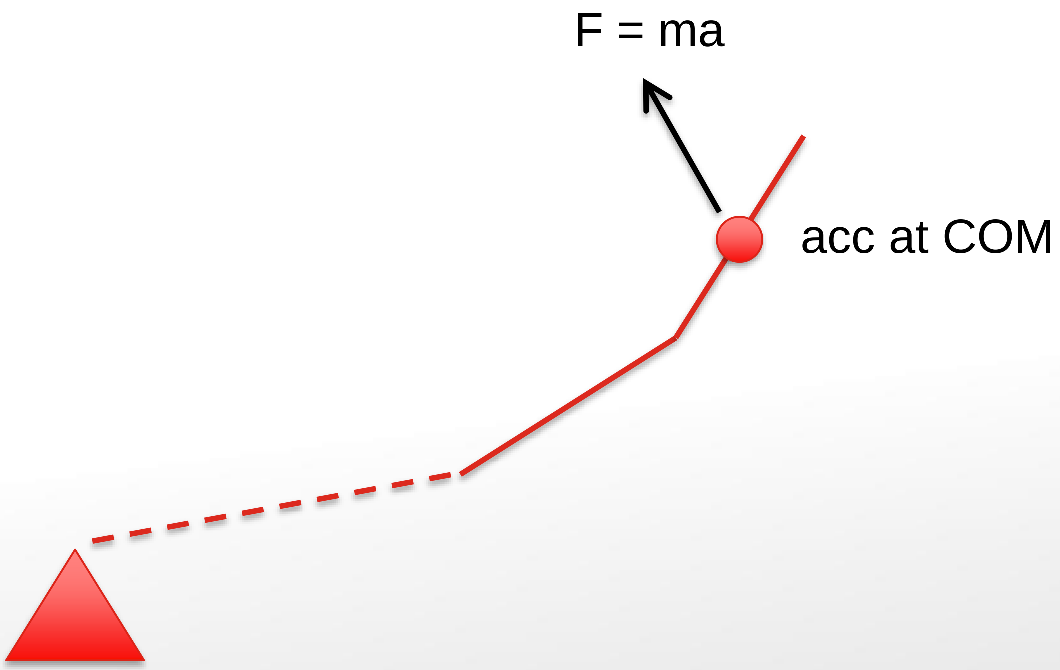
Iterative Newton-Euler Formulation

- Next, the acceleration at frame i+1 can be propagated to the centre of mass.



