Part V: Linearization

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Outline

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Introduction

- One of the approaches to obtain the models for the control system design is based on analysis of the system dynamics using first principles, such as mass balance, Newton's laws, current law and voltage law. The majority of these types of models are nonlinear in nature.
- Thus, in order to use them for the PID controller design or other linear time invariant controller design, these nonlinear models need to be linearized around the operating conditions of the system.

The General Principle

- Assume that the nonlinear models have the general form:

$$\dot{x}(t) = f[x(t), u(t), t] \tag{1}$$

  where $f[.]$ is a nonlinear function. The purpose of linearization is to find a linear function (a set of linear functions) to describe the dynamics of the nonliear model at a given operating condition.
- Note that this linear model is obtained at a given operating condition.

Linearization of Nonlinear Functions (i)

- We will use Taylor series expansion to approximate a nonlinear function.
- A single variable case. A function with variable x, $f(x)$ can be expressed in terms of Taylor series expansion as

$$f(x) = f(x^0) + \frac{df(x)}{dx}|_{x=x^0}(x - x^0) + \frac{1}{2}\frac{d^2f(x)}{dx^2}(x - x^0)^2 + \dots \tag{2}$$

  if the function $f(x)$ is smooth and its derivatives exist for all the orders.
- Using first two terms in the Taylor series expansion leads to the approximation of the original function $f(x)$ at the specific point x^0,

$$f(x) \approx f(x^0) + \frac{df(x)}{dx}|_{x=x^0}(x - x^0) \tag{3}$$

- This first order Taylor series approximates the original nonlinear function $f(x)$ using the function evaluated at x^0 and its first derivative at $x = x^0$.
- The approximation holds well in the vicinity of $x = x^0$.

Illustration of Linearization

Figure 1 illustrates an example of linear approximation of a nonlinear function where $x^0 = 5.3$, $f(x^0) = 140$ and $\frac{df(x)}{dx}|_{x=x^0} = 85$. It is seen that within the region where x is close to x^0, $f(x)$ is closely approximated by the first order Taylor series expansion (3).

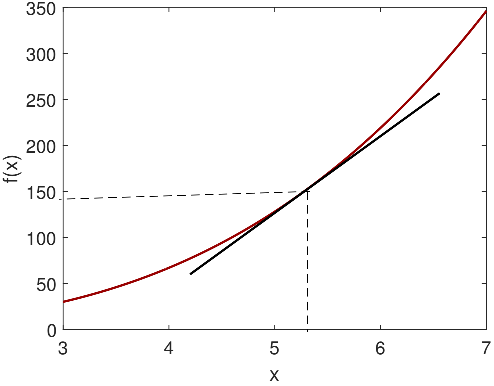


Figure 1: Approximation of a nonlinear function at $x^0 = 5.3$.

Linearization of Nonlinear Functions (ii)

- Intuitively, we can think of the original variable x as a 'large' variable because it covers a large region, and the perturbed variable $x - x^0$ as a 'small' variable because it covers a small region around x^0.
- The term of 'linear' comes from the second term of the right-hand side of the equation for its linear relationship between $f(x)$ and $x - x^0$.
- The first term is a constant, $f(x_0)$. If it is not zero, then it is not truly linear because it violates the homogeneity and additivity conditions required for linearity. In this case, on the $(\dot{x} - x^0)$ and $x - x^0$ plane, the function is a straight line between $(\dot{x} - x^0)$ and $x - x^0$, but it will not pass through the origin.

Linearization of Nonlinear Functions with Multiple Variables (i)

If the nonlinear function $f(x)$ contains n variables, meaning that $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ is a vector with dimension n, then the function is approximated using the first $n+1$ terms in the multivariable Taylor series expansion as

$$\begin{aligned} f(x_1, x_2, x_3, \dots, x_n) &\approx f(x_1^0, x_2^0, x_3^0, \dots, x_n^0) + \frac{\partial f(x)}{\partial x_1}|_{x_1=x_1^0, x_2=x_2^0, \dots}(x_1 - x_1^0) \\ &+ \frac{\partial f(x)}{\partial x_2}|_{x_1=x_1^0, x_2=x_2^0, \dots}(x_2 - x_2^0) + \dots + \frac{\partial f(x)}{\partial x_n}|_{x_1=x_1^0, x_2=x_2^0, \dots}(x_n - x_n^0) \end{aligned} \tag{4}$$

