Week 7 – Manipulator Dynamics

Advanced Robotic Systems – MANU2453

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Explicit Form

- Using Lagrangian formulation, we derived the dynamic equations by differentiating the kinetic and potential energy.
- It turns out that we can even skip calculating the energies!
  - We can write out the dynamic equations just by looking at the structure of the manipulator!
  - → Explicit form.
- Recap: The Lagrangian formulation is given by:

Lagrangian ⇒
$$\frac{d}{dt}\frac{\partial k}{\partial \dot{q}} - \frac{\partial k}{\partial q} + \frac{\partial u}{\partial q} = \tau$$

- Recap: The structure of manipulator's dynamic equations is:

General ⇒
$$M(q)\ddot{q} + V(q,\dot{q}) + G(q) = \tau$$

- We will try to find some explicit relationship between the two equations.

How to get $V(q,\dot{q})$ from $M(q)$
How to get $M(q)$ from Jacobians, mass/Inertia

Explicit Form – Mass Matrix

How to get V from M

- The total kinetic energy of the manipulator, in terms of joint velocities, is: $\Rightarrow$ $k = \frac{1}{2}\dot{q}^T M(q)\dot{q}$

$\frac{1}{2}mv^2$

J. vel → Link vel

- Differentiating this in accordance to the Lagrangian formulation gives:

II → $$\frac{\partial k}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}\dot{q}^T M(q)\dot{q}\right) = M(q)\dot{q}$$

I → $$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}}\right) = \frac{d}{dt}\left(M(q)\dot{q}\right) = M(q)\ddot{q} + \dot{M}(q)\dot{q}$$

III → $$\frac{\partial k}{\partial q} = \frac{\partial}{\partial q}\left(\frac{1}{2}\dot{q}^T M(q)\dot{q}\right) = \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M(q)}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M(q)}{\partial q_n}\dot{q} \end{bmatrix}$$

- Therefore:

I II III

$$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}}\right) - \frac{\partial k}{\partial q} = M(q)\ddot{q} + \underbrace{\dot{M}(q)\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M(q)}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M(q)}{\partial q_n}\dot{q} \end{bmatrix}}_{\text{Centrifugal \& Coriolis}}$$

V

$M(q)\ddot{q} + V + G = \tau$

1

$\frac{1}{2}mv^2$

$\frac{\partial}{\partial v}\left(\frac{1}{2}mv^2\right) = mv$

Explicit Form – Mass Matrix

$M(q) \Rightarrow V$

- Thus we see that if we have the mass matrix M(q), we can immediately obtain the inertial forces, centrifugal forces and the Coriolis forces.
- But what is this M(q) matrix?
- We know that the kinetic energy of each link is:

Joint Velocities

$$\Rightarrow \quad k_i = \frac{1}{2} m_i \upsilon_{c_i}^T \upsilon_{c_i} + \frac{1}{2} \cdot {}^i\omega_i^T \cdot {}^{C_i}I_i \cdot {}^i\omega_i$$

$$\dot{q}^T = [\dot{q}_1, \cdots, \dot{q}_n]$$

- And the total kinetic energy of the manipulator is:

$$\Rightarrow \quad k = \sum_{i=1}^{n} k_i$$

$$k = \frac{1}{2} \dot{q}^T M(q) \dot{q}$$

- The kinetic energy of the manipulator, based on the joint velocities (previous slide), and based on the sum of each link (above), must be the same.
- Thus:

$$\Rightarrow \quad k = \frac{1}{2} \dot{q}^T M(q) \dot{q} = \frac{1}{2} \sum_{i=1}^{n} \left(m_i \upsilon_{c_i}^T \upsilon_{c_i} + {}^i\omega_i^T \cdot {}^{C_i}I_i \cdot {}^i\omega_i \right)$$

$\underline{v}$ $\underline{\omega}$

Explicit Form – Mass Matrix

- Now, the link velocities and the joint velocities are related as:

J_v & J_ω $\Rightarrow$
$$\begin{cases} \upsilon_{C_i} = \underline{J_{V_i}}\dot{q} \\ {}^i\omega_{C_i} = \underline{{}^iJ_{\omega_i}}\dot{q} \end{cases}$$