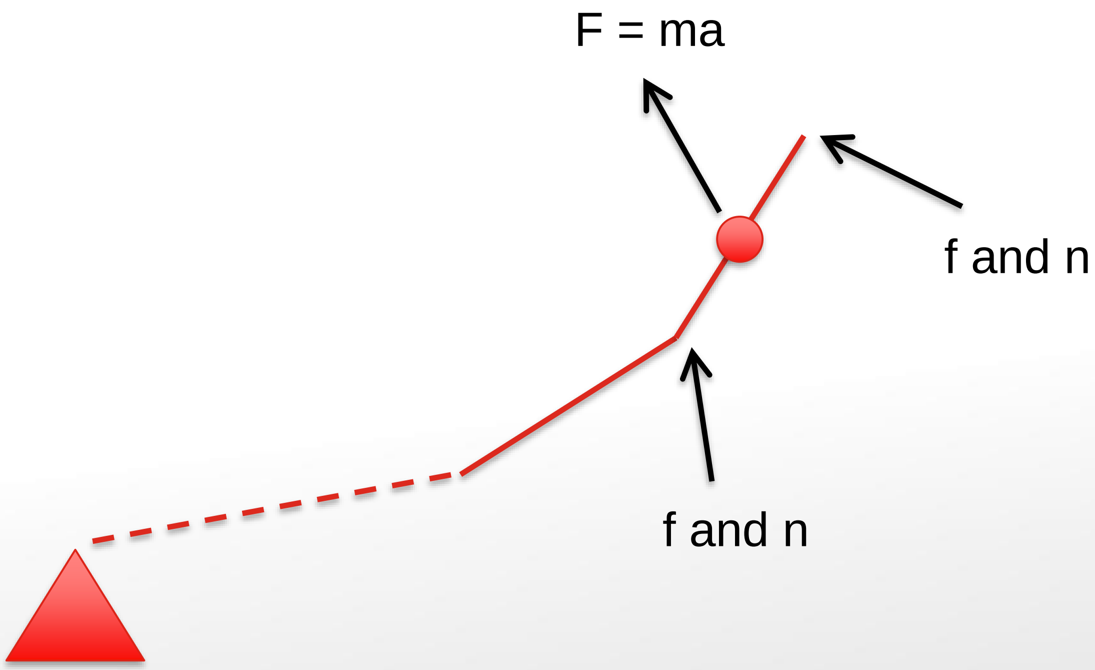
Iterative Newton-Euler Formulation

- Once the acceleration at centre of mass is known, then we also know the force acting on the centre of mass since F = ma.



Iterative Newton-Euler Formulation

- But what "creates" F? It would be the forces / torques caused by the motors at both ends of the link, as well as contact force at the end-effector.



Iterative Newton-Euler Formulation

- Algorithm:
- Start with: $^{0}\omega_0 = 0$ $\quad ^{0}\dot{\omega}_0 = 0$ $\quad ^{0}\dot{\upsilon}_0 = \text{depends}$
- Outward iterations: i = 0 → 5

Note: Multiplication before cross product

Frame vel & acc

$$^{i+1}\omega_{i+1} = \left(^{i+1}_{i}R \cdot {}^{i}\omega_i\right) + \left(\dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right)$$

$$^{i+1}\dot{\omega}_{i+1} = \left(^{i+1}_{i}R \cdot {}^{i}\dot{\omega}_i\right) + \left(^{i+1}_{i}R \cdot {}^{i}\omega_i \times \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left(\ddot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right)$$

$$^{i+1}\dot{\upsilon}_{i+1} = \left(^{i+1}_{i}R \cdot {}^{i}\dot{\upsilon}_i\right) + \left(2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left(\ddot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + {}^{i+1}_{i}R \cdot \left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right)\right)$$

COM

$$^{i+1}\dot{\upsilon}_{C_{i+1}} = \left(^{i+1}\dot{\upsilon}_{i+1}\right) + \left(^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + \left(^{i+1}\omega_{i+1} \times \left(^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right)\right)$$

F = ma

$$^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{\upsilon}_{C_{i+1}}$$

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1} \cdot {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1} \cdot {}^{i+1}\omega_{i+1}$$

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Iterative Newton-Euler Formulation

- Algorithm (Continued):
- Inward iterations: i = 6 → 1

Joint force & torque

$$^{i}f_i = {}^{i}_{i+1}R \cdot {}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{i}n_i = {}^{i}_{i+1}R \cdot {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R \cdot {}^{i+1}f_{i+1} + {}^{i}N_i$$

$$\tau_i = {}^{i}n_i^T \cdot {}^{i}\hat{Z}_i$$ (Revolute)

$$\tau_i = {}^{i}f_i^T \cdot {}^{i}\hat{Z}_i$$ (Prismatic)

Inclusion of Gravity Forces

- The effect of gravity forces can be included by setting:

$$ {}^{0}\dot{\upsilon}_{0} = G $$

  - where G has the magnitude of gravity vector but points in the opposite direction.

- This can be interpreted as the base moving upwards with 1g acceleration.

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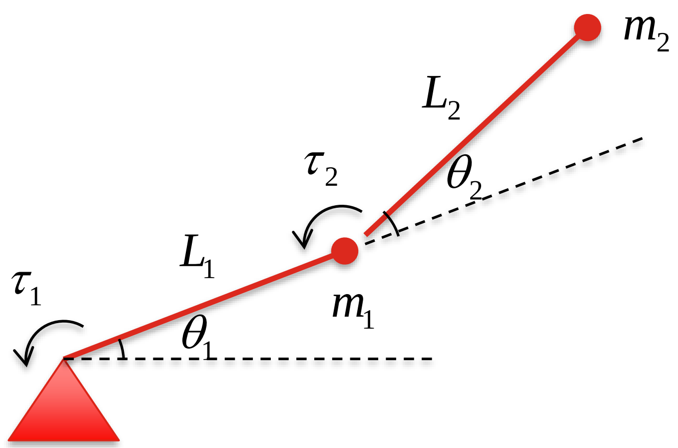
Example

