QS. 1

Thursday, 6 August 2020 2:00 PM

- **Question 1:**
  - A vector ^{A}P is rotated about Z_A by θ degrees.
  - It is subsequently rotated about X_A by Φ degrees.
  - Give the rotation matrix that accomplishes these rotations in the specified order.

Recall. Fixed Angles
- Physical Rot. Order eg. $X \to Y \to Z$
- Rot. Matrix opposite order $R_z R_y R_x$

The question →
- Physical order $Z \to X$
- Rot. Matrix $R_x R_z$?

$$R = R_x(\phi) R_z(\theta)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & -S\phi \\ 0 & S\phi & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta & -S\theta & 0 \\ C\phi S\theta & C\phi C\theta & -S\phi \\ S\phi S\theta & S\phi C\theta & C\phi \end{bmatrix}$$

QS. 2

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- **Question 2:**
  - A vector ^{A}P is rotated about Y_A by 30 degrees.
  - It is subsequently rotated about X_A by 45 degrees.
  - Give the rotation matrix that accomplishes these rotations in the specified order.

Fixed Angles → { Physical order $Y_A(30) \rightarrow X_A(45)$; Rot. Matrix $R_x(45) \times R_y(30)$ }

$$R = R_x(45) \cdot R_y(30)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C45 & -S45 \\ 0 & S45 & C45 \end{bmatrix} \begin{bmatrix} C30 & 0 & S30 \\ 0 & 1 & 0 \\ -S30 & 0 & C30 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & -0.7071 \\ 0 & 0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 0.866 & 0 & 0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0.866 \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & 0 & 0.5 \\ 0.3536 & 0.7071 & -0.6123 \\ -0.3536 & 0.7071 & 0.6123 \end{bmatrix}$$

QS. 3

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- **Question 3:**
  - A frame {B} is originally coincident with frame {A}.
  - We first rotate {B} about Z_B by θ degrees.
  - Then we rotate the resulting frame about X_B by Φ degrees.
  - Give the rotation matrix that will change the descriptions of vectors from ^BP to ^AP.

Recall: Euler Angles → { we Rotate the resulting frame about the new axes.

{ Physical Rot. order e.g. $X \to Y^* \to Z^{**}$

order of Rot. Matrix. $R_x R_y R_z$

This question → $Z_B \to X_B$

$R = R_Z R_x$

$$R = R_Z(\theta) R_x(\varphi)$$

$$= \begin{bmatrix} C\varphi & -S\varphi & 0 \\ S\varphi & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

$$= \begin{bmatrix} C\theta & -S\theta C\varphi & S\theta S\varphi \\ S\theta & C\theta C\varphi & -C\theta S\varphi \\ 0 & S\varphi & C\varphi \end{bmatrix}$$

QS 4.

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- **Question 4:**
  - The axis of a particular rotation is K = [2, 1, 2]. (NOTE: this is not yet a unit vector).
  - This axis passes through the origin.
  - The angle of rotation is 45 degrees.
  - Derive the rotation matrix based on equivalent angle-axis representation.

Recall: Equivalent Angle-Axis

K should be a unit vector $\rightarrow k_x^2 + k_y^2 + k_z^2 = 1$

First $\Rightarrow$ divide $\vec{K}$ by its magnitude $|K|$

$$|K| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$$

The $\Rightarrow$ $\hat{K} = \dfrac{K}{|K|} = [2/3, 1/3, 2/3]^T$

$$\underset{k_x}{2/3},\quad \underset{k_y}{1/3},\quad \underset{k_z}{2/3}$$

This Question $\hat{K}$ & $\theta = 45^\circ$

$S\theta = 0.7071 \quad C\theta = 0.7071$

$V\theta = 1 - C\theta = 0.2929$

$$R = \begin{bmatrix} k_x^2 V\theta + C\theta & k_x k_y V\theta - k_z S\theta & k_x k_z V\theta + k_y S\theta \\ k_x k_y V\theta + k_z S\theta & k_y^2 V\theta + C\theta & k_y k_z V\theta - k_x S\theta \\ k_x k_z V\theta - k_y S\theta & k_y k_z V\theta + k_x S\theta & k_z^2 V\theta + C\theta \end{bmatrix}$$

$$R = \begin{bmatrix} 0.8373 & -0.4063 & 0.3659 \\ 0.5365 & 0.7396 & -0.4063 \\ -0.1055 & 0.5365 & 0.8373 \end{bmatrix}$$

QS 5.

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- **Question 5**:
  - For the same rotations as in Question 4, derive the rotation matrix based on Euler parameters.

