Lecture 1

FINITE ELEMENT METHOD IN VIBRATIONAL ANALYSIS

1.1 INTRODUCTION

The increasing complexity of structures and sophistication of digital computers have been instrumental in the development of new methods of analysis, particularly of the finite element method.

The idea behind the finite element method is to provide a formulation which can exploit computer automation for the analysis of irregular systems. To this end, the method regards a complex structure as an assemblage of finite elements, where every such element is part of a continuous structural member. By requiring that the displacements be compatible and the internal forces in balance at certain points shared by several elements, where the points are known as *nodes*, the entire structure is compelled to act as one entity.

1.2 HISTORICAL GALLERY

Charles HERMITE (1822-1901), a French mathematician, Foreign Corresponding Member (since 1857) and Foreign Honourable Member (since 1895) of the St.Petersburg Academy of Sciences. His work in the theory of functions includes the application of elliptic functions to provide the first solution to the general equation of the fifth degree, the quintic equation.

He was appointed as professor at the Ècole Normale, Paris in 1869. In 1970 he became professor of higher algebra at the Sorbonne.

In 1873 Hermite published the first proof that e is a transcendental number; i.e. it is not the root of any algebraic equation with rational coefficients. Hermite was a major figure in the development of the theory of algebraic forms, the arithmetical theory of quadratic forms, and the theories of elliptic and Abelian functions. He first studied the representation of integers in what are now called Hermitian forms. His famous solution of the general quintic equation appeared in 1858. Many late-19th-century mathematicians first gained recognition for their work largely through the encouragement and publicity supplied by Hermite.

In our days the Hermite polynomials are widely used in the finite element methods.

1.3 VIBRATION OF ONE-DIMENSIONAL TRUSS ELEMENTS

1.3.1 Stiffness Matrix

Let us consider a one-dimensional truss element in axial vibration, such as shown in Fig. 1.1. It has the following nodal degrees-of-freedom: $\{u_1(t) \quad u_2(t)\}^T$.

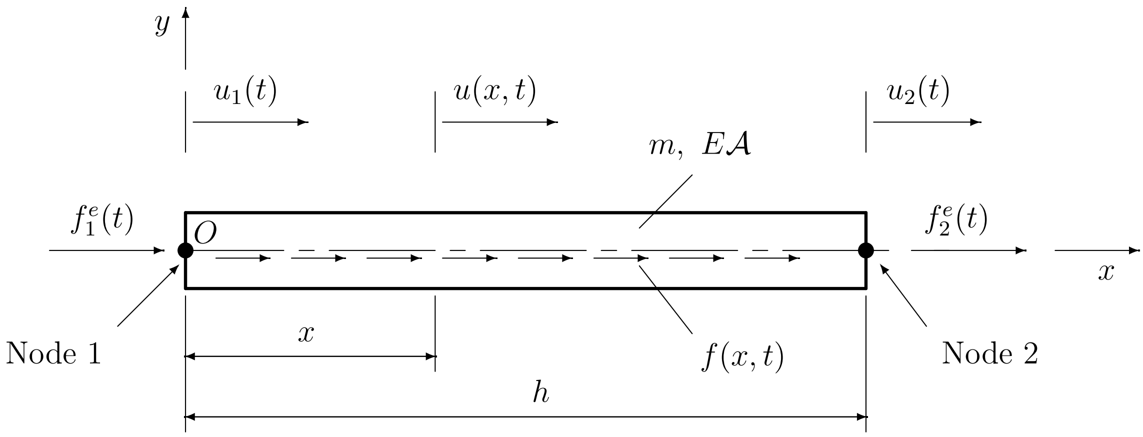

Figure 1.1: A uniform truss finite element (bar element for axial vibration).

The axial stiffness within the particular element can be assumed to be constant, so that the differential equation for the axial displacement $u(x)$ is

$$E\mathcal{A}\,\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \qquad (0 < x < h). \tag{1.1}$$

Integrating Eq. (1.1) twice, we obtain

$$u(x,t) = c_1(t)\,x + c_2(t), \tag{1.2}$$

where $c_1(t)$ and $c_2(t)$ are some functions of time to be determined. It can be seen from Fig. 1.1 that at $x = 0$ the axial displacement $u(x,t)$ is equal to the nodal displacement $u_1(t)$ and at $x = h$ it is equal to the nodal displacement $u_2(t)$. Hence, using Eq. (1.2), we can write

$$u(0,t) = u_1(t) = c_2(t), \qquad u(h,t) = c_1(t)\,h + c_2(t) = u_2(t). \tag{1.3}$$

Equations (1.3) have the solution

$$c_1(t) = \frac{u_2(t) - u_1(t)}{h}; \qquad c_2(t) = u_1(t), \tag{1.4}$$

so that inserting the functions $c_1(t)$ and $c_2(t)$ just obtained into Eq. (1.2), we obtain the expression for the axial displacement

$$\boxed{u(x,t) = \left(1 - \frac{x}{h}\right) u_1(t) + \frac{x}{h}\, u_2(t).} \tag{1.5}$$

Eq. (1.5) can be rewritten in the form

$$u(x,t) = H_1(x)\, u_1(t) + H_2(x)\, u_2(t), \qquad \text{where} \tag{1.6}$$

$$H_1(x) = 1 - \frac{x}{h}; \qquad H_2(x) = \frac{x}{h} \tag{1.7}$$

are known as **shape functions**, or **interpolation functions**.

They are plotted in Fig. 1.2.

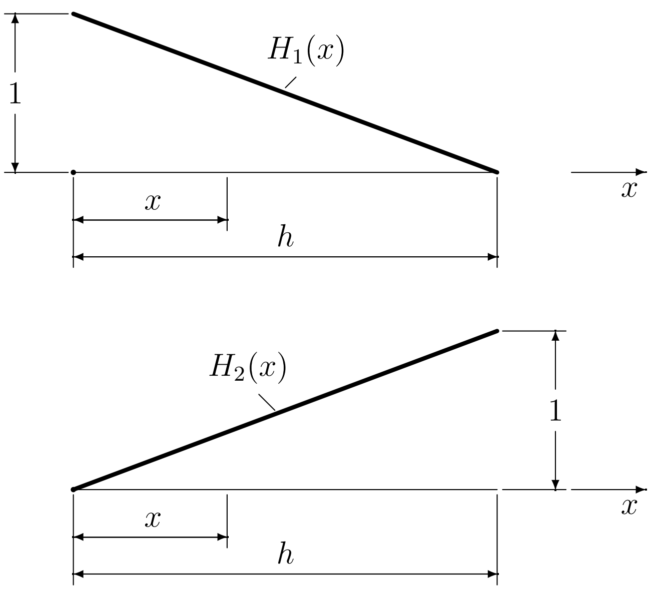

Figure 1.2: Shape functions for a truss finite element.

The term interpolation functions can be justified, as the functions $H_1(x)$ and $H_2(x)$ permit us to determine the displacement at any distance x from the left end through an interpolation between the nodal displacements u_1 and u_2.

The displacement $u(x)$ is related to the nodal forces through the boundary conditions

$$E\mathcal{A}\, \frac{\partial u(x,t)}{\partial x}\bigg|_{x=0} = -f_1^e(t); \qquad E\mathcal{A}\, \frac{\partial u(x,t)}{\partial x}\bigg|_{x=h} = f_2^e(t), \tag{1.8}$$

so that, using Eq. (1.5), we have

$$E\mathcal{A}\,\frac{u_2(t)-u_1(t)}{h} = -f_1^e(t); \qquad E\mathcal{A}\,\frac{u_2(t)-u_1(t)}{h} = f_2^e(t). \tag{1.9}$$

Eqs. (1.9) can be rewritten in the matrix form

$$[k^e]\{d\} = \{f^e\} \tag{1.10}$$

where superscript e denotes the *element level* and

$$\{u^e\} = \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} \qquad \{f^e\} = \begin{Bmatrix} f_1^e(t) \\ f_2^e(t) \end{Bmatrix} \tag{1.11}$$

are the nodal displacement vector and nodal force vector, respectively, and

Stiffness Matrix for a 1D Truss FE:
(Global and FE Local Coordinate Systems)

$$[k^e] = \frac{E\mathcal{A}}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.12}$$

is the desired element stiffness matrix.

The used technique is known as the direct method, but the same result can be obtained if we employ energy relationships. This alternative technique is presented below.

The general expression for potential energy of the finite element can be written in the matrix form as

$$\mathcal{V}^e = \frac{1}{2}\{u^e(t)\}^T[k^e]\{u^e(t)\}. \tag{1.13}$$

Also, this strain energy of the finite element can be explicitly calculated, using Eq. (1.5), as follows:

$$\begin{aligned}
\mathcal{V}^e(t) &= \frac{1}{2}\int_0^h E\mathcal{A}\left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx = \frac{1}{2}\int_0^h E\mathcal{A}\left[-\frac{1}{h}\,u_1(t) + \frac{1}{h}\,u_2(t)\right]^2 dx = \\
&= \frac{1}{2}\frac{E\mathcal{A}}{h}\left(u_1^2 - 2u_1u_2 + u_2^2\right) = \frac{1}{2}\frac{E\mathcal{A}}{h}\left[(u_1-u_2)u_1 + (-u_1+u_2)u_2\right] = \\
&= \frac{1}{2}\frac{E\mathcal{A}}{h}[u_1 \quad u_2]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},
\end{aligned}$$

or

$$\mathcal{V}^e(t) = \frac{1}{2}\left\{\begin{array}{c} u_1 \\ u_2 \end{array}\right\}^T \left[\ \frac{E\mathcal{A}}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\ \right]\left\{\begin{array}{c} u_1 \\ u_2 \end{array}\right\}. \tag{1.14}$$

Comparing Eqs. (1.13) and (1.14) the stiffness matrix $[k^e]$ can be again identified as

$$[k^e] = \frac{E\mathcal{A}}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

1.3.2 Mass Matrix

Let us write down the general expression for the kinetic energy of the finite element in the matrix form

$$\mathcal{T}^e = \frac{1}{2}\{d(t)\}^T[m^e]\{d(t)\}. \tag{1.15}$$

From the other side, the kinetic energy $\mathcal{T}^e$ can be calculated, using expressions (1.5) as follows:

$$\begin{aligned}
\mathcal{T}^e(t) &= \frac{1}{2}\int_0^h m\left[\frac{\partial u(x,t)}{\partial t}\right]^2 dx = \frac{1}{2}\int_0^h m\left[\left(1-\frac{x}{h}\right)\frac{du_1(t)}{dt} + \left(\frac{x}{h}\right)\frac{du_2(t)}{dt}\right]^2 dx = \\
&= \frac{1}{2}\frac{mh}{3}\left(\dot{u}_1^2 + \dot{u}_1\dot{u}_2 + \dot{u}_2^2\right) = \frac{1}{2}\frac{mh}{3}\frac{1}{2}\left[\left(2\dot{u}_1 + \dot{u}_2\right)\dot{u}_1 + \left(\dot{u}_1 + 2\dot{u}_2\right)\dot{u}_2\right] = \\
&= \frac{1}{2}\frac{mh}{3}\frac{1}{2}\left[\dot{u}_1 \quad \dot{u}_2\right]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\left\{\begin{array}{c} \dot{u}_1 \\ \dot{u}_2 \end{array}\right\},
\end{aligned}$$

or

$$\mathcal{T}^e(t) = \frac{1}{2}\left\{\begin{array}{c} \dot{u}_1 \\ \dot{u}_2 \end{array}\right\}^T \left[\ \frac{mh}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\ \right]\left\{\begin{array}{c} \dot{u}_1 \\ \dot{u}_2 \end{array}\right\}. \tag{1.16}$$

If we compare Eqs. (1.16) and (1.15), the mass matrix $[m]$ can be identified as

Consistent Mass Matrix for a 1D Truss FE:
(Global and FE Local Coordinate Systems)

$$[m^e] = \frac{mh}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{1.17}$$

1.3.3 Force Vector

The force vector

$$\{f^e(t)\} = \left\{ \begin{array}{c} f_1^e(t) \\ f_2^e(t) \end{array} \right\}$$

can be derived from the virtual work expression. For the truss element, loaded with the distributed force $f(x,t)$, the virtual work δW^e is calculated as

$$\begin{aligned} \delta W^e(t) &= \int_0^h f(x,t)\, \delta u(x,t)\, dx = \\ &= \int_0^h f(x,t) \left[\left(1 - \frac{x}{h}\right) \delta u_1(t) + \left(\frac{x}{h}\right) \delta u_2(t) \right] dx = \\ &= \left[\int_0^h f(x,t) \left(1 - \frac{x}{h}\right) dx \right] \delta u_1(t) + \left[\int_0^h f(x,t) \left(\frac{x}{h}\right) dx \right] \delta u_2(t). \end{aligned}$$

By expressing this result in matrix form as

$$\begin{aligned} \delta W(t) &= \{\delta d(t)\}^T \{f^e(t)\} = \\ &= f_1^e(t)\, \delta u_1(t) + f_2^e(t)\, \delta u_2(t), \end{aligned}$$

the equivalent nodal forces can be identified as

Nodal Forces for a 1D Truss FE:

$$\begin{aligned} f_1^e(t) &= \int_0^h f(x,t) \left(1 - \frac{x}{h}\right) dx = \int_0^h f(x,t)\, H_1(x)\, dx \\ f_2^e(t) &= \int_0^h f(x,t) \left(\frac{x}{h}\right) dx = \int_0^h f(x,t)\, H_2(x)\, dx \end{aligned} \tag{1.18}$$

1.4 VIBRATION OF STRUCTURES MODELLED WITH 2D TRUSS ELEMENTS

1.4.1 Degrees of Freedom and Coordinate Transformations

One-dimensional truss elements can be connected by pin joints. Each member of such a structure supports the external load through its axial force and it does not undergo the bending deformation. An example of a two-dimensional truss structure is presented in Figure 1.3.

