Tutorial TWO.

Q.1. $$G(s) = \frac{b}{s+a} = \frac{B(s)}{A(s)}$$

$$C(s) = \frac{c_1 s + c_0}{s} = \frac{P(s)}{L(s)}.$$

Actual closed-loop polynomial is

$$A(s)L(s) + B(s)P(s)$$

$$= (s+a)s + b(c_1 s + c_0)$$

$$= s^2 + (a + bc_1)s + bc_0$$

The desired closed-loop polynomial is

$$s^2 + 2\zeta w_n s + w_n^2 = s^2 + 2 \times 0.707 \times a s + a^2.$$

~~Actual~~ where $w_n = a$.

Comparing the actual closed-loop polynomial with the desired closed-loop polynomial leads to

s^2: $1 = 1$

s: $a + bc_1 = 2\xi\omega_n = 2 \times 0.707 \times a$.

s^0: $bc_0 = a^2$.

$$c_1 = \frac{1.4014a - a}{b} = \frac{0.4014 \times 0.1}{3}$$

$$c_0 = \frac{a^2}{b} = \frac{0.01}{3}.$$

where $a = 0.1$, $b = 3$.

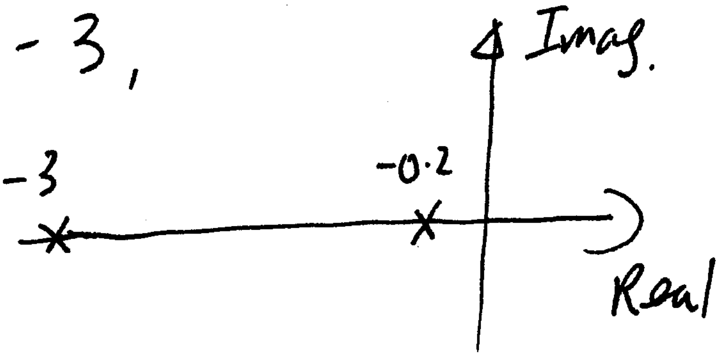
$$K_c = c_1, \quad \tau_I = \frac{c_1}{c_0}.$$

0.2.

1. We first determine the approximate model for PI controller.

design. Since the transfer function has two poles at

-0.2 and -3,



The pole at -0.2 is the dominant pole. We will neglect the pole at -3 to obtain the approximate model.

Write the transfer function in the constant form:

$$G(s) = \frac{0.1}{(s+0.2)(s+3)} = \frac{0.1}{0.2 \times 3 \left(\frac{1}{0.2}s+1\right)\left(\frac{1}{3}s+1\right)}$$

Note that $\frac{1}{3}$ is more than 10 times less than $\frac{1}{0.2}$. Thus, we approximate

$$\frac{1}{\frac{1}{3}s+1} \approx 1.$$

Therefore,

$$G(s) = \frac{0.1}{0.2\times3\left(\frac{1}{0.2}s+1\right)\left(\frac{1}{3}s+1\right)} \approx \frac{0.1}{0.2\times3\left(\frac{1}{0.2}s+1\right)}$$

$$= \frac{0.1/3}{s+0.2} = \frac{B(s)}{A(s)}$$

The PI controller parameters then are obtained with $\omega_n = 5\times a = 5\times 0.2 = 1.$ and $\xi = 0.707$, as,

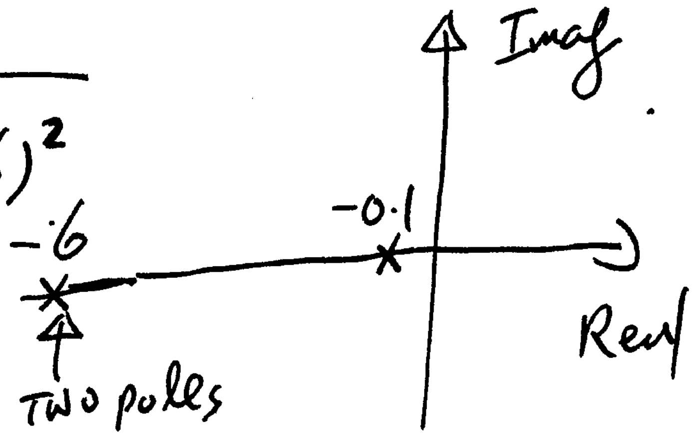
$$c_1 = \frac{1.4014\times\omega_n - 0.2}{0.1/3} = \frac{3\times(1.4014-0.2)}{0.1}$$

$$c_0 = \frac{\omega_n^2}{0.1/3} = \frac{3}{0.1} = 30.$$

$$K_c = c_1, \quad \tau_I = \frac{c_1}{c_0}.$$

6.2, (2).

