Part VII: Sensitivity Functions and Frequency Response Analysis

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Outline

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Learning Objectives

- One degree of freedom and two degrees of freedom control systems.
- The sensitivity functions and the relationship between the complementary sensitivity function and sensitivity function.
- Understand how sensitivity functions are used to describe closed-loop performance in terms of reference following, disturbance rejection and measurement noise attenuation.
- Understand how closed-loop bandwidth is used in a trade-off relationship between disturbance rejection and measurement noise attenuation.

One Degree of Freedom Control System Structure

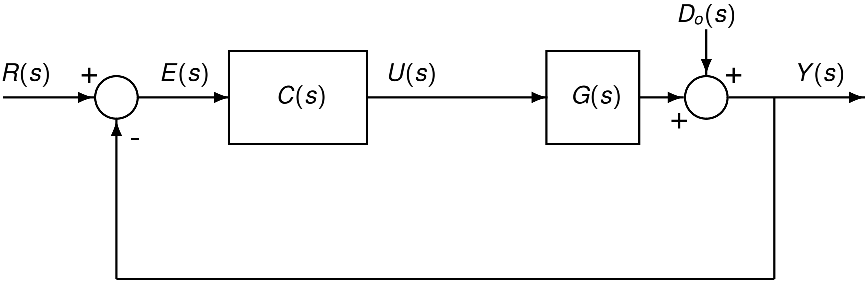


Figure 1: One-degree of freedom control system structure. Only one-degree of freedom is available in the controller structure to influence the output response $Y(s)$ to the reference signal $R(s)$ and to the disturbance $D_o(s)$.

Closed-loop Signals

- The Laplace transform of the error signal $E(s)$ is expressed as

$$\begin{aligned} E(s) &= R(s) - Y(s) = R(s) - G(s)U(s) - D_o(s) \\ &= R(s) - G(s)C(s)E(s) - D_o(s) \end{aligned} \quad (1)$$

Therefore, the error signal is expressed as

$$E(s) = \frac{R(s)}{1 + G(s)C(s)} - \frac{D_o(s)}{1 + G(s)C(s)} \quad (2)$$

- Then, the output of the control system is

$$\begin{aligned} Y(s) &= R(s) - E(s) = (1 - \frac{1}{1 + G(s)C(s)})R(s) + \frac{D_o(s)}{1 + G(s)C(s)} \\ &= \frac{G(s)C(s)}{1 + G(s)C(s)}R(s) + \frac{D_o(s)}{1 + G(s)C(s)} \end{aligned} \quad (3)$$

- The control signal is

$$U(s) = C(s)E(s) = \frac{C(s)}{1 + G(s)C(s)}R(s) - \frac{C(s)}{1 + G(s)C(s)}D_o(s) \quad (4)$$

The Closed-loop Transfer Functions

- The transfer function between the set-point signal and the plant output is

$$\frac{Y(s)}{R(s)} = \frac{G(s)C(s)}{1+G(s)C(s)} \tag{5}$$

- The set-point signal and the control signal is

$$\frac{U(s)}{R(s)} = \frac{C(s)}{1+G(s)C(s)} \tag{6}$$

- The transfer functions between the output disturbance and the output, and the output disturbance and the control signal are

$$\frac{Y(s)}{D_o(s)} = \frac{1}{1+G(s)C(s)} \tag{7}$$

$$\frac{U(s)}{D_o(s)} = -\frac{C(s)}{1+G(s)C(s)} \tag{8}$$

One-degree of Freedom Design:Summary

In this controller structure, once the controller $C(s)$ is selected, all four closed-loop transfer functions are fixed, only one-degree of freedom is available to influence the output response $Y(s)$ to the reference signal $R(s)$ and to the disturbance $D_o(s)$. This is called one-degree of freedom design.

Two Degrees of Freedom Control System Structure

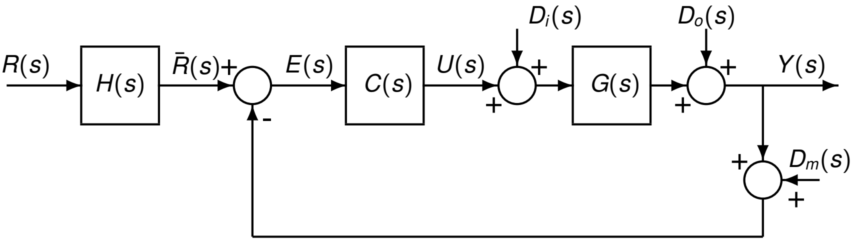


Figure 2: Two-degrees of freedom control system structure. An extra component $H(s)$ is placed after the reference signal $R(s)$, which will be used in the design.

Closed-loop Transfer Functions

- With the assumption that $D_i(s) = 0$ and $D_m(s) = 0$, we calculate the output response $Y(s)$ in relation to the reference signal $R(s)$ and the output disturbance $D_o(s)$,

$$Y(s) = \frac{G(s)C(s)H(s)}{1 + G(s)C(s)} R(s) + \frac{D_o(s)}{1 + G(s)C(s)} \tag{9}$$

- From this, we have the two transfer functions

$$\frac{Y(s)}{R(s)} = \frac{G(s)C(s)H(s)}{1 + G(s)C(s)} \tag{10}$$

$$\frac{Y(s)}{D_o(s)} = \frac{1}{1 + G(s)C(s)} \tag{11}$$

(12)

- Here, in comparison to one degree of freedom control system, we have the transfer function $H(s)$ act on the reference signal $R(s)$.

Two Degrees of Freedom Control System: Summary

- Transfer function $H(s)$ provides one more degree of freedom to shape the output response to the reference signal $R(s)$.
- This extra degree of freedom plus the original one degree of freedom gives the two degrees of freedom in the design.
- If the control system is configured as a two degrees of freedom, then we can shape, independently, the output response to the reference signal and to the disturbance.

Two Degrees of Freedom Design of PI Controllers (i)

- The Laplace transform of the control signal, $U(s)$, is expressed as the function of feedback error signal $E(s)$ using the relation

$$U(s) = C(s)E(s) = \frac{c_1 s + c_0}{s} E(s) \tag{13}$$

where $E(s) = R(s) - Y(s)$.

