Week 3 – Manipulator Kinematics: Forward & Inverse

Advanced Robotic Systems – MANU2453

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Lectures

Wk	Date	Lecture (NOTE: video recording)	Maths Difficulty	Hands-on Activity	Related Assessment
1	24/7	• Introduction to the Course • Spatial Descriptions & Transformations			
2	31/7	• Spatial Descriptions & Transformations • Robot Cell Design			Robot Cell Design Assignment
3	7/8	• Forward Kinematics • Inverse Kinematics			
4	14/8	• ABB Robot Programming via Teaching Pendant • ABB RobotStudio Offline Programming		ABB RobotStudio Offline Programming	Offline Programming Assignment
5	21/8	• Jacobians: Velocities and Static Forces			
6	28/8	• Manipulator Dynamics			
7	11/9	• Manipulator Dynamics		MATLAB Simulink Simulation	
8	18/9	• Robotic Vision		MATLAB Simulation	Robotic Vision Assignment
9	25/9	• Robotic Vision		MATLAB Simulation	
10	2/10	• Trajectory Generation			
11	9/10	• Linear & Nonlinear Control		MATLAB Simulink Simulation	
12	16/10	• Introduction to I4.0 • Revision			Final Exam

Content

- Forward Kinematics
  - Introduction
  - Denavit-Hartenberg Parameters
  - Introduction to DH Parameters, Link Frame Attachment, Forward Kinematics through Example 1
  - More Examples
- Inverse Kinematics
  - Introduction
  - Algebraic Approach
  - Geometric Approach

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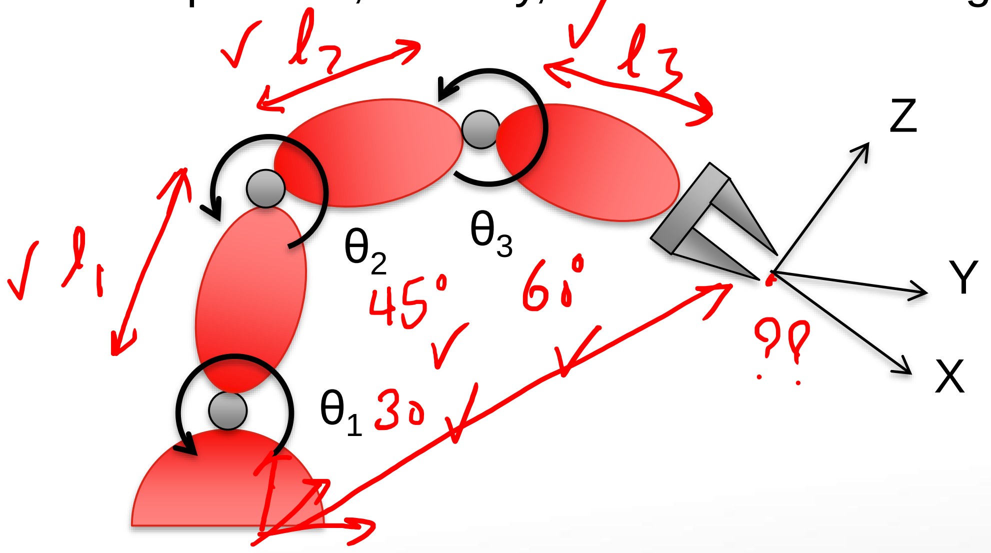
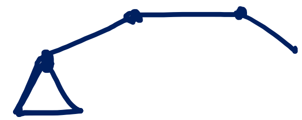
Introduction

- Kinematics is the study of motion without regard to forces which causes it.
- Of interests are position, velocity, acceleration and higher order derivatives.

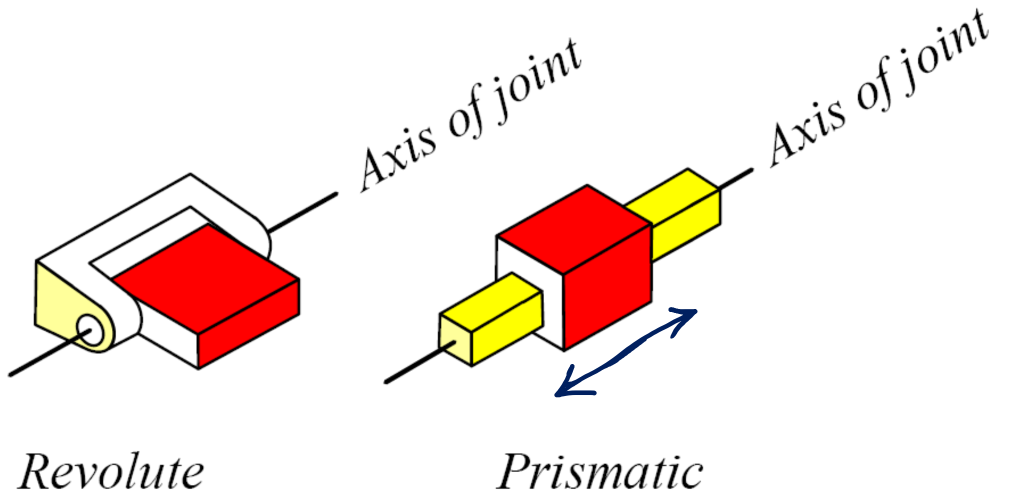


- In this lecture, position and orientation in static situations will be discussed:
  - Given the joint space parameters (angles for revolute joints, or offsets for prismatic joints), as well as the lengths of the links, what is the position and orientation of the end-effector in Cartesian space?

Introduction

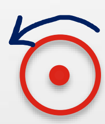


- Industrial robots are mostly made by arms / links connected by joints.
  - The joints could be revolute (R) or prismatic (P).

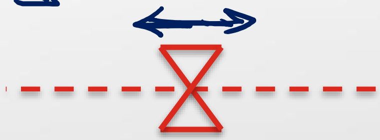


- Schematics:

R (axis in plane)



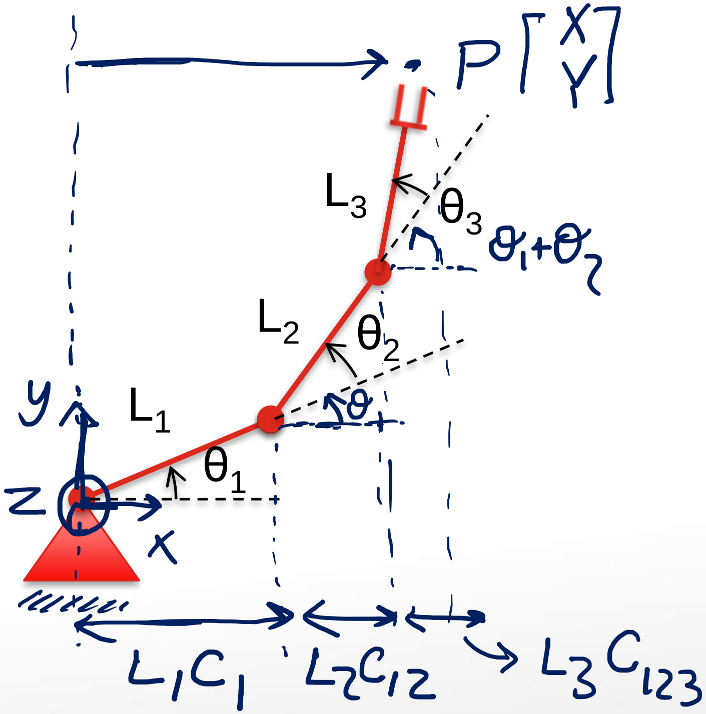
R (axis out of plane)



P (axis in plane)

A Simple Example

- Consider a 3-link planar robot as shown:



$$Y = L_1 S_1 + L_2 S_{12} + L_3 S_{123}$$

$$C_1 = \cos\theta_1$$

$$C_{12} = \cos(\theta_1+\theta_2)$$

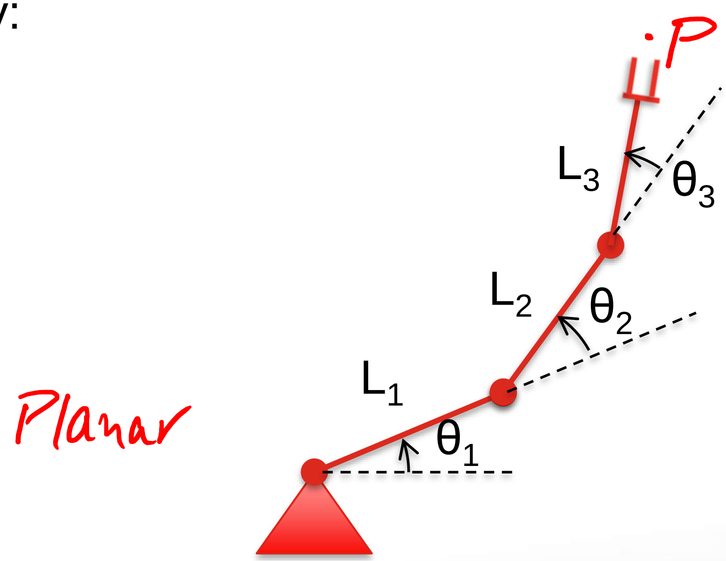
$$C_{123} = \cos(\theta_1+\theta_2+\theta_3)$$

$$X = L_1C_1 + L_2C_{12} + L_3C_{123}$$

- The link lengths and the joint angles are known.
- What is the position and orientation of the end-effector?

A Simple Example

- In this simple example, the position can be calculated based on trigonometry:



$$
\begin{cases}
x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3) \\
y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3)
\end{cases}
$$

- The orientation is also straightforward: $\theta_{total} = \theta_1 + \theta_2 + \theta_3$

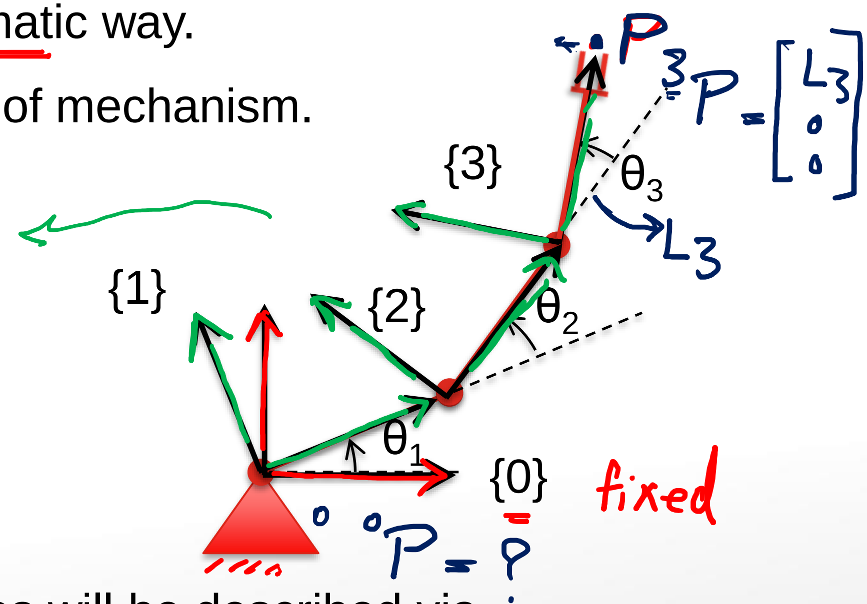
For More General Cases

- For more general cases (e.g. more links, prismatic joints, non-planar robots), it is important to do this in a more systematic way.
- Frames will be attached to various parts of mechanism.

Compound Transf.