Linearization of Nonlinear Functions with Multiple Variables (ii)

- (4) is not linear, because it has a constant offset, which violates the homogeneity and additivity conditions required for linearity.
- In some applications, by appropriate selection of operating conditions, the constant is equal to zero.
- If this constant is not zero, it is regarded as a constant disturbance.
- This is one of the important reasons why integrator is often required in a feedback control system, which will overcome the effect of the offset in the system.

Linearization of Nonlinear Model(i)

- The nonlinear models obtained from using first principles of the physical laws are differential equations.
- We assume that the nonlinear differential equation used to describe a physical system takes the general form:

$$\dot{x}(t) = f(x(t), u(t), t) \tag{5}$$

  where $x(t)$ is a vector that represents the state variables of dimension n and $u(t)$ is a vector for the control signals of dimension m.

Linearization of Nonlinear Model(ii)

- In the linearization of a nonlinear dynamic system, we will firstly choose the constant vectors $x^0 = \begin{bmatrix} x_1^0 & x_2^0 & \ldots & x_n^0 \end{bmatrix}^T$, and $u^0 = \begin{bmatrix} u_1^0 & u_2^0 & \ldots & u_m^0 \end{bmatrix}^T$, and apply the linearization procedure of the nonlinear functions as outlined in the previous section.
- The linearization of differential equations is basically to apply the linearization of functions as outlined in the previous section to each term in the differential equation.

Steady-state Solutions

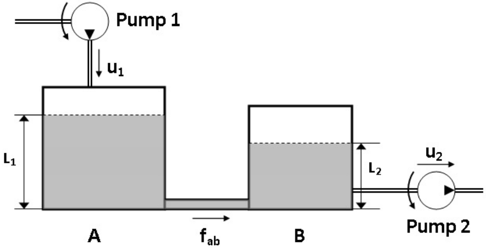
- The constant vectors x^0 and u^0 play an important role in the linearized model.
- To make the linearized system truly linear, these vectors need to be selected carefully. The point of interest is called an equilibrium point These equilibrium points in control system design and implementation are often referred to as stationary points, which represent a steady-state solution to the dynamic equation (5).
- The equilibrium points satisfy the following steady-state solution of the nonlinear differential equation (5):

$$\dot{x}(t) = f(x^0, u^0) = 0 \tag{6}$$

Outline

Case Study: Nonlinear Water Tank Model (i)

Two cubic water tanks are connected in series. Water flows into the first tank and flows out from the second tank. A pump controls the water in-flow rate $u_1(t)$ (m^3/sec) to the first tank; and another pump controls the water out-flow rate $u_2(t)$ (m^3/sec) from the second tank. Water flows from tank A to tank B, with a flow rate $f_{ab}(t)$. The units for the flow rate is m^3/sec and the units for the water level is m.



Case Study: Nonlinear Water Tank Model (ii)

- Using mass balance, the rate of change of water volume $V_1(t)$ in tank A is

$$\frac{dV_1(t)}{dt} = u_1(t) - f_{ab}(t) \tag{7}$$

- The water volume can also be expressed as $V_1(t) = S_1 L_1(t)$, where S_1 is the cross-sectional area of the tank A, and $L_1(t)$ is the water level in tank A.
- The dynamic equation to describe the rate of change in the water level $L_1(t)$ (tank A) is

$$S_1 \frac{dL_1(t)}{dt} = u_1(t) - f_{ab}(t) \tag{8}$$

- Likewise, the rate of change in the water level $L_2(t)$ is

$$S_2 \frac{dL_2(t)}{dt} = f_{ab}(t) - u_2(t) \tag{9}$$

where S_2 is the cross-sectional area for tank B.

Case Study: Nonlinear of Water Tank Model (iii)

- For a small orifice with a cross-sectional area a_s (m^2), $f_{ab}(t)$ is linked to the tank levels $L_1(t)$ and $L_2(t)$ with the following relationship:

$$f_{ab}(t) = a_s \sqrt{2g(L_1(t) - L_2(t))} \tag{10}$$

- where g is acceleration due to gravity ($= 9.81 m/sec^2$); f_{ab} is the flow rate (m^3/sec).