$$G(s) = \frac{-5}{(s+0.1)(s+6)^2}$$



The two poles at -6 are much further away from the imaginary axis. Thus we neglect them to obtain the approximate model.

$$G(s) = \frac{-5}{0.1\times 6^2\left(\frac{1}{0.1}s+\text{[redacted]}\right)\left(\frac{1}{6}s+1\right)^2} \approx \frac{-5/36}{(s+0.1)}$$

where $\left(\frac{1}{\frac{1}{6}s+1}\right)^2 \approx 1$ $\qquad$ $\frac{1}{6}$ is much smaller ~~be in~~

comparison with $\frac{1}{0.1}$.

Thus, $G(s) = \dfrac{-5/36}{(s+0.1)} = \dfrac{b}{s+a}$.

$a = 0.1$, $b = -5/36$.

$c_1 = \dfrac{2\xi\omega_n - a}{b}$, $c_0 = \dfrac{\omega_n^2}{b}$.

$K_c = c_1 = \dfrac{2\times0.707\times0.1\times5 - 0.1}{-5/36}$.

$\tau_I = \dfrac{c_1}{c_0} = \dfrac{2\times0.707\times0.1\times5 - 0.5}{(5\times0.1)^2}$.

0.2 (3) excercise.

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0.3. (1)

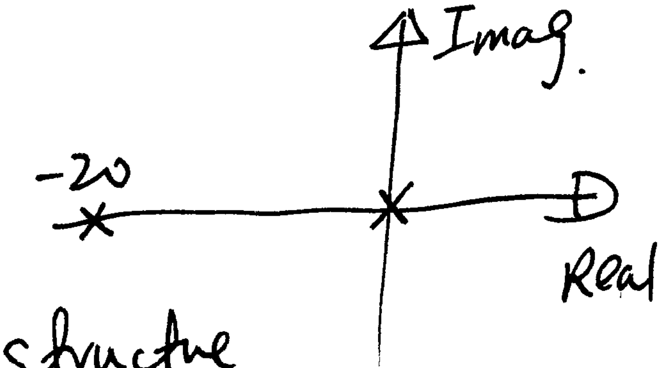
$C(s) = K_c + \dfrac{K_c}{\tau_I s} + K_c \tau_D s$ is an ideal PID, and we write it as

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s}$$

To design a PID controller, we need to use a second order model.

In Q3(1), it is a second order model.

To simplify the computation, we will use pole-zero cancellation. There are two poles $s_1 = 0$, $s_2 = -20$.



We will choose the PID controller structure to cancel the pole at -20.

Note that

$$G(s) = \frac{10}{s(s+20)} = \frac{B(s)}{A(s)}.$$

$$C(s) = \frac{c_2(s+\gamma_1)(s+20)}{s} = \frac{P(s)}{L(s)}$$

$$\frac{B(s)}{A(s)}\,\frac{P(s)}{L(s)} = \frac{10}{s\,\cancel{(s+20)}}\;\frac{c_2(s+\gamma_1)\,\cancel{(s+20)}}{s}$$

$$= \frac{10\,c_2(s+\gamma_1)}{s^2}$$

The actual closed-loop polynomial with pole-zero cancellation is

$$s^2 + 10\,c_2(s+\gamma_1) = s^2 + 2\xi\omega_n s + \omega_n^2.$$

$$= s^2 + 2\times 0.707\times 5s + 25$$

s^2: $1 = 1$

s: $10c_2 = 7.07$ $\qquad c_2 = \dfrac{7.07}{10}$

s^0: $10c_2\gamma_1 = 25$ $\qquad \gamma_1 = \dfrac{25}{7.07}.$

The controller transfer function is

$$C(s) = \frac{c_2(s+\gamma_1)(s+20)}{s} = \frac{c_2 s^2 + c_2(\gamma_1+20)s + 20\times c_2\times\gamma_1}{s}$$

