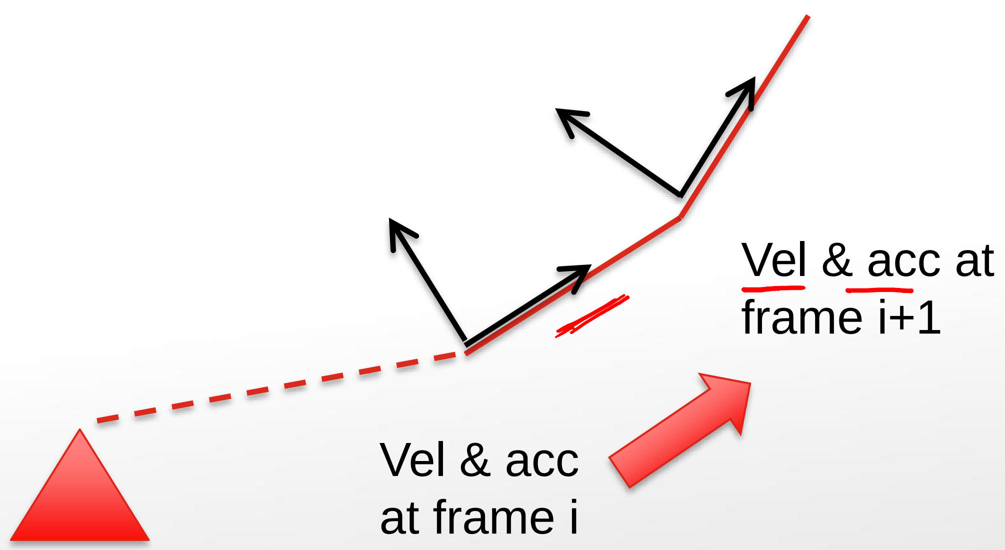
Content

- Introduction & Structure of Manipulator's Dynamic Equations
- Mass Distribution
- Newton-Euler Formulation

$M\ddot{q} + V + G = \tau$

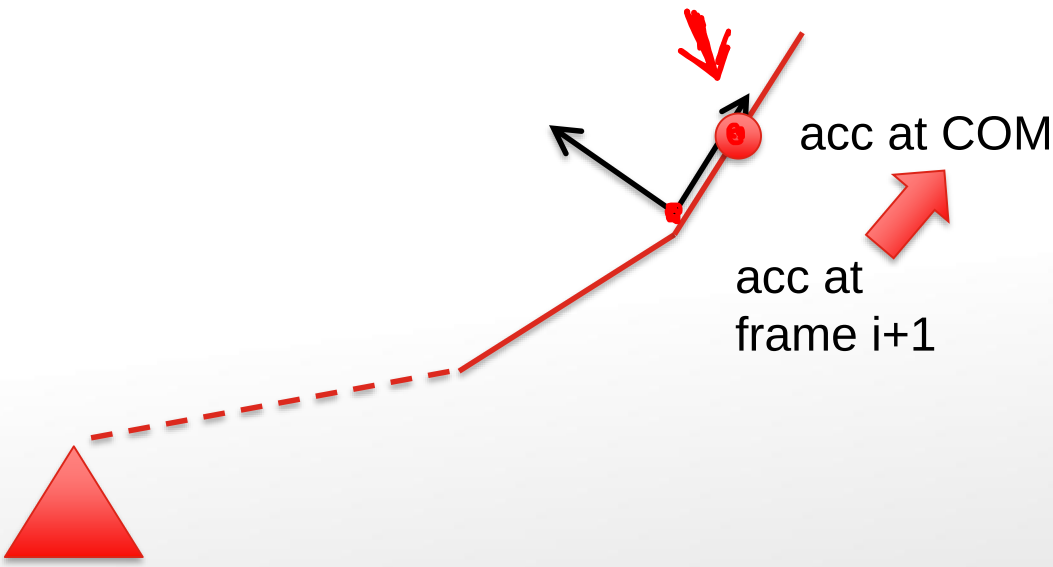
Iterative Newton-Euler Formulation

- Basic idea:
- Firstly, similar to velocity propagation which you learnt last week, acceleration can also be propagated from lower frame to upper frame.