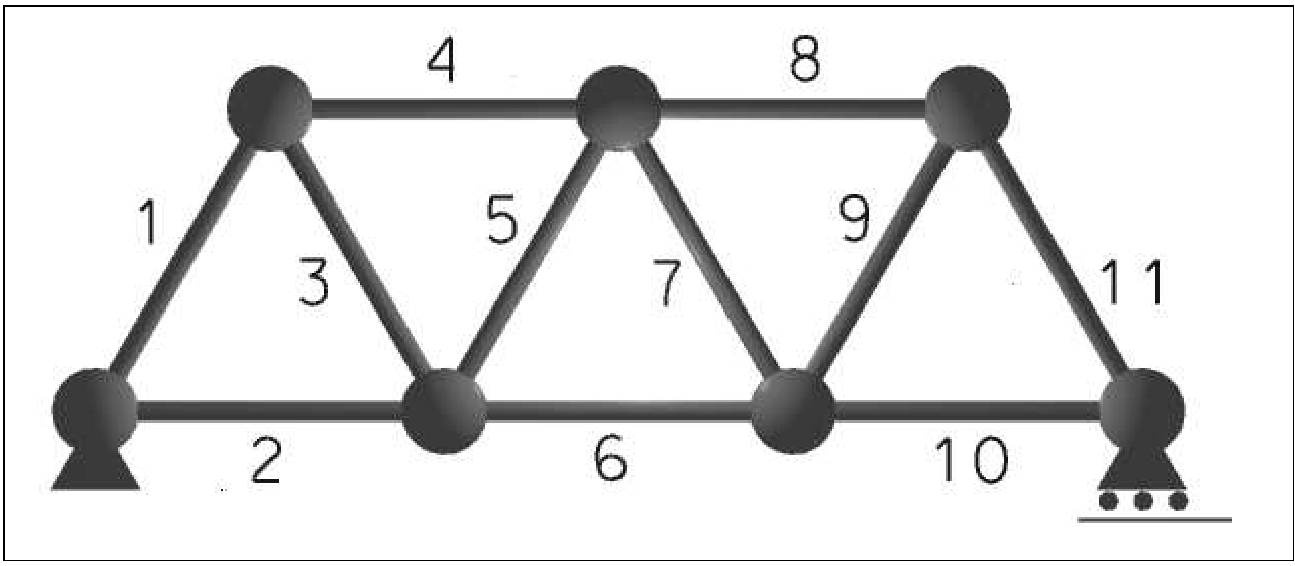

Figure 1.3: An example of a plane truss structure.

The stiffness matrix for a truss member is still given in Eq. (1.12), but the number of degrees-of-freedom for a typical element used in a 2D truss structure should be extended to allow the model to take into account all possible displacements of its nodes, including the nodal displacements which are not necessarily along the truss element. Therefore, the minimal set of required degrees-of-freedom is

$$\{u^e(t)\} = \{u_1(t) \quad u_2(t) \quad u_3(t) \quad u_4(t)\}^T$$

where superscript e denotes the element level and the meaning of all displacements u_1, u_2, u_3, and u_4 is explained in Fig. 1.4.

Important remark: It should be noted, that as an alternative it was possible to use symbol u for the horizontal displacements and symbol v for the vertical displacements. In this case we could denote the element degrees of freedom as u_1, v_1, u_2 and v_2. But keeping in mind that serious problems can only be solved using the computer power, we sacrifice the convenience in the physical meaning and in favour of the programming convenience we denote all element displacements with the same symbol u, distinguishing different DOF with the appropriate subscripts 1, 2, 3 and 4. It is because from the programming point of view, all DOF in the structure are "equal in rights", and with the used convention the processing of all DOF in the element can be done within one loop where the number of cycles is equal to the number of DOF. This convention will be used in this text in the other cases, for example, for beam and frame finite elements.

The corresponding to this set of degrees-of-freedom stiffness matrix is

$$[k^e] = \frac{E\mathcal{A}}{h} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.19}$$

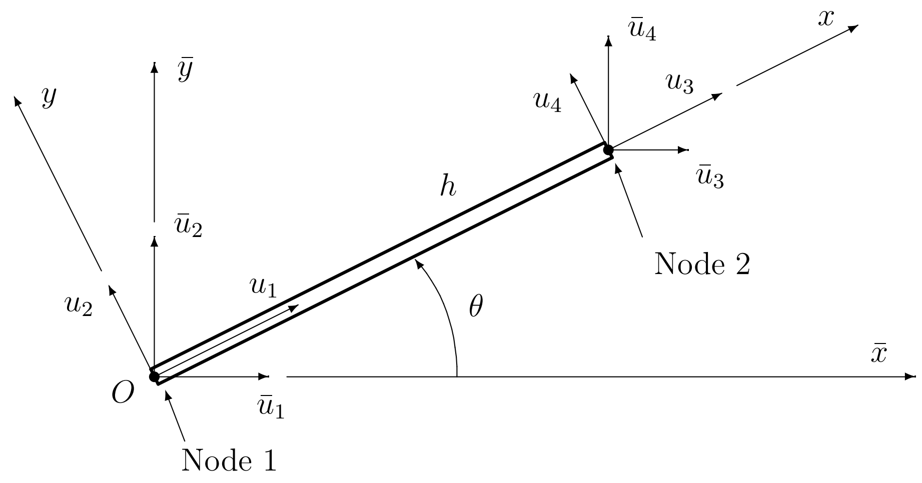

Figure 1.4: A two-dimensional truss element.

The second and fourth columns and rows of the stiffness matrix associated with the transverse displacements u_2 and u_4 are zeros since the truss member has axial deformation only.

For the description of motion of all finite elements in truss structure with different axial orientation of truss members, the transformation of nodal displacements expressed in *local coordinate systems* xy to the common *global coordinate system* $\bar{x}\bar{y}$ is required.

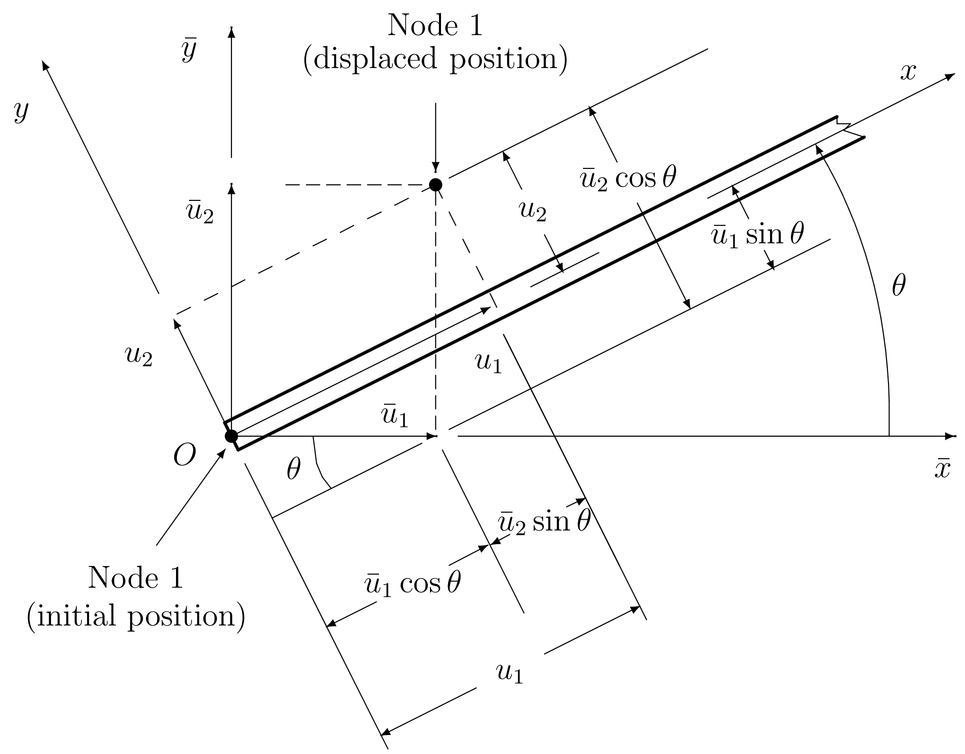

Figure 1.5: Displacements of the Node 1 in the local xy and global $\bar{x}\bar{y}$ coordinate systems.

As shown in Fig. 1.5 at the example of the displacements for the Node-1 the two coordinate systems are related by the *transformation equations*:

$$\begin{aligned} u_1 &= \bar{u}_1 \cos\theta + \bar{u}_2 \sin\theta; \\ u_2 &= \bar{u}_2 \cos\theta - \bar{u}_1 \sin\theta. \end{aligned}$$

The same relationships hold for the second set of nodal displacements u_3, u_4, $\bar{u}_3$, and $\bar{u}_4$:

$$\begin{aligned} u_3 &= \bar{u}_3 \cos\theta + \bar{u}_4 \sin\theta; \\ u_4 &= \bar{u}_4 \cos\theta - \bar{u}_3 \sin\theta. \end{aligned}$$

All these formulas for the coordinate transformation between the local coordinate system parameters (finite element displacements in xy) and global coordinate system parameters (finite element displacements in $\bar{x}\bar{y}$) can be easily put in the matrix form:

$$\left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \right\} = \left[\begin{array}{rrrr} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{array} \right] \left\{ \begin{array}{c} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{array} \right\}, \tag{1.20}$$

where $c = \cos\theta$ and $s = \sin\theta$. It is convenient to write this equation in the compact matrix form as

$$\{u^e(t)\} = [T]\{\bar{u}^e(t)\}.$$

It should be mentioned one remarkable property of the matrix $[T]$: because it represents a transformation between two orthogonal systems, $[T]^{-1} = [T]^T$ (it is said that matrix [T] is orthonormal). Therefore, the inverse transformation between $\bar{x}\bar{y}$ and xy is given with the matrix equation:

$$\left\{ \begin{array}{c} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \end{array} \right\} = \left[\begin{array}{rrrr} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{array} \right]^{-1} \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \right\} = \left[\begin{array}{rrrr} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \right\},$$

or in the compact form:

$$\{\bar{u}^e(t)\} = [T]^{-1}\{u^e(t)\} = [T]^T\{u^e(t)\}. \tag{1.21}$$

1.4.2 FE Stiffness Matrix in the Global Coordinate System

To transform the element stiffness matrix from xy coordinate system to $\bar{x}\bar{y}$ coordinate system, consider the concept of strain energy. The strain energy is expressed as

$$\mathcal{V}^e = \tfrac{1}{2}\{u^e\}^T [k^e] \{u^e\} \tag{1.22}$$

in terms of the xy-coordinate system. If we substitute Eq. (1.21) into Eq. (1.22), we obtain

$$\mathcal{V}^e = \tfrac{1}{2}\{\bar{u}^e\}^T \ [T]^T \ [k^e] \ [T] \ \{\bar{u}^e\}. \tag{1.23}$$

The strain energy is now expressed in terms of the $\bar{x}\bar{y}$-coordinate system.

$$\mathcal{V}^e = \tfrac{1}{2}\{\bar{u}^e\}^T \left[\bar{k}^e\right] \{\bar{u}^e\}, \tag{1.24}$$

in which $\left[\bar{k}^e\right]$ is the transformed element stiffness matrix in terms of the $\bar{x}\bar{y}$-coordinate system. The strain energy in Eq. (1.24) should be the same as that in Eq. (1.23) because the strain energy is independent of the coordinate system. Equating Eq. (1.23) to Eq. (1.24) shows that

$$\left[\bar{k}^e\right] = [T]^T \ [k^e] \, [T]. \tag{1.25}$$

Substitution of Eqs. (1.19) and (1.20) into Eq. (1.25) gives

$$\left[\bar{k}^e\right] = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}^T \left[\frac{E\mathcal{A}}{h} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right] \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}.$$

The resulting transformed stiffness matrix has the following form:

Stiffness Matrix for a 2D Truss FE:

(Global Coordinate System)

$$\left[\bar{k}^e\right] = \frac{E\mathcal{A}}{h} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \tag{1.26}$$

for the nodal degrees-of-freedom

$$\{\bar{u}^e(t)\} = \{\bar{u}_1(t) \quad \bar{u}_2(t) \quad \bar{u}_3(t) \quad \bar{u}_4(t)\}^T.$$

The element stiffness matrix can be assembled into the global matrix as usual for the shared nodal points.

