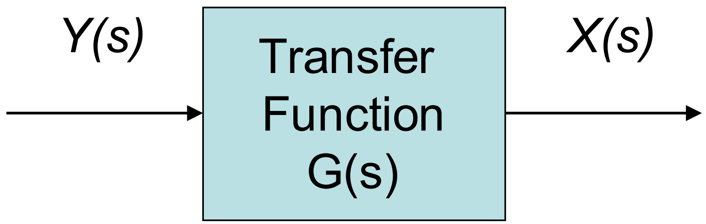
Mechatronics Design
MIET 2362

Topic 6
Transfer Function

Definition

Time domain

s domain



$$G = gain = \frac{output}{input}$$

$$G(s) = Transfer\ \ Function = \frac{Laplace\ transform\ of\ \ output}{Laplace\ transform\ of\ \ input}$$

Laplace Transform Definitions

One (Easy) Way to Solve ODEs

Laplace transform

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

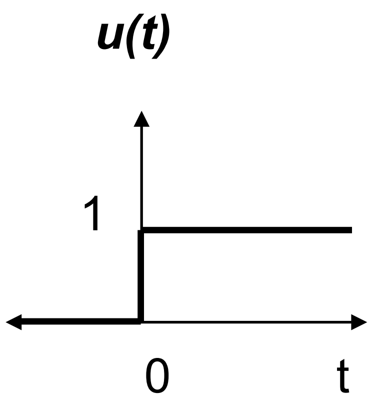
$$F(s) = L\{f(t)\}$$

Inverse Laplace transform

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - jw}^{\sigma + jw} F(s) e^{st} ds$$

$$f(t) = L^{-1}\{F(s)\}$$

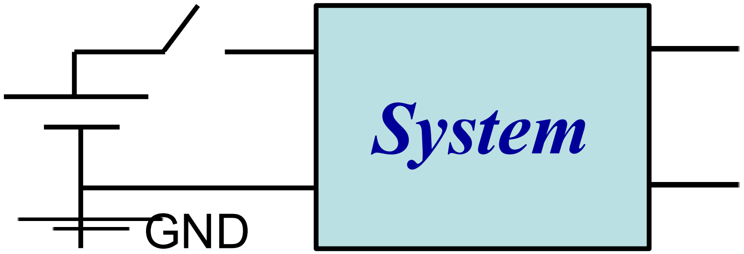
The Unit Step Function



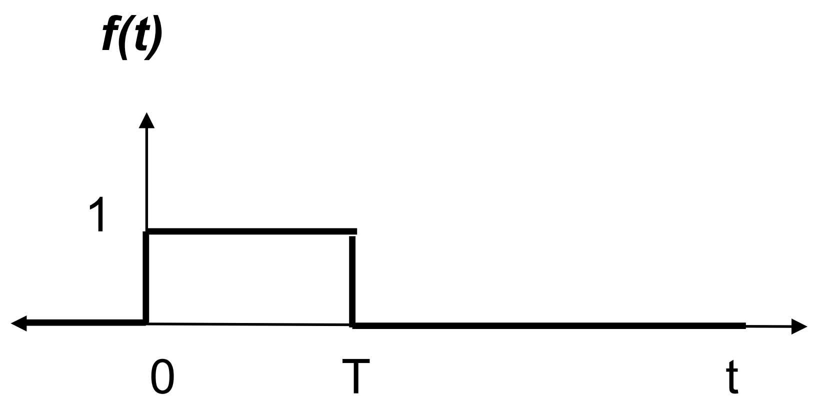
$$u(t) = 0,\ t < 0$$
$$u(t) = 1,\ t \geq 0$$

$$L\{u(t)\} = \int_0^\infty e^{-st} u(t)dt = \int_0^\infty 1e^{-st} dt = -\frac{1}{s}[e^{-st}]_0^\infty = \frac{1}{s}$$

$$L\{u(t)\} = U(s) = \frac{1}{s}$$

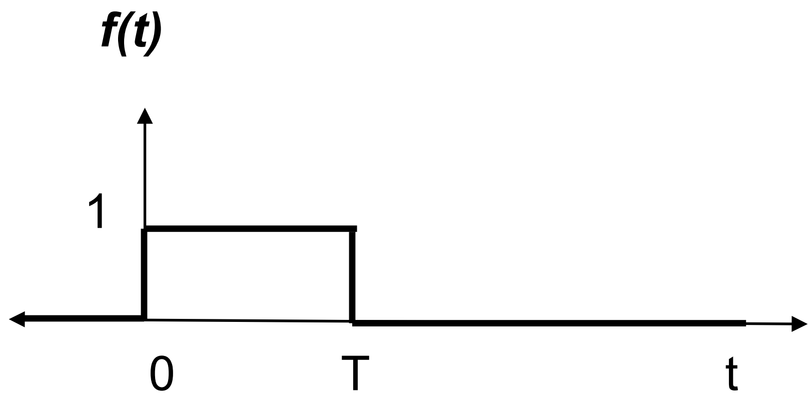


Digital Signal



Find the Laplace transform of this signal using LT definition integral

Solution



$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

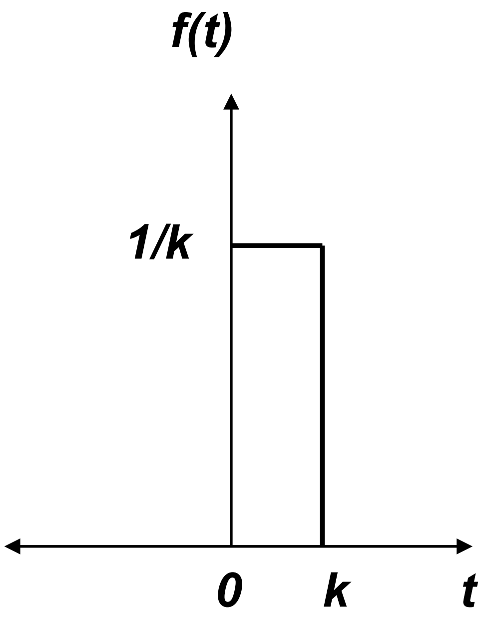
$$\int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt =$$

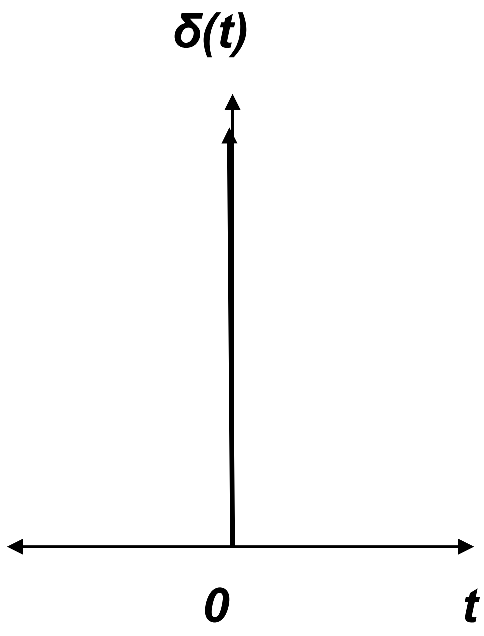
$$\int_0^T e^{-st} * 1dt + 0 = \frac{1}{-s}\left[e^{-st}\right]_0^T$$

$$L = \frac{1}{s}(1 - e^{-sT})$$

f(t) = 0, t < 0
f(t) = 1, 0 ≤ t ≤ T
f(t) = 0, t > T

Impulse Function





$$f(t) = \frac{1}{k} \; for \; 0 \le t < k$$

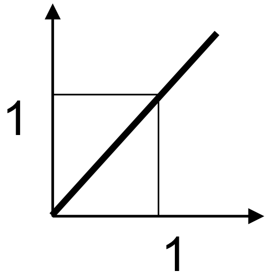
$$f(t) = 0 \; for \; t > k$$

$$F(s) = 1$$

Laplace Transform of Some Common Functions

$\delta(t)$, *unit impulse*	1
$\delta(t-T)$, *delayed unit impulse*	e^{-sT}
$u(t)$, *a unit step*	$\dfrac{1}{s}$
$u(t-T)$, *a delayed unit step*	$\dfrac{e^{-sT}}{s}$

Laplace Transform of Some Common Functions



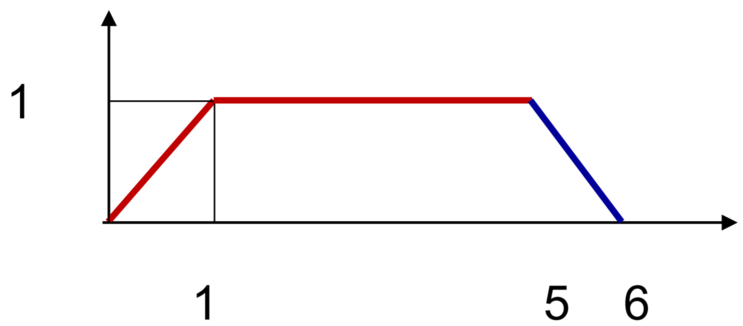
$t,\ a\ unit\ ramp$ $$F(s)=\int_0^\infty te^{-st}dt=\left[\frac{te^{-st}}{-s}\right]_0^\infty+\frac{1}{s}\int_0^\infty e^{-st}dt=\frac{1}{s^2}$$

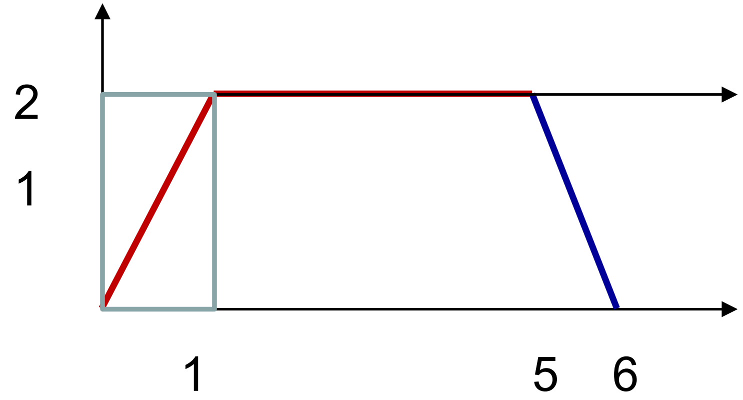
$t^n,\ n-th\ order\ ramp$ $$\frac{n!}{s^{n+1}}$$

e^{-at}, exponential decay $$\frac{1}{s+a}$$

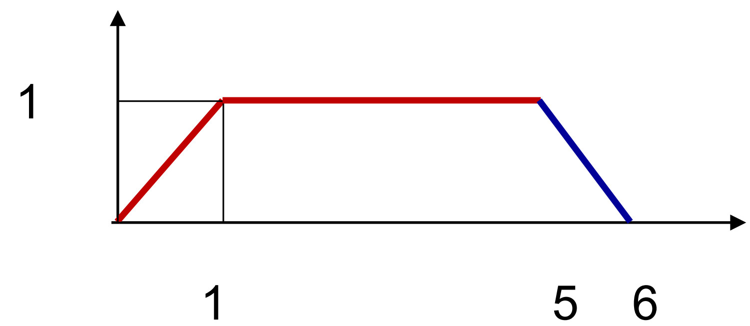
$1-e^{-at}$, exponential growth $$\frac{a}{s(s+a)}$$

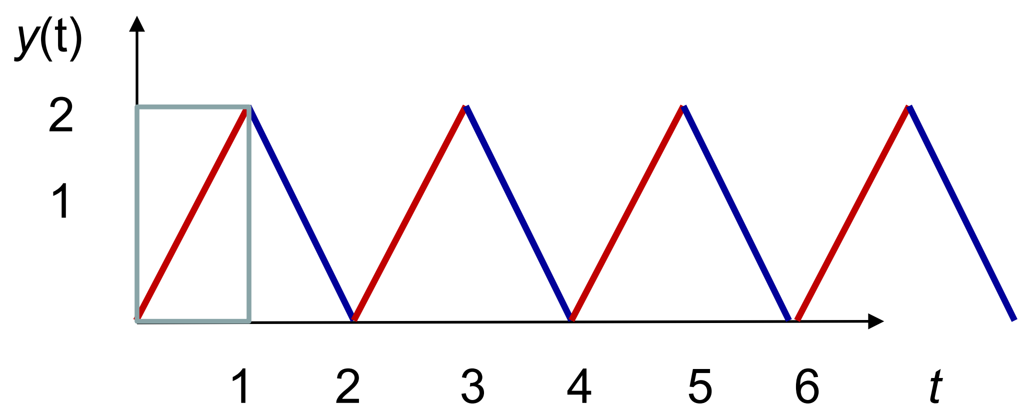
Laplace Transform of Some Common Functions



