RMIT University

OENG-1116: MODELLING & SIMUALTION OF ENGINEERING SYSTEMS

Week 2: Part-2

Simplified Introduction into FEM. Simulation of Multi-DOF Systems. Simulation of Non-Linear Systems.

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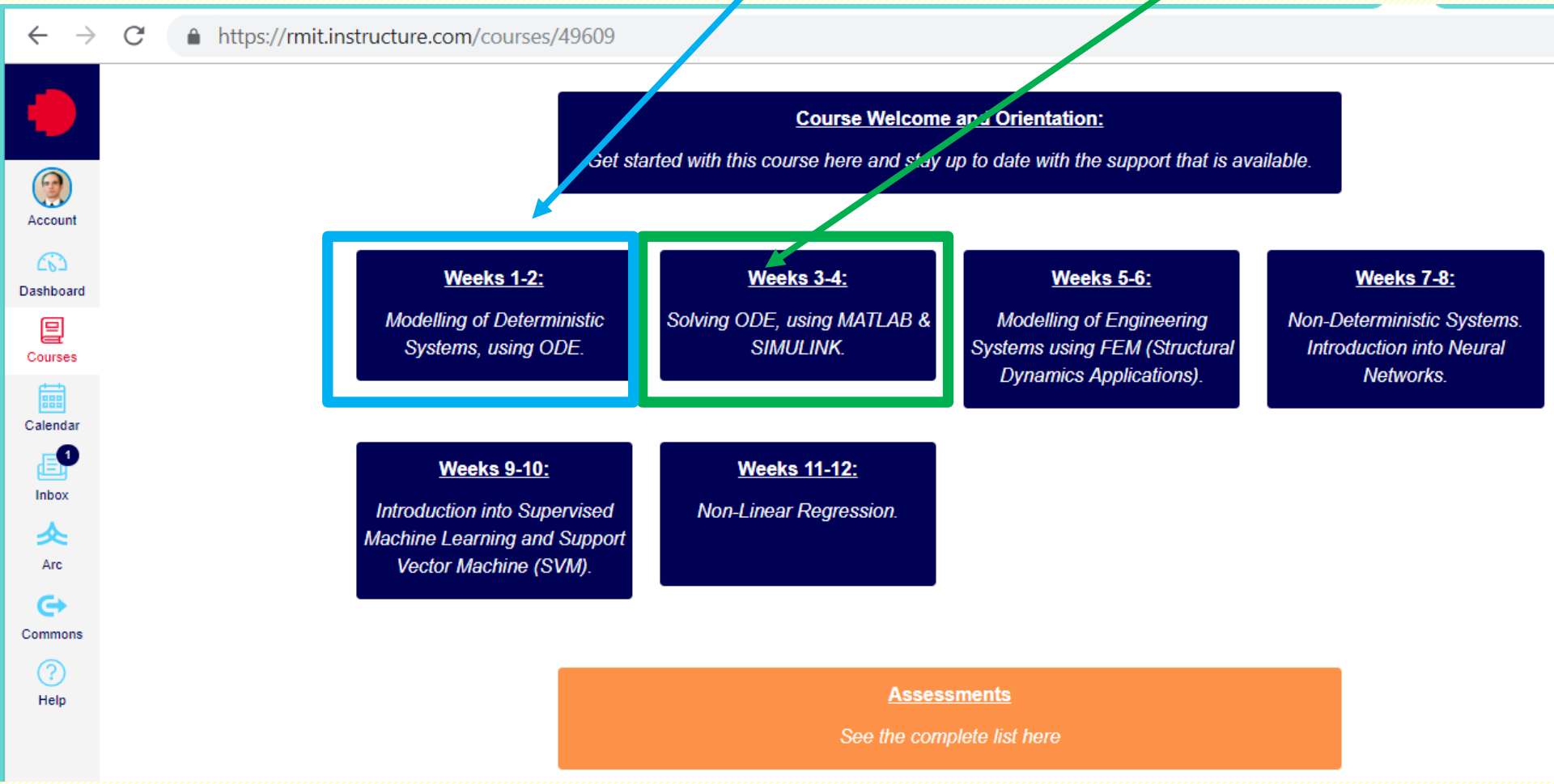
FEEDBACK:

QUESTIONS
HOME WORK
OBSERVATIONS
GENERAL DISCUSSIONS

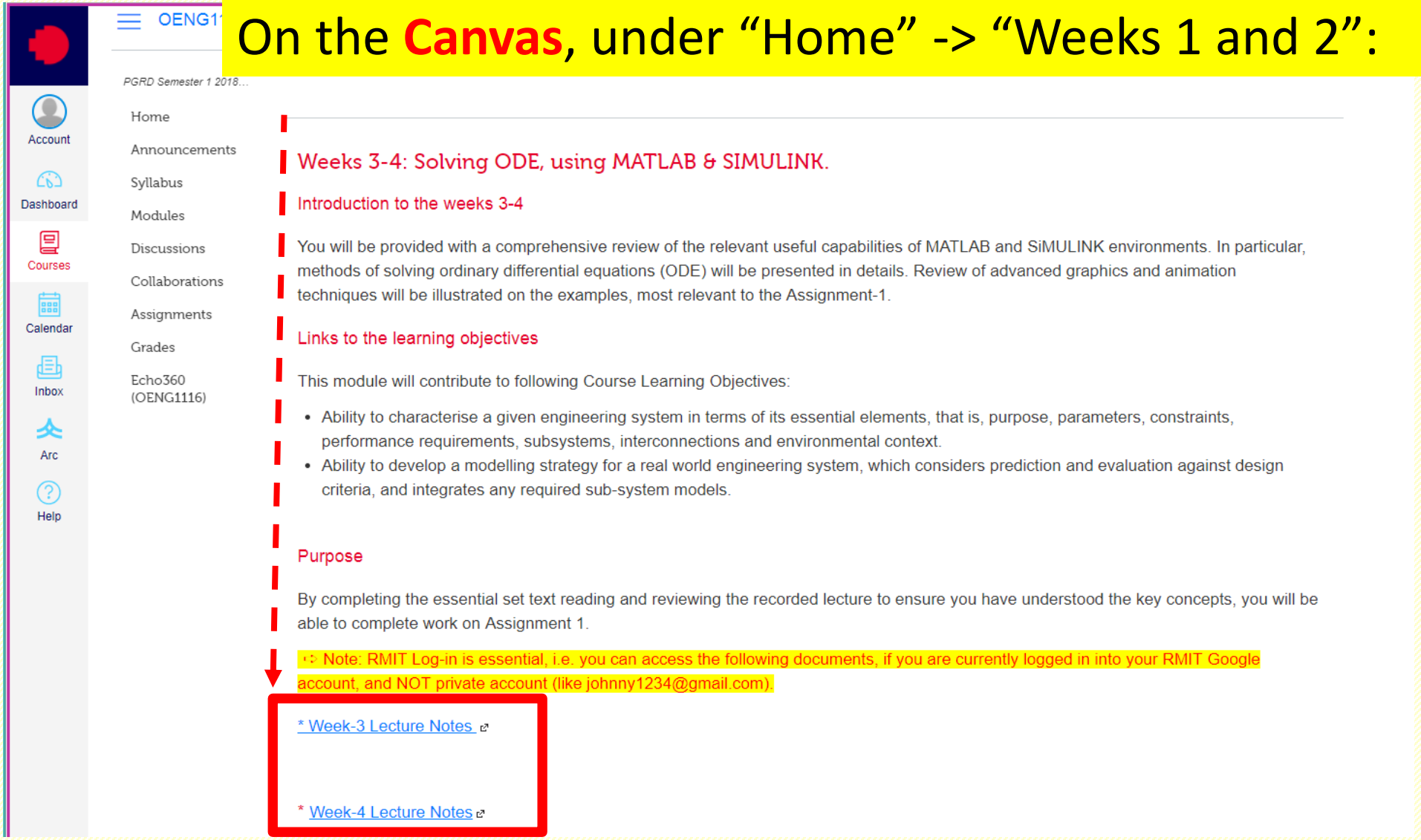
CANVAS MATERIALS:

On the **Canvas**, under "**Week 1a, 1b**", "**Week 2a, 2b**":



CANVAS Lecture Notes :

On the **Canvas**, under “Home” -> “Weeks 1 and 2”:



ASSIGNMENT-1, Pt1

Released: individual links sent vial emails (should reach every student!)

Deadline for A1, Pt1:

The deadline for the submission of the ***Individual Variants*** **Assignment-1, Pt1** (OENG1116) is end of **Week-5, Friday, 03 Apr 2020.**

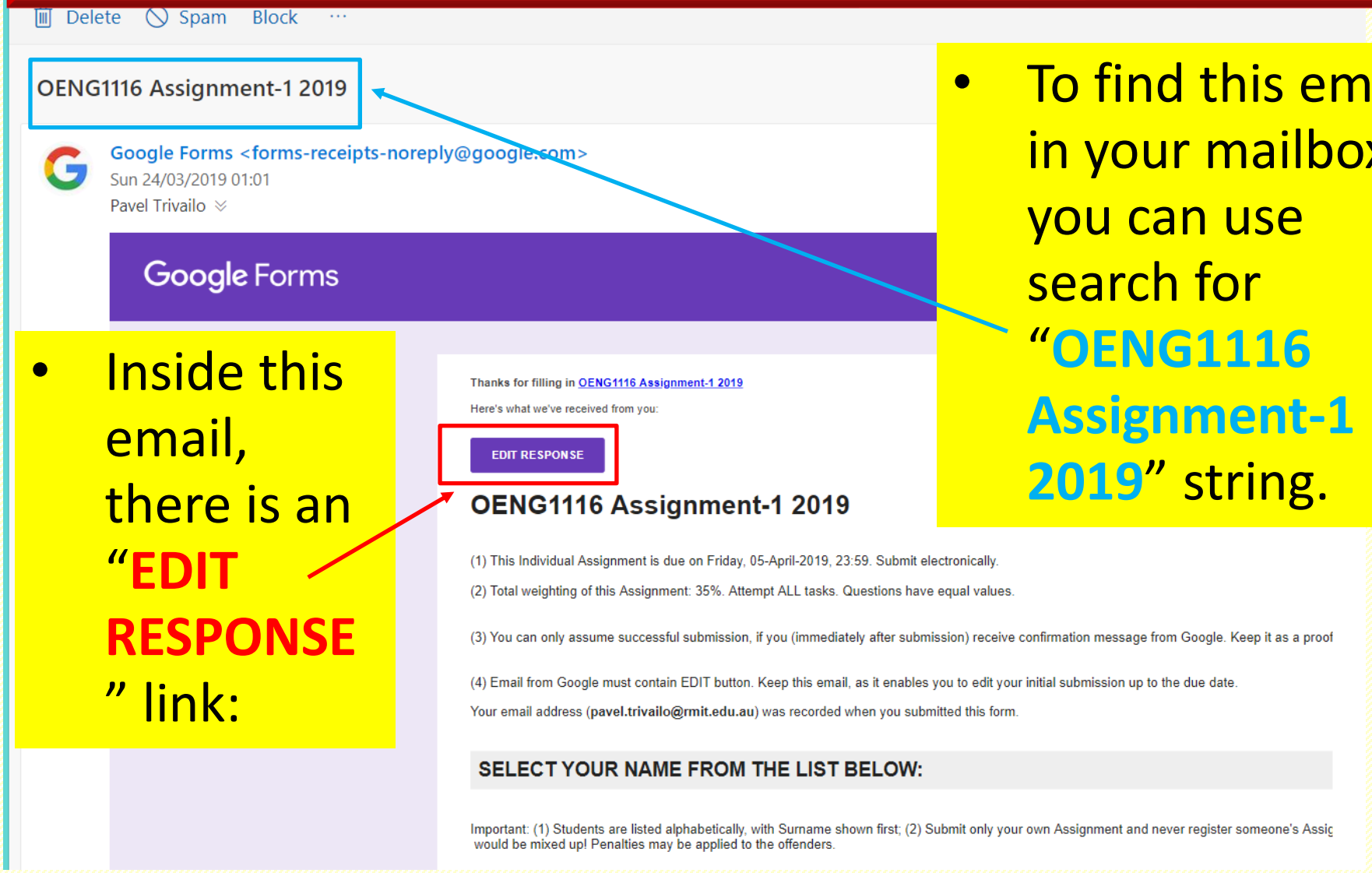
The aim of this submission IS:

for each Master to receive a feedback from the Research Supervisor by the end of Week-6 to bring you certainty, peace of mind and to multiply your confidence!

Web Links to the Google Forms with your ***Individual Variants*** **Assignment-1** (OENG1116) will be sent via your Student Email.

PLEASE, IMMEDIATELY REPORT TO ME (via email) IF YOU DID NOT RECEIVE THIS EMAIL FROM ME.

COMMENTS ON FINDING Assignment-1 GOOGLE FORM

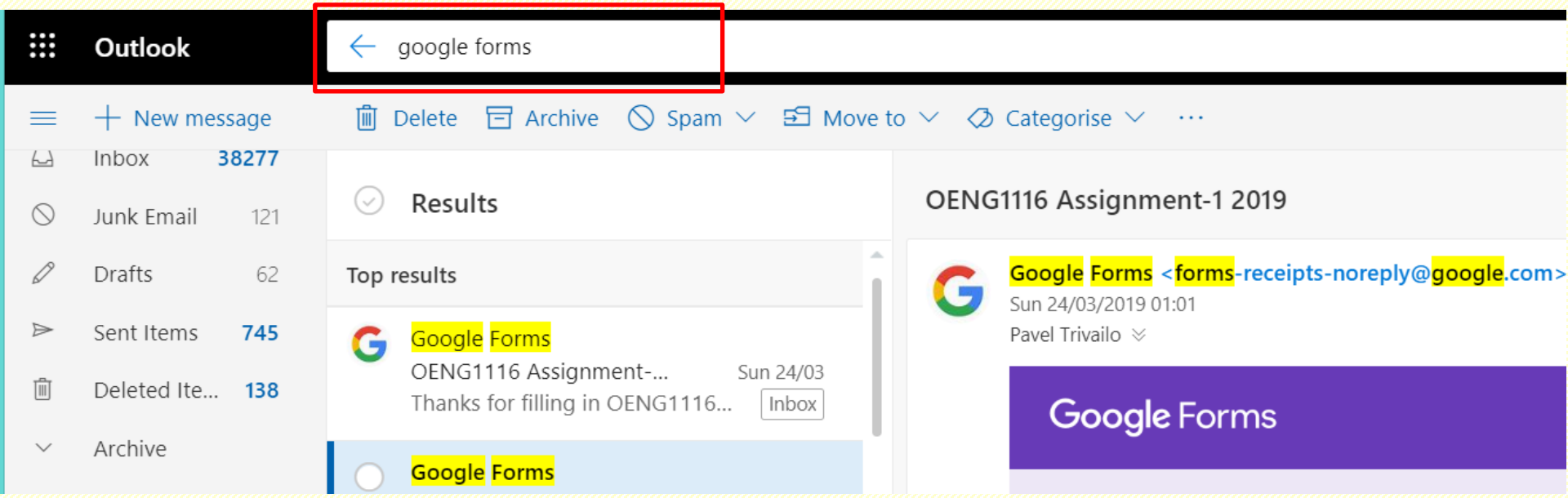


- To find this email in your mailbox, you can use search for "**OENG1116 Assignment-1 2019**" string.

