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Tutorial # 03

Advanced Control Systems

$$G(s) = \frac{b_0}{s^2 + a_1 s + a_0} \leftarrow \text{2nd Order System}$$

As we need to design a PID controller

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s} \leftarrow \text{(Polynomial form)}$$

$c_2, c_1 \,\&\, c_0 \rightarrow 3$ unknowns

$$1 + CG = 0$$

$$1 + \left(\frac{c_2 s^2 + c_1 s + c_0}{s}\right)\left(\frac{b_0}{s^2 + a_1 s + a_0}\right) = 0$$

$$s^3 + a_1 s^2 + a_0 s + b_0 c_2 s^2 + b_0 c_1 s + b_0 c_0 = 0$$

$$s^3 + s^2(a_1 + b_0 c_2) + s(a_0 + b_0 c_1) + b_0 c_0 = 0$$

Compare poles with $(s+\lambda)^3 = 0$

$(s+\lambda)^n$ means that all "n" poles are at "$-\lambda$"

$$s^3 + 3\lambda s^2 + 3\lambda^2 s + \lambda^3 = 0$$

(s^2)	(s)	(const)
$a_1 + b_0 c_2 = 3\lambda$	$a_0 + b_0 c_1 = 3\lambda^2$	$b_0 c_0 = \lambda^3$
$\boxed{c_2 = \frac{3\lambda - a_1}{b_0}}$	$\boxed{c_1 = \frac{3\lambda^2 - a_0}{b_0}}$	$\boxed{c_0 = \frac{\lambda^3}{b_0}}$

Conversion of polynomial form to standard form

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s}$$

$$C(s) = \underbrace{\boxed{c_2 s}}_{D} + \underbrace{\boxed{c_1}}_{P} + \underbrace{\boxed{\frac{c_0}{s}}}_{I}$$

$$C(s) = c_1 + \frac{c_0}{s} + c_2 s$$

Standard $\leftarrow C(s) = K_c + \frac{K_c}{T_i}\frac{1}{s} + K_c T_D s$

$c_1 = K_c \quad ; \quad c_0 = \frac{K_c}{T_i} \quad ; \quad c_2 = K_c T_D$

$c_1 = K_c \quad ; \quad c_0 = \frac{c_1}{T_i} \quad ; \quad c_2 = c_1 T_D$

$$\boxed{K_c = c_1 \quad ; \quad T_i = \frac{c_1}{c_0} \quad ; \quad T_D = \frac{c_2}{c_1}}$$

Q1 Part 2

$$G(s) = \frac{-3}{s^2 + 3^2} = \frac{-3}{s^2 + \boxed{0}s + 9}$$

$b_0 = -3 \quad ; \quad a_1 = 0 \quad ; \quad a_0 = 9$

λ is given as "6" $\longrightarrow \lambda = 6$

$c_2 = \frac{3\lambda - a_1}{b_0}$	$c_1 = \frac{3\lambda^2 - a_0}{b_0}$	$c_0 = \frac{\lambda^3}{b_0}$
$c_2 = \frac{18}{-3}$	$c_1 = \frac{99}{-3}$	$c_0 = \frac{216}{-3}$
$\boxed{c_2 = -6}$	$\boxed{c_1 = -33}$	$\boxed{c_0 = -72}$

$$\boxed{K_c = -36 \quad ; \quad T_i = 0.148 \quad ; \quad T_D = 0.182}$$

$$\boxed{C(s) = \frac{-6s^2 - 33s - 72}{s}}$$

We know that all the closed loop poles of the closed-loop system are on LHS of s-plane. So, the system is stable.

$$\lim_{s\to 0} \underbrace{sY(s)} = R_0$$ ← Apply final value theorem

Considering LHS:

$$\lim_{s\to 0} \overbrace{s\frac{Y(s)}{R(s)} \times R(s)}$$

$$\lim_{s\to 0} s\left(\frac{CG}{1+CG}\right) R(s)$$ ← Step

$$\lim_{s\to 0} s\left(\frac{\frac{(s+6)^3}{s(s^2+9)} - 1}{\frac{(s+6)^3}{s(s^2+9)}}\right) R(s)$$

$$\lim_{s\to 0} \cancel{s}\left(\frac{(s+6)^3 - s(s^2+9)}{(s+6)^3}\right)\frac{(-3)}{\cancel{s}}$$