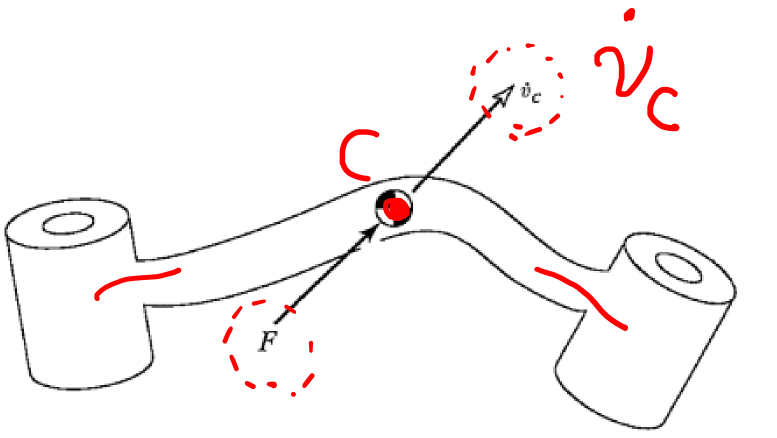
Iterative Newton-Euler Formulation

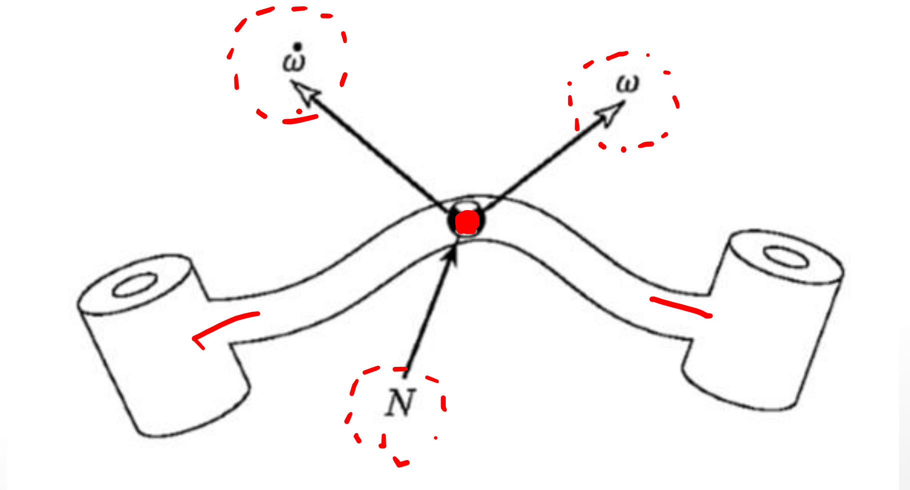
- Next, the acceleration at frame i+1 can be propagated to the centre of mass.



Iterative Newton-Euler Formulation

- Once the acceleration at centre of mass is known, then we also know the force acting on the centre of mass since F = ma.





$$F = m\,\dot{v}_C$$

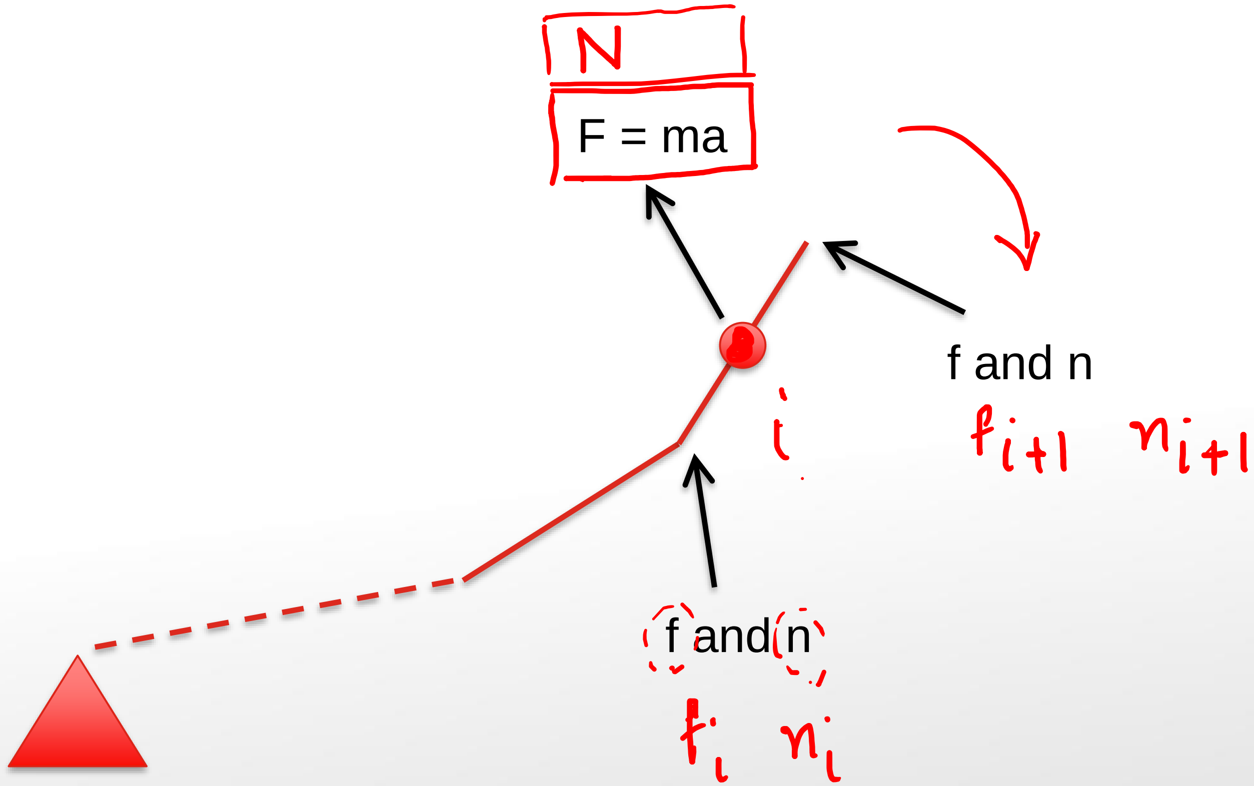
Newton's Eq.