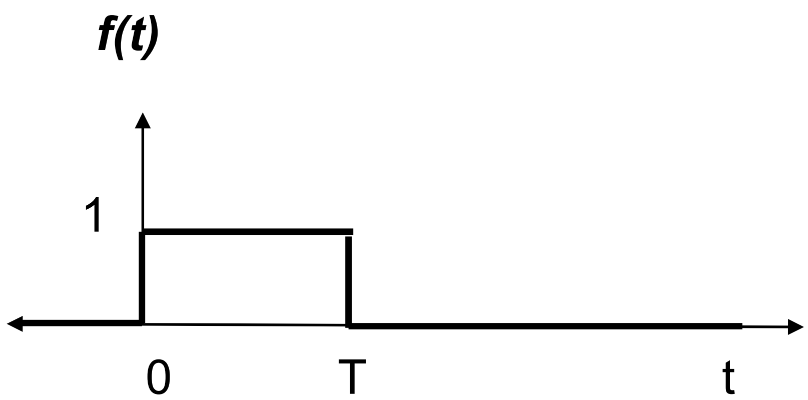


Laplace Transform of Some Common Functions



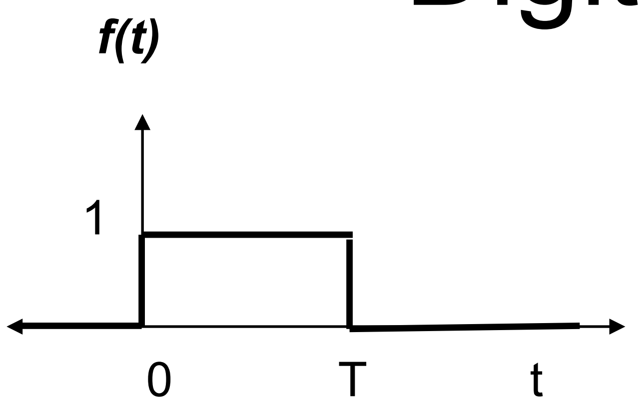


Digital Signal



Find the Laplace transform using common signals

Digital Signal Again



$u(t)$, a unit step $\quad \dfrac{1}{s}$

$u(t-T)$, a delayed unit step $\quad \dfrac{e^{-sT}}{s}$

$u(t)-u(t-T) \Rightarrow \quad \Delta U(s) = \dfrac{1}{s}(1-e^{-sT})$

Laplace Transform Properties

- Linearity

$$L\{af(t)+bg(t)\}=aLf(t)+bLg(t)$$

- Shifting in s – domain

$$L\{e^{at}f(t)\}=F(s-a)$$

- Time domain shifting

$$L\{f(t-T)u(t-T)\}=e^{-sT}F(s)$$

- Periodic functions

$$f(t)=f(t+T), \quad Lf(t)=\frac{1}{1-e^{-sT}}F_1(s)$$

F_1 is Laplace transform for the first period only

Laplace Transform Properties

- Initial and final values
- Derivatives
- Integrals

First Order System

$$a_1 \frac{dx}{dt} + a_0 x = b_0 y$$

$\mathrm{a}_1\ \mathrm{a}_0\ \mathrm{b}_0$ are constants, y *and* x are input and output

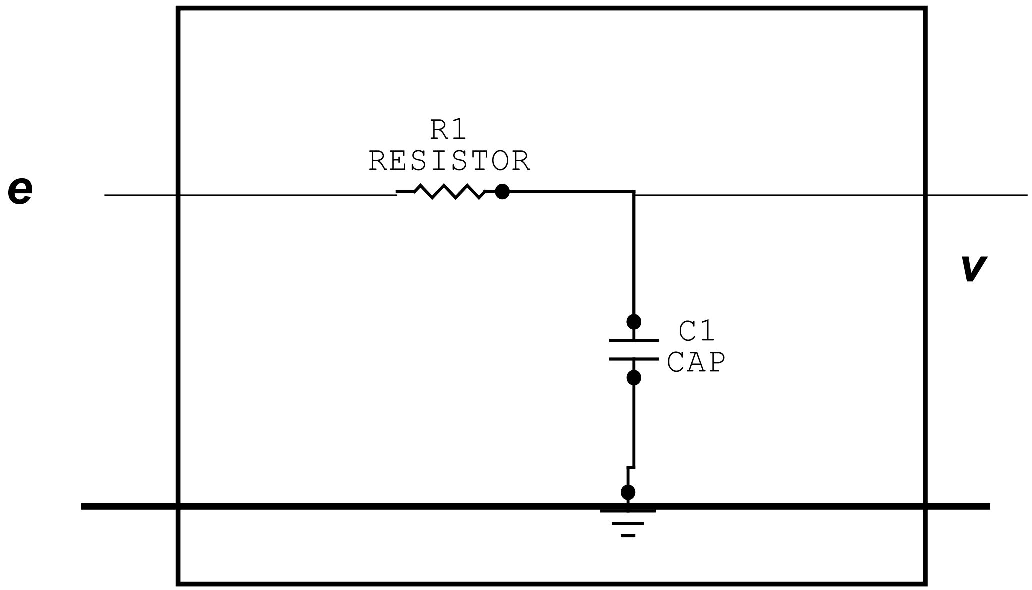
Laplace transform with all initial conditions zero is

$$\mathrm{a}_1 sX(s) + \mathrm{a}_0 X(s) = \mathrm{b}_0 Y(s)$$

$$G(s) = \frac{X(s)}{Y(s)} = \frac{b_0}{a_1 s + a_0}$$

$$G(s) = \frac{b_0 / a_0}{(a_1 / a_0)s + 1} = \frac{G}{\tau s + 1}$$

A First Order System



$$e = iR + v; \quad i = C\frac{dv}{dt}$$

$$e = C\frac{dv}{dt}R + v$$

$$E(s) = RCsV(s) + V(s)$$

$$\frac{V(s)}{E(s)} = \frac{V(s)}{sRCV(s) + V(s)}$$

$$\frac{V(s)}{E(s)} = \frac{1}{sRC + 1} = \frac{1}{\tau s + 1}$$

$$\tau = RC = \text{time constant}$$

Second Order System

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_0 y$$

$$G(s) = ?$$

Second Order System

$$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_0 y$$

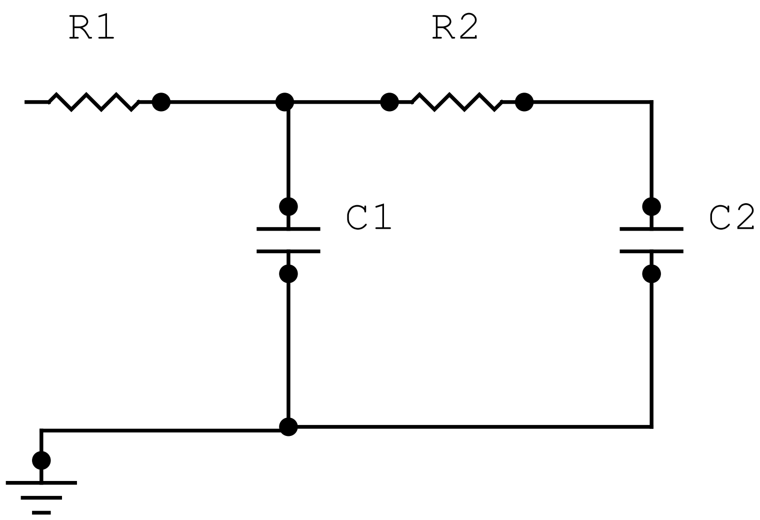
$a_2\ a_1\ a_0\ b_0$ are constants y is the input, x is the output

Laplace transform with all initial conditions zero is

$$a_2 s^2 X(s) + a_1 s X(s) + a_0 X(s) = b_0 Y(s)$$

$$G(s) = \frac{X(s)}{Y(s)} = \frac{b_0}{a_2 s^2 + a_1 s + a_0}$$

A Second Order System



$$\frac{V(s)}{E(s)} = \frac{1}{(1+\tau_1 s)(1+\tau_2 s)}$$

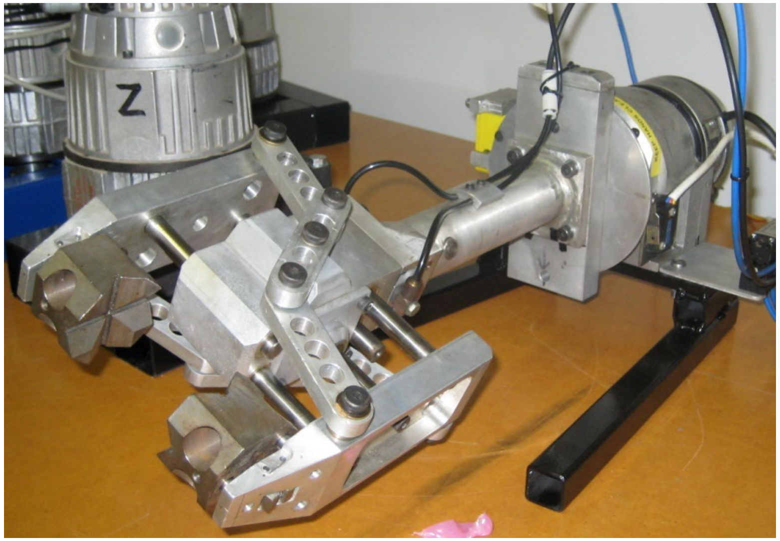
$$\frac{V(s)}{E(s)} = \frac{A}{1+\tau_1 s} + \frac{B}{1+\tau_2 s}$$

A Second Order System Example

A robot arm has following transfer function:

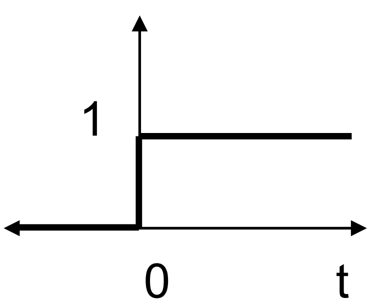
$$G(s) = \frac{K}{(s+3)^2}$$

If an unit step input is applied, what will be the output?



A Second Order System
Exam Question Example

$$X(s) = G(s)Y(s) = \frac{K}{(s+3)^2} \times ?$$



Find the response in time domain and draw the output function

Solution

$$X(s) = G(s)Y(s) = \frac{K}{(s+3)^2} \times \frac{1}{s}$$

Using partial fractions we can get

$$X(s) = \frac{K}{9s} - \frac{K}{9(s+3)} - \frac{K}{3(s+3)^2}$$

The inverse transform is

$$x(t) = \frac{1}{9}K - \frac{1}{9}Ke^{-3t} - \frac{1}{3}Kte^{-3t}$$

Solution - Explained

$$X(s) = \frac{K}{(s+3)^2} \times \frac{1}{s} = \frac{A}{s} + \frac{Bs+C}{(s+3)^2} = \frac{A}{s} + \frac{C}{(s+3)} + \frac{D}{(s+3)^2}$$

$$\frac{K}{(s+3)^2} \times \frac{1}{s} = \frac{A}{s} + \frac{C}{(s+3)} + \frac{D}{(s+3)^2}$$

$$K = A(s+3)^2 + Cs(s+3) + Ds$$

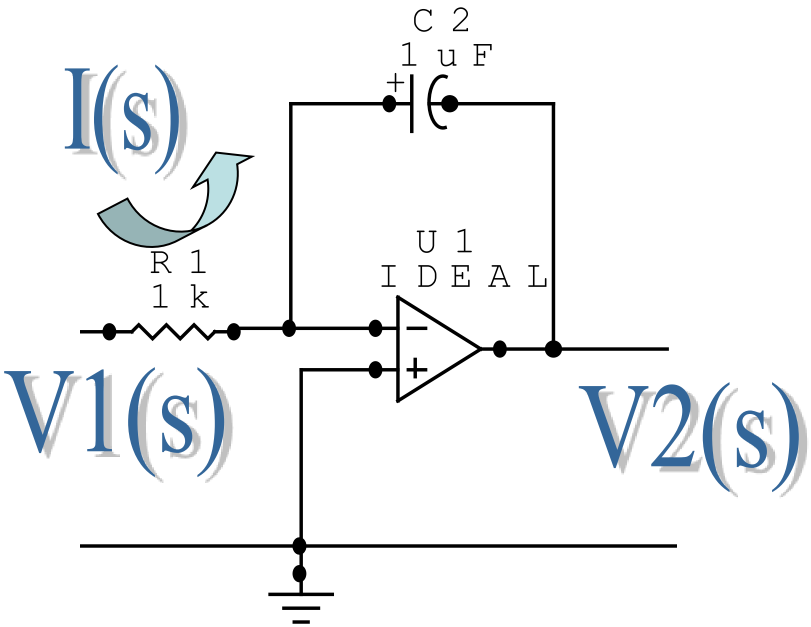
$$A = \frac{K}{9}; \qquad C = -\frac{K}{9}; \quad D = -\frac{K}{3}$$

The Method of Partial Fractions With Laplace Transform

http://www.math.oregonstate.edu/home/programs/undergrad/CalculusQuestStudyGuides/ode/laplace/pf/pf.html