  - where:

$$\begin{cases} J_{V_i} = \begin{bmatrix} \underbrace{\dfrac{\partial P_{C_i}}{\partial q_1} & \cdots & \dfrac{\partial P_{C_i}}{\partial q_i}} & \underbrace{0 & \cdots & 0}_{0} \end{bmatrix} \\ {}^iJ_{\omega_i} = \begin{bmatrix} \bar{\varepsilon}_1 {}^iZ_1 & \cdots & \bar{\varepsilon}_i {}^iZ_i & 0 & \cdots & 0 \end{bmatrix} \end{cases}$$

  - (Why do we stop at i and have zeros thereafter?)
  - (Because the center of mass of link i is affected by joints 1 to i only!)

Explicit Form – Mass Matrix

- Therefore:

$$
\begin{aligned}
\frac{1}{2}\dot{q}^T M(q)\dot{q} &= \frac{1}{2}\sum_{i=1}^{n}\left(m_i \upsilon_{c_i}^T \upsilon_{c_i} + {}^i\omega_i^T \cdot {}^{C_i}I_i \cdot {}^i\omega_i\right) \\
&= \frac{1}{2}\sum_{i=1}^{n}\left(m_i \dot{q}_i^T J_{V_i}^T J_{V_i}\dot{q}_i + \dot{q}_i \cdot {}^iJ_{\omega_i}^T \cdot {}^{C_i}I_i \cdot {}^iJ_{\omega_i}\dot{q}_i\right) \\
&= \frac{1}{2}\dot{q}^T \underbrace{\left(\sum_{i=1}^{n}\left(m_i J_{V_i}^T J_{V_i} + {}^iJ_{\omega_i}^T \cdot {}^{C_i}I_i \cdot {}^iJ_{\omega_i}\right)\right)}_{M(q)}\dot{q}
\end{aligned}
$$

2

$$M(q) \Leftrightarrow J_{v_i} \ \& \ J_{\omega_i} \ \& \ m_i \ \& \ {}^{C_i}I_i$$

Explicit Form – Mass Matrix

- Summary:
  - Calculate M(q) using Jacobians, mass and inertia tensor of each link:

2 $\Rightarrow$

$$M(q) = \sum_{i=1}^{n} \left(m_i J_{V_i}^T J_{V_i} + {}^{i}J_{\omega_i}^T \cdot {}^{C_i}I_i \cdot {}^{i}J_{\omega_i} \right)$$

  - The inertial, centrifugal and Coriolis forces can directly be calculated as:

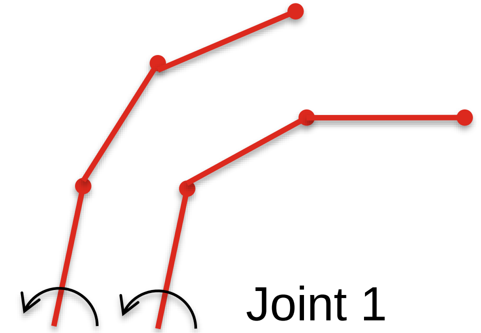
1 $\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}}\right) - \frac{\partial k}{\partial q} = M(q)\ddot{q} + \underbrace{\dot{M}(q)\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \dfrac{\partial M(q)}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \dfrac{\partial M(q)}{\partial q_n}\dot{q} \end{bmatrix}}_{V(q,\dot{q})}$$

Side Notes – M(q) Matrix

- Let's try to understand the M(q) matrix more.
- M(q) is an n x n matrix:

$$M(q) = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$



Joint 1

  - m_{11} = "perceived inertia" at joint 1, when all other joints are locked. It is a function of q_2 to q_n.
  - m_{22} = "perceived inertia" at joint 2, when all other joints are locked. It is a function of q_3 to q_n.
  - ……
  - $m_{(n-1)(n-1)}$ = "perceived inertia" at joint n-1, when all other joints are locked. It is a function of q_n.
  - m_{nn} = "perceived inertia" at joint n, when all other joints are locked. It is a constant!