1.4.3 FE Mass Matrix in the Global Coordinate System

The element mass matrix for the plane truss member can be calculated using the same coordinate transformation. Using the kinetic energy expression, as similar to the strain energy expression for derivation of the element stiffness matrix, we can get

$$[\bar{m}^e] = [T]^T \; [m^e] \, [T].$$

where the mass element matrix $[m^e]$ has the form similar to the one, given by Eq.(1.17), but adjusted for the current DOF vector

$$\{u^e(t)\} = \{\bar{u}_1(t) \quad \bar{u}_2(t) \quad \bar{u}_3(t) \quad \bar{u}_4(t)\}^T.$$

Carrying out this matrix multiplication, we obtain

$$[\bar{m}^e] = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}^T \left[\frac{mh}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right] \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}$$

Performing all required multiplications, we obtain:

Consistent Mass Matrix for a 2D Truss FE:

(Global Coordinate System)

$$[\bar{m}^e] = \frac{mh}{6} \begin{bmatrix} 2c^2 & 2cs & c^2 & cs \\ 2cs & 2s^2 & cs & s^2 \\ c^2 & cs & 2c^2 & 2cs \\ cs & s^2 & 2cs & 2s^2 \end{bmatrix} \tag{1.27}$$

1.4.4 Force Vector

The force vector

$$\{f^e(t)\} = \begin{Bmatrix} f_1^e(t) \\ f_2^e(t) \\ f_3^e(t) \\ f_4^e(t) \end{Bmatrix}$$

can be derived from the virtual work expression. For the truss element, loaded with the distributed force $f(x,t)$, the virtual work δW^e is calculated as

$$\begin{aligned} \delta W^e(t) &= \int_0^h f(x,t)\,\delta u(x,t)\,dx = \\ &= \int_0^h f(x,t)\left[\left(1-\frac{x}{h}\right)\,\delta u_1(t) + 0\cdot\delta u_2(t) + \left(\frac{x}{h}\right)\,\delta u_3(t) + 0\cdot\delta u_4(t)\right]\,dx = \\ &= \left[\int_0^h f(x,t)\left(1-\frac{x}{h}\right)\,dx\right]\,\delta u_1(t) + \left[\int_0^h f(x,t)\left(\frac{x}{h}\right)\,dx\right]\,\delta u_2(t). \end{aligned}$$

By expressing this result in matrix form as

$$\begin{aligned} \delta W^e(t) &= \{\delta u^e(t)\}^T\,\{f^e(t)\} = \\ &= f_1^e(t)\,\delta u_1(t) + f_2^e(t)\,\delta u_2(t) + f_3^e(t)\,\delta u_3(t) + f_4^e(t)\,\delta u_4(t), \end{aligned}$$

the equivalent nodal forces can be identified as

<u>Nodal Forces for a 2D Truss FE:</u>

$$f_1^e(t) = \int_0^h f(x,t)\left(1-\frac{x}{h}\right)\,dx = \int_0^h f(x,t)\,H_1(x)\,dx$$

$$f_3^e(t) = \int_0^h f(x,t)\left(\frac{x}{h}\right)\,dx = \int_0^h f(x,t)\,H_2(x)\,dx \tag{1.28}$$

$$f_2^e(t) = f_4^e(t) = 0$$

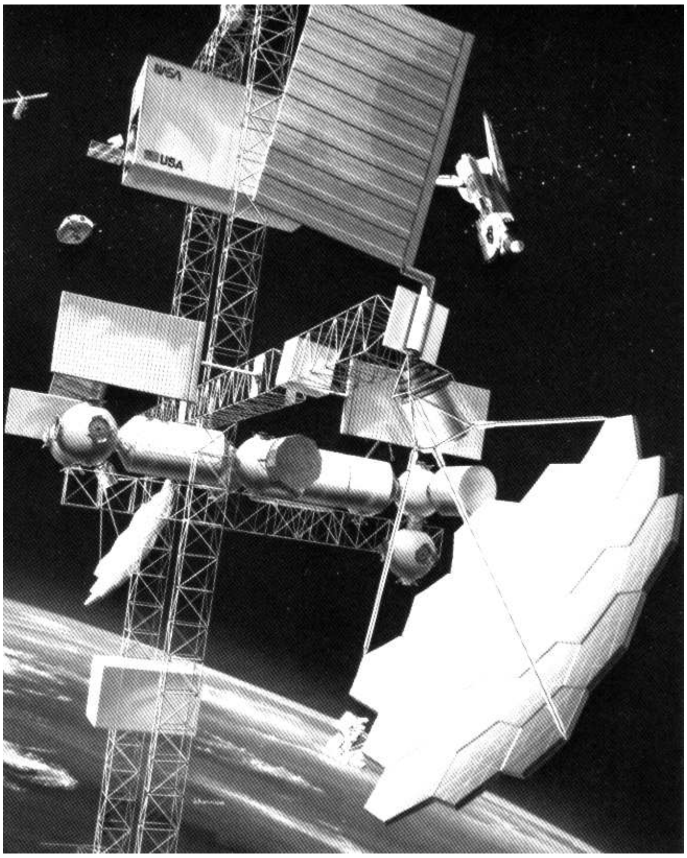

Figure 1.6: Project of a future space station.

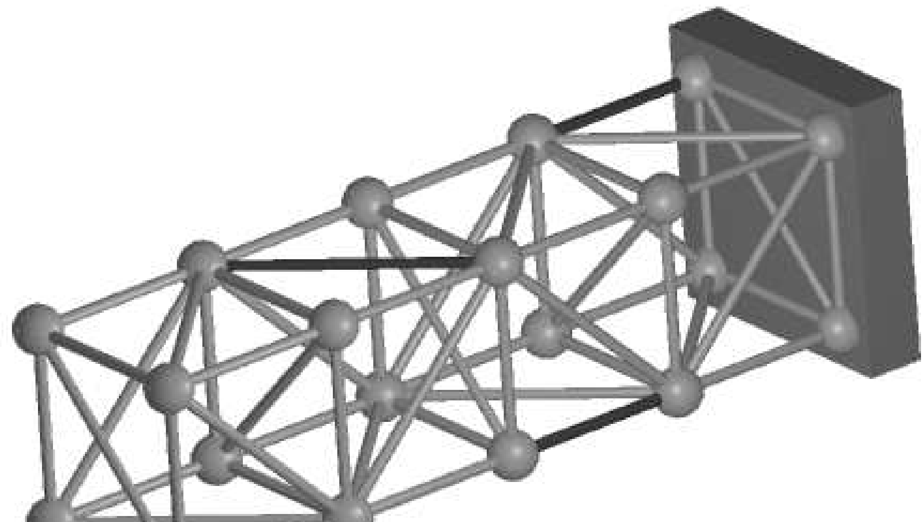

Figure 1.7: 48-DOF finite model of a four-bay segment of the Space Station Freedom [**?**].

1.5 VIBRATION OF STRUCTURES MODELLED WITH 3D TRUSS ELEMENTS

1.5.1 Example of 3D Truss Structure: Space Station "Freedom"

Three-dimensional structures such as these are called space trusses if they have joint that do not exsert couples on the members (that is, joints behave like ball and socket supports) and they are loaded and supported at their joints. Space trusses are analyzed by the same methods we described for 2D trusses. The only difference is the need to cope with the more complicated geometry. Many of the space structures (*see*Fig.1.6) can be efficiently modelled with 3D truss finite elements. A typical example of a 3D truss structure – a four-bay segment of the Space Station Freedom – is presented in Fig. 1.7. If it is clamped at one side as shown in figure, it's dynamics can be studied with a 48-DOF 3D truss model having 20 nodes and 57 truss elements [?].

1.5.2 Degrees of Freedom and Coordinate Transformations

Development of the element stiffness matrix for the space truss member is similar to that for the plane truss member. The element stiffness matrix in terms of the global Cartesian coordinate system is obtained in the same way as given in Eq. (1.25). However, the sizes of both the transformation matrix and the element stiffness matrix in terms of the body coordinate system are 6×6 for the space truss member. Here, the body coordinate system denotes the coordinate system one of whose axes lies along the member direction. The stiffness matrix in terms of the body coordinate system is

$$[k^e] = \frac{E\mathcal{A}}{h} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.29}$$

for the nodal degrees of freedom of

$$\{u^e(t)\} = \{u_1(t) \quad u_2(t) \quad u_3(t) \quad u_4(t) \quad u_5(t) \quad u_6(t)\}^T,$$

where u_1 and u_4 are the displacement along the x-axis in the element's local coordinate system as shown in in Fig. 1.8a.

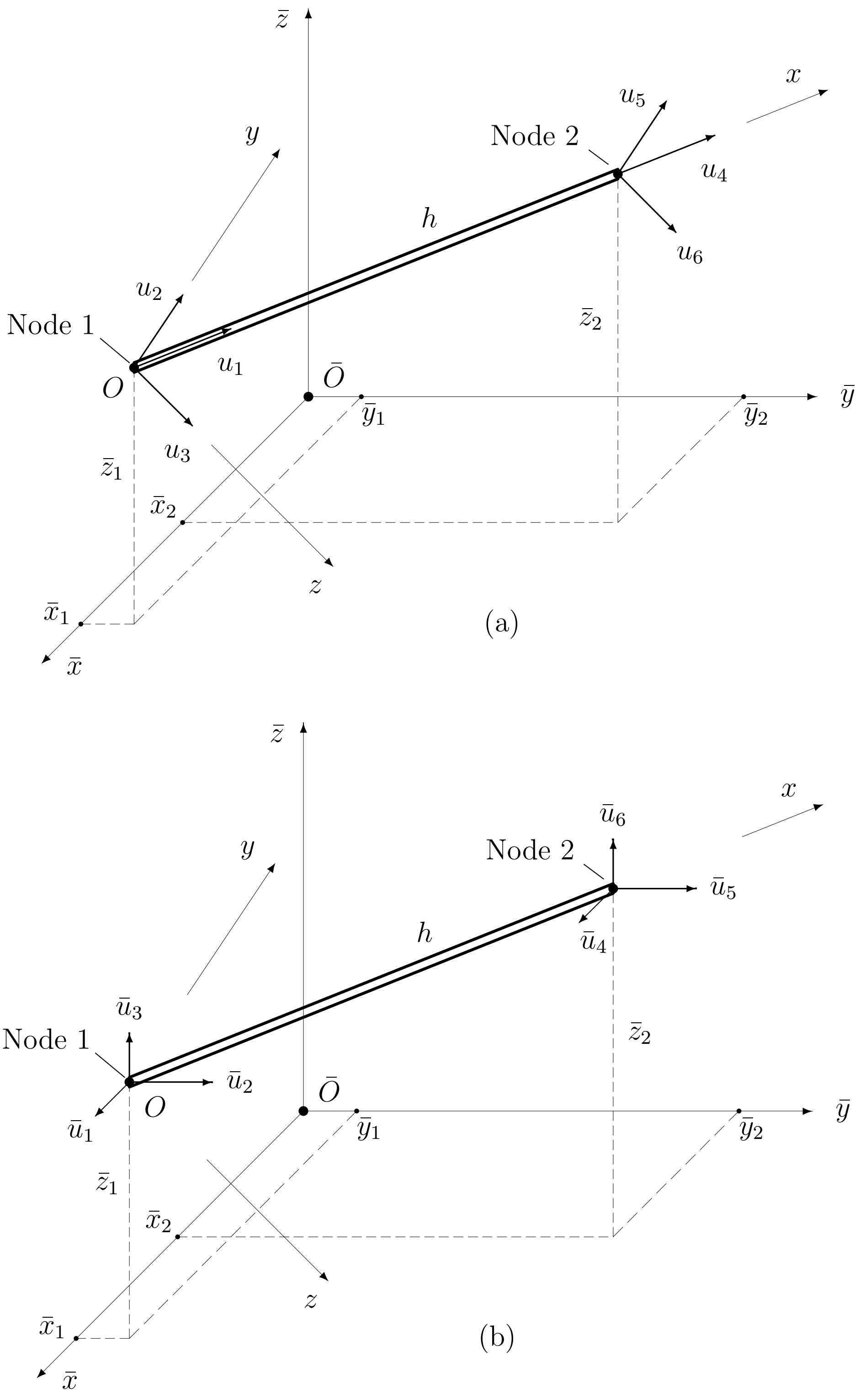

Figure 1.8: A three-dimensional truss element where: (a) node displacements are resolved in the local coordinate system; (b) node displacements are resolved in the global coordinate system.