Apply limit

$$= \lim_{s\to 0}\left(\frac{(s+6)^3 - s(s^2+9)}{(s+6)^3}\right)(-3)$$

$$= \left(\frac{6^3 - 0}{6^3}\right)(-3)$$

$$= \left(\frac{\cancel{216}}{\cancel{216}}\right)(-3)$$

$$\lim_{s\to 0} sY(s) = -3$$ (steady state value of $Y(s)$ will be $\boxed{-3}$)

OR

When enough time has passed the output will come at the value of $\boxed{}$ i-e- the reference signal.

Q no. 2

Design a resonant controller

$$G(s) = \frac{b}{s+a}$$ ← (first order system)

$$C(s) = \frac{C_2 s^2 + C_1 s + C_0}{s^2 + \omega_0^2}$$

→ In order to obtain steady state error of zero, one has to have the controller denominator equal to that of the reference signal. i-e- infinite gain at reference frequency

PI $\rightarrow C(s) = \dfrac{c_1 s + c_0}{s}$ infinity at $s=0$

PID $\rightarrow C(s) = \dfrac{c_2 s^2 + c_1 s + c_0}{s}$ infinity at $\Rightarrow s=0$

→ What would it take to have infinity gain at 50 Hz or 100π rad/sec ?

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{\left(s^2 + (100\pi)^2\right)}$$ infinity at $s = 50$ Hz

→ because other than zero, all reference frequencies come in complementary pole pairs.

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{(s^2 + \omega_0^2)}$$ (Simplest Resonant controller)

— Resonant Controller —

$$G(s) = \frac{b}{s+a}$$

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s^2 + \omega_0^2}$$ } numerator degree should be equal to the denominator

$$1 + C(s)G(s) = 0$$

$$1 + \left(\frac{c_2 s^2 + c_1 s + c_0}{s^2 + \omega_0^2}\right)\left(\frac{b}{s+a}\right) = 0$$

$$(s+a)(s^2+\omega_0^2) + bc_2 s^2 + bc_1 s + bc_0 = 0$$

$$s^3 + s\omega_0^2 + as^2 + a\omega_0^2 + bc_2 s^2 + bc_1 s + bc_0 = 0$$

$$s^3 + s^2(a + bc_2) + s(\omega_0^2 + bc_1) + a\omega_0^2 + bc_0 = 0$$

$$(s+\lambda)^3 = 0$$

$$s^3 + 3\lambda s^2 + 3\lambda^2 s + \lambda^3 = 0$$

(s^2)	(s)	(Constt.)
$a + bc_2 = 3\lambda$	$\omega_0^2 + bc_1 = 3\lambda^2$	$a\omega_0^2 + bc_0 = \lambda^3$
$c_2 = \dfrac{3\lambda - a}{b}$	$c_1 = \dfrac{3\lambda^2 - \omega_0^2}{b}$	$c_0 = \dfrac{\lambda^3 - a\omega_0^2}{b}$

→ For Resonant controllers, we like to prove that steady-state error is zero at $t \to \infty$. because $y(t)$ is changing all the time.

$$\lim_{s \to 0} sE(s) = 0$$

$$E(s) = R(s) - Y(s)$$

$$E(s) = \left(1 - \frac{Y(s)}{R(s)}\right) R(s)$$

$$E(s) = \left(1 - \frac{CG}{1 + CG}\right) R(s)$$

$$E(s) = \left(\frac{1 + \cancel{CG} - \cancel{CG}}{1 + CG}\right) R(s)$$

$$E(s) = \left(\frac{1}{1+CG}\right) R(s)$$

$$1 + CG = \frac{s(s^2+\omega_o^2) + b(c_2 s^2 + c_1 s + c_0)}{s(s^2+\omega_o^2)}$$

