Week 7 – Manipulator Dynamics

Advanced Robotic Systems – MANU2453

Dr Ehsan Asadi, School of Engineering
RMIT University, Victoria, Australia
Email: ehsan.asadi@rmit.edu.au

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Lectures

Wk	Date	Lecture (NOTE: video recording)	Maths Difficulty	Hands-on Activity	Related Assessment
1	24/7	• Introduction to the Course • Spatial Descriptions & Transformations			
2	31/7	• Spatial Descriptions & Transformations • Robot Cell Design			Robot Cell Design Assignment
3	7/8	• Forward Kinematics • Inverse Kinematics			
4	14/8	• ABB Robot Programming via Teaching Pendant • ABB RobotStudio Offline Programming		ABB RobotStudio Offline Programming	Offline Programming Assignment
5	21/8	• Jacobians: Velocities and Static Forces			
6	28/8	• Manipulator Dynamics			
7	11/9	• Manipulator Dynamics		MATLAB Simulink Simulation	
8	18/9	• Robotic Vision		MATLAB Simulation	Robotic Vision Assignment
9	25/9	• Robotic Vision		MATLAB Simulation	
10	2/10	• Trajectory Generation			
11	9/10	• Linear & Nonlinear Control		MATLAB Simulink Simulation	
12	16/10	• Introduction to I4.0 • Revision			Final Exam

Content

- Lagrangian Formulation
- Inclusion of Non-Rigid Body Effects
- Explicit Form

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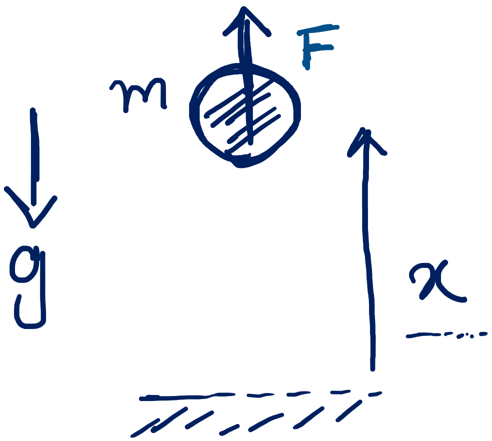
Lagrangian Formulation

- Apart from Newton-Euler's method, there are other approaches to obtain the manipulator's dynamic equation as well.
- The Lagrangian formulation is one such method.
  - It is an "energy-based" approach.
  - The dynamic equations will be derived from the kinetic energy and the potential energy of the manipulator.
- Another approach is the "Explicit Form" method.
  - The V and G vectors can be derived directly from M matrix.

Lagrangian Formulation



KE: $K = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2$

PE: $U = mgh = mgx$

Lagrangian Eq:

$$\frac{d}{dt}\left(\frac{dK}{d\dot{x}}\right) - \frac{dK}{dx} + \frac{dU}{dx} = F$$

$dK/d\dot{x} = m\dot{x}$

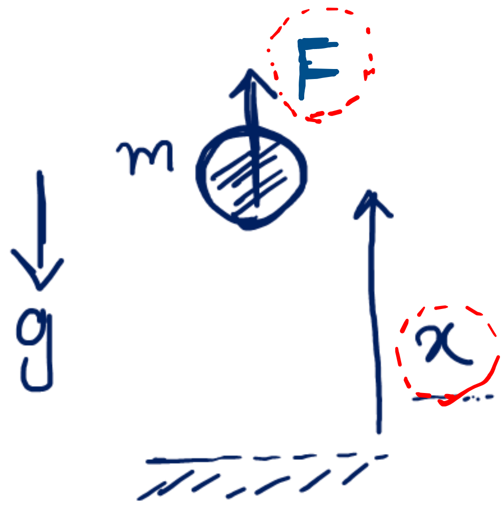
$$\frac{d}{dt}\left(\frac{dK}{d\dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$$

$dK/dx = 0$

$dU/dx = mg$

$m\ddot{x} - 0 + mg = F \Rightarrow$

Lagrangian Formulation

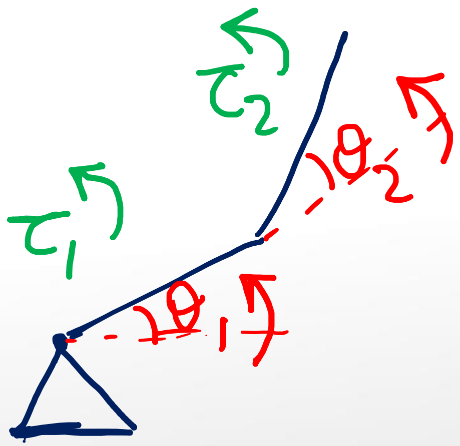


$\underline{\underline{F}}$ is in the same direction as $\underline{\underline{x}}$

$$m\ddot{x} - 0 + mg = F$$

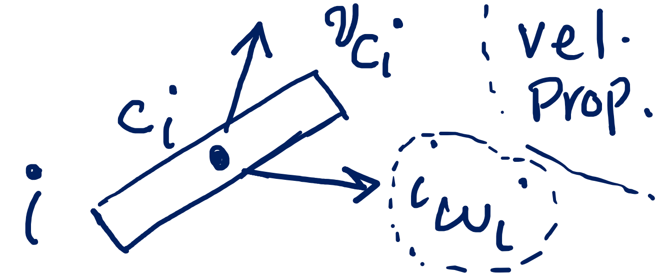
$$m\ddot{x} = F - mg$$

$$\Sigma F = ma$$



τ & (θ or d) are in the same direction

Lagrangian Formulation



- The kinetic energy of each link is:

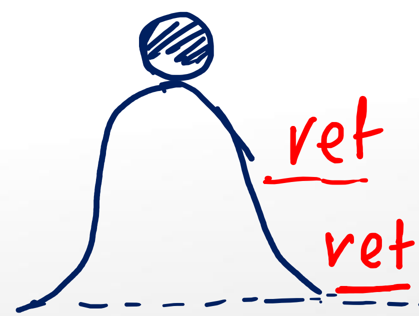
$$i: \quad K_i = \tfrac{1}{2} m_i \, v_{c_i}^T v_{c_i} + \tfrac{1}{2} \, {}^i\omega_i \cdot {}^{c_i}I_i \cdot {}^i\omega_i$$

and the total kinetic energy of the whole manipulator is $k = \sum_{i=1}^{n} k_i$

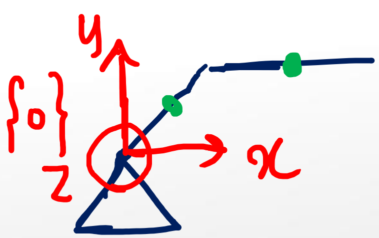
- The potential energy of each link is:

$$i: \quad U_i = -m_i \cdot {}^0g^T \cdot {}^0P_{c_i} + u_{ref}$$

Constant



$g \downarrow$



$${}^0g = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

$$* \; \frac{du_{ref}}{dx} = 0$$

The total potential energy of the manipulator is then: $u = \sum_{i=1}^{n} u_i$

Lagrangian Formulation

- The kinetic energy of each link is:

$$k_i = \frac{1}{2} m_i \upsilon_{c_i}^T \upsilon_{c_i} + \frac{1}{2} \cdot {}^i\omega_i^T \cdot {}^{C_i} I_i \cdot {}^i\omega_i$$

  - and the total kinetic energy of the whole manipulator is: $k = \sum_{i=1}^{n} k_i$

- The potential energy of each link is:

$$u_i = -m_i \cdot {}^0 g^T \cdot {}^0 P_{C_i} + u_{ref_i}$$

  - where ${}^0g$ is the 3 x 1 gravity vector, ${}^0P_{Ci}$ is the vector representing the position of the centre of the mass of the ith link, and u_{refi} is a constant so that the minimum of u_i is zero.
  - The total potential energy of the manipulator is then: $u = \sum_{i=1}^{n} u_i$

Lagrangian Formulation

- Lagrangian is the difference between the kinetic and potential energy of a mechanical system:

$$L = k - u$$

- The equation of motion for the manipulator is then:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \tau$$

- Because the potential energy, u, is independent of velocity, the equation can be written as:

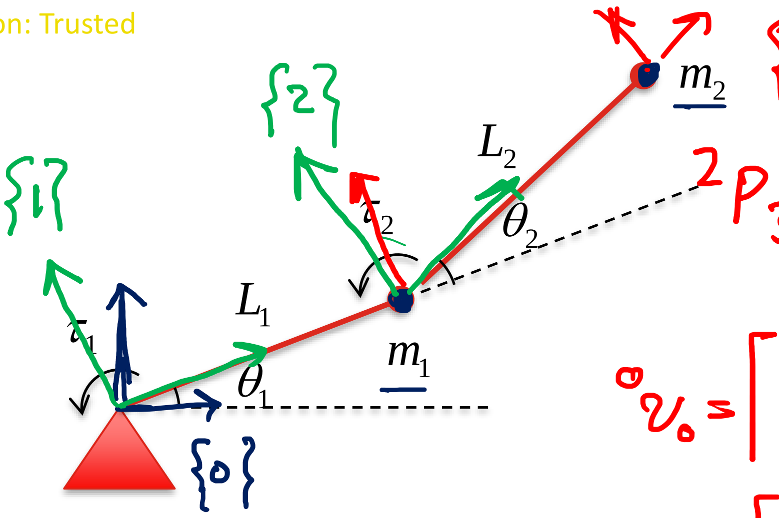
$$\frac{d}{dt}\frac{\partial k}{\partial \dot{q}} - \frac{\partial (k-u)}{\partial q} = \tau$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{q}} - \frac{\partial k}{\partial q} + \frac{\partial u}{\partial q} = \tau$$

$$\frac{d}{dt}\left(\frac{dk}{d\dot{q}}\right) - \frac{dk}{dq} + \frac{du}{dq} = \tau$$