Case Study: Nonliner Water Tank Model (iv)

By substituting (10) into (8) and (9), we obtain

$$\frac{dL_1(t)}{dt} = -\frac{a_s}{S_1}\sqrt{2g(L_1(t) - L_2(t))} + \frac{1}{S_1}u_1(t) \quad (11)$$

$$\frac{dL_2(t)}{dt} = \frac{a_s}{S_2}\sqrt{2g(L_1(t) - L_2(t))} - \frac{1}{S_2}u_2(t) \quad (12)$$

Both of these models are nonlinear.

Solution: Linearization of Water Tank Model (i)

- In the linearization, the independent variables are $L_1(t)$, $L_2(t)$, $u_1(t)$ and $u_2(t)$. We will linearize the two equations (11) and (12) separately in terms of those independent variables.
- We let L_1^0 and L_2^0 denote the operating points for the tanks.
- The coefficients $\gamma_1 = \frac{a_s\sqrt{2g}}{S_1}$ and $\gamma_2 = \frac{a_s\sqrt{2g}}{S_2}$ are used to simplify the notation in both (11) and (12).

Solution: Linearization of Water Tank Model (ii)

- The first term in (11) is approximated by the first order Taylor series expansion as

$$
\begin{aligned}
\gamma_1\sqrt{L_1(t)-L_2(t)} &\approx \gamma_1\sqrt{L_1^0-L_2^0} \\
&+ \quad \gamma_1\frac{\partial(\sqrt{L_1(t)-L_2(t)})}{\partial L_1}|_{L_1^0,L_2^0}(L_1(t)-L_1^0) \\
&+ \quad \gamma_1\frac{\partial(\sqrt{L_1(t)-L_2(t)})}{\partial L_2}|_{L_1^0,L_2^0}(L_2(t)-L_2^0) \qquad (13)
\end{aligned}
$$

- Note that

$$
\frac{\partial(\sqrt{L_1(t)-L_2(t)})}{\partial L_1}|_{L_1^0,L_2^0} = \frac{1}{2}\frac{1}{\sqrt{L_1^0-L_2^0}} \qquad (14)
$$

$$
\frac{\partial(\sqrt{L_1(t)-L_2(t)})}{\partial L_2}|_{L_1^0,L_2^0} = -\frac{1}{2}\frac{1}{\sqrt{L_1^0-L_2^0}} \qquad (15)
$$

Solution: Linearization of Water Tank Model (iii)

- Therefore, (13) is written as

$$\gamma_1\sqrt{L_1(t)-L_2(t)} = \gamma_1\sqrt{L_1^0-L_2^0}+\frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0-L_2^0}}(L_1(t)-L_1^0)$$
$$-\quad\frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0-L_2^0}}(L_2(t)-L_2^0) \tag{16}$$

- The second term in the differential equation (11) is already linear in relation to $u_1(t)$, therefore, we keep it unchanged.

Solution: Linearization of Water Tank Model (iv)

- By substituting the Taylor series approximation (16) into the differential equation (11), we obtain the linearized model for water tank A (do not forget that there is a negative sign):

$$
\begin{aligned}
\frac{dL_1(t)}{dt} &= -\gamma_1\sqrt{L_1^0 - L_2^0} - \frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}(L_1(t) - L_1^0) \\
&+ \frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}(L_2(t) - L_2^0) + \frac{1}{S_1}u_1(t) \qquad (17)
\end{aligned}
$$

- Firstly, we notice that in order for the linearization to be valid, the operating points $L_1^0 > L_2^0$.
- Secondly, the first term is a constant that is not zero because $L_1^0 \neq L_2^0$.
- We can choose the steady-state value of $u_1(t)$ according to this constant.