Joint n

Side Notes – M(q) Matrix

- M(q) is positive definite:
  - Kinetic energy, $k = \frac{1}{2}\dot{q}^T M(q)\dot{q}$ is always greater or equal to zero.
    - Equals to zero only when velocity is zero.
    - Object cannot have zero mass.
- M(q) is symmetrical.
  - $m_{12} = m_{21}$, and so on.
  - The off-diagonal terms represent couplings between links.

Centrifugal & Coriolis

- We already know:

$$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}}\right)-\frac{\partial k}{\partial q}=\underbrace{M(q)\ddot{q}}_{\text{Inertial forces}}+\underbrace{\dot{M}(q)\dot{q}-\frac{1}{2}\begin{bmatrix}\dot{q}^T\dfrac{\partial M(q)}{\partial q_1}\dot{q}\\ \vdots \\ \dot{q}^T\dfrac{\partial M(q)}{\partial q_n}\dot{q}\end{bmatrix}}_{\text{Centrifugal \& Coriolis}}$$

- Let's look at the Centrifugal and Coriolis forces closer.
- For simplicity, we consider a two-link robot first, and make generalization later.
- M(q) is therefore 2 x 2:

$$M(q)=\begin{bmatrix}m_{11}(q) & m_{12}(q)\\ m_{12}(q) & m_{22}(q)\end{bmatrix}$$

Centrifugal & Coriolis

2-Link Robot $: M(q) = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}$

- Hence:

$$V(q,\dot{q}) = \dot{M}(q)\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \dfrac{\partial M(q)}{\partial q_1}\dot{q} \\ \dot{q}^T \dfrac{\partial M(q)}{\partial q_2}\dot{q} \end{bmatrix}$$

①

$$m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$$

$$\text{e.g. } m_{121} = \frac{\partial m_{12}}{\partial q_1}$$

$$= \begin{bmatrix} \dot{m}_{11}(q) & \dot{m}_{12}(q) \\ \dot{m}_{12}(q) & \dot{m}_{22}(q) \end{bmatrix}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix}\dot{q} \\ \dot{q}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix}\dot{q} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial m_{11}}{\partial q_1}\dot{q}_1 + \dfrac{\partial m_{11}}{\partial q_2}\dot{q}_2 & \dfrac{\partial m_{12}}{\partial q_1}\dot{q}_1 + \dfrac{\partial m_{12}}{\partial q_2}\dot{q}_2 \\ \dfrac{\partial m_{12}}{\partial q_1}\dot{q}_1 + \dfrac{\partial m_{12}}{\partial q_2}\dot{q}_2 & \dfrac{\partial m_{22}}{\partial q_1}\dot{q}_1 + \dfrac{\partial m_{22}}{\partial q_2}\dot{q}_2 \end{bmatrix}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix}\dot{q} \\ \dot{q}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix}\dot{q} \end{bmatrix}$$

$$= \begin{bmatrix} m_{111}\dot{q}_1 + m_{112}\dot{q}_2 & m_{121}\dot{q}_1 + m_{122}\dot{q}_2 \\ m_{121}\dot{q}_1 + m_{122}\dot{q}_2 & m_{221}\dot{q}_1 + m_{222}\dot{q}_2 \end{bmatrix}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix}\dot{q} \\ \dot{q}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix}\dot{q} \end{bmatrix}$$

Centrifugal & Coriolis

- Expand and simplify gives:

$$V(q,\dot{q}) = \begin{bmatrix} m_{111}\dot{q}_1 + m_{112}\dot{q}_2 & m_{121}\dot{q}_1 + m_{122}\dot{q}_2 \\ m_{121}\dot{q}_1 + m_{122}\dot{q}_2 & m_{221}\dot{q}_1 + m_{222}\dot{q}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \dot{q}^T \begin{bmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{bmatrix} \dot{q} \\ \dot{q}^T \begin{bmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{bmatrix} \dot{q} \end{bmatrix}$$