→ known ${}^0_1T,\ {}^1_2T,\ {}^2_3T$

→ calculate ${}^0_3T = {}^0_1T \cdot {}^1_2T \cdot {}^2_3T$



- Finally, relationship between these frames will be described via Homogeneous Transform.
  - ${}^0_1T, {}^1_2T, {}^2_3T$ known individually.
  - Then calculate ${}^0_3T = {}^0_1T \cdot {}^1_2T \cdot {}^2_3T$ to get the position of frame {3}.

→ ${}^3P = [L_3 \cdot 0 \quad 0]^T$ → ${}^0P = {}^0_3T \cdot {}^3P$

For More General Cases

- For more general cases (e.g. more links, prismatic joints, non-planar robots), it is important to do this in a more systematic way.
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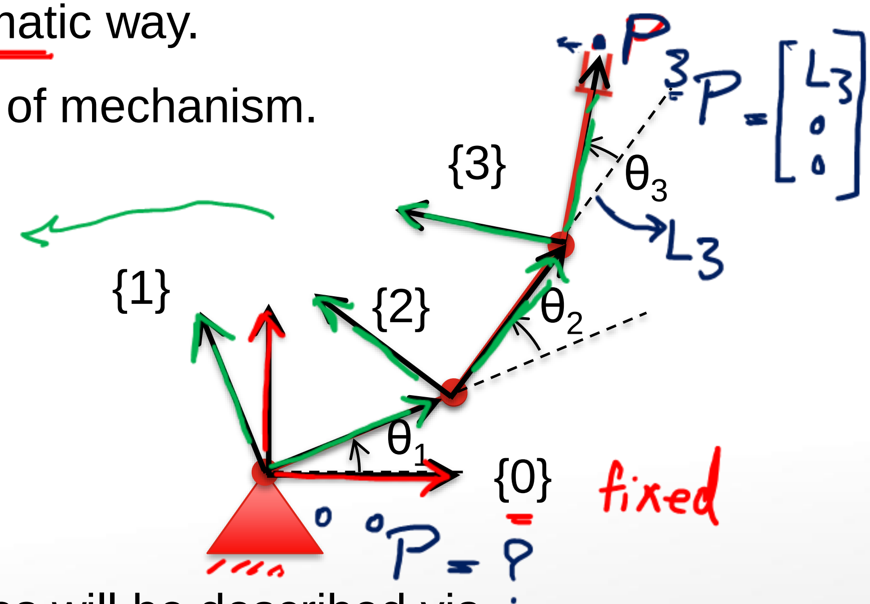

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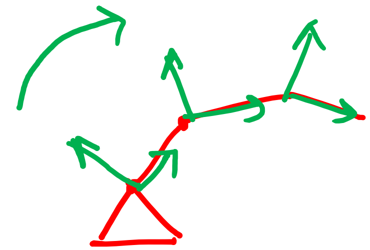
Content



(DH) Parameters

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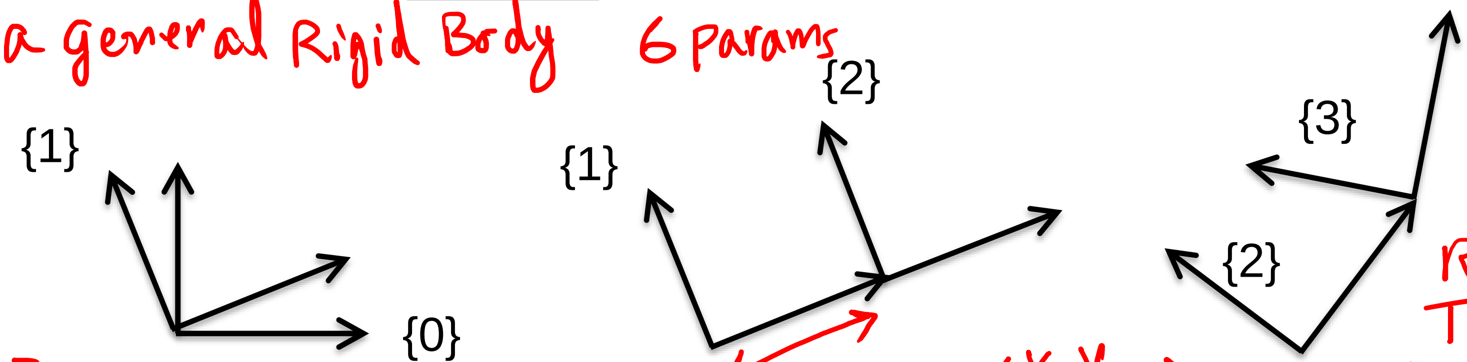
Denavit-Hartenberg Parameters

- We mentioned that ${}^0_1T, {}^1_2T, {}^2_3T$ can be obtained individually.

a general Rigid Body 6 params

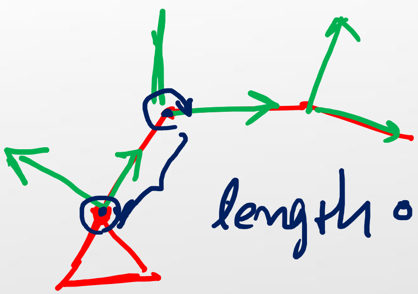


3 Params (α, β, γ) 3 Parms (X, Y, Z) Rot Trans. 6 Params

- Question: How many parameters are needed to describe each transformation? In general transformations, six: three for position and three for orientation.

we don't need 6 Params

- However, in robots, certain values are fixed.
  - E.g. the distance between origins of {1} and {2} – constrained by the link length.
  - Also, each joint is constrained to only 1 DOF.



length of links

Denavit-Hartenberg Parameters

- Therefore, for robotic manipulators, we only need 4 parameters to describe the transformation between each links.
- They are:
  - Link lengths $a_{i\text{-}1}$
  - Link twists $\alpha_{i\text{-}1}$
  - Joint angles θ_i
  - Link offsets d_i
- The above four parameters are called Denavit-Hartenberg Parameters, or in short "DH Parameters".
- We will learn about the DH Parameters, how to attach the frames, and how to get the Cartesian description of the last link, through examples in the next sections.

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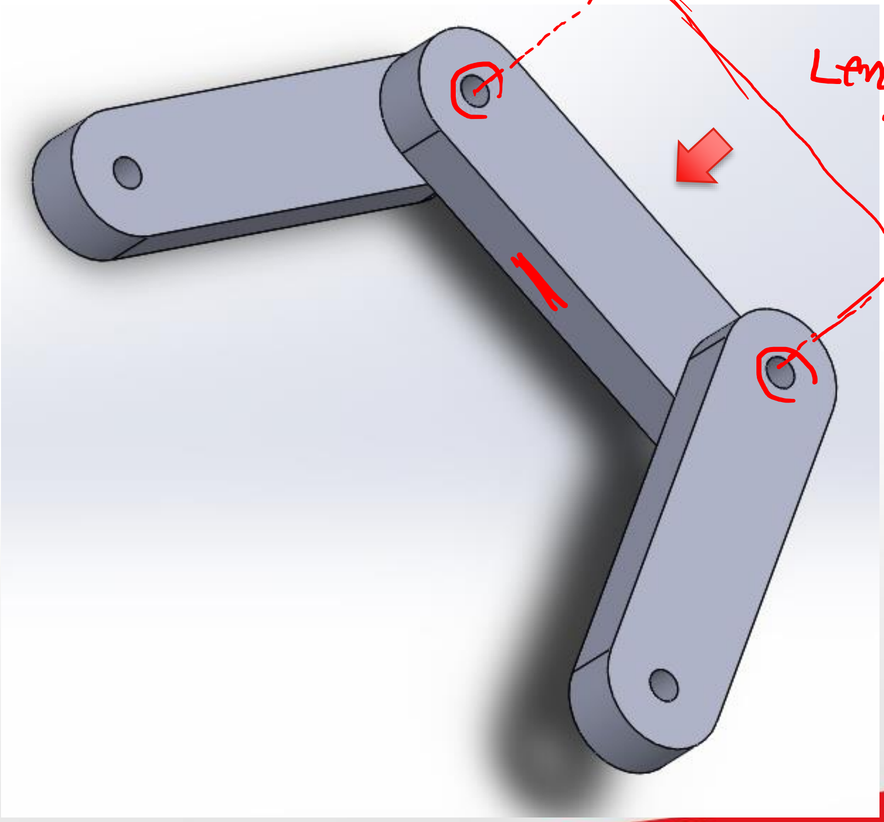
Link Length

- A link is basically just a rigid body between two joint axes.
- It turns out that a link can be fully described by its length and its twist.
- What is its length?

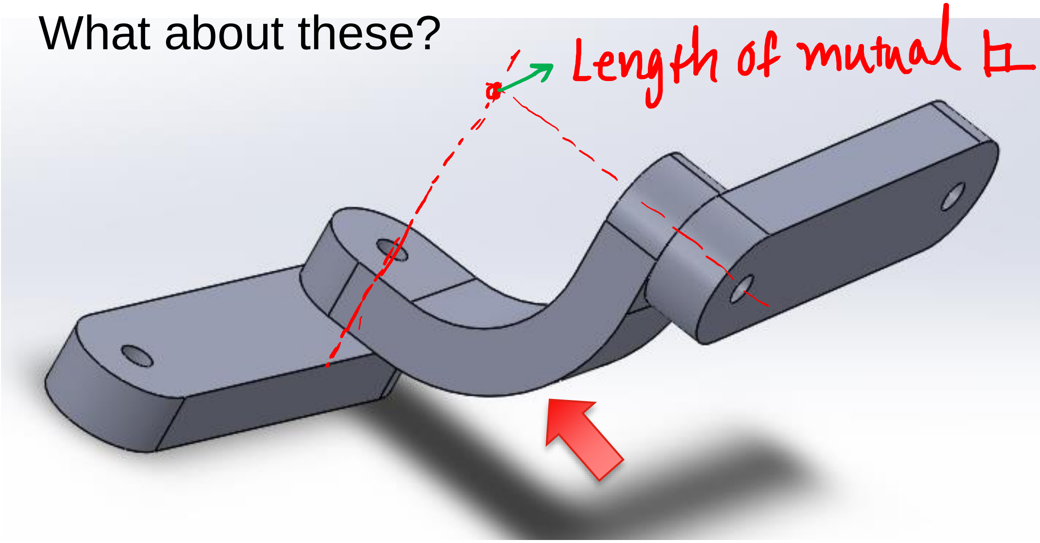
Length of mutual ⊥

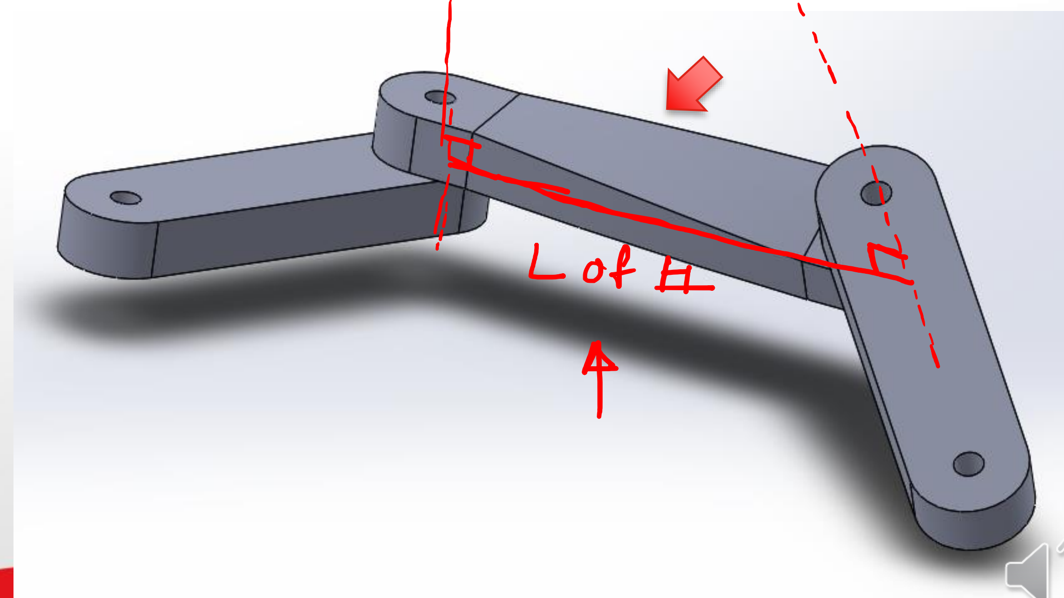
L



Link Length

- What about these?