Linearized Model for Water Tank

- For this purpose, we re-write (17) as

$$\begin{aligned}\frac{dL_1(t)}{dt} &= -\frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}(L_1(t) - L_1^0) \\ &+ \frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}(L_2(t) - L_2^0) + \frac{1}{S_1}(u_1(t) - S_1\gamma_1\sqrt{L_1^0 - L_2^0}) \quad (18)\end{aligned}$$

- To find the small signal model for the Tank A, we define the deviation variables as

$$\tilde{L}_1(t) = L_1(t) - L_1^0; \;\; \tilde{L}_2(t) = L_2(t) - L_2^0; \;\; \tilde{u}_1(t) = u_1(t) - S_1\gamma_1\sqrt{L_1^0 - L_2^0}$$

- This leads to the linearized model for the Tank A as

$$\frac{d\tilde{L}_1(t)}{dt} = -\frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}\tilde{L}_1(t) + \frac{\gamma_1}{2}\frac{1}{\sqrt{L_1^0 - L_2^0}}\tilde{L}_2(t) + \frac{1}{S_1}\tilde{u}_1(t) \qquad (19)$$

- Linearization of Tank B is left as an exercise

Discussions: Linearization of Water Tank Model

- The coefficients to represent the operating conditions of the two tanks must be positive and $L_1^0 > L_2^0$ in order for the linear models to be valid.
- Note that the steady-state value of the control signal $S_1\gamma_1\sqrt{L_1^0 - L_2^0}$ is a function of the system parameters S_1, γ_1. If there are errors in these parameters, then there is an error in the steady-state value of the control signal. This error could be modelled as an input disturbance. This is one of the important reasons why integrator is needed in control system.

Outline

Ball and Plate Balancing System

This case study is based on a final year project performed by Mr John Lee, who was previously a fourth year electrical engineering student in RMIT University, Australia.

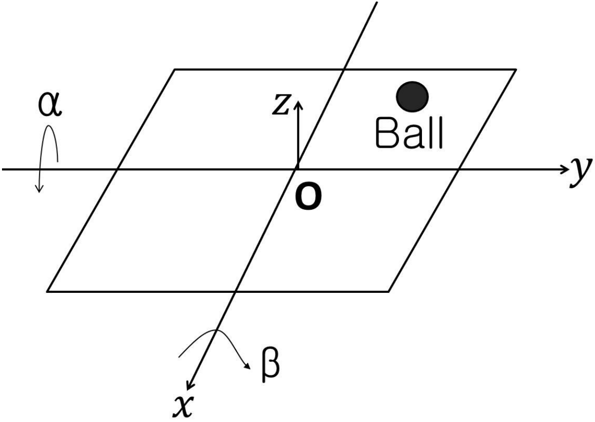


Figure 3: Schematic of ball and plate balancing system.

Input and Output Variable

Control objective

The position of the ball on the top of the plate is controlled by manipulating the inclination of the plate about its x and y axes.

Input and output variables

- The first pair corresponds to the position of the ball in the x and y axes denoted by x and y.
- The second pair corresponds to the inclination of the plate in the x and y axes captured by the angles of the plate θ_x and θ_y from the x and y axes.
- There are two DC motor drives used to control the system.

Dynamic Model A (i)

The relationship between the motor torque forces and the inclination of the plate:

$$
\begin{aligned}
(J_p + J_b + mx^2)\ddot{\theta}_x &+ 2mx\dot{x}\dot{\theta}_x + mxy\ddot{\theta}_y + m\dot{x}y\dot{\theta}_y + mx\dot{y}\dot{\theta}_y \\
&+ mgx\cos(\theta_x) = \tau_x \qquad (20) \\
(J_p + J_b + my^2)\ddot{\theta}_y &+ 2my\dot{y}\dot{\theta}_y + mxy\ddot{\theta}_x + m\dot{x}y\dot{\theta}_x + mx\dot{y}\dot{\theta}_x \\
&+ mgy\cos(\theta_y) = \tau_y \qquad (21)
\end{aligned}
$$

where J_p and J_b are the mass moment of inertia of the plate and the ball respectively, m is the mass of the ball and g is gravity constant ($g = 9.8m/(sec)^2$). The variables τ_x and τ_y are the torque forces on the x and y directions.

