Tut #02 Advanced Control

by Junaid Saeed

$$G(s) = \frac{b}{s+a}$$ (Generalized form of 1st order T.F.)

$$C(s) = K_c + \frac{K_c}{\tau_i s} \quad \text{—— (I)}$$

C(s) can also be of the form,

$$C(s) = \frac{c_1 s + c_0}{s} \quad \text{—— (II)}$$

Equivalence of (I) & (II)

Let's take II

$$C(s) = c_1 + c_0/s$$

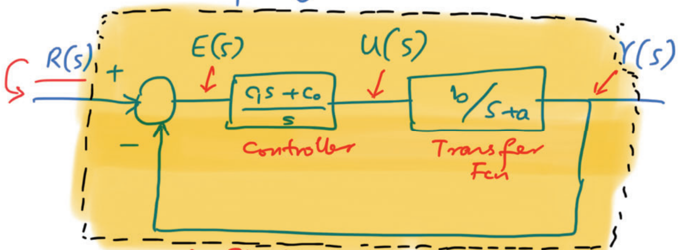
Compare with (I)

$$K_c = c_1 \quad ; \quad \tau_i = \frac{c_1}{c_0}$$

Any representation can be used and is correct. We will use (II)

$$C(s) = \frac{c_1 s + c_0}{s}$$

Closed loop system



Block Diagram of a system with PI control

Closed loop T.F. for this system is the output of the yellow box divided by input i.e. $Y(s)/R(s)$

$$\frac{Y}{R} = \frac{CG}{1+CG}$$

closed loop poles of the system

$$1 + CG = 0$$

$$1 + \left(\frac{c_1 s + c_0}{s}\right)\left(\frac{b}{s+a}\right) = 0$$

$$\frac{s(s+a) + bc_1 s + bc_0}{\boxed{s(s+a)}} = 0$$

falls into the abyss!

$$s^2 + (a + bc_1)s + bc_0 = 0 \quad \text{—— III}$$

This is a second-order system. Question tells us to compare this closed loop denominator polynomial with the given $s^2 + 2\zeta\omega_n s + \omega_n^2$ — IV

Compare coefficients for III & IV

$\boxed{\text{s - term}} \rightarrow 2\zeta\omega_n = a + bc_1$

$\boxed{\text{const-term}} \rightarrow \omega_n^2 = bc_0$

Recall Algebra, two equations, two unknowns. (In our case, unknowns are c_1 & c_0 because we are trying to design a controller).

$$\Rightarrow \quad c_0 = \frac{\omega_n^2}{b} \quad ; \quad c_1 = \frac{2\xi\omega_n - a}{b}$$

In case, you are interested to find out K_c & τ_i

$$K_c = c_1 \quad ; \quad \tau_i = c_1/c_0$$

You are already given "ω_n", "a", "b", "ξ". so we used them to find out values of our controller parameters.

Q # 1

part 2

$$G(s) = \frac{-0.1}{s+10}$$

$\rightarrow \boxed{\omega_n = a}$

$$c_0 = \frac{a^2}{b} = \frac{100}{-0.1} = -1000$$

$$c_1 = \frac{2(0.707)10 - 10}{-0.1}$$

$$c_1 = \quad 4.14 \quad = -41.4$$

$$\overline{-0.1}$$

$$K_c = -41.4 \quad ; \quad \tau_i = 0.0414$$

$\rightarrow \boxed{\omega_n = 5a}$

$$c_0 = \frac{\omega_n^2}{b} = \frac{25(100)}{-0.1} = -25000$$

$$c_1 = \frac{70.7 - 10}{-0.1} = \frac{-60.7}{0.1} = -607$$

$$K_c = -607 \quad ; \quad \tau_i = 0.02428$$

$\rightarrow \boxed{\omega_n = 10a}$

$$c_0 = \frac{(100)^2}{-0.1} = -100000$$

$$c_1 = \frac{131.4}{-0.1} = -1314$$

$$K_c = -1314 \quad ; \quad \tau_i = 0.01314$$

	$\omega_n = a$	$\omega_n = 5a$	$\omega_n = 10a$
K_c	-41.4	-607	-1314
τ_i	0.414	0.02428	0.01314

The proportional gain varies directly with "ω_n". However, τ_i varies inversely.