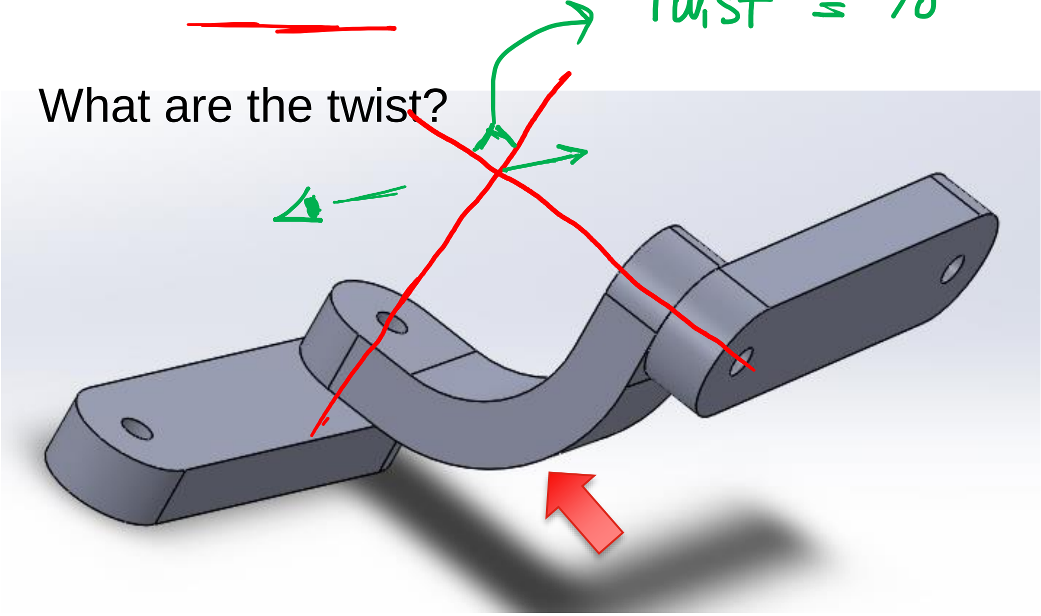




Link Twist

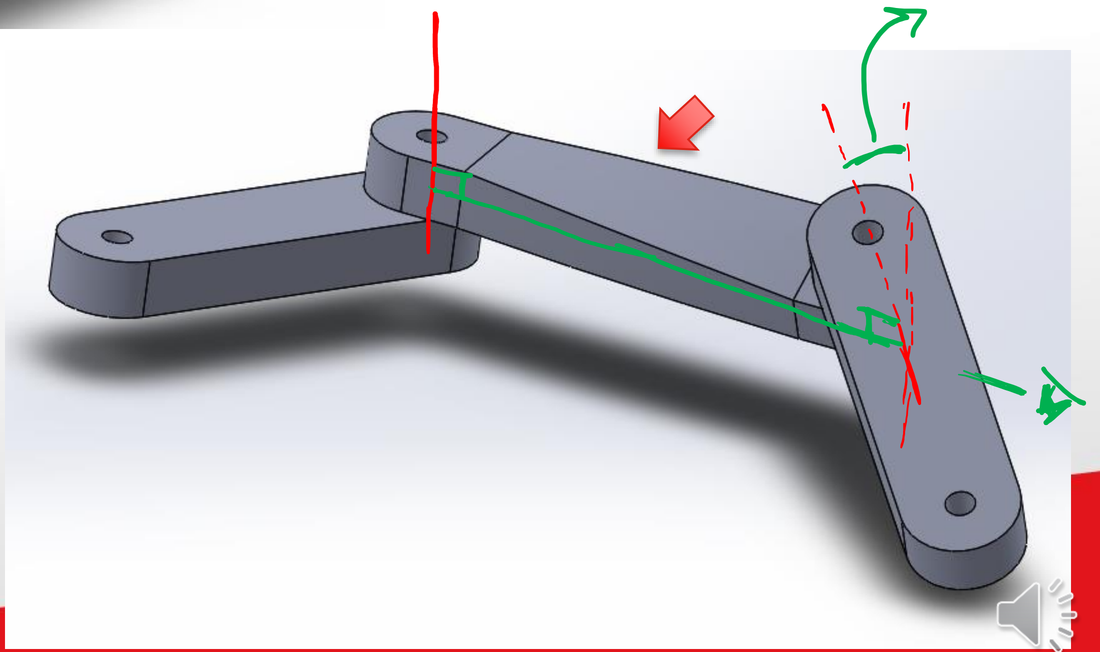
twist = 90°

- What are the twist?



30°

twist



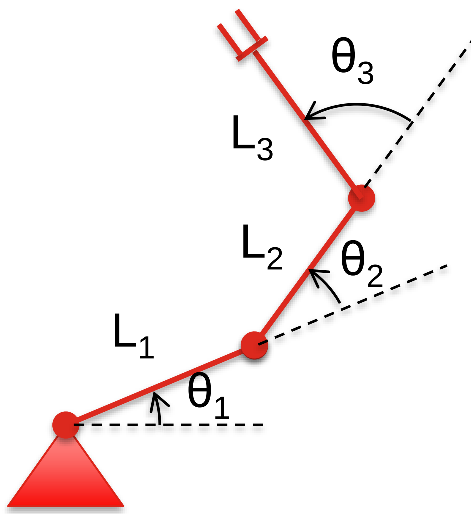
Example 1

FK { Compound Transf
DH Convention

FK →

- 3-link RRR planar robot:

→



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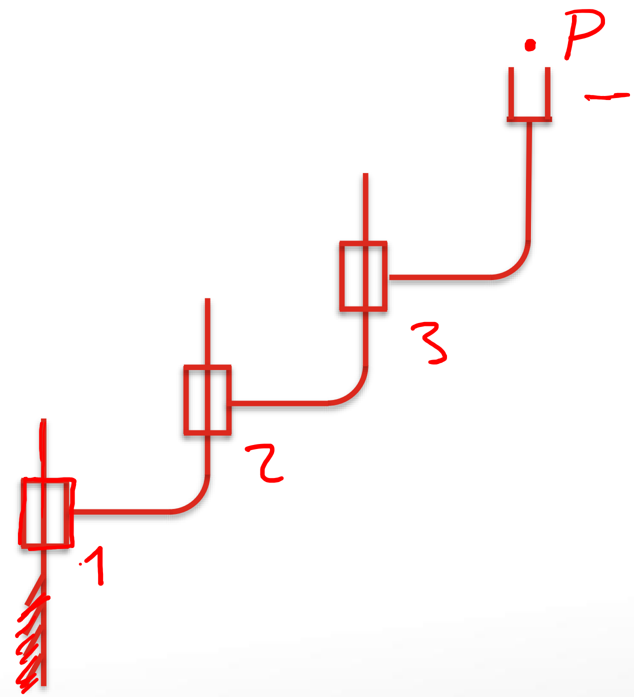


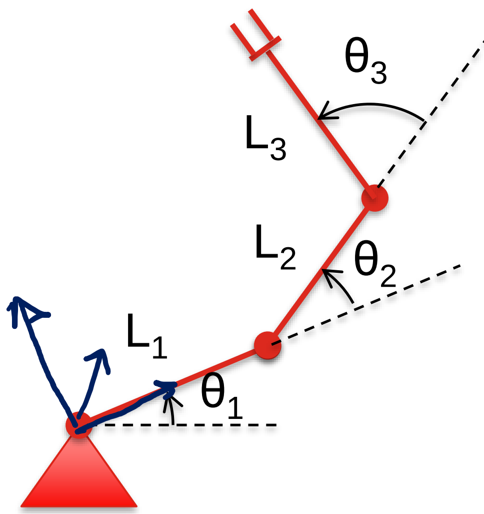
←

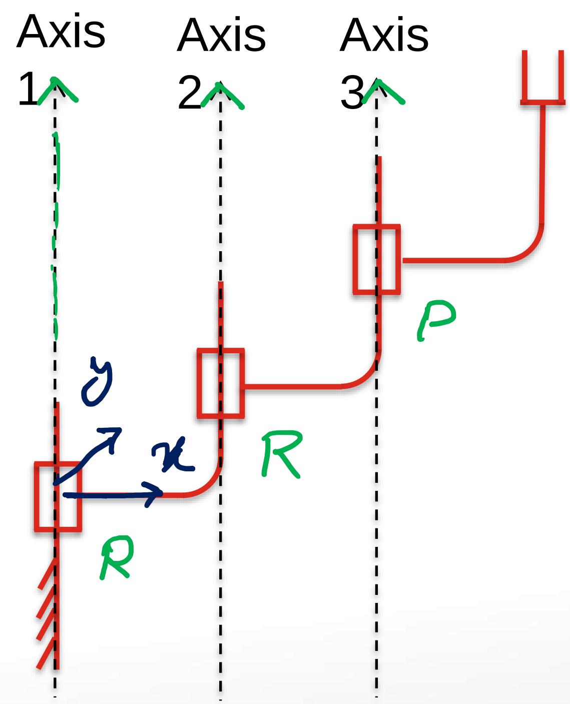
→ step by step → FK

Example 1

You can use any direction for the arrows/axis

- 3-link RRR planar robot:



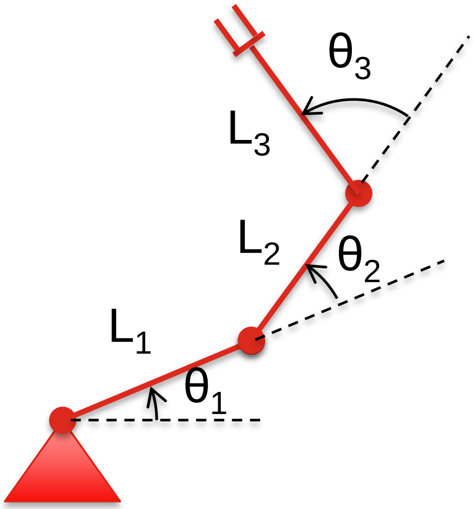


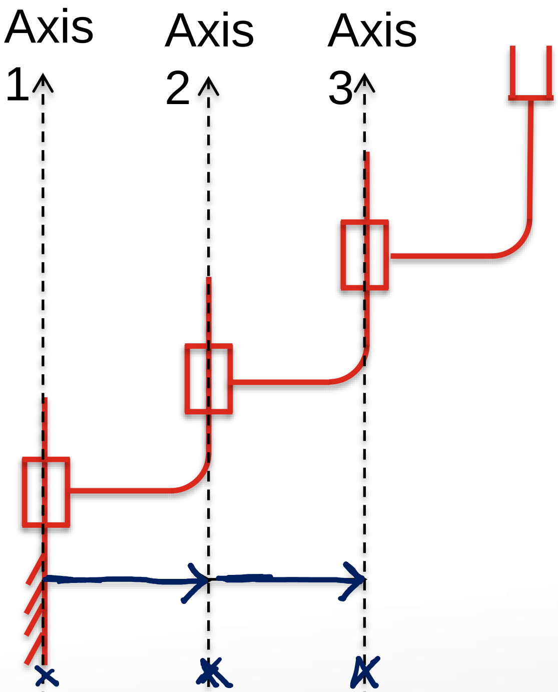
However you must use the direction & follow through the whole calculation