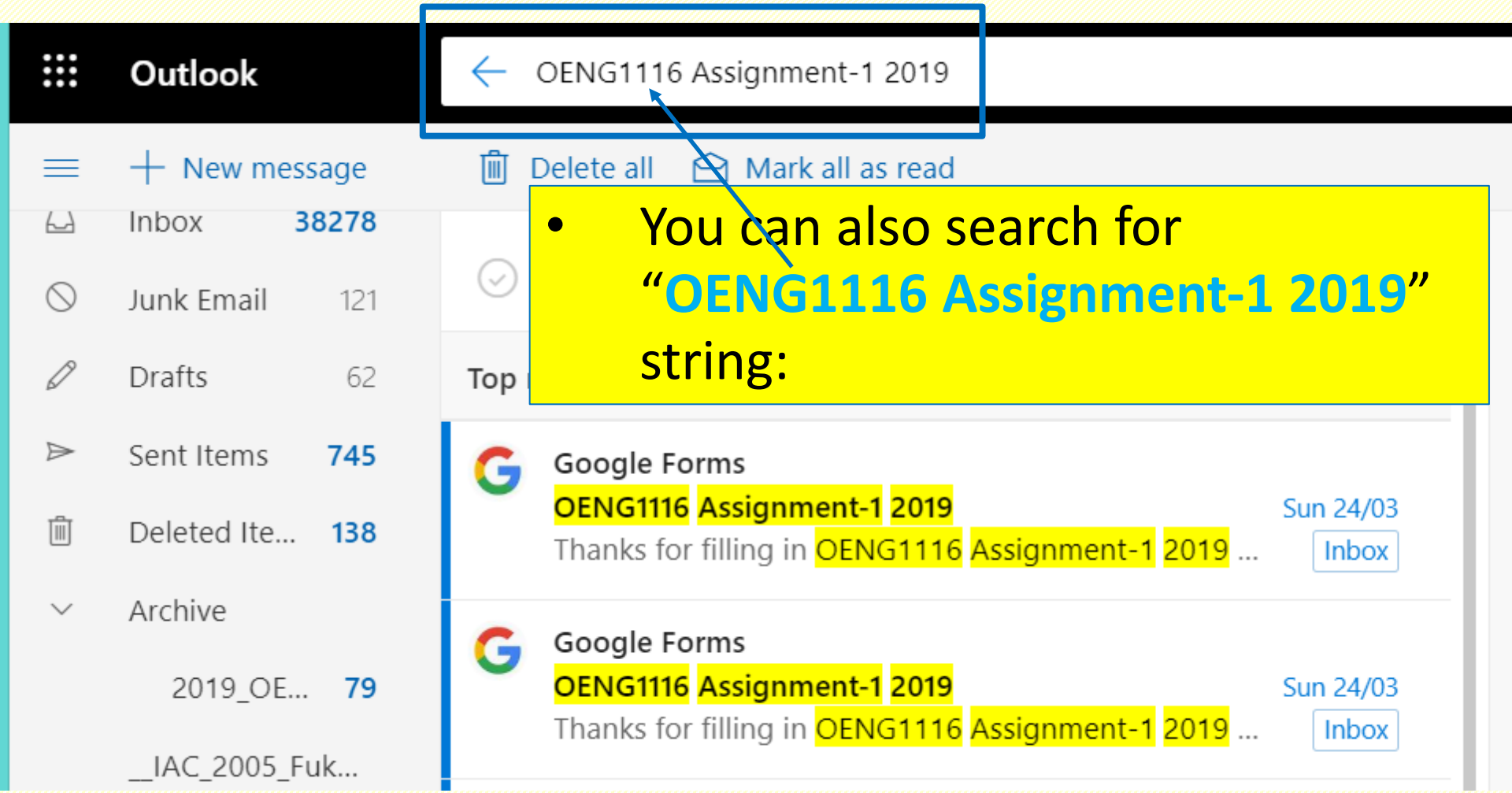
- Inside this email, there is an "**EDIT RESPONSE**" link:

COMMENTS ON FINDING Assignmen GOOGLE FORM

- To find this email in your mailbox, you can use search for "**Google Forms**" string:



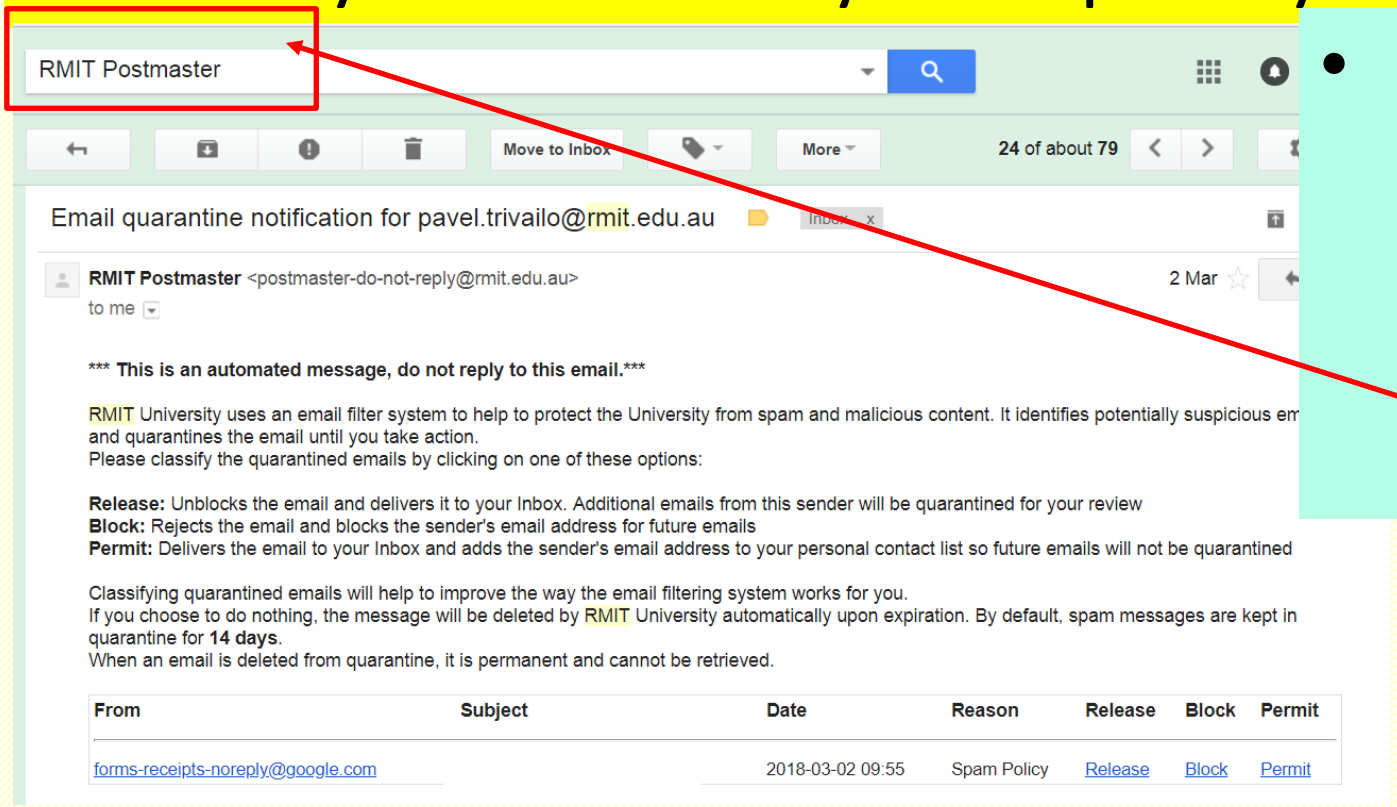
COMMENTS ON FINDING Assignmen GOOGLE FORM



- You can also search for "**OENG1116 Assignment-1 2019**" string:

KNOWN ISSUES: Missing Google emails

- There were several cases reported in other Course, when the confirmation email was filtered out by the University anti-spam system.



- You can search for these using "**RMIT Postmaster**"

A1 : STRUCTURE, INSTRUCTIONS:

This is an individual assignment and its submission is via Google Form only.

In order to access the Assignment, you must be logged in into RMIT account.

If you attempt to do this from your private Gmail account, system would not be able to recognise your association with RMIT and access would not be granted.

Please, carefully read instructions on Assignment, which are also reproduced below for your convenience:

(1) This Individual Assignment is due on Friday, 03-April-2020, 23:59. Submit electronically.

(2) Total weighting of this Assign-1, Pt1: 20%. Attempt ALL tasks. Questions have shown values.

(3) **You can only assume successful submission, if you (immediately after submission) receive confirmation message from Google. Keep it as a proof.**

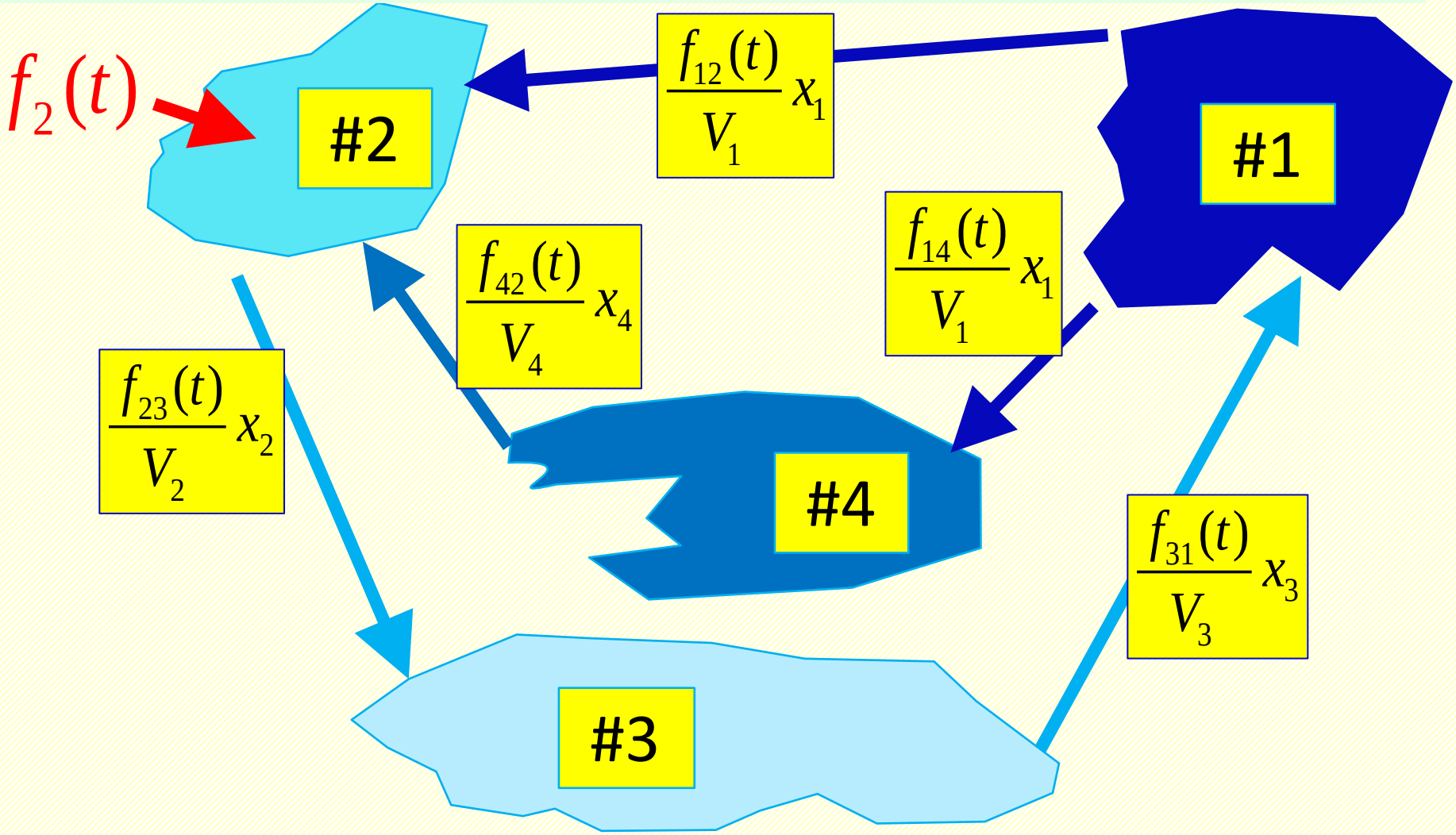
(4) Email from Google must contain **EDIT** button. Keep this email, as it enables you to edit your initial submission up to the due date.

A1, Pt2 Assist:

MATLAB MINI-TUTORIAL:

(2) LAKES: 1st order system; ode45

Consider four ponds connected by streams.
The second pond has a pollution source f_2=2 kg/h, which spreads via the connecting streams to the other ponds.
Plot the amount of pollutant in each pond as a function of time.



Symbol *f(t)* is the pollutant flow rate into pond 2 (kg/h).

- Symbols f_1, f_2, f_3, f_4 denote the pollutant flow rates out of ponds 1, 2, 3, 4 respectively (m^3/h).
- It is assumed that the pollutant is well-mixed in each pond.
- The ponds have volumes V_1, V_2, V_3, V_4 (m^3), which remain constant.
- Symbols $x_1(t), x_2(t), x_3(t), x_4(t)$ denote the amount (kg) of pollutant in ponds 1, 2, 3, 4 respectively.

The pollutant flux is the flow rate times the pollutant concentration, e.g., pond 1 is emptied with flux f_{12} times $x_1(t)/V_1$ plus flux f_{14} times $x_1(t)/V_1$.

The diagram plus compartment analysis gives the following differential equations:

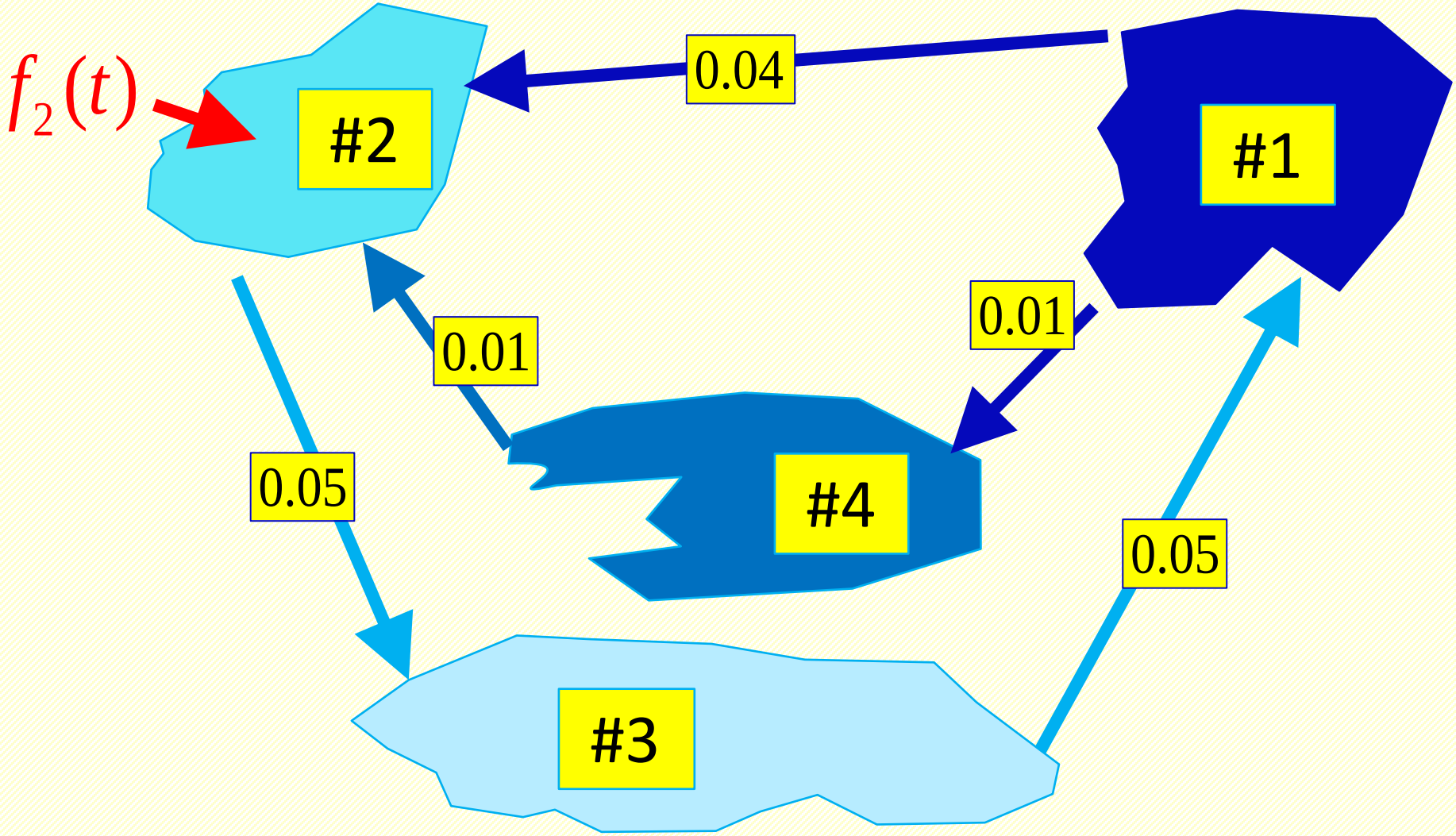
$$\frac{dx_1(t)}{dt} = -\frac{f_{12}}{V_1}x_1(t) - \frac{f_{14}}{V_1}x_1(t) + \frac{f_3}{V_3}x_3(t)$$

$$\frac{dx_2(t)}{dt} = \frac{f_{12}}{V_1}x_1(t) - \frac{f_{23}}{V_2}x_2(t) + \frac{f_{42}}{V_4}x_4(t) + f_2(t)$$