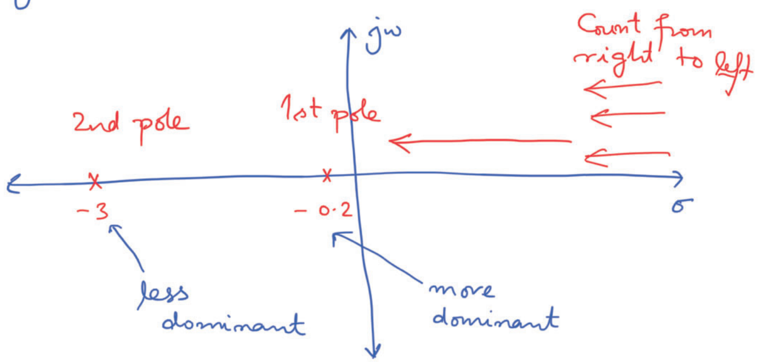
Q # 0.2

$$G(s) = \frac{0.1}{(s+0.2)(s+3)}$$

→ PI controller is designed for a first order system. However, it can be designed for higher order systems by using their first order approximation.

$$G(s) = \frac{0.1}{(s+0.2)(s+3)}$$

The approximation is done by selecting a most dominant pole & ignoring the rest. The most dominant pole is the first pole that you see on a pole-zero map counting from right hand side.



Keep the most dominant pole intact and convert the rest to a simple gain by placing $s=0$ in those poles.

The resulting $G(s)$

$$G(s) = \frac{0.1}{(s+0.2)(0+3)}$$

$$G(s) = \frac{0.1/3}{(s+0.2)} \Rightarrow \begin{array}{l} b = 0.1/3 \\ a = 0.2 \end{array}$$

$$c_0 = \omega_n^2/b \quad ; \quad c_1 = \frac{2\zeta\omega_n - a}{b}$$

→ $\underline{\omega_n = 5a \; ; \; \zeta = 0.707}$

$$c_0 = \frac{1}{0.1/3} = 30 \quad ; \quad c_1 = \frac{1.414 - 0.2}{0.1/3}$$

$$c_1 = 36.42$$

$$\boxed{K_c = 36.42 \quad ; \quad T_i = 1.214}$$

Q no. 2

Part 2

$$G(s) = \frac{-5/6^2}{s+0.1}$$

Part 3

$$G(s) = \frac{e^{-4s}}{(s+0.1)}$$

This term represents a time delay.

Using Padé Approximation, any time delay can be approximated as

$$e^{-ds} \cong \frac{-ds+2}{ds+2}$$

In this case

$$e^{-4s} \simeq \frac{-4s+2}{4s+2} = \frac{-s+0.5}{s+0.5}$$

Replace

$$G(s) = \frac{(-s+0.5)}{(s+0.1)(s+0.5)}$$

Ignoring the least dominant pole

$$G(s) = \frac{-(s-0.5)}{(s+0.1)\,0.5}$$

$$G(s) = \frac{-2(s-0.5)}{(s+0.1)}$$

closed loop denominator

$$1 + CG = 0$$

$$1 + \left(\frac{c_1 s + c_0}{s}\right)\left(\frac{-2(s-0.5)}{s+0.1}\right) = 0$$

$$s^2 + 0.1s - (2s-1)(c_1 s + c_0) = 0$$

$$s^2 + 0.1s - 2c_1 s^2 - 2c_0 s + c_1 s + c_0 = 0$$

$$(1-2c_1)s^2 + (0.1 + c_1 - 2c_0)s + c_0 = 0$$

$$s^2 + \left(\frac{0.1 + c_1 - 2c_0}{1-2c_1}\right)s + \frac{c_0}{1-2c_1} = 0$$

Compare this with

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$0.1 + c_1 - 2c_0$ ... 2ξ ... (s term)

