Part II: Model Based PID Control System Design

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Outline

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Model based controller design

- Desired closed-loop performance specification;
- Model (first order for PI and second order for PID);
- Parameters of the controller;
- Simulation and validation of the closed-loop control system.

Specification of Desired Closed-loop Performance

- In the PI controller case, a second order transfer function is used in the specification,

$$T(s) = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2} \tag{1}$$

  where w_n and ξ are the natural frequency and damping coefficient for the second order transfer function. These are the free parameters to be selected by the designer as desired performance specification.
- This is the desired closed-loop response from set-point signal to output signal.

Desired closed-loop poles

The parameter ξ is often chosen as 1 or 0.707. When $\xi = 1$, the poles of the desired closed-loop transfer function (1) are the solutions of the polynomial equation,

$$s^2 + 2w_n s + w_n^2 = 0 \tag{2}$$

which are $s_1 = s_2 = -w_n$. Namely, we have two identical poles when $\xi = 1$. With the second choice of $\xi = 0.707$, the poles are a pair of complex-conjugate numbers determined by

$$s_{1,2} = \frac{-2\xi w_n \pm \sqrt{4\xi^2 w_n^2 - 4w_n^2}}{2} = -0.707w_n \pm j0.707w_n \tag{3}$$

which are on the trajectories of the two straight-lines on the complex plane, shown in Figure 1.

Desired closed-loop poles-cont'

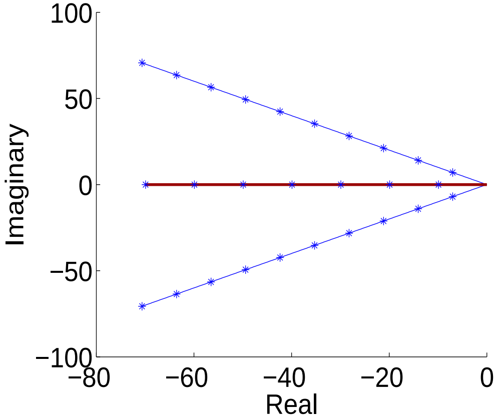


Figure 1: Trajectories of the poles for $\xi = 0.707$ (solid-line) and trajectories of the poles for $\xi = 1$ (darker-solid-line)

Choice of w_n-from time domain

From the simulation of a step response (1) (see Figure 2), the total response time T_{total} with respect to the parameter w_n is estimated as

$$T_{total} \approx \frac{7}{w_n} \quad (4)$$

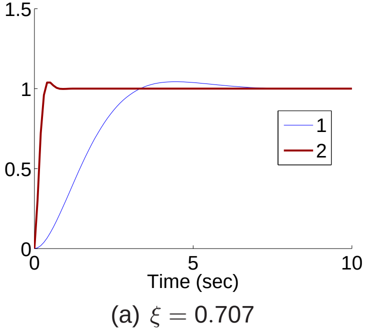


Figure 2: Step response of the desired closed-loop transfer function. Key: line (1) $w_n = 1$, line (2) $w_n = 10$.

Choice of w_n from bandlimit

Here the parameter w_n is related to the bandlimit of the desired closed-loop control system. The bandlimit is defined according to the cut-off frequency ω_c, where the parameter ω_c is chosen such that $|T(j\omega)| = \frac{T(0)}{\sqrt{2}}$.

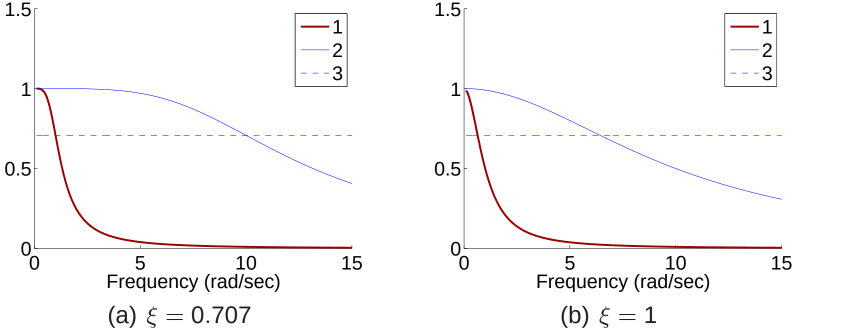


Figure 3: Magnitude of the frequency response of transfer function (1). Key: line (1) $w_n = 1$; line (2) $w_n = 10$; line (3) indicator ($\frac{1}{\sqrt{2}}$) for the bandlimit.

Choice of w_n: summary

With the parameter ξ chosen (either 1 or 0.707), the natural frequency w_n becomes a closed-loop performance parameter that the user specifies according to the desired closed-loop response requirement. In general, when w_n is larger, the closed-loop response speed is faster and the system is more sensitive to measurement noise.

Model and Controller Structures

- First order model for PI
$$G(s) = \frac{K}{\tau s + 1} \quad (5)$$
which can also be expressed in the pole-zero form,
$$G(s) = \frac{b}{s + a} \quad (6)$$
where $a = 1/\tau$ and $b = K/\tau$.
-
$$C(s) = K_c(1 + \frac{1}{\tau_I s}) \quad (7)$$
$$C(s) = \frac{c_1 s + c_0}{s} \quad (8)$$
where $K_c = c_1$ and $\tau_I = \frac{c_1}{c_0}$.