$q(\theta, d)$, $\dot{q}(\dot{\theta}, \dot{d})$, $\tau(n, f)$

Example



$${}^2P_3 = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \quad {}^0v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad {}^0\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Let's try this method for the two-link robot from week 6.
- From lecture 5, we had:

$${}^1v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad {}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \quad {}^2v_2 = \begin{bmatrix} L_1\dot{\theta}_1 S_1 \\ L_1\dot{\theta}_1 C_1 \\ 0 \end{bmatrix} \quad {}^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

- We also need the speed of the origin of frame {3} which is at the tip of manipulator, having the same orientation of frame {2}.
- Using ${}^{i+1}\upsilon_{i+1} = {}^{i+1}_{i}R \cdot \left({}^{i}\upsilon_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right)$, we get:

${}^2P_3$

$${}^3\upsilon_3 = \underbrace{{}^3_2R}_{I} \cdot \left({}^2\upsilon_2 + {}^2\omega_2 \times {}^2P_3\right) = \begin{bmatrix} L_1 s_2 \dot{\theta}_1 \\ L_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 s_2 \dot{\theta}_1 \\ L_1 c_2 \dot{\theta}_1 + L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ 0 \end{bmatrix}$$

${}^2v_2 + {}^2\omega_2 \times {}^2P_3 =$

Example

Prop. Vel. to COM

- Note that the center of mass for 1st link is the origin of frame {2}, and center of mass for 2nd link is the origin of frame {3} / tip of manipulator.
- Therefore:

C_1 & ORG of {2} are the same

$$^2\upsilon_{C_1} = {}^2\upsilon_2 = \begin{bmatrix} L_1 s_2 \dot{\theta}_1 \\ L_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix} \qquad ^3\upsilon_{C_2} = {}^3\upsilon_3 = \begin{bmatrix} L_1 s_2 \dot{\theta}_1 \\ L_1 c_2 \dot{\theta}_1 + L_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$

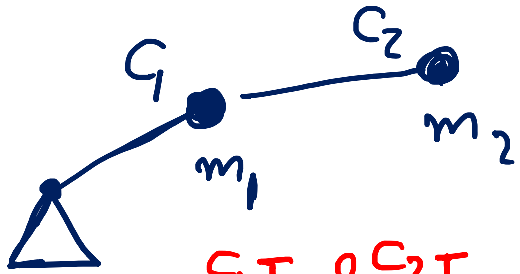
C_2 & ORG of {3} are the same

$$\Rightarrow \quad ^1\omega_{C_1} = {}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad ^2\omega_{C_2} = {}^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

- Looking at the kinetic equation again:

$$\Rightarrow \quad k_i = \frac{1}{2} m_i \upsilon_{c_i}^T \upsilon_{c_i} + \frac{1}{2} \cdot {}^i\omega_i^T \cdot {}^{C_i}I_i \cdot {}^i\omega_i \quad \begin{cases} K_1 \\ K_2 \end{cases} \Rightarrow K = K_1 + K_2$$

Example



$^{C_1}I_1$ & $^{C_2}I_2 = 0$

- The total kinetic energy of the manipulator is thus:

$$k = \overbrace{\frac{1}{2}m_1\upsilon_{c_1}^T\upsilon_{c_1} + \frac{1}{2}\cdot{}^1\omega_1^T\,\underbrace{^{C_1}I_1}_{0}\,{}^1\omega_1}^{K_1} + \overbrace{\frac{1}{2}m_2\upsilon_{c_2}^T\upsilon_{c_2} + \frac{1}{2}\cdot{}^2\omega_2^T\,\underbrace{^{C_2}I_2}_{0}\,{}^2\omega_2}^{K_2}$$

$$K = \frac{1}{2}m_1\begin{bmatrix} L_1 s_2\dot{\theta}_1 \\ L_1 c_2\dot{\theta}_1 \\ 0 \end{bmatrix}^T \begin{bmatrix} L_1 s_2\dot{\theta}_1 \\ L_1 c_2\dot{\theta}_1 \\ 0 \end{bmatrix} + \frac{1}{2}m_2\begin{bmatrix} L_1 s_2\dot{\theta}_1 \\ L_1 c_2\dot{\theta}_1 + L_2\left(\dot{\theta}_1+\dot{\theta}_2\right) \\ 0 \end{bmatrix}^T \begin{bmatrix} L_1 s_2\dot{\theta}_1 \\ L_1 c_2\dot{\theta}_1 + L_2\left(\dot{\theta}_1+\dot{\theta}_2\right) \\ 0 \end{bmatrix}$$