- The closed- loop transfer function between the set-point signal $R(s)$ and the output signal $Y(s)$ is then

$$\frac{Y(s)}{R(s)} = \frac{G(s)C(s)}{1 + G(s)C(s)} = \frac{b(c_1 s + c_0)}{s(s+a) + b(c_1 s + c_0)} \tag{14}$$

where $G(s)$ is the first order transfer function $\frac{b}{s+a}$.

- The closed-loop transfer function becomes:

$$\frac{Y(s)}{R(s)} = \frac{(2\xi w_n - a)s + w_n^2}{s^2 + 2\xi w_n s + w_n^2} = \frac{w_n^2(\frac{2\xi w_n - a}{w_n^2} s + 1)}{s^2 + 2\xi w_n s + w_n^2}$$

Two Degrees of Freedom Design of PI Controllers (ii)

- By choosing the reference filter $H(s) = \frac{1}{\frac{2\xi w_n - a}{w_n^2} s+1)}$, with the two degrees of freedom structure, the relationship between the reference signal and the output response is

$$\frac{Y(s)}{R(s)} = \frac{w_n^2}{s^2 + 2\xi w_n s + w_n^2} \tag{15}$$

- Note that the parameter $\frac{2\xi w_n - a}{w_n^2} = \tau_I$, thus, the set-point filter is $H(s) = \frac{1}{\tau_I s+1)}$.
- This closed-loop transfer function is equivalent to what we obtained from the alternative PI controller structure.
- However, in the two degree of freedom design, we can choose any reference filter $H(s)$, as long as it is stable, to reflect the requirement of desired reference following.

Two Degrees of Freedom Design of PI Controllers (iii)

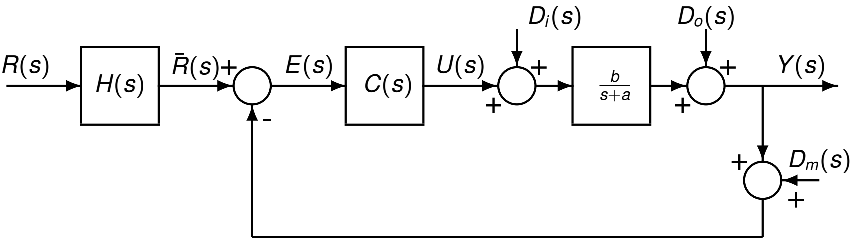


Figure 3: Two-degrees of freedom PI control system structure: $H(s) = \frac{1}{\tau_I s+1}$, $C(s) = K_c(1 + \frac{1}{\tau_I s})$.

Outline

Sensitivity Functions (i)

- We calculate the feedback error of the closed-loop system firstly as

$$\begin{aligned} E(s) &= H(s)R(s) - (Y(s) + D_m(s)) \\ &= H(s)R(s) - [G(s)(U(s) + D_i(s)) + D_o(s) + D_m(s)] \\ &= H(s)R(s) - G(s)C(s)E(s) - G(s)D_i(s) - D_o(s) - D_m(s) \quad (16) \end{aligned}$$

- The closed-loop feedback error by re-arranging (16) is

$$E(s) = \frac{H(s)}{1 + G(s)C(s)}R(s) - \frac{G(s)}{1 + G(s)C(s)}D_i(s) - \frac{D_o(s)}{1 + G(s)C(s)} - \frac{D_m(s)}{1 + G(s)C(s)}$$

- The expression of the closed-loop output $Y(s)$ is

$$\begin{aligned} Y(s) &= \frac{G(s)C(s)H(s)}{1 + G(s)C(s)}R(s) + \frac{D_o(s)}{1 + G(s)C(s)} + \frac{G(s)}{1 + G(s)C(s)}D_i(s) \\ &- \frac{G(s)C(s)}{1 + G(s)C(s)}D_m(s) \quad (17) \end{aligned}$$

Sensitivity Functions (ii)

Also, from the feedback error (16), we calculate the closed-loop control signal as

$$\begin{aligned} U(s) &= C(s)E(s) = \frac{C(s)H(s)}{1+G(s)C(s)}R(s) - \frac{C(s)G(s)}{1+G(s)C(s)}D_i(s) \\ &- \frac{C(s)}{1+G(s)C(s)}D_o(s) - \frac{C(s)}{1+G(s)C(s)}D_m(s) \end{aligned} \tag{18}$$

Definitions of Sensitivity Functions

Based on these relationships, the following sensitivity functions are defined:

- **Sensitivity function.**

$$S(s) = \frac{1}{1 + G(s)C(s)}$$

- **Complementary sensitivity function.**

$$T(s) = \frac{G(s)C(s)}{1 + G(s)C(s)}$$

- **Input disturbance sensitivity.**

$$S_i(s) = \frac{G(s)}{1 + G(s)C(s)}$$

- **Control sensitivity.**

$$S_u(s) = \frac{C(s)}{1 + G(s)C(s)}$$

Relationships between Sensitivity Functions

- The sensitivity plus complementary sensitivity equals to one:

$$S(s) + T(s) = \frac{1}{1 + G(s)C(s)} + \frac{G(s)C(s)}{1 + G(s)C(s)} = 1 \tag{19}$$

- The input disturbance sensitivity is related to sensitivity:

$$S_i(s) = \frac{G(s)}{1 + G(s)C(s)} = S(s)G(s) \tag{20}$$

- The control sensitivity is related to sensitivity:

$$S_u(s) = \frac{C(s)}{1 + G(s)C(s)} = S(s)C(s) \tag{21}$$

Expressions of Input and Output

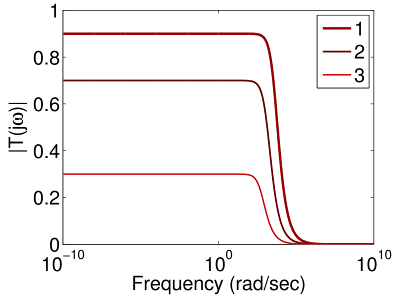
With the sensitivity functions, we re-write the output of the closed-loop system (17) as

$$Y(s) = H(s)T(s)R(s) + +S(s)D_o(s) + S_i(s)D_i(s) - T(s)D_m(s) \tag{22}$$

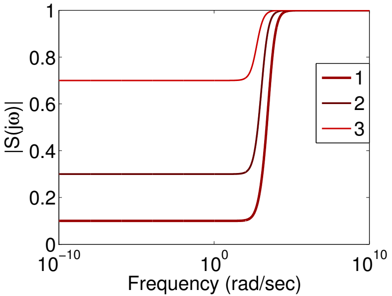
and the control signal (18) as

$$U(s) = H(s)S_u(s)R(s) - S_u(s)D_o(s) - S_u(s)G(s)D_i(s) - D_m(s) \tag{23}$$

Example of Sensitivity Functions (Proportional Control)



(a) Complementary sensitivity



(b) Sensitivity

Figure 4: Magnitude of complementary sensitivity and sensitivity functions using proportional control. Key: line(1) steady-state gain $\alpha = 0.9$, line(2) $\alpha = 0.7$, line(3) $\alpha = 0.3$.