The solution of PI controller parameters

- The key to the solution of the PI controller parameters is to equate the desired closed-loop poles to the actual closed-loop poles.
- This controller design technique is called pole-assignment controller design.
- It is a general controller design technique that can be applied to any model structure.

Closed-loop characteristic equation

We calculate the actual closed-loop system using the design model (6) and the controller model (8):

$$\begin{aligned} T_{cl} &= \frac{G(s)C(s)}{1+G(s)C(s)} = \frac{\frac{b}{s+a}\frac{c_1 s+c_0}{s}}{1+\frac{b}{s+a}\frac{c_1 s+c_0}{s}} \\ &= \frac{b(c_1 s+c_0)}{s(s+a)+b(c_1 s+c_0)} \end{aligned} \tag{9}$$

The closed-loop poles of the actual system are the solutions of the polynomial equation with respect to s

$$s(s+a)+b(c_1 s+c_0)=0 \tag{10}$$

Equation (10) is called closed-loop characteristic equation.

The key to the solution of the parameters

Since the model parameters a and b are given, the free parameters in (10) are the controller parameters c_1 and c_0. To find the controller parameters c_1 and c_0, the following polynomial equation is set,

$$s(s+a) + b(c_1 s + c_0) = s^2 + 2\xi w_n s + w_n^2 \tag{11}$$

where the left-hand side of the equation (11) is the polynomial that determines the actual closed-loop poles and the right-hand side is the polynomial that determines the desired closed-loop poles.

Solution I

- By equating these two polynomials, the actual closed-loop poles are assigned to the desired closed-loop poles.
- Now, we compare the coefficients of the polynomial equation (11) on both sides:

$$\begin{aligned} left - hand & \quad right - hand && (12)\\ s^2 \quad : & \quad 1 = 1 && (13)\\ s \quad : & \quad a + bc_1 = 2\xi w_n && (14)\\ s^0 \quad : & \quad bc_0 = w_n^2 && (15) \end{aligned}$$

$$c_1 = \frac{2\xi w_n - a}{b} \qquad (16)$$

$$c_0 = \frac{w_n^2}{b} \qquad (17)$$

Solution II

With the relationships between c_1, c_0 and K_c, τ_I (see Equation (8)), we find the PI controller parameters as

$$K_c = c_1 = \frac{2\xi w_n - a}{b} \tag{18}$$

$$\tau_I = \frac{c_1}{c_0} = \frac{2\xi w_n - a}{w_n^2} \tag{19}$$

Example

A first order system is used to describe the dynamic relationship between voltage change and velocity of a DC motor. Assume that a particular motor has the Laplace transfer function

$$G(s) = \frac{0.1}{10s + 1} \tag{20}$$

Find the PI controller parameters for velocity control, where the desired closed-loop performance is specified by two performance levels: one fast response $w_n = 5$ and one slow response $w_n = 0.5$, $\xi = 0.707$ in both cases. Simulate the closed-loop step responses with proportional control on the output only and compare the results.

Solution I

The model parameters needed for the PI controller design are $a = \frac{1}{10} = 0.1$ and $b = \frac{0.1}{10} = 0.01$. With these parameters and the closed-loop performance specification, based on equations (18-19) we find the controller parameters as, for $w_n = 5$ and $\xi = 0.707$

$$K_c = 697; \tau_I = 0.2788,$$

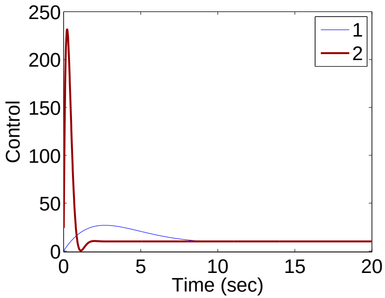
for $w_n = 0.5$ and $\xi = 0.707$

$$K_c = 60.7; \tau_I = 2.43.$$

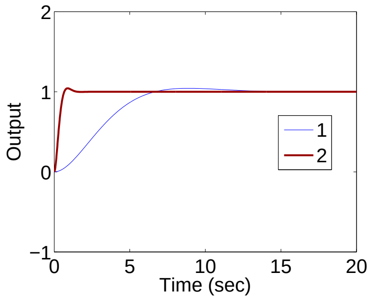
As we can see, when w_n is larger, the proportional gain K_c is larger whilst the integral time constant τ_I is smaller.

Solution II

We simulate the closed-loop unit step responses with the results shown in Figure 4. The closed-loop response speed is much faster when $w_n = 5$ in comparing with the case of $w_n = 0.5$, also the control signal has a much larger amplitude.



(a) Control signal



(b) Output response

Figure 4: Closed-loop response. Key: line (1) $w_n = 0.5$; line (2) $w_n = 5$.

Model Based Design for PID Controllers

Models for PID controllers

$$Model\ A: \quad G(s) = \frac{b_0}{s^2 + a_1 s + a_0} \tag{21}$$

$$Model\ B: \quad G(s) = \frac{K_p e^{-ds}}{\tau_p s + 1} \tag{22}$$

$$Model\ C: \quad G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \tag{23}$$

We will exclusively use the techniques of pole-assignment controller design to find the PID controller parameters.

Configurations of PID controllers

- The general form of PID controller is defined by the transfer function

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s(s + l_0)} \tag{24}$$

  where l_0 is a parameter that needs to be utilized as part of the derivative filter.
- However, the industrial control systems are often defined in terms of the PID controller parameters, K_c, τ_I, τ_D and a filter time constant τ_f.
- This τ_f previously was linked to the derivative gain τ_D as $\tau_f = \beta \tau_D$.