$$= \frac{c_2 s^2 + c_2(\gamma_1 + 20)s + 20 c_2 \gamma_1}{s} = \frac{c_2 s^2 + c_1 s + c_0}{s}$$

$$K_c = c_2(\gamma_1 + 20) = c_1$$

$$\tau_I = \frac{c_2(\gamma_1 + 20)}{20 c_2 \gamma_1} = \frac{\gamma_1 + 20}{20} = \frac{c_1}{c_0}$$

$$\tau_D = \frac{c_2}{c_1} = \frac{c_2}{c_2(\gamma_1 + 20)} = \frac{1}{\gamma_1 + 20}$$

The closed-loop poles are at

$$s_{1,2} = -\xi\omega_n \pm j\,\omega_n\sqrt{1-\xi^2}$$

and $s_3 = -20$

To verify the answer, we consider the closed-loop ~~transfer function~~ polynomial without pole zero cancellation.

~~G(s)~~

$$A(s)L(s) + B(s)P(s)$$

$$= s(s+20)s + 10\,C_2(s+\gamma_1)(s+20)$$

$$= (s+20)\left(s^2 + 10C_2(s+\gamma_1)\right)$$

$$= (s+20)(s^2 + 2\xi\omega_n s + \omega_n^2).$$

0.3 (2) — exercise.

0.3 (3). Pade approximation of the time delay leads to

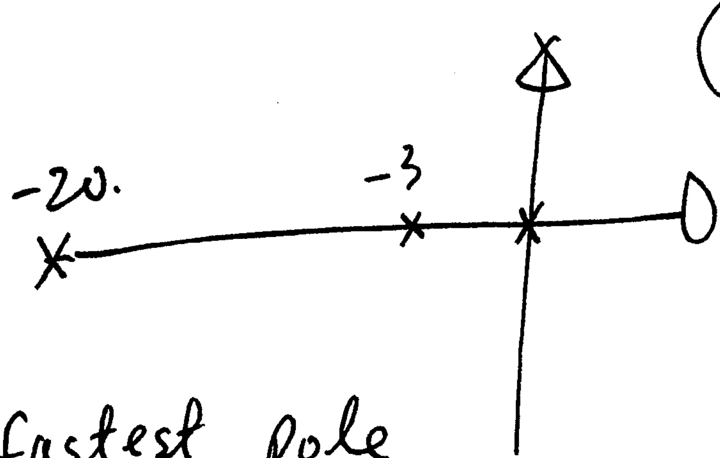
$$G(s) = \frac{e^{-0.1s}}{s(s+3)} \approx \frac{1}{s(s+3)} \cdot \frac{-0.1s+2}{0.1s+2}$$

$$= \frac{10(-0.1s+2)}{s(s+3)(s+20)}$$

Three poles:

$s_1 = 0$, $s_2 = -3$,

$s_3 = -20$.



We neglect the fastest pole at $s = -20$. However, we need to work out the steady-state condition.

$$G(s) = \frac{10(-0.1s+2)}{3 \times 20 s\left(\frac{1}{3}s+1\right)\left(\frac{1}{20}s+1\right)} = \frac{10(-0.1s+2)}{3 \times 20 s\left(\frac{1}{3}s+1\right)}.$$

where $\dfrac{1}{\frac{1}{20}s+1} \approx 1$ because $\frac{1}{20} \ll \frac{1}{3}$

$\dfrac{1}{s+0}$ ← the time constant for this is infinite.

The approximate model is

$$G(s) = \frac{0.5(-0.1s+2)}{s(s+3)} = \frac{B(s)}{A(s)}$$

We will cancel the ~~closed-loop~~ model's pole at -3. to simplify the computation. To this end, we choose the PID controller as

$$C(s) = \frac{c_2 (s+\gamma_1)(s+3)}{s} = \frac{P(s)}{L(s)}.$$

$$\frac{B(s)}{A(s)} \frac{P(s)}{L(s)} = \frac{0.5(-0.1s+2)}{s\cancel{(s+3)}} \cdot \frac{c_2(s+\gamma_1)\cancel{(s+3)}}{s}$$