The transformation matrix between the two coordinate systems is given by:

$$[T] = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_3 & 0 & 0 & 0 \\ \zeta_1 & \zeta_2 & \zeta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 \\ 0 & 0 & 0 & \eta_1 & \eta_2 & \eta_3 \\ 0 & 0 & 0 & \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix}. \tag{1.30}$$

where $\{\xi_1 \quad \eta_1 \quad \zeta_1\}$ is the *direction cosines* of $\bar{x}$- axis with respect to the xyz -coordinate system. Similarly, $\{\xi_2 \quad \eta_2 \quad \zeta_2\}$ and $\{\xi_3 \quad \eta_3 \quad \zeta_3\}$ are the *direction cosines* of $\bar{y}$- and $\bar{z}$-axis with respect to xyz coordinate system respectively:

$$\xi_1 = \cos(x\bar{x}) = \frac{\bar{x}_2 - \bar{x}_1}{h}; \quad \xi_2 = \cos(x\bar{y}) = \frac{\bar{y}_2 - \bar{y}_1}{h}; \quad \xi_3 = \cos(x\bar{z}) = \frac{\bar{z}_2 - \bar{z}_1}{h}.$$

1.5.3 FE Stiffness Matrix in the Global Coordinate System

Following the procedure, explained in previous sections, we can get the expression for the calculation of the element mass matrix in the global coordinate system

$$[\bar{k}^e] = [T]^T \; [k^e] \; [T] =$$

$$= \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_3 & 0 & 0 & 0 \\ \zeta_1 & \zeta_2 & \zeta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 \\ 0 & 0 & 0 & \eta_1 & \eta_2 & \eta_3 \\ 0 & 0 & 0 & \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix}^T \frac{E\mathcal{A}}{h} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_3 & 0 & 0 & 0 \\ \zeta_1 & \zeta_2 & \zeta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 \\ 0 & 0 & 0 & \eta_1 & \eta_2 & \eta_3 \\ 0 & 0 & 0 & \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix}.$$

Conducting the matrix manipulation yields:

Stiffness Matrix for a 3D Truss FE:
(Global Coordinate System)

$$\left[\bar{k}^e\right] = \frac{E\mathcal{A}}{h} \begin{bmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 & -\xi_1^2 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 & -\xi_1\xi_2 & -\xi_2^2 & -\xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 & -\xi_1\xi_3 & -\xi_2\xi_3 & -\xi_3^2 \\ -\xi_1^2 & -\xi_1\xi_2 & -\xi_1\xi_3 & \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ -\xi_1\xi_2 & -\xi_2^2 & -\xi_2\xi_3 & \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ -\xi_1\xi_3 & -\xi_2\xi_3 & -\xi_3^2 & \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{bmatrix}. \tag{1.31}$$

The corresponding nodal degrees of freedom are

$$\{\bar{u}^e(t)\} = \{\bar{u}_1(t) \quad \bar{u}_2(t) \quad \bar{u}_3(t) \quad \bar{u}_4(t) \quad \bar{u}_5(t) \quad \bar{u}_6(t)\}^T.$$

1.5.4 FE Mass Matrix in the Global Coordinate System

Similarly we can get the expression for the FE mass matrix in the global coordinate system.

$$
\begin{aligned}
[\bar{m}^e] &= [T]^T\,[m^e]\,[T] = \\
&= \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_3 & 0 & 0 & 0 \\ \zeta_1 & \zeta_2 & \zeta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 \\ 0 & 0 & 0 & \eta_1 & \eta_2 & \eta_3 \\ 0 & 0 & 0 & \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix}^T \frac{mh}{6} \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_3 & 0 & 0 & 0 \\ \zeta_1 & \zeta_2 & \zeta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 \\ 0 & 0 & 0 & \eta_1 & \eta_2 & \eta_3 \\ 0 & 0 & 0 & \zeta_1 & \zeta_2 & \zeta_3 \end{bmatrix}.
\end{aligned}
$$

<u>Consistent Mass Matrix for a 3D Truss FE:</u>
<u>(Global Coordinate System)</u>

$$
[\bar{m}^e] = \frac{mh}{6} \begin{bmatrix} 2\xi_1^2 & 2\xi_1\xi_2 & 2\xi_1\xi_3 & \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ 2\xi_1\xi_2 & 2\xi_2^2 & 2\xi_2\xi_3 & \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ 2\xi_1\xi_3 & 2\xi_2\xi_3 & 2\xi_3^2 & \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \\ \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 & 2\xi_1^2 & 2\xi_1\xi_2 & 2\xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 & 2\xi_1\xi_2 & 2\xi_2^2 & 2\xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 & 2\xi_1\xi_3 & 2\xi_2\xi_3 & 2\xi_3^2 \end{bmatrix}. \tag{1.32}
$$

Sometimes it is convenient to simplify the model by lumping the distributed masses within the element into the concentrated masses at two nodal points. In this case we have a *lumped mass matrix*:

<u>Lumped Mass Matrix for a 3D Truss FE:</u>
<u>(Global Coordinate System)</u>

$$
[\bar{m}^e] = \frac{mh}{2} \begin{bmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 & 0 & 0 & 0 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 & 0 & 0 & 0 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ 0 & 0 & 0 & \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ 0 & 0 & 0 & \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{bmatrix}. \tag{1.33}
$$

1.5.5 Force Vector

The force vector

$$\{f^e(t)\} = \begin{Bmatrix} f_1^e(t) \\ f_2^e(t) \\ f_3^e(t) \\ f_4^e(t) \\ f_5^e(t) \\ f_6^e(t) \end{Bmatrix}$$

can be derived from the virtual work expression. For the truss element, loaded with the distributed force $f(x,t)$, the virtual work δW^e is calculated as

$$\begin{aligned} \delta W^e(t) &= \int_0^h f(x,t)\,\delta u(x,t)\,dx = \\ &= \int_0^h f(x,t)\left[\left(1-\frac{x}{h}\right)\delta u_1(t) + 0\cdot\delta u_2(t) + 0\cdot\delta u_3(t)\right] dx + \\ &+ \int_0^h f(x,t)\left[\left(\frac{x}{h}\right)\delta u_4(t) + 0\cdot\delta u_5(t) + 0\cdot\delta u_6(t)\right] dx = \\ &= \left[\int_0^h f(x,t)\left(1-\frac{x}{h}\right) dx\right]\delta u_1(t) + \left[\int_0^h f(x,t)\left(\frac{x}{h}\right) dx\right]\delta u_4(t). \end{aligned}$$

By expressing this result in matrix form as

$$\begin{aligned} \delta W^e(t) &= \{\delta u^e(t)\}^T\{f^e(t)\} = \\ &= f_1^e(t)\,\delta u_1(t) + f_2^e(t)\,\delta u_2(t) + f_3^e(t)\,\delta u_3(t) + \\ &+ f_4^e(t)\,\delta u_4(t) + f_5^e(t)\,\delta u_5(t) + f_6^e(t)\,\delta u_6(t), \end{aligned}$$

the equivalent nodal forces can be identified as

<u>Nodal Forces for a 3D Truss FE:</u>

$$f_1^e(t) = \int_0^h f(x,t)\left(1-\frac{x}{h}\right) dx = \int_0^h f(x,t)\,H_1(x)\,dx$$

$$f_4^e(t) = \int_0^h f(x,t)\left(\frac{x}{h}\right) dx = \int_0^h f(x,t)\,H_2(x)\,dx \qquad (1.34)$$

$$f_2^e(t) = f_5^e(t) = f_3^e(t) = f_6^e(t) = 0$$

1.5.6 FE Modelling and 3D Animation With MATLAB

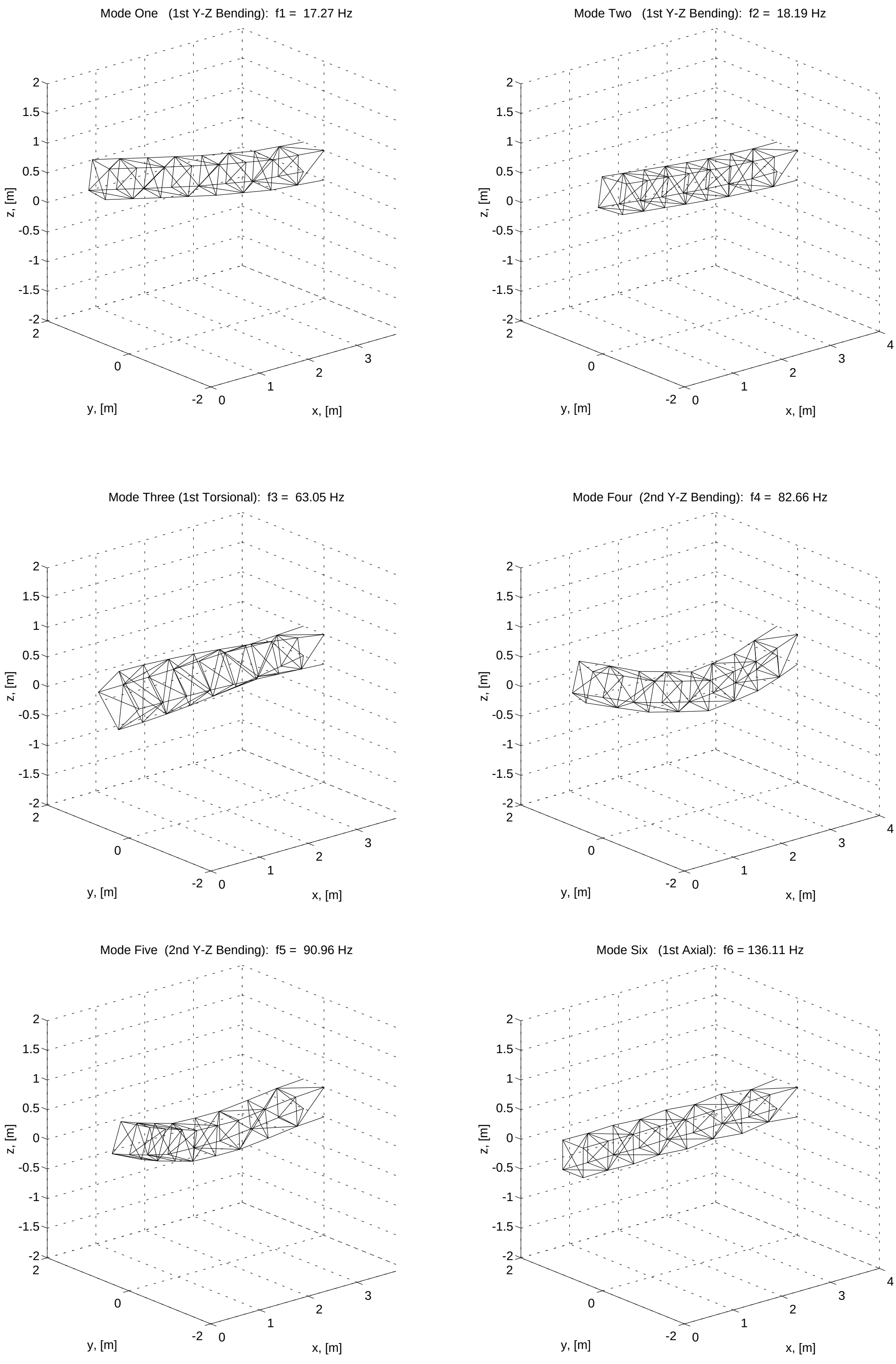

Figure 1.9: The first natural modes of vibration of the 8-Bay 3D truss structure [?].

1.6 VIBRATION OF STRUCTURES MODELLED WITH 2D BEAM ELEMENTS

1.6.1 Stiffness Matrix

Another case of particular interest is **the bar in bending vibration**. We propose to use an approach, known as the direct approach, to derive the corresponding element stiffness matrix. To this end, we consider the element, shown in Fig. 1.10.