So,

$$E(s) = \left(\frac{s(s^2+\omega_o^2)}{s(s^2+\omega_o^2) + b(c_2 s^2 + c_1 s + c_0)}\right) R(s)$$

Now

$$r(t) = \sin \omega_o t$$

$$\updownarrow$$

$$R(s) = \frac{\omega_o}{s^2+\omega_o^2}$$

To calculate S.S.E

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} s \times \frac{s\cancel{(s^2+\omega_o^2)}}{s(s^2+\omega_o^2)+b(c_2 s^2+c_1 s+c_0)} \times \frac{\omega_o}{\cancel{s^2+\omega_o^2}}$$

$$= \lim_{s\to 0} \frac{s^2 \; \omega_o}{s(s^2+\omega_o^2)+b(c_2 s^2+c_1 s+c_0)}$$

$$\boxed{\lim_{s\to 0} sE(s) = 0}$$

So, in case of a sinusoidal reference at ω_o frequency, the system will be able to follow the reference with S.S.E = $\boxed{\text{zero.}}$

Q no. 2 Part 3

$$G(s) = \frac{1}{s-1}$$

$b = 1$

$a = -1$

$\omega_o = 2$; & $\lambda = 1$

$$c_2 = \frac{3\lambda - a}{b} = \frac{3+1}{1} = 4$$

$$c_1 = \frac{3\lambda^2 - \omega_o^2}{b} = \frac{3-4}{1} = -1$$

$$c_0 = \frac{\lambda^3 - a\omega_o^2}{b} = \frac{1+4}{1} = 5$$

$$\boxed{C(s) = \frac{4s^2 - s + 5}{s^2+2^2}}$$

(Resonant controller)

→ The steady state error is going to be zero as proved before.

Q no. 3

$$G(s) = \frac{b}{s+a}$$

step – reference & sinusoid reference

$$C(s) = \frac{c_3 s^3 + c_2 s^2 + c_1 s + c_0}{s(s^2+\omega_o^2)}$$

→ degree 3

→ degree 3

4 unknowns

$$1 + C(s)G(s) = 0$$

$$s(s+a)(s^2+\omega_o^2) + bc_3 s^3 + bc_2 s^2 + bc_1 s + bc_0 = 0$$

$$s^4 + as^3 + \omega_o^2 s^2 + a\omega_o^2 s + bc_3 s^3 + bc_2 s^2 + bc_1 s + bc_0 = 0$$

$$s^4 + s^3(a+bc_3) + s^2(\omega_o^2 + bc_2) + s(a\omega_o^2 + bc_1) + bc_0 = 0$$

$$(s+\lambda)^4 = 0$$

$$(s+\lambda)^n = \sum_{i=0}^{n} {}^nC_i\, s^{n-i} \lambda^i$$

$$(s+\lambda)^4 = \underbrace{{}^4C_0}_{1} s^4 + \underbrace{{}^4C_1}_{4} s^3\lambda + \underbrace{{}^4C_2}_{6} s^2\lambda^2 + \underbrace{{}^4C_3}_{4} s\lambda^3 + \underbrace{{}^4C_4}_{1}\lambda^4$$

$$(s+\lambda)^4 = s^4 + 4s^3\lambda + 6s^2\lambda^2 + 4s\lambda^3 + \lambda^4$$

— Compare coefficients —

$a + bc_3 = 4\lambda$	$\omega_o^2 + bc_2 = 6\lambda^2$	$a\omega_o^2 + bc_1 = 4\lambda^3$	$bc_0 = \lambda^4$
$c_3 = \frac{4\lambda - a}{b}$	$c_2 = \frac{6\lambda^2 - \omega_o^2}{b}$	$c_1 = \frac{4\lambda^3 - a\omega_o^2}{b}$	$c_0 = \frac{\lambda^4}{b}$