The closed-loop transfer function with cancelled pole becomes

$$\frac{\dfrac{0.5 c_2(-0.1s+2)(s+\gamma_1)}{s^2}}{1 + \dfrac{0.5(-0.1s+2)c_2(s+\gamma_1)}{s^2}} =$$

$$= \frac{0.5C_2(-0.1s+2)(s+\gamma_1)}{s^2+0.5C_2(-0.1s+2)(s+\gamma_1)}$$

$$= \frac{0.5C_2(-0.1s+2)(s+\gamma_1)}{s^2+0.5C_2(-0.1s^2+(2-0.1\gamma_1)s+2\gamma_1)}$$

$$= \frac{0.5C_2(-0.1s+2)(s+\gamma_1)}{(1-0.05C_2)s^2+0.5C_2(2-0.1\gamma_1)s+0.5C_2\times 2\gamma_1}$$

$$= \frac{\frac{0.5C_2}{1-0.05C_2}(-0.1s+2)(s+\gamma_1)}{s^2+\frac{0.5C_2(2-0.1\gamma_1)}{1-0.05C_2}s+\frac{C_2\gamma_1}{1-0.05C_2}}$$

where we assume $(1-0.05C_2)>0$.

Here, the closed-loop polynomial with pole-zero cancellation becomes:

$$s^2+\frac{0.5C_2(2-0.1\gamma_1)}{1-0.05C_2}s+\frac{C_2\gamma_1}{1-0.05C_2}$$

which is equal to the desired closed-loop polynomial:

$$s^2 + 2\xi\omega_n s + \omega_n^2.$$

s^2: $\quad 1 = 1.$

s: $$\frac{0.5c_2(2-0.1\gamma_1)}{1-0.05c_2} = 2\xi\omega_n. \quad (1)$$

s^0: $$\frac{c_2\gamma_1}{1-0.05c_2} = \omega_n^2. \quad (2)$$

From (1)

$$c_2 - 0.05c_2\gamma_1 = 2\xi\omega_n - 0.1\xi\omega_n c_2 \quad \cdots (3)$$

or $$(1+0.1\xi\omega_n)c_2 - 0.05c_2\gamma_1 = 2\xi\omega_n \quad (4)$$

From (2)

$$c_2\gamma_1 = \omega_n^2 - 0.05\omega_n^2 c_2 \quad (5)$$

substituting (5) into (4) gives

$$(1+0.1\xi\omega_n)C_2 - 0.05\left(\omega_n^2 - 0.05\omega_n^2 C_2\right) = 2\xi\omega_n$$

$$(1+0.1\xi\omega_n + 0.05^2\omega_n^2)C_2 = 2\xi\omega_n + 0.05\omega_n^2$$

$$C_2 = \frac{2\xi\omega_n + 0.05\omega_n^2}{1+0.1\xi\omega_n + 0.05^2\omega_n^2} > 0 \quad \checkmark$$

From 5, we find

$$c_2\gamma_1 = \omega_n^2 - 0.05\,\omega_n^2\, C_2$$

The PID cutroller ~~shou~~ is

$$C(s) = \frac{c_2(s+\gamma_1)(s+3)}{s}$$

We need to check the assumption

$$1 - 0.05C_2 > 0$$

$$= 1 - \frac{0.05\,(2\xi\omega_n + 0.05\omega_n^2)}{1+0.1\xi\omega_n + 0.05^2\omega_n^2}$$

$$= \frac{1 + \cancel{0.1\zeta\omega_n} + \cancel{0.05^2\omega_n^2} - \cancel{0.1\zeta\omega_n} - \cancel{0.05^2\omega_n^2}}{1 + 0.1\zeta\omega_n + 0.05^2\omega_n^2}$$

$$= \frac{1}{1 + 0.1\zeta\omega_n + 0.05^2\omega_n^2} > 0.$$

The closed-loop poles are determined by the zeros of the polynomial (with páde approximation):

$$s(s+3)(s+20)s + (c_2 s^2 + c_1 s + c_0)(10(-0.1s+2))$$

We have four closed-loop poles. We ~~They are close~~ can verify them using MATLAB function roots([ ])

0.3(4) ~ exercise.