Dynamic Model A (ii)

- Because in the control system implementation two DC motors are used as the actuators with their angular positions under control from the manufacturer, the dynamics from the actuators are neglected.
- Instead, steady-state relationships between the angular positions of the DC motors and the inclination of the plate (governed by θ_x and θ_y) are found.

Dynamic Model B

The movement of the ball on the plate is described the following two equations:

$$(m + \frac{J_b}{R^2})\ddot{x} - mx(\dot{\theta}_x)^2 - my\dot{\theta}_x\dot{\theta}_y = -mg\sin(\theta_x) \quad (22)$$

$$(m + \frac{J_b}{R^2})\ddot{y} - my(\dot{\theta}_y)^2 - mx\dot{\theta}_x\dot{\theta}_y = -mg\sin(\theta_y) \quad (23)$$

where R is the radius of the ball.

Simplified Model B

In the control system implementation, a touch screen is used as the sensor to measure the ball's position on the plate. As a result, a heavy ball is used, which gives the inertial parameter J_b as

$$J_b = \frac{2}{5} mR^2$$

Thus, the plate dynamic equations are re-written as

$$\frac{7m}{5}\ddot{x} - mx(\dot{\theta}_x)^2 - my\dot{\theta}_x\dot{\theta}_y = -mg\sin(\theta_x) \tag{24}$$

$$\frac{7m}{5}\ddot{y} - my(\dot{\theta}_y)^2 - mx\dot{\theta}_x\dot{\theta}_y = -mg\sin(\theta_y) \tag{25}$$

Linearization of Nonlinear Model B (i)

Operating conditions

1. At the equilibrium, the ball is stable at the center of the plate, which is $x^0 = 0$ and $y^0 = 0$.
2. The angle of the plate is zero in both x and y axes, which is translated to $\theta_x^0 = \theta_y^0 = 0$.
3. The angle of the plate is not changing, which leads to $\dot{\theta}_x^0 = \dot{\theta}_y^0 = 0$.

Linearization of Nonlinear Model B (ii)

The first term in (24) is linear by itself and does not require linearization. The nonlinear function in the second term is approximated by Taylor series expansion as

$$
\begin{aligned}
x\dot{\theta}_x^2 &\approx x^0(\dot{\theta}_x^0)^2 + \frac{\partial(x\dot{\theta}_x^2)}{\partial x}|_{x=x^0,\dot{\theta}_x=\dot{\theta}_x^0}(x - x^0) + \frac{\partial(x\dot{\theta}_x^2)}{\partial\dot{\theta}_x}|_{x=x^0,\dot{\theta}_x=\dot{\theta}_x^0}(\dot{\theta}_x - \dot{\theta}_x^0) \\
&= x^0(\dot{\theta}_x^0)^2 + \dot{\theta}_x^2|_{x=x^0,\dot{\theta}_x=\dot{\theta}_x^0}(x - x^0) + 2x\dot{\theta}_x|_{x=x^0,\dot{\theta}_x=\dot{\theta}_x^0}(\dot{\theta}_x - \dot{\theta}_x^0) \\
&= 0
\end{aligned}
$$

because $x^0 = \dot{\theta}_x^0 = 0$.

Linearization of Nonlinear Model B (iii)

The quantity in the third term of (24) is approximated by Taylor series expansion as

$$\begin{aligned} y\dot{\theta}_x\dot{\theta}_y &\approx y^0\dot{\theta}_x^0\dot{\theta}_y^0 + \frac{\partial(y\dot{\theta}_x\dot{\theta}_y)}{\partial y}\Big|_{y=y^0,\dot{\theta}_x=\dot{\theta}_x^0,\dot{\theta}_y=\dot{\theta}_y^0}(y-y^0) \\ &+ \frac{\partial(y\dot{\theta}_x\dot{\theta}_y)}{\partial \dot{\theta}_x}\Big|_{y=y^0,\dot{\theta}_x=\dot{\theta}_x^0,\dot{\theta}_y=\dot{\theta}_y^0}(\dot{\theta}_x-\dot{\theta}_x^0) \\ &+ \frac{\partial(y\dot{\theta}_x\dot{\theta}_y)}{\partial \dot{\theta}_y}\Big|_{y=y^0,\dot{\theta}_x=\dot{\theta}_x^0,\dot{\theta}_y=\dot{\theta}_y^0}(\dot{\theta}_y-\dot{\theta}_y^0) = 0 \end{aligned}$$

because $y^0 = \dot{\theta}_x^0 = \dot{\theta}_y^0 = 0$.