$$= \begin{bmatrix} m_{111}\dot{q}_1^2 + m_{112}\dot{q}_1\dot{q}_2 + m_{121}\dot{q}_1\dot{q}_2 + m_{122}\dot{q}_2^2 \\ m_{121}\dot{q}_1^2 + m_{122}\dot{q}_1\dot{q}_2 + m_{221}\dot{q}_1\dot{q}_2 + m_{222}\dot{q}_2^2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} m_{111}\dot{q}_1^2 + m_{121}\dot{q}_1\dot{q}_2 + m_{121}\dot{q}_1\dot{q}_2 + m_{221}\dot{q}_2^2 \\ m_{112}\dot{q}_1^2 + m_{122}\dot{q}_1\dot{q}_2 + m_{122}\dot{q}_1\dot{q}_2 + m_{222}\dot{q}_2^2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} m_{111}\dot{q}_1^2 + (2m_{122} - m_{221})\dot{q}_2^2 \\ (2m_{121} - m_{112})\dot{q}_1^2 + m_{222}\dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112}\dot{q}_1\dot{q}_2 \\ m_{221}\dot{q}_1\dot{q}_2 \end{bmatrix}$$

$$V(q,\dot{q}) = \frac{1}{2} \begin{bmatrix} \underline{m_{111}} & 2m_{122} - m_{221} \\ 2m_{121} - m_{112} & m_{222} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112}\dot{q}_1\dot{q}_2 \\ m_{221}\dot{q}_1\dot{q}_2 \end{bmatrix}$$

Centrifugal & Coriolis

- The equation can be written in a special form:

$$V(q,\dot{q}) = \frac{1}{2}\begin{bmatrix} m_{111} & 2m_{122}-m_{221} \\ 2m_{121}-m_{112} & m_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112}\dot{q}_1\dot{q}_2 \\ m_{221}\dot{q}_1\dot{q}_2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} m_{111}+m_{111}-m_{111} & m_{122}+m_{122}-m_{221} \\ m_{121}+m_{121}-m_{112} & m_{222}+m_{222}-m_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112}+m_{121}-m_{121} \\ m_{212}+m_{221}-m_{122} \end{bmatrix}\dot{q}_1\dot{q}_2$$

This will cancel off because $m_{12} = m_{21}$

- Introducing the "Christoffel Symbols":

$$b_{ijk} = \frac{1}{2}\left(m_{ijk} + m_{ikj} - m_{jki}\right)$$

Centrifugal & Coriolis

- Using the Christoffel Symbols, the V matrix can now be written as:

$$
\begin{aligned}
V(q,\dot{q}) &= \frac{1}{2}\begin{bmatrix} m_{111}+m_{111}-m_{111} & m_{122}+m_{122}-m_{221} \\ m_{121}+m_{121}-m_{112} & m_{222}+m_{222}-m_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112}+m_{121}-m_{121} \\ m_{212}+m_{221}-m_{122} \end{bmatrix}\dot{q}_1\dot{q}_2 \\
&= \underbrace{\begin{bmatrix} b_{111} & b_{122} \\ b_{121} & b_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}}_{\text{C, Centrifugal}} + \underbrace{\begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix}\dot{q}_1\dot{q}_2}_{\text{B, Coriolis}}
\end{aligned}
$$

Centrifugal & Coriolis

- We can finally generalize the results to:

$$\underbrace{C(q)}_{n\times n}\underbrace{\left(\dot{q}^2\right)}_{n\times 1}=\begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

$$\underbrace{B(q)}_{n\times\frac{(n-1)n}{2}}\underbrace{\left(\dot{q}\dot{q}\right)}_{\frac{(n-1)n}{2}\times 1}=\begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \ddots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix}\begin{bmatrix} \dot{q}_1\dot{q}_2 \\ \dot{q}_1\dot{q}_3 \\ \vdots \\ \dot{q}_{n-1}\dot{q}_n \end{bmatrix}$$

Centrifugal & Coriolis

- Another summary:
  - Calculate M(q) using Jacobians, mass and inertia tensor of each link:

2 ⇒

$$M(q)=\sum_{i=1}^{n}\left(m_i J_{V_i}^T J_{V_i} + {}^{i}J_{\omega_i}^T \cdot {}^{C_i}I_i \cdot {}^{i}J_{\omega_i}\right)$$

  - The inertial, centrifugal and Coriolis forces can directly be calculated as:

1 ⇒

$$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}}\right)-\frac{\partial k}{\partial q}=M(q)\ddot{q}+\dot{M}(q)\dot{q}-\frac{1}{2}\begin{bmatrix}\dot{q}^T \dfrac{\partial M(q)}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \dfrac{\partial M(q)}{\partial q_n}\dot{q}\end{bmatrix}$$

  - Use the expressions for B and C on previous slide to get the last two terms.