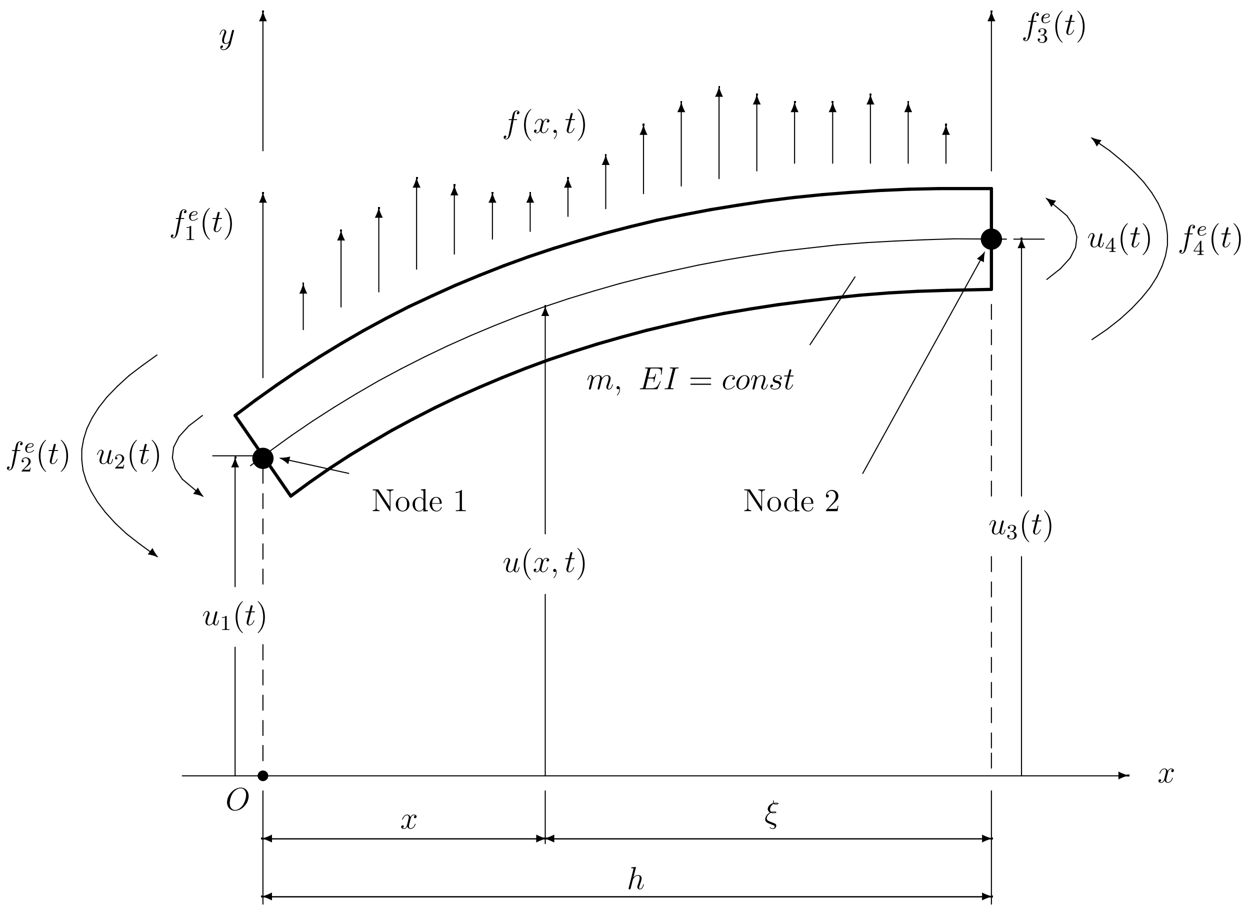

Figure 1.10: A 2D two-noded beam finite element.

At this stage of our derivations we shall concentrate on the stiffness effect, while ignoring the inertia effects. The mass influence will be included in the later sections, when the mass matrix will be introduced in the analysis. In view of this remarks for *uniform* bending stiffness, the differential equation for the displacement $u(x,t)$ is

$$EI\,\frac{\partial^4 u(x,t)}{\partial x^4} = 0, \qquad (0 < x < h). \tag{1.35}$$

Integrating Eq. (1.35) four times, we have

$$u(x,t) = \tfrac{1}{6}\, c_1(t)\, x^3 + \tfrac{1}{2}\, c_2(t)\, x^2 + c_3(t)\, x + c_4(t), \tag{1.36}$$

where $c_i(t)\,(i = 1, 2, 3, 4)$ are time-dependent functions, which can be found from the boundary conditions for the finite element. To do this, we refer to Fig. 1.10 and write

$$\begin{aligned} u(0,t) &= u_1(t); &\qquad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} &= u_2(t);\\ u(h,t) &= u_3(t); &\qquad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=h} &= u_4(t), \end{aligned} \tag{1.37}$$

where $u_1(t)$ and $u_3(t)$ are nodal *translational* displacements and $u_2(t)$ and $u_4(t)$ are nodal rotations, or nodal *angular*, or rotational displacements. Introducing Eq. (1.36) into Eqs. (1.37), we obtain

$$\begin{aligned} & u(0,t) = c_4(t) = u_1(t);\\ & \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = c_3(t) = u_2(t);\\ & u(h,t) = \tfrac{1}{6}\, c_1(t)\, h^3 + \tfrac{1}{2}\, c_2(t)\, h^2 + c_3(t)\, h + c_4(t) = u_3(t);\\ & \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=h} = \tfrac{1}{2}\, c_1(t)\, h^2 + c_2(t)\, h + c_3(t) = u_4(t), \end{aligned} \tag{1.38}$$

which have the solution

$$\begin{aligned} c_1(t) &= \frac{6}{h^3}\left[2u_1(t) + hu_2(t) - 2u_3(t) + hu_4(t)\right];\\ c_2(t) &= \frac{2}{h^3}\left[-3u_1(t) - 2hu_2(t) + 3u_3(t) - hu_4(t)\right];\\ c_3(t) &= u_2(t); \qquad c_4(t) = u_1(t). \end{aligned} \tag{1.39}$$

Hence, introducing Eq. (1.39) into Eqs. (1.36), we obtain the expression for the bending displacement

$$\begin{aligned} u(x,t) &= \left[1 - 3\left(\frac{x}{h}\right)^2 + 2\left(\frac{x}{h}\right)^3\right] u_1(t) + h\left[\left(\frac{x}{h}\right) - 2\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right] u_2(t) +\\ &+ \left[3\left(\frac{x}{h}\right)^2 - 2\left(\frac{x}{h}\right)^3\right] u_3(t) + h\left[-\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right] u_4(t). \end{aligned} \tag{1.40}$$

The actual finite element is subjected to the transverse distributed force $f(x,t)$. Let us assume that we can replace this load with the "equivalent" set of element's nodal loads $f_i^e(t)$ $(i = 1, 2, 3, 4)$, ensuring the identical nodal displacements. (We do not discuss this procedure here – it will be explained later.) It should be noted that nodal forces $f_1^e(t)$ and $f_3^e(t)$ are associated with nodal *translational* displacements $u_1(t)$ and $u_3(t)$, correspondingly, shown in Fig. 1.10. And nodal forces $f_2^e(t)$ and $f_4^e(t)$ are associated with nodal *rotational* displacements $u_2(t)$ and $u_4(t)$, also shown in Fig. 1.10.

The bending displacements are related to the nodal forces $f_1^e(t)$, $f_2^e(t)$, $f_3^e(t)$ and $f_4^e(t)$ as follows:

$$\begin{aligned}
EI \left.\frac{\partial^3 u(x,t)}{\partial x^3}\right|_{x=0} &= f_1^e(t); \qquad & EI \left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=0} &= f_2^e(t) \\
EI \left.\frac{\partial^3 u(x,t)}{\partial x^3}\right|_{x=h} &= -f_3^e(t) \qquad & EI \left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=h} &= f_4^e(t),
\end{aligned} \tag{1.41}$$

which yield

$$\begin{aligned}
f_1^e(t) &= \frac{EI}{h^3}\left[\ 12u_1(t) + 6hu_2(t) - 12u_3(t) + 6hu_4(t)\right] \\
f_2^e(t) &= \frac{EI}{h^2}\left[\ 6u_1(t) + 4hu_2(t) - 6u_3(t) + 2hu_4(t)\right] \\
f_3^e(t) &= \frac{EI}{h^3}\left[-12u_1(t) - 6hu_2(t) + 12u_3(t) - 6hu_4(t)\right] \\
f_4^e(t) &= \frac{EI}{h^2}\left[\ 6u_1(t) + 2hu_2(t) - 6u_3(t) + 4hu_4(t)\right].
\end{aligned} \tag{1.42}$$

Equations (1.42) have the matrix form

$$[k^e]\{u\} = \{f^e\}, \qquad \text{where} \tag{1.43}$$

$$\{u(t)\} = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{Bmatrix} \qquad \{f^e\} = \begin{Bmatrix} f_1^e(t) \\ f_2^e(t) \\ f_3^e(t) \\ f_4^e(t) \end{Bmatrix} \tag{1.44}$$

are the nodal displacement vector and nodal force vector, respectively, and $[k]$ is the desired element stiffness matrix in the local FE coordinate system:

Stiffness Matrix for a 2D Beam FE:
(Local FE Coordinate System)

$$[k] = \frac{EI}{h^3}\begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix} \tag{1.45}$$

Eq. (1.40) can be rewritten in the form

$$u(x,t) \quad = \quad H_1(x)\,u_1(t) + H_2(x)\,u_2(t) + H_3(x)\,u_3(t) + H_4(x)\,u_4(t), \tag{1.46}$$

where the interpolation functions

Hermite Interpolation Functions for a 2D Beam FE:
(Local FE Coordinate System)

$$\begin{aligned} H_1(x) &= 1 - 3\left(\frac{x}{h}\right)^2 + 2\left(\frac{x}{h}\right)^3; \\ H_2(x) &= h\left[\left(\frac{x}{h}\right) - 2\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right]; \\ H_3(x) &= 3\left(\frac{x}{h}\right)^2 - 2\left(\frac{x}{h}\right)^3; \\ H_4(x) &= h\left[-\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right] \end{aligned} \tag{1.47}$$

are known as **Hermite cubics**. They are plotted in Fig. 1.11.

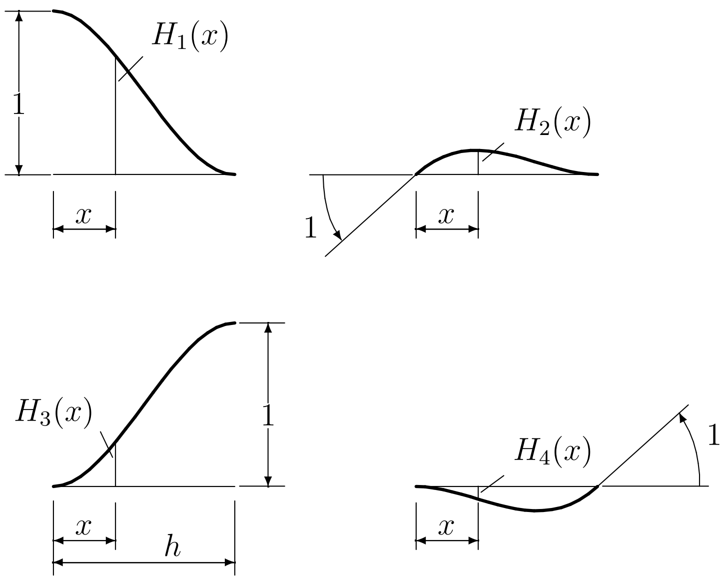

Figure 1.11: Hermitian beam functions.

The interpolation functions and the stiffness matrices were derived on the basis of the static deformation pattern under nodal forces. It turns out that the interpolation functions are *not unique* and other choices are possible. The interpolation functions derived here, however, represent the *lowest-degree polynomials* that can be used for fourth-order problems.

It is interesting to note that the same expression for the stiffness element matrix $[k^e]$

could be obtained based on the analysis of the potential energy $\mathcal{V}^e(t)$ of the finite element. This another, alternative, approach is explored below.

$$\mathcal{V}^e(t) \;=\; \frac{1}{2}\int_0^h EI(x)\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]^2 dx = \tfrac{1}{2}\left\{u(t)\right\}^T\left[k^e\right]\left\{u(t)\right\}, \tag{1.48}$$

where the 4×4 *element stiffness matrix* has the expression

$$[k^e] = \int_0^h EI(x)\,\{H''(x)\}\,\{H''(x)\}^T\,dx, \tag{1.49}$$

Here we use the following matrix notations:

$$\{H(x)\} = \left\{\begin{array}{c} 1-3\left(\dfrac{x}{h}\right)^2+2\left(\dfrac{x}{h}\right)^3 \\ h\left[\left(\dfrac{x}{h}\right)-2\left(\dfrac{x}{h}\right)^2+\left(\dfrac{x}{h}\right)^3\right] \\ 3\left(\dfrac{x}{h}\right)^2-2\left(\dfrac{x}{h}\right)^3 \\ h\left[-\left(\dfrac{x}{h}\right)^2+\left(\dfrac{x}{h}\right)^3\right] \end{array}\right\} \tag{1.50}$$

$$\{H''(x)\} = \frac{d^2}{dx^2}\{H(x)\} = \frac{2}{h^2}\left\{\begin{array}{c} -3+6\,\dfrac{x}{h} \\ h\left[-2+3\,\dfrac{x}{h}\right] \\ 3-6\,\dfrac{x}{h} \\ h\left[-1+3\,\dfrac{x}{h}\right] \end{array}\right\} \tag{1.51}$$

Introducing Eq. (1.51) into Eq. (1.49), we obtain the element mass matrix

$$[k^e] = \frac{4EI}{h^4}\int_0^h \left\{\begin{array}{c} -3+6\,\dfrac{x}{h} \\ h\left[-2+3\,\dfrac{x}{h}\right] \\ 3-6\,\dfrac{x}{h} \\ h\left[-1+3\,\dfrac{x}{h}\right] \end{array}\right\}\left\{\begin{array}{c} -3+6\,\dfrac{x}{h} \\ h\left[-2+3\,\dfrac{x}{h}\right] \\ 3-6\,\dfrac{x}{h} \\ h\left[-1+3\,\dfrac{x}{h}\right] \end{array}\right\}^T dx. \tag{1.52}$$

In case when the beam rigidity EI is not constant within a beam element, the integral given by Eq. (1.52) must be evaluated considering EI as a function of x. If the element is relatively short, for example in the refined mesh, the average value of EI for the element may be used with Eq. (1.52) for a simple and reasonable approximation. In this case, after time-consuming integration, we obtain

$$[k^e] = \frac{EI}{h^3} \begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix},$$

which is the same matrix as that given by Eq. (1.45).