Equivalent structure I

We need to find PID controller parameters that lead to a completely identical configuration between the controller $C(s)$ defined by (24) and an industrial PID controller. With this in mind, we will choose the PID controller parameters such that

$$C(s) = K_c(1 + \frac{1}{\tau_I s} + \frac{\tau_D s}{\tau_f s + 1}) \tag{25}$$

is identical to the PID controller in (24).

Equivalent structure II

The problem is solved using reverse engineering by expressing (25) as

$$C(s) = \frac{K_c(\tau_I s(\tau_f s + 1) + (\tau_f s + 1) + \tau_I \tau_D s^2)}{\tau_I s(\tau_f s + 1)} \tag{26}$$

which should exactly equal to (24). By comparing these two expressions, we obtain,

$$c_2 = \frac{K_c(\tau_I \tau_D + \tau_I \tau_f)}{\tau_I \tau_f} \tag{27}$$

$$c_1 = \frac{K_c(\tau_I + \tau_f)}{\tau_I \tau_f} \tag{28}$$

$$c_0 = \frac{K_c}{\tau_I \tau_f} \tag{29}$$

$$l_0 = \frac{1}{\tau_f} \tag{30}$$

Equivalent structure III

We solve PID controller parameters using these four linear equations and their values are,

$$\tau_f = \frac{1}{l_0} \tag{31}$$

$$\tau_I = \frac{c_1}{c_0} - \tau_f \tag{32}$$

$$K_c = \tau_I \tau_f c_0 \tag{33}$$

$$\tau_D = \frac{c_2 \tau_I \tau_f - K_c \tau_I \tau_f}{K_c \tau_I} \tag{34}$$

Pole-zero Cancelation Technique

Overview

- Pole-zero cancelation technique is frequently used to obtain the simple analytical solutions for PID controller parameters;
- We should not cancel unstable poles or zeros.
- The plant poles cancelled will re-appear as the closed-loop poles from input disturbance to output response. Therefore, we do not cancel the poles corresponding to slow dynamic response.

Basic idea

The numerator of the controller $C(s)$ is factored:

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s} = \frac{c_2(s+\gamma_1)(s+\gamma_2)}{s(s+l_0)} \tag{35}$$

By choosing the zero of the controller $-\gamma_2$ equal to the pole of the model $-\alpha_2$ (i.e. $\gamma_2 = \alpha_2$), we cancelled the pole in the model with the zero in the controller.
We will use first order plus delay system to illustrate the pole-zero cancelation design idea.

PID Design for First Order Plus Delay Model (1)

With the polynomial approximation (called first order Pade approximation), the transfer function model becomes

$$G(s) = \frac{K_p e^{-ds}}{\tau_p s + 1} \approx \frac{K_p(-ds + 2)}{(\tau_p s + 1)(ds + 2)} \tag{36}$$

$$G(s) = \frac{b_1 s + b_0}{(s + \alpha_1)(s + \alpha_2)} \tag{37}$$

where $b_1 = -\frac{K_p}{\tau_p}$, $b_0 = \frac{2K_p}{\tau_p d}$. If $\frac{1}{\tau_p} < \frac{2}{d}$, then we choose $\alpha_1 = \frac{1}{\tau_p}$ and $\alpha_2 = \frac{2}{d}$. In the case of a plant with a dominant time delay when $\frac{1}{\tau_p} > \frac{2}{d}$, we let $\alpha_1 = \frac{2}{d}$ and $\alpha_2 = \frac{1}{\tau_p}$.

PID Design for First Order Plus Delay Model (2)

Because the zero is unstable located at $s = \frac{2}{d}$, this zero should not be cancelled in the controller design. However, we will cancel the pole $-\alpha_2$, which has a faster dynamics response. The PID controller structure has the transfer function form,

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s(s + l_0)} \tag{38}$$

which is an ideal PID controller with a filter. The filter pole $-l_0$ will be used in the design. We also assume that the PID with filter has the zeros located at $-\gamma_1$ and $-\alpha_2$, where α_2 corresponds to one of the poles in the model. The open- loop transfer function is the quantity,

$$\begin{aligned} L(s) = G(s)C(s) &= \frac{b_1 s + b_0}{(s + \alpha_1)(s + \alpha_2)} \frac{c_2(s + \gamma_1)(s + \alpha_2)}{s(s + l_0)} \\ &= \frac{c_2(b_1 s + b_0)(s + \gamma_1)}{s(s + \alpha_1)(s + l_0)} \end{aligned} \tag{39}$$

PID Design for First Order Plus Delay Model (3)

The closed-loop transfer function is

$$T(s) = \frac{L(s)}{1+L(s)} = \frac{c_2(b_1 s + b_0)(s+\gamma_1)}{s(s+\alpha_1)(s+l_0) + c_2(b_1 s + b_0)(s+\gamma_1)} \tag{40}$$

Note that the denominator of (40) is a third order polynomial and it has three unknown controller parameters, l_0, c_2 and γ_1. Thus, the desired closed-loop polynomial $A_{cl}(s)$ must be a third order polynomial with its order to match the denominator of (40). To this end, we select

$$A_{cl}(s) = (s^2 + 2\xi w_n s + w_n^2)(s+\lambda_1) \tag{41}$$

where $\lambda_1 > 0$ is a positive parameter.