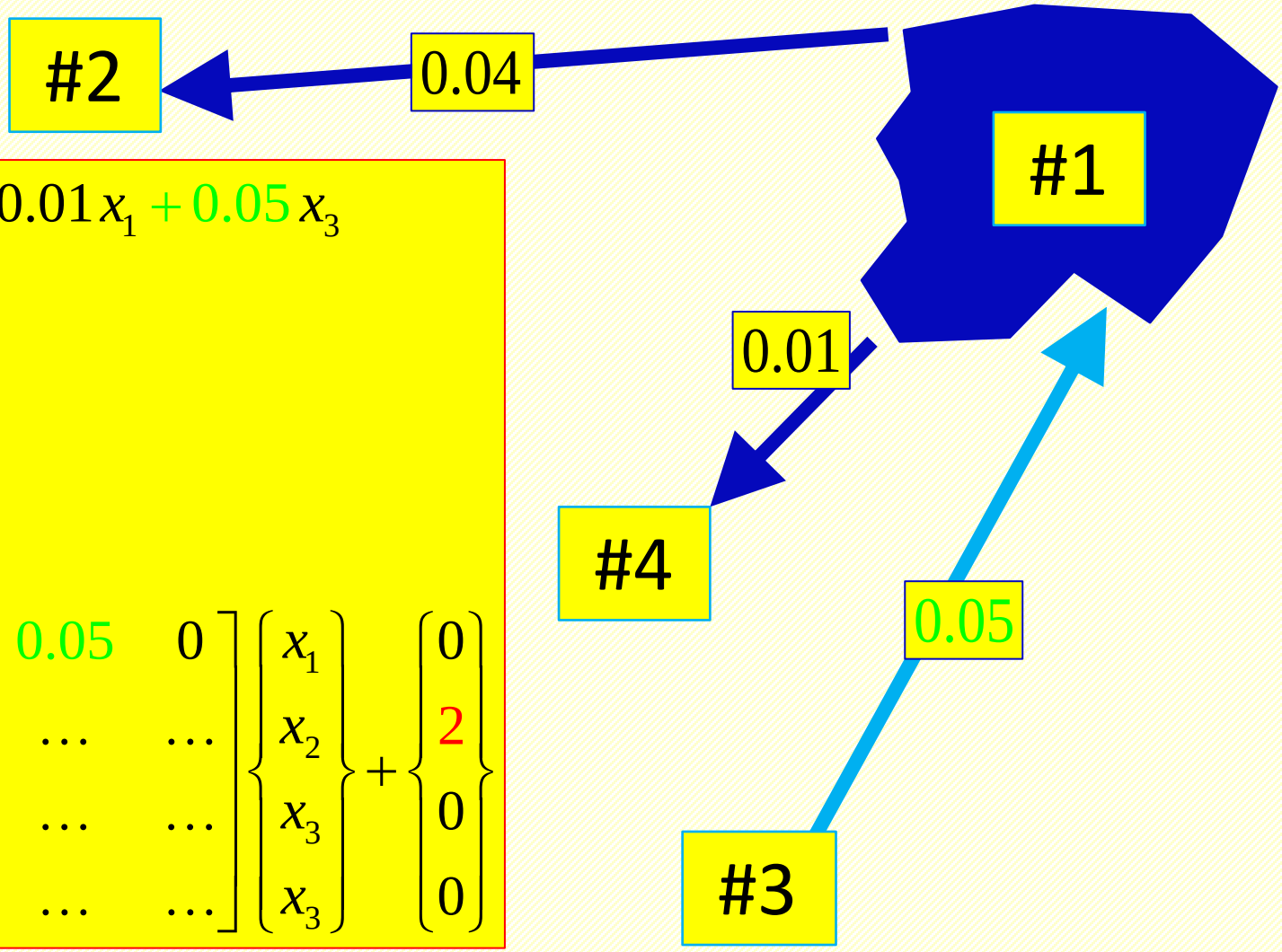
$$\frac{dx_3(t)}{dt} = \frac{f_{23}}{V_2}x_2(t) - \frac{f_{31}}{V_3}x_3(t)$$

$$\frac{dx_4(t)}{dt} = \frac{f_{14}}{V_1}x_1(t) - \frac{f_{41}}{V_4}x_4(t)$$

For a specific numerical example, take $f_{14}/V_1 = f_{42}/V_4 = 0.01$, $f_{12}/V_1 = 0.04$, $f_{23}/V_2 = f_{31}/V_3 = 0.05$ [all in 1/h] and let $f_2(t) = 2$ kg/h for the first 48 hours, thereafter $f(t) = 0$.
We expect due to uniform mixing that after a long time there will be 2*48 = 96 kg of pollutant uniformly deposited. Initially, $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0$, i.e. the ponds were pristine.



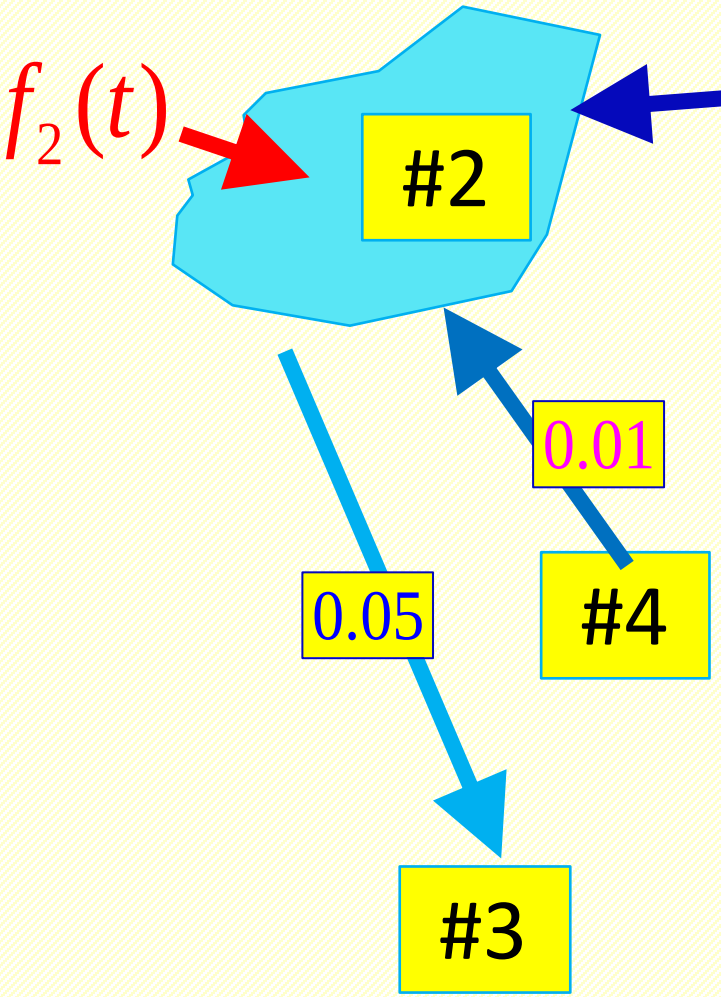
Consider Pond-1 only to set 1st Equation:



$$\begin{cases} \dot{x}_1 = -0.04\,x_1 - 0.01\,x_1 + 0.05\,x_3 \\ \dot{x}_2 = \dots \\ \dot{x}_3 = \dots \\ \dot{x}_4 = \dots \end{cases}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{Bmatrix}$$

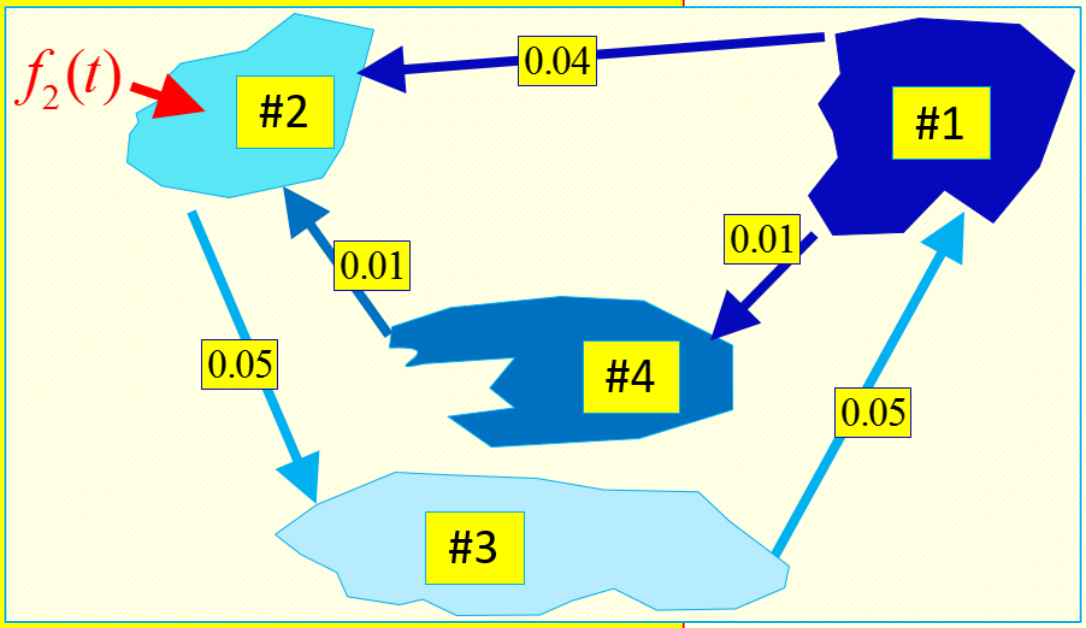
Consider Pond-2 only to set 2nd Equation:



$$\begin{cases} \dot{x}_1 = \ldots \\ \dot{x}_2 = 0.04\,x_1 - 0.05\,x_2 + 0.01\,x_4 + 2 \\ \dot{x}_3 = \ldots \\ \dot{x}_4 = \ldots \end{cases}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots \\ 0.04 & -0.05 & 0 & 0.01 \\ 0 & 0.05 & -0.05 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{Bmatrix}$$

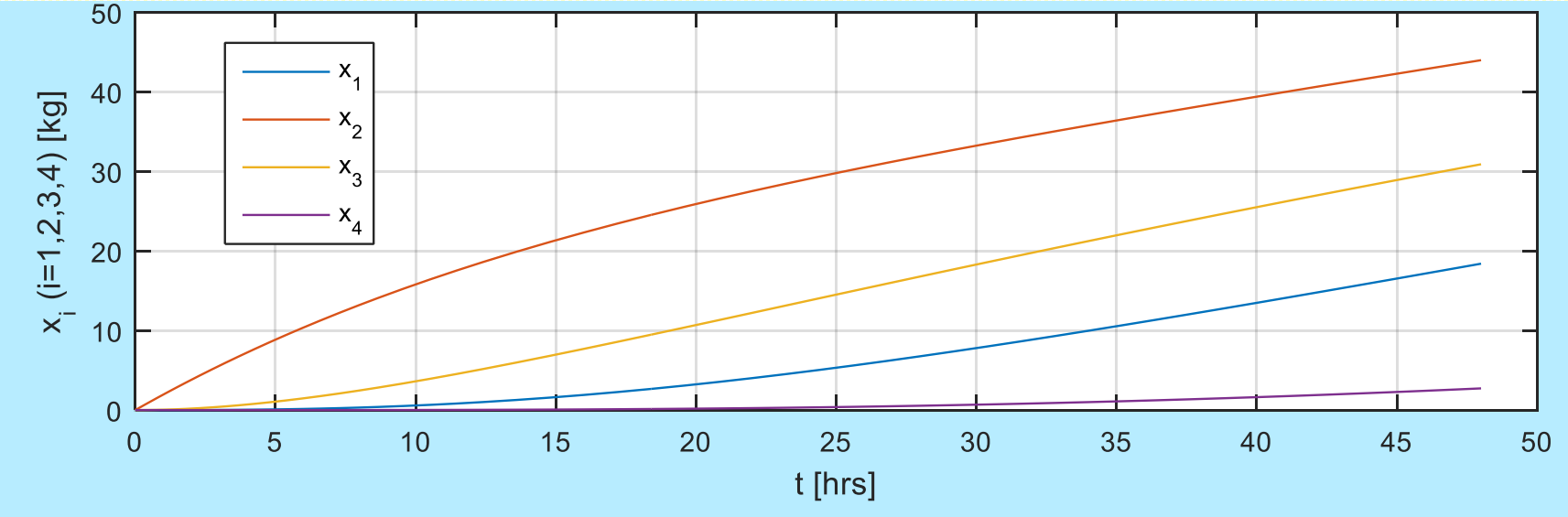
$$
\begin{cases}
\dot{x}_1 = -0.04\,x_1 - 0.01\,x_1 + 0.05\,x_3 \\
\dot{x}_2 = 0.04\,x_1 - 0.05\,x_2 + 0.01\,x_4 + 2 \\
\dot{x}_3 = 0.05\,x_2 - 0.05\,x_3 \\
\dot{x}_4 = 0.01\,x_1 - 0.01\,x_4
\end{cases}
$$



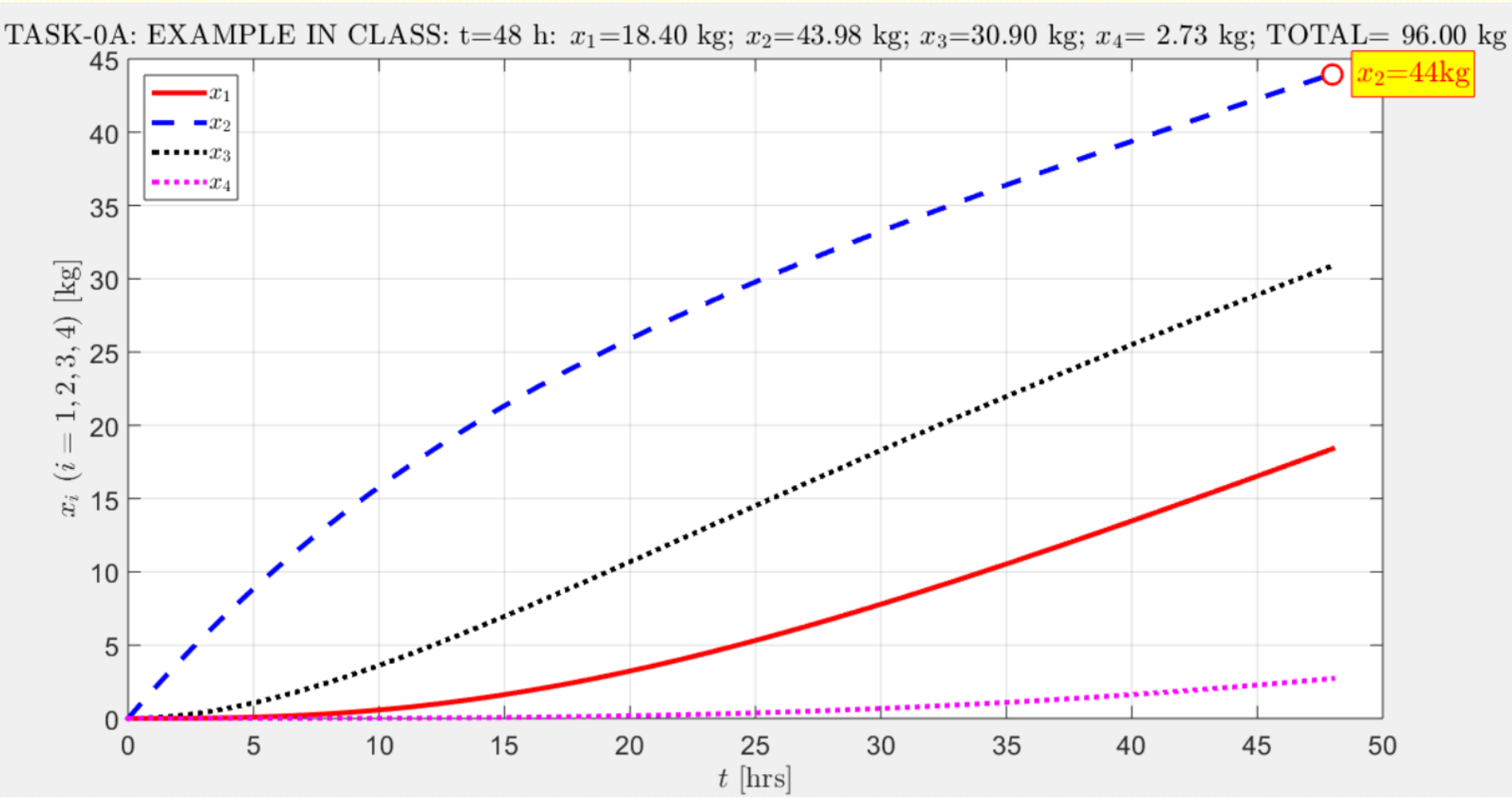
$$
\dot{\mathbf{x}} = \begin{bmatrix}
-0.05 & 0 & 0.05 & 0 \\
0.04 & -0.05 & 0 & 0.01 \\
0 & 0.05 & -0.05 & 0 \\
0.01 & 0 & 0 & -0.01
\end{bmatrix}
\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{Bmatrix}
+ \begin{Bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{Bmatrix}
$$