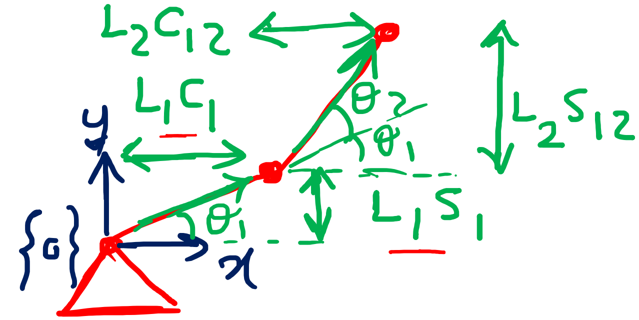
$$\Rightarrow = \frac{1}{2}m_1 L_1^2 s_2^2\dot{\theta}_1^2 + \frac{1}{2}m_1 L_1^2 c_2^2\dot{\theta}_1^2 + \frac{1}{2}m_2 L_1^2 s_2^2\dot{\theta}_1^2 + \frac{1}{2}m_2 L_1^2 c_2^2\dot{\theta}_1^2 + \frac{1}{2}m_2 L_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + m_2 L_1 L_2 c_2\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)$$

$$K = \frac{1}{2}(m_1+m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2 L_2^2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + m_2 L_1 L_2 c_2\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)$$

$\cos\theta_2$

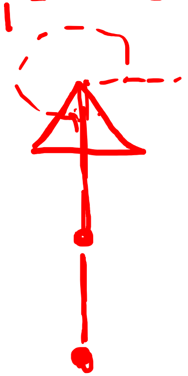
Example

$${}^0g = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} \quad g \downarrow$$



- The potential energy of the manipulator is:

θ1=270



$$u = \overbrace{-m_1 \cdot {}^0g^T \cdot {}^0P_{C_1} + u_{ref_1}}^{u_1} \overbrace{-m_2 \cdot {}^0g^T \cdot {}^0P_{C_2} + u_{ref_2}}^{u_2} \qquad {}^0P_{C_2}$$

$$= -m_1\begin{bmatrix}0 & -g & 0\end{bmatrix}\begin{bmatrix} L_1c_1 \\ L_1s_1 \\ 0 \end{bmatrix} + \underbrace{u_{ref_1}}_{\text{Const}} - m_2\begin{bmatrix}0 & -g & 0\end{bmatrix}\begin{bmatrix} L_1c_1 + L_2c_{12} \\ L_1s_1 + L_2s_{12} \\ 0 \end{bmatrix} + \underbrace{u_{ref_2}}_{\text{Const}}$$

$$= m_1gL_1s_1 + u_{ref_1} + m_2g\left(L_1s_1 + L_2s_{12}\right) + u_{ref_2}$$

- As mentioned, $uref_i$ is chosen such that the minimum of potential energy is zero.
  - For link 1, the minimum of $m_1gL_1s_1$ is $-m_1gL_1$ when θ_1 = 270 deg. Therefore: $u_{ref_1} = m_1gL_1$
  - Following the same argument, we have: $u_{ref_2} = m_2g\left(L_1 + L_2\right)$
- Therefore:

$$\Rightarrow \quad u = m_1gL_1s_1 + \underbrace{m_1gL_1}_{\text{Const}} + m_2g\left(L_1s_1 + L_2s_{12}\right) + \underbrace{m_2g\left(L_1 + L_2\right)}_{\text{Const}}$$

Example

$\cos\theta_2$

- The kinetic and potential energies are repeated here for convenience sake:

$$k = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + m_2L_1L_2c_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)$$

$$u = m_1gL_1s_1 + m_1gL_1 + m_2g\left(L_1s_1 + L_2s_{12}\right) + m_2g\left(L_1 + L_2\right)$$

- Now, apply the formula:

$$\frac{d}{dt}\frac{\partial k}{\partial\dot{\theta}} - \frac{\partial k}{\partial\theta} + \frac{\partial u}{\partial\theta} = \tau$$

$(\theta_1, \theta_2) \Rightarrow$

$$\frac{d}{dt}\begin{bmatrix}\dfrac{\partial k}{\partial\dot{\theta}_1}\\[2mm] \dfrac{\partial k}{\partial\dot{\theta}_2}\end{bmatrix} - \begin{bmatrix}\dfrac{\partial k}{\partial\theta_1}\\[2mm] \dfrac{\partial k}{\partial\theta_2}\end{bmatrix} + \begin{bmatrix}\dfrac{\partial u}{\partial\theta_1}\\[2mm] \dfrac{\partial u}{\partial\theta_2}\end{bmatrix} = \tau$$

$$\frac{d}{dt}\begin{bmatrix}(m_1 + m_2)L_1^2\dot{\theta}_1 + m_2L_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right) + 2m_2L_1L_2c_2\dot{\theta}_1 + m_2L_1L_2c_2\dot{\theta}_2\\ m_2L_2^2\left(\dot{\theta}_1 + \dot{\theta}_2\right) + m_2L_1L_2c_2\dot{\theta}_1\end{bmatrix}$$

$$-\begin{bmatrix}0\\ -m_2L_1L_2s_2\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\end{bmatrix} + \begin{bmatrix}m_1gL_1c_1 + m_2gL_1c_1 + m_2gL_2c_{12}\\ m_2gL_2c_{12}\end{bmatrix} = \tau$$