$$\frac{\quad}{1-2c_1} = 2\xi\omega_n \quad \text{—— (s term)}$$

$$+\frac{c_0}{1-2c_1} = \omega_n^2 \quad \text{—— (constt term)}$$

$$\frac{0.1 + c_1}{1-2c_1} - 2\frac{c_0}{1-2c_1} = 2\xi\omega_n$$

$$\frac{0.1 + c_1}{1-2c_1} + 2\omega_n^2 = 2\xi\omega_n$$

$$\frac{0.1 + c_1}{1-2c_1} = 2\omega_n(\xi - \omega_n)$$

$$0.1 + c_1 = 2\omega_n(\xi - \omega_n)(1-2c_1)$$

$$0.1 = 2\omega_n(\xi - \omega_n) - 4c_1\omega_n(\xi - \omega_n) - c_1$$

$$\boxed{\frac{2\omega_n(\xi - \omega_n) - 0.1}{1 + 4\omega_n(\xi - \omega_n)} = c_1}$$

$$\boxed{c_0 = (1-2c_1)\omega_n^2}$$

Assuming $\omega_n = 5a = 5 \times 0.1 = 0.5$

& $\xi = 0.707$

c_1 & c_0 values can be calculated.

Q no. 3

Different forms of a PID controller

— $C(s) = K_c\left(1 + \frac{1}{\tau_i s} + \tau_D s\right) \longrightarrow$ Standard

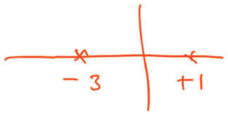
$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s} \longrightarrow \text{Polynomial}$$

$$C(s) = \frac{c_2 (s+r_1)(s+r_2)}{s} \longrightarrow \text{Pole-zero form}$$

$\longrightarrow$ Any form can be used and would be considered correct.

Part II

$$G(s) = \frac{2}{(s+3)(s-1)}$$



Assuming

$$C(s) = \frac{c_2 (s+r_1)(s+r_2)}{s}$$

Here, we will use pole-zero cancellation technique.

$\longrightarrow$ If the least dominant pole of $G(s)$ is stable, we take it equal to one of the zeros of the controller.

Hence $\longrightarrow r_2 = 3$

$$C(s) = \frac{c_2 (s+r_1)(s+3)}{s}$$

closed-loop poles

$$1 + CG = 0$$

$$1 + \frac{2c_2(s+r_1)}{s(s-1)} = 0$$

$$s^2 - s + 2c_2 s + 2c_2 r_1 = 0$$

$$s^2 + (2c_2 - 1)s + 2c_2 r_1 = 0$$

Compare with

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$-1 + 2c_2 = 2\xi\omega_n$	$2c_2 r_1 = \omega_n^2$
$c_2 = \frac{2\xi\omega_n + 1}{2}$	$r_1 = \frac{\omega_n^2}{2c_2}$
$c_2 = \frac{2(0.707)+1}{2}$	$r_1 = \frac{1}{2(1.207)}$
$\boxed{c_2 = 1.207}$	$\boxed{r_1 = 0.414}$

$$\boxed{C(s) = \frac{1.207(s+0.414)(s+3)}{s}}$$

Resulting PID controller

To verify the answer, consider the system without pole-zero cancellation.

$$1 + C(s)G(s) = 0$$

$$1 + \frac{c_2(s+r_1)(s+r_2)}{s} \times \frac{2}{(s+3)(s-1)} = 0$$

$$s(s-1)(s+3) + 2c_2(s+r_1)(s+r_2) = 0$$

$$s(s-1)(s+3) + 2c_2(s+r_1)(s+3) = 0$$

$$(s+3)\left(s(s-1) + 2c_2 s + 2c_2 r_1 \right) = 0$$

$$(s+3)\left(s^2 - s + 2.414 s + 1 \right) = 0$$

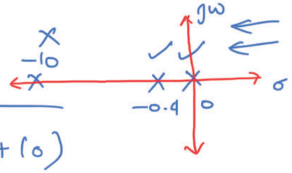
$$(s+3)\left(s^2 + \underbrace{1.414}_{2\zeta\omega_n} s + \underbrace{1}_{\omega_n^2} \right) = 0$$

$$(s+3)\left(s^2 + 2\zeta\omega_n s + \omega_n^2 \right) = 0$$

closed-loop poles of the system

Part #04

$$G(s) = \frac{s-3}{s(s+0.4)(s+10)}$$



We need only two poles for designing PID control. So, convert $s+10$ to a gain because it is the weakest pole.