MATLAB SCRIPT (no annotations)

```
%% Designed by Prof Pavel M.Trivailo (C) 2020
% LAKES: Simple Script
clear; close all; clc;
r=[ -5 0 5 0; 4 -5 0 1 ; 0 5 -5 0; 1 0 0 -1]*0.01;
D=[0; 2; 0; 0]; tmax=24*2; % hrs
xd = @(t, x) r*x+D;
[tt,zz]=ode45(xd,[0 tmax],[0;0;0;0]);
plot(tt,zz); grid on; legend('x_1','x_2','x_3','x_4');
xl=xlabel('t [hrs]'); yl=ylabel('x_i (i=1,2,3,4) [kg]');
```



MATLAB annotated plot

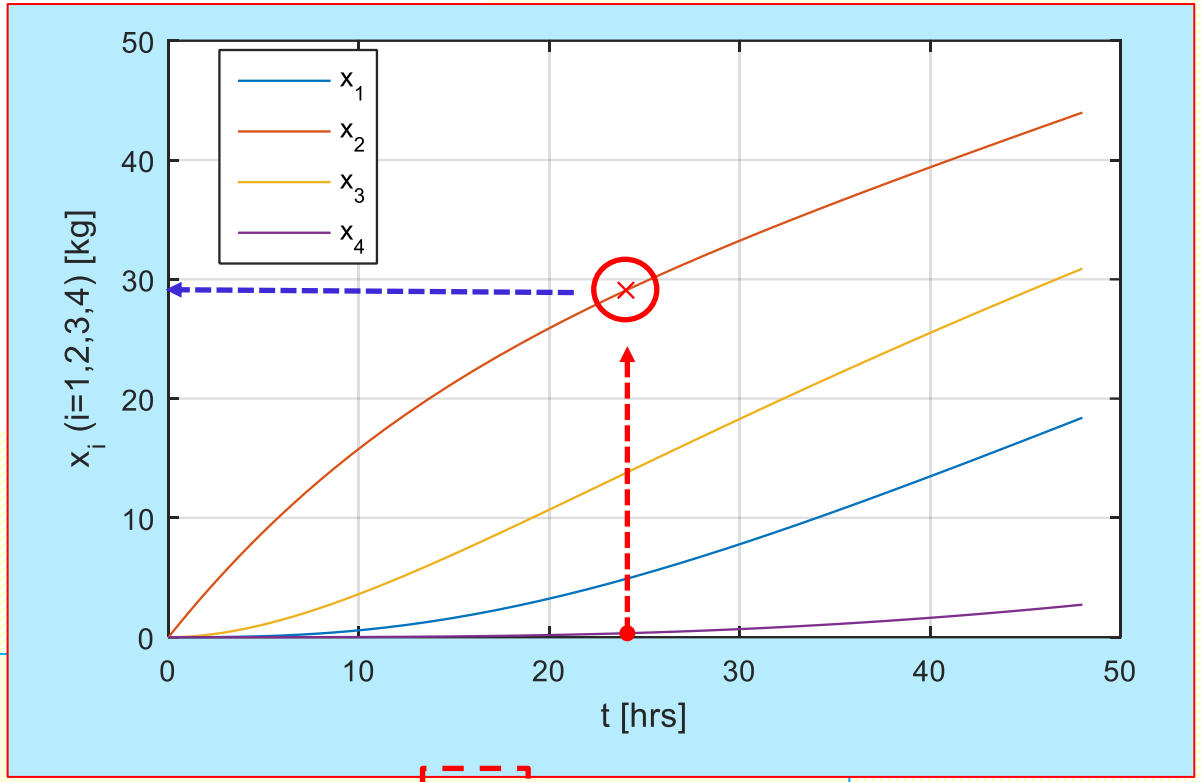


Extra Task-1: find x_2 for t=24 hrs

Command Window

New to MATLAB? See resources for Getting Started.

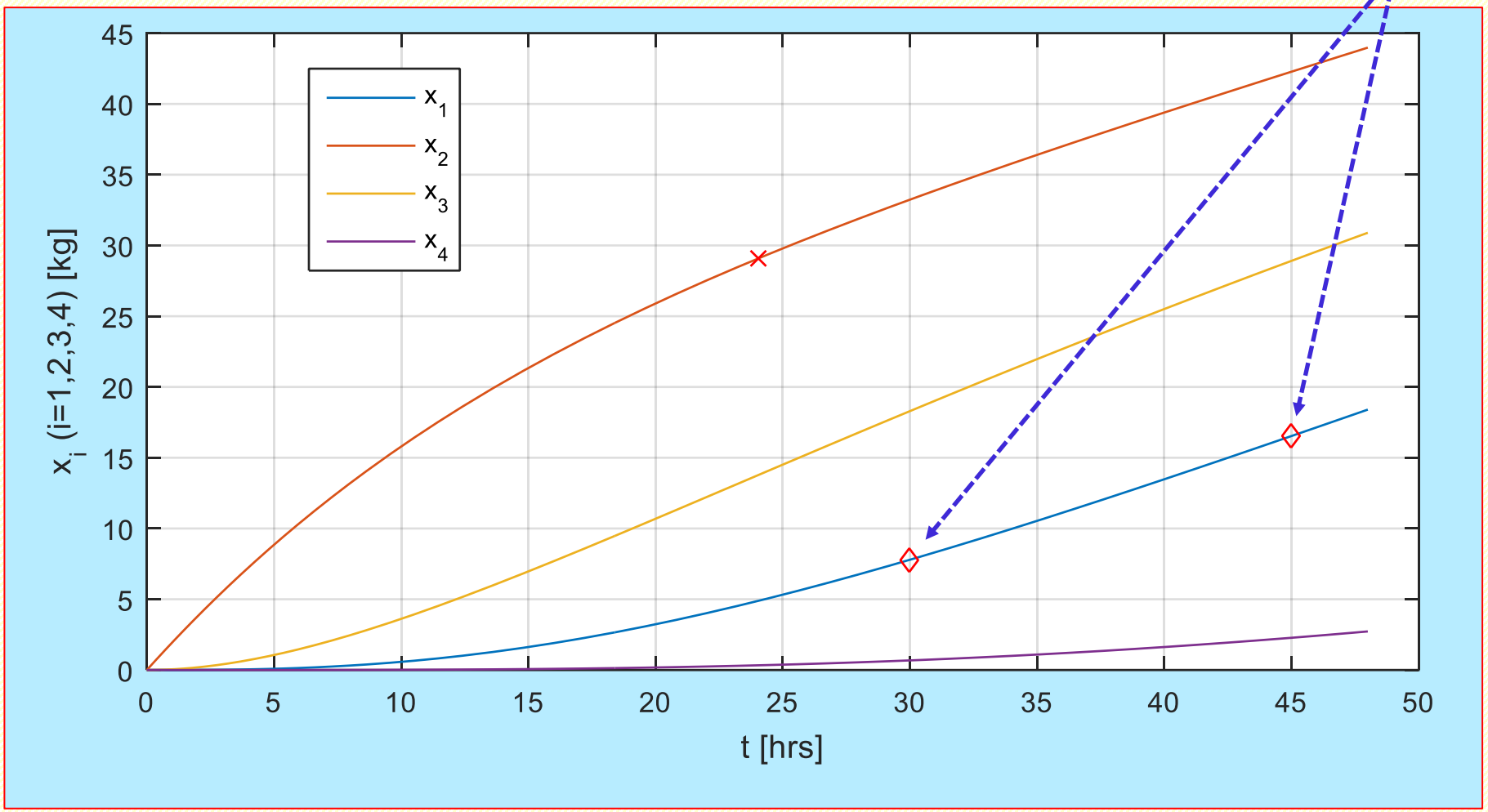
```
>> interp1(tt, xx(:, 2), 24)

ans =

   29.0509
```




```
hold on;
plot(24, interp1(tt, xx(:, 2), 24), 'rx');
```

Extra Task-2: find x_1 for t=[30 45] hrs

```
p2=plot([30 45], interp1(tt, xx(:, 1), [30 45]), 'rd');
```

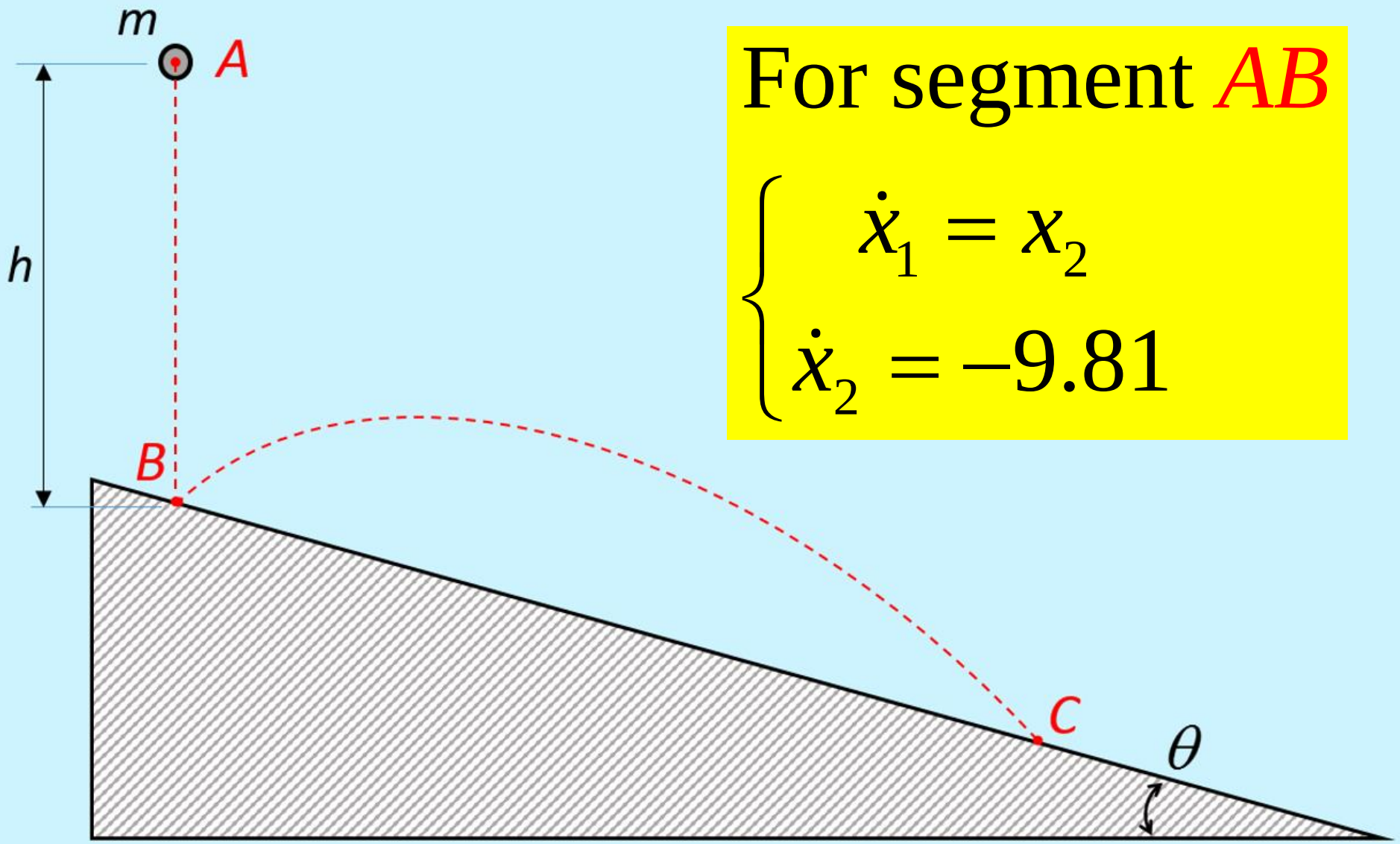


A1, Pt2 Assist:

MATLAB MINI-TUTORIAL:

(2) FALLING MASS: 2nd order system; ode45

T3 in Assignment-1

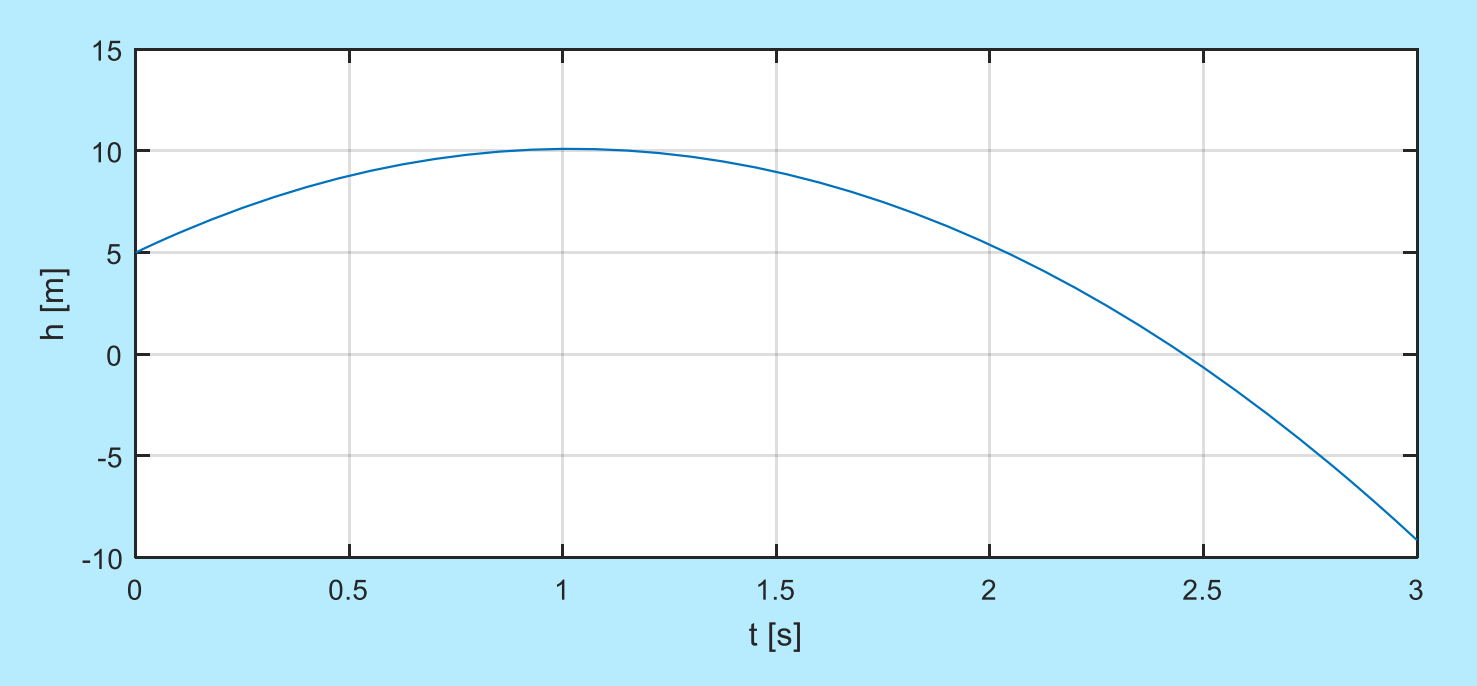


For segment *AB*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -9.81 \end{cases}$$

MATLAB SCRIPT (no annotations)

```
%% Designed by Prof Pavel M.Trivailo (C) 2019
% Falling mass: simple script
clear; close all; clc;
f = @(t,x) [x(2); -9.81];
[tt,xx]=ode45(f,[0 3],[5 10]);
plot(tt,xx(:,1));
xlabel('t [s]'); ylabel('h [m]'); grid on
```



Extra Task-1: find *t* when *h*=0

Command Window

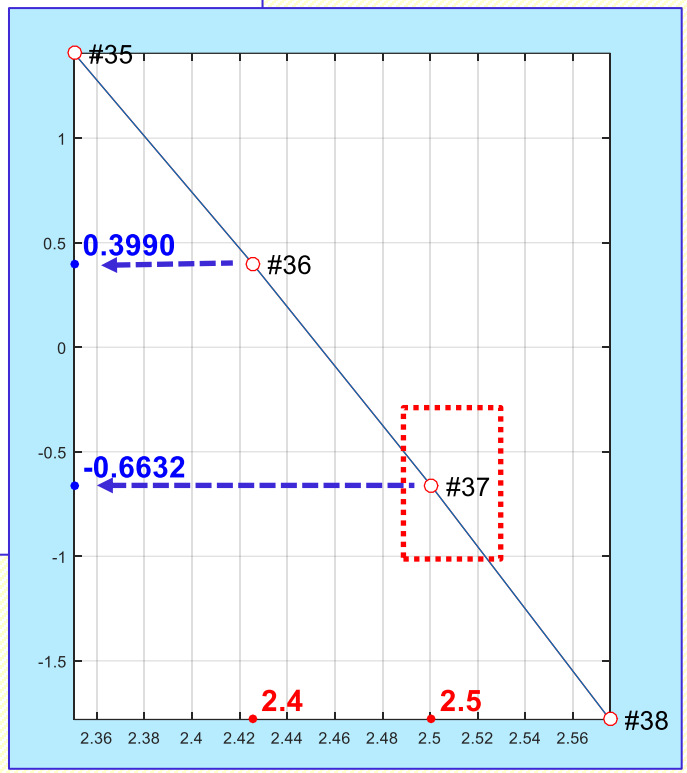
New to MATLAB? See resources for Getting Started.

```
>> find(xx(:,1)<0)

ans =

    37
    38
    39
    40
    41
    42
    43
    44
    45
```