Implications of the Sensitivity Functions

- The complementary sensitivity function $T(s)$ represents the effect of both reference signal and measurement noise on the output. If we want a fast response speed to a reference signal, then the closed-loop bandwidth will be wider (larger w_n). As a consequence, the closed-loop control system will amplify the measurement noise.
- The sensitivity $S(s)$ represents the effect of output disturbance on the output.
- The input sensitivity $S_i(s)$ represents the effect of input disturbance on the output.

Disturbance Rejection and Noise Attenuation (i)

- There are both noise and disturbance existed in a physical system. A good closed-loop performance requires minimization of the effects of both disturbance rejection and noise. Or we call it disturbance rejection and noise attenuation.
- For minimization of the effects of both input and output disturbances, we will make the magnitude of the output in frequency response

$$|Y_d(j\omega)| = |S(j\omega)(D_o(j\omega) + G(j\omega)D_i(j\omega))| \tag{24}$$

  as small as possible.
- For minimization of the measurement noise, we will make the magnitude of the output in frequency response

$$|Y_m(j\omega)| = |T(j\omega)D_m(j\omega)| \tag{25}$$

  as small as possible.

Disturbance Rejection and Noise Attenuation (ii)

We can not alter the disturbances and noise, because they already existed in the system. What we will do is to make

- the magnitude of sensitivity $S(j\omega)$ ($|S(j\omega)|$) small for disturbance rejection;
- the magnitude of complementary sensitivity $T(j\omega)$ ($|T(j\omega)|$) small for noise attenuation.

These are the basic design principles for control systems.

Disturbance Rejection and Noise Attenuation (iii)

- Noting that the relationship between the sensitivity and complementary sensitivity is constrained by

$$S(j\omega) + T(j\omega) = 1 \tag{26}$$

  which says that we can not make both $|S(j\omega)|$ and $|T(j\omega)|$ small over the same frequency bands.
- In other words, if the disturbance is minimized in a given frequency region where $|S(j\omega)|$ is small, then inevitably the measurement noise is not attenuated in the same frequency region where $|T(j\omega)|$ is large.

Disturbance Rejection and Noise Attenuation (iv)

- So how are we going to design a closed-loop control system that will minimize the effects of disturbance and the measurement noise?
- Note that the disturbances existed in the system correspond to slow movement of the variables or slow changes, therefore, the frequency contents of the disturbance term $|D_o(j\omega) + G(j\omega)D_i(j\omega)|$ are concentrated in the low frequency region.
- In contrast, the measurement noise corresponds to fast movement of the variables or fast and frequent changes of the variables, therefore, the frequency contents of the measurement noise $|D_m(j\omega)|$ are concentrated in the higher frequency region.

Disturbance Rejection and Noise Attenuation (v)

The strategies for disturbance rejection and noise attenuation are

- to achieve disturbance rejection by choosing the sensitivity function $S(j\omega) \approx 0$ at the low frequency region, implying $T(j\omega) \approx 1$ at the low frequency region, because $S(j\omega) + T(j\omega) = 1$.
- This is not too bad for noise attenuation because $|D_m(j\omega)|$ is small in the low frequency region.
- At the high frequency region, to avoid the amplifcation of measurement noise, we choose $|T(j\omega)| \approx 0$, which implies $|S(j\omega)| \approx 1$.
- This is not too bad for disturbance rejection because $|D_o(j\omega) + G(j\omega)D_i(j\omega)|$ is small in the high frequency region.
- Essentially, we will adjust the closed-loop bandwidth to compromise disturbance rejection with measurement noise attenuation.

Summary of Disturbance Rejection and Noise Attenuation

- The parameter w_n corresponds to the desired closed-loop bandwidth.
- Larger w_n leads to a wider bandwidth for $|T(j\omega)|$, which implies a faster disturbance rejection and higher noise amplification.
- On the other hand, smaller w_n leads to a more narrow bandwidth for $|T(j\omega)|$, which implies a slower disturbance rejection and lower noise amplification.
- Larger w_n will also lead to faster reference response.

Outline

Nyquist Stability Criterion (i)

Nyquist diagram

When using the Nyquist stability criterion, we examine the frequency response of the loop transfer function ($M(j\omega) = C(j\omega)G(j\omega)$) that contains the transfer functions for the plant and the controller. We plot this frequency response in a complex plane.

Stability Criterion

The criterion states that a feedback control system with single input and single output is stable *if and only if*, for the frequency response of the loop transfer function, number of counter clockwise encirclements of the $(-1, 0)$ point is equal to the number of poles of this loop transfer function with positive real parts.

Nyquist Stability Criterion (ii)

- For majority of the PID control problems, this loop transfer function $M(s)$ does not contain any poles that have positive real parts.
- For this type of systems, the closed-loop system will be stable if and only if the frequency response $M(j\omega)$ is not to encircle $(-1, 0)$ point on the complex plane.

Nyquist Diagram

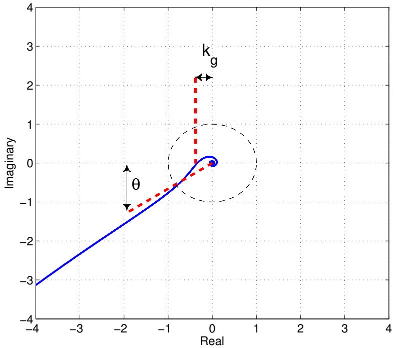


Figure 5: Nyquist plot with a unit circle for illustration of gain margin and phase margin. Solid line: Nyquist loci; dashed lines: pointers for the gain margin and phase margin. $M(s) = \frac{3187.4}{s+3355.2} \frac{594.9}{s+1.956} \frac{0.0271s+0.2737}{s} e^{-0.03s}$

Gain Margin

Definition

Gain margin is defined as $GM = \frac{1}{k_g}$, where k_g is the distance between the origin of the complex plane and the point that $M(j\omega)$ intersects the real axis (see Figure 5.