$$N = {}^{C}I\,\dot{\omega} + \omega \times {}^{C}I\,\omega$$

Euler's Eq.

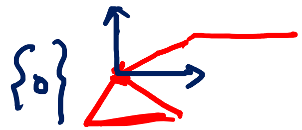
Iterative Newton-Euler Formulation

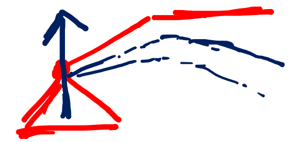
- But what "creates" F? It would be the forces / torques caused by the motors at both ends of the link, as well as contact force at the end-effector.



Full →

Iterative Newton-Euler Formulation





- Algorithm:
- Start with: ${}^{0}\omega_0 = 0$ ${}^{0}\dot{\omega}_0 = 0$ ${}^{0}\dot{\upsilon}_0 = \text{depends}$ ${}^{0}\dot{v}_0 = g$
- Outward iterations: i = 0 → 5

Note: Multiplication before cross product

vel. Prop.

Frame vel & acc

$$ {}^{i+1}\omega_{i+1} = \left({}^{i+1}_{i}R \cdot {}^{i}\omega_i\right) + \left(\dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) $$

$$ {}^{i+1}\dot{\omega}_{i+1} = \left({}^{i+1}_{i}R \cdot {}^{i}\dot{\omega}_i\right) + \left({}^{i+1}_{i}R \cdot {}^{i}\omega_i \times \dot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left(\ddot{\theta}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) $$

$$ {}^{i+1}\dot{\upsilon}_{i+1} = \left({}^{i+1}_{i}R \cdot {}^{i}\dot{\upsilon}_i\right) + \left(2\,{}^{i+1}\omega_{i+1} \times \dot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + \left(\ddot{d}_{i+1} \cdot {}^{i+1}\hat{Z}_{i+1}\right) + {}^{i+1}_{i}R \cdot \left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right)\right) $$

Prismatic

COM

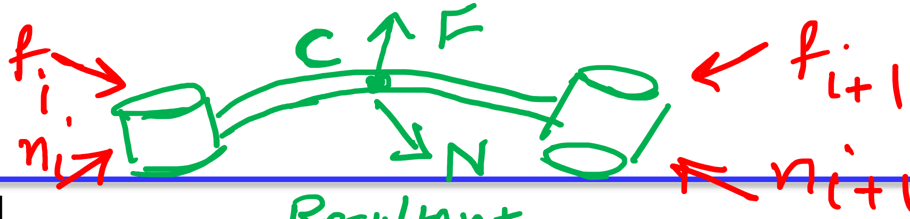
$$ {}^{i+1}\dot{\upsilon}_{C_{i+1}} = \left({}^{i+1}\dot{\upsilon}_{i+1}\right) + \left({}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + \left({}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right)\right) $$

F = ma

$$ {}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{\upsilon}_{C_{i+1}} $$

$$ {}^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1} \cdot {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1} \cdot {}^{i+1}\omega_{i+1} $$

Iterative Newton-Euler Formulation



- Algorithm (Continued):
- Inward iterations: i = 6 → 1

Joint force & torque

$${}^i f_i = {}^i_{i+1}R \cdot {}^{i+1} f_{i+1} + {}^i F_i$$

$${}^i n_i = {}^i_{i+1}R \cdot {}^{i+1} n_{i+1} + {}^i P_{C_i} \times {}^i F_i + {}^i P_{i+1} \times {}^i_{i+1}R \cdot {}^{i+1} f_{i+1} + {}^i N_i$$

