QS. 1

Wednesday, 14 October 2020 7:00 PM

- **Question 1:**
  - Determine the motion of a mass-spring-damper system if parameter values are m = 2, b = 6 and k = 4, and the mass (initially at rest) is released from the position x = 1.

$t:0 \rightarrow x(0) = 1$

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$2\ddot{x} + 6\dot{x} + 4x = 0$$

characteristic equation: $2\lambda^2 + 6\lambda + 4 = 0$

two roots: $\lambda = \frac{-6 \pm \sqrt{36-32}}{4} = \frac{-6 \pm \sqrt{4}}{4} = \underbrace{-1 \,\&\, -2}_{\text{2 distinct real roots}}$

recall from lecture:

The general equation (harmonic solution) is:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$x(t) = c_1 e^{-t} + c_2 e^{-2t} \qquad (1)$$

c_1 & c_2 can be determined from the initial conditions.

we know: at $t=0$, $x(0) = 1 \overset{(1)}{=} c_1 + c_2$ (I)

we know: at $t=0$, $\dot{x}(0) = 0$ (mas at rest)

$\frac{d}{dt}(1)$: $\dot{x}(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$

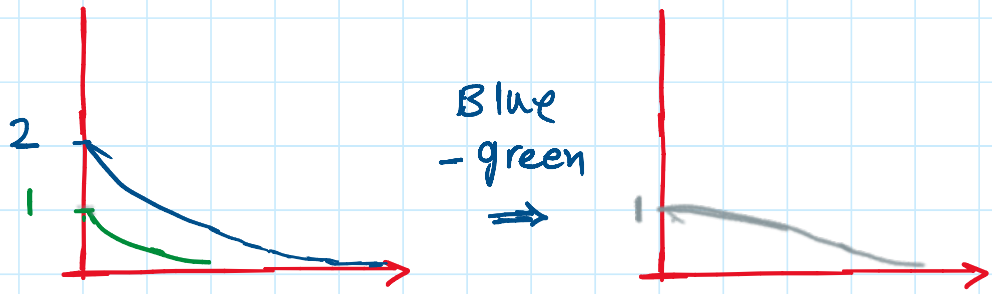
at $t:0$ $\dot{x}(0) = -c_1 - 2c_2 = 0$ (II)

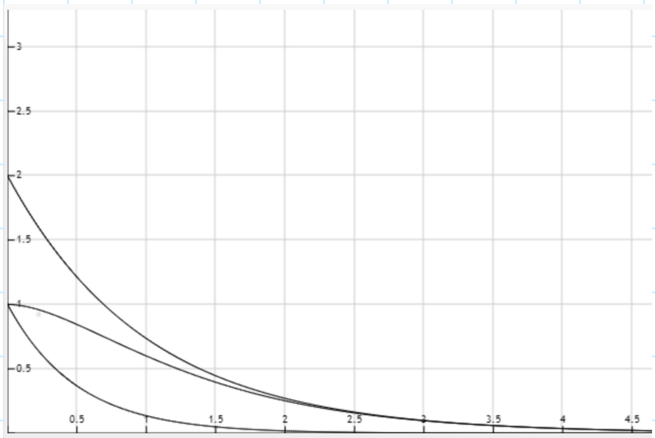
$\hookrightarrow$ at $t:0 \quad \dot{x}(0) = -c_1 - 2c_2 = 0$ (II)

$$\text{I} \,\&\, \text{II} \rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad x(t) = 2e^{-t} - e^{-2t}$$





QS. 2

Wednesday, 14 October 2020 7:18 PM

- **Question 2:**
  - Determine the motion of a mass-spring-damper system if parameter values are m = 1, b = 2 and k = 1, and the mass (initially at rest) is released from the position x = 4.

$\dot{x}(0) = 0$

$x(0) = 4$

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + 2\dot{x} + x = 0$$

characteristic equation: $\lambda^2 + 2\lambda + 1 = 0$

two roots: $\lambda = \frac{-2 \pm \sqrt{4-4}}{2} = \frac{-2 \pm 0}{2} = \underbrace{-1 \; \& \; -1}_{\text{2 same real roots}}$

recall from lecture:

The general equation (harmonic solution) is:

$$x(t) = [c_1 + c_2 t] e^{\lambda t}$$

$$x(t) = [c_1 + c_2 t] e^{-t} \quad (2)$$

c_1 & c_2 can be determined from the initial conditions.

we know: at $t=0$, $x(0) = \underline{c_1 = 4}$ (I)

we know: at $t=0$, $\dot{x}(0) = 0$ (mas at rest)

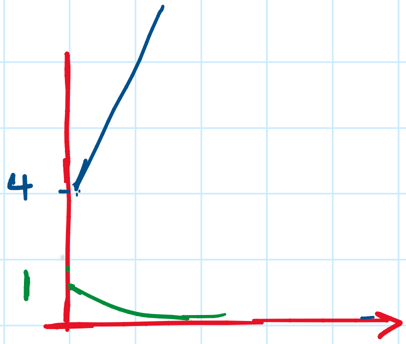
$\hookrightarrow \frac{d}{dt}(2)$: $\dot{x}(t) = -[c_1 + c_2 t]e^{-t} + c_2 e^{-t}$