More Transfer Functions

Filter, Integrating Circuit



$$v1(t) = R \times i(t)$$

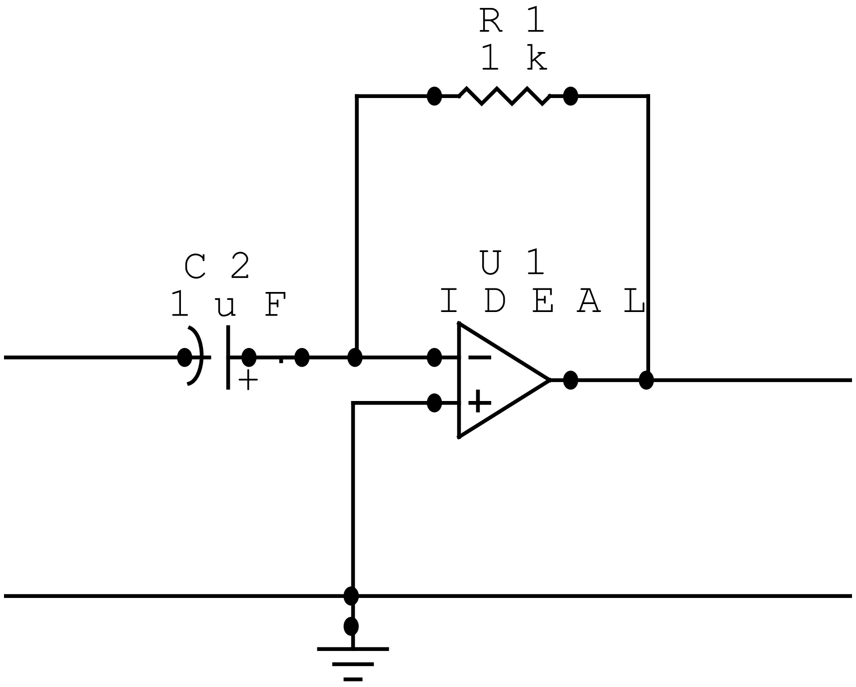
$$V1(s) = R \times I(s)$$

$$v2(t) = -\frac{1}{C}\int i(t)dt$$

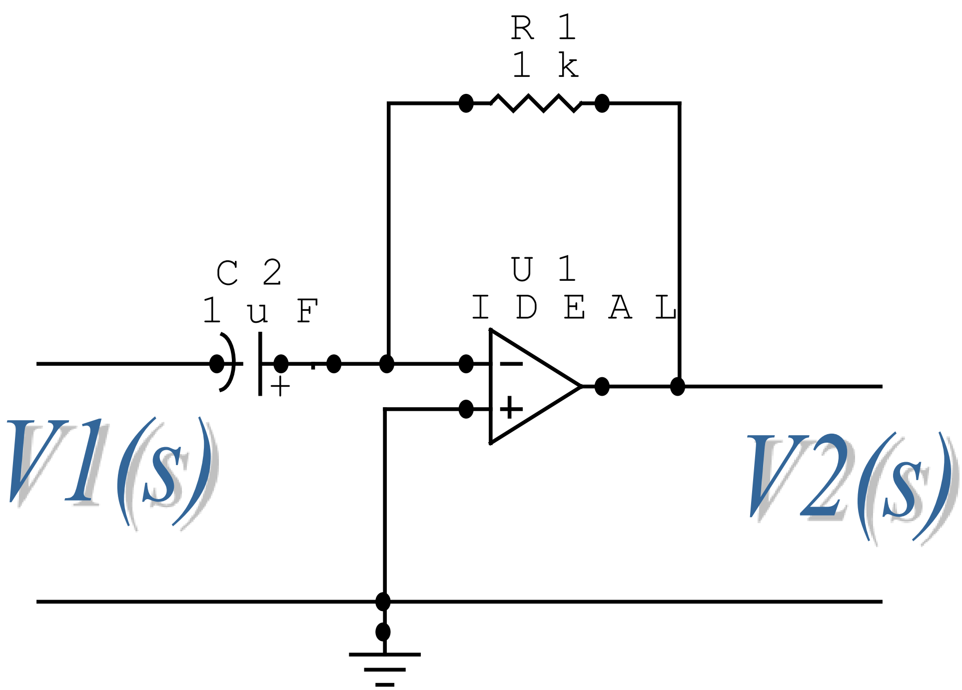
$$V2(s) = -\frac{1}{sC} \times I(s)$$

$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$

Differentiating Circuit



Differentiating Circuit

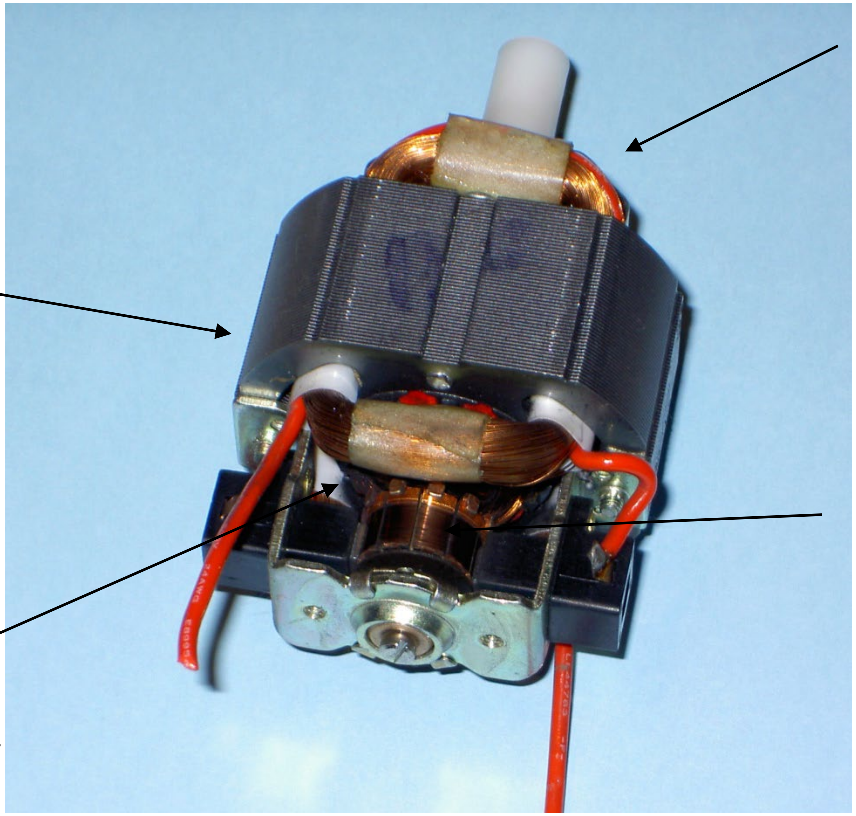


$$\frac{V_2(s)}{V_1(s)} = -RCs$$

DC Motor Physical Structure

Field coils

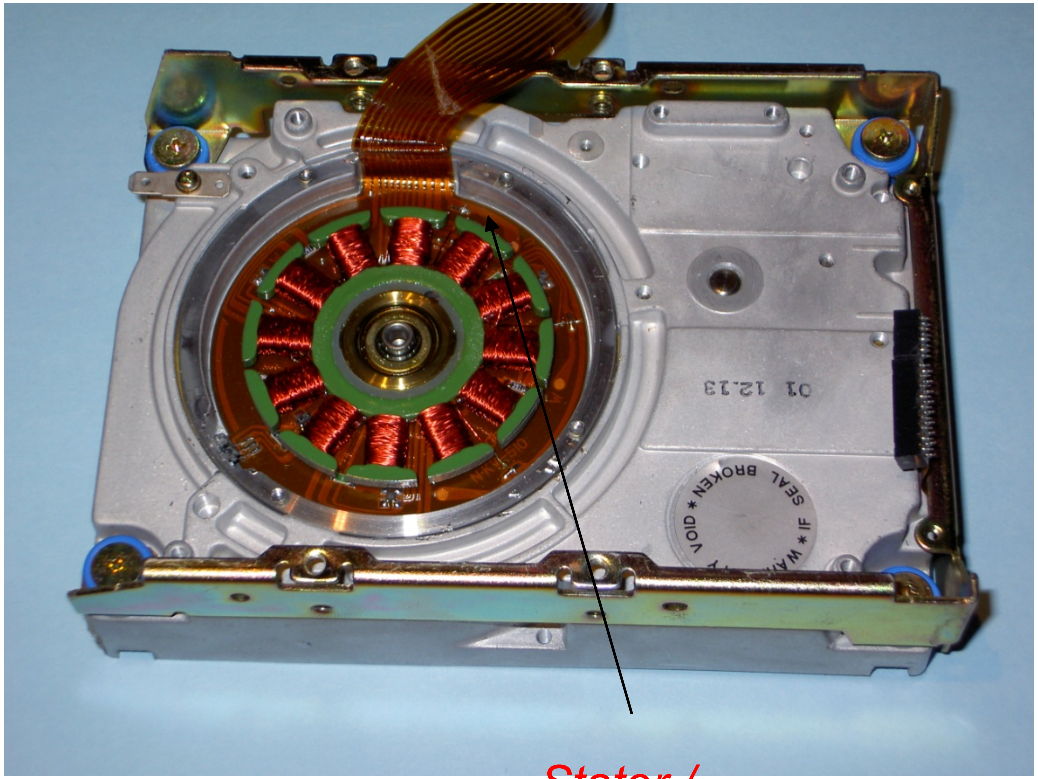
Stator
Field Frame
or yoke,
Field Pole



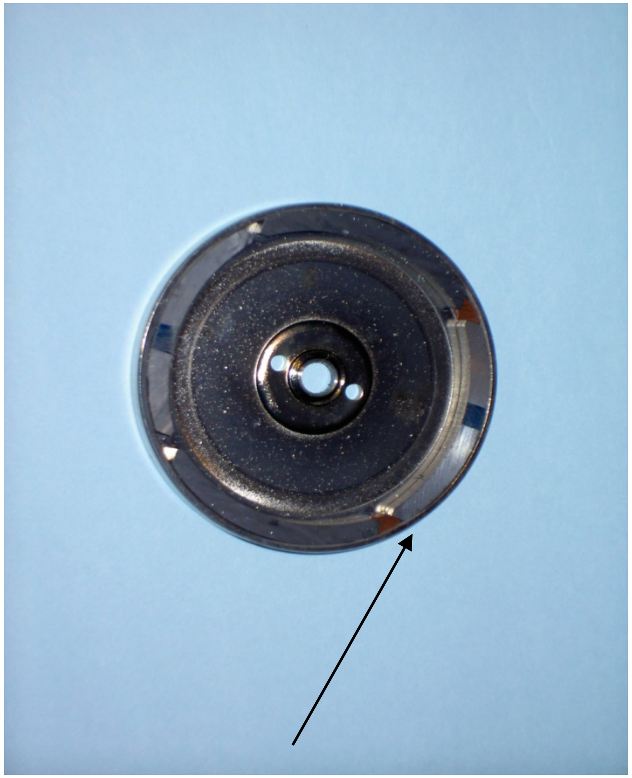
Commutator

Rotor
Armature and
Commutator

BLDC -Hard Disk Spindle Motor



Stator / Field Pole



Permanent Magnet Rotor

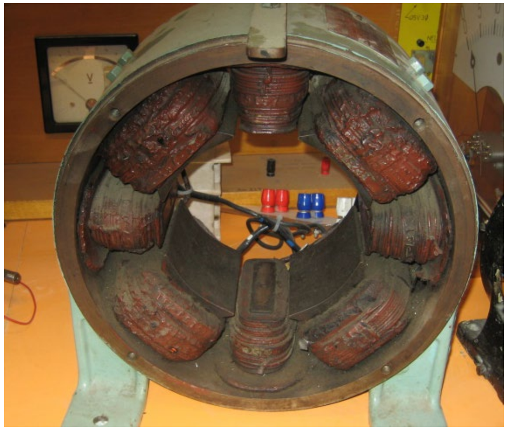
A DC Motor Components: Armature / Rotor and Stator