PID Design for First Order Plus Delay Model (4)

- Same as before, we select the damping coefficient $\xi = 0.707$ and the natural frequency w_n to reflect the design requirements such as closed-loop response time and bandwidth. The extra pole located at $-\lambda_1$ is often chosen to be away from the pair of dominate poles $-\xi w_n \pm j\xi w_n$.
- The Diophantine equation is expressed as

$$s(s+\alpha_1)(s+l_0)+c_2(b_1 s+b_0)(s+\gamma_1)=(s^2+2\xi w_n s+w_n^2)(s+\lambda_1) \qquad (42)$$

where the left-hand side of this equation is the closed-loop polynomial and the right-hand side is the desired closed-loop polynomial.

PID Design for First Order Plus Delay Model (5)

$$
\begin{aligned}
s^3 \quad & +(\alpha_1 + l_0 + c_2 b_1)s^2 + (\alpha_1 l_0 + c_2(b_0 + b_1\gamma_1))s + c_2 b_0 \gamma_1 \\
s^3 \quad & +(2\xi w_n + \lambda_1)s^2 + (w_n^2 + 2\lambda_1 \xi w_n)s + \lambda_1 w_n^2
\end{aligned} \tag{43}
$$

By comparing the coefficients of the both sides of the polynomials, three linear equations are obtained,

$$
s^2 \quad : \quad \alpha_1 + l_0 + c_2 b_1 = 2\xi w_n + \lambda_1 \tag{44}
$$

$$
s \quad : \quad \alpha_1 l_0 + c_2 b_0 + c_2 b_1 \gamma_1 = w_n^2 + 2\lambda_1 \xi w_n \tag{45}
$$

$$
s^0 \quad : \quad c_2 b_0 \gamma_1 = \lambda_1 w_n^2 \tag{46}
$$

PID Design for First Order Plus Delay Model (6)

We solve for $c_2\gamma_1$ based on (46):

$$c_2\gamma_1 = \frac{\lambda_1 w_n^2}{b_0} \tag{47}$$

Then, the value of $c_2\gamma_1$ is substituted into (45), which becomes,

$$\alpha_1 l_0 + c_2 b_0 = w_n^2 + 2\lambda_1 \xi w_n - \frac{b_1 \lambda_1 w_n^2}{b_0} \tag{48}$$

Note that both (44) and (48) contain the same pair of unknown variables (α_1, c_2), so we will solve these two together using these two equations. From (44), we find the value of l_0,

$$l_0 = -c_2 b_1 + 2\xi w_n + \lambda_1 - \alpha_1 \tag{49}$$

PID Design for First Order Plus Delay Model (7)

Substituting this l_0 into (48) and collecting the terms, we find c_2 as

$$c_2 = \frac{-2\xi w_n \alpha_1 - \lambda_1 \alpha_1 + \alpha_1^2 + w_n^2 + 2\lambda_1 \xi w_n - \frac{b_1 \lambda_1 w_n^2}{b_0}}{b_0 - \alpha_1 b_1} \quad (50)$$

where we assume that $b_0 - \alpha_1 b_1 \neq 0$. The value of l_0 is found using (49) with c_2 given by (50). From c_2, we also find γ_1 as

$$\gamma_1 = \frac{\lambda_1 w_n^2}{b_0 c_2} \quad (51)$$

PID Design for First Order Plus Delay Model (8)

With the parameters c_2, γ_1 and l_0 calculated, the PID controller with filter is re-constructed as,

$$C(s) = \frac{c_2(s+\gamma_1)(s+\alpha_2)}{s(s+l_0)} \tag{52}$$

where α_2 corresponds to the location of the pole in the model that we chose to cancel. Equivalently, (52) is expressed in the more general form,

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s(s+l_0)} \tag{53}$$

where c_2 is calculated using (50), $c_1 = c_2(\gamma_1 + \alpha_2)$ and $c_0 = c_2 \alpha_2 \gamma_1$.

Example

Given a first order plus delay system with the transfer function

$$G(s) = \frac{10e^{-5s}}{10s+1} \tag{54}$$

find the PID controller parameters using pole assignment design technique. The desired closed-loop performance is specified by $\xi = 0.707$, $\lambda_1 = 1$. To understand that the approximation of time delay using the transfer function model causes error between the actual plant and the model used for the design, find the PID controller parameters for $w_n = 0.4$ and then reducing it to $w_n = 0.2$, and simulate the closed-loop performance with a unit step set-point signal and disturbance rejection of step signal with amplitude 0.2.

Solution I

The first order plus delay model is approximated using Pade approximation, leading to

$$G(s) \approx \frac{-s + 0.4}{(s + 0.1)(s + 0.4)} \tag{55}$$

In the design, we cancel the pole from the time delay, and assign the values of α_1 and α_2 as $\alpha_1 = 0.1$ and $\alpha_2 = 0.4$. Also from (55), we find $b_1 = -1$ and $b_0 = 0.4$.