Example

- (Continued) $\partial/\partial t \left[\cdots \right]$

$$\begin{bmatrix} (m_1+m_2)L_1^2\ddot{\theta}_1 + m_2L_2^2(\ddot{\theta}_1+\ddot{\theta}_2) + 2m_2L_1L_2c_2\ddot{\theta}_1 - 2m_2L_1L_2s_2\dot{\theta}_1\dot{\theta}_2 + m_2L_1L_2c_2\ddot{\theta}_2 - m_2L_1L_2s_2\dot{\theta}_2^2 \\ m_2L_2^2(\ddot{\theta}_1+\ddot{\theta}_2) + m_2L_1L_2c_2\ddot{\theta}_1 - m_2L_1L_2s_2\dot{\theta}_1\dot{\theta}_2 \end{bmatrix}$$
$$- \begin{bmatrix} 0 \\ -m_2L_1L_2s_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2) \end{bmatrix} + \begin{bmatrix} m_1gL_1c_1 + m_2gL_1c_1 + m_2gL_2c_{12} \\ m_2gL_2c_{12} \end{bmatrix} = \tau$$

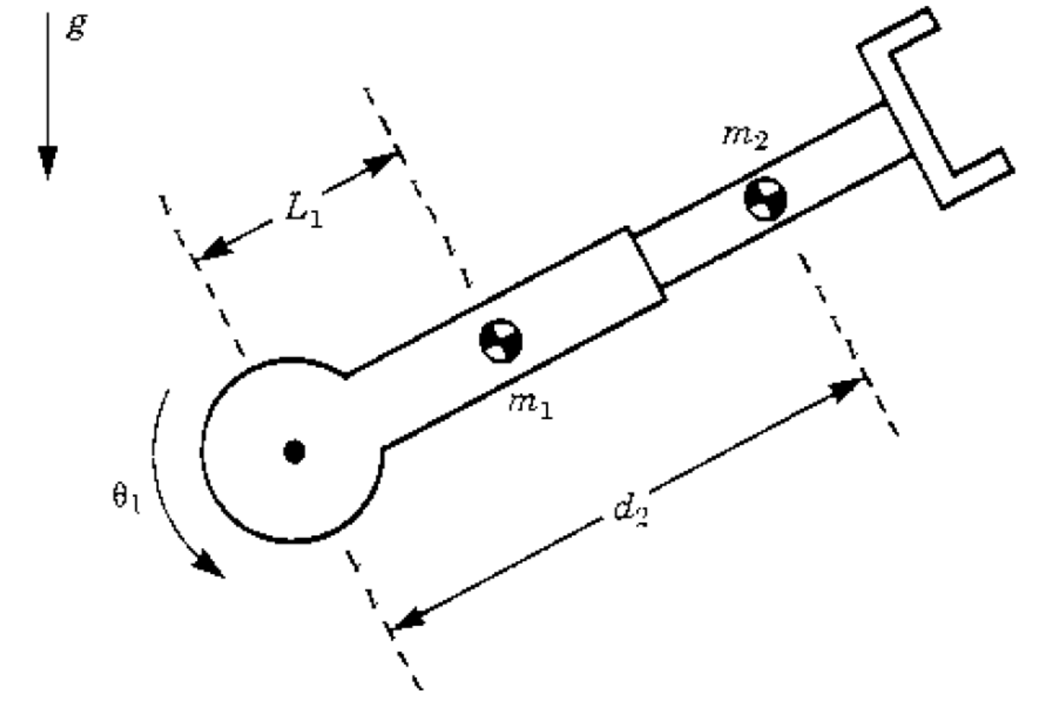
- This gives the following dynamic equation:

$$\tau_1 = m_2L_2^2(\ddot{\theta}_1+\ddot{\theta}_2) + m_2L_1L_2c_2(2\ddot{\theta}_1+\ddot{\theta}_2) + (m_1+m_2)L_1^2\ddot{\theta}_1 - m_2L_1L_2s_2\dot{\theta}_2^2$$
$$-2m_2L_1L_2s_2\dot{\theta}_1\dot{\theta}_2 + (m_1+m_2)gL_1c_1 + m_2gL_2c_{12}$$
$$\tau_2 = m_2L_1L_2c_2\ddot{\theta}_1 + m_2L_1L_2s_2\dot{\theta}_1^2 + m_2gL_2c_{12} + m_2L_2^2(\ddot{\theta}_1+\ddot{\theta}_2)$$

- which is exactly the same as the ones derived from Newton-Euler formulation.