1

Rotor

2

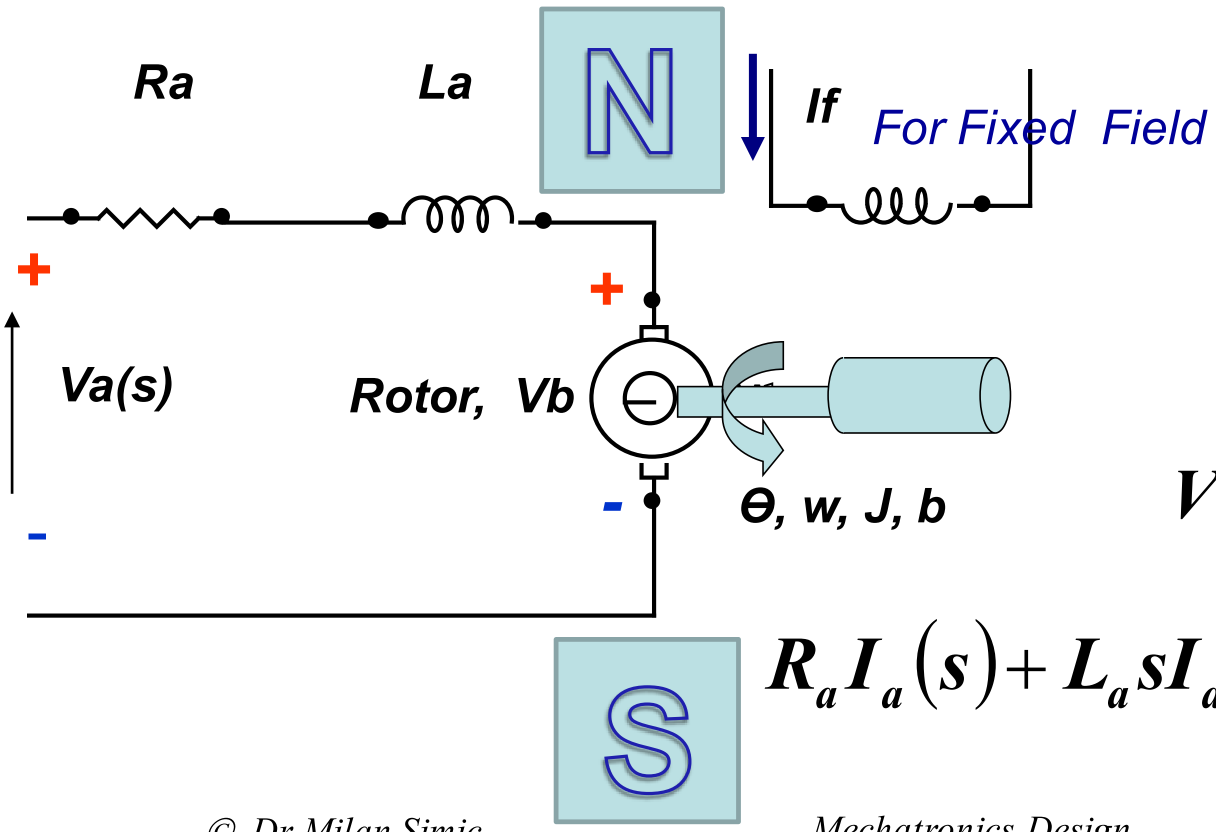
Stator Permanent Magnet,
or Electromagnet like this one



Field Winding /

Field Magnet

DC Motor Armature Controlled with Permanent Magnet, or Fixed Field



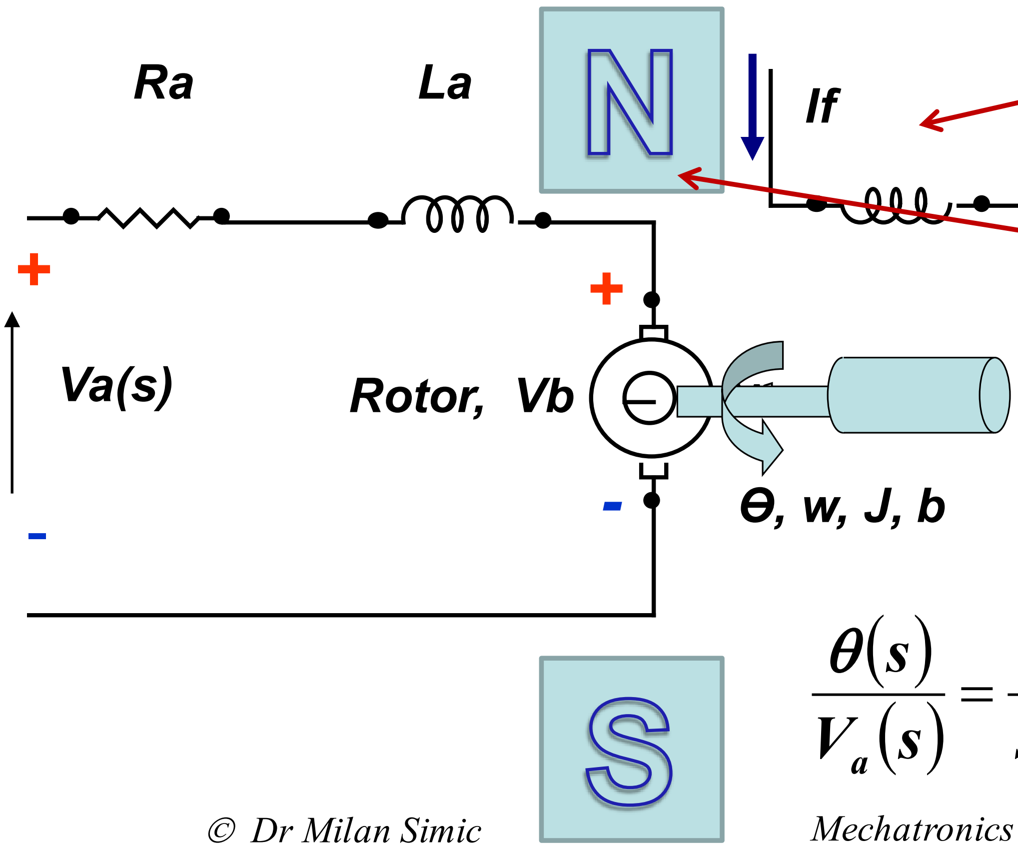
$$e = -\frac{d\phi(t)}{dt}$$

$$v_b(t) = K_b \frac{d\theta(t)}{dt}$$

$$V_b(s) = K_b s\theta(s)$$

$$R_a I_a(s) + L_a s I_a(s) - V_b(s) = V_a(s)$$

DC Motor Armature Controlled with Permanent Magnet, or Fixed Field



Stator

Fixed Field (Electromagnet) ,

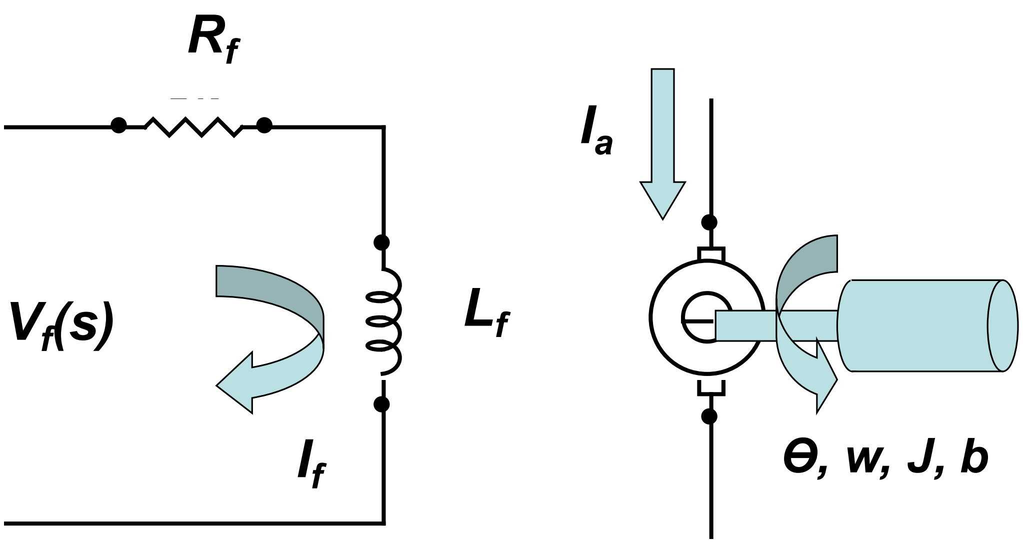
Field Coil (Winding)

Or , Permanent Magnet (PM)

J = Inertia
b = Viscous Friction Coefficient

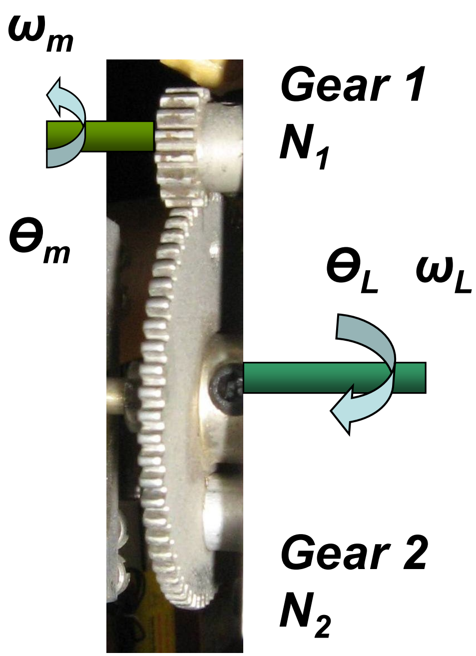
$$\frac{\theta(s)}{V_a(s)} = \frac{K_m}{s[(R_a + L_a s)(Js + b) + K_b K_m]}$$

DC Motor Field Controlled



$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js+b)(L_f s + R_f)}$$

Gear Train, Rotational Transformer



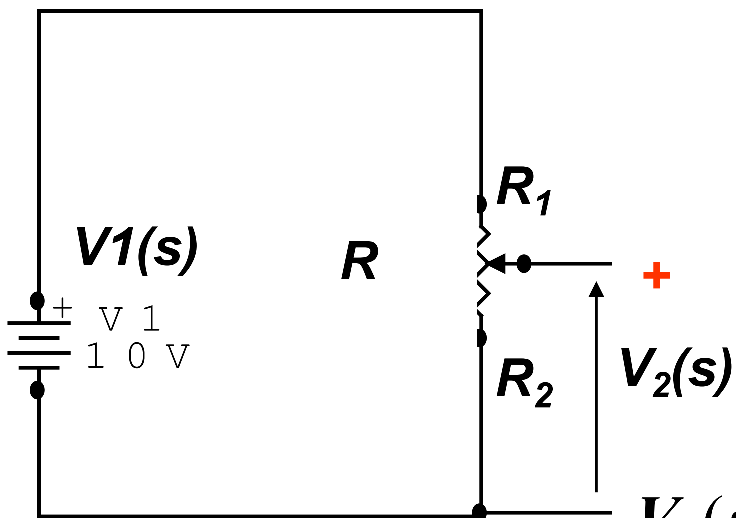
$$\text{Gear Ratio} = \boldsymbol{n} = \frac{N_1}{N_2}$$

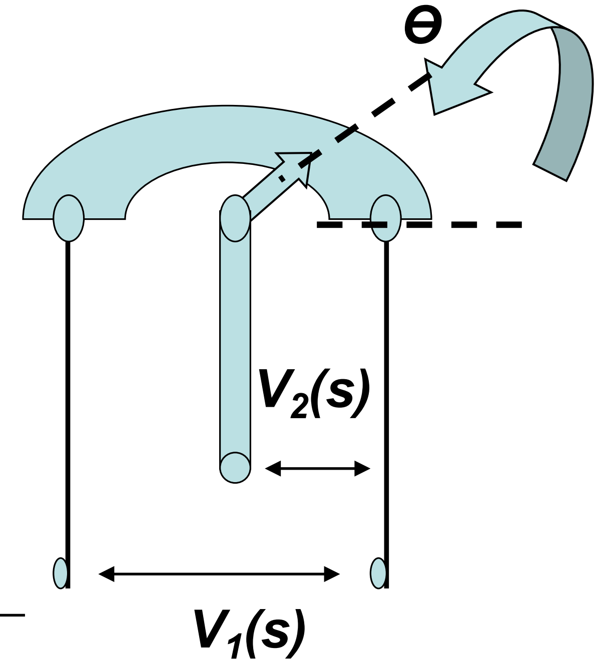
$$N_2 \theta_L = N_1 \theta_m$$

$$\theta_L = n\theta_m$$

$$\omega_L = n\omega_m$$

Potentiometer, Voltage Control





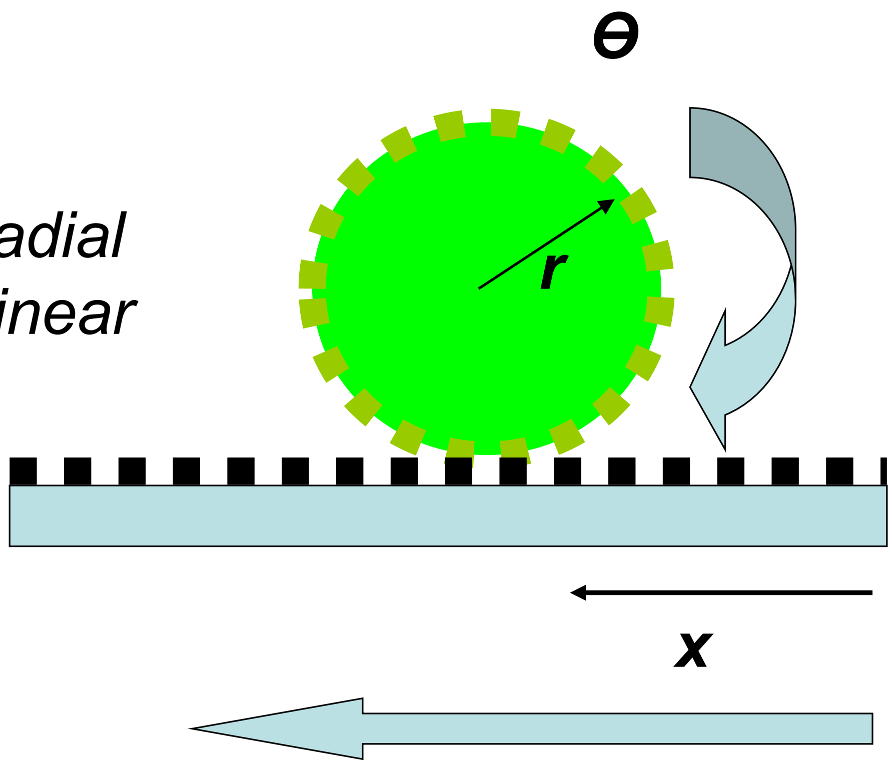
$$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R} = \frac{R_2}{R_1 + R_2}$$