1.6.2 Mass Matrix

Let us assume that the axial displacements of the fourth-order system depicted in Fig. 1.1 can be written in the form

$$\begin{aligned} u(x,t) &= H_1(x)\,u_1(x) + H_2(x)\,u_2(x) + H_3(x)\,u_3(x) + H_4(x)\,u_4(x) = \\ &= \{H(x)\}^T\,\{u(t)\}, \end{aligned} \tag{1.53}$$

where $\{H(x)\}$ is a four-dimensional vector of interpolation functions and $\{u(t)\}$ is the four-dimensional vector of nodal displacements.

The kinetic energy for the element has the form

$$\mathcal{T}(t) = \frac{1}{2}\int_0^h m(x)\left[\frac{\partial u(x,t)}{\partial t}\right]^2 dx = \tfrac{1}{2}\,\{\dot{u}(t)\}^T\,[m^e]\,\{\dot{u}(t)\}\,, \tag{1.54}$$

where the 4×4 *element mass matrix* is

$$[m^e] = \int_0^h m(x)\,\{H(x)\}\,\{H(x)\}^T\,dx \tag{1.55}$$

Introducing Eq. (1.50) into Eq. (1.55), and assuming the uniform mass distribution within the finite element, we obtain the element mass matrix

$$[m] = m\int_0^h \left\{\begin{matrix} 1 - 3\left(\frac{x}{h}\right)^2 + 2\left(\frac{x}{h}\right)^3 \\ h\left[\left(\frac{x}{h}\right) - 2\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right] \\ 3\left(\frac{x}{h}\right)^2 - 2\left(\frac{x}{h}\right)^3 \\ h\left[-\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right] \end{matrix}\right\} \left\{\begin{matrix} 1 - 3\left(\frac{x}{h}\right)^2 + 2\left(\frac{x}{h}\right)^3 \\ h\left[\left(\frac{x}{h}\right) - 2\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right] \\ 3\left(\frac{x}{h}\right)^2 - 2\left(\frac{x}{h}\right)^3 \\ h\left[-\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right] \end{matrix}\right\}^T dx.$$

After time-consuming integration, we finally obtain

Consistent Mass Matrix
for Translational Inertia for a 2D Beam FE:
(Local FE Coordinate System)

$$[m^e] = \frac{mh}{420}\begin{bmatrix} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{bmatrix} \tag{1.56}$$

The mass matrix given by Eq. (1.56) is known as the *consistent* mass matrix. The sum of its all components, associated with only the nodal translational displacements (i.e. u_1 and u_3), yields mh, the total mass of the beam element. It is said that the beam element conserves the mass in terms of its translational degrees-of-freedom.

In many aerospace application the rotational inertia of a vibrating beam element and shear deformations are significant and should be taken into consideration. It happens, for example, during the analysis of the highest modes of a vibrating rocket vehicle. In this case, in addition to the consistent mass matrix for translational inertia, we have to introduce the consistent mass matrix for and rotational inertia and shearing deformations, contributing to the total mass matrix.

The first of them is presented below:

Consistent Mass Matrix
for Rotational Inertia for a 2D Beam FE:
(Local FE Coordinate System)

$$[m^e] = \frac{mI}{30\mathcal{A}h}\begin{bmatrix} 36 & 3h & -36 & 3h \\ 3h & 4h^2 & -3h & -h^2 \\ -36 & -3h & 36 & -3h \\ 3h & -h^2 & -3h & 4h^2 \end{bmatrix} \tag{1.57}$$

Here I is the moment of inertia (second moment of area) of the cross section with respect to neutral axis:

$$I = \int_{\mathcal{A}} y^2\, \mathcal{A}.$$

1.6.3 Force Vector

To derive the nodal force vector, we turn to the virtual work expression. Assuming that the element is subjected to the distributed axial nonconservative force $f(x,t)$ and considering Eq.(1.46), we can write

$$\begin{aligned}
\overline{\delta W}(t) &= \int_0^h f(x,t)\,\delta u(x,t)\,dx = \\
&= \int_0^h f(x,t)\ \{H(x)\}^T\ \{\delta u(t)\}\ dx = \\
&= \{f(t)\}^T\ \{\delta u(t)\}\,,
\end{aligned} \tag{1.58}$$

where

$$\{f(t)\} = \int_0^h f(x,t)\ \{H(x)\}\ dx \tag{1.59}$$

is the *nodal nonconservative force vector*, which is a four-dimensional vector and includes forces ($f_1^e(t)$ and $f_3^e(t)$) and moments ($f_2^e(t)$ and $f_4^e(t)$). This result can be rewritten in the scalar form, convenient for calculations, as follows:

Nodal Forces for a 2D Beam FE:
(Local FE Coordinate System)

$$\begin{aligned}
f_1^e(t) &= \int_0^h f(x,t)\,H_1(x)\,dx = \int_0^h f(x,t)\ \left[1 - 3\left(\frac{x}{h}\right)^2 + 2\left(\frac{x}{h}\right)^3\right] dx \\
f_2^e(t) &= \int_0^h f(x,t)\,H_2(x)\,dx = \int_0^h f(x,t)\,h\left[\left(\frac{x}{h}\right) - 2\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right] dx \\
f_3^e(t) &= \int_0^h f(x,t)\,H_3(x)\,dx = \int_0^h f(x,t)\ \left[3\left(\frac{x}{h}\right)^2 - 2\left(\frac{x}{h}\right)^3\right] dx \\
f_4^e(t) &= \int_0^h f(x,t)\,H_4(x)\,dx = \int_0^h f(x,t)\,h\left[-\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right] dx
\end{aligned} \tag{1.60}$$

To demonstrate the described procedure in detail, we consider as an example the case when the finite element is loaded with the concentrated force $P(t)$ applied at $x = 2h/3$. To compute the nodal force vector $\{f^e(t)\}$ for this particular example, we first express the concentrated force in the distributed form

$$f(x,t) = P(t)\,\delta\left(x - \frac{2h}{3}\right), \tag{1.61}$$

where $\delta\,(x - 2h/3)$ is a spatial Dirac delta function. Introducing (1.61) into Eq. (1.59), we obtain

$$\{f(t)\} = P(t)\int_0^h \delta\left(x - \frac{2h}{3}\right) \left\{ \begin{array}{c} 1 - 3\left(\frac{x}{h}\right)^2 + 2\left(\frac{x}{h}\right)^3 \\ h\left[\left(\frac{x}{h}\right) - 2\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right] \\ 3\left(\frac{x}{h}\right)^2 - 2\left(\frac{x}{h}\right)^3 \\ h\left[-\left(\frac{x}{h}\right)^2 + \left(\frac{x}{h}\right)^3\right] \end{array} \right\} dx =$$

$$= P(t) \left\{ \begin{array}{c} 1 - 3\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right)^3 \\ h\left[\left(\frac{2}{3}\right) - 2\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3\right] \\ 3\left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right)^3 \\ h\left[-\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3\right] \end{array} \right\} dx = \frac{1}{27} P(t) \left\{ \begin{array}{c} 7 \\ 2h \\ 20 \\ -4h \end{array} \right\}.$$

1.6.4 Reference Systems

The displacements of finite elements, considered in the previous sections, were expressed in a coordinate system particular to every such element. Such system is often referred to as *local coordinate system*.

But, in order to match displacements at a given node for differently oriented elements, it is advisable to work with displacement components in a single set of coordinates (or *global reference system*), while retaining the advantages of identifying the displacement components of any one element with the directions most convenient for that particular element.

The coordinate transformation can be introduced with the following expression

$$\{u\} = [T]\,\{\overline{u}\}, \tag{1.62}$$

where $\{u\}$ and $\{\overline{u}\}$ are column matrices with displacements of FE in the local and global coordinate systems respectively, and $[T]$ is the transformation matrix, defined for the planar case as follows

$$[T] = \left[\begin{array}{ccc|ccc} c & -s & 0 & 0 & 0 & 0 \\ s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

where c and s is the *direction cosines* of $\bar{x}$ and $\bar{y}$ - axis with respect to the xy - coordinate system, respectively.

In view of Eq.(1.62) the kinetic energy of the finite element can be expressed in the global coordinate system

$$\begin{aligned} \mathcal{T}^e &= \tfrac{1}{2}\{\dot{u}\}^T [m^e] \{\dot{u}\} = \\ &= \tfrac{1}{2}\{\dot{\overline{u}}\}^T [T]^T [m^e] [T] \{\dot{\overline{u}}\} = \tfrac{1}{2}\{\dot{\overline{u}}\}^T [\overline{m}^e] \{\dot{\overline{u}}\}, \end{aligned} \tag{1.63}$$

where

$$[\overline{m}^e] = [T]^T [m^e] [T] \tag{1.64}$$

is the inertia matrix of the element in terms of the global coordinates $\overline{x}, \overline{y}, \overline{z}$.

Similarly, we can write the potential energy in the global coordinate system as

$$\begin{aligned} \mathcal{V}^e &= \tfrac{1}{2}\{u\}^T [k^e] \{u\} = \\ &= \tfrac{1}{2}\{\overline{u}\}^T [T]^T [k^e] [T] \{\overline{u}\} = \tfrac{1}{2}\{\overline{u}\}^T [\overline{k}^e] \{\overline{u}\}, \end{aligned} \tag{1.65}$$

where

$$[\overline{k}^e] = [T]^T [k^e] [T] \tag{1.66}$$

is the stiffness matrix of the element in terms of the global coordinates.

We can also write expression for virtual work

$$\overline{\delta W}^e = \{\delta\overline{u}\}^T [T]^T \{f^e\} = \{\delta\overline{u}\}^T \{\overline{f}^e\}, \qquad \text{where} \tag{1.67}$$

$$\{\overline{f}^e\} = [T]^T \{f^e\} \tag{1.68}$$

is recognized as the vector of the nodal forces in terms of global components.

1.6.5 The Assembling Process

The equations of motion for the complete system can be obtained by an *assembling process* that amounts to expressing the kinetic energy, the potential energy and the virtual work in terms of contributions from the individual elements.

$$\begin{aligned} \mathcal{T} &= \tfrac{1}{2}\textstyle\sum_{e=1}^{E}\{\dot{\overline{u}}^{e}\}^{T}\,[\overline{m}^{e}]\,\{\dot{\overline{u}}^{e}\} = \\ &= \tfrac{1}{2}\textstyle\sum_{e=1}^{E}\{\dot{\overline{U}}^{e}\}^{T}\,[\overline{M}^{e}]\,\{\dot{\overline{U}}^{e}\} = \tfrac{1}{2}\{\dot{\overline{U}}\}^{T}\,[\overline{M}]\,\{\dot{\overline{U}}\}, \end{aligned} \tag{1.69}$$

where

$$[\overline{M}] = \sum_{e=1}^{E}[\overline{M}^{e}] \tag{1.70}$$

is the symmetric *mass matrix for the complete system*, which is obtained by simple addition of the extended element mass matrices (this process is clearly illustrated in Fig. 1.12), and $\{\overline{U}^{e}\}$ is the *extended element nodal displacement vector*, which is obtained by adding to the vector $\{\overline{u}^{e}\}$ as many zero components as to make the dimension of the vector $\{\overline{U}^{e}\}$ equal to N - dimension of displacements for the complete system.

Similarly

$$\mathcal{V} = \tfrac{1}{2}\textstyle\sum_{e=1}^{E}\{\overline{u}^{e}\}^{T}\,[\overline{k}^{e}]\,\{\overline{u}^{e}\} = \tfrac{1}{2}\textstyle\sum_{e=1}^{E}\{\overline{U}^{e}\}^{T}\,[\overline{K}^{e}]\,\{\overline{U}^{e}\} = \tfrac{1}{2}\{\overline{U}\}^{T}\,[\overline{K}]\,\{\overline{U}\}, \tag{1.71}$$

where

$$[\overline{K}] = \sum_{e=1}^{E}[\overline{K}^{e}] \tag{1.72}$$

is the symmetric *stiffness matrix for the complete system.*

The virtual work expression has the form

$$\overline{\delta W} \;=\; \sum_{e=1}^{E}\{\overline{f}^{e}\}^{T}\,\{\delta\overline{u}^{e}\} = \sum_{e=1}^{E}\{\overline{F}^{e}\}^{T}\,\{\delta\overline{U}^{e}\} = \{\overline{F}\}^{T}\,\{\delta\overline{U}\}, \quad \text{where} \tag{1.73}$$

$$\{\overline{F}\} = \sum_{e=1}^{E}\{\overline{F}^{e}\} \tag{1.74}$$

is the *vector of nodal nonconservative forces for the complete system.*

Lagrange's equations of motion for the complete structure can be presented in the matrix form:

$$[\overline{M}]\,\{\ddot{\overline{U}}\} + [\overline{K}]\,\{\overline{U}\} = \{\overline{F}\} \tag{1.75}$$

If complete structure is unrestrained and capable of rigid-body motion, the matrix $[\overline{K}]$ is singular. A simple way of treating the problem in which $[\overline{K}]$ is singular and the structure is supported so that a given number of joint displacements are zero is to eliminate from matrices $[\overline{M}]$, $[\overline{K}]$ and $[\overline{F}]$ the corresponding number of rows and columns.