Another Example

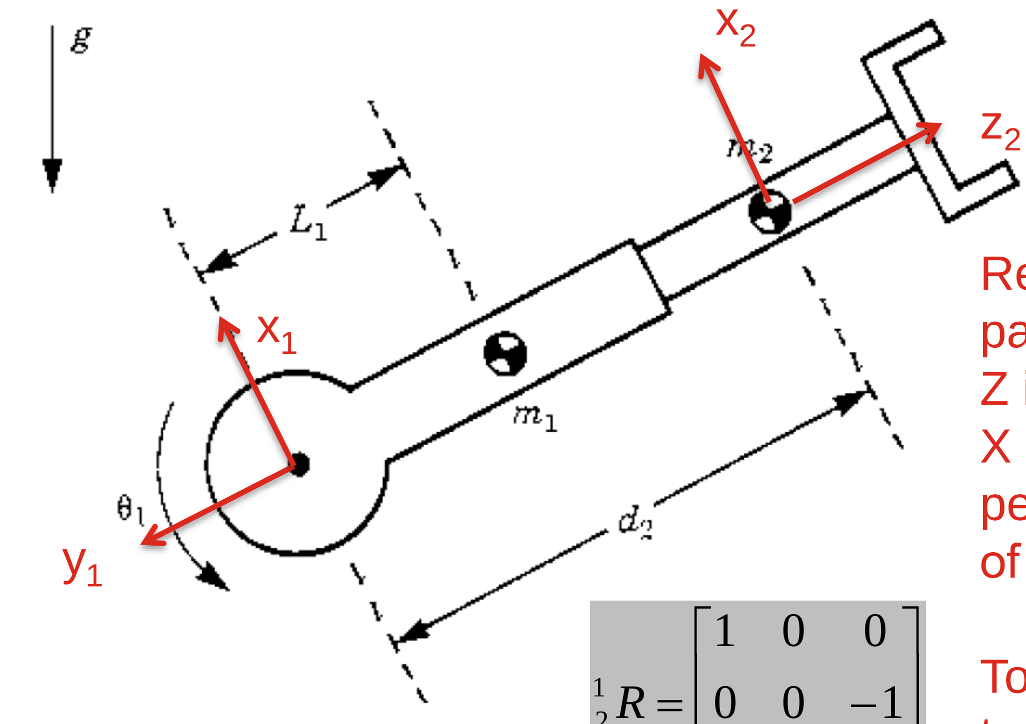
- Consider the following RP manipulator:



- Mass and dimensions are shown in the figure.

Another Example

- The frames are:



$$ {}^1_2R = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{Rotate 90deg about X}} $$

Remember DH parameters? Z is along axis, X is mutual perpendicular of two Z's.

To get from {1} to {2}, rotate 90deg along x1

Another Example

- Let its inertial tensors be:

$$^{C_1}I_1 = \begin{bmatrix} I_{xx_1} & 0 & 0 \\ 0 & I_{yy_1} & 0 \\ 0 & 0 & I_{zz_1} \end{bmatrix} \qquad ^{C_2}I_2 = \begin{bmatrix} I_{xx_2} & 0 & 0 \\ 0 & I_{yy_2} & 0 \\ 0 & 0 & I_{zz_2} \end{bmatrix}$$

- We calculate the velocities propagation as shown in Lecture 5:

$$^0\omega_0 = 0 \qquad ^0\upsilon_0 = 0$$

$$^1\omega_1 = {}^1_0R \cdot \underbrace{^0\omega_0}_{0} + \dot{\theta}_1 \cdot {}^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad ^1\upsilon_1 = {}^1_0R \cdot \left(\underbrace{^0\upsilon_0}_{0} + \underbrace{^0\omega_0}_{0} \times {}^0P_1 \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Another Example

- Assume we fix a frame {C$_1$} at center of mass of link 1. Its velocity propagated from frame {1} is then:

$$ {}^{C_1}\upsilon_{C_1} = \underbrace{{}^{C_1}_{1}R}_{I} \cdot \left({}^{1}\upsilon_1 + {}^{1}\omega_1 \times {}^{1}P_{C_1} \right) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix} $$

- For link 2, we have:

$$ {}^{2}\omega_2 = {}^{2}_{1}R \cdot {}^{1}\omega_1 = {}^{1}_{2}R^T \cdot {}^{1}\omega_1 $$
$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} $$

- Assume that the frame of link 2 is located at the center of mass of link 2. Thus:

$$ {}^{C_2}\upsilon_{C_2} = {}^{2}\upsilon_2 = {}^{2}_{1}R \cdot \left({}^{1}\upsilon_1 + {}^{1}\omega_1 \times {}^{1}P_{C_2} \right) + \dot{d}_2 \cdot {}^{2}\hat{Z}_2 $$
$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -d_2 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_2 \end{bmatrix} = \begin{bmatrix} d_2\dot{\theta}_1 \\ 0 \\ \dot{d}_2 \end{bmatrix} $$

Another Example

- The total kinetic energy is therefore:

$$k = \frac{1}{2} m_1 \upsilon_{c_1}^T \upsilon_{c_1} + \frac{1}{2} {}^1\omega_1^T \, {}^{C_1}I_1 \, {}^1\omega_1 + \frac{1}{2} m_2 \upsilon_{c_2}^T \upsilon_{c_2} + \frac{1}{2} {}^2\omega_2^T \, {}^{C_2}I_2 \, {}^2\omega_2$$

$$= \frac{1}{2} m_1 \begin{bmatrix} l_1\dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} l_1\dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}^T \begin{bmatrix} I_{xx_1} & 0 & 0 \\ 0 & I_{yy_1} & 0 \\ 0 & 0 & I_{zz_1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$$+ \frac{1}{2} m_2 \begin{bmatrix} d_2\dot{\theta}_1 \\ 0 \\ \dot{d}_2 \end{bmatrix}^T \begin{bmatrix} d_2\dot{\theta}_1 \\ 0 \\ \dot{d}_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}^T \begin{bmatrix} I_{xx_2} & 0 & 0 \\ 0 & I_{yy_2} & 0 \\ 0 & 0 & I_{zz_2} \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_{zz_1} \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2 \right) + \frac{1}{2} I_{yy_2} \dot{\theta}_1^2$$