- Two link robot, where the mass of each link is a point mass at the end of the link:



- The vectors that locate the center of mass for each link are:

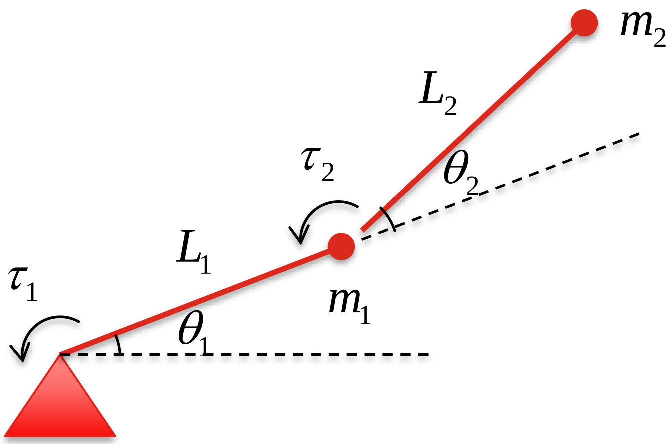
$$^{1}P_{C_1} = L_1 \hat{X}_1 \qquad ^{2}P_{C_2} = L_2 \hat{X}_2$$

- Because the mass of each link is point mass, the inertia tensor at the center of mass is zero:

$$^{C_1}I_1 = 0 \qquad ^{C_2}I_2 = 0$$

Example

- Furthermore, the rotation matrices between successive links are:



$$ {}^{i}_{i+1}R = \begin{bmatrix} c_{i+1} & -s_{i+1} & 0 \\ s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{i+1}_{i}R = \begin{bmatrix} c_{i+1} & s_{i+1} & 0 \\ -s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Example – Outward Iteration

- Now we use the Iterative Newton-Euler algorithm.
- First, we start with:

$$^{0}\omega_0 = 0 \qquad ^{0}\dot{\omega}_0 = 0 \qquad ^{0}\dot{\upsilon}_0 = g\hat{Y}_0$$

- Then, the outward iterations for link 1 give:

$$^{1}\omega_1 = \left({}^{1}_{0}R \cdot {}^{0}\omega_0\right) + \left(\dot{\theta}_1 \cdot {}^{1}\hat{Z}_1\right) = \dot{\theta}_1 \cdot {}^{1}\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$^{1}\dot{\omega}_1 = \left({}^{1}_{0}R \cdot {}^{0}\dot{\omega}_0\right) + \left({}^{1}_{0}R \cdot {}^{0}\omega_0 \times \dot{\theta}_1 \cdot {}^{1}\hat{Z}_1\right) + \left(\ddot{\theta}_1 \cdot {}^{1}\hat{Z}_1\right) = \ddot{\theta}_1 \cdot {}^{1}\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}$$

Example – Outward Iteration

- (continued):

$$
\begin{aligned}
{}^{1}\dot{\upsilon}_1 &= \left({}^{1}_{0}R \cdot {}^{0}\dot{\upsilon}_0\right) + \left(2\,{}^{1}\omega_1 \times \dot{d}_1 \cdot {}^{1}\hat{Z}_1\right) + \left(\ddot{d}_1 \cdot {}^{1}\hat{Z}_1\right) \\
&\quad + {}^{1}_{0}R \cdot \left({}^{0}\dot{\omega}_0 \times {}^{0}P_1 + {}^{0}\omega_0 \times {}^{0}\omega_0 \times {}^{0}P_1\right) \\
&= \left({}^{1}_{0}R \cdot {}^{0}\dot{\upsilon}_0\right) \\
&= \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}\dot{\upsilon}_{C_1} &= \left({}^{1}\dot{\upsilon}_1\right) + \left({}^{1}\dot{\omega}_1 \times {}^{1}P_{C_1}\right) + \left({}^{1}\omega_1 \times {}^{1}\omega_1 \times {}^{1}P_{C_1}\right) \\
&= \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ L_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -L_1\dot{\theta}_1^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -L_1\dot{\theta}_1^2 + gs_1 \\ L_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix}
\end{aligned}
$$