$$G(s) = \frac{(s-3)}{s(s+0.4)(\overset{=0}{\cancel{s}}+10)} = \frac{0.1(s-3)}{s(s+0.4)}$$

$$C(s) = \frac{c_2 (s+r_1)(s+r_2)}{s}$$

Using pole-zero cancellation technique,

we take $r_2 = 0.4$ → the least dominant pole in approximated $G(s)$

$$C(s) = \frac{c_2 (s+r_1)(s+0.4)}{s}$$

closed-loop poles

$$1 + C(s)G(s) = 0$$

$$1 + \frac{c_2(s+r_1)}{s} \times \frac{(s-3)0.1}{s} = 0$$

$$s^2 + 0.1c_2(s-3)(s+r_1) = 0$$

$$s^2 + 0.1c_2\left(s^2 + (r_1-3)s - 3r_1\right) = 0$$

$$(1+0.1c_2)s^2 + 0.1c_2(r_1-3)s - 0.3c_2 r_1 = 0$$

$$s^2 + \frac{0.1c_2(r_1-3)}{1+0.1c_2} - \frac{0.3c_2 r_1}{1+0.1c_2} = 0$$

Compare coeff. with

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\frac{0.1c_2(r_1-3)}{1+0.1c_2} = 2\zeta\omega_n \qquad \frac{-0.3c_2 r_1}{1+0.1c_2} = \omega_n^2$$

$$\frac{0.1c_2 r_1}{1+0.1c_2} - \frac{0.3c_2}{1+0.1c_2} = 2\zeta\omega_n \qquad \frac{0.1c_2 r_1}{1+0.1c_2} = -\frac{\omega_n^2}{3}$$

Replace

$$-\frac{\omega_n^2}{3} - \frac{0.3c_2}{1+0.1c_2} = 2\zeta\omega_n$$

$$-\frac{0.3c_2}{1+0.1c_2} = \underbrace{2\zeta\omega_n + \frac{\omega_n^2}{3}}$$ → Assume $= F$

$$-\frac{0.3c_2}{1+0.1c_2} = F$$

$1 + 0.1 c_2$

$$-0.3 c_2 = F + 0.1 F c_2$$

$$-0.3 c_2 - 0.1 F c_2 = F$$

$$c_2 \left(-0.3 - 0.1F\right) = F$$

$$c_2 = \frac{-F}{0.3 + 0.1F}$$

$$\boxed{c_2 = \frac{-2\zeta\omega_n - \omega_n^2/3}{0.3 + 0.1\left(2\zeta\omega_n + \frac{\omega_n^2}{3}\right)}}$$

$$\boxed{\gamma_1 = \frac{-\omega_n^2/3\,\left(1 + 0.1 c_2\right)}{0.1\, c_2}}$$

Put the values of ζ & ω_n to get the values of c_2 & γ_1

Q # 0.4

Part 1

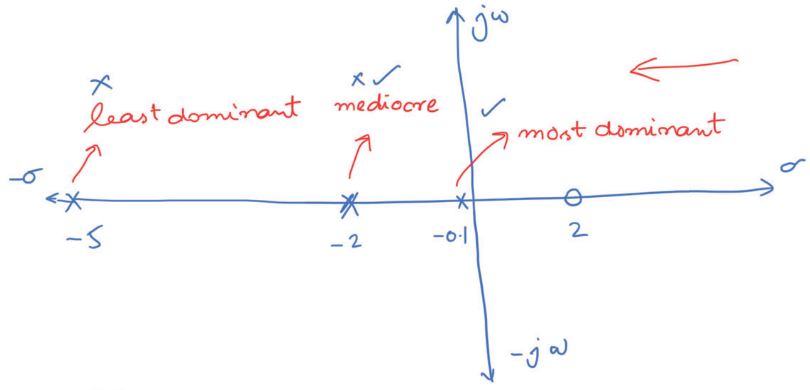
$$e^{-s} \cong \frac{-s+2}{s+2} \qquad G(s) = \frac{\boxed{e^{-s}}}{(s+5)(s+2)(s+0.1)}$$