1.6.6 Example: Assembling Procedure for the Simplified Finite Element Model of a Wing

Consider the assembling procedure for the continuos aircraft wing, modelled as a five-span finite element model, shown in Fig. 1.12. The modelled translational and rotational degrees-of-freedom for this simplified model are shown by positive arrows and double arrows correspondingly. These DOF are marked as $1, 2, \ldots, 12$. The numbering system for the finite elements ①, ②, ..., ⑤ id also presented in Fig. 1.12.

As there six nodes (four joints) for this system and each node has two possible degrees-of-freedom (vertical translational and rotational displacements), the orders of the global (overall joint) mass and stiffness matrices are both 12×12.

As no other DOF in the system, except 1, 2, 3, and 4, produces an effect on FE (or member) ①, it follows that the FE mass and member stiffness matrices for this member will occupy the upper left position of the global (joint) mass and stiffness matrices correspondingly.

Proceeding the element ②, we find the FE mass and FE stiffness matrices for this portion are determined by the following DOF: 3, 4, 5, and 6. When these FE mass and stiffness matrices are placed in the *global* mass and stiffness matrices, there is an overlapping of elements from FE ①. In other words, the four elements of the member mass and stiffness matrices for FE ② associated with deformations 3 and 4 are superimposed on the four elements of the FE mass and stiffness matrices for member ① associated with the same deformations.

This pattern is then repeated from FE to FE until the complete global (joint) mass and stiffness matrices for the system are formed. The superposition of FE matrix components is necessary, as any set of deformations for an interior node (joint) will produce mass and stiffness influence coefficients in all finite elements associated with this joint. This pattern of assembling global stiffness matrix for the wing structure is shown schematically in Fig. 1.13

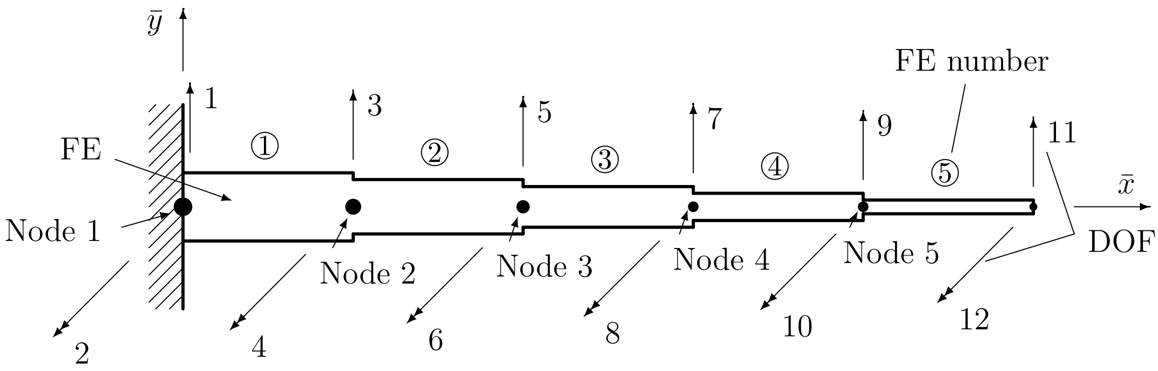

Figure 1.12: Simplified finite element model of an aircraft wing.

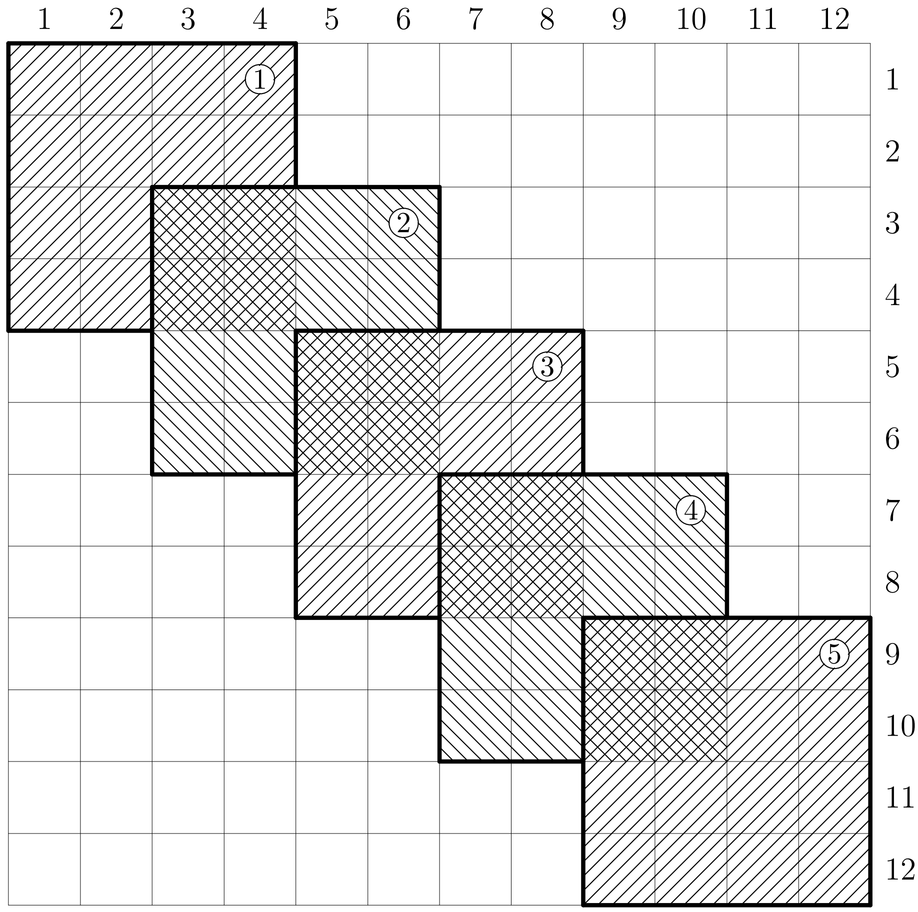

Figure 1.13: Demonstration of the assembling procedure of the global stiffness matrix.

1.6.7 Example: Uniform Bar in Bending

Consider the uniform bar in bending clamped at $x = 0$ and free at $x = L$, as shown in Fig. 1.14, represent it as a two-finite-element model and derive the equations for the free vibration of the system.

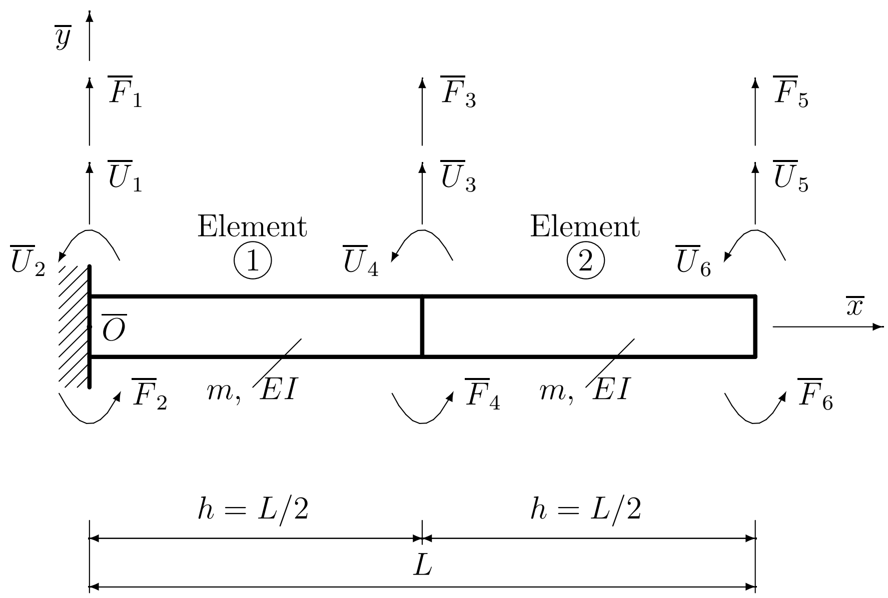

Figure 1.14: Two-finite-element model of the beam.

Solution

In this case, when the sytstem has constant parameters m and EI and when it is modelled with the same length finite elements, it is convenient to rewrite equations of motion for the system in terms of the force vector $\{f^e\} = \{f_1^e;\quad f_2^e/h;\quad f_3^e;\quad f_4^e/h\}$ and the element mass and element stiffness matrices in the form with nondimensional components in brackets as follows:

$$[m^e] = [\overline{m}^e] = [\overline{m}] = \frac{mh}{420}\begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}.$$

$$[k^e] = [\overline{k}^e] = [\overline{k}] = \frac{EI}{h^3}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}.$$

Hence, the extended element mass and stiffness matrices are

$$\left[\overline{M}^{①}\right] = \frac{mh}{420}\begin{bmatrix} 156 & 22 & 54 & -13 & 0 & 0 \\ 22 & 4 & 13 & -3 & 0 & 0 \\ 54 & 13 & 156 & -22 & 0 & 0 \\ -13 & -3 & -22 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\left[\overline{M}^{②}\right] = \frac{mh}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 156 & 22 & 54 & -13 \\ 0 & 0 & 22 & 4 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix}.$$

$$\left[\overline{K}^{①}\right] = \frac{EI}{h^3}\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 12 & -6 & 0 & 0 \\ 6 & 2 & -6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\left[\overline{K}^{②}\right] = \frac{EI}{h^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 6 & -12 & 6 \\ 0 & 0 & 6 & 4 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}.$$

Using Eqs.(1.70) and (1.72), we obtain the matrices for the complete structure

$$\left[\overline{M}\right] = \sum_{e=1}^{2}\left[\overline{M}^{e}\right] = \frac{mh}{420}\begin{bmatrix} 156 & 22 & 54 & -13 & 0 & 0 \\ 22 & 4 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix}.$$

$$\left[\overline{K}\right] = \sum_{e=1}^{2}\left[\overline{K}^{e}\right] = \frac{EI}{h^3}\begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}.$$

Because the bar is clamped at $x = 0$, the translational and rotational displacements must be zero, $\overline{U}^{①} = 0$ and $\overline{U}^{②} = 0$. Hence, deleting the first and second rows and columns in $\left[\overline{M}\right]$ and $\left[\overline{K}\right]$, we can write the equations of motion

$$\frac{mL}{840}\begin{bmatrix} 312 & 0 & 54 & -13 \\ 0 & 8 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{\overline{U}}_3 \\ h\ddot{\overline{U}}_4 \\ \ddot{\overline{U}}_5 \\ h\ddot{\overline{U}}_6 \end{Bmatrix} +$$

$$+\frac{8EI}{L^3}\begin{bmatrix} 24 & 0 & -12 & 6 \\ 0 & 8 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}\begin{Bmatrix} \overline{U}_3 \\ h\overline{U}_4 \\ \overline{U}_5 \\ h\overline{U}_6 \end{Bmatrix} = \begin{Bmatrix} \overline{F}_3 \\ \overline{F}_4/h \\ \overline{F}_5 \\ \overline{F}_6/h \end{Bmatrix}$$

1.6.8 Example: Saturn-5 Three Stage Launch Vehicle

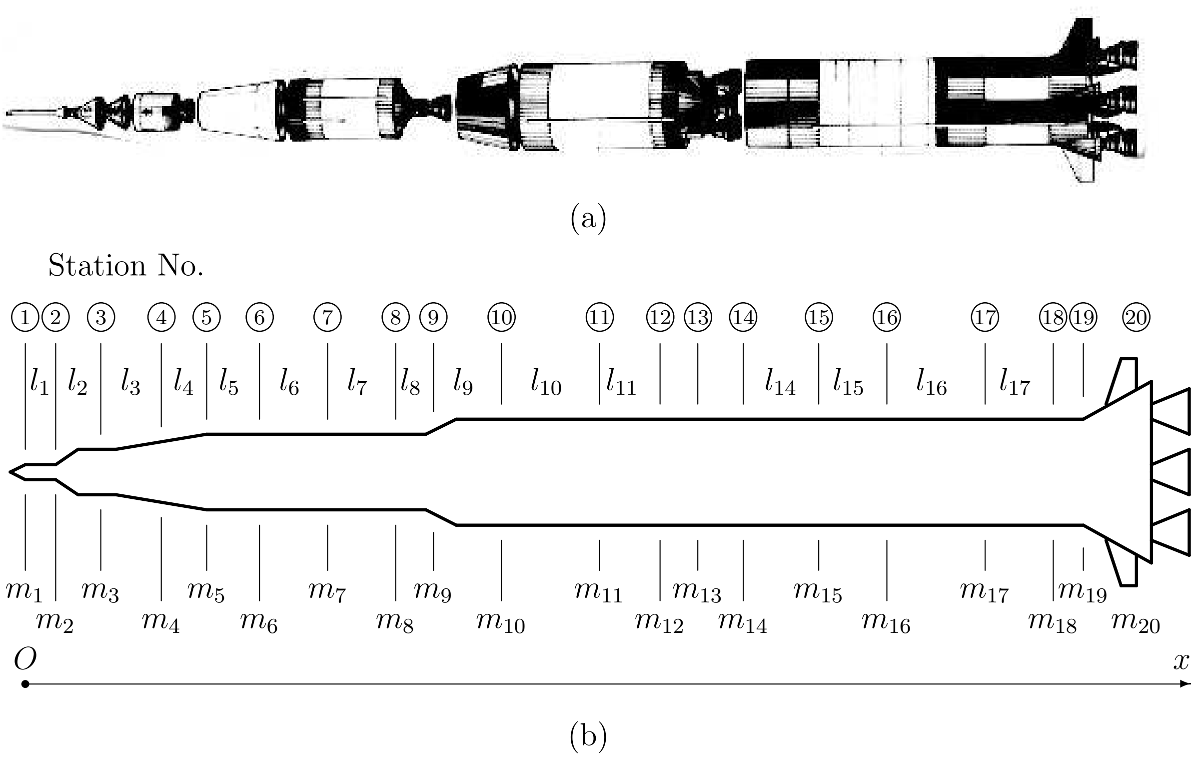

Figure 1.15: Saturn-5 rocket: (a) Photo; (b) Notations for the data.