Example – Outward Iteration

- (continued):

$$ {}^{1}F_1 = m_1 {}^{1}\dot{\upsilon}_{C_1} = m_1 \begin{bmatrix} -L_1\dot{\theta}_1^2 + gs_1 \\ L_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -m_1L_1\dot{\theta}_1^2 + m_1gs_1 \\ m_1L_1\ddot{\theta}_1 + m_1gc_1 \\ 0 \end{bmatrix} $$

$$ {}^{1}N_1 = {}^{C_1}I_1 \cdot {}^{1}\dot{\omega}_1 + {}^{1}\omega_1 \times {}^{C_1}I_1 \cdot {}^{1}\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

Example – Outward Iteration

- The outward iterations for link 2 give:

$$
\begin{aligned}
{}^{2}\omega_2 &= \left({}^{2}_{1}R \cdot {}^{1}\omega_1\right) + \left(\dot{\theta}_2 \cdot {}^{2}\hat{Z}_2\right) \\
&= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{2}\dot{\omega}_2 &= \left({}^{2}_{1}R \cdot {}^{1}\dot{\omega}_1\right) + \left({}^{2}_{1}R \cdot {}^{1}\omega_1 \times \dot{\theta}_2 \cdot {}^{2}\hat{Z}_2\right) + \left(\ddot{\theta}_2 \cdot {}^{2}\hat{Z}_2\right) \\
&= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}
\end{aligned}
$$

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Example – Outward Iteration

- (Continued):

$$
\begin{aligned}
{}^{2}\dot{\upsilon}_2 &= \left({}^{2}_{1}R \cdot {}^{1}\dot{\upsilon}_1\right) + \left(2\,{}^{2}\omega_2 \times \dot{d}_2 \cdot {}^{2}\hat{Z}_2\right) + \left(\ddot{d}_2 \cdot {}^{2}\hat{Z}_2\right) + {}^{2}_{1}R \cdot \left({}^{1}\dot{\omega}_1 \times {}^{1}P_2 + {}^{1}\omega_1 \times {}^{1}\omega_1 \times {}^{1}P_2\right) \\
&= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} + 0 + 0 + \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} -L_1 c_2 \dot{\theta}_1^2 + L_1 s_2 \ddot{\theta}_1 + gs_{12} \\ L_1 s_2 \dot{\theta}_1^2 + L_1 c_2 \ddot{\theta}_1 + gc_{12} \\ 0 \end{bmatrix}
\end{aligned}
$$

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Example – Outward Iteration

- (Continued):

$$
\begin{aligned}
{}^2\dot{\upsilon}_{C_2} &= \left({}^2\dot{\upsilon}_2\right) + \left({}^2\dot{\omega}_2 \times {}^2 P_{C_2}\right) + \left({}^2\omega_2 \times {}^2\omega_2 \times {}^2 P_{C_2}\right) \\
&= \begin{bmatrix} -L_1 c_2 \dot{\theta}_1^2 + L_1 s_2 \ddot{\theta}_1 + g s_{12} \\ L_1 s_2 \dot{\theta}_1^2 + L_1 c_2 \ddot{\theta}_1 + g c_{12} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -L_1 c_2 \dot{\theta}_1^2 + L_1 s_2 \ddot{\theta}_1 + g s_{12} - L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ L_1 s_2 \dot{\theta}_1^2 + L_1 c_2 \ddot{\theta}_1 + g c_{12} + L_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \\ 0 \end{bmatrix}
\end{aligned}
$$