Delay term

Pole-zero form $\quad C(s) = \dfrac{c_2(s+\gamma_1)(s+\gamma_2)}{s(s+l_0)} \leftarrow$ (PID with Filter)

First of all, $G(s)$ needs a bit of work. Using Padé approximation,

$$G(s) = \frac{-s+2}{(s+5)(s+2)(s+0.1)}$$



We are just looking for a total of two poles to design a PID. Therefore,

$$G(s) = \frac{-s+2}{(s+0.1)(s+2)(\cancel{s}+2)_{=0}(\cancel{s}+5)_{=0}}$$

$$G(s) = \frac{-0.1\;(s-2)}{(s+0.1)(s+2)}$$

Now $G(s)$ is ready for further work. Assuming γ_2 in $C(s)$ equal to 2,

$$C(s) = \frac{c_2\,(s+\gamma_1)(s+2)}{s(s+l_0)}$$

$$1 + CG = 0$$

$$1 - \frac{0.1\; c_2(s+\gamma_1)(s-2)}{s(s+0.1)(s+l_0)} = 0$$

$$s(s+0.1)(s+l_0) - 0.1\; c_2(s+\gamma_1)(s-2) = 0$$

$$s\left(s^2+(0.1+l_0)s+0.1l_0\right)-0.1\,c_2\left(s^2+(r_1-2)s-2r_1\right)=0$$

$$s^3+(0.1+l_0)s^2+0.1l_0s-0.1c_2s^2-0.1c_2(r_1-2)s+0.2c_2r_1=0$$

$$s^3+(0.1+l_0-0.1c_2)s^2+(0.1l_0+0.2c_2-0.1c_2r_1)s+0.2c_2r_1=0$$

Comparing with

$$(s+\lambda)(s^2+2\zeta\omega_n s+\omega_n^2)=0$$

$$\lambda=\omega_n$$

$$s^3+(2\zeta+1)\omega_n s^2+(2\zeta+1)\omega_n^2 s+\omega_n^3=0$$

$\boxed{s^2}$

$$0.1+l_0-0.1c_2=(2\zeta+1)\omega_n$$

$$\boxed{l_0=(2\zeta+1)\omega_n-0.1+0.1c_2}$$

$\boxed{s}$

Replace

$$0.1l_0+0.2c_2-0.1c_2r_1=(2\zeta+1)\omega_n^2$$

$\boxed{\text{Constt}}$

Replace

$$0.2c_2r_1=\omega_n^3$$

$$\boxed{0.1c_2r_1=\frac{\omega_n^3}{2}}$$

$$0.2c_2=(2\zeta+1)\omega_n^2+0.1c_2r_1-0.1l_0$$

$$0.2c_2=(2\zeta+1)\omega_n^2+\frac{\omega_n^3}{2}-0.1\left((2\zeta+1)\omega_n-0.1+0.1c_2\right)$$

$$0.2c_2=(2\zeta+1)\omega_n^2+\frac{\omega_n^3}{2}-0.1(2\zeta+1)\omega_n+0.01-0.01c_2$$

$$\boxed{c_2=\frac{(2\zeta+1)\omega_n^2+\omega_n^3/2-0.1(2\zeta+1)\omega_n+0.01}{0.21}}$$

$$\boxed{c_2=12.77}$$

$$r_1=\frac{\omega_n^3}{0.2c_2}=0.392$$

$$\boxed{r_1=0.392}$$

$$l_0=(2\zeta+1)\omega_n-0.1+0.1c_2$$

$$\boxed{l_0=3.59}$$

$$\boxed{C(s)=\frac{12.77(s+0.392)(s+2)}{s(s+3.59)}}$$