What does it mean

It means that if the loop gain were to exceed the reciprocal of k_g, then the closed-loop system would become unstable.

Phase Margin

Definition

It is the angle between the negative real axis and the line that intersects the circle $|M(j\omega)| = 1$ (see Figure 5).

What does it mean

Phase margin indicates the additional phase lag that could be associated with $M(j\omega)$ before the closed-loop system became unstable.

Good design

A good design should have reasonable gain and phase margins to ensure that the closed-loop system is robustly stable in the presence of factors known and unknown.

Delay Margin

- Although phase margin θ represents how much additional phase lag that can be added to the feedback control system before it became unstable, it does not directly convey the size of maximum time delay that can be added to the system.
- To determine the maximum time delay that can be tolerated, we let

$$e^{-j\theta} = e^{-jd_m\omega_p}$$

  where d_m is the delay margin or the maximum delay to be tolerated and ω_p is the frequency when the unit circle intersects with the Nyquist loci.
- This yields

$$d_m = \frac{\theta}{\omega_p}$$

  Clearly, a larger ω_p would lead to a smaller delay margin given the same phase margin θ.

Determining ω_p

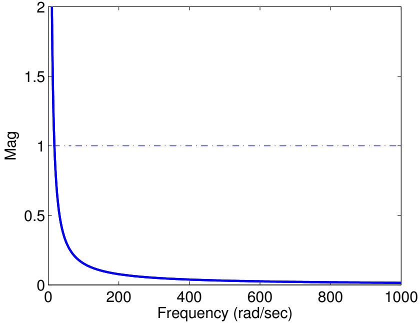


Figure 6: Magnitude of $M(j\omega)$ (solid line) together with dashed line to determine ω_p.

Example: Examining PI Controllers from Using Tuning Rules

We re-examine the PI controllers presented in Part V on tuning rules, where the continuous-time plant has the transfer function

$$G(s) = \frac{0.5e^{-20s}}{(30s+1)^3} \tag{27}$$

Three sets of PI controller parameters were obtained by using the tuning rules of Ziegler-Nichols, Cohen-Coon and Wang and Cluett, as shown in Table 7 (see Part V).

The Nyquist Plots of Original System

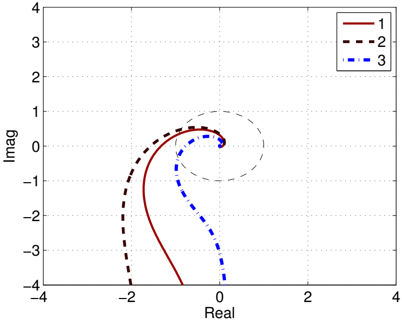


Figure 7: Nyquist plots with a unit circle. Key: line 1- $M(j\omega)$ using Ziegler-Nichols; line 2- Cohen-Coon tuning rules; and line 3- using Wang-Cluett tuning rules.

Unstable systems

The original PID control systems

Clearly the Ziegler-Nichols and Cohen-Coon tuning rules lead to unstable closed-loop system, which was confirmed by the simulation results.

Modifications

From the Nyquist diagrams, if we half the proportional controller gain for the controllers using Ziegler-Nichols and Cohen-Coon tuning rules, then the closed-loop systems will be stable.

Nyquist Plots of Modified Systems

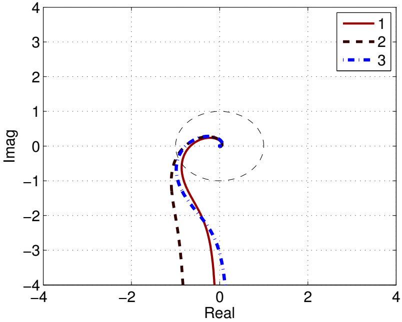


Figure 8: Nyquist plots for the modified controller with a unit circle. Key: line 1- using Ziegler-Nichols with half of the original K_c; line 2- Cohen-Coon tuning rules with half of the original K_c; and line 3- using Wang-Cluett tuning rules.

Gain, Phase and Delay Margins

	K_c	τ_I	Gain margin	Phase margin	Delay margin
Ziegler-Nichols	3.2	108	1.4914	0.5305	23.5778
Cohen-Coon	3.26	75.9231	1.258	0.2879	11.9461
Wang-Cluett	3.8867	127.2154	1.2191	0.3184	12.585

Table 1: Modified PI controller parameters with gain margin, phase margin and delay margin

Closed-loop Response of Modified System

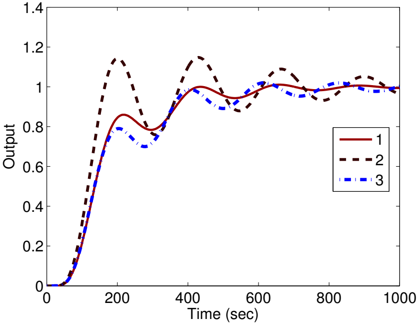


Figure 9: Comparison of closed-loop step responses. Key: line 1- Ziegler-Nichols with reduced K_c; line 2- Cohen-Coon tuning rules with reduced K_c; and line 3- using Wang-Cluett tuning rules.

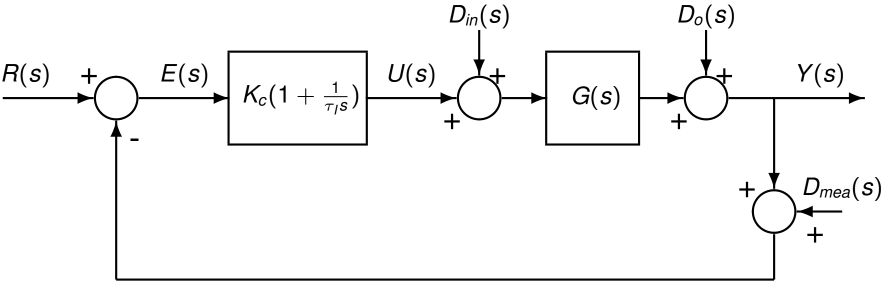
Summary

- Configuration of control systems in terms of one degree of freedom and two degrees of freedom design;
- Sensitivity functions, the relationship between them and their roles in control system analysis;
- The trade-off relationship between disturbance rejection and noise attenuation.
- Nyquist stability criterion with gain margin and phase margin.