$\tau_i = {}^i n_i^T \cdot {}^i \hat{Z}_i$ (Revolute) 3rd column of n

$\tau_i = {}^i f_i^T \cdot {}^i \hat{Z}_i$ (Prismatic) 3rd column of f

$M\ddot{q} + V + G = \tau$ $\left\{ \begin{array}{l} n \\ f \end{array} \right.$

Inclusion of Gravity Forces

${}^0\dot{v}_0 = \hat{G}$

- The effect of gravity forces can be included by setting:

$\{0\}$ y x ${}^0\dot{v}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$

$${}^0\dot{\upsilon}_0 = G$$

Z x ${}^0\dot{v}_0 = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$

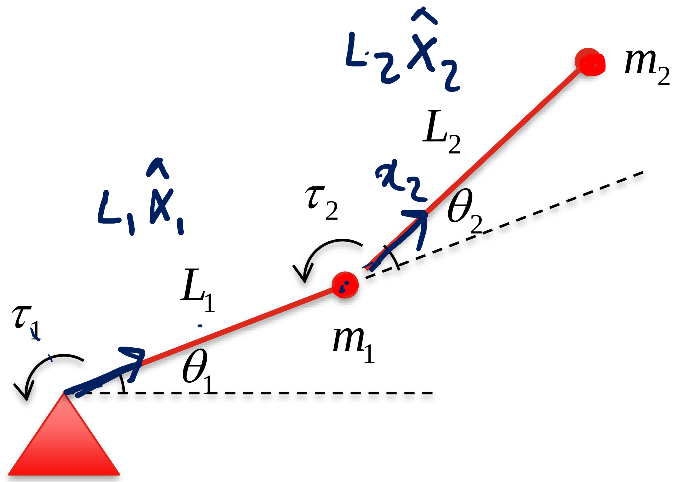
  - where G has the magnitude of gravity vector but points in the opposite direction.

- This can be interpreted as the base moving upwards with 1g acceleration.

Example

- Two link robot, where the mass of each link is a point mass at the end of the link:

^{C}I & P_c for each Link



- The vectors that locate the center of mass for each link are:

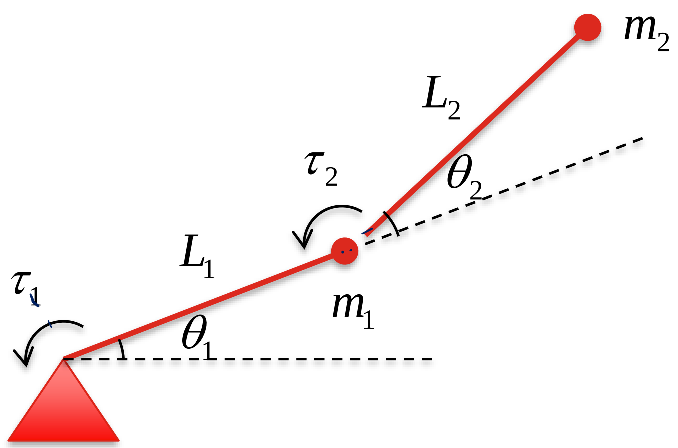
$$^{1}P_{C_1} = L_1 \hat{X}_1 \qquad ^{2}P_{C_2} = L_2 \hat{X}_2$$

- Because the mass of each link is point mass, the inertia tensor at the center of mass is zero:

$$^{C_1}I_1 = 0 \qquad ^{C_2}I_2 = 0$$

Example

- Two link robot, where the mass of each link is a point mass at the end of the link:



- The vectors that locate the center of mass for each link are:

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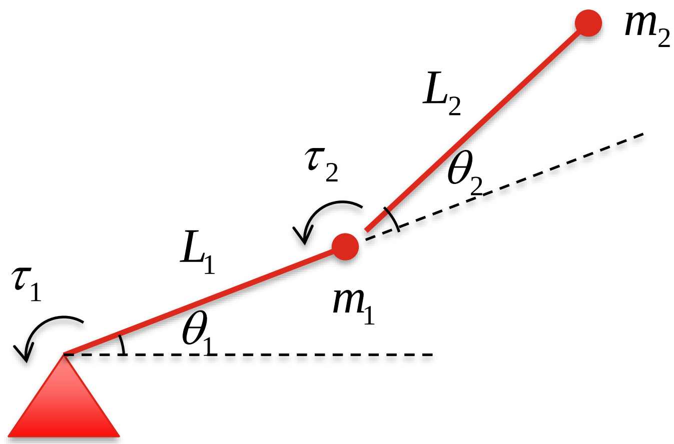
$$ {}^{C_1}I_1 = 0 \qquad {}^{C_2}I_2 = 0 $$

Example

- Furthermore, the rotation matrices between successive links are:



$\Rightarrow$

$$ {}^{i}_{i+1}R = \begin{bmatrix} c_{i+1} & -s_{i+1} & 0 \\ s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^{i+1}_{i}R = \begin{bmatrix} c_{i+1} & s_{i+1} & 0 \\ -s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Example – Outward Iteration

- Now we use the Iterative Newton-Euler algorithm.
- First, we start with:

$\{0\}$

$$^{0}\omega_0 = 0 \qquad ^{0}\dot{\omega}_0 = 0 \qquad ^{0}\dot{v}_0 = g\hat{Y}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$$

- Then, the outward iterations for link 1 give:

$$^{1}Z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

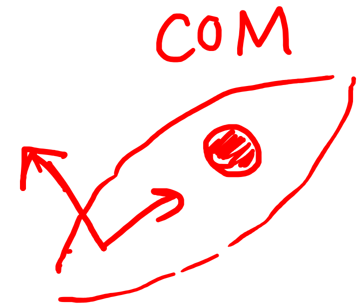
$$^{1}\omega_1 = \left({}^{1}_{0}R \cdot \underbrace{\cancel{^{0}\omega_0}}_{0}\right) + \left(\dot{\theta}_1 \cdot {}^{1}\hat{Z}_1\right) = \dot{\theta}_1 \cdot {}^{1}\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$^{1}\dot{\omega}_1 = \left({}^{1}_{0}R \cdot \underbrace{\cancel{^{0}\dot{\omega}_0}}_{0}\right) + \left({}^{1}_{0}R \cdot \underbrace{\cancel{^{0}\omega_0}}_{0} \times \dot{\theta}_1 \cdot {}^{1}\hat{Z}_1\right) + \left(\ddot{\theta}_1 \cdot {}^{1}\hat{Z}_1\right) = \ddot{\theta}_1 \cdot {}^{1}\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}$$

Example – Outward Iteration

- (continued):

$$
\begin{aligned}
{}^1\dot{\upsilon}_1 &= \boxed{\left({}^1_0R \cdot {}^0\dot{\upsilon}_0\right)} + \overbrace{\left(2\,{}^1\omega_1 \times \dot{d}_1 \cdot {}^1\hat{Z}_1\right)}^{0} + \overbrace{\left(\ddot{d}_1 \cdot {}^1\hat{Z}_1\right)}^{0} \\
&\quad + {}^1_0R \cdot \underbrace{\left({}^0\dot{\omega}_0 \times {}^0P_1 + {}^0\omega_0 \times {}^0\omega_0 \times {}^0P_1\right)}_{0} \\
&= \left({}^1_0R \cdot {}^0\dot{\upsilon}_0\right) \\
&= \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix}
\end{aligned}
$$



$$
\begin{aligned}
{}^1\dot{\upsilon}_{C_1} &= \left({}^1\dot{\upsilon}_1\right) + \left({}^1\dot{\omega}_1 \times {}^1P_{C_1}\right) + \left({}^1\omega_1 \times {}^1\omega_1 \times {}^1P_{C_1}\right) \\
&= \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ L_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -L_1\dot{\theta}_1^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -L_1\dot{\theta}_1^2 + gs_1 \\ L_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix}
\end{aligned}
$$

$${}^1P_{C_1} = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix}$$

Example – Outward Iteration

- (continued):

$$ {}^{1}F_1 = m_1 \, {}^{1}\dot{v}_{C_1} = m_1 \begin{bmatrix} -L_1\dot{\theta}_1^2 + gs_1 \\ L_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -m_1L_1\dot{\theta}_1^2 + m_1 g s_1 \\ m_1 L_1\ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix} $$

$$ {}^{1}N_1 = \underbrace{{}^{C_1}I_1}_{0} \cdot {}^{1}\dot{\omega}_1 + {}^{1}\omega_1 \times \underbrace{{}^{C_1}I_1}_{0} \cdot {}^{1}\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$^{C_1}I_1 = 0$

Example – Outward Iteration

- The outward iterations for link 2 give:

$$
{}^{2}\omega_{2} = \left({}^{2}_{1}R \cdot {}^{1}\omega_{1}\right) + \left(\dot{\theta}_{2} \cdot {}^{2}\hat{Z}_{2}\right)
$$

$$
= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}
$$

$$
{}^{2}\dot{\omega}_{2} = \left({}^{2}_{1}R \cdot {}^{1}\dot{\omega}_{1}\right) + \left({}^{2}_{1}R \cdot {}^{1}\omega_{1} \times \dot{\theta}_{2} \cdot {}^{2}\hat{Z}_{2}\right) + \left(\ddot{\theta}_{2} \cdot {}^{2}\hat{Z}_{2}\right)
$$

$$
= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}
$$

Example – Outward Iteration

- (Continued):