Solution II

In the calculation, we first use $w_n = 0.4$. We calculate the value of c_2 as

$$c_2 = \frac{-2\xi w_n \alpha_1 - \lambda_1 \alpha_1 + \alpha_1^2 + w_n^2 + 2\lambda_1 \xi w_n - \frac{b_1 \lambda_1 w_n^2}{b_0}}{b_0 - \alpha_1 b_1} = 1.9581 \quad (56)$$

l_0 as

$$l_0 = -c_2 b_1 + 2\xi w_n + \lambda_1 - \alpha_1 = 3.4237 \quad (57)$$

$c_1 = c_2(\gamma_1 + \alpha_2) = 1.1832$ and $c_0 = c_2 \alpha_2 \gamma_1 = 0.16$. From these parameters, we calculate the PID controller parameters using (31) -(34):

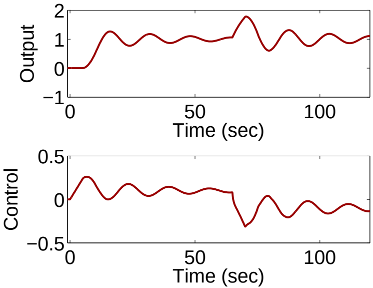
$$K_c = 0.332; \ \tau_I = 7.1; \ \tau_D = 1.43; \ \tau_f = 0.292.$$

We obtain the PID controller parameters for $w_n = 0.2$, as

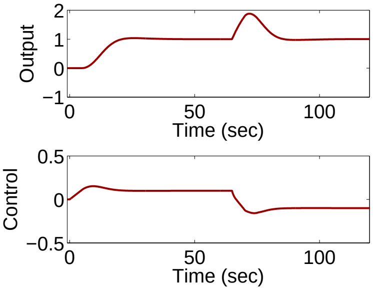
$$K_c = .1793; \ \tau_I = 8.0323; \ \tau_D = 1.3375; \tau_f = 0.5581.$$

Response

Figure 5a shows the closed-loop response. It is seen that both control signal and the plant output signal are oscillatory, which is due to the modelling error introduced by the approximation of the time delay. Figure 5b shows the closed-loop response with this reduced w_n. It is seen indeed that the closed-loop oscillation is eliminated.



(a) PID control with $\xi = 0.707$, $\lambda_1 = 1$ and $w_n = 0.4$



(b) PID control with $\xi = 0.707$, $\lambda_1 = 1$ and $w_n = 0.2$

Figure 5: Closed-loop response of PID control system

Outline

Reasons for not using pole-zero cancellation technique

- When the plant is underdamped, the pole-zero cancellation technique we used before will be avoided for the reason that the plant pole that was cancelled in the design will re-appear in the closed-loop system.
- We should not cancel an unstable pole.

General design technique-I

We assume that the PID controller has the form,

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s(s + l_0)} \tag{58}$$

and the second order model has the form,

$$G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \tag{59}$$

Without pole-zero cancellation, the open-loop transfer function is

$$L(s) = C(s)G(s) = \frac{c_2 s^2 + c_1 s + c_0}{s(s + l_0)} \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \tag{60}$$

General design technique-II

The closed-loop transfer function is

$$T(s) = \frac{L(s)}{1 + L(s)} = \frac{(c_2 s^2 + c_1 s + c_0)(b_1 s + b_0)}{s(s + l_0)(s^2 + a_1 s + a_0) + (c_2 s^2 + c_1 s + c_0)(b_1 s + b_0)} \qquad (61)$$

Note that the denominator of the closed-loop transfer function is a fourth order polynomial and there are four unknown controller parameters to be determined in the design. Thus, the desired closed-loop polynomial $A_{cl}(s)$ must be a fourth order polynomial with all zeros on the left half of the complex plane.

The desired closed-loop polynomial

We can assume that $A_{cl}(s)$ has the form ($\xi, w_n, \lambda_1 > 0$)

$$A_{cl}(s) = (s^2 + 2\xi w_n s + w_n^2)(s + \lambda_1)^2 \quad (62)$$

where the dominant poles are $-\xi w_n \pm j\xi w_n$ ($\xi < 1$), and $\lambda_1 \geq w_n$. For simplicity, the desired closed-loop polynomial $A_{cl}(s)$ is denoted as $s^4 + t_3 s^3 + t_2 s^2 + t_1 s + t_0$.

The Diophantine equation

To assign the closed-loop poles to the desired locations, we solve the Diophantine equation,

$$s(s+l_0)(s^2+a_1s+a_0)+(c_2s^2+c_1s+c_0)(b_1s+b_0)=s^4+t_3s^3+t_2s^2+t_1s+t_0$$

By multiplication and collecting terms, we find the exact quantity on the left hand side of equation:

$$\begin{aligned} s^4 &+ (b_1c_2+a_1+l_0)s^3+(b_1c_1+b_0c_2+a_0+a_1l_0)s^2+(b_1c_0+b_0c_1+l_0a_0)s \\ &+ b_0c_0=s^4+t_3s^3+t_2s^2+t_1s+t_0 \end{aligned} \tag{63}$$

Solution of the Diophantine equation I

By comparing both sides of (63), a set of linear equations is formed,

$$
\begin{aligned}
s^3 &: \quad b_1c_2 + a_1 + l_0 = t_3 && (64)\\
s^2 &: \quad b_1c_1 + b_0c_2 + a_0 + a_1l_0 = t_2 && (65)\\
s &: \quad b_1c_0 + b_0c_1 + l_0a_0 = t_1 && (66)\\
s^0 &: \quad b_0c_0 = t_0 && (67)
\end{aligned}
$$

Solution of the Diophantine equation II

This set of linear equations is expressed in matrix and vector form for convenience of solution,

$$\overbrace{\begin{bmatrix} 1 & b_1 & 0 & 0 \\ a_1 & b_0 & b_1 & 0 \\ a_0 & 0 & b_0 & b_1 \\ 0 & 0 & 0 & b_0 \end{bmatrix}}^{S_y} \begin{bmatrix} l_0 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} t_3 - a_1 \\ t_2 - a_0 \\ t_1 \\ t_0 \end{bmatrix} \tag{68}$$

where we assume that the square matrix S_y (called Sylvester matrix) is invertible.