Linearization of Nonlinear Model B (iv)

The nonlinear quantity on the right hand of (24) is approximated as

$$\begin{aligned}\sin\theta_x &\approx \sin\theta_x^0 + \frac{d\sin\theta_x}{d\theta_x}\Big|_{\theta_x=\theta_x^0}(\theta_x-\theta_x^0)\\ &= \sin\theta_x^0 + \cos\theta_x|_{\theta_x=\theta_x^0}(\theta_x-\theta_x^0) = \theta_x\end{aligned}$$

Linearization of Nonlinear Model B (v)

Combining all the linearized quantities together yields the linear model that describes the dynamics of the ball and plate balancing system at its operating condition as,

$$\frac{7m}{5}\ddot{x} = -mg\theta_x \tag{26}$$

which says that at the equilibrium point, the ball and plate balancing system is a double integrator system. The input to system is angle of the plate θ_x and the output is the position of the ball x on the plate.

Linearization of Nonlinear Model B (vi)

The dynamic model for the y-axis is obtained through the linearization of the nonlinear model (25) as

$$\frac{7m}{5}\ddot{y} = -mg\theta_y \tag{27}$$

It is seen that the linearized models are equal, also the coupling relationships in the original nonlinear models are gone, meaning that two identical PID controllers can be used to control the x and y axes separately.

Transfer Function Model

Because all the steady-state variables are zero in the ball and plate balancing system, the Laplace transfer function of (26) for the x-axis becomes

$$\frac{X(s)}{\Theta_x(s)} = -\frac{5}{7} g \frac{1}{s^2} \tag{28}$$

PID Controller Design (i)

The controller structure is selected as

$$C(s) = \frac{c_2 s^2 + c_1 s + c_0}{s(s + l_0)} \tag{29}$$

and the desired closed-loop polynomial is selected as

$$\begin{aligned} A_{cl} &= (s^2 + 2 \times 0.707 w_n s + w_n^2)(s + w_n)^2 \\ &= s^4 + t_3 s^3 + t_2 s^2 + t_1 s + t_0 \end{aligned}$$

where the parameter w_n is a tuning parameter for the closed-loop performance.

PID Controller Design (ii)

With the second order model,

$$G(s) = \frac{b_0}{s^2}$$

where $b_0 = -\frac{5}{7}g$, the polynomial equation for solving the PID controller parameters becomes

$$\begin{aligned} s^4 &+ l_0 s^3 + b_0 c_2 s^2 + b_0 c_1 s + b_0 c_0 \\ &= s^4 + t_3 s^3 + t_2 s^2 + t_1 s + t_0 \end{aligned} \tag{30}$$

The solution of (30) gives

$$l_0 = t_3;\ c_2 = \frac{t_2}{b_0};\ c_1 = \frac{t_1}{b_0};\ c_0 = \frac{t_0}{b_0}$$

PID Controller Design (iii)

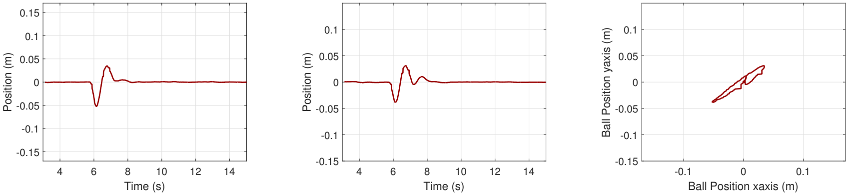
- To determine the performance parameter w_n, using Simulink simulation, nonlinear system simulators were built with consideration of nonlinear plant and actuator dynamics for various reference signals.
- It was found that $w_n = 3$ is a satisfactory choice from the simulation studies based on the nonlinear simulators, and this w_n was used in the actual implementation with a small adjustment for special cases.