Another Example

- As for the potential energy, we have:

$$u_1 = m_1 l_1 g \sin(\theta_1) + m_1 l_1 g$$

$$u_2 = m_2 d_2 g \sin(\theta_1) + m_2 d_{2\max} g$$

  - where d_{2max} is the maximum extension of joint 2.
- The total potential energy is thus:

$$u = m_1 l_1 g \sin(\theta_1) + m_1 l_1 g + m_2 d_2 g \sin(\theta_1) + m_2 d_{2\max} g$$

Another Example

- The kinetic energy and potential energy are repeated here:

$$k = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_{zz_1} \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2 \right) + \frac{1}{2} I_{yy_2} \dot{\theta}_1^2$$

$$u = m_1 l_1 g \sin(\theta_1) + m_1 l_1 g + m_2 d_2 g \sin(\theta_1) + m_2 d_{2\max} g$$

- Applying the Lagrangian formula:

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{q}} - \frac{\partial k}{\partial q} + \frac{\partial u}{\partial q} = \tau$$

$$\frac{d}{dt}\begin{bmatrix} \frac{\partial k}{\partial \dot{q}_1} \\ \frac{\partial k}{\partial \dot{q}_2} \end{bmatrix} - \begin{bmatrix} \frac{\partial k}{\partial q_1} \\ \frac{\partial k}{\partial q_2} \end{bmatrix} + \begin{bmatrix} \frac{\partial u}{\partial q_1} \\ \frac{\partial u}{\partial q_2} \end{bmatrix} = \tau$$

- with $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ d_2 \end{bmatrix}$

- gives (next page)

Another Example

$$\frac{d}{dt}\begin{bmatrix} m_1 l_1^2\dot{\theta}_1 + I_{zz_1}\dot{\theta}_1 + m_2 d_2^2\dot{\theta}_1 + I_{yy_2}\dot{\theta}_1 \\ m_2\dot{d}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ m_2 d_2\dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} m_1 l_1 g c_1 + m_2 d_2 g c_1 \\ m_2 g s_1 \end{bmatrix} = \tau$$

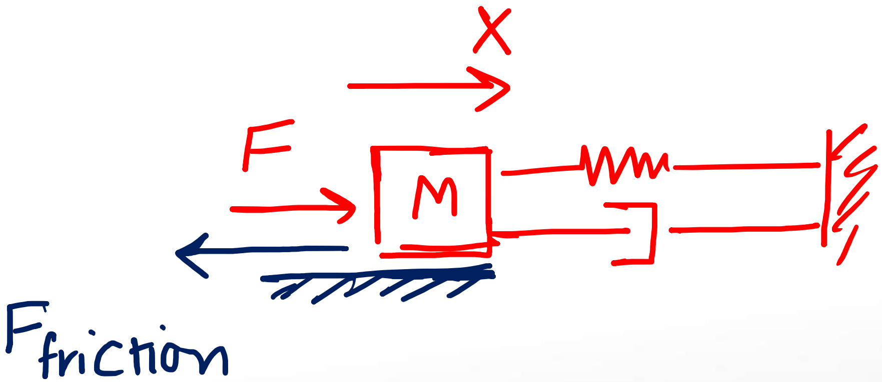
$$\begin{bmatrix} m_1 l_1^2\ddot{\theta}_1 + I_{zz_1}\ddot{\theta}_1 + m_2 d_2^2\ddot{\theta}_1 + 2m_2 d_2\dot{d}_2\dot{\theta}_1 + I_{yy_2}\ddot{\theta}_1 \\ m_2\ddot{d}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ m_2 d_2\dot{\theta}_1^2 \end{bmatrix} + \begin{bmatrix} m_1 l_1 g c_1 + m_2 d_2 g c_1 \\ m_2 g s_1 \end{bmatrix} = \tau$$

- This gives the structure:

$$\underbrace{\begin{bmatrix} m_1 l_1^2 + I_{zz_1} + m_2 d_2^2 + I_{yy_2} & 0 \\ 0 & m_2 \end{bmatrix}}_{M(q)}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2m_2 d_2\dot{d}_2\dot{\theta}_1 \\ -m_2 d_2\dot{\theta}_1^2 \end{bmatrix}}_{V(q,\dot{q})} + \underbrace{\begin{bmatrix} m_1 l_1 g c_1 + m_2 d_2 g c_1 \\ m_2 g s_1 \end{bmatrix}}_{G(q)} = \tau$$