An interested reader can use the theory presented in the Section and data in the Table 1.1 to calculate the natural frequencies for the Saturn-5 rocket.

Table 1.1: Saturn-5 Three Stage Launch Vehicle: Mass and Stiffness Data [?].

STATION	*PARAMETER*				
	l_i, *cm*	m_i, *kg*	P_i, *kg·cm*^{2}	$I_i \times 10^{-5}$, *cm*^{4}	$\mathcal{A}_i$, *cm*^{2}
1		3596.604	22853.650		
	226.06			4.162	9.677
2		12640.564	120388.380		
	431.80			12.487	32.258
3		11372.987	321035.680		
	459.74			34.963	90.322
4		11494.526	1866097.360		
	396.24			166.493	235.483
5		2618.514	2588427.640		
	401.32			270.550	248.387
6		15169.237	5096596.360		
	596.90			478.666	438.709
7		58244.070	10895535.740		
	541.02			967.738	558.063
8		4223.912	11236636.150		
	335.28			1498.433	593.547
9		21812.017	17135821.710		
	516.89			2081.157	822.579
10		63257.635	36880135.790		
	852.17			2081.157	822.579
11		313765.104	36278193.890		
	467.36			2247.650	887.095
12		60549.105	18500300.820		
	312.42			2247.650	887.095
13		10088.424	15008882.860		
	355.60			2164.403	854.837
14		326880.440	29295435.440		
	621.03			2871.997	1145.159
15		577533.441	45829315.250		
	621.03			3288.228	1309.675
16		469821.139	47675347.880		
	787.40			2206.027	870.966
17		339557.956	42358039.490		
	601.98			2705.504	1070.966
18		269399.753	26345842.660		
	215.90			3413.098	1348.384
19		89783.047	29195111.790		
	457.20			4453.676	1761.287
20		89783.047	29195111.790		

1.7 VIBRATION OF STRUCTURES MODELLED WITH 3D FRAME ELEMENTS

1.7.1 Degrees of Freedom and Coordinate Transformations

A 3D element subjected to axial, bending and torsional loads and working on stretching/compression, bending and torsion is known as *3D truss element.* The corresponding 12 DOF, defined in the FE local coordinate system xyz are shown in Fig. 1.16a, can be represented as an element's DOF vector:

$$\begin{aligned}\{u^e(t)\} = \{&u_1(t) \quad u_2(t) \quad u_3(t) \quad u_4(t) \quad u_5(t) \quad u_6(t) \\ &u_7(t) \quad u_8(t) \quad u_9(t) \quad u_{10}(t) \quad u_{11}(t) \quad u_{12}(t)\}^T,\end{aligned}$$

where
u_1 and u_7 are the displacement along the x-axis in the element's local coordinate system;
u_2 and u_8 are the displacement along the y-axis;
u_3 and u_9 are the displacement along the z-axis;
u_4 and u_{10} are the rotational displacements about the x-axis;
u_5 and u_{11} are the rotational displacements about the y-axis;
u_6 and u_{12} are the rotational displacements about the z-axis, as shown in in Fig. 1.16a.
It should be noted that u_2 and u_8 are associated with the twisting moment, whereas $u_3 - u_6$ and $u_9 - u_{12}$ are associated with bending moments.

The stiffness matrix in terms of the local (or body) coordinate system is given with the Eq. (1.76).

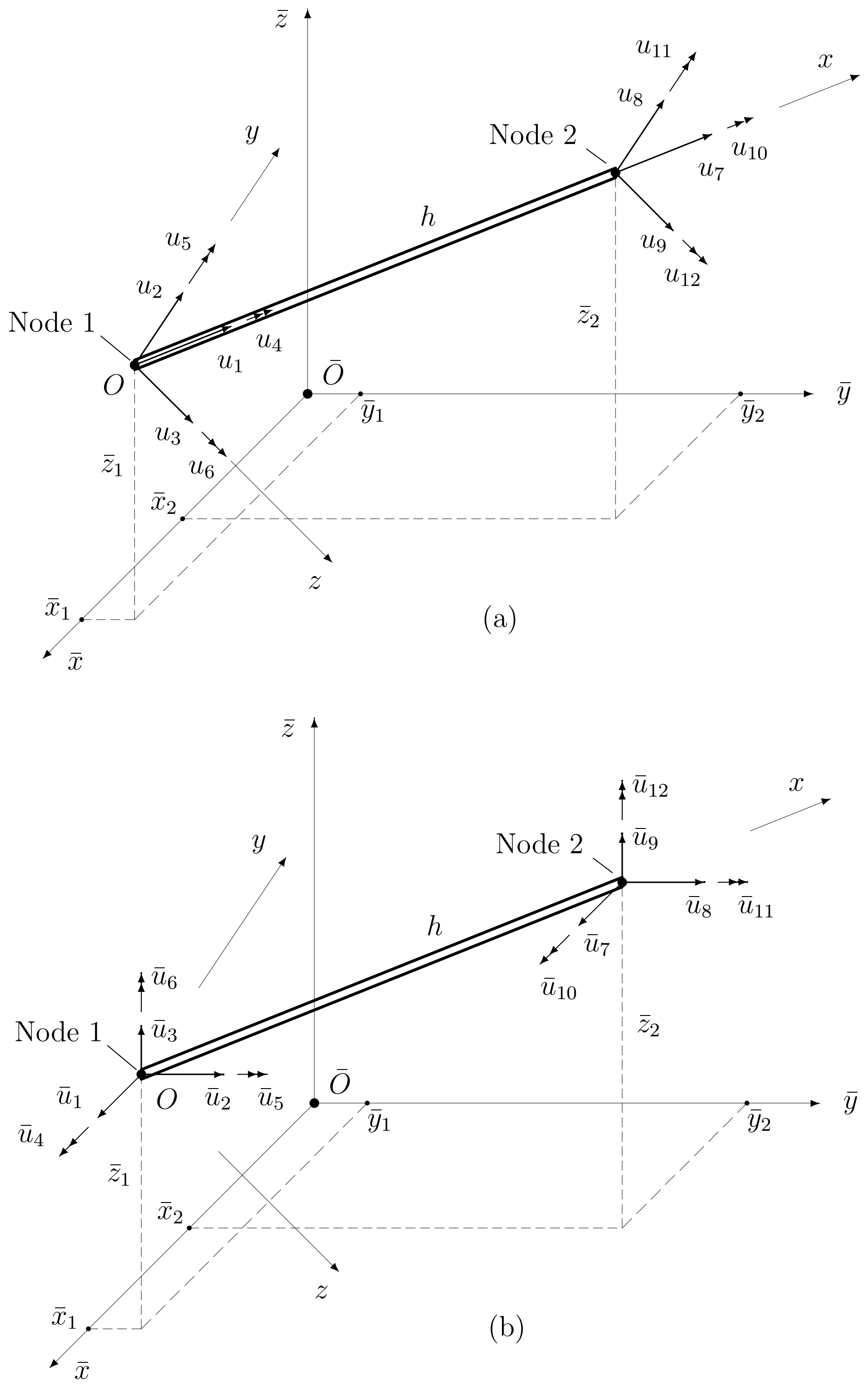

Figure 1.16: A three-dimensional frame element where: (a) node displacements are resolved in the local coordinate system; (b) node displacements are resolved in the global coordinate system.

Stiffness Matrix for a 3D Frame Finite Element:

(FE Local Coordinate System)

$$
[k^e] = \left[\begin{array}{cccccc|cccccc}
\frac{E\mathcal{A}}{h} & 0 & 0 & 0 & 0 & 0 & -\frac{E\mathcal{A}}{h} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_z}{h^3} & 0 & 0 & 0 & \frac{6EI_z}{h^2} & 0 & -\frac{12EI_z}{h^3} & 0 & 0 & 0 & \frac{6EI_z}{h^2} \\
0 & 0 & \frac{12EI_y}{h^3} & 0 & -\frac{6EI_y}{h^2} & 0 & 0 & 0 & -\frac{12EI_y}{h^3} & 0 & -\frac{6EI_y}{h^2} & 0 \\
0 & 0 & 0 & \frac{GJ}{h} & 0 & 0 & 0 & 0 & 0 & -\frac{GJ}{h} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{h^2} & 0 & \frac{4EI_y}{h} & 0 & 0 & 0 & \frac{6EI_y}{h^2} & 0 & \frac{2EI_y}{h} & 0 \\
0 & \frac{6EI_z}{h^2} & 0 & 0 & 0 & \frac{4EI_z}{h} & 0 & -\frac{6EI_z}{h^2} & 0 & 0 & 0 & \frac{2EI_z}{h} \\
\hdashline
-\frac{E\mathcal{A}}{h} & 0 & 0 & 0 & 0 & 0 & \frac{E\mathcal{A}}{h} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12EI_z}{h^3} & 0 & 0 & 0 & -\frac{6EI_z}{h^2} & 0 & \frac{12EI_z}{h^3} & 0 & 0 & 0 & -\frac{6EI_z}{h^2} \\
0 & 0 & -\frac{12EI_y}{h^3} & 0 & \frac{6EI_y}{h^2} & 0 & 0 & 0 & \frac{12EI_y}{h^3} & 0 & \frac{6EI_y}{h^2} & 0 \\
0 & 0 & 0 & -\frac{GJ}{h} & 0 & 0 & 0 & 0 & 0 & \frac{GJ}{h} & 0 & 0 \\
0 & 0 & -\frac{6EI_y}{h^2} & 0 & \frac{2EI_y}{h} & 0 & 0 & 0 & \frac{6EI_y}{h^2} & 0 & \frac{4EI_y}{h} & 0 \\
0 & \frac{6EI_z}{h^2} & 0 & 0 & 0 & \frac{2EI_z}{h} & 0 & -\frac{6EI_z}{h^2} & 0 & 0 & 0 & \frac{4EI_z}{h}
\end{array}\right] \tag{1.76}
$$

Consistent Mass Matrix for a 3D Frame Finite Element:

(FE Local Coordinate System)

$$
[m^e] = \frac{mh}{420}
\left[
\begin{array}{cccccc|cccccc}
140 & 0 & 0 & 0 & 0 & 0 & 70 & 0 & 0 & 0 & 0 & 0 \\
0 & 156 & 0 & 0 & 0 & 22h & 0 & 54 & 0 & 0 & 0 & -13h \\
0 & 0 & 156 & 0 & -22h & 0 & 0 & 0 & 54 & 0 & 13h & 0 \\
0 & 0 & 0 & \frac{140\, I_p}{\mathcal{A}} & 0 & 0 & 0 & 0 & 0 & \frac{70\, I_p}{\mathcal{A}} & 0 & 0 \\
0 & 0 & -22h & 0 & 4h^2 & 0 & 0 & 0 & -13h & 0 & -3h^2 & 0 \\
0 & 22h & 0 & 0 & 0 & 4h^2 & 0 & 13h & 0 & 0 & 0 & -3h^2 \\
\hline
70 & 0 & 0 & 0 & 0 & 0 & 140 & 0 & 0 & 0 & 0 & 0 \\
0 & 54 & 0 & 0 & 0 & 13h & 0 & 156 & 0 & 0 & 0 & -22h \\
0 & 0 & 54 & 0 & -13h & 0 & 0 & 0 & 156 & 0 & 22h & 0 \\
0 & 0 & 0 & \frac{70\, I_p}{\mathcal{A}} & 0 & 0 & 0 & 0 & 0 & \frac{140\, I_p}{\mathcal{A}} & 0 & 0 \\
0 & 0 & 13h & 0 & -3h^2 & 0 & 0 & 0 & 22h & 0 & 4h^2 & 0 \\
0 & -13h & 0 & 0 & 0 & -3h^2 & 0 & -22h & 0 & 0 & 0 & 4h^2
\end{array}
\right]
$$

(1.77)