Q no. 3 part 2

$$G(s) = \frac{1}{5s+3}$$

$$G(s) = \frac{1/5}{s + 3/5} = \frac{0.2}{s+0.6}$$

$0.2 \to b$, $0.6 \to a$

$\omega_o = 0.1$ & $\lambda = 2$

$$c_3 = \frac{4(2) - 0.6}{0.2} = \frac{7.4}{0.2} = 37$$

$$c_2 = \frac{6(4) - 0.01}{0.2} = \frac{23.99}{0.2} = 119.95$$

$$c_1 = \frac{4(8) - (0.6)(0.01)}{0.2} = 159.97$$

$$c_0 = \frac{16}{0.2} = 80$$

$$\boxed{C(s) = \frac{37s^3 + 119.95s^2 + 159.97s + 80}{s(s^2+0.01)}}$$

→ Resonant controller to follow both step & resonant references

Proof

$$r(t) = \sin\omega_o t + 1$$

$$R(s) = \frac{\omega_o}{s^2+\omega_o^2} + \frac{1}{s}$$

$$\boxed{R(s) = \frac{s\omega_o + s^2 + \omega_o^2}{s(s^2+\omega_o^2)}}$$

$$\lim_{s\to 0} sE(s) = 0 \longrightarrow \text{To prove}$$

$$E(s) = \left(\frac{1}{1+CG}\right) R(s)$$

$$E(s) = \left(\frac{\cancel{s(s^2+\omega_0^2)}(s+a)}{(s+\lambda)^4}\right)\left(\frac{s\omega_0 + s^2 + \omega_0^2}{\cancel{s(s^2+\omega_0^2)}}\right)$$

$$sE(s) = s\,\boxed{\frac{(s+a)(s\omega_0+s^2+\omega_0^2)}{(s+\lambda)^4}}$$

Apply limit

$$\lim_{s\to 0} E(s) = 0\,\boxed{} = 0$$

→ So the designed controller will follow the sinusoidal & step reference regardless of their amplitudes.

Q no. 4

$$G(s) = 1/s^2$$

$$C(s) = \frac{C_2 s^2 + C_1 s + C_0}{s^2 + \omega_0^2}$$

$$1 + C(s)G(s) = 0$$

$$1 + \left(\frac{C_2 s^2 + C_1 s + C_0}{s^2+\omega_0^2}\right)\frac{1}{s^2} = 0$$

$$s^2(s^2+\omega_0^2) + C_2 s^2 + C_1 s + C_0 = 0$$

$$s^4 + 0s^3 + s^2(\omega_0^2 + C_2) + C_1 s + C_0 = 0$$

$$s^4 + 4\lambda s^3 + 6\lambda^2 s^2 + 4s\lambda^3 + \lambda^4 = 0$$

How about s^3 – term ? Is $0 = 4\lambda$?

If $\lambda \neq 0$, s^3 term cannot be zero.

This means the structure of the controller is not right.

How about

$$C(s) = \frac{C_3 s^3 + C_2 s^2 + C_1 s + C_0}{s(s^2+\omega_0^2)}$$

Same problem will happen for s^4 term. This means that the controller is not right again.

This is a peculiar plant, called double integrator plant ; it has two poles at zero. Conventional resonant control does not work on this plant. We have to select a special controller with a filter.

$$C(s) = \frac{c_3 s^3 + c_2 s^2 + c_1 s + c_0}{\underbrace{(s^2+\omega_0^2)}_{\text{Resonant term}}\underbrace{(s+l_0)}_{\text{filter}}}$$

Note: 5 unknown

c_3, c_2, c_1, c_0 & l_0

$1 + C(s)G(s)$

$$1 + \frac{c_3 s^3 + c_2 s^2 + c_1 s + c_0}{(s^2+\omega_0^2)(s+l_0)} \times \frac{1}{s^2} = 0$$

$$s^2(s+l_0)(s^2+\omega_0^2) + c_3 s^3 + c_2 s^2 + c_1 s + c_0 = 0$$

$$s^2(s^3 + l_0 s^2 + \omega_0^2 s + l_0\omega_0^2) + c_3 s^3 + c_2 s^2 + c_1 s + c_0 = 0$$