Command Window

New to MATLAB? See resources for Getting Started.

```
>> xx(36:37,1)

ans =

    0.3990
   -0.6632
```

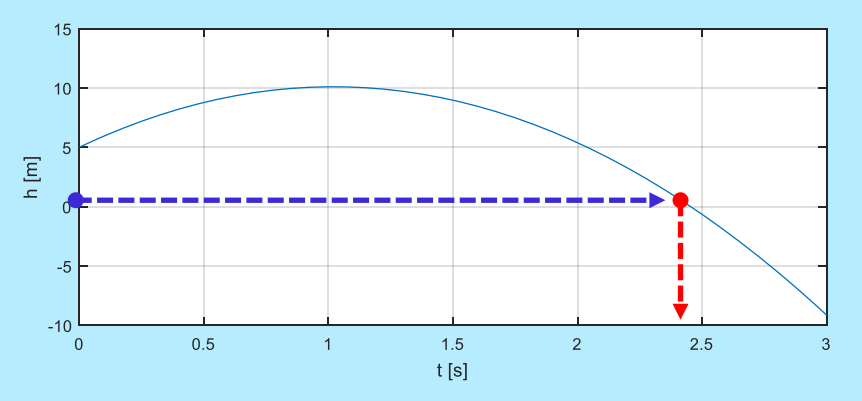


Extra Task-1: find *t* when *h*=0

Command Window

New to MATLAB? See resources for Getting Started.

```
>> t_B=interp1(xx(idx1:idx2),tt(idx1:idx2),0)

t_B =

    2.4536

>> sprintf('Time, when mass hits the ground is: %6.4f s',t_B)

ans =

Time, when mass hits the ground is: 2.4536 s
```

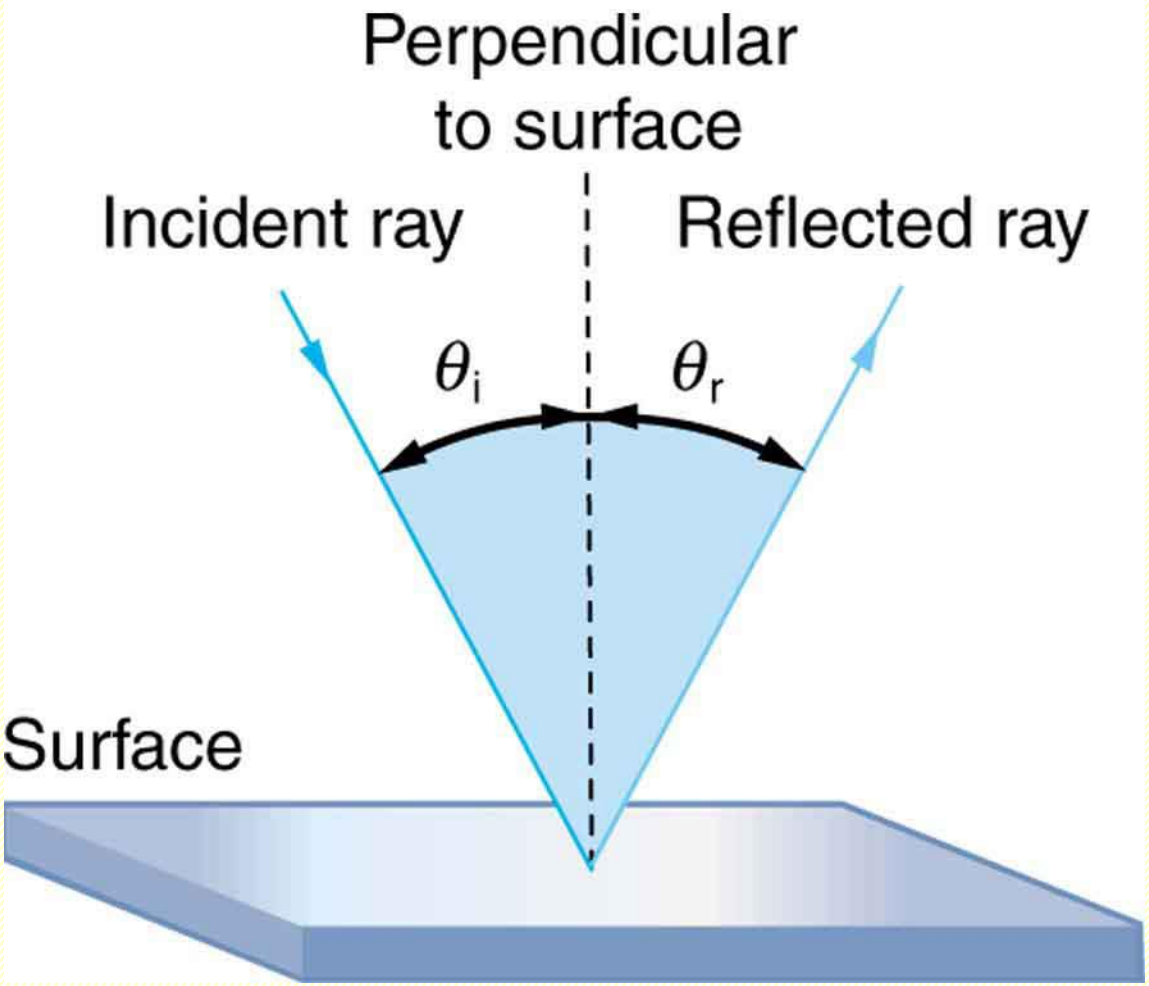


A-1

HINTS:

Previous & New Examples

Task in Assignment-1: Optics Analogy for Bouncing



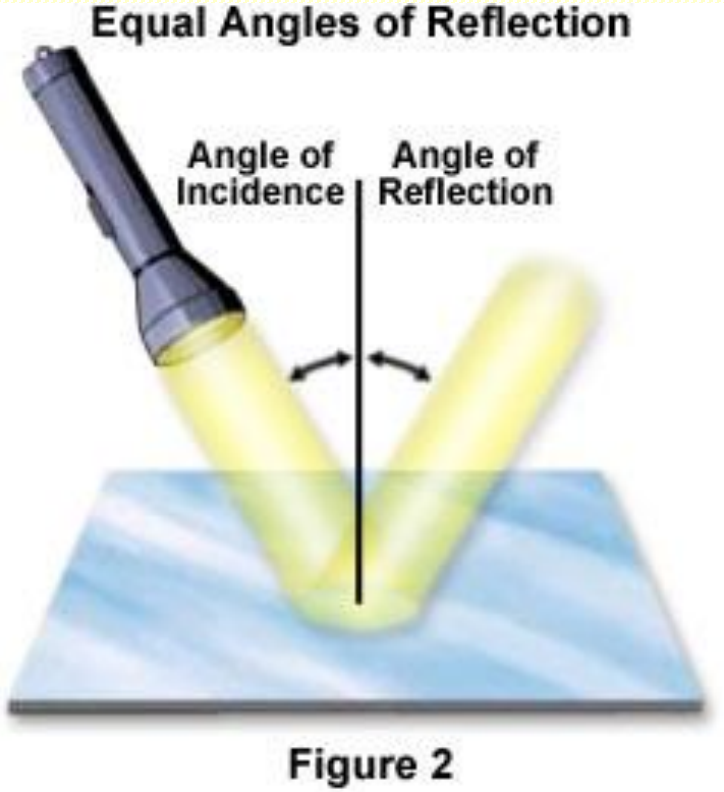
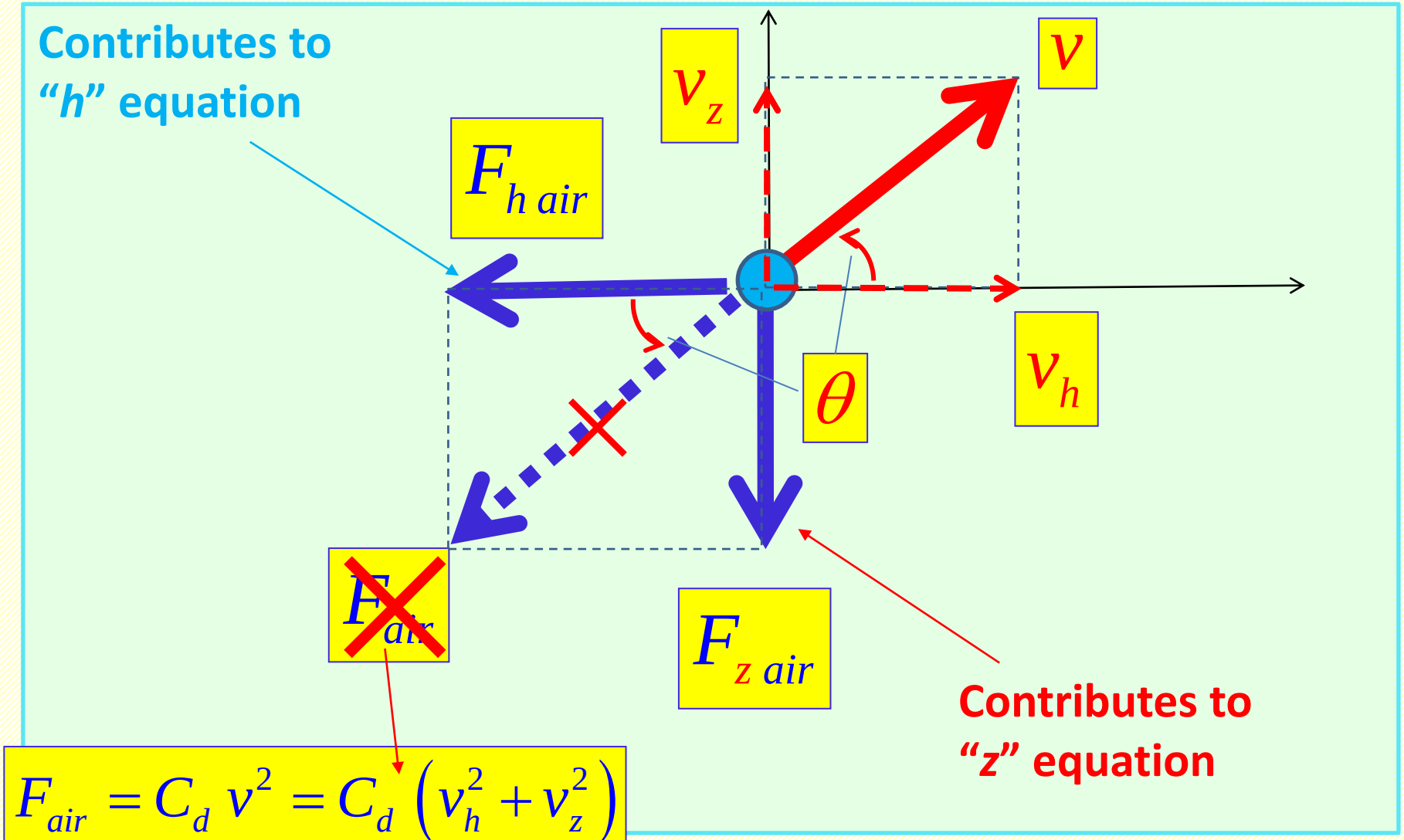


Figure 2

PROJECTILE:

INCLUDING AIR RESISTANCE !!!!!!!!!!!!

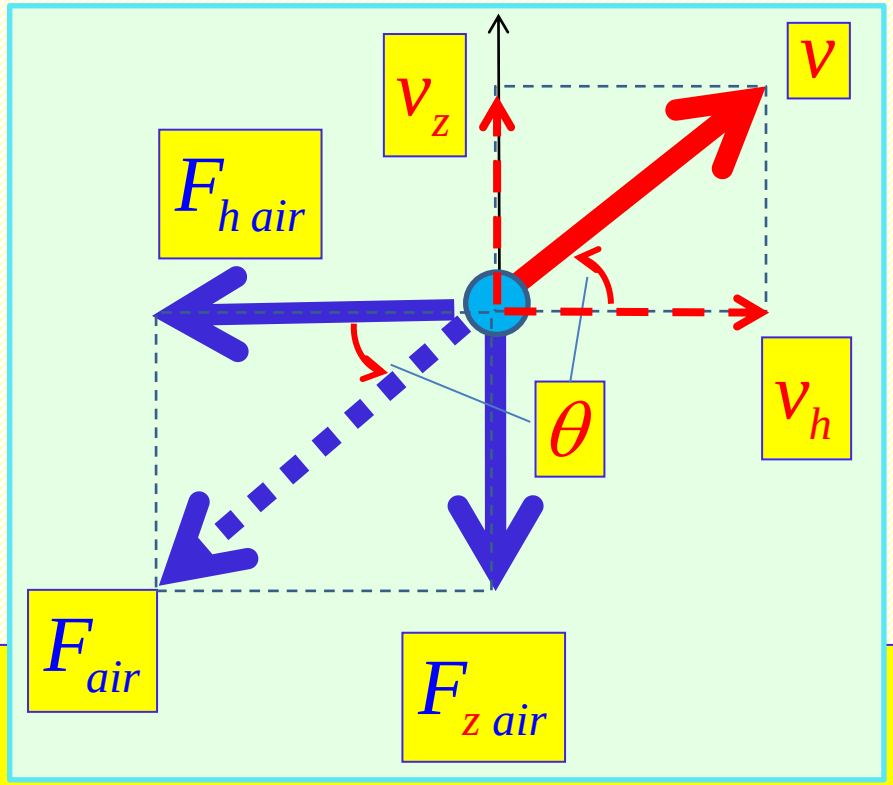
Hint for Assignment-1: Air Resistance



$$F_{air} = C_d\, v^2 = C_d \left(v_h^2 + v_z^2 \right)$$

Hint for Assignment-1: Air Resistance

IMPORTANT ANALYTICAL DERIVATIONS!!!



$$F_{air} = C_d\, v^2;$$

$$F_{z\,air} = F_{air} \sin\theta = F_{air} \frac{v_z}{v} = C_d\, v^2 \frac{v_z}{v} = C_d\, v_z\, v = C_d\, v_z \sqrt{v_h^2 + v_z^2};$$

$$F_{h\,air} = F_{air} \cos\theta = F_{air} \frac{v_h}{v} = C_d\, v^2 \frac{v_h}{v} = C_d\, v_h\, v = C_d\, v_h \sqrt{v_h^2 + v_z^2};$$

MATLAB SCRIPT (no annotations)

```
%% PROJECTILE (WITH AIR RESISTANCE) EXAMPLE
% Designed by Prof P.M.Trivailo (C) 2019
% Feature: ALL COMMANDS ARE IN ONE FILE!!!
%---------------------------------------
m=10; coeff=0.6; % F=coeff*Vsq;
th=45*pi/180;    % rad      ^ z
v0=10;           % m/s      |
z0=0; h0=0       % m        +--------------> h
g=9.81;
t=[0:0.01:1]*1.3;
v0h=v0*cos(th); v0z=v0*sin(th);

figure; grid on; hold on; axis equal;
pr_xdot = @(t, x) ([x(2); -9.81-coeff*x(2)*sqrt(x(2)^2+x(4)^2)/m;...
    x(4); -coeff*x(4)*sqrt(x(2)^2+x(4)^2)/m]);
[tt,zz]=ode45(pr_xdot,t,[z0; v0z; h0; v0h]);
plot(zz(:,3), zz(:,1),'LineWidth',3,'Color',[0 0.5 1]);
xlabel('$h$ [m]','Interpreter','LaTeX');
ylabel('$z$ [m]','Interpreter','LaTeX');
title('\bf Projectile (With Air Drag): Time History');
set(gca, 'FontSize',18);
set(gcf,'Position', [20 80 1270  680]);

g=line('XData',h0,'YData',z0,'Marker','.','MarkerSize',48);
for i=1:length(tt),
    set(g,'XData',zz(i,3),'YData',zz(i,1)); drawnow;   pause(0.01);
end
```

Definition of states:

$$\begin{cases} x_1 = z \\ x_2 = \dot{z} \\ x_3 = h \\ x_4 = \dot{h} \end{cases}$$