Outline

Example (i)

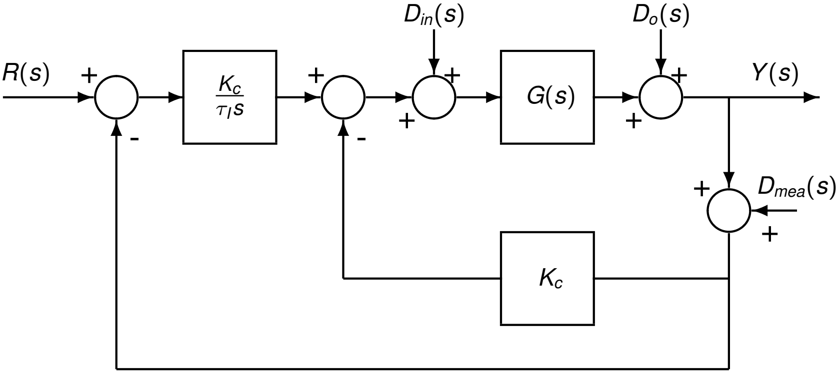
Derive closed-loop transfer functions for the following PI control system.



$R(s)$ is the reference signal
$D_{in}(s)$ is the input disturbance
$D_o(s)$ is the output disturbance
$D_{mea}(s)$ is the measurement noise
$Y(s)$ is the output signal

Example (ii)

Second configuration of the PI controller



Example (iii)

Given $K_c = 0.56$, $\tau_I = 8$
and the plant transfer function $G(s) = \frac{1}{s(s+3)^3}$.

Find the following transfer functions:

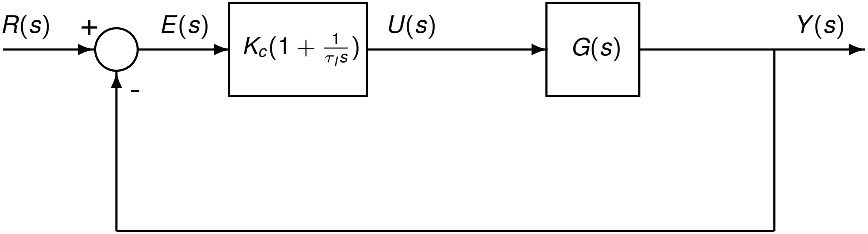
$$\frac{Y(s)}{R(s)}, \frac{Y(s)}{D_{in}(s)}, \frac{Y(s)}{D_{out}(s)}, \frac{Y(s)}{D_{mea}(s)}$$

for both controller configurations.

Discuss the differences.

Solution I

System diagram for reference and output signals in first configuration.

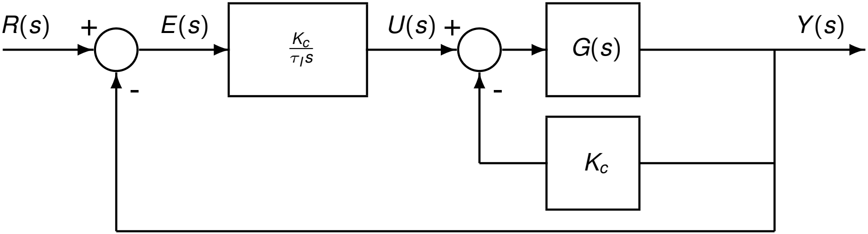


The transfer function

$$
\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{C(s)G(s)}{1+C(s)G(s)} = \frac{\frac{K_c\tau_I s+K_c}{\tau_I s}G(s)}{1+\frac{K_c\tau_I s+K_c}{\tau_I s}G(s)} = \frac{K_c\tau_I sG(s)+K_cG(s)}{\tau_I s+K_c\tau_I sG(s)+K_cG(s)} \\
&= \frac{K_c\tau_I s\frac{1}{s(s+1)^3}+K_c\frac{1}{s(s+1)^3}}{\tau_I s+K_c\tau_I s\frac{1}{s(s+1)^3}+K_c\frac{1}{s(s+1)^3}} = \frac{K_c\tau_I s+K_c}{\tau_I s^2(s+1)^3+K_c\tau_I s+K_c} \\
&= \frac{4.48s+0.56}{8s^5+24s^4+24s^3+8s^2+4.48s+0.56}
\end{aligned}
$$

Solution II

Block Diagram of $Y(s)$ and $R(s)$ in the second configuration of the PI controller



The transfer function is shown as following

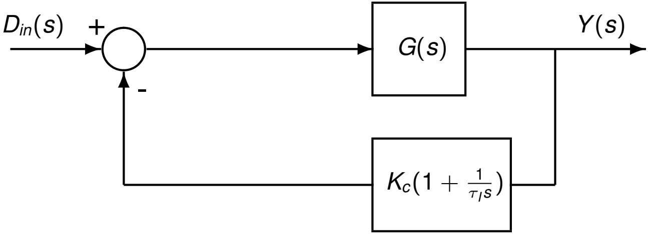
$$\frac{Y(s)}{R(s)} = \frac{\frac{K_c}{\tau_I s}\frac{G(s)}{1+K_cG(s)}}{1+\frac{K_c}{\tau_I s}\frac{G(s)}{1+K_cG(s)}} = \frac{K_cG(s)}{\tau_I s + K_c\tau_I sG(s) + K_cG(s)}$$

$$= \frac{K_c\frac{1}{s(s+1)^3}}{\tau_I s + K_c\tau_I s\frac{1}{s(s+1)^3} + K_c\frac{1}{s(s+1)^3}} = \frac{K_c}{\tau_I s^2(s+1)^3 + K_c\tau_I s + K_c}$$

$$= \frac{0.56}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}$$

Solution III

System diagram for input disturbance and output signals in first configuration.