Explicit Form – Gravity Terms

$M(q)\ddot{q} + V(q,\dot{q}) + G_q = \tau$

- So far, we already have the explicit forms for the terms related to kinetic energy.
- What can we say about the potential energy? Const.
- Potential energy of each link was: $u_i = -m_i \cdot {}^0g^T \cdot {}^0P_{C_i} + u_{ref_i}$
- The gravity terms in the Lagrangian formulation is obtained by: $\frac{\partial u}{\partial q}$ $\quad u = \sum_{i=1}^{n} u_i$
- Hence for each link, the gravity term is:

Explicit Form $\Rightarrow$ $$G_j = \frac{\partial u}{\partial q_j} = -\sum_{i=1}^{n}\left(m_i \cdot {}^0g^T \cdot \frac{\partial\, {}^0P_{C_i}}{\partial q_j}\right)$$ $\rightarrow J_{vi}$

- Thus we have:

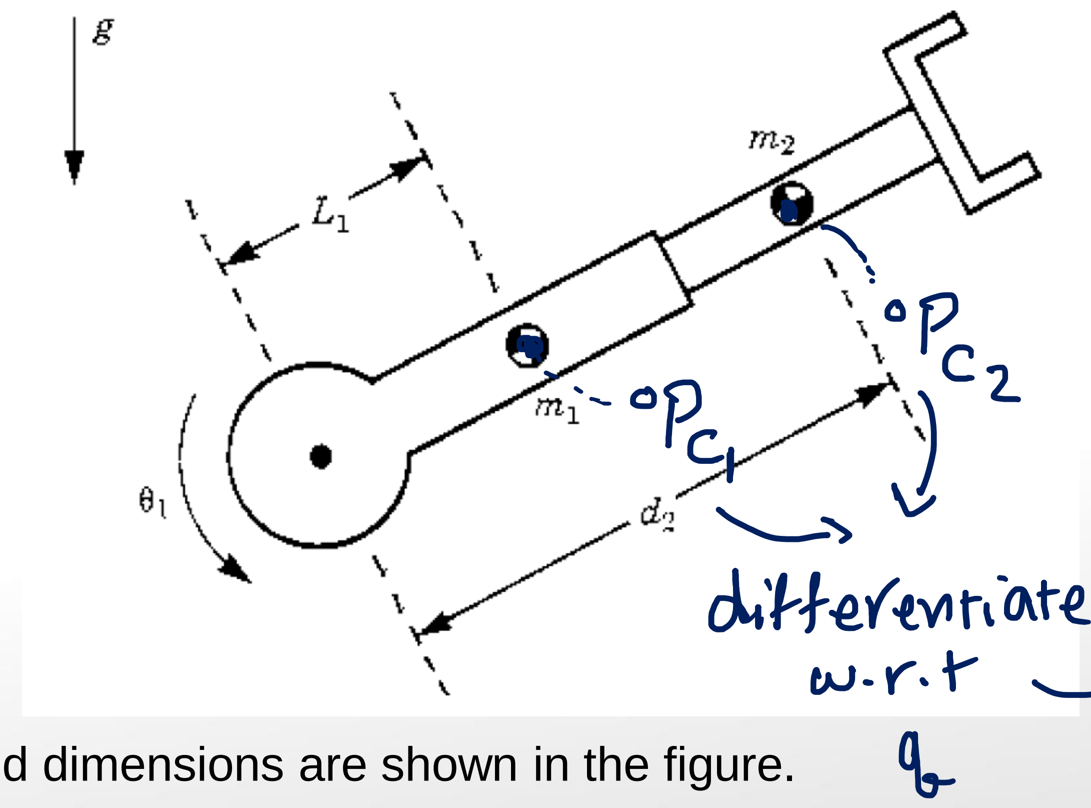
$$G = -\begin{bmatrix} J_{V1}^T & J_{V2}^T & \cdots & J_{Vn}^T \end{bmatrix} \begin{bmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{bmatrix}$$

3

Example

D. Eqs. of M. using the Explicit form

- Consider the same RP manipulator:



Jacobians

CoM

$J_{\omega_{ci}}$

$J_{v_{ci}}$

- Mass and dimensions are shown in the figure.