$$\frac{R_2}{R} = \frac{\theta}{\theta_{\max}}$$

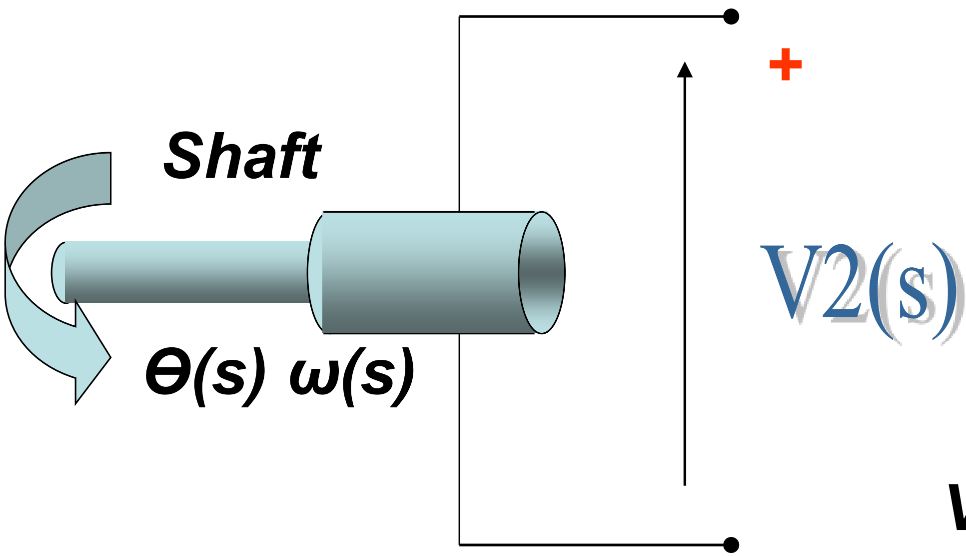
Rack and Pinion

x* = *rθ

Converts radial motion to linear motion



Tachometer, Velocity Sensor



$V2(s)=K_t w(s)=K_t s \Theta(s)$

K_t = constant

Block Diagram Models

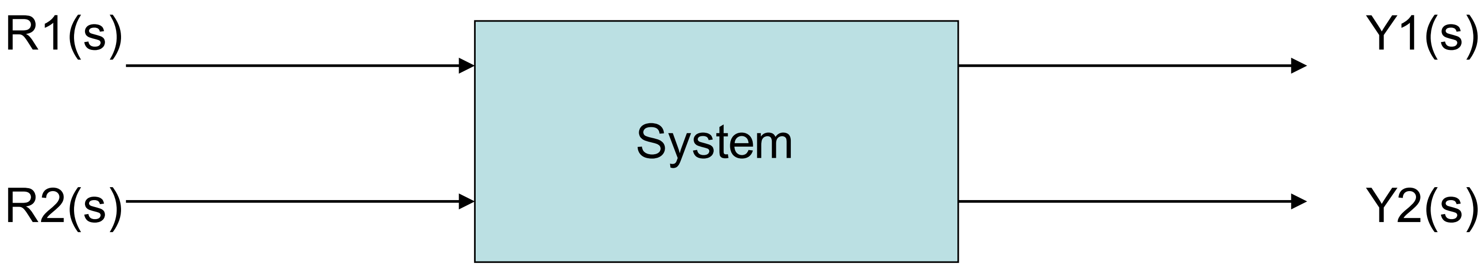
- Dynamic systems that contain automatic control sub-systems can mathematically be represented by a set of simultaneous differential equations.
- Application of Laplace transform simplifies solutions to the domain of linear algebraic equations.
- The block diagram representation of the control system is widely used in the system design.

Block Diagram

- Block diagram consists of unidirectional operational blocks that represent transfer functions of the variables involved.
- A block diagram of previously analysed DC motor (field controlled) is shown below.

$$V_f(s) \longrightarrow \boxed{G(s) = \frac{K_m}{s(Js+b)(L_f+R_f)}} \longrightarrow \theta(s)$$

Complex System



$$Y1(s)=G11(s)R1(s)+G12(s)R2(s)$$
$$Y2(s)=G21(s)R1(s)+G22(s)R2(s)$$

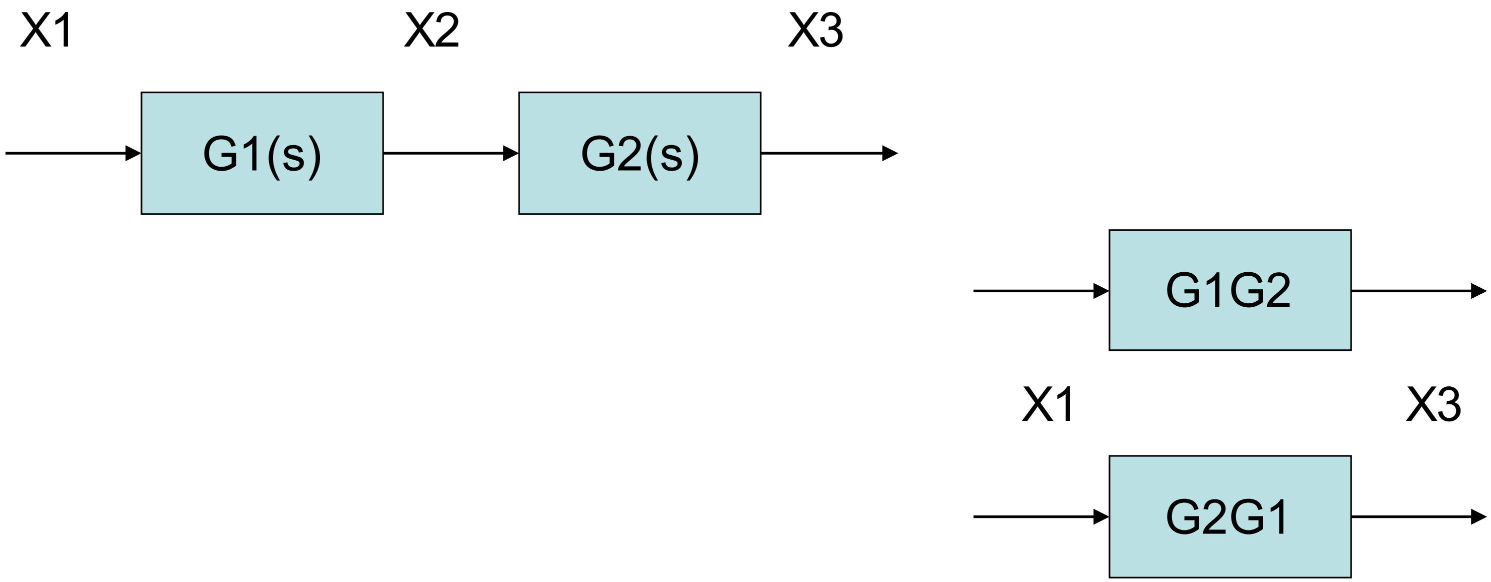
We can have ***m*** inputs and ***n*** outputs

Y=GR

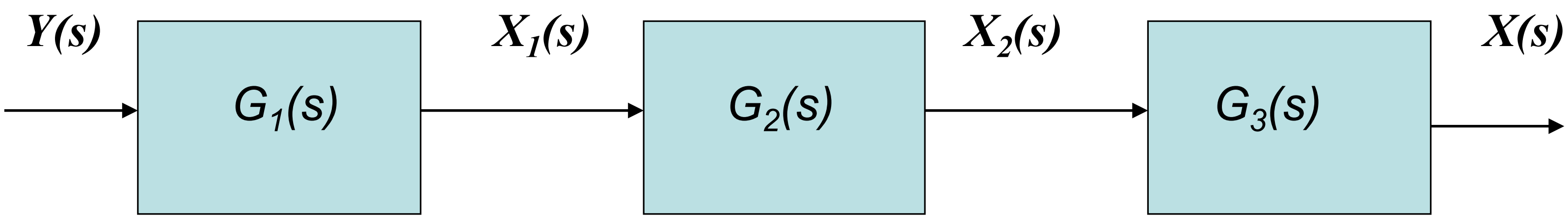
where *G* is a *mxn transfer function* matrix and

Y and **R** are column matrices

Transformations



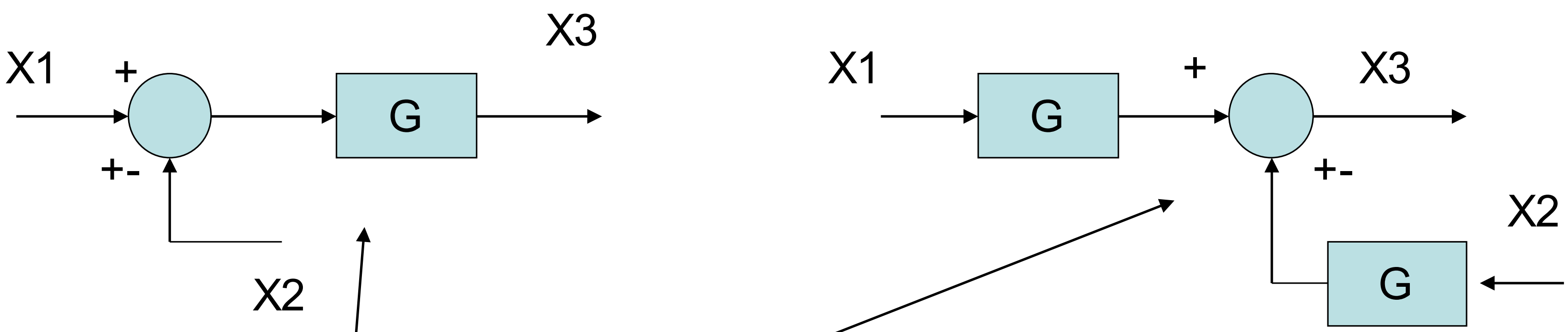
Systems in Series



$$G(s)=\frac{X(s)}{Y(s)}=\frac{X_1(s)}{Y(s)}\times\frac{X_2(s)}{X_1(s)}\times\frac{X(s)}{X_2(s)}$$

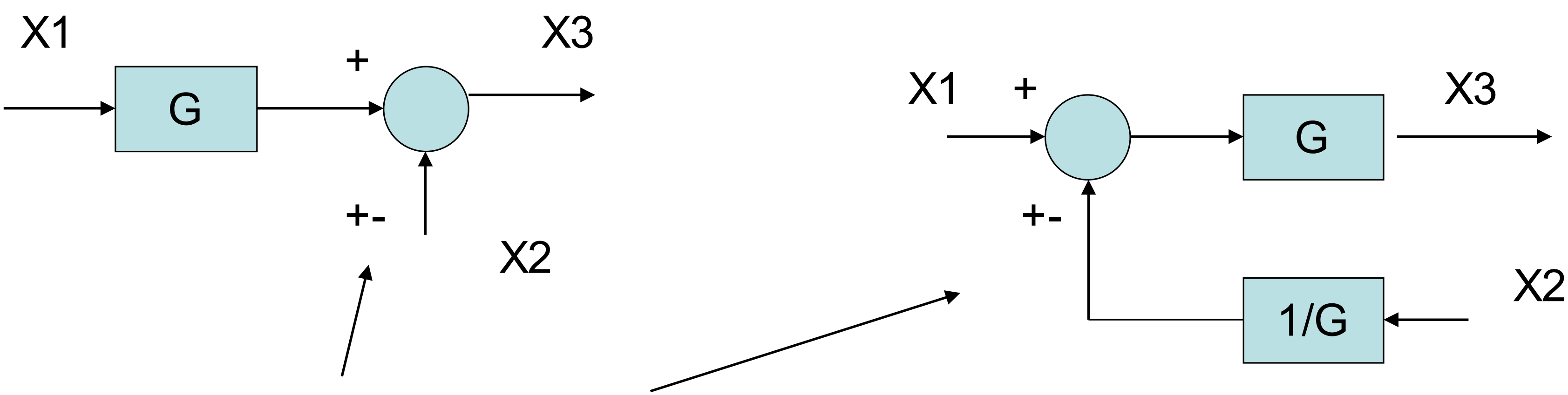
$$G(s)=G_1(s)\times G_2(s)\times G_3(s)$$