MATLAB SCRIPT (with annotations)

```
%% PROJECTILE (WITH AIR RESISTANCE) EXAMPLE
% Designed by Prof P.M.Trivailo (C) 2019
% Feature: ALL COMMANDS ARE IN ONE FILE!!!
%-------------------------------------
m=10; coeff=0.6; % F=coeff*Vsq;
th=45*pi/180;    % rad          ^ z
v0=10;           % m/s
z0=0; h0=0       % m
g=9.81;
t=[0:0.01:1]*1.3;
v0h=v0*cos(th); v0z=v0*sin(th);

figure; grid on; hold on; axis equal;
pr_xdot = @(t, x) ([x(2); -9.81-coeff*x(2)*sqrt(x(2)^2+x(4)^2)/m;...
    x(4); -coeff*x(4)*sqrt(x(2)^2+x(4)^2)/m]);

[tt,zz]=ode45(pr_xdot,t,[z0; v0z; h0; v0h]);
plot(zz(:,3), zz(:,1),'LineWidth',3,'Color',[0 0.5 1]);
xlabel('$h$ [m]','Interpreter','LaTeX');
ylabel('$z$ [m]','Interpreter','LaTeX');
title('\bf Projectile (With Air Drag): Time History');
set(gca, 'FontSize',18);
set(gcf,'Position', [20 80 1270  680]);

g=line('XData',h0,'YData',z0,'Marker','.','MarkerSize',48);
for i=1:length(tt),
    set(g,'XData',zz
end
```

1. Input of the data

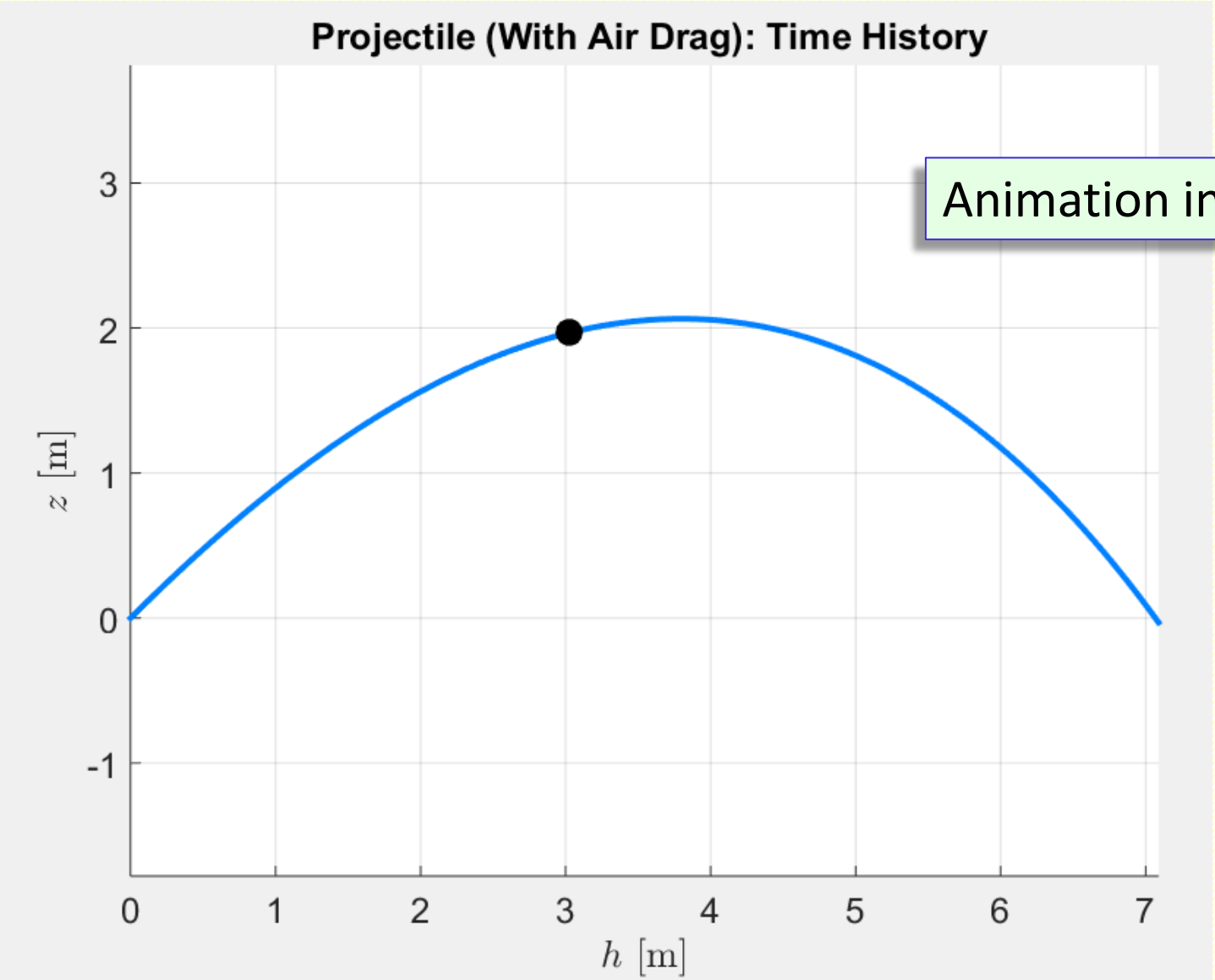
2. EOM

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g - \dfrac{1}{m} c v^2 \sin\theta \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\dfrac{1}{m} c v^2 \cos\theta \end{cases}$$

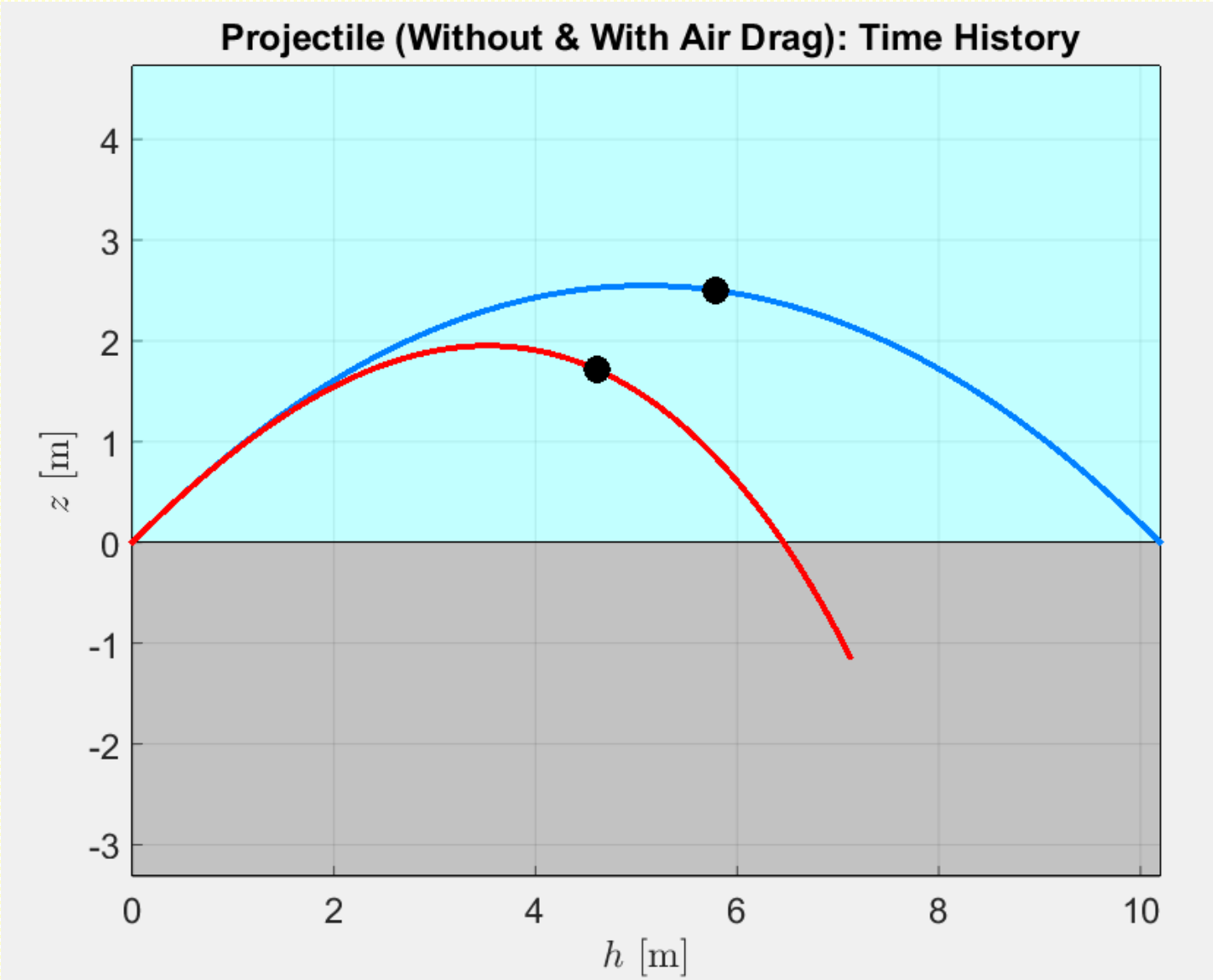
3. Calling ode45 procedure !

4. Plotting results

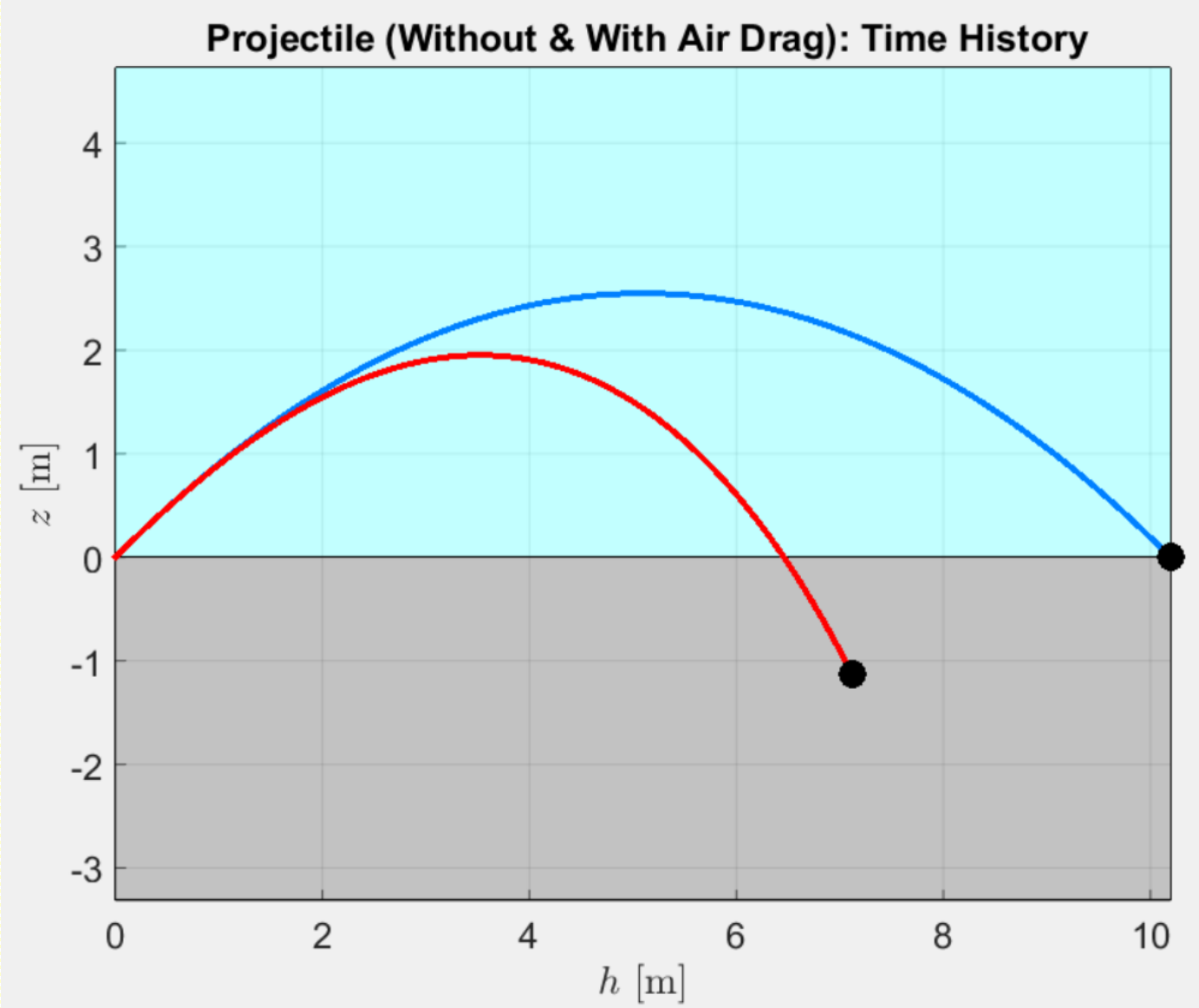
Projectile: Trajectory for Air-Resistance Case



Air-resistance versus No-air-resistance Cases



Air-resistance versus No-air-resistance Cases



Simplified Introduction into FEM:

Mass-Spring System Examples

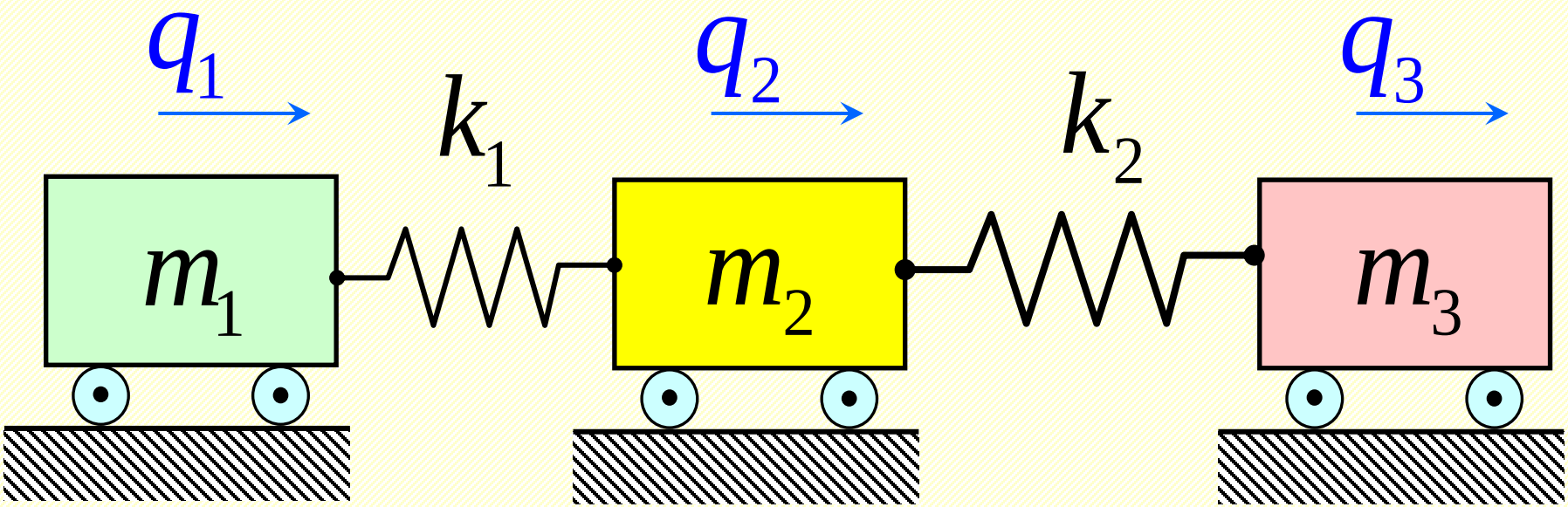
Multi-DOF mass-spring systems:

Derivation of EOM using an FEM approach: assembling the global stiffness matrix.