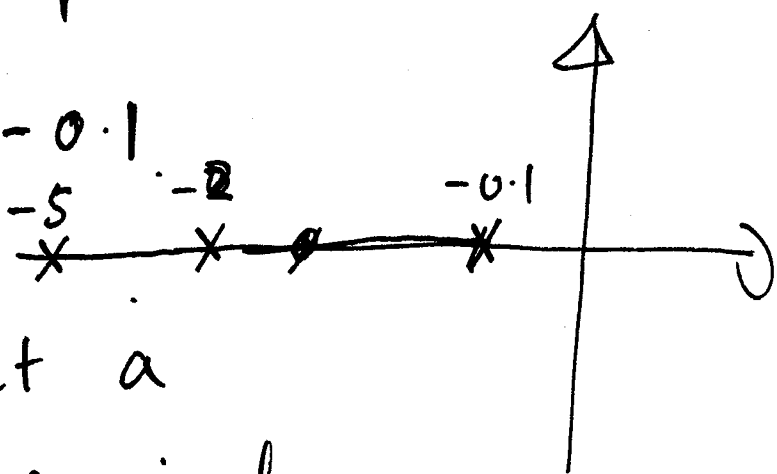
0.4(1). Páde approximation

$$G(s) = \frac{e^{-s}}{(s+5)(s+2)(s+0.1)}$$

$$\approx \frac{1}{(s+5)(s+2)(s+0.1)} \frac{-s+2}{s+2}.$$

The open-loop poles are

$-5,\ -2,\ -2,\ -0.1$



We need to get a second order approximate model, so we neglect the poles at -5,

and one pole at -2.

$$G(s) = \frac{-s+2.}{0.1 \times 2^2 \times 5 \left(\frac{1}{5}s+1\right)\left(\frac{1}{2}s+1\right)\left(\frac{1}{2}s+1\right)\left(\frac{1}{0.1}s+1\right)}$$

We need to be careful in selecting the desired closed-loop poles because ~~the~~ one of the poles neglected is equal to the pole in the design model.

We can check stability using Routh-Hurwitz stability criterion.

$$G(s) \approx \frac{-s+2}{(s+2)(s+0.1)} = \frac{B(s)}{A(s)}$$

For pole-zero cancellation, we chose controller with filter

$$C(s) = \frac{c_2(s+\gamma_1)(s+2)}{s(s+l_0)} = \frac{P(s)}{L(s)}$$

$$\frac{P(s)}{L(s)}\frac{B(s)}{A(s)} = \frac{c_2(s+\gamma_1)\cancel{(s+2)}}{s(s+l_0)}\frac{-s+2}{\cancel{(s+2)}(s+0.1)}$$

The closed-loop polynomial with pole-zero cancellation becomes

$$s(s+l_0)(s+0.1) + c_2\,(s+\gamma_1)(-s+2).$$

$$= s^3 + (l_0+0.1)s^2 + 0.1s - c_2 s^2 + c_2(2-\gamma_1)s + 2c_2\gamma_1$$

The desired closed-loop polynomial is selected as

$$(s^2 + 2\xi\omega_n s + \omega_n^2)(s+\omega_n).$$

$$= s^3 + (2\xi+1)\,\omega_n s^2 + (2\xi+1)\,\omega_n^2 s + \omega_n^3$$

s^3: $1 = 1.$

s^2 $l_0 + 0.1 - c_2 = (2\xi+1)\,\omega_n.$

s^1: $0.1 + 2c_2 - c_2\gamma_1 = (2\xi+1)\,\omega_n^2.$

s^0: $2c_2\gamma_1 = \omega_n^3$

$$c_2\gamma_1 = \frac{\omega_n^3}{2.}$$

then c_2

$$c_2 = \frac{(2\xi+1)\omega_n^2 + c_2\gamma_1 - 0.1}{2.}$$

$$= \frac{(2\xi+1)\omega_n^2 + \frac{\omega_n^3}{2} - 0.1.}{2}$$

$$l_0 = (2\xi+1)\omega_n + c_2 - 0.1.$$

$$= (2\xi+1)\omega_n + \frac{(2\xi+1)\omega_n^2 + \frac{\omega_n^3}{2} - 0.1}{2} - 0.1.$$

leave as exercise to check closed-loop stability with pádé approximate using Routh-Harwitz stability criterion.

0.4 (2) Exercise -