Example – Outward Iteration

- (Continued):

$$
\begin{aligned}
{}^{2}F_{2} &= m_{2}\,{}^{2}\dot{\upsilon}_{C_{2}} \\
&= \begin{bmatrix} -m_{2}L_{1}c_{2}\dot{\theta}_{1}^{2} + m_{2}L_{1}s_{2}\ddot{\theta}_{1} + m_{2}gs_{12} - m_{2}L_{2}\left(\dot{\theta}_{1} + \dot{\theta}_{2}\right)^{2} \\ m_{2}L_{1}s_{2}\dot{\theta}_{1}^{2} + m_{2}L_{1}c_{2}\ddot{\theta}_{1} + m_{2}gc_{12} + m_{2}L_{2}\left(\ddot{\theta}_{1} + \ddot{\theta}_{2}\right) \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{2}N_{2} &= {}^{C_{2}}I_{2} \cdot {}^{2}\dot{\omega}_{2} + {}^{2}\omega_{2} \times {}^{C_{2}}I_{2} \cdot {}^{2}\omega_{2} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Example – Inward Iteration

- We have completed the outward iteration.
- Now let's continue with the inward iteration.
- The inward iteration for link 2 are as follows:
- Because the end-effector is not in contact with the environment, we start with:

$$^{3}f_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ^{3}n_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Then:

$$^{2}f_2 = \underbrace{^{2}_{3}R}_{I} \cdot {}^{3}f_3 + {}^{2}F_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -m_2L_1c_2\dot{\theta}_1^2 + m_2L_1s_2\ddot{\theta}_1 + m_2gs_{12} - m_2L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ m_2L_1s_2\dot{\theta}_1^2 + m_2L_1c_2\ddot{\theta}_1 + m_2gc_{12} + m_2L_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \\ 0 \end{bmatrix}$$

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Example – Inward Iteration

- (Continued)

$$
\begin{aligned}
{}^{2}n_2 &= \underbrace{{}^{2}_{3}R}_{I}\cdot\underbrace{{}^{3}n_3}_{0} + {}^{2}P_{C_2}\times{}^{2}F_2 + {}^{2}P_3\times{}^{2}_{3}R\cdot\underbrace{{}^{3}f_3}_{0} + \underbrace{{}^{2}N_2}_{0} \\
&= \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix}\times\begin{bmatrix} -m_2L_1c_2\dot{\theta}_1^2 + m_2L_1s_2\ddot{\theta}_1 + m_2gs_{12} - m_2L_2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 \\ m_2L_1s_2\dot{\theta}_1^2 + m_2L_1c_2\ddot{\theta}_1 + m_2gc_{12} + m_2L_2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ m_2L_1L_2s_2\dot{\theta}_1^2 + m_2L_1L_2c_2\ddot{\theta}_1 + m_2gL_2c_{12} + m_2L_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) \end{bmatrix}
\end{aligned}
$$

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Example – Inward Iteration

- The inward iteration for link 1 gives:

$$
\begin{aligned}
{}^{1}f_1 &= {}^{1}_{2}R \cdot {}^{2}f_2 + {}^{1}F_1 \\
&= \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} -m_2L_1c_2\dot{\theta}_1^2 + m_2L_1s_2\ddot{\theta}_1 + m_2gs_{12} - m_2L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ m_2L_1s_2\dot{\theta}_1^2 + m_2L_1c_2\ddot{\theta}_1 + m_2gc_{12} + m_2L_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \\ 0 \end{bmatrix}
+ \begin{bmatrix} -m_1L_1\dot{\theta}_1^2 + m_1gs_1 \\ m_1L_1\ddot{\theta}_1 + m_1gc_1 \\ 0 \end{bmatrix}
\end{aligned}
$$

Example – Inward Iteration

- (Continued)

$$
\begin{aligned}
{}^{1}n_1 &= {}^{1}_{2}R \cdot {}^{2}n_2 + {}^{1}P_{C_1} \times {}^{1}F_1 + {}^{1}P_2 \times {}^{1}_{2}R \cdot {}^{2}f_2 + {}^{1}N_1 \\
&= \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ m_2 L_1 L_2 s_2 \dot{\theta}_1^2 + m_2 L_1 L_2 c_2 \ddot{\theta}_1 + m_2 g L_2 c_{12} + m_2 L_2^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \end{bmatrix} \\
&+ \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} -m_1 L_1 \dot{\theta}_1^2 + m_1 g s_1 \\ m_1 L_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix} \\
&+ \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -m_2 L_1 c_2 \dot{\theta}_1^2 + m_2 L_1 s_2 \ddot{\theta}_1 + m_2 g s_{12} - m_2 L_2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ m_2 L_1 s_2 \dot{\theta}_1^2 + m_2 L_1 c_2 \ddot{\theta}_1 + m_2 g c_{12} + m_2 L_2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

- This leads to the expression on the following page.