- Step 1, draw the axes.
    - For rotary joint: About the rotation
    - For prismatic joint: Along the translation

Example 1

- 3-link RRR planar robot:



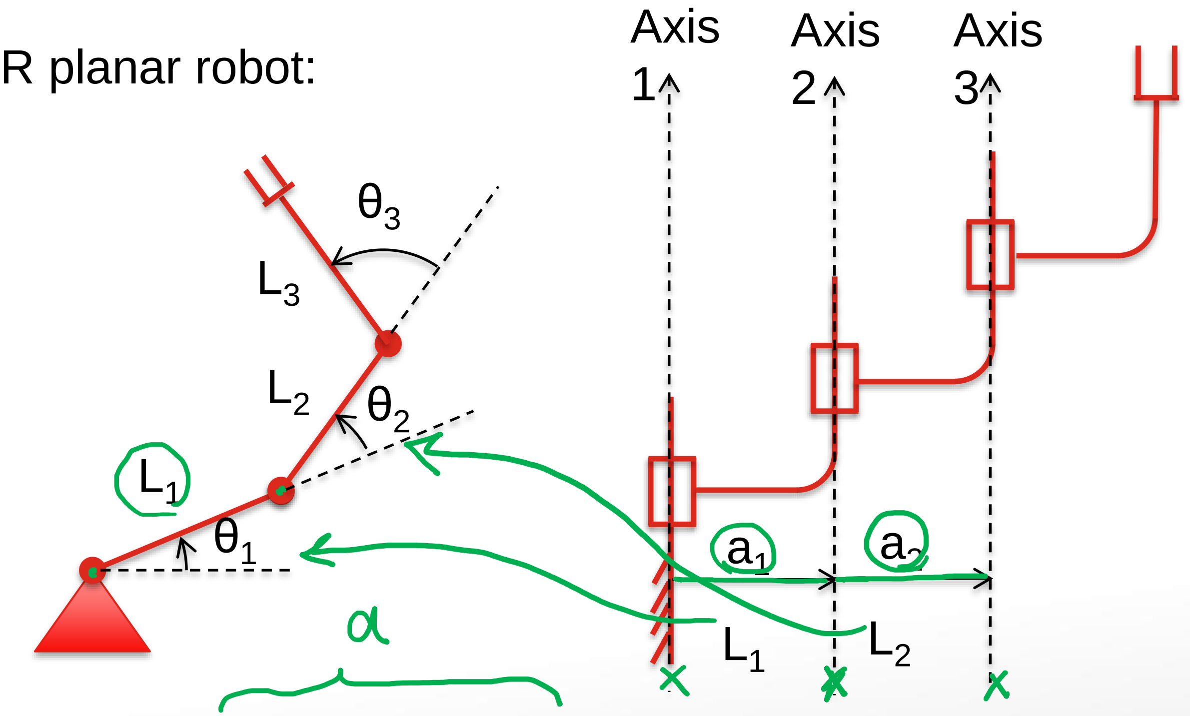


- Step 2, we draw the mutually perpendicular lines between axes.
  - Draw from lower axis to higher axis, e.g. 1 to 2, 2 to 3…
  - Because the axes are parallel, the mutual perpendicular can be placed in an arbitrary position.
  - We will put them in the same plane.

Example 1

- 3-link RRR planar robot:

n=3



- Step 3, put in the link lengths a_{i-1}.
- Definition: a_{i-1} = length of mutual perpendicular, from axis i-1 to axis i.

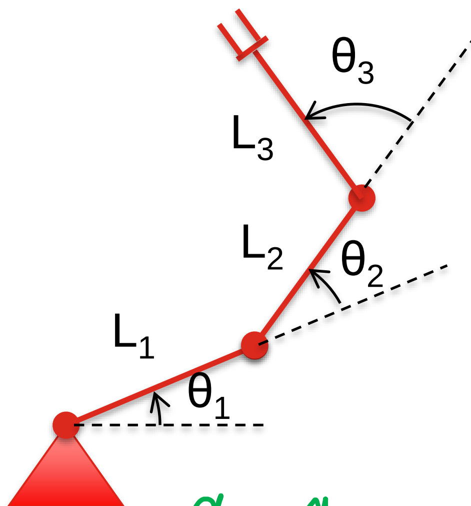
Do 1 to n-1

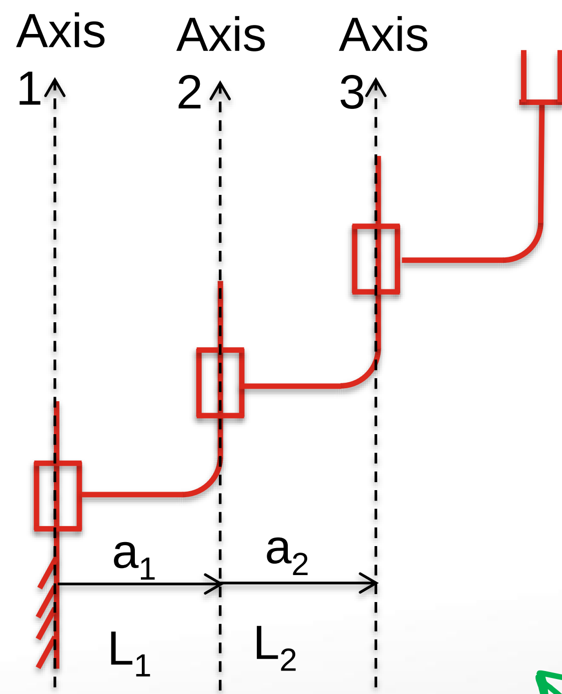
a_1 to a_2

- a_1 = length of mutual perpendicular from axis 1 to 2 = L_1.
- a_2 = length of mutual perpendicular from axis 2 to 3 = L_2.

Example 1

- 3-link RRR planar robot:





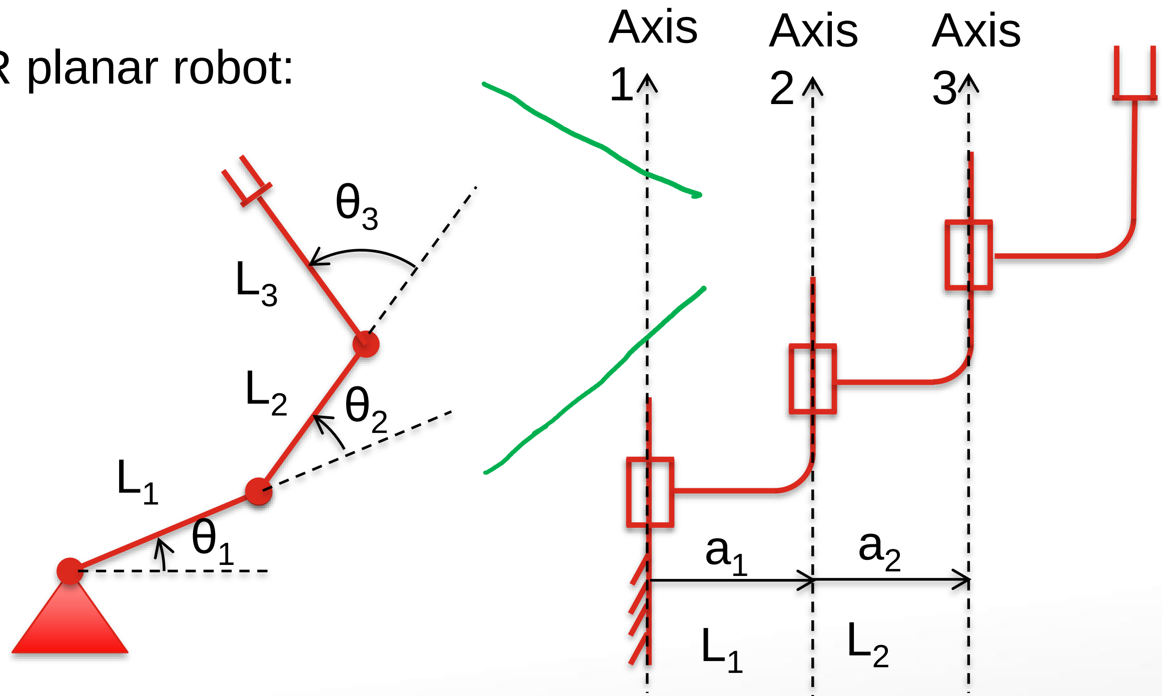
a_1, a_2

- What about a_0? a_{i-1} $(i=1) \rightarrow a_0$
  - a_0 = length of mutual perpendicular from axis 0 to 1. However, axis 0 is not known yet.

Example 1

Axis 0 intersecting with

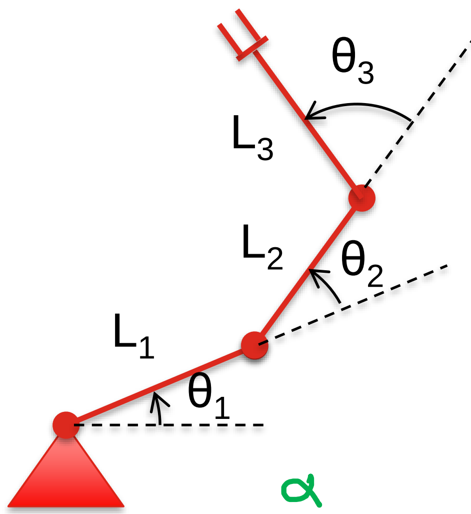
- 3-link RRR planar robot:



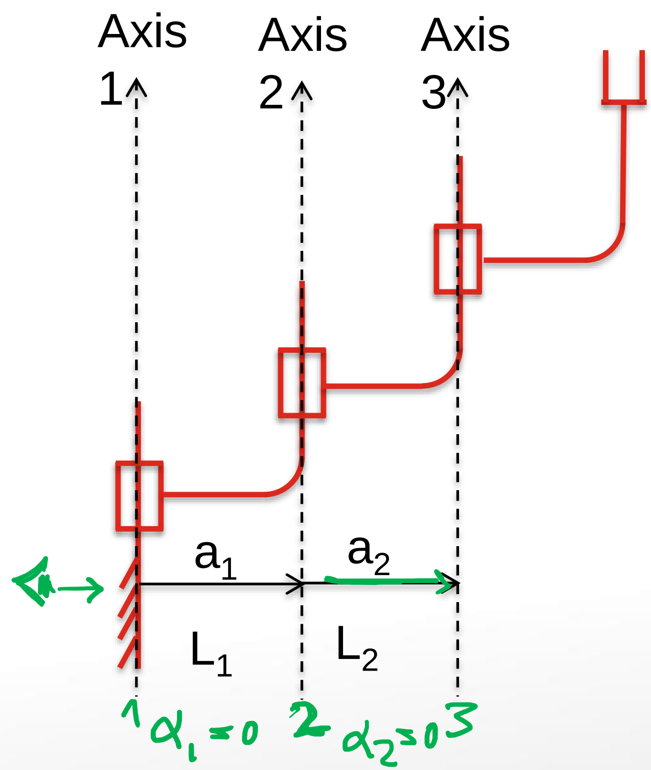
- What about a_0?
  - a_0 = length of mutual perpendicular from axis 0 to 1. However, axis 0 is not known yet.
  - $\boxed{\text{By convention, } a_0 = 0.}$ $a_0 = 0$ → mean
  - This means: Axis 0 and Axis 1 intersect with each other.