Example

An inverted pendulum on a cart has external force as input $f(t)$ and its output is the angular position $\theta(t)$. The Laplace transfer function to describe the dynamics of a laboratory pendulum test bed has the form,

$$G(s) = \frac{-0.1}{(s-1)(s+1)} \tag{69}$$

Design a PID controller for this test bed, where the closed-loop performance is specified by $\xi = 0.707$, and $w_n = \lambda_1 = 10$. Simulate closed-loop step response using PID controller with the configuration of derivative term on output while the proportional and integral terms on the error signal, and with the configuration of both derivative term and proportional term on the output while only the integral term on the error signal.

Solution I

For this example, $b_1 = 0$, $b_0 = -0.1$, $a_1 = 0$ and $a_0 = -1$. From (68), the linear equation formulated based on the pole-assignment design technique is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.1 & 0 & 0 \\ 1 & 0 & -0.1 & 0 \\ 0 & 0 & 0 & -0.1 \end{bmatrix} \begin{bmatrix} l_0 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 34 \\ 482 \\ 3414 \\ 10000 \end{bmatrix} \tag{70}$$

Solution II

Solution of this set of linear equations gives the controller designed using the pole-assignment technique: $l_0 = 34.14$, $c_2 = -4818$, $c_1 = -33799$, $c_0 = -100000$. We convert these parameters into PID controller parameters as

$$\tau_f = \frac{1}{l_0} = 0.0293 \quad (71)$$

$$\tau_I = \frac{c_1}{c_0} - \tau_f = 0.3087 \quad (72)$$

$$K_c = \tau_I \tau_f c_0 = -904.2028 \quad (73)$$

$$\tau_D = \frac{c_2 \tau_I \tau_f - K_c \tau_I \tau_f}{K_c \tau_I} = 0.1268 \quad (74)$$

Solution III

The closed-loop step response is simulated for both structures. Figure 6 shows both control signal and output signal for a unit step change at time $t = 0$. In comparison, the alternative structure by implementing the proportional term on the output only significantly reduced the overshoot exhibited in the original structure.

Solution IV

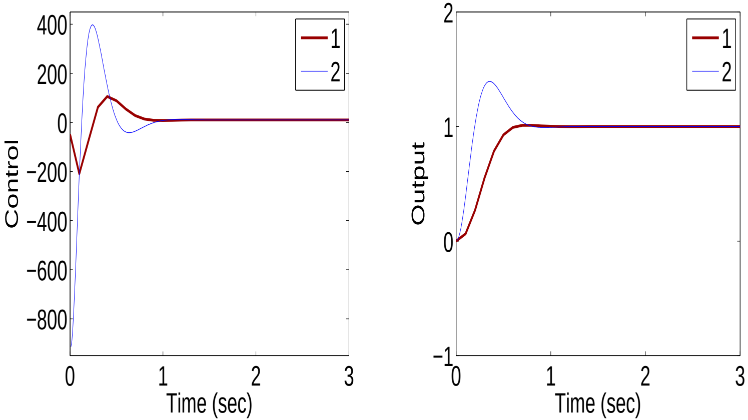


Figure 6: Closed-loop response. Key: line (1) The alternative PID controller structure; line (2) The original PID controller structure

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Outline

Resonant Controller Design (i)

- In the applications of control systems to power electronics, aerospace and mechanical engineering, it is often required for the output of the closed-loop control system to track sinusoidal reference signal or to reject a sinusoidal disturbance. In these applications, if the sinusoidal reference signal is $r(t) = A_m cos(\omega_0 t + \theta)$ then the feedback controller will embed the mode $\frac{1}{s^2+\omega_0^2}$ into the controller structure.
- Similarly, for disturbance rejection, if the disturbance signal is $d(t) = D_m cos(\omega_0 t + \theta)$, where the frequency ω_0 is known but the amplitude of the disturbance D_m and the phase of the sinusoidal signal are unknown, then the feedback controller will also embed the mode $\frac{1}{s^2+\omega_0^2}$.

Resonant Controller Design (ii)

- With the embedded mode, the closed-loop feedback control system is designed to be stable, and at the steady-state, the output of the control system will completely track the sinusoidal signal and/or reject a sinusoidal disturbance signal without any steady-state errors. In the literature, this type of controller is also called resonant controller.

Design (i)

- Consider a first order transfer function that is used to describe the dynamics of an AC motor with the form,

$$G(s) = \frac{b}{s+a} = \frac{B(s)}{A(s)} \tag{75}$$

- where the input is the torque current and output is velocity. The task is to design a controller $C(s)$ to reject a sinusoidal disturbance with frequency ω_0(rad/sec).
- Pole-assignment controller design will be used here.

Design(ii)

For the first order system, the controller structure is chosen as

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s^2 + \omega_0^2} = \frac{P(s)}{L(s)} \tag{76}$$

Here, the denominator of the controller is second order (s^2), the numerator is also chosen to be second order to allow proportional control action to be embedded. This choice leads to three unknown coefficients c_2, c_1 and c_0 to be determined.