PID Controller Implementation

- In the implementation of the control system, the parameters in controller (29) are converted to K_c, τ_I, τ_D and τ_f for the discretization.
- A two-degrees of freedom PID controller is implemented, which is to put the proportional control and the derivative control on the output only.
- To protect the equipment, constraints on the derivative of the control signal and the amplitude of the control signal are imposed with anti-windup mechanisms.
- The sampling interval used in the implementation is $\Delta t = 0.01$ (sec).

Experimental Results-Disturbance Rejection

Figure 4 (c) shows the *x*-*y* plane plot for the disturbance rejection. It is seen that the PID control system has successfully rejected the disturbance without steady-state errors.

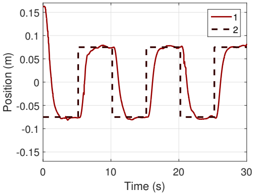


(a) x-axis response (b) y-axis response (c) Ball position

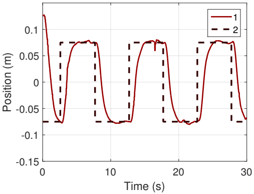
Figure 4: Disturbance rejection

Experimental Results- Square Movement

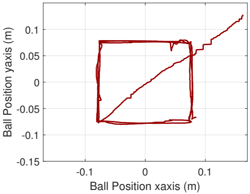
Figure 5 (c) shows the ball movement on the plate, which is seen as a square trajectory.



(a) x-axis response



(b) y-axis response



(c) Ball position

Figure 5: Making a square movement. Key: line (1) output response; line (2) reference signal.

Experimental Results-Circle Movement (i)

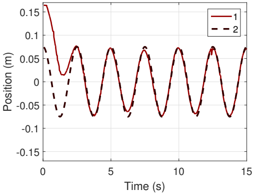
To make a circle movement, the reference signals to the x -axis and y-axis are chosen to be

$$\begin{aligned} x^*(t) &= 0.075\cos(\frac{2\pi}{2.5}t) \\ y^*(t) &= 0.075\sin(\frac{2\pi}{2.5}t) \end{aligned}$$

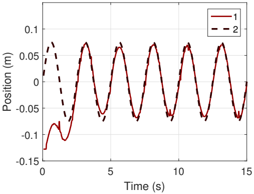
Here, the desired angular velocity of the ball movement is $\frac{2\pi}{2.5}$ rad/sec and radius of the circle is 0.075 (m).

Experimental Results-Circle Movement (ii)

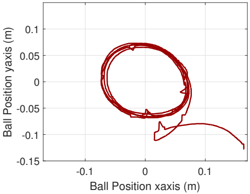
Figure 6 (c) shows the *x*-*y* plane movement of the ball, which is seen to make a circle movement.



(a) x-axis response



(b) y-axis response



(c) Ball position

Figure 6: Making a circle movement. Key: line (1) output response; line (2) desired reference signal.

Steps in Linearization of Nonlinear Plant Model

- Choose the operating conditions for the plant model.
- Use Taylor series to approximate each nonlinear term in the plant model by taking the derivative of the nonlinear function and calculate its value at the operating points.
- Collecting all the approximated linear terms to form the linearized model.

Exercise: Linearization of PMS Motor

A Permanent Magnetic Synchronous Motor (PMSM) is described by the differential equations in the d-q rotating reference frame

$$\frac{di_d(t)}{dt} = \frac{1}{L_d}(v_d(t) - Ri_d(t) + \omega_e(t)L_q i_q(t)) \tag{31}$$

$$\frac{di_q(t)}{dt} = \frac{1}{L_q}(v_q(t) - Ri_q(t) - \omega_e(t)L_d i_d(t) - \omega_e(t)\phi_{mg}) \tag{32}$$

$$\frac{d\omega_e(t)}{dt} = \frac{p}{J}(T_e - \frac{B}{p}\omega_e(t) - T_L) \tag{33}$$

$$T_e = \frac{3}{2}p\phi_{mg}i_q \tag{34}$$

where ω_e is the electrical speed and is related to the rotor speed by $\omega_e = p\omega_m$ with p denoting the number of pole pairs, v_d and v_q represent the stator voltages in the d-q frame, i_d and i_q represent the stator currents in this frame, and T_L is load torque that is assumed to be zero if no load is attached to the motor.