$\hookrightarrow$ at $t:0$ $\dot{x}(0) = -c_1 + c_2 = 0$ $\leftarrow c_1 = 4$

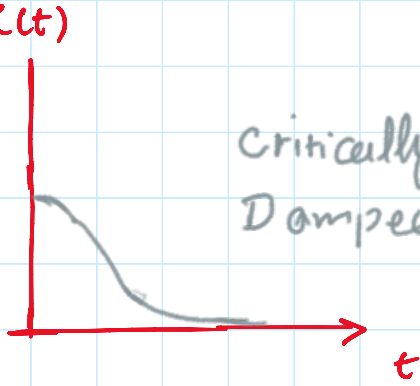
$$-4 + c_2 = 0$$

$$\underline{c_2 = 4}$$

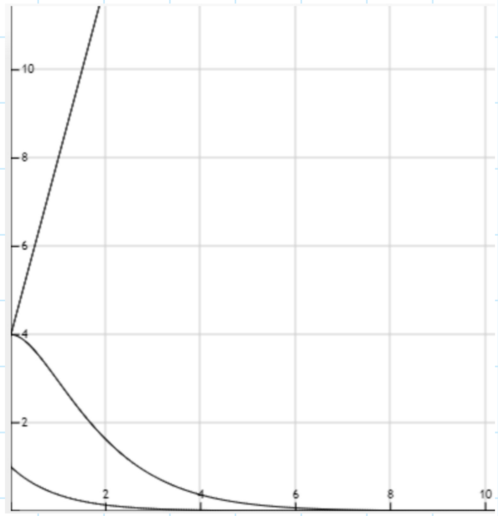
Finally $x(t) = [4 + 4t]\, e^{-t}$



Blue times Green $\Rightarrow$



Critically Damped



QS. 3

Wednesday, 14 October 2020 8:30 PM

- **Question 3:**
  - Determine the motion of a mass-spring-damper system if parameter values are m = 1, b = 4 and k = 5, and the mass (initially at rest) is released from the position x = 2.

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\ddot{x} + 4\dot{x} + 5x = 0$$

characteristic equation: $\lambda^2 + 4\lambda + 5 = 0$

two roots: $\lambda = \dfrac{-2 \pm \sqrt{16 - 20}}{2} = \dfrac{-4 \pm \sqrt{-4}}{2} = \underbrace{-2 \pm i}_{\text{complex roots}}$

recall from lecture:

The general equation (harmonic solution) is:

$$x(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t \qquad (3)$$

c_1 & c_2 can be determined from the initial conditions.

we know: at $t = 0$, $x(0) = \underline{c_1 = 2}$ (I)

we know: at $t = 0$, $\dot{x}(0) = 0$ (mas at rest)

$\frac{d}{dt}$ (3): $\dot{x}(t) = -c_1 e^{-2t} \sin t - 2c_1 e^{-2t} \cos t$

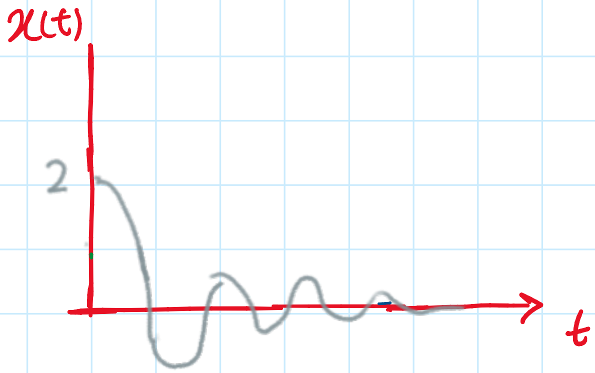
$+ c_2 e^{-2t} \cos t - 2c_2 e^{-2t} \sin t$

at $t:0$ $\quad \dot{x}(0) = 0 - 2C_1 + C_2 + 0$

$$= -2C_1 + C_2 = 0 \qquad C_1 = 2$$

$$\Rightarrow \underline{C_2 = 4}$$

Finally $x(t) = 2e^{-2t}\cos(t) + 4e^{-2t}\sin(t)$



QS. 4

Wednesday, 14 October 2020 8:49 PM

- **Question 4:**
  - Consider a mass-spring-damper system with parameter values m = 1, b = 4 and k = 5.
  - The system is known to possess an unmodeled resonance at ω_{res} = 6 rad/sec.
  - Determine the gains k_D and k_p which will critically damp the system with as high a stiffness as reasonable.