Moving a Summing Point, 1



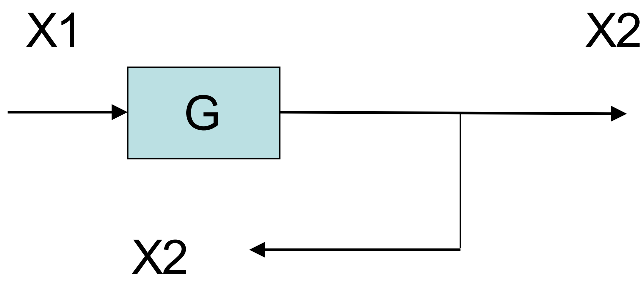
X3=G(X1±X2)=GX1±GX2

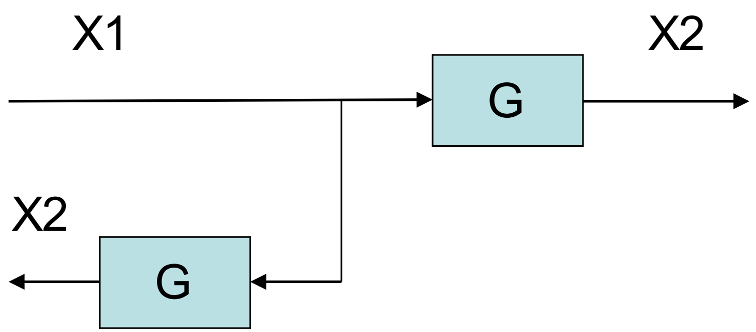
Moving a Summing Point, 2



$X3=GX1 \pm X2=G(X1 \pm X2*1/G)$

Moving a Pickoff Point, 1

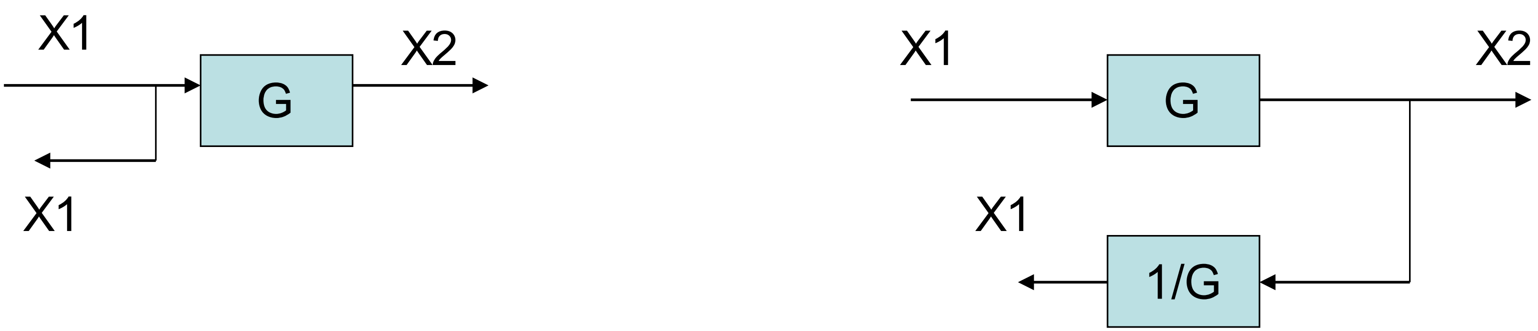




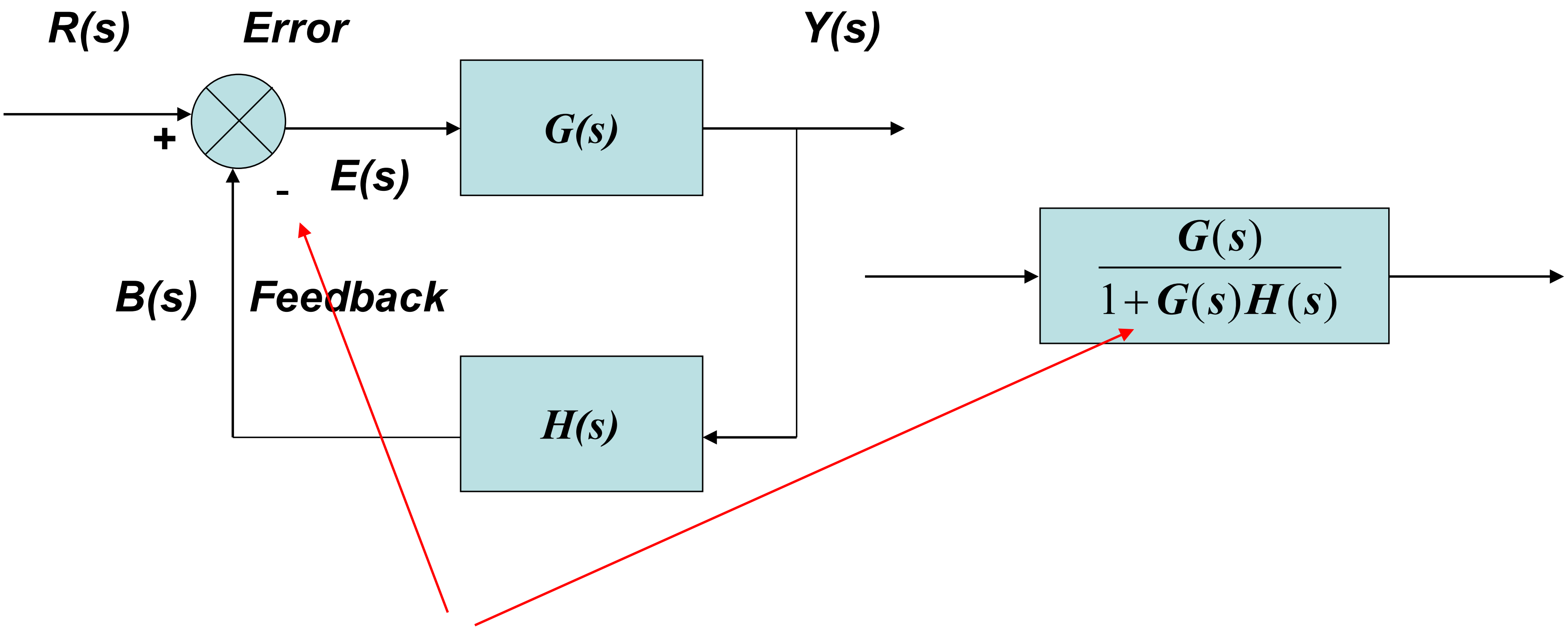
X2=GX1

X2=GX1
X2=GX1

Moving a Pickoff Point, 2



Negative Feedback Loops



Feedback Loops - Equations

$$E(s) = R(s) - B(s) = R(s) - H(s)Y(s)$$

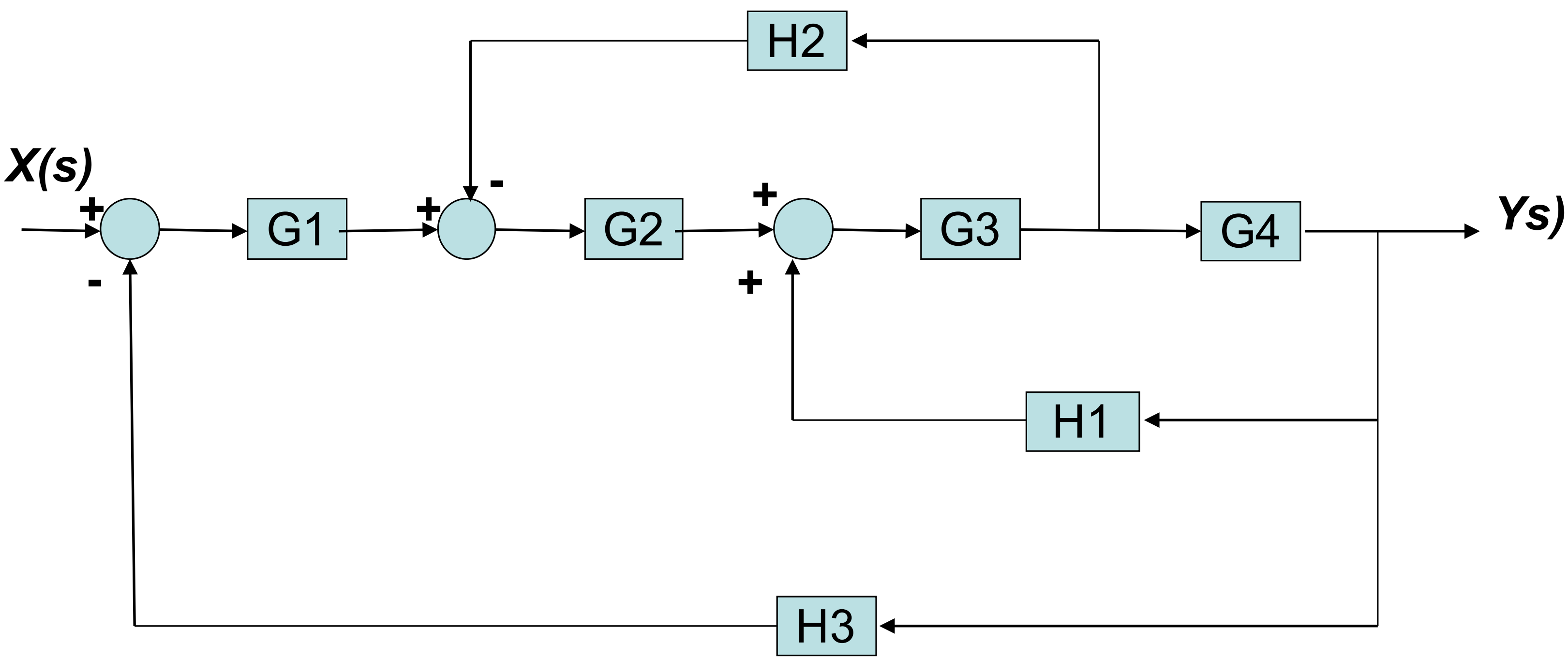
$$Y(s) = G(s)E(s)$$

$$Y(s) = G(s)[R(s) - H(s)Y(s)]$$

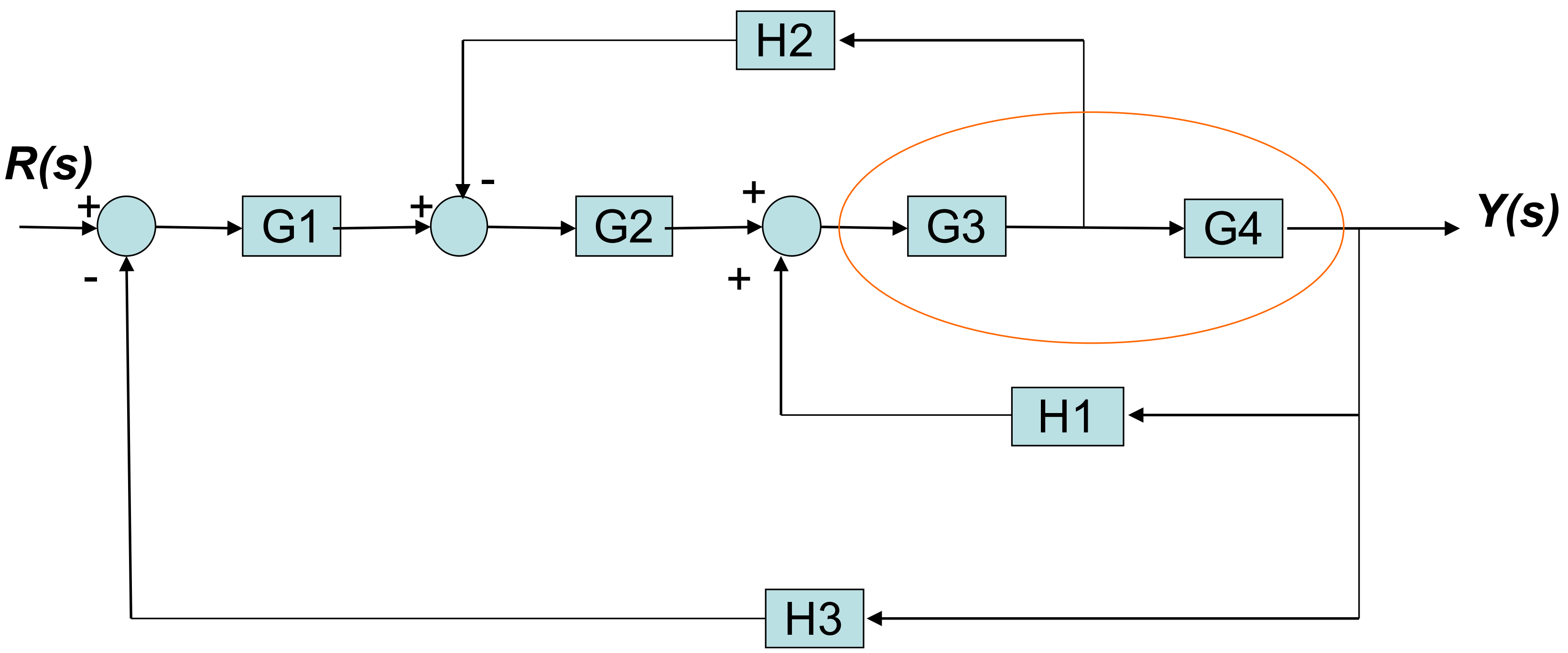
$$Y(s)[1 + G(s)H(s)] = G(s)R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Block Diagram Reduction