Example 1

- 3-link RRR planar robot:



Axis 0 intersecting with



- Step 4, put in link twists α_{i-1}.
- Definition: α_{i-1} = angle between axis i-1 and axis i, in the right hand sense about a_{i-1}

n=3

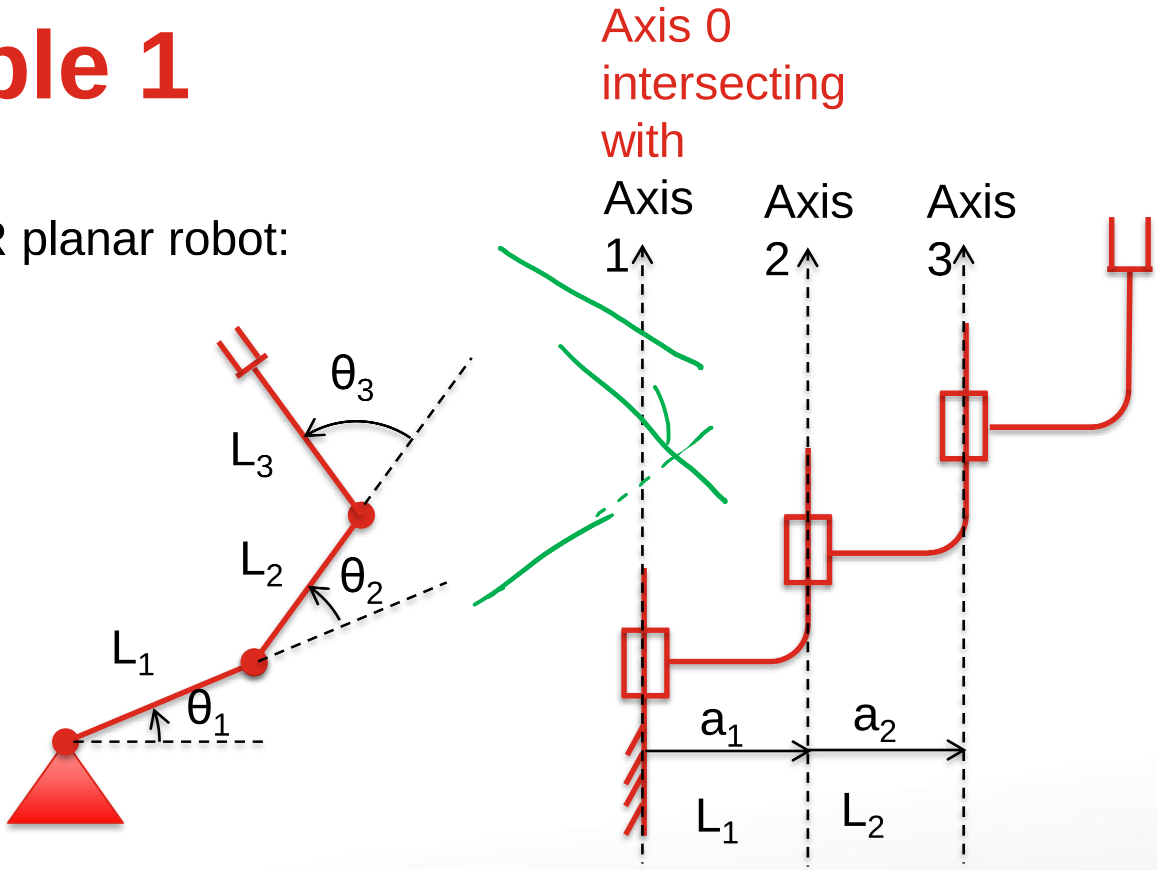
Do 1 to n-1

α_1, α_2

- α_1 = angle between axis 1 and axis 2, about a_1 = 0deg.
- α_2 = angle between axis 2 and axis 3, about a_2 = 0deg.

Example 1

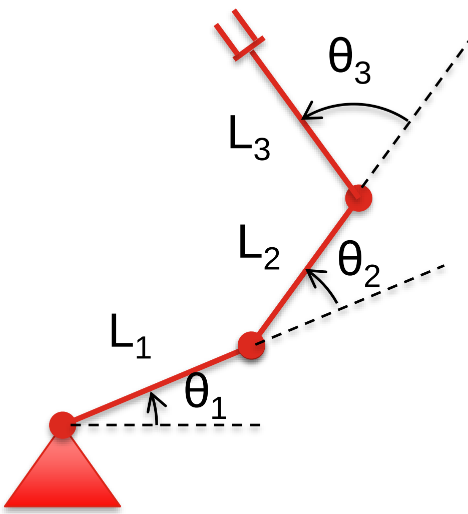
- 3-link RRR planar robot:

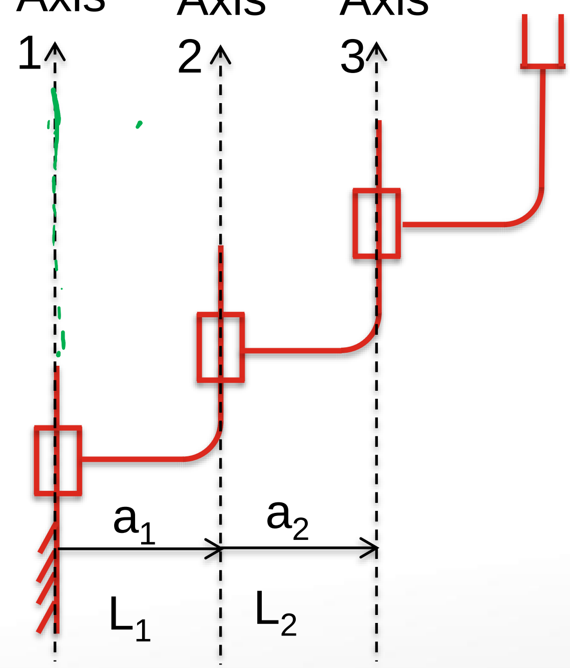


- What about α_0?
  - α_0 = angle between axis 0 and axis 1, about a_0 . However, axis 0 is not fully known yet.

Example 1

- 3-link RRR planar robot:

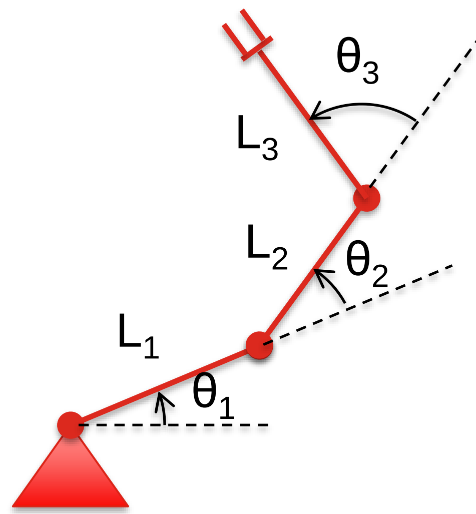


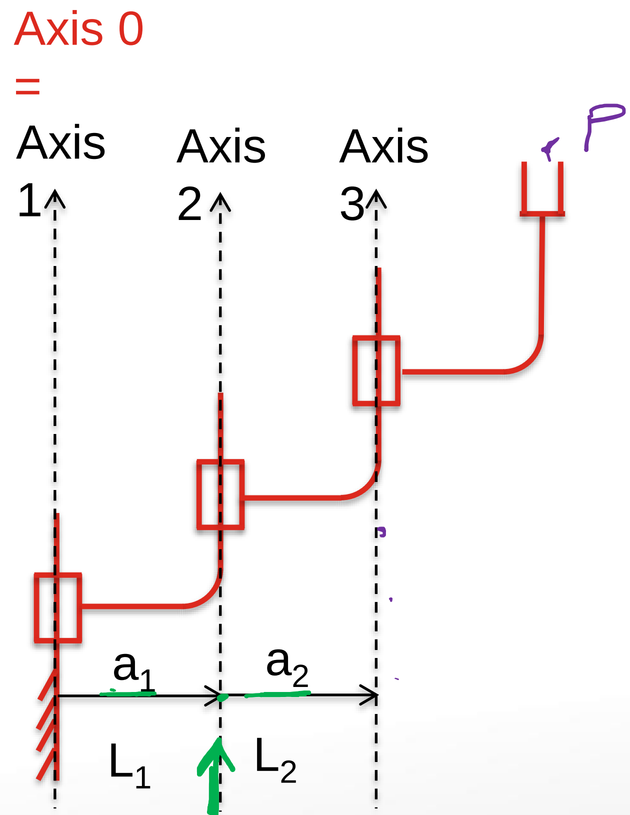


- What about α_0?
    - α_0 = angle between axis 0 and axis 1, about a_0 . However, axis 0 is not fully known yet.
    - $\boxed{\text{By convention, } \alpha_0 = 0.}$ $\alpha_0 = 0$
    - This means: Axis 0 and Axis 1 are the same.

Example 1

- 3-link RRR planar robot:





$d_2 = 0$

- Step 5, write down the link offsets d_i.

x

- $\boxed{\text{Definition: } d_i = \text{distance from } a_{i-1} \text{ to } a_i\text{, along axis i.}}$
  - d_2 = distance from a1 to a2, along axis 2 = 0.

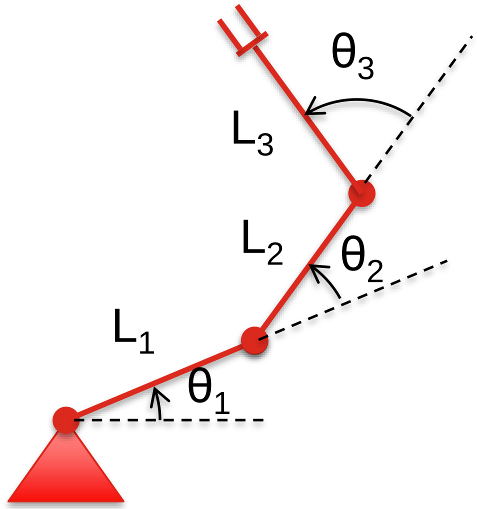
Do 2 to n-1

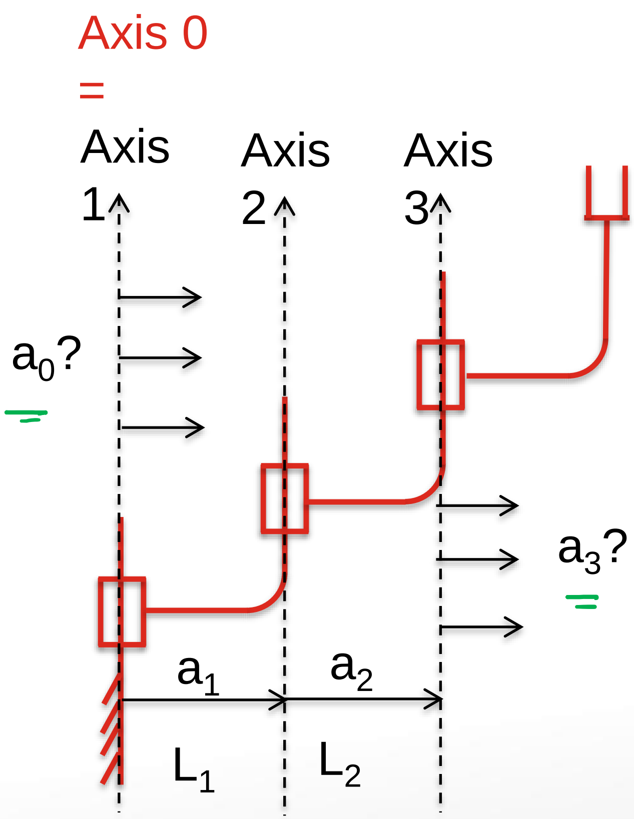
d_2

n-3

Example 1

- 3-link RRR planar robot:

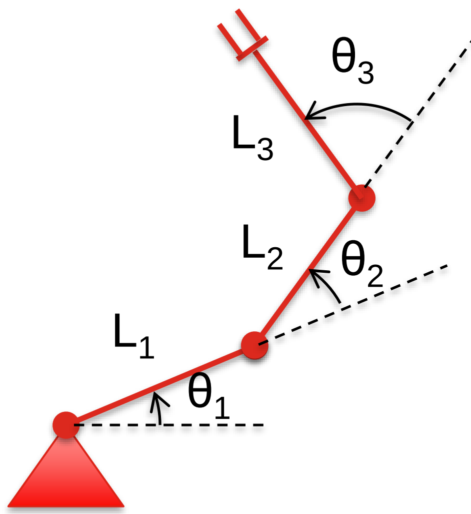


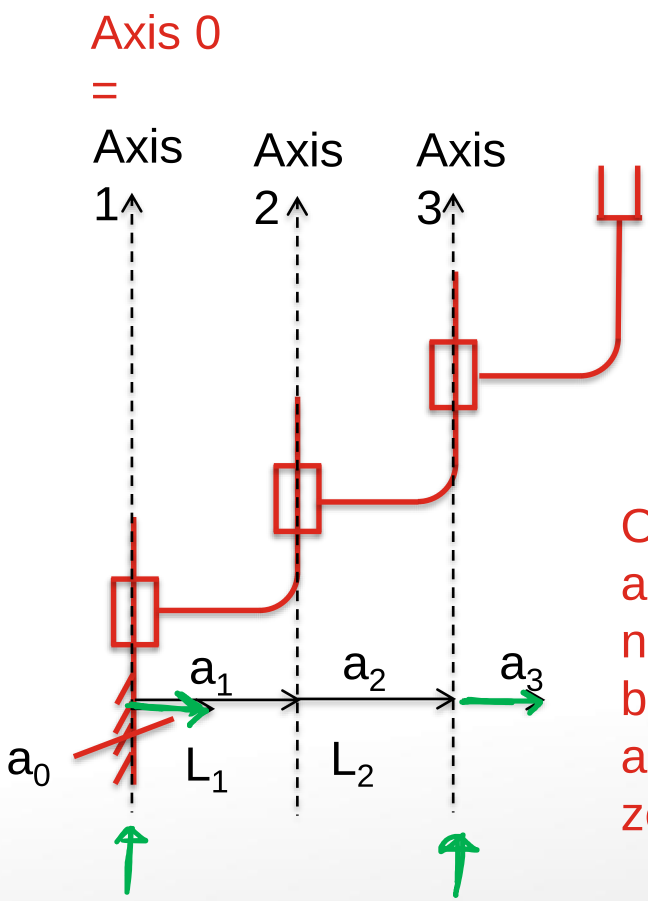


- What about d_1 and d_3?
  - d_1 = distance from a_0 to a_1, along axis 1.
  - d_3 = distance from a_2 to a_3, along axis 3.
  - But where exactly are a_0 and a_3?

Example 1

- 3-link RRR planar robot:



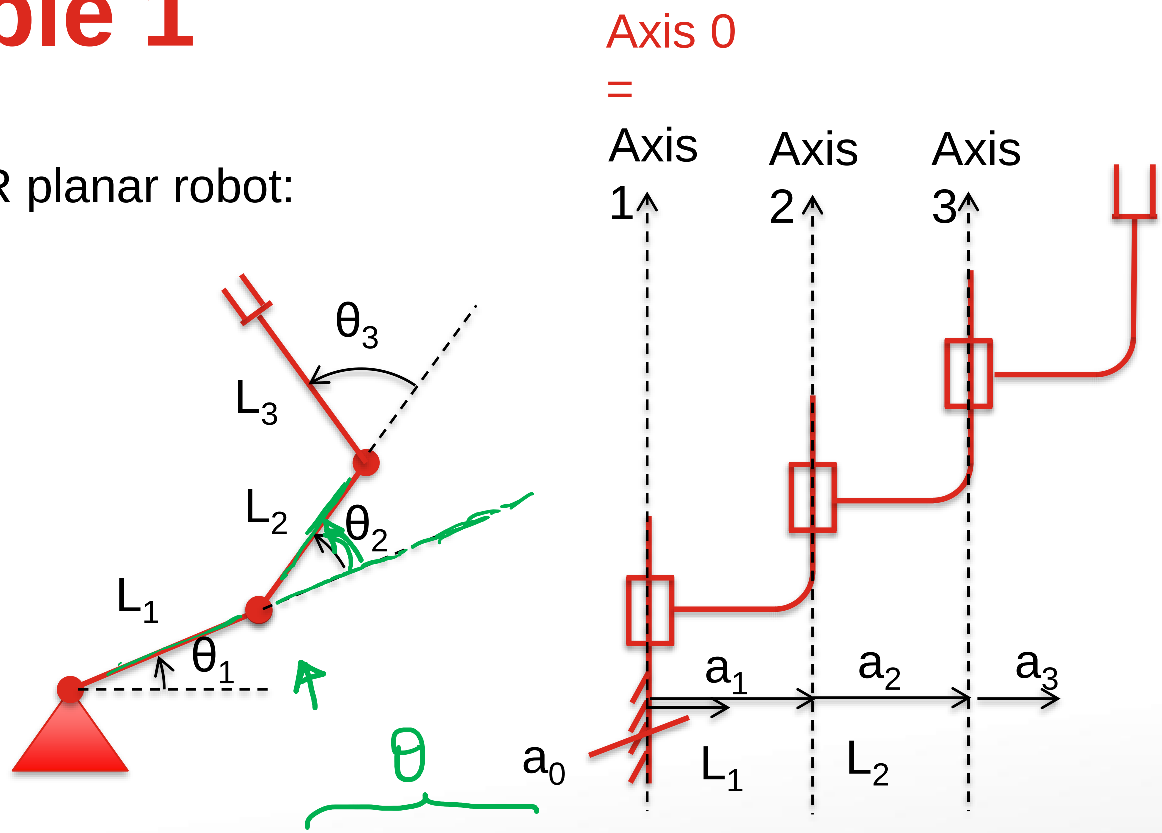


Origins of a_0 and a_3 are now known, because d_1 and d_3 are zero.

- What about d_1 and d_3?
  - d_1 = distance from a_0 to a_1, along axis 1.
  - d_3 = distance from a_2 to a_3, along axis 3.
  - By convention: Zero for revolute joint, variable for prismatic joint
  - So in this case, d_1 and d_3 are both zero.

Example 1

- 3-link RRR planar robot:



- Step 6, write down the joint angle θ_i.
  - Definition: θ_i is the angle between the (extension of a_{i-1}) and a_i, measured about the axis i.

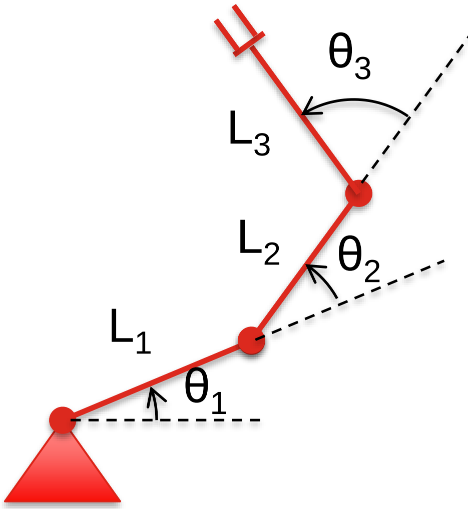
Do 2 to n-1

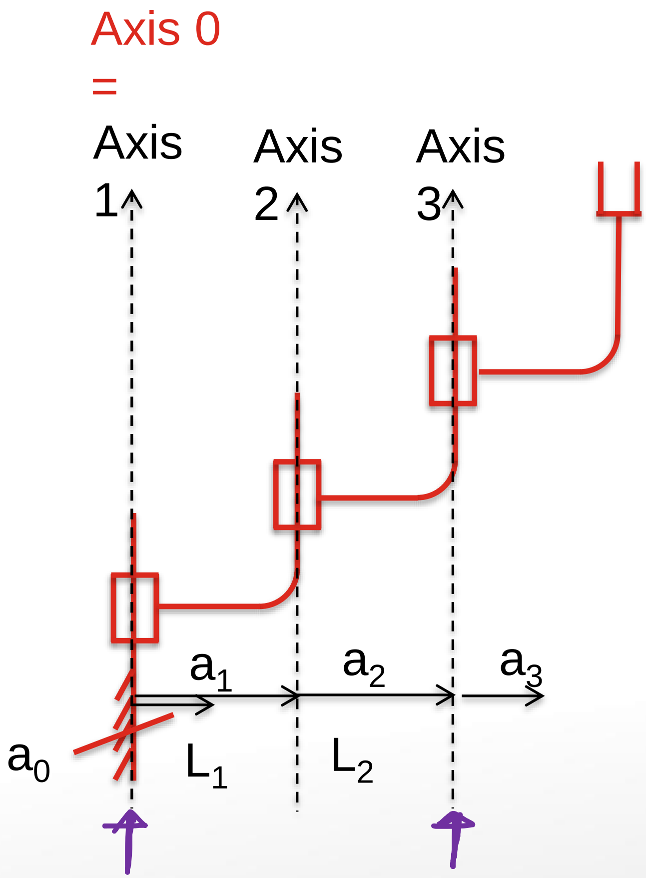
θ_2

- θ_2 = angle between (extension of a1) and a2, about axis 2.
- It is a variable because the joint is revolute.

Example 1

- 3-link RRR planar robot:

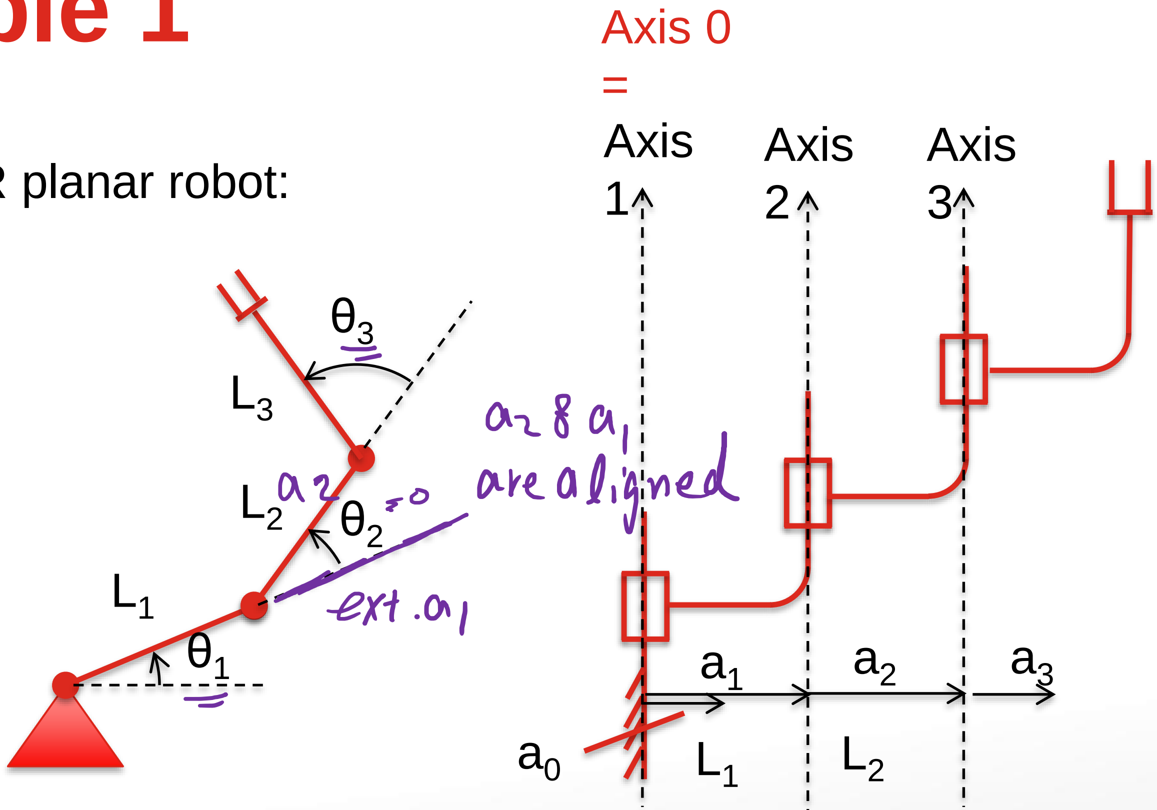




- What about θ_1 and θ_3?
  - θ_1 = angle between (extension of a0) and a1, about axis 1.
  - θ_3 = angle between (extension of a2) and a3, about axis 3.
  - By convention: Zero for prismatic joint, variable for revolute joint
  - So in this case, θ_1 and θ_3 are both variables.