Example – Inward Iteration

- (Continued)

$$
\begin{aligned}
{}^{1}n_1 = &\begin{bmatrix} 0 \\ 0 \\ m_2L_1L_2s_2\dot{\theta}_1^2 + m_2L_1L_2c_2\ddot{\theta}_1 + m_2gL_2c_{12} + m_2L_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ m_1L_1^2\ddot{\theta}_1 + m_1gL_1c_1 \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ 0 \\ -m_2L_1L_2s_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + m_2L_1^2\ddot{\theta}_1 + m_2gL_1s_2s_{12} + m_2L_1L_2c_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2gL_1c_2c_{12} \end{bmatrix}
\end{aligned}
$$

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Example – Inward Iteration

- Finally, we obtain:

$$
\begin{aligned}
\tau_1 &= {}^1n_1^T \cdot {}^1\hat{Z}_1 \\
&= {}^1n_1^T \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= m_2 L_1 L_2 s_2 \dot{\theta}_1^2 + m_2 L_1 L_2 c_2 \ddot{\theta}_1 + m_2 g L_2 c_{12} + m_2 L_2^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_1 L_1^2 \ddot{\theta}_1 + m_1 g L_1 c_1 \\
&\quad + -m_2 L_1 L_2 s_2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + m_2 L_1^2 \ddot{\theta}_1 + m_2 g L_1 s_2 s_{12} + m_2 L_1 L_2 c_2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2 g L_1 c_2 c_{12} \\
&= m_2 L_2^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2 L_1 L_2 c_2 \left(2\ddot{\theta}_1 + \ddot{\theta}_2\right) + \left(m_1 + m_2\right) L_1^2 \ddot{\theta}_1 - m_2 L_1 L_2 s_2 \dot{\theta}_2^2 - 2 m_2 L_1 L_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\
&\quad + m_2 g L_2 c_{12} + \left(m_1 + m_2\right) g L_1 c_1
\end{aligned}
$$

$$
\begin{aligned}
\tau_2 &= {}^2n_2^T \cdot {}^2\hat{Z}_2 \\
&= m_2 L_1 L_2 s_2 \dot{\theta}_1^2 + m_2 L_1 L_2 c_2 \ddot{\theta}_1 + m_2 g L_2 c_{12} + m_2 L_2^2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right)
\end{aligned}
$$

Example - Structure

- Recall that the manipulator's dynamic equation has the following structure:

$$M(q)\ddot{q}+V(q,\dot{q})+G(q)=\tau$$

- For the case of the 2-link manipulator, which is:

$$\tau_1 = m_2L_2^2(\ddot{\theta}_1+\ddot{\theta}_2)+m_2L_1L_2c_2(2\ddot{\theta}_1+\ddot{\theta}_2)+(m_1+m_2)L_1^2\ddot{\theta}_1 - m_2L_1L_2s_2\dot{\theta}_2^2 - 2m_2L_1L_2s_2\dot{\theta}_1\dot{\theta}_2 + m_2gL_2c_{12}+(m_1+m_2)gL_1c_1$$

$$\tau_2 = m_2L_1L_2s_2\dot{\theta}_1^2 + m_2L_1L_2c_2\ddot{\theta}_1 + m_2gL_2c_{12} + m_2L_2^2(\ddot{\theta}_1+\ddot{\theta}_2)$$

- we can write:

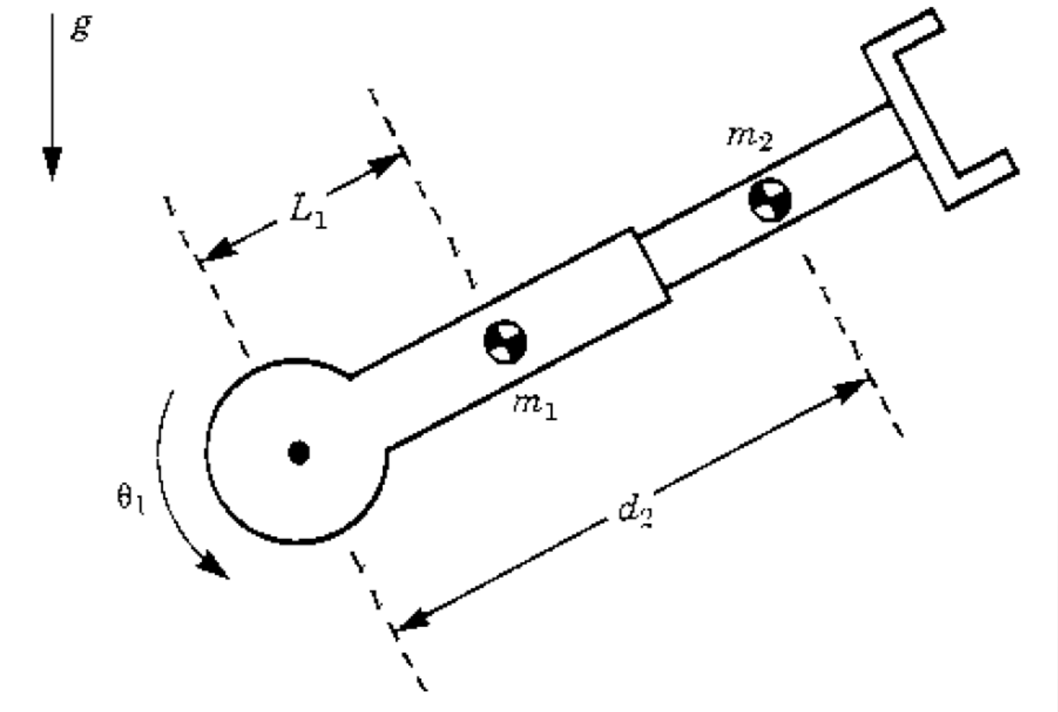
$$\underbrace{\begin{bmatrix}(m_1+m_2)L_1^2+m_2L_2^2+2m_2L_1L_2c_2 & m_2L_2^2+m_2L_1L_2c_2 \\ m_2L_2^2+m_2L_1L_2c_2 & m_2L_2^2\end{bmatrix}}_{M(q)}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \underbrace{\underbrace{\begin{bmatrix}-m_2L_1L_2s_2\dot{\theta}_2^2\\ m_2L_1L_2s_2\dot{\theta}_1^2\end{bmatrix}}_{\text{Centrifugal}} + \underbrace{\begin{bmatrix}-2m_2L_1L_2s_1\dot{\theta}_1\dot{\theta}_2\\ 0\end{bmatrix}}_{\text{Coriolis}}}_{V(q,\dot{q})} + \underbrace{\begin{bmatrix}m_2gL_2c_{12}+(m_1+m_2)gL_1c_1\\ m_2gL_2c_{12}\end{bmatrix}}_{G(q)} = \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}$$

Tutorial Assignments

- **Question 1:**
  - Find the inertia tensor of a right cylinder of homogenous density, with respect to a frame with origin at the center of mass of the body.
  - What is its inertia tensor with respect to a frame at one far end of the cylinder?

Tutorial Assignments

- **Question 2:**
  - Consider the following robot with:



$$ {}^{C_1}I_1 = \begin{bmatrix} I_{xx_1} & 0 & 0 \\ 0 & I_{yy_1} & 0 \\ 0 & 0 & I_{zz_1} \end{bmatrix} $$

$$ {}^{C_2}I_2 = \begin{bmatrix} I_{xx_2} & 0 & 0 \\ 0 & I_{yy_2} & 0 \\ 0 & 0 & I_{zz_2} \end{bmatrix} $$

  - Derive its dynamic equations.

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Thank you!

Have a good evening.