Example

- First, we need to obtain the mass matrix M(q) from:

$$M(q)=\sum_{i=1}^{n}\left(m_i J_{V_i}^T J_{V_i} + {}^i J_{\omega_i}^T \cdot {}^{C_i} I_i \cdot {}^i J_{\omega_i}\right)$$

- To do this, we need the Jacobians.
- The positions of the centres of mass are:

$$ {}^0P_{C_1} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix} \qquad {}^0P_{C_2} = \begin{bmatrix} d_2 c_1 \\ d_2 s_1 \\ 0 \end{bmatrix} $$

$$ {}^0V_{C_1} = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 \\ l_1 c_1 \dot{\theta}_1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}}_{{}^0J_{V_1}} \begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix} $$

- Therefore:

$$ {}^0J_{V_1} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix} \qquad {}^0J_{V_2} = \begin{bmatrix} -d_2 s_1 & c_1 \\ d_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix} $$

(Frame doesn't matter for linear kinetic energy)

Up to i=1 Add zero column

Example

- This yields:

$$\Rightarrow \quad m_1 J_{V_1}^T J_{V_1} = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$m_2 J_{V_2}^T J_{V_2} = \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$

- As for the rotational terms, we have: $^1J_{\omega_1} = \begin{bmatrix} \bar{\varepsilon}_1 {}^1Z_1 & 0 \end{bmatrix}$ $\quad$ $^2J_{\omega_2} = \begin{bmatrix} \bar{\varepsilon}_1 {}^2Z_1 & \bar{\varepsilon}_2 {}^2Z_2 \end{bmatrix}$

Z_1 in {2}
= y-axis of {2}

- For joint 1, the Jacobian is: $^1J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$ since $^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}}^{^1J_{\omega_1}} \overbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}}^{\dot{q}}$

- For joint 2, the Jacobian is: $^2J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$ since $^2\omega_2 = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}}\begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix}$ $\quad {}^2J_{\omega_2}\dot{q}$

Example

- Therefore:

$$\Rightarrow \quad {}^{1}J_{\omega_1}^{T} \cdot {}^{C_1}I_1 \cdot {}^{1}J_{\omega_1} = \begin{bmatrix} I_{zz_1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$ {}^{2}J_{\omega_2}^{T} \cdot {}^{C_2}I_2 \cdot {}^{2}J_{\omega_2} = \begin{bmatrix} I_{yy_2} & 0 \\ 0 & 0 \end{bmatrix}$$

- Finally, the mass matrix is:

2 $\Rightarrow$

$$\underline{M(q)} = \sum_{i=1}^{n} \left(m_i J_{V_i}^{T} J_{V_i} + {}^{i}J_{\omega_i}^{T} \cdot {}^{C_i}I_i \cdot {}^{i}J_{\omega_i} \right)$$

$$= \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} I_{zz_1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I_{yy_2} & 0 \\ 0 & 0 \end{bmatrix}$$

M(q) $$= \begin{bmatrix} m_1 l_1^2 + I_{zz_1} + m_2 d_2^2 + I_{yy_2} & 0 \\ 0 & m_2 \end{bmatrix}$$

Example

$M(q) \Rightarrow V \begin{cases} B \\ C \end{cases}$

- Next, we will use the Christoffel Symbols to calculate the centrifugal and Coriolis forces:

$$M(q) = \begin{bmatrix} m_1 l_1^2 + I_{zz_1} + m_2 d_2^2 + I_{yy_2} & 0 \\ 0 & m_2 \end{bmatrix}$$

$$b_{ijk} = \frac{1}{2}\left(m_{ijk} + m_{ikj} - m_{jki}\right)$$

$$\Rightarrow \underbrace{C(q)}_{n\times n}\underbrace{\left(\dot{q}^2\right)}_{n\times 1} = \begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

$$\Rightarrow \underbrace{B(q)}_{n\times\frac{(n-1)n}{2}}\underbrace{\left(\dot{q}\dot{q}\right)}_{\frac{(n-1)n}{2}\times 1} = \begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \ddots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix} \begin{bmatrix} \dot{q}_1\dot{q}_2 \\ \dot{q}_1\dot{q}_3 \\ \vdots \\ \dot{q}_{n-1}\dot{q}_n \end{bmatrix}$$