Example 1

- 3-link RRR planar robot:



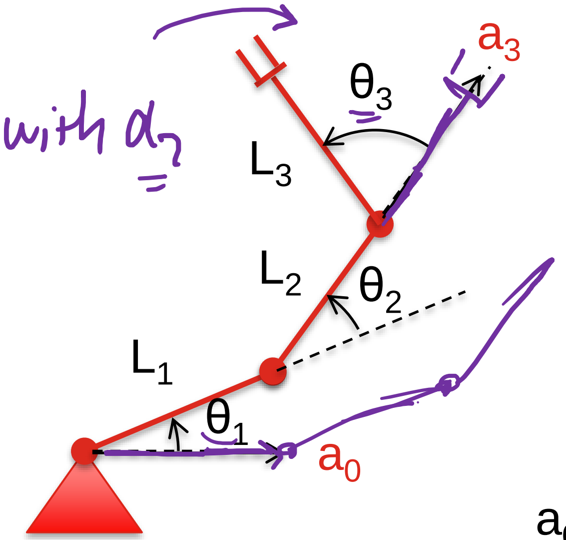
- We still have a problem. Since θ_1 and θ_3 are both variables, we need to determine their “zero”-angle position.

Example 1

- 3-link RRR planar robot:

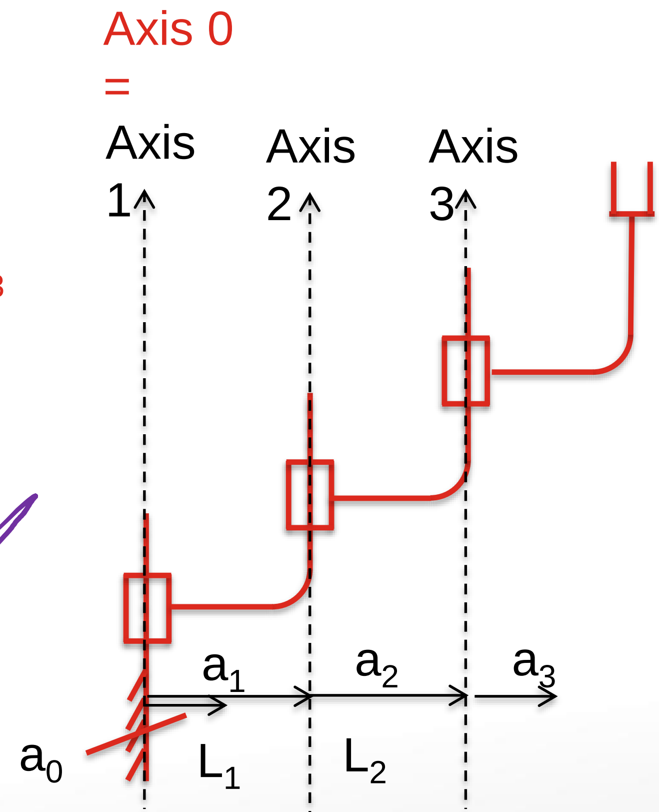
$\theta_3 = 0$

a_3 is align with a_2



$\theta_1 = 0$

a_0 is align with a_1



- For convenience, align a_0 with a_1 when the joint variable 1 is zero.
- As for joint n:
  - Revolute: align a_n with a_{n-1} when $\theta_n = 0$.
  - Prismatic: align a_n with a_{n-1} when $d_n = 0$.

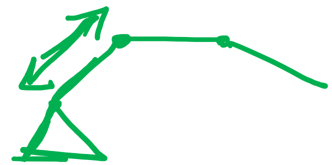
Aside: DH Parameters

6 Steps

- In summary, any robot can be described kinematically using the four parameters we have just learnt:
  - Link length a_{i-1}.
  - Link twist α_{i-1}.

    Const.

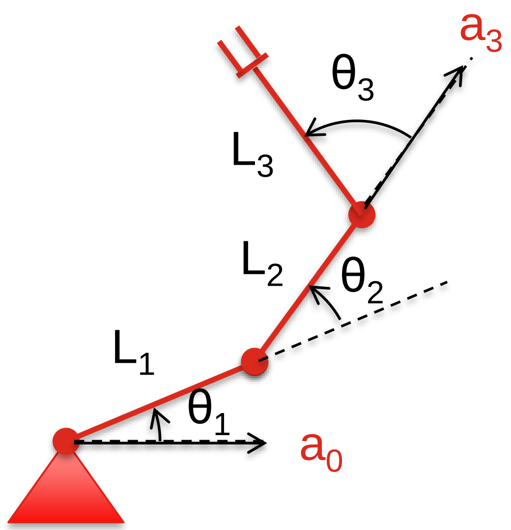
  - Link offset d_i.
  - Joint angle θ_i.

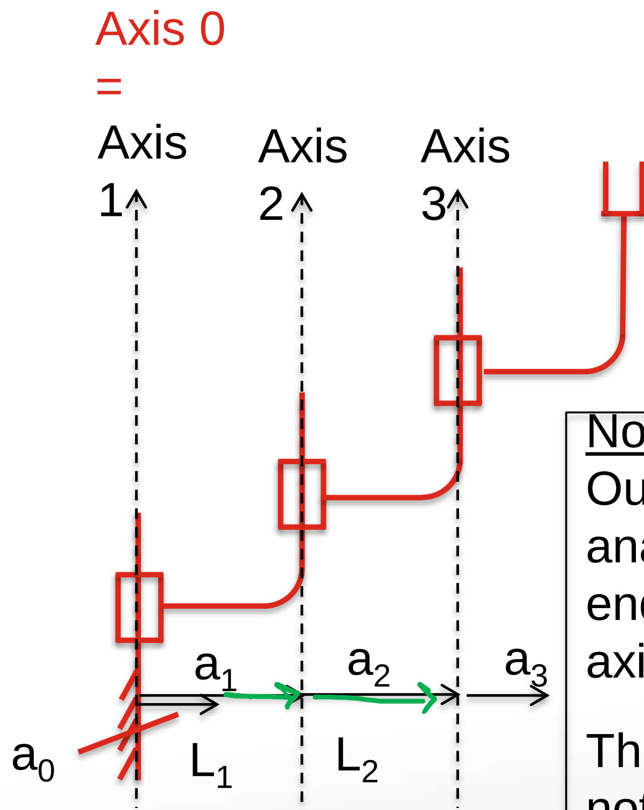


- Important Notes:
  - a_{i-1}, α_{i-1} are always constants
  - Revolute joint:
    - d_i constants
    - θ_i variable.
  - Prismatic joint:
    - θ_i constants
    - d_i variable.

Example 1

- 3-link RRR planar robot:





Note:
Our kinematic analysis always ends at last joint axis.

Therefore L_3 does not appear in the table.

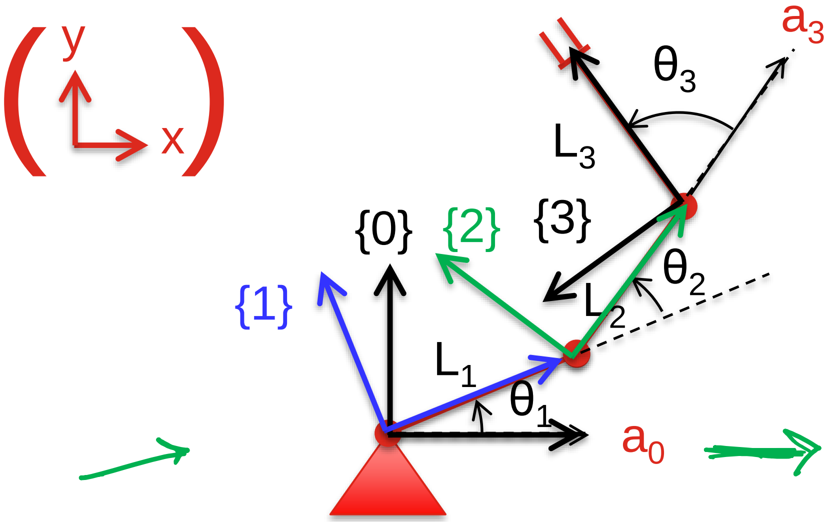
It is a final offset which can be added later.

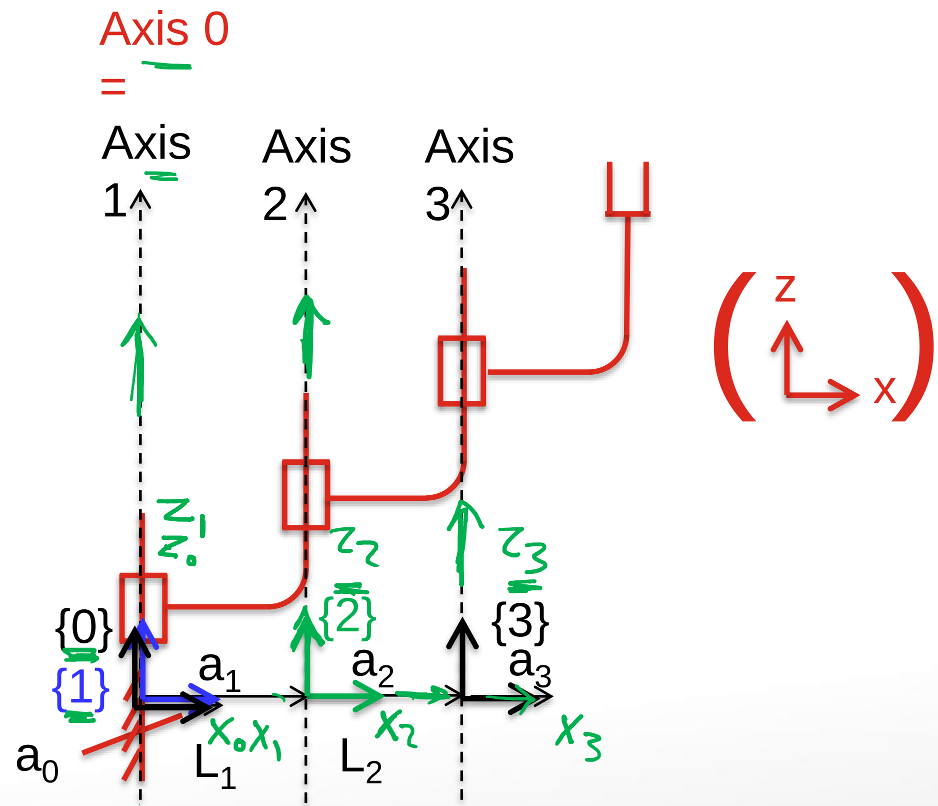
- Step 7, transfer to a DH-table:

i →	α_{i-1}	a_{i-1}	d_i	θ_i → Var
1	0 $\xleftarrow{c}$ α_1	0 $\xleftarrow{c}$ a_0	0 $\xleftarrow{c}$ d_1	θ_1 θ_1
2	0 ← α_1	L_1 ← a_1	0 ← d_2	θ_2 θ_2
3	0 ← α_2	L_2 ← a_2	0 $\xleftarrow{c}$ d_3	θ_3 θ_3

Example 1

- 3-link RRR planar robot:





- Step 8, insert the frames. Rules:
  - Z-axis of frame {i}, i.e. Z_i, is coincident with joint axis i.
  - Origin of frame {i} is where the a_i intersects the joint i axis.
  - X-axis of frame {i}, i.e. X_i, is coincident with a_i.