$$
\begin{aligned}
{}^2\dot{\upsilon}_2 &= \left({}^2_1R\cdot{}^1\dot{\upsilon}_1\right)+\left(2\,{}^2\omega_2\times\dot{d}_2\cdot{}^2\hat{Z}_2\right)+\left(\ddot{d}_2\cdot{}^2\hat{Z}_2\right)+{}^2_1R\cdot\left({}^1\dot{\omega}_1\times{}^1P_2+{}^1\omega_1\times{}^1\omega_1\times{}^1P_2\right)\\
&=\begin{bmatrix} c_2 & s_2 & 0\\ -s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} gs_1\\ gc_1\\ 0\end{bmatrix}+0+0+\begin{bmatrix} c_2 & s_2 & 0\\ -s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix}\times\begin{bmatrix}L_1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}\times\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}\times\begin{bmatrix}L_1\\0\\0\end{bmatrix}\right)\\
&=\begin{bmatrix} -L_1c_2\dot{\theta}_1^2+L_1s_2\ddot{\theta}_1+gs_{12}\\ L_1s_2\dot{\theta}_1^2+L_1c_2\ddot{\theta}_1+gc_{12}\\ 0\end{bmatrix}
\end{aligned}
$$

Example – Outward Iteration

- (Continued):

$$
\begin{aligned}
{}^{2}\dot{\upsilon}_{C_2} &= \left({}^{2}\dot{\upsilon}_2\right) + \left({}^{2}\dot{\omega}_2 \times {}^{2}P_{C_2}\right) + \left({}^{2}\omega_2 \times {}^{2}\omega_2 \times {}^{2}P_{C_2}\right) \\
&= \begin{bmatrix} -L_1 c_2 \dot{\theta}_1^2 + L_1 s_2 \ddot{\theta}_1 + g s_{12} \\ L_1 s_2 \dot{\theta}_1^2 + L_1 c_2 \ddot{\theta}_1 + g c_{12} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -L_1 c_2 \dot{\theta}_1^2 + L_1 s_2 \ddot{\theta}_1 + g s_{12} - L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ L_1 s_2 \dot{\theta}_1^2 + L_1 c_2 \ddot{\theta}_1 + g c_{12} + L_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \\ 0 \end{bmatrix}
\end{aligned}
$$

Example – Outward Iteration

- (Continued):

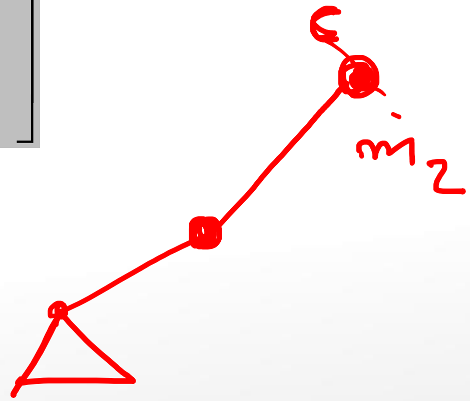
$$
{}^2F_2 = m_2\,{}^2\dot{v}_{C_2}
$$

$$
= \begin{bmatrix} -m_2L_1c_2\dot{\theta}_1^2 + m_2L_1s_2\ddot{\theta}_1 + m_2gs_{12} - m_2L_2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 \\ m_2L_1s_2\dot{\theta}_1^2 + m_2L_1c_2\ddot{\theta}_1 + m_2gc_{12} + m_2L_2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) \\ 0 \end{bmatrix}
$$

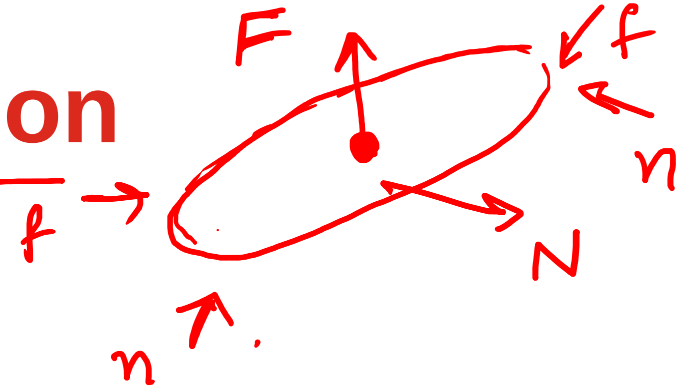
$$
{}^2N_2 = \overbrace{{}^{C_2}I_2 \cdot {}^2\dot{\omega}_2}^{0} + \overbrace{{}^2\omega_2 \times {}^{C_2}I_2 \cdot {}^2\omega_2}^{0}
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$^{C_2}I_2 = 0$