Block Diagram Reduction



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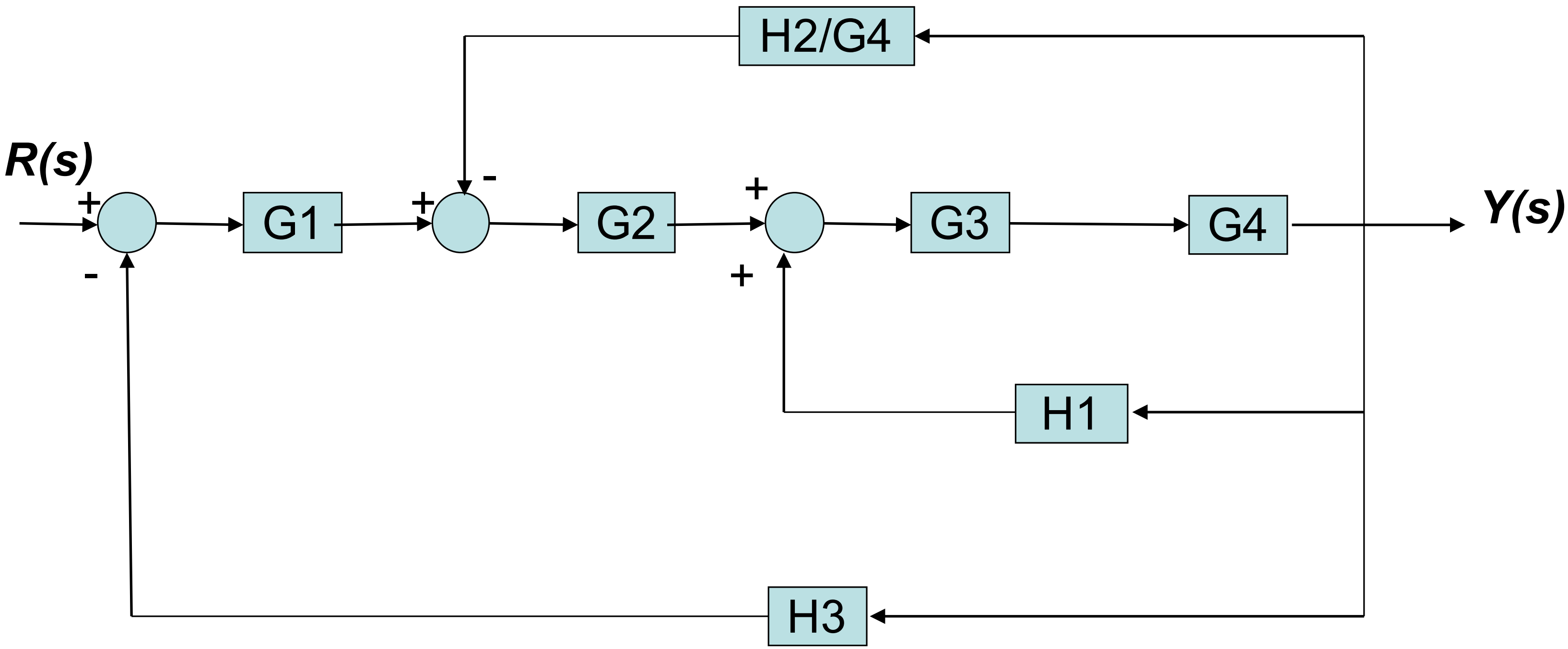

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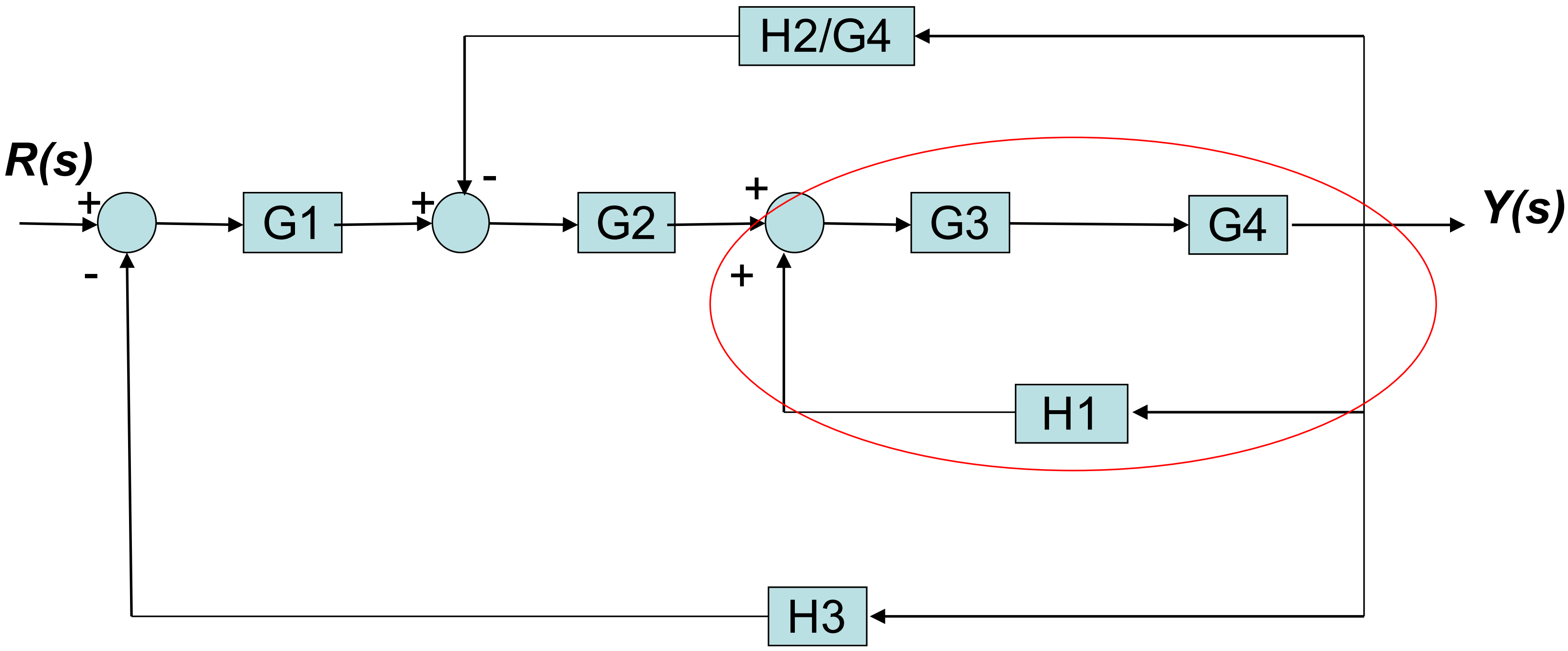

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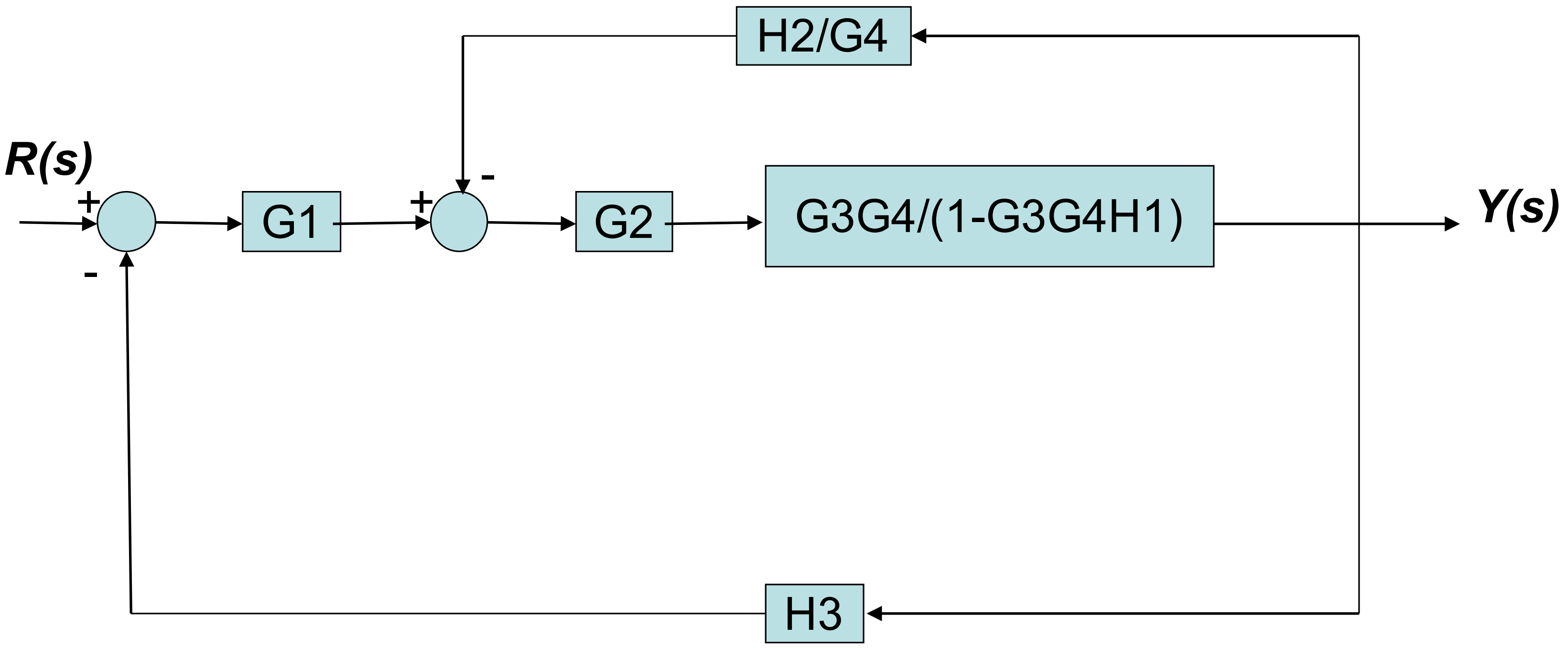

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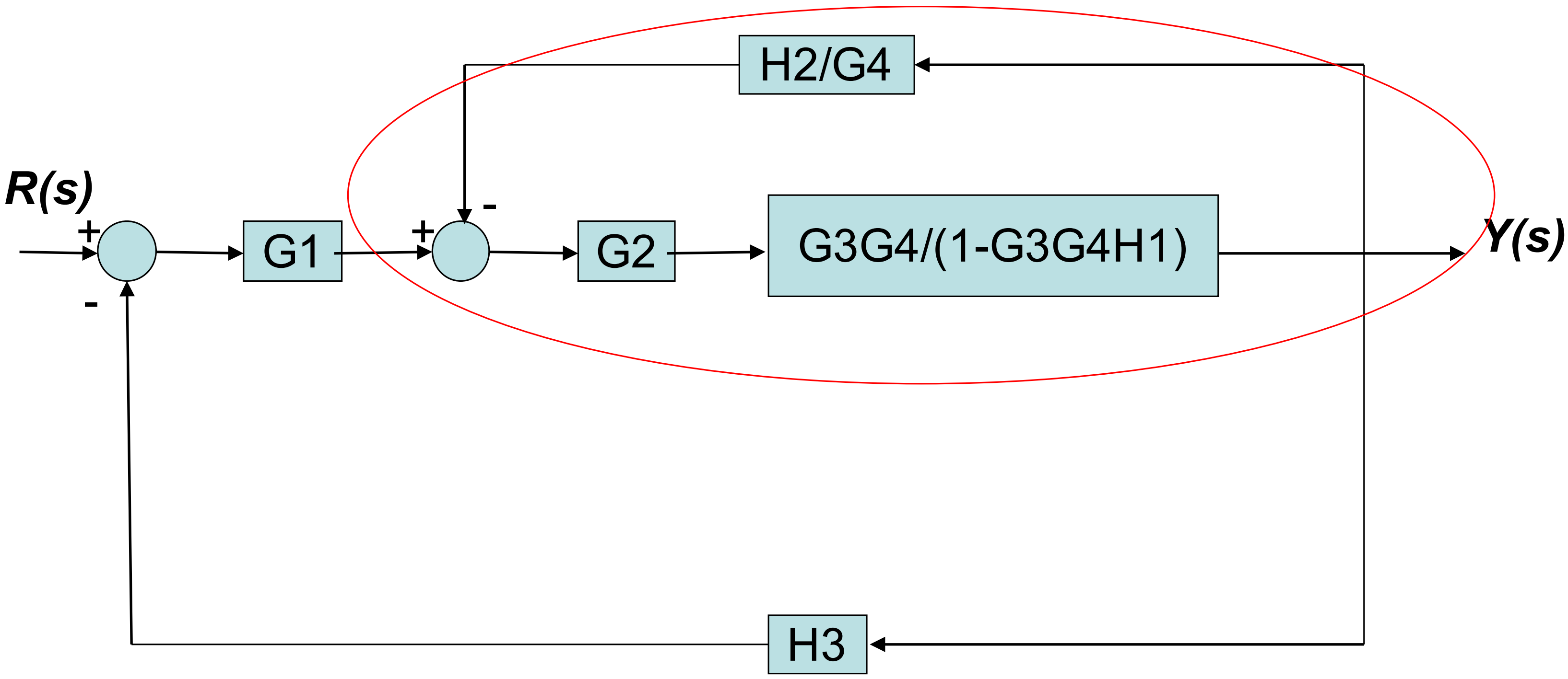