Aside: Link Transformation

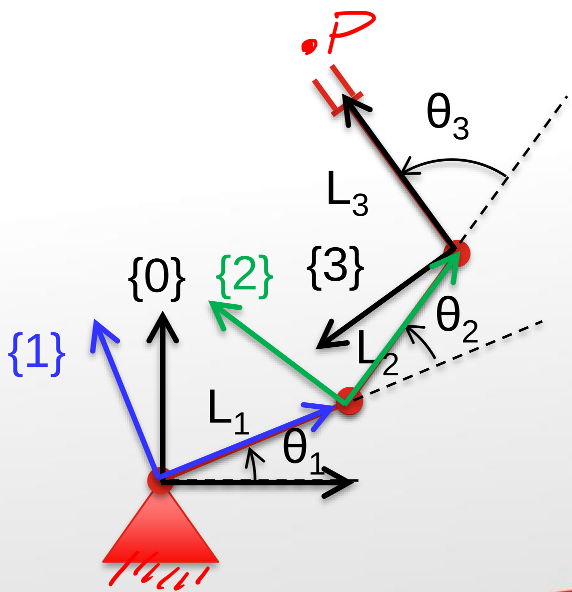
- It should be quite clear now why we attached frames to the links, and also why we determined the DH parameters for the links and joints.
  - We want to able to do transformation from frame {0} to {1}, then {1} to {2}, then {2} to {3} and so on.
  - Then by compound transformation, we can get the transformation from link the base to end-effector.

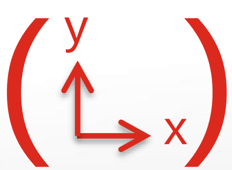
Forward Kinematics!

${}^0_1T, {}^1_2T, {}^2_3T$

General Form

DH $\rightarrow$ ${}^0_1T, {}^1_2T, {}^2_3T$





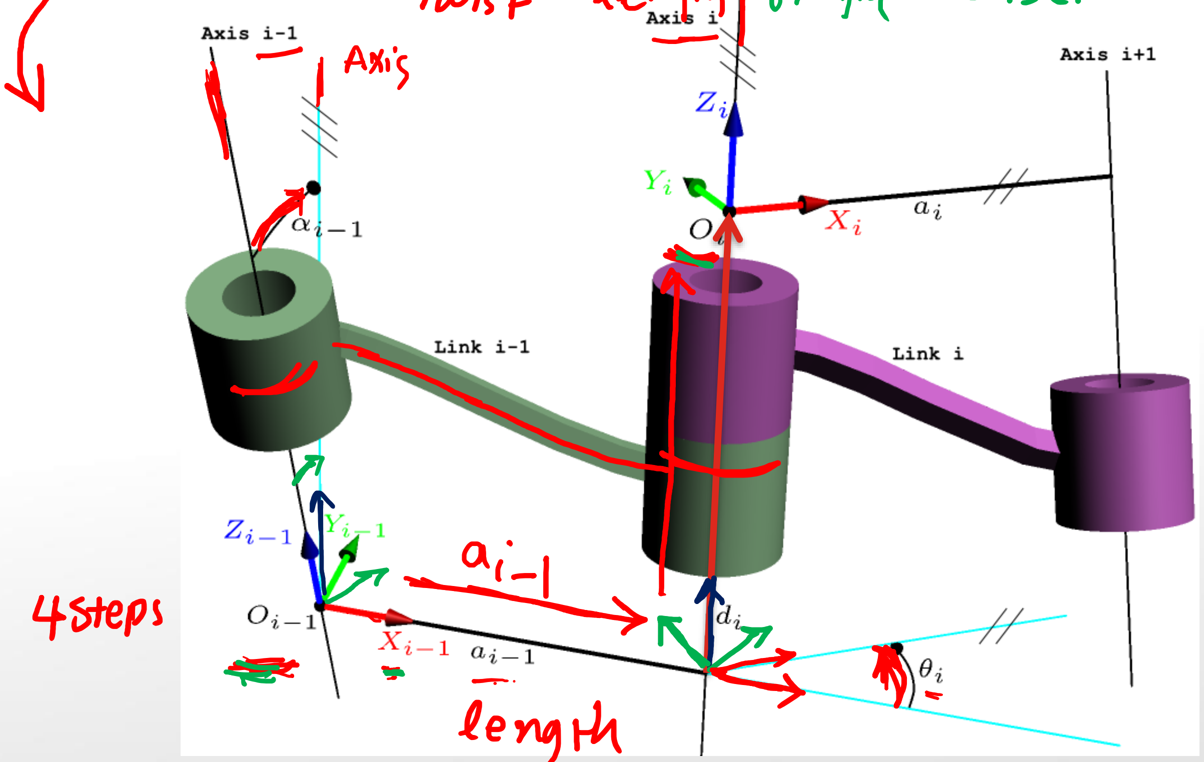
Aside: Link Transformation

4 param → DH → T → 4 steps

- Each of the transformation can be simplified to 4 sub-problems:

$$\underbrace{{}^{i-1}_{i}T}_{\text{4 steps}} = \underbrace{R_x(\alpha_{i-1})}_{\text{twist}} \cdot \underbrace{D_x(a_{i-1})}_{\text{length}} \cdot \underbrace{R_z(\theta_i)}_{\text{angle}} \cdot \underbrace{D_z(d_i)}_{\text{offset}}$$

→ ${}^0_1T$, ${}^1_2T$, ${}^2_3T$



Aside: Link Transformation

$$
{}^{i-1}_{i}T = R_x(\alpha_{i-1})\cdot D_x(a_{i-1})\cdot R_z(\theta_i)\cdot D_z(d_i)
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_{i-1} & -s\alpha_{i-1} & 0 \\ 0 & s\alpha_{i-1} & c\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & c\alpha_{i-1} & -s\alpha_{i-1} & 0 \\ 0 & s\alpha_{i-1} & c\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ c\alpha_{i-1}s\theta_i & c\alpha_{i-1}c\theta_i & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\alpha_{i-1}s\theta_i & s\alpha_{i-1}c\theta_i & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

${}^{0}_{1}T$

${}^{1}_{2}T$

${}^{2}_{3}T$

- Finally: ${}^{0}_{N}T = {}^{0}_{1}T \cdot {}^{1}_{2}T \cdot {}^{2}_{3}T \cdots {}^{N-1}_{N}T$

Example 1

8

- Step 9 (Final step!), calculate the transformations.

Known

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0 *	0 *	0 *	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ c\alpha_{i-1}s\theta_i & c\alpha_{i-1}c\theta_i & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\alpha_{i-1}s\theta_i & s\alpha_{i-1}c\theta_i & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{1}T = \begin{bmatrix} \boxed{c\theta_1} & \boxed{-s\theta_1} & 0 & 0 \\ \boxed{s\theta_1} & \boxed{c\theta_1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{1}_{2}T = \begin{bmatrix} \boxed{c\theta_2} & \boxed{-s\theta_2} & 0 & \boxed{L_1} \\ \boxed{s\theta_2} & \boxed{c\theta_2} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & L_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{3}T = {}^{0}_{1}T \cdot {}^{1}_{2}T \cdot {}^{2}_{3}T = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & L_2 c\theta_{12} + L_1 c\theta_1 \\ s\theta_{123} & c\theta_{123} & 0 & L_2 s\theta_{12} + L_1 s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

where

$$ \theta_{12} = \theta_1 + \theta_2 $$
$$ \theta_{123} = \theta_1 + \theta_2 + \theta_3 $$

Example 1

- Step 10. ^{0}P the position of endeffector w.r.t Base{0}

The end-effector, with reference to frame {3}, has position $[L_3, 0, 0]^T$, and same orientation as frame {3}. Therefore:

^{3}P

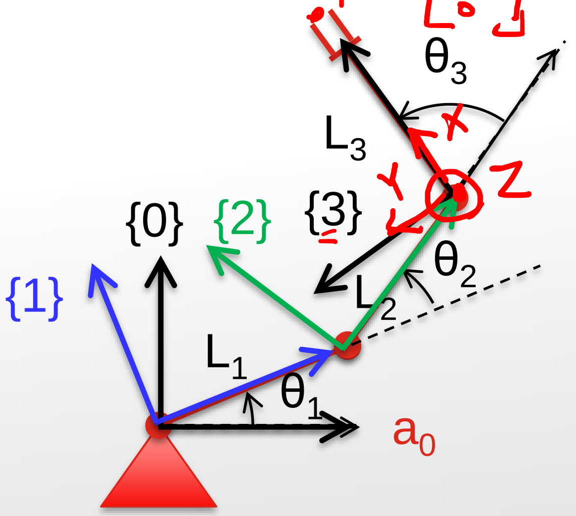
$$^{0}P = {}^{0}_{3}T \cdot {}^{3}P = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & L_2c\theta_{12} + L_1c\theta_1 \\ s\theta_{123} & c\theta_{123} & 0 & L_2s\theta_{12} + L_1s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} L_3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} L_3c\theta_{123} + L_2c\theta_{12} + L_1c\theta_1 \\ L_3s\theta_{123} + L_2s\theta_{12} + L_1s\theta_1 \\ 0 \\ 1 \end{bmatrix}$$

Orientation

Position Vector ^{0}P

$$^{3}P = \begin{bmatrix} L_3 \\ 0 \\ 0 \end{bmatrix}$$



Example 1

- Verification: At the beginning of this lecture, we showed that the position and orientation of the end-effector are:

$$\begin{cases} x = L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) + L_3\cos(\theta_1+\theta_2+\theta_3) \\ y = L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) + L_3\sin(\theta_1+\theta_2+\theta_3) \end{cases}$$

$$\theta_{total} = \theta_1 + \theta_2 + \theta_3$$

- Let's verify using the frame transformation method:

$$ {}^0P = {}^0_3T \cdot {}^3P = \begin{bmatrix} \boxed{\begin{matrix} c\theta_{123} & -s\theta_{123} & 0 \\ s\theta_{123} & c\theta_{123} & 0 \\ 0 & 0 & 1 \end{matrix}} & \begin{matrix} L_2c\theta_{12} + L_1c\theta_1 \\ L_2s\theta_{12} + L_1s\theta_1 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \cdot \begin{bmatrix} L_3 \\ 0 \\ 0 \\ 1 \end{bmatrix} $$

Orientation

$\checkmark$

$$ = \begin{bmatrix} \boxed{\begin{matrix} L_3c\theta_{123} + L_2c\theta_{12} + L_1c\theta_1 \\ L_3s\theta_{123} + L_2s\theta_{12} + L_1s\theta_1 \\ 0 \end{matrix}} \\ 1 \end{bmatrix} $$

Position

$\checkmark$

Thank you!

Have a good evening.