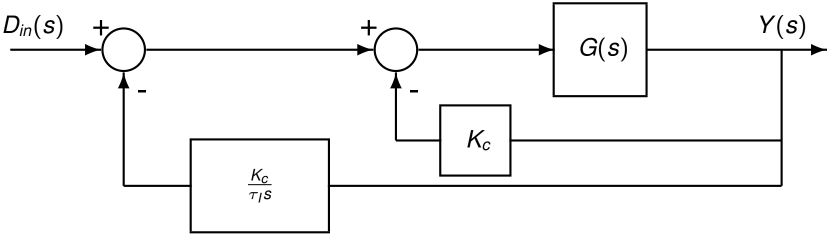


The transfer function of input disturbance is

$$
\begin{aligned}
\frac{Y(s)}{D_{in}(s)} &= \frac{G(s)}{1+\frac{K_c\tau_I s+K_c}{\tau_I s}G(s)} = \frac{\tau_I s G(s)}{\tau_I s + K_c\tau_I s G(s) + K_c G(s)} \\
&= \frac{\tau_I s \frac{1}{s(s+1)^3}}{\tau_I s + K_c\tau_I s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}} = \frac{\tau_I s}{\tau_I s^2(s+1)^3 + K_c\tau_I s + K_c} \\
&= \frac{8s}{8s^5+24s^4+24s^3+8s^2+4.48s+0.56}
\end{aligned}
$$

Solution IV

Block Diagram of $Y(s)$ and $D_{in}(s)$ in the second configuration of the PI controller

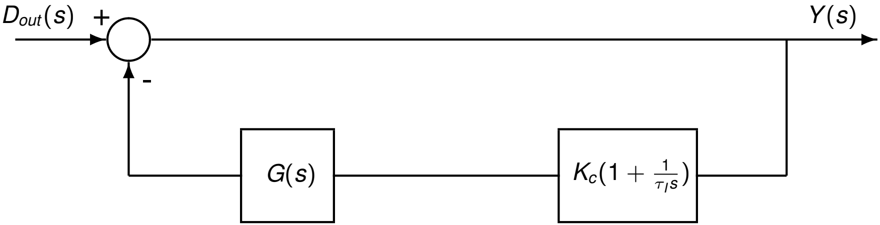


The transfer function of input disturbance in second configuration is

$$
\begin{aligned}
\frac{Y(s)}{D_{in}(s)} &= \frac{\frac{G(s)}{1+G(s)K_c}}{1+\frac{G(s)}{1+G(s)K_c}\frac{K_c}{\tau_I s}} = \frac{\tau_I s G(s)}{\tau_I s + K_c \tau_I s G(s) + K_c G(s)} \\
&= \frac{\tau_I s \frac{1}{s(s+1)^3}}{\tau_I s + K_c \tau_I s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}} = \frac{\tau_I s}{\tau_I s^2 (s+1)^3 + K_c \tau_I s + K_c} \\
&= \frac{8s}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}
\end{aligned}
$$

Solution V

System block diagram for output disturbance and output signals in first configuration.



The transfer function in this case is

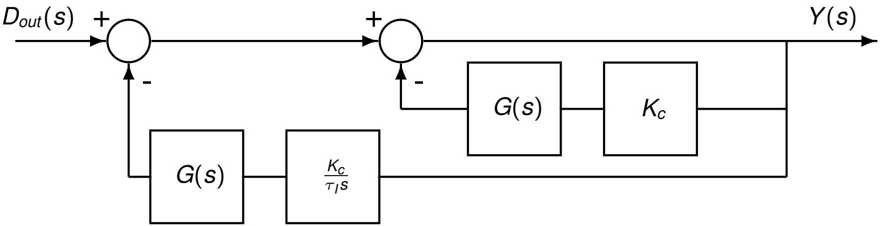
$$\frac{Y(s)}{D_{out}(s)} = \frac{1}{1 + G(s)C(s)} = \frac{1}{1 + \frac{K_c \tau_I s + K_c}{\tau_I s} G(s)} = \frac{\tau_I s}{\tau_I s + K_c \tau_I s G(s) + K_c G(s)}$$

$$= \frac{\tau_I s}{\tau_I s + K_c \tau_I s \frac{1}{s(s+1)^3} + K_c \frac{1}{s(s+1)^3}} = \frac{\tau_I s^2 (s+1)^3}{\tau_I s^2 (s+1)^3 + K_c \tau_I s + K_c}$$

$$= \frac{8s^5 + 24s^4 + 24s^3 + 8s^2}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}$$

Solution VI

After simplifying, the system block diagram for $D_{out}(s)$ and $Y(s)$ in second configuration is shown below



The transfer function in this case is

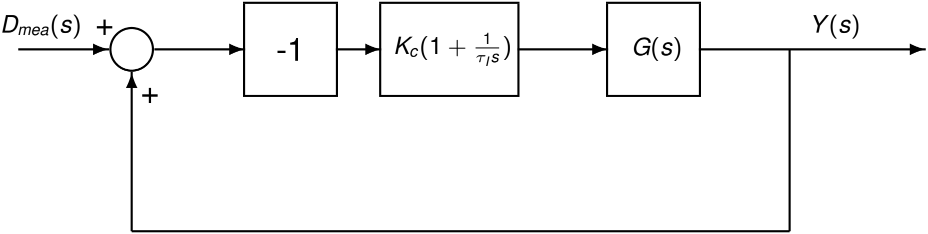
$$\frac{Y(s)}{D_{out}(s)} = \frac{\frac{1}{1+K_cG(s)}}{1+\frac{1}{1+K_cG(s)}G(s)\frac{K_c}{\tau_I s}} = \frac{\tau_I s}{\tau_I s + K_c\tau_I sG(s) + K_cG(s)}$$

$$= \frac{\tau_I s}{\tau_I s + K_c\tau_I s\frac{1}{s(s+1)^3} + K_c\frac{1}{s(s+1)^3}} = \frac{\tau_I s^2(s+1)^3}{\tau_I s^2(s+1)^3 + K_c\tau_I s + K_c}$$

$$= \frac{8s^5 + 24s^4 + 24s^3 + 8s^2}{8s^5 + 24s^4 + 24s^3 + 8s^2 + 4.48s + 0.56}$$

Solution VII

System block diagram for measurement noise and output signals in first configuration.