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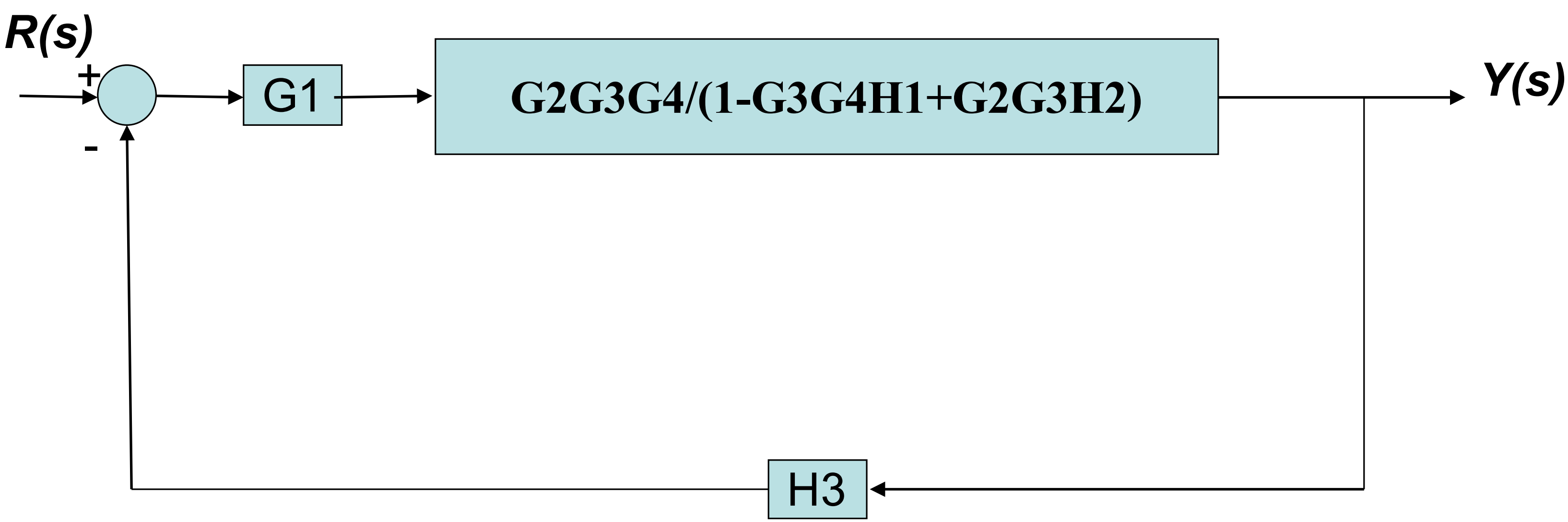

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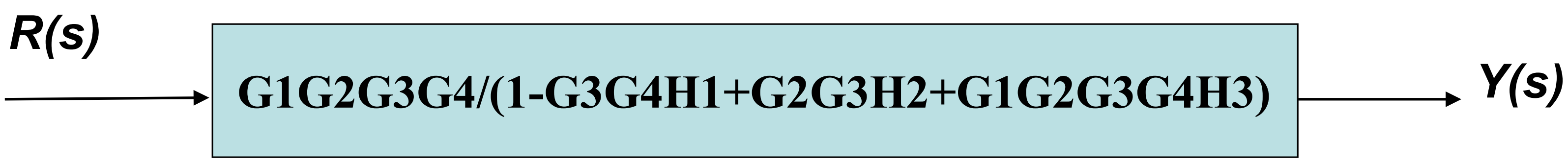
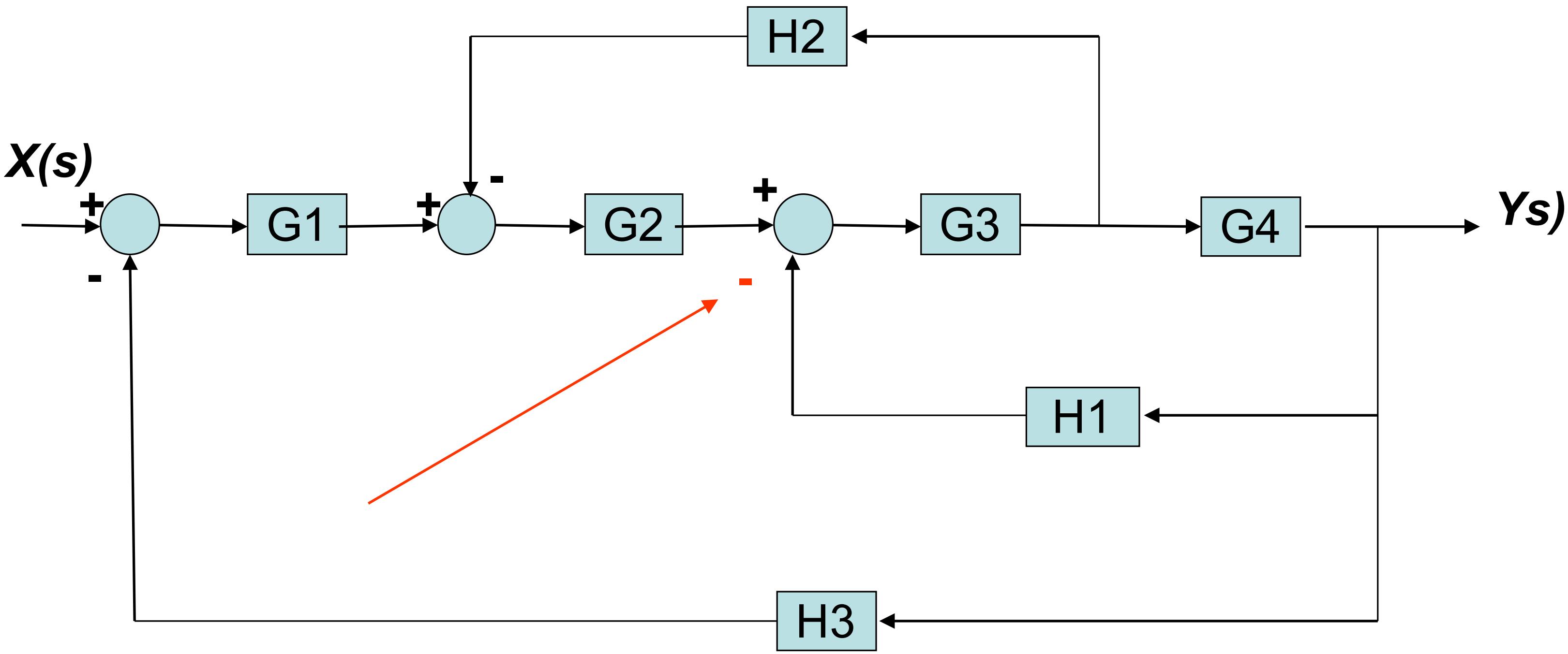


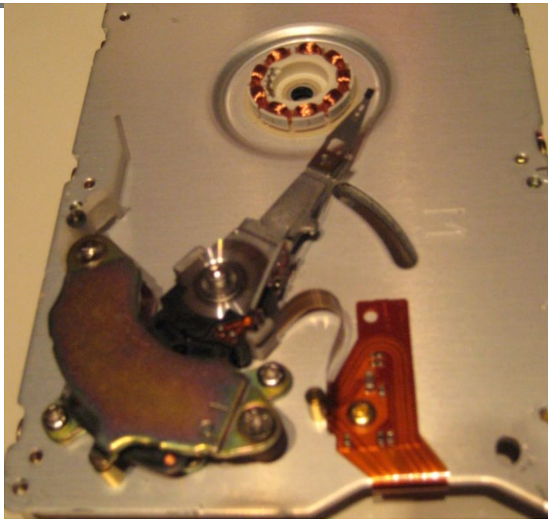
Exam Question



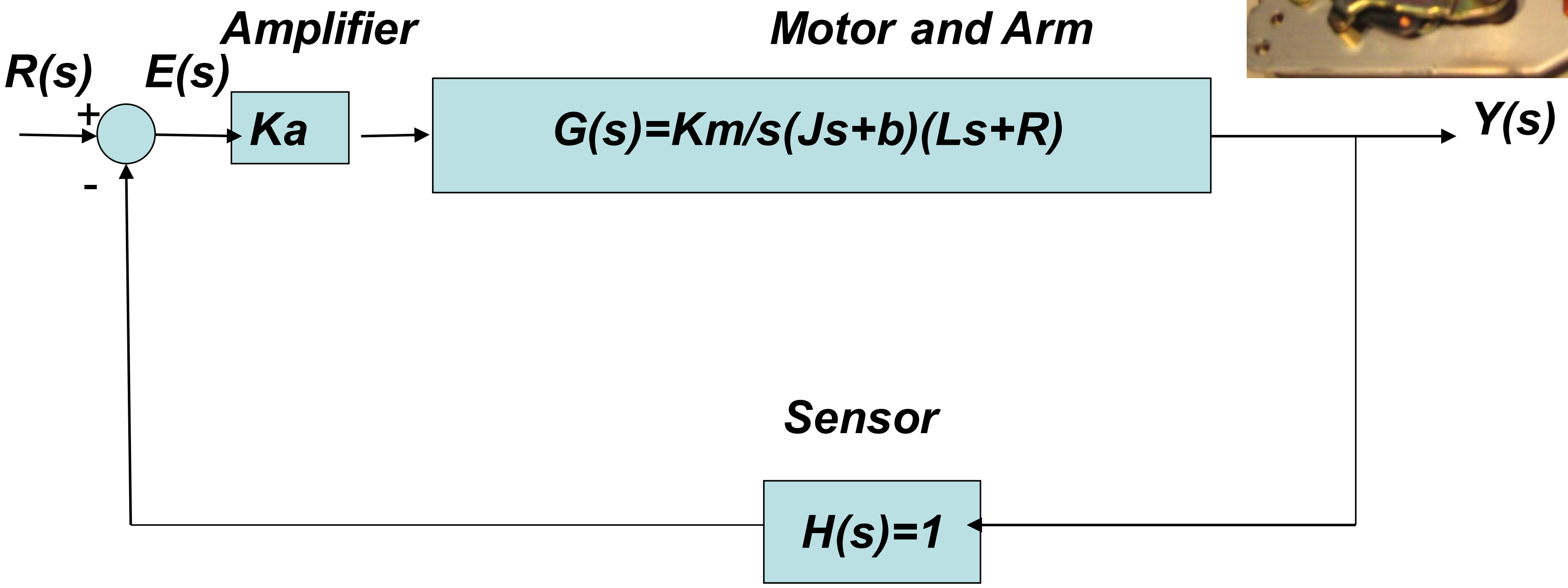
Solution

$$R(s) \longrightarrow \boxed{G1G2G3G4/(1+G3G4H1+G2G3H2+G1G2G3G4H3)} \longrightarrow Y(s)$$

Example



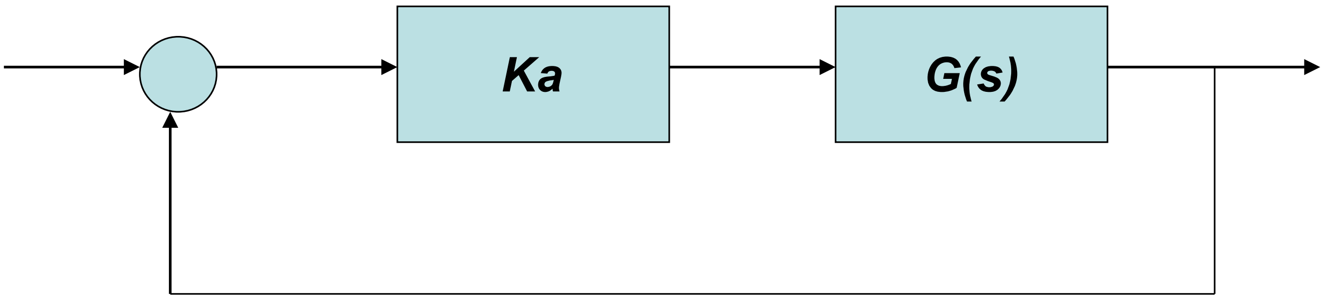
Disk drive R/W system block diagram



Find transfer function of this system

Solution

$$\frac{Y(s)}{R(s)} = \frac{K_a G(s)}{1 + K_a G(s)}$$



Resources

- De Silva, C. W. *Mechatronics: an integrated approach*, CRC Press, 2005.
- Necsulescu, D. *Mechatronics,* Prentice-Hall, 2007.
- Bishop, R.H. *LabVIEW 8, Student Edition*, Pearson Prentice-Hall, 2007.
- **Online@RMIT (Learning Hub)** **http://www.rmit.edu.au/online**
  - Lecture Notes, Labs, Project, Assessment
- Engineering Journal (You will create it during the course)

Thank you,
Questions