Recall: $\xi_1 = k_x \sin(\theta/2)$

$\xi_2 = k_y \sin(\theta/2)$

$\xi_3 = k_z \sin(\theta/2)$

$\xi_4 = \cos(\theta/2)$

Qs. 4

$k_x = 2/3$

$k_y = 1/3$

$k_z = 2/3$

$\theta = 45°$

$\Rightarrow$ $\xi_1 = 0.2551$

$\xi_2 = 0.1276$

$\xi_3 = 0.2551$

$\xi_4 = 0.9239$

check if $\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 \overset{?}{=} 1$

$$R = \begin{bmatrix} 1-2\xi_2^2-2\xi_3^2 & 2(\xi_1\xi_2-\xi_3\xi_4) & 2(\xi_1\xi_3+\xi_2\xi_4) \\ 2(\xi_1\xi_2+\xi_3\xi_4) & 1-2\xi_1^2-2\xi_3^2 & 2(\xi_2\xi_3-\xi_1\xi_4) \\ 2(\xi_1\xi_3-\xi_2\xi_4) & 2(\xi_2\xi_3+\xi_1\xi_4) & 1-2\xi_1^2-2\xi_2^2 \end{bmatrix}$$

$$R = \begin{bmatrix} 0.8373 & -0.4063 & 0.3659 \\ 0.5365 & 0.7397 & -0.4063 \\ -0.1055 & 0.5365 & 0.8373 \end{bmatrix}$$

QS. 6

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- **Question 6:**
  - A rotation matrix is given by:

$$R = \begin{bmatrix} 0.8373 & -0.4063 & 0.3659 \\ 0.5365 & 0.7397 & -0.4063 \\ -0.1055 & 0.5365 & 0.8373 \end{bmatrix}$$

  - Find out all the parameters in terms of:
    - X-Y-Z fixed angles
    - Z-Y-X Euler angles
    - Equivalent Angle-Axis representation
    - Euler parameters

Recall

It is hard to visualize the Rot. Matrix via Projection

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

1) X-Y-Z fixed Angles

$$\beta = \arctan2\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right)$$
$$= \arctan2\left(0.1055, \sqrt{0.8373^2 + 0.5365^2}\right)$$
$$= \arctan2(0.1055, 0.9944) = \underline{6.0561^\circ} \checkmark$$

$$\alpha = \arctan2(r_{12}/C\beta, r_{11}/C\beta)$$
$$= \arctan2(0.5365/0.9944, 0.8373/0.9944)$$
$$= \arctan2(0.5395, 0.8420) = \underline{32.6492^\circ} \checkmark$$

$$\gamma = \arctan2(r_{32}/C\beta, r_{33}/C\beta)$$
$$= \arctan2(0.5365/0.9944, 0.8373/0.9944)$$
$$= \arctan2(0.5395, 0.8420) = \underline{32.6492^\circ} \checkmark$$

$$\text{In summary, the Rot. matrix is interpreted as}\left\{\begin{array}{l}\text{1st : Rotate } \overset{\gamma}{32.6492^\circ} \text{ about } X_A \\ \text{2nd : Rotate } \overset{\beta}{6.0561^\circ} \text{ about } Y_A \\ \text{3rd : Rotate } \underset{\alpha}{32.6492^\circ} \text{ about } Z_A\end{array}\right.$$

2. Z-Y-X Euler Angles

Euler Angles → Just opposite order of Fixed Angles

1st : Rotate 32.6492° about Z_B (α)

2nd : Rotate 6.0561° about Y_B (β)

3rd : Rotate 32.6492° about X_B (γ)

3. Equivalent Angle-Axis (K, θ)

$$\theta = \arccos\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)$$

$$= \arccos\left[\tfrac{1}{2}(0.8373+0.7397+0.8373-1)\right]$$

$$= \arccos(0.7071) = \underline{45^\circ} \checkmark$$

$$\hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12}\end{bmatrix} = \frac{1}{2\sin(45)}\begin{bmatrix} 0.5365-(-0.4063) \\ 0.3659-(-0.1055) \\ 0.5365-(-0.4063)\end{bmatrix}$$

$$\frac{1}{2(0.7071)}\begin{bmatrix} 0.9428 \\ 0.4714 \\ 0.9428\end{bmatrix} = \begin{bmatrix} 0.6666 \\ 0.3333 \\ 0.6666\end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3\end{bmatrix}$$

4. Euler parameters

$$\xi_4 = \tfrac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}$$

$$= \tfrac{1}{2}\sqrt{1 + 0.8373 + 0.7397 + 0.8373}$$

$$\xi_4 = \underline{0.9239} \checkmark$$

$$\xi_1 = \frac{r_{32} - r_{23}}{4\xi_4} = 0.2551$$

$$\xi_2 = \frac{r_{13} - r_{31}}{4\xi_4} = 0.1276$$

$$\xi_3 = \frac{r_{21} - r_{12}}{4\xi_4} = 0.2551$$

QS. 7

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- **Question 7:**
  - The following frame definitions are given as known.