Design(iii)

The actual closed-loop polynomial is

$$L(s)A(s) + P(s)B(s) = (s^2 + \omega_0^2)(s + a) + b(c_2 s^2 + c_1 s + c_0) \tag{77}$$

This is a third order polynomial. Therefore, the desired closed- loop polynomial should be third order and the number of desired closed-loop poles should be 3 accordingly. For instance, we can assume that the desired closed-loop polynomial $A_{cl}(s)$ has the form
$(\xi, w_n, \lambda_1 > 0)$

$$A_{cl}(s) = (s^2 + 2\xi w_n s + w_n^2)(s + \lambda_1) \tag{78}$$

where the dominant poles are $-\xi w_n \pm j\xi w_n$ $(\xi < 1)$, and $\lambda_1 \geq w_n$. For simplicity, the desired closed-loop polynomial $A_{cl}(s)$ is denoted as $s^3 + t_2 s^2 + t_1 s + t_0$.

Design (iv)

With the pole-assignment controller design technique, we let the actual closed-loop polynomial equal to the desired closed-loop polynomial, which leads to

$$L(s)A(s) + P(s)B(s) = A_{cl}(s) \tag{79}$$

Equation (79) is called the Diophantine equation. By substituting the expressions of $L(s)$, $A(s)$, $P(s)$, $B(s)$ and $A_{cl}(s)$ into equation (79), the Diophantine equation is written as:

$$s^3 + (a + bc_2)s^2 + (\omega_0^2 + bc_1)s + (a\omega_0^2 + bc_0) = s^3 + t_2 s^2 + t_1 s + t_0 \tag{80}$$

Design (v)

- In order for the left-hand side of the equation to be equal to the right-hand side of the equation, we have the following linear equations:

$$\begin{aligned} s^2 &: \quad a + bc_2 = t_2 && (81) \\ s &: \quad \omega_0^2 + bc_1 = t_1 && (82) \\ s^0 &: \quad a\omega_0^2 + bc_0 = t_0 && (83) \end{aligned}$$

- Solving these linear equations gives the coefficients of the controller as

$$\begin{aligned} c_2 &= \frac{t_2 - a}{b} && (84) \\ c_1 &= \frac{t_1 - \omega_0^2}{b} && (85) \\ c_0 &= \frac{t_0 - a\omega_0^2}{b} && (86) \end{aligned}$$

Steady-state Error Analysis (i)

- To show that the output of the closed-loop control system will follow a sinusoidal signal with frequency ω_0, we calculate the closed-loop feedback error signal $E(s) = R(s) - Y(s)$ in relation to the sinusoidal reference signal $R(s)$.
- Here, the control signal is

$$U(s) = \frac{c_2 s^2 + c_1 s + c_0}{s^2 + \omega_0^2} E(s) \tag{87}$$

and the output signal is

$$Y(s) = \frac{b}{s+a} U(s) = \frac{b}{s+a} \frac{c_2 s^2 + c_1 s + c_0}{s^2 + \omega_0^2} E(s) \tag{88}$$

Steady-state Error Analysis (ii)

- Noting that $Y(s) = R(s) - E(s)$, by substituting this into (88), we obtain

$$(1 + \frac{b}{s+a}\frac{c_2 s^2 + c_1 s + c_0}{s^2 + \omega_0^2})E(s) = R(s) \tag{89}$$

- The relationship between the reference signal $R(s)$ and the output signal $Y(s)$ is

$$\begin{aligned} E(s) &= \frac{(s+a)(s^2+\omega_0^2)}{(s^2+\omega_0^2)(s+a) + b(c_2 s^2 + c_1 s + c_0)} R(s) \\ &= \frac{(s+a)(s^2+\omega_0^2)}{(s^2 + 2\xi w_n s + w_n^2)(s+\lambda_1)} R(s) \end{aligned} \tag{90}$$

where we have used the Diophantine equation (79).

Steady-state Error Analysis (iii)

- When the set-point signal $r(t) = R_m sin(\omega_0 t + \theta_0)$, its Laplace transform is $R(s) = \frac{R_m\omega_0}{s^2+\omega_0^2}$.
- From (90), we have the Laplace transform of the feedback error signal,

$$
\begin{aligned}
E(s) &= \frac{(s+a)(s^2+\omega_0^2)}{(s^2+2\xi w_n s + w_n^2)(s+\lambda_1)} \frac{R_m\omega_0}{s^2+\omega_0^2} \\
&= \frac{(s+a)R_m\omega_0}{(s^2+2\xi w_n s + w_n^2)(s+\lambda_1)} \qquad (91)
\end{aligned}
$$

where we have cancelled the factor $s^2 + \omega_0^2$.

Steady-state Error Analysis (iv)

- Since the denominator of (91) contains all zeros on the left half of the complex plane, by applying final value theorem, we obtain

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\frac{(s+a)R_m\omega_0}{(s^2+2\xi w_n s+w_n^2)(s+\lambda_1)} = 0 \tag{92}$$

- Because $e(t) = r(t) - y(t)$ and $\lim_{t\to\infty} e(t) = 0$, we conclude that the output $y(t)$ will converge to the set-point signal $r(t)$.