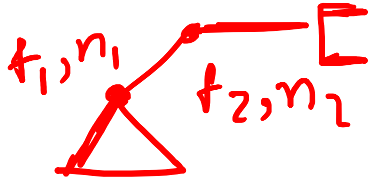


Example – Inward Iteration



- We have completed the outward iteration.
- Now let's continue with the inward iteration.
- The inward iteration for link 2 are as follows:
- Because the end-effector is not in contact with the environment, we start with:

$$ {}^{3}f_{3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^{3}n_{3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$



$f_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$n_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

- Then:

$$ {}^{2}f_{2} = \underbrace{{}^{2}_{3}R}_{I} \overbrace{\cdot\, {}^{3}f_{3}}^{0} + {}^{2}F_{2} $$

$$ = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -m_2 L_1 c_2 \dot{\theta}_1^2 + m_2 L_1 s_2 \ddot{\theta}_1 + m_2 g s_{12} - m_2 L_2 \left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ m_2 L_1 s_2 \dot{\theta}_1^2 + m_2 L_1 c_2 \ddot{\theta}_1 + m_2 g c_{12} + m_2 L_2 \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \\ 0 \end{bmatrix} = \begin{bmatrix} {}^{2}F_{2x} \\ {}^{2}F_{2y} \\ {}^{2}F_{2z} \end{bmatrix} $$

Example – Inward Iteration

- (Continued)

$$
{}^2n_2 = \underbrace{{}^2_3R}_{I}\cdot \underbrace{{}^3n_3}_{0} + {}^2P_{C_2}\times {}^2F_2 + {}^2P_3\times {}^2_3R\cdot \underbrace{{}^3f_3}_{0} + \underbrace{{}^2N_2}_{0}
$$

$$
= \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} -m_2L_1c_2\dot{\theta}_1^2 + m_2L_1s_2\ddot{\theta}_1 + m_2gs_{12} - m_2L_2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 \\ m_2L_1s_2\dot{\theta}_1^2 + m_2L_1c_2\ddot{\theta}_1 + m_2gc_{12} + m_2L_2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) \\ 0 \end{bmatrix}
$$

$$
: \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} {}^2F_{2x} \\ {}^2F_{2y} \\ 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ m_2L_1L_2s_2\dot{\theta}_1^2 + m_2L_1L_2c_2\ddot{\theta}_1 + m_2gL_2c_{12} + m_2L_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) \end{bmatrix}
$$

Example – Inward Iteration

- The inward iteration for link 1 gives:

$$
{}^{1}f_1 = {}^{1}_{2}R \cdot {}^{2}f_2 + {}^{1}F_1
$$

$$
= \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -m_2L_1c_2\dot{\theta}_1^2 + m_2L_1s_2\ddot{\theta}_1 + m_2gs_{12} - m_2L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ m_2L_1s_2\dot{\theta}_1^2 + m_2L_1c_2\ddot{\theta}_1 + m_2gc_{12} + m_2L_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \\ 0 \end{bmatrix} + \begin{bmatrix} -m_1L_1\dot{\theta}_1^2 + m_1gs_1 \\ m_1L_1\ddot{\theta}_1 + m_1gc_1 \\ 0 \end{bmatrix}
$$

Example – Inward Iteration

- (Continued)

$$
\begin{aligned}
{}^{1}n_1 &= {}^{1}_{2}R \cdot {}^{2}n_2 + {}^{1}P_{C_1} \times {}^{1}F_1 + {}^{1}P_2 \times {}^{1}_{2}R \cdot {}^{2}f_2 + {}^{1}N_1 \\
&= \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ m_2L_1L_2s_2\dot{\theta}_1^2 + m_2L_1L_2c_2\ddot{\theta}_1 + m_2gL_2c_{12} + m_2L_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \end{bmatrix} \\
&\quad + \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} -m_1L_1\dot{\theta}_1^2 + m_1gs_1 \\ m_1L_1\ddot{\theta}_1 + m_1gc_1 \\ 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -m_2L_1c_2\dot{\theta}_1^2 + m_2L_1s_2\ddot{\theta}_1 + m_2gs_{12} - m_2L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ m_2L_1s_2\dot{\theta}_1^2 + m_2L_1c_2\ddot{\theta}_1 + m_2gc_{12} + m_2L_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

- This leads to the expression on the following page.