Example

$M(q)$

- This gives:

$$\underbrace{C(q)}_{n\times n}\underbrace{(\dot{q}^2)}_{n\times 1}=\begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

$$m_{111}=\frac{\partial m_{11}}{\partial q_1}$$

$$=\frac{1}{2}\begin{bmatrix} m_{111}+m_{111}-m_{111} & m_{122}+m_{122}-m_{221} \\ m_{211}+m_{211}-m_{112} & m_{222}+m_{222}-m_{222} \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

Joint 2 only sees centrifugal force

$$\Rightarrow \underbrace{B(q)}_{n\times\frac{(n-1)n}{2}}\underbrace{(\dot{q}\dot{q})}_{\frac{(n-1)n}{2}\times 1}=\begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \ddots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix}\begin{bmatrix} \dot{q}_1\dot{q}_2 \\ \dot{q}_1\dot{q}_3 \\ \vdots \\ \dot{q}_{n-1}\dot{q}_n \end{bmatrix}$$

$$=\begin{bmatrix} m_{112}+m_{121}-m_{121} \\ m_{212}+m_{221}-m_{122} \end{bmatrix}\dot{q}_1\dot{q}_2=\begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}\dot{q}_1\dot{q}_2$$

Joint 1 only sees Coriolis force

Example

- Finally, we shall derive the gravity term.
- We use the formula:

3 $\Rightarrow$

$$G = -\begin{bmatrix} J_{V1}^T & J_{V2}^T & \cdots & J_{Vn}^T \end{bmatrix} \begin{bmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{bmatrix} = -J_{V1}^T m_1 g - J_{V2}^T m_2 g$$

- In frame {0}. the gravity vector is:

$${}^0 g = \begin{bmatrix} 0 & -g & 0 \end{bmatrix}^T$$

- Therefore:

$$G = -J_{V1}^T m_1 g - J_{V2}^T m_2 g$$

$$\Rightarrow \quad = -m_1 \begin{bmatrix} -l_1 s_1 & l_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} - m_2 \begin{bmatrix} -d_2 s_1 & d_2 c_1 & 0 \\ c_1 & s_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

$$G = \begin{bmatrix} m_1 g l_1 c_1 + m_2 g d_2 c_1 \\ m_1 g s_1 \end{bmatrix}$$

Example

$M(q)\ddot{q} + V(q,\dot{q}) + G = \tau$

- Combining all the results (for M, C, B and G), we arrive at:

$$\underbrace{\begin{bmatrix} m_1 l_1^2 + I_{zz_1} + m_2 d_2^2 + I_{yy_2} & 0 \\ 0 & m_2 \end{bmatrix}}_{M(q)} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} + \underbrace{\underbrace{\begin{bmatrix} 0 \\ -m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}}_{C(q)\dot{q}^2} + \underbrace{\begin{bmatrix} 2m_2 d_2 \dot{d}_2 \dot{\theta}_1 \\ 0 \end{bmatrix}}_{B(q)(\dot{q}\dot{q})}}_{V} + \underbrace{\begin{bmatrix} m_1 l_1 g c_1 + m_2 d_2 g c_1 \\ m_2 g s_1 \end{bmatrix}}_{G(q)} = \tau$$

- which is exactly the same as what we had from differentiation of Lagrangian:

$$\underbrace{\begin{bmatrix} m_1 l_1^2 + I_{zz_1} + m_2 d_2^2 + I_{yy_2} & 0 \\ 0 & m_2 \end{bmatrix}}_{M(q)} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2m_2 d_2 \dot{d}_2 \dot{\theta}_1 \\ -m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}}_{V(q,\dot{q})} + \underbrace{\begin{bmatrix} m_1 l_1 g c_1 + m_2 d_2 g c_1 \\ m_2 g s_1 \end{bmatrix}}_{G(q)} = \tau$$

Thank you!

Have a good evening.