$$\underbrace{{}^U_AR}_{} \qquad {}^UP_{AORG}$$

$$ {}^U_AT = \begin{bmatrix} 0.866 & -0.5 & 0 & 11 \\ 0.5 & 0.866 & 0 & -1 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^B_AT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 10 \\ 0 & 0.5 & 0.866 & -20 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^C_UR \qquad {}^CP_{UORG}$$

$${}^C_UT = \begin{bmatrix} 0.866 & -0.5 & 0 & -3 \\ 0.433 & 0.75 & -0.5 & -3 \\ 0.25 & 0.433 & 0.866 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Solve for: ${}^B_CT$

given ? ?

$${}^B_CT = {}^B_AT \cdot {}^A_UT \cdot {}^U_CT$$

$$= {}^B_AT \cdot \underbrace{{}^U_AT^{-1}} \cdot \underbrace{{}^C_UT^{-1}}$$

inverse

$${}^U_AT^{-1} = \left[\begin{array}{c|c} {}^U_AR^T & -{}^U_AR^T\,{}^UP_{AORG} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

$${}^U_AT^{-1} = \left[\begin{array}{ccc|c} 0.866 & 0.5 & 0 & -\begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 11 \\ -1 \\ 8 \end{bmatrix} \\ -0.5 & 0.866 & 0 & \\ 0 & 0 & 1 & \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 0.866 & 0.5 & 0 & -9.026 \\ -0.5 & 0.866 & 0 & 6.366 \\ 0 & 0 & 1 & -8 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \checkmark$$

$${}^{C}_{U}T^{-1} = \left[\begin{array}{c|c} {}^{C}_{U}R^{T} & -{}^{C}_{U}R^{T}\,{}^{C}P_{UORG} \\ \hline 0\;0\;0 & 1 \end{array}\right]$$

$${}^{C}_{U}T^{-1} = \left[\begin{array}{ccc|c} 0.866 & 0.433 & 0.25 & \\ -0.5 & 0.75 & 0.433 & -\begin{bmatrix} 0.866 & 0.433 & 0.25 \\ -0.5 & 0.75 & 0.433 \\ 0 & -0.5 & 0.866 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix} \\ 0 & -0.5 & 0.866 & \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 0.866 & 0.433 & 0.25 & 3.147 \\ -0.5 & 0.75 & 0.433 & -0.549 \\ 0 & -0.5 & 0.866 & -4.098 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \checkmark$$

Finally plug them into $\rightarrow$ ${}^{B}_{C}T = {}^{B}_{A}T \cdot {}^{U}_{A}T^{-1} \cdot {}^{C}_{U}T^{-1}$

to get the final numerical ans.

QS. 8

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- **Question 8:**
  - For sufficiently small rotations so that the approximations $\sin\theta = \theta$ $\cos\theta = 1$ $\theta^2 = 0$
  - Derive the rotation matrix equivalent to a rotation of θ about a general axis, $\hat{K}$
  - Show that two infinitesimal rotations commute (i.e. the order in which the rotations are performed is not important).

- Replace $\begin{cases} S\theta = \theta \\ C\theta = 1 \\ \theta^2 = 0 \\ V\theta = 1 - C\theta = 0 \end{cases}$ → Equivalent Angle–Axis R

Then The R simplified $= \begin{bmatrix} 1 & -K_z\theta & K_y\theta \\ K_z\theta & 1 & -K_x\theta \\ -K_y\theta & K_x\theta & 1 \end{bmatrix}$

Mathematically, $AB \neq BA$ except in special cases

For extremely small rotations

1- assume R_1 is specified by $\hat{K}$ & θ $\Rightarrow R_1 = \begin{bmatrix} 1 & -K_z\theta & K_y\theta \\ K_z\theta & 1 & -K_x\theta \\ -K_y\theta & K_x\theta & 1 \end{bmatrix}$

2- assume R_2 is specified by $\hat{J}$ & α $\Rightarrow R_2 = \begin{bmatrix} 1 & -J_z\alpha & J_y\alpha \\ J_z\alpha & 1 & -J_x\alpha \\ -J_y\alpha & J_x\alpha & 1 \end{bmatrix}$

Then calculate $R_1 R_2$ & $R_2 R_1$

$$R_1 R_2 = \begin{bmatrix} 1 & -J_z\alpha - K_z\theta & J_y\alpha + K_y\theta \\ K_z\theta + J_z\alpha & 1 & -J_x\alpha - K_x\theta \\ -K_y\theta - J_y\alpha & K_x\theta + J_x\alpha & 1 \end{bmatrix}$$

<u>Note:</u> θ & α are very small then $\underline{\alpha\theta = 0}$

if you calculate $R_2 R_1$ you will get Same Matrix

Therefore, conclusion $\rightarrow$ $R_1 R_2 = R_2 R_1$

only for very small angles