Analysis for Disturbance Rejection (i)

- To show that the closed-loop control system will completely reject a sinusoidal disturbance, we will find the relationship between the input disturbance and the output.
- Here, the transfer function between the input disturbance $D_{in}(s)$ and the output $Y(s)$ is

$$
\begin{aligned}
\frac{Y(s)}{D_{in}(s)} &= \frac{G(s)}{1 + G(s)C(s)} \\
&= \frac{(s^2 + \omega_0^2)b}{(s^2 + \omega_0^2)(s + a) + b(c_2 s^2 + c_1 s + c_0)} \\
&= \frac{b(s^2 + \omega_0^2)}{(s^2 + 2\xi w_n s + w_n^2)(s + \lambda_1)} \qquad (93)
\end{aligned}
$$

Analysis for Disturbance Rejection (ii)

- Assume that the disturbance signal is a sinusoidal signal $d_{in}(t) = d_m sin(\omega_0 t + \theta)$ with unknown amplitude d_m and unknown phase θ, and it has the Laplace transform $D_{in}(s) = \frac{d_m \omega_0}{s^2+\omega_0^2}$.
- Thus, the output in response to the input disturbance $D_{in}(s)$ is

$$\begin{aligned} Y(s) &= \frac{b(s^2+\omega_0^2)}{(s^2+2\xi w_n s + w_n^2)(s+\lambda_1)} D_{in}(s) \\ &= \frac{b(s^2+\omega_0^2)}{(s^2+2\xi w_n s + w_n^2)(s+\lambda_1)} \frac{d_m \omega_0}{s^2+\omega_0^2} \\ &= \frac{b d_m \omega_0}{(s^2+2\xi w_n s + w_n^2)(s+\lambda_1)} \end{aligned} \tag{94}$$

- The zeros of denominator are all on the left-half of the complex plane as the parameters $\lambda_1 > 0$, $\xi = 0.707$ or 1 and $w_n > 0$.

Analysis for Disturbance Rejection (iii)

- Thus, by applying final value theorem, we obtain

$$\begin{aligned} \lim_{t \to \infty} y(t) &= \lim_{s \to 0} sY(s) \\ &= \lim_{s \to 0} \frac{sbd_m \omega_0}{(s^2 + 2\xi w_n s + w_n^2)(s + \lambda_1)} = 0 \end{aligned} \tag{95}$$

- Therefore, the output $y(t)$ in response to the input disturbance $d_{in}(t)$ is zero at the steady-state. This means that the input disturbance $d_{in}(t)$ will be completely rejected by the closed- loop feedback control system.

Example

For an AC motor with parameters $a = 0.01$ and $b = 0.05$, if the disturbance is at frequency $\omega_0 = 0.1$, and all three desired closed-loop poles are selected as -0.1, design the feedback controller for this AC motor so that the closed-loop control signal will reject the sinusoidal disturbance signal.

Solution (i)

The desired closed-loop polynomial is

$$A_{cl}(s) = (s+0.1)^3 = s^3 + 0.3s^2 + 0.03s + 0.001 \tag{96}$$

Here, $t_2 = 0.3$, $t_1 = 0.03$ and $t_0 = 0.001$. From the equations given in (86), the we find the controller parameters as

$$c_2 = \frac{t_2 - a}{b} = \frac{0.3 - 0.01}{0.05} = 5.8 \tag{97}$$

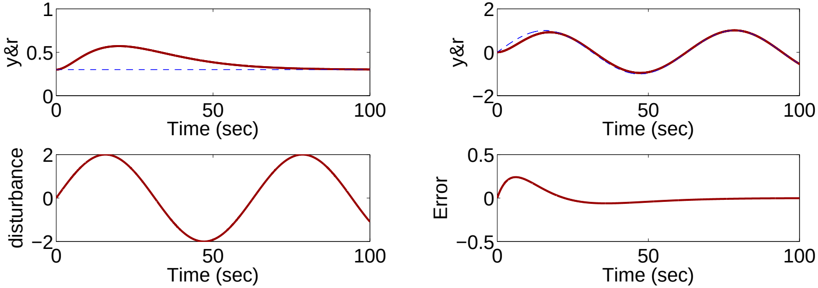
$$c_1 = \frac{t_1 - \omega_0^2}{b} = \frac{0.03 - 0.01}{0.05} = 0.4 \tag{98}$$

$$c_0 = \frac{t_0 - a\omega_0^2}{b} = \frac{0.001 - (0.01)^2}{0.05} = 0.018 \tag{99}$$

Solution (ii): Simulation Results

- First, we simulate the closed-loop disturbance rejection by operating the system at the steady-state value 0.3 and at time $t = 0$, inject a disturbance signal $din(t) = 2sin(0.1t)$. Figure 7a shows the output response to the disturbance signal. It is seen that the disturbance is completely rejected and the output returns to the reference signal.
- The same control system will also follow a sinusoidal set-point signal, as shown in Figure 7b, which shows that the output tracks a sinusoidal signal $r(t) = sin(0.1t)$. Thus, the control system we designed for sinusoidal disturbance rejection will automatically follow a sinusoidal signal with the same frequency.

Responses



(a) Disturbance rejection (b) Reference following

Figure 7: Closed-loop response of resonant control

Key Points

- For reference following or/and disturbance rejection of a sinusoidal signal with frequency ω_0, this frequency information should be included in the the denominator of the controller structure as $(s^2 + \omega_0^2)$.
- We will select the rest of the controller structure for the unique solution of the Diophantine equation

$$A(s)L(s) + B(s)P(s) = A_{cl}(s)$$

- $A_{cl}(s)$ is the desired closed-loop polynomial.