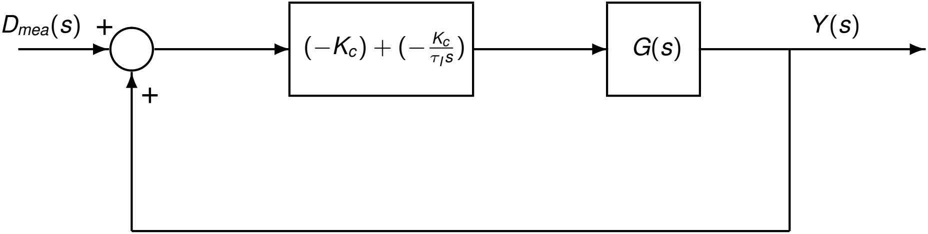


The transfer function in this case is

$$
\begin{aligned}
\frac{Y(s)}{D_{mea}(s)} &= \frac{-C(s)G(s)}{1-(-C(s)G(s))} = \frac{-\frac{K_c\tau_I s+K_c}{\tau_I s}G(s)}{1+\frac{K_c\tau_I s+K_c}{\tau_I s}G(s)} = -\frac{K_c\tau_I sG(s)+K_cG(s)}{\tau_I s+K_c\tau_I sG(s)+K_cG(s)} \\
&= -\frac{K_c\tau_I s\frac{1}{s(s+1)^3}+K_c\frac{1}{s(s+1)^3}}{\tau_I s+K_c\tau_I s\frac{1}{s(s+1)^3}+K_c\frac{1}{s(s+1)^3}} = -\frac{K_c\tau_I s+K_c}{\tau_I s^2(s+1)^3+K_c\tau_I s+K_c} \\
&= -\frac{4.48s+0.56}{8s^5+24s^4+24s^3+8s^2+4.48s+0.56}
\end{aligned}
$$

Solution VIII

After the simplifying rule of the block diagram, the system for measurement noise and output signals in second configuration become



The transfer function is identical to the previous case

$$\frac{Y(s)}{D_{mea}(s)} = \frac{-\frac{K_c\tau_I s+K_c}{\tau_I s}G(s)}{1+\frac{K_c\tau_I s+K_c}{\tau_I s}G(s)} = -\frac{K_c\tau_I sG(s)+K_cG(s)}{\tau_I s+K_c\tau_I sG(s)+K_cG(s)}$$

$$= -\frac{K_c\tau_I s\frac{1}{s(s+1)^3}+K_c\frac{1}{s(s+1)^3}}{\tau_I s+K_c\tau_I s\frac{1}{s(s+1)^3}+K_c\frac{1}{s(s+1)^3}} = -\frac{K_c\tau_I s+K_c}{\tau_I s^2(s+1)^3+K_c\tau_I s+K_c}$$

$$= -\frac{4.48s+0.56}{8s^5+24s^4+24s^3+8s^2+4.48s+0.56}$$

Conclusions

- From the transfer functions before, we can obtain that the transfer functions for both controller configurations are different only for the Reference signal, other disturbance cases have identical transfer function.
- That's because the second configuration of PI controller is designed to reduce the overshoot of the set-point change during the transient.
- However, the second configuration does NOT affect the transfer function of the disturbances, which means the configuration will only change the response of the reference signal change.

Exercise

Given $K_c = -4.5$, $\tau_I = 1.68$
and the plant transfer function $G(s) = \frac{s-2}{(s+1)(s+2)(s+3)}$.

Find the following transfer functions:

$$\frac{Y(s)}{R(s)}, \ \frac{Y(s)}{D_{in}(s)}, \ \frac{Y(s)}{D_{out}(s)}, \ \frac{Y(s)}{D_{mea}(s)}$$

for both controller configurations.

Discuss the differences.

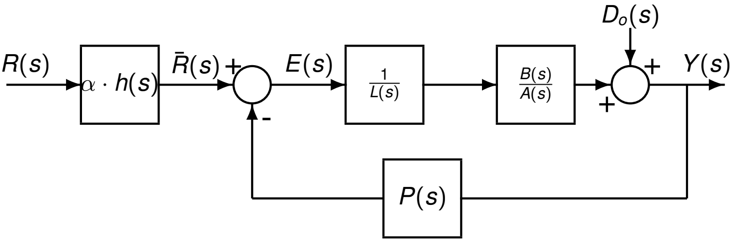
Outline

Two degree of freedom Design Example

Two degrees of freedom pole-assignment controller design.
Suppose that the plant transfer function is given by:

$$G(s) = \frac{3}{s^2 - s} = \frac{3}{s(s-1)}$$

The block diagram of the control system is shown as following:



Two degree of freedom Design Example

we wish to obtain the output response to a set-point input as

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where we choose that $\omega_n^2 = 2.44$ and $\xi = 0.768$.
Therefore,

$$\begin{aligned} T(s) &= \frac{2.44}{s^2 + 2.4s + 2.44} \\ &= \frac{2.44}{(s + 1.2 - j1)(s + 1.2 + j1)} \end{aligned}$$

The closed-loop transfer function is derived from the

$$\frac{Y(s)}{R(s)} = \frac{\alpha B(s)h(s)}{L(s)A(s) + B(s)P(s)} = \frac{\alpha B(s)}{\hat{A}_{cl}(s)}$$

Here $\hat{A}_{cl}(s) = s^2 + 2\xi\omega_n s + \omega_n^2$.

Two degree of freedom Design Example

By applying the Final Value Theorem,

$$\alpha \times \frac{B(0)}{\hat{A}_{cl}(0)} = 1$$

Therefore,

$$\alpha = \frac{\hat{A}_{cl}(0)}{B(0)} = \frac{\omega_n^2}{3} = \frac{2.44}{3} = 0.813$$

$h(s)$ does not affect set-point response, but affects disturbance rejection. Let $h(s) = s + h_0$.

Question: How h_0 affects the controller $L(s)$ and $P(s)$ for $h_0 = 0.2$, 2 and 20? (where $C(s) = \frac{p_1 s + p_0}{s + l_0}$)

Two degree of freedom Design Example

Case A: $h_0 = 0.2$
The Diophantine Equation:

$$\begin{aligned} L(s)A(s) + B(s)P(s) &= \hat{A}_{cl}(s)h(s) \\ (s + l_0)(s(s-1)) + 3(p_1 s + p_0) &= (s^2 + 2.4s + 2.44)(s + 0.2) \\ s^3 + (l_0 - 1)s^2 - l_0 s + 3p_1 s + 3p_0 &= s^3 + 2.6s^2 + 2.92s + 0.488 \end{aligned}$$

By comparing the coefficients, we have

$$\begin{aligned} l_0 - 1 &= 2.6 \\ -l_0 + 3p_1 &= 2.92 \\ 3p_0 &= 0.488 \end{aligned}$$

Therefore, the solution of the controller design is:

$$\begin{cases} l_0 = 3.6 \\ p_1 = 2.17 \\ p_0 = 0.163 \end{cases}$$

Two degree of freedom Design Example

Case B: $h_0 = 2$
The Diophantine Equation:

$$\begin{aligned} L(s)A(s) + B(s)P(s) &= \hat{A}_{cl}(s)h(s) \\ (s + l_0)(s(s-1)) + 3(p_1 s + p_0) &= (s^2 + 2.4s + 2.44)(s+2) \end{aligned}$$