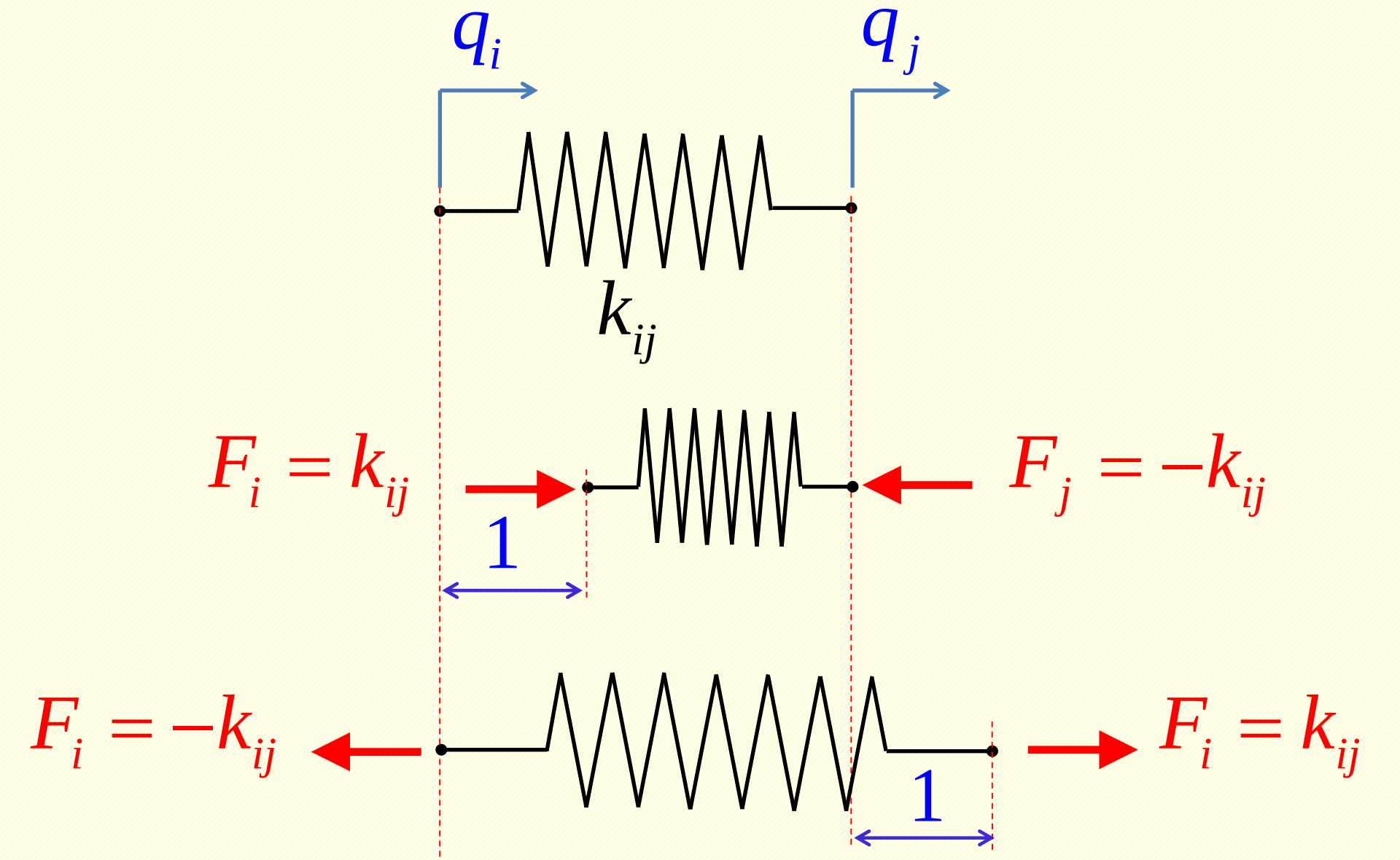
Introduction into FEM:

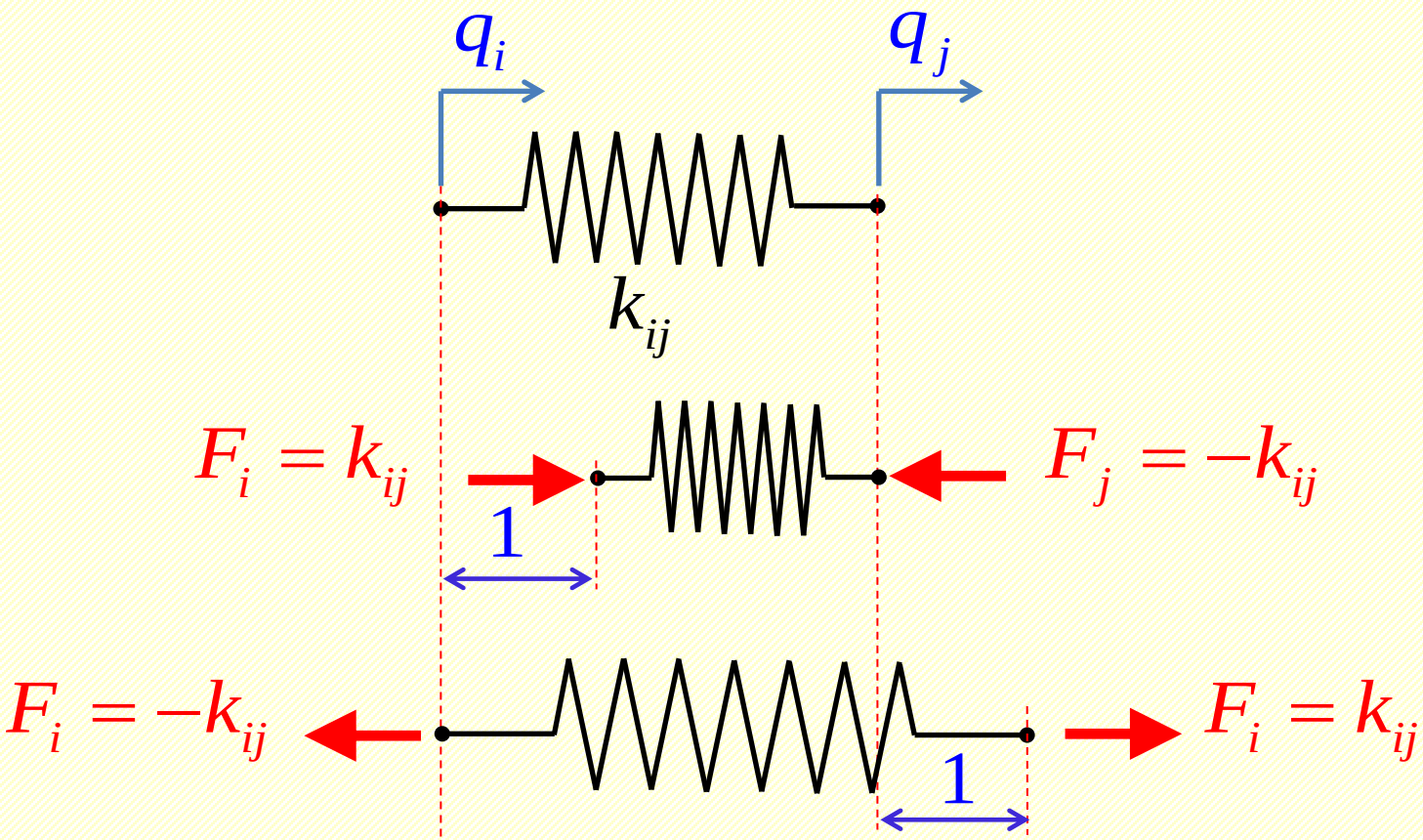
Free Boundary Case

Case-1 Study: 3-DOF System



Element Stiffness Matrix: Derivation



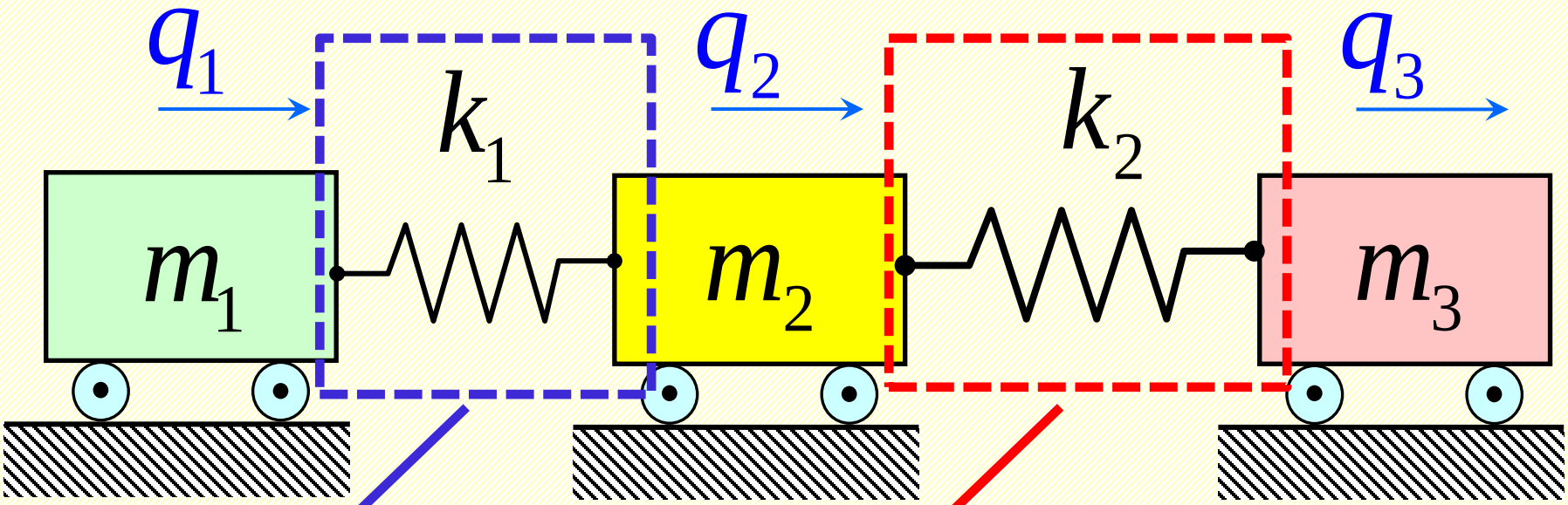


$$\begin{bmatrix} k_{ij} & -k_{ij} \\ -k_{ij} & k_{ij} \end{bmatrix} \begin{Bmatrix} q_i \\ q_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}$$

$$\begin{bmatrix} k_{ij} & -k_{ij} \\ -k_{ij} & k_{ij} \end{bmatrix} \begin{Bmatrix} q_i \\ q_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ F_j \end{Bmatrix}$$

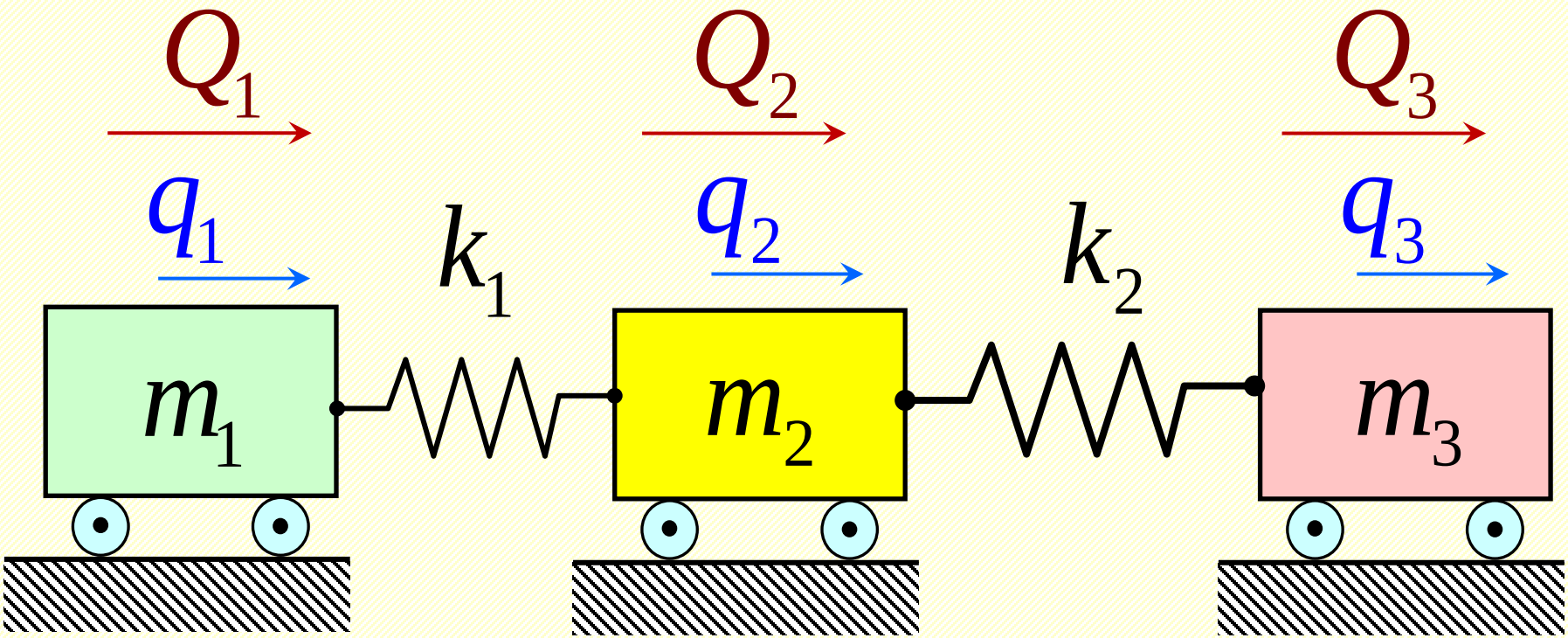
$$\left[k_{\text{FElement }(ij-\text{DOF})} \right] = \begin{bmatrix} k_{ij} & -k_{ij} \\ -k_{ij} & k_{ij} \end{bmatrix}$$

Case-1 Study: Assembling Global [k]



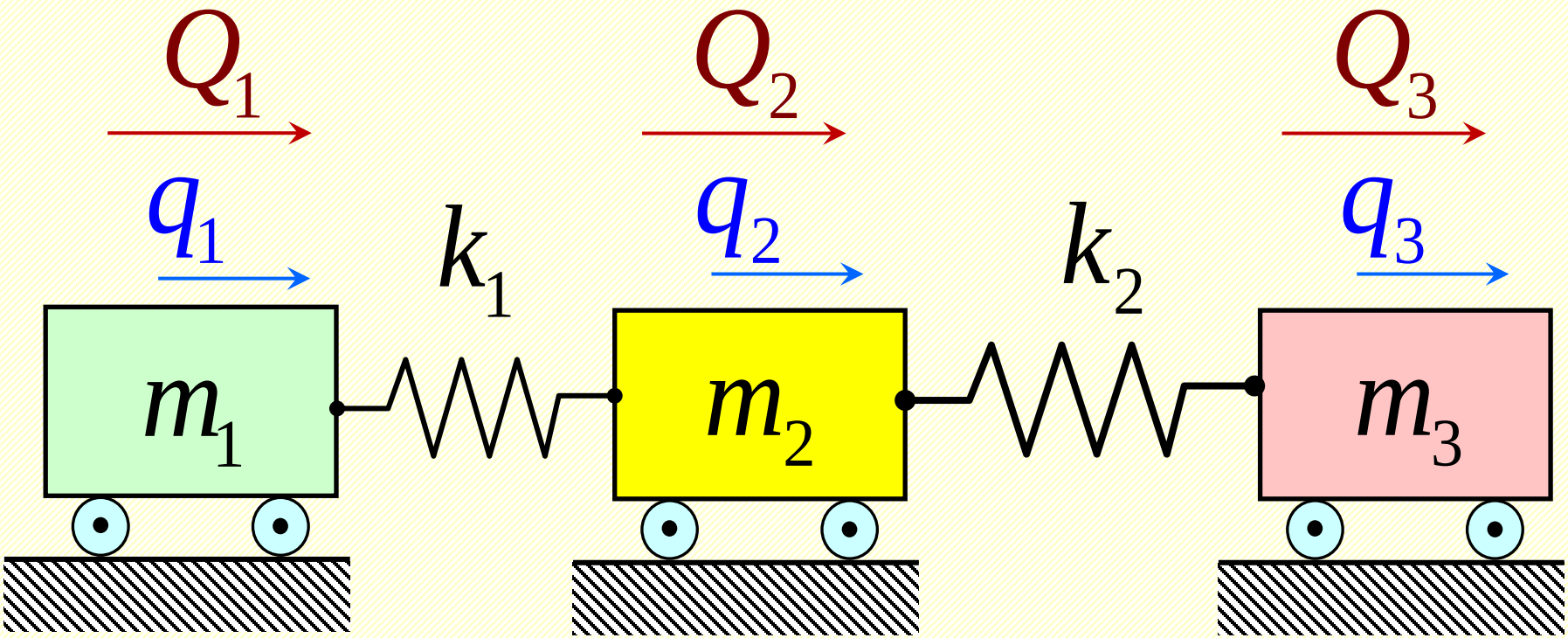
$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