$$s^5 + l_0 s^4 + \omega_0^2 s^3 + l_0 \omega_0^2 s^2 + c_3 s^3 + c_2 s^2 + c_1 s + c_0 = 0$$

$$s^5 + l_0 s^4 + s^3(\omega_0^2 + c_3) + s^2(l_0\omega_0^2 + c_2) + s c_1 + c_0 = 0$$

$$(s+\lambda)^5 = \sum_{i=0}^{5} {}^5C_i\, s^{5-i} \lambda^i$$

$$(s+\lambda)^5 = \boxed{1} s^5 + \boxed{5} \lambda s^4 + \boxed{10} \lambda^2 s^3 + \boxed{10} \lambda^3 s^2 + \boxed{5} \lambda^4 s + \boxed{1} \lambda^5$$

$l_0 = 5$ | $\omega_0^2 + c_3 = 10\lambda^2$, $c_3 = 10\lambda^2 - \omega_0^2$ | $l_0\omega_0^2 + c_2 = 10\lambda^3$, $c_2 = 10\lambda^3 - l_0\omega_0^2$ | $c_1 = 5\lambda^4$ | $c_0 = \lambda^5$

$\omega_0 = 2$; $\lambda = 1$

$l_0 = 5$

$c_3 = 10 - 4 = 6$

$c_2 = 10 - 20 = -10$

$c_1 = 5$

$c_0 = 1$

$$\boxed{C(s) = \frac{6s^3 - 10s^2 + 5s + 1}{(s^2+4)(s+5)}}$$

→ Proof that $E(s)$ final value will be zero

$$sE(s) = s\,\frac{1}{1+CG}\,R(s)$$

$$sE(s) = \frac{s^3(s^2+\omega_0^2)(s+l_0)}{(s+\lambda)^5}\,R(s)$$

$r(t) = \sin\left(2t + \frac{\pi}{3}\right)$

$r(t) = \sin 2t \cos\frac{\pi}{3} + \cos 2t \sin\frac{\pi}{3}$

$r(t) = 0.5\sin 2t + \frac{\sqrt{3}}{2}\cos 2t$

$$R(s) = \frac{\frac{1}{2}(2)}{s^2+2^2} + \frac{\frac{\sqrt{3}}{2}\, s}{s^2+2^2}$$

$$R(s) = \frac{1 + \frac{\sqrt{3}}{2} s}{s^2+2^2} = \frac{1+0.866s}{s^2+4}$$

$$sE(s) = \frac{s^3 \cancel{(s^2+4)}(s+5)}{(s+1)^5} \times \frac{1+0.866s}{\cancel{(s^2+4)}}$$

Apply limit

$$\lim_{s\to 0} sE(s) = (0)^3 \boxed{} = 0$$

→ Zero steady state error means, it will follow the given reference

→ Proof that a sinusoidal disturbance will be rejected

$$d_i(t) = \sin(2t - \pi)$$

$$= \sin 2t \cos\pi - \cos 2\pi \sin\pi \quad (\sin\pi = 0)$$

$$= (-1)\sin 2t - 0$$

$$d_i(t) = -\sin 2t$$

$$D_i(s) = \frac{-2}{s^2+4}$$

$$\frac{Y(s)}{D_i(s)} = \frac{G(s)}{1 + C(s)G(s)}$$

$$Y(s) = \frac{G(s)}{1+CG} D_i(s)$$

$$Y(s) = \frac{\cancel{1/s^2}}{\frac{(s+\lambda)^5}{\cancel{s^2}(s^2+4)(s+5)}} D_i(s)$$

$$Y(s) = \frac{(s^2+4)(s+5)}{(s+\lambda)^5} D_i(s)$$

$$Y(s) = \frac{\cancel{(s^2+4)}(s+5)}{(s+\lambda)^5} \times \frac{-2}{\cancel{(s^2+4)}}$$

$$sY(s) = s\,\frac{-2(s+5)}{(s+\lambda)^5}$$

Apply limit

$$\lim_{s\to 0} sY(s) = 0 \boxed{} = 0$$

Given enough time, the disturbance $D_i(s)$ will have **ZERO** effect on the system output.