Content

- Lagrangian Formulation
- Inclusion of Non-Rigid Body Effects
- Explicit Form



$$m\ddot{x} + b\dot{x} + Kx = F - F_{friction}$$

Friction

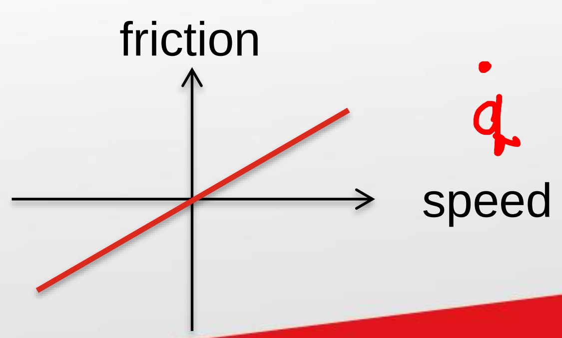
- All mechanisms are affected by friction.
- The manipulator's joint motors need to provide torque to overcome the friction, in addition to all other forces we have seen just now.
- The effect of friction to the manipulator's dynamic can be included in the dynamic equation:

$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau - \tau_{friction}$$

- But how do we model frictional forces?
- One simple model is the viscous friction:

$$\tau_{friction} = k\dot{q}$$

  - k is the viscous-friction constant.



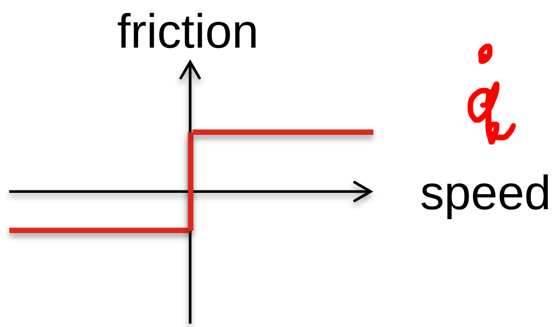
Friction

$\operatorname{sgn}(\dot{q}) \begin{cases} +1 & \text{if } \dot{q} > 0 \\ -1 & \text{if } \dot{q} < 0 \end{cases}$

- Another simple model is the Coulomb-friction: μN

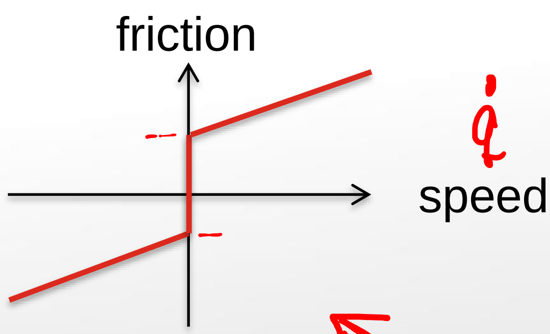
$$\tau_{friction} = c\,\mathrm{sgn}(\dot{q})$$

  - where c is the Coulomb-friction constant.

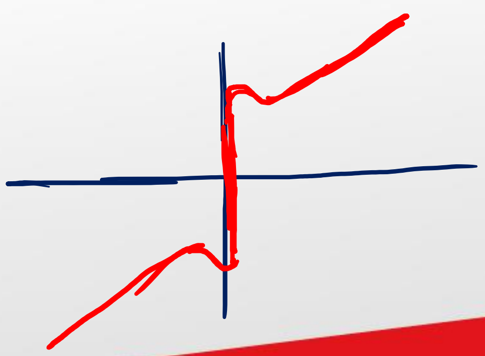


- A better model would be combination of both viscous and Coulomb friction:

$$\tau_{friction} = c\,\mathrm{sgn}(\dot{q}) + k\dot{q}$$

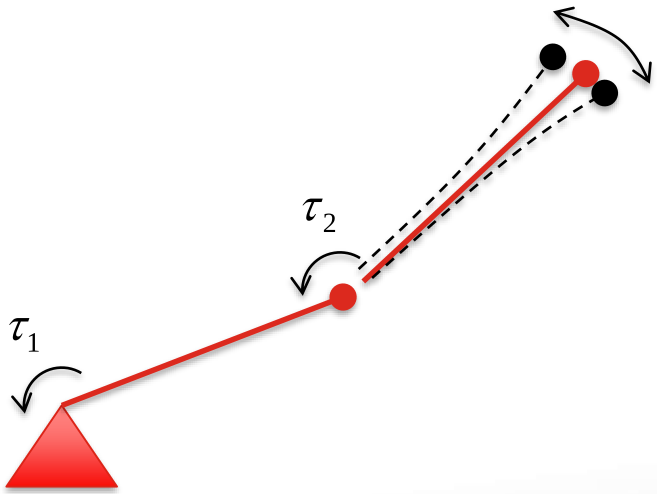


- There are even more accurate models, for e.g. including Stribeck effect or joint-position-dependent friction.



Resonance Modes

- There are also bending effects and resonance in actual robots.



- However, these are very difficult to model and thus will be ignored in this course.