By comparing the coefficients, the solution of the controller design is derived:

$$\begin{cases} l_0 = 5.4 \\ p_1 = 4.21 \\ p_0 = 1.63 \end{cases}$$

Case C: $h_0 = 20$
The Diophantine Equation:

$$\begin{aligned} L(s)A(s) + B(s)P(s) &= \hat{A}_{cl}(s)h(s) \\ (s + l_0)(s(s-1)) + 3(p_1 s + p_0) &= (s^2 + 2.4s + 2.44)(s+20) \end{aligned}$$

Two degree of freedom Design Example

By comparing the coefficients, the solution of the controller design is derived:

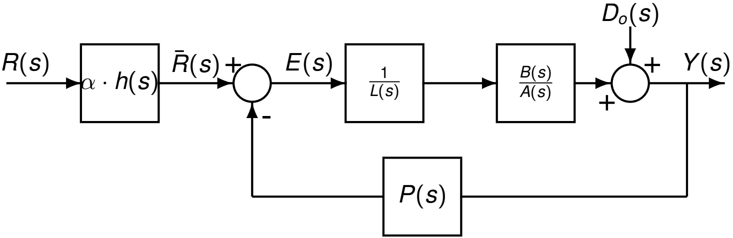
$$\begin{cases} l_0 = 23.4 \\ p_1 = 24.61 \\ p_0 = 16.26 \end{cases}$$

By comparing the controller parameters solution from case A, B and C, we see that as h_0 increases, l_0, p_1 and p_0 increases.

In other words, as we move the closed-loop poles h_0 further away from the origin of the complex plane, the gain of the controller increases. Higher controller gain will result in the faster disturbance rejection response.

Two degree of freedom Design Example

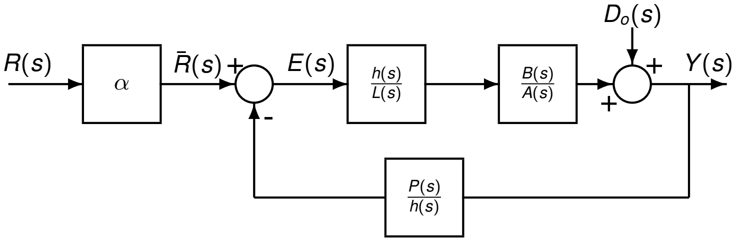
Implementation of two-degree of freedom control system.
Previously, we proposed the design based on the diagram as following:



However, this structure of the feedback control system is not good for implementation. That's because the sectors of $h(s) = s + h_0$ and $P(s) = p_n s^n + p_{n-1} s^{n-1} + \ldots p_0$ involve the differentiation operation.
Question: How can we modify the configuration so that the control system is realizable?

Two degree of freedom Design Example

Solution: The modified configuration is displayed as following:



Proof This is equivalent because the closed-loop transfer functions for output from both reference signal and output disturbance signal are the same as before.

$$
\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{\alpha \frac{h(s)}{L(s)} \frac{B(s)}{A(s)}}{1 + \frac{h(s)}{L(s)} \frac{P(s)}{h(s)} \frac{B(s)}{A(s)}} \\
&= \frac{\alpha h(s) B(s)}{L(s)A(s) + P(s)B(s)} = \frac{\alpha h(s) B(s)}{\hat{A}_{cl}(s) h(s)} \\
&= \frac{\alpha B(s)}{\hat{A}_{cl}(s)}
\end{aligned}
$$

Two degree of freedom Design Example

$$
\begin{aligned}
\frac{Y(s)}{D_o(s)} &= \frac{1}{1 + \frac{h(s)}{L(s)} \frac{P(s)}{h(s)} \frac{B(s)}{A(s)}} \\
&= \frac{L(s)A(s)}{L(s)A(s) + P(s)B(s)} \\
&= \frac{L(s)A(s)}{\hat{A}_{cl}(s)h(s)}
\end{aligned}
$$

Use of Sensitivity Function in Design

Suppose that a model of plant has Laplace transfer function:

$$G(s) = \frac{4}{(s+1)(s+2)^2}$$

the plant has output disturbance

$$d_o(t) = k + d_v(t)$$

where $d_v(t)$ is a zero mean signal with energy in the band $B_d : [0, 4]$ rad/s, k is a constant.
If we choose the complementary sensitivity

$$T(s) = \frac{\alpha}{(s^2 + 1.2\omega_n s + \omega_n^2)(\tau s + 1)^2}$$

Question: How should we design a feedback controller to reject the disturbance by choosing α, ω_n and τ?

Use of Sensitivity Function in Design

Solution:

(1) The constant disturbance with unknown amplitude k has frequency content at $\omega = 0$. In order to reject a constant disturbance we want to choose $S(0) = 0$, which makes

$$Y_d(0) = S(0)D_o(0) = 0$$

at zero frequency.
This choice implies $T(0) = 1$, because of $S(j\omega) + T(j\omega) = 1$.
Therefore, $\alpha = \omega_n^2$.

(2) How do we reject $d_v(t)$?
We want $S(j\omega)$ to be small in $[0, 4]$ rad/s to reject $d_v(t)$.
For a second order system, this means we need to choose $\omega_n \gg 4$ rad/s.
Let us select $\omega_n = 10$ rad/s.

Use of Sensitivity Function in Design

(3)How to choose τ?
We select τ value to be small, so that, it's not to interfere with ω_n.
So we have $\tau = 0.01$.

Therefore, the complementary sensitivity transfer function is:

$$T(s) = \frac{100}{(s^2 + 12s + 100)(0.01s + 1)^2}$$

Exercise

If the complementary sensitivity is given as:

$$\begin{aligned} T(s) &= \frac{C(s)G(s)}{1 + C(s)G(s)} \\ C(s)G(s) &= \frac{T(s)}{1 - T(s)} \\ C(s) &= \frac{T(s)}{1 - T(s)} G^{-1}(s) \end{aligned}$$

By using the information given in the part A, find the controller structure and parameters based on the last equation above.

Question:

(1) Does $C(s)$ have an integrator? If yes, where was it generated?

(2) What are the orders of the numerator and denominator of the controller? Do you think the orders are reasonable choices?