Model: $m\ddot{x} + b\dot{x} + kx = F$ Control signal

Ctrl step1: $F = m\alpha + b\dot{x} + kx$

Ctrl step2: $\alpha = \ddot{x}_d + k_D(\dot{x}_d - \dot{x}) + k_p(x_d - x)$

Note: in some cases we should limit k_p

$$\begin{cases} \ddot{x} + k_D\dot{x} + k_p x = 0 \\ s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \end{cases} \Rightarrow k_p = \omega_n^2 \quad (1)$$

another way to write 2nd order system

7.4.1 Control Gain Limitations

The higher k_p is, the better the disturbance rejection becomes. However, control gains are limited by various factors involving structural flexibilities in the mechanism, time-delays in actuators and sensing, and sampling rates. An increase of k_p results in an increase of the closed-loop frequency ω. As this frequency approaches the the first unmodeled resonant frequency, $\omega_{\text{low-resonant}}$, the corresponding mode can be excited. It is thus important to keep ω well below this frequency. In addition, ω must be remain below the frequency corresponding to the largest time delay, $\omega_{\text{large-delay}}$. The frequency associated with the sampling rate, $\omega_{\text{sampling-rate}}$ also imposes a limitation on ω. Typically ω is selected as

$$\begin{aligned} \omega &< \frac{1}{2}\omega_{\text{low-resonant}} \quad \xRightarrow{(1)} \quad k_p < \frac{1}{4}\omega_{res}^2 \\ \omega &< \frac{1}{3}\omega_{\text{large-delay}} \\ \omega &< \frac{1}{5}\omega_{\text{sampling-rate}} \end{aligned}$$

To avoid exciting the resonance, we should limit the Kp

$$K_p \leq \frac{1}{4}\omega_{res}^2 \Rightarrow K_p \leq \frac{1}{4}(6)^2 = 9$$

So let $K_p = \underline{\underline{9}}$

Then to get critically damped response, set

$$K_D = 2\sqrt{K_p} = 2\sqrt{9} = \underline{\underline{6}}$$

In summary: Controller should be:

$$F = 1 \times [\ddot{x}_d + 6(\dot{x}_d - \dot{x}) + 9(x_d - x)] + 4\dot{x} + 5x$$

QS. 5

Thursday, 15 October 2020 8:58 AM

- **Question 5:**
  - Give the nonlinear control equations for the system:

$$\left(2\sqrt{\theta}+1\right)\ddot{\theta}+3\dot{\theta}^2-\sin(\theta)=\tau$$

  - Choose gains so that this system is always critically damped with closed-loop stiffness K_{CL} of 10.

Model: $(2\sqrt{\theta}+1)\ddot{\theta}+3\dot{\theta}^2-\sin(\theta)=\tau$

Ctrl step 1: $\tau=(2\sqrt{\theta}+1)\alpha+3\dot{\theta}^2-\sin(\theta)$

Ctrl step 2: $\alpha=\ddot{\theta}_d+K_D(\dot{\theta}_d-\dot{\theta})+K_P(\theta_d-\theta)$

set $K_P=K_{CL}=10 \Rightarrow K_D=2\sqrt{K_P}=2\sqrt{10}$

QS. 6

Thursday, 15 October 2020 9:05 AM

- **Question 6:**
  - Give the nonlinear control equations for the system:

$$2\ddot{\theta} + 5\theta\dot{\theta} - 13\dot{\theta}^3 + 5 = \tau$$

  - Choose gains so that this system is always critically damped with closed-loop stiffness K_{CL} of 10.

Model: $2\ddot{\theta} + 5\theta\dot{\theta} - 13\dot{\theta}^3 + 5 = \tau$

Ctrl step 1: $\tau = 2\alpha + 5\theta\dot{\theta} - 13\dot{\theta}^3 + 5$

Ctrl step 2: $\alpha = \ddot{\theta}_d + K_D(\dot{\theta}_d - \dot{\theta}) + K_P(\theta_d - \theta)$

set $K_P = K_{CL} = \underline{\underline{10}}$ $\Rightarrow$ $K_D = 2\sqrt{K_P} = \underline{\underline{2\sqrt{10}}}$

QS. 7

Thursday, 15 October 2020 9:12 AM

- **Question 7:**
  - Design a trajectory-following control system for a system with the following dynamic equations:

$$m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 = \tau_1$$
$$m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + v_2 \dot{\theta}_2 = \tau_2$$

Model robot into M, V & G form

$$\begin{bmatrix} m_1 l_1^2 & 0 \\ m_2 l_2^2 & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} m_1 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \\ v_2 \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Ctrl Step 1:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_1 l_1^2 & 0 \\ m_2 l_2^2 & m_2 l_2^2 \end{bmatrix} \overbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}}^{\alpha} + \begin{bmatrix} m_1 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \\ v_2 \dot{\theta}_2 \end{bmatrix} \quad (1)$$

Ctrl Step 2:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \ddot{\theta}_{1d} \\ \ddot{\theta}_{2d} \end{bmatrix} + \begin{bmatrix} k_{D1} & 0 \\ 0 & k_{D2} \end{bmatrix} \begin{bmatrix} \dot{\theta}_{1d} - \dot{\theta}_1 \\ \dot{\theta}_{2d} - \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_{P1} & 0 \\ 0 & k_{P2} \end{bmatrix} \begin{bmatrix} \theta_{1d} - \theta_1 \\ \theta_{2d} - \theta_2 \end{bmatrix} \quad (2)$$

Finally Put (2) into (1)