Example – Inward Iteration

- (Continued)

$$\Rightarrow {}^1n_1 = \begin{bmatrix} 0 \\ 0 \\ m_2L_1L_2s_2\dot{\theta}_1^2 + m_2L_1L_2c_2\ddot{\theta}_1 + m_2gL_2c_{12} + m_2L_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ m_1L_1^2\ddot{\theta}_1 + m_1gL_1c_1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ -m_2L_1L_2s_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + m_2L_1^2\ddot{\theta}_1 + m_2gL_1s_2s_{12} + m_2L_1L_2c_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2gL_1c_2c_{12} \end{bmatrix}$$

$${}^2f_2,\ {}^2n_2,\ {}^1n_1,\ {}^1f_1 \Rightarrow \begin{cases} \tau_1 = ? \\ \tau_2 = ? \end{cases}$$

Example – Inward Iteration

- Finally, we obtain:

$$\tau_1 = {}^1n_1^T \cdot {}^1\hat{Z}_1$$

$$= {}^1n_1^T \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= m_2L_1L_2s_2\dot{\theta}_1^2 + m_2L_1L_2c_2\ddot{\theta}_1 + m_2gL_2c_{12} + m_2L_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_1L_1^2\ddot{\theta}_1 + m_1gL_1c_1$$
$$+ -m_2L_1L_2s_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + m_2L_1^2\ddot{\theta}_1 + m_2gL_1s_2s_{12} + m_2L_1L_2c_2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2gL_1c_2c_{12}$$

$$\tau_1 = m_2L_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + m_2L_1L_2c_2\left(2\ddot{\theta}_1 + \ddot{\theta}_2\right) + \left(m_1 + m_2\right)L_1^2\ddot{\theta}_1 - m_2L_1L_2s_2\dot{\theta}_2^2 - 2m_2L_1L_2s_2\dot{\theta}_1\dot{\theta}_2$$
$$+ m_2gL_2c_{12} + \left(m_1 + m_2\right)gL_1c_1$$

Revolute: 3rd column of ${}^1n_1^T$: [¦ ¦0]

3rd row at ${}^1n_1$: 0 → τ_1

$$\tau_2 = {}^2n_2^T \cdot {}^2\hat{Z}_2$$
$$= m_2L_1L_2s_2\dot{\theta}_1^2 + m_2L_1L_2c_2\ddot{\theta}_1 + m_2gL_2c_{12} + m_2L_2^2\left(\ddot{\theta}_1 + \ddot{\theta}_2\right)$$

Rev. 3rd row of ${}^2n_2$

τ_1 & τ_2

Example - Structure

$$\begin{bmatrix} m_2L_2^2+(m_1+m_2)L_1^2 + 2m_2L_1L_2C_2 & m_2L_2^2 + m_2L_1L_2C_2 \\ m_2L_1L_2C_2 + m_2L_2^2 & m_2L_2^2 \end{bmatrix}\begin{bmatrix}\ddot\theta_1 \\ \ddot\theta_2\end{bmatrix}$$

- Recall that the manipulator's dynamic equation has the following structure:

$$\boxed{M(q)\ddot q} + \boxed{V(q,\dot q)} + \boxed{G(q)} = \tau$$

τ_1
τ_2

- For the case of the 2-link manipulator, which is:

$$\tau_1 = m_2L_2^2(\ddot\theta_1+\ddot\theta_2) + m_2L_1L_2c_2(2\ddot\theta_1+\ddot\theta_2) + (m_1+m_2)L_1^2\ddot\theta_1 - m_2L_1L_2s_2\dot\theta_2^2 - 2m_2L_1L_2s_2\dot\theta_1\dot\theta_2 + m_2gL_2c_{12} + (m_1+m_2)gL_1c_1$$

$$\tau_2 = m_2L_1L_2s_2\dot\theta_1^2 + m_2L_1L_2c_2\ddot\theta_1 + m_2gL_2c_{12} + m_2L_2^2(\ddot\theta_1+\ddot\theta_2)$$

$V(q,\dot q)$

- we can write:

$$\underbrace{\begin{bmatrix}(m_1+m_2)L_1^2+m_2L_2^2+2m_2L_1L_2c_2 & m_2L_2^2+m_2L_1L_2c_2 \\ m_2L_2^2+m_2L_1L_2c_2 & m_2L_2^2\end{bmatrix}}_{M(q)}\begin{bmatrix}\ddot\theta_1\\ \ddot\theta_2\end{bmatrix} + \underbrace{\underbrace{\begin{bmatrix}-m_2L_1L_2s_2\dot\theta_2^2 \\ m_2L_1L_2s_2\dot\theta_1^2\end{bmatrix}}_{\text{Centrifugal}} + \underbrace{\begin{bmatrix}-2m_2L_1L_2s_2\dot\theta_1\dot\theta_2 \\ 0\end{bmatrix}}_{\text{Coriolis}}}_{V(q,\dot q)} + \underbrace{\begin{bmatrix}m_2gL_2c_{12}+(m_1+m_2)gL_1c_1 \\ m_2gL_2c_{12}\end{bmatrix}}_{G(q)} = \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}$$

Thank you!

Have a good evening.