Case-1 Study: Stiffness Static Equation



$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

Case-1 Study: Equations of Motion

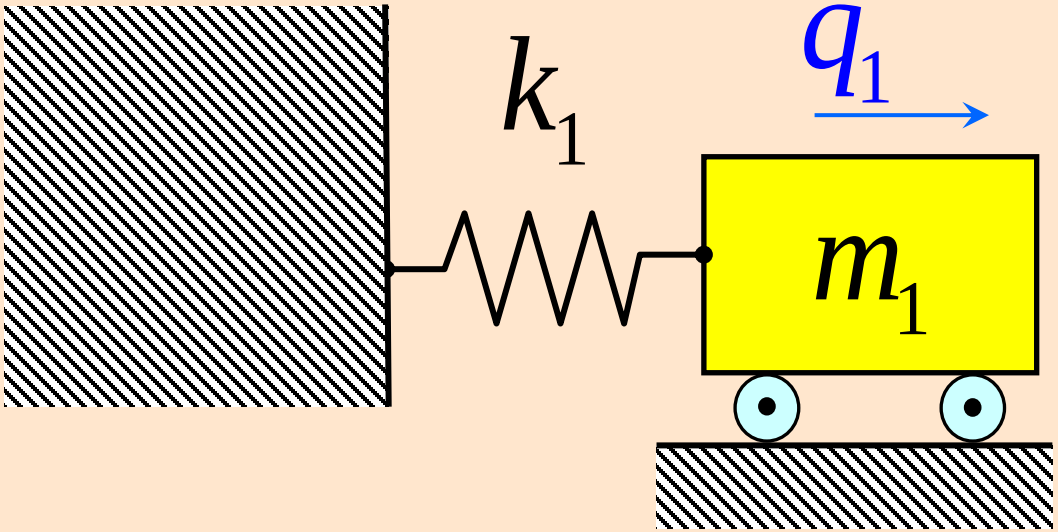


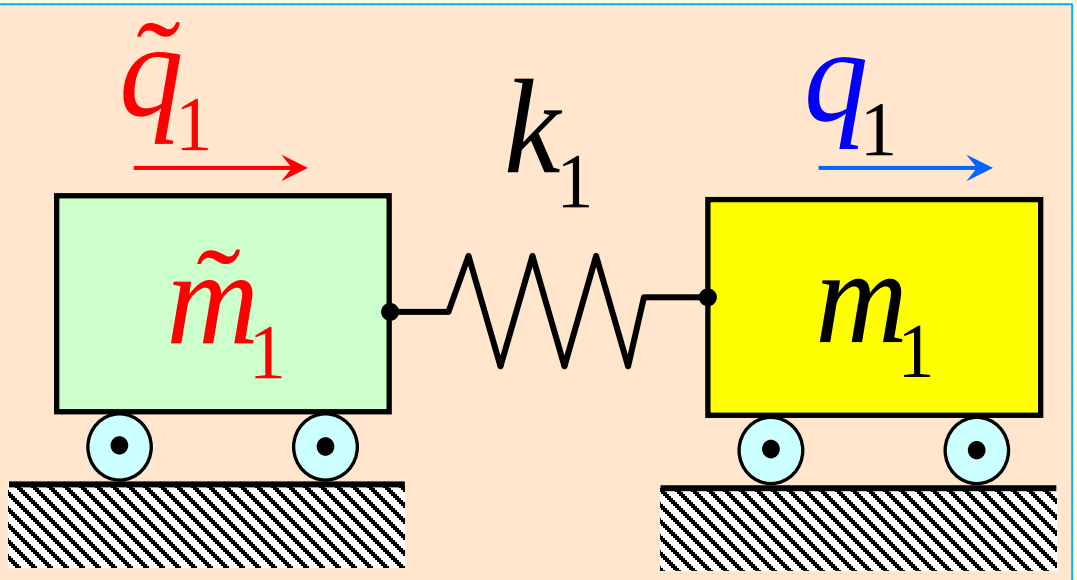
$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

Introduction into FEM:

Fixed Boundary Case, 1-mass Example

Original & Supplementary Systems

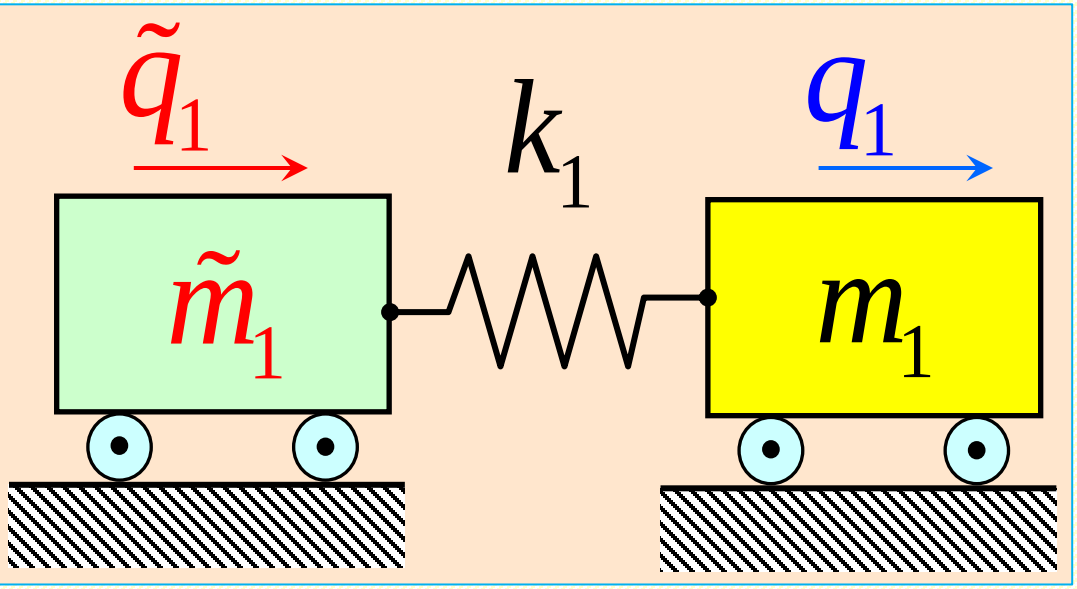


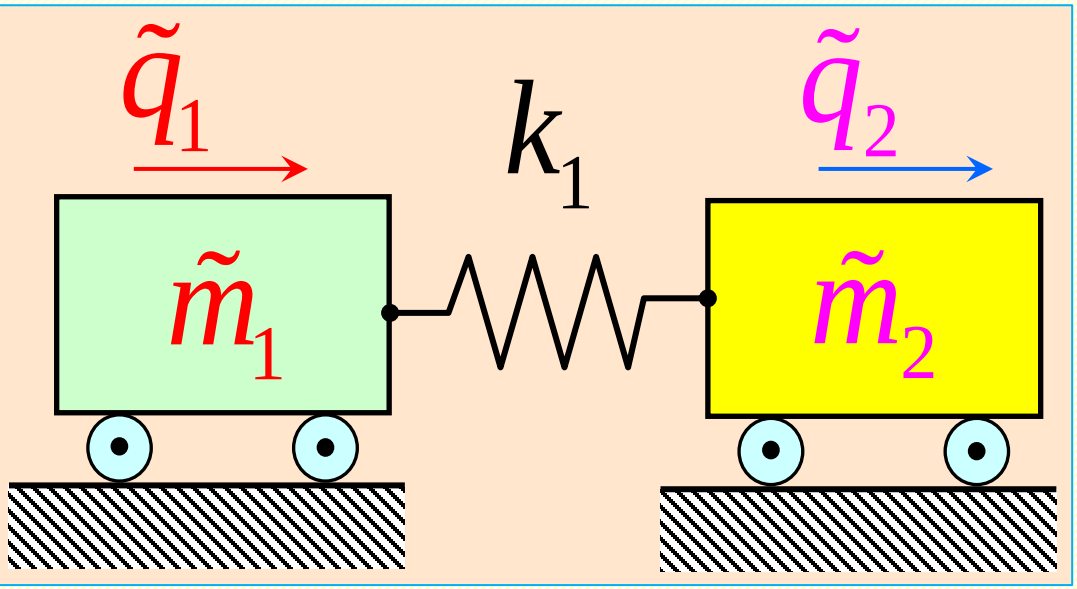


To model 1-mass and 1-spring case (as in Assignment, for N=1),

we start with a SUPPLEMENTARY system, having ONE spring and TWO masses, by adding to the ORIGINAL system a DUMMY mass, replicating left wall.

Rename Supplementary Parameters



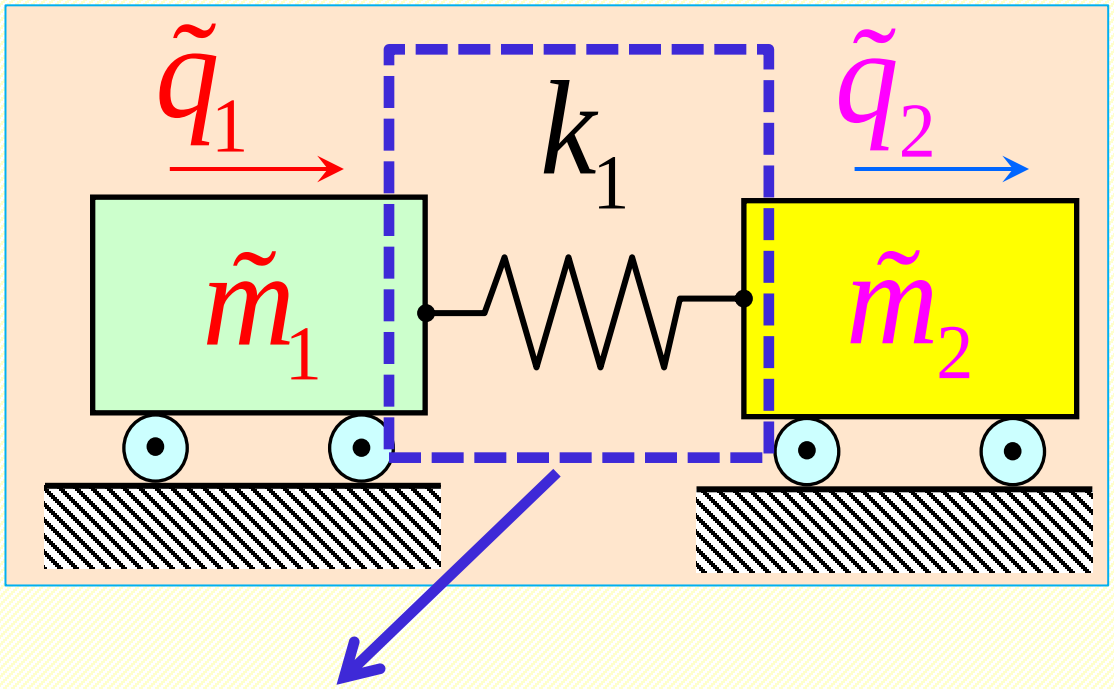


In the SUPPLEMENTARY system

we then rename notations (this is done for consistency and programming convenience):

q_1	is renamed to	$\tilde{q}_2$
m_1	is renamed to	$\tilde{m}_2$

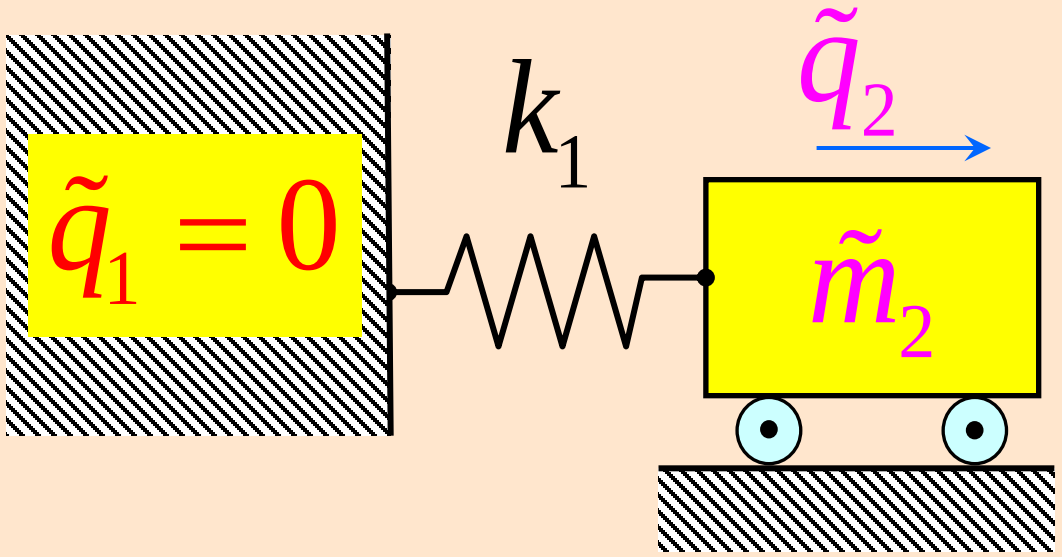
Assembling Stiffness Matrix



GLOBAL STIFFNESS MATRIX for SUPPLEMENTARY System in this case is equal to the elementary stiffness matrix:

$$[\tilde{K}] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

Constraining Supplementary System



Now we are returning back to the ORIGINAL System, replacing the dummy mass in the SUPPLEMENTARY system with a wall.

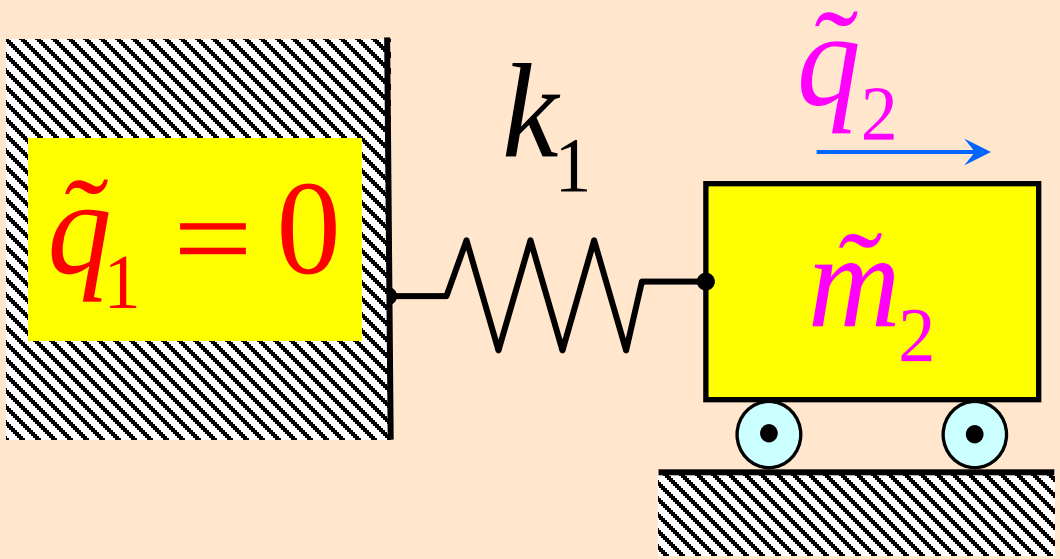
Mathematically, this is equivalent to setting $\tilde{q}_1 = 0$, which means that in the global matrix of the system first row and column can be crossed out:

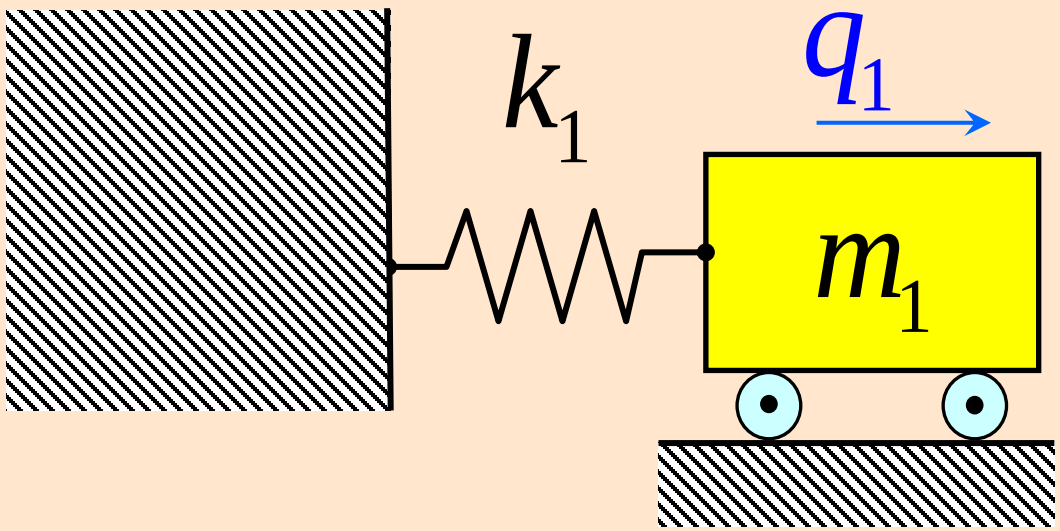
$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

reducing GLOBAL stiffness matrix to one number, k_1

$$[k_1]$$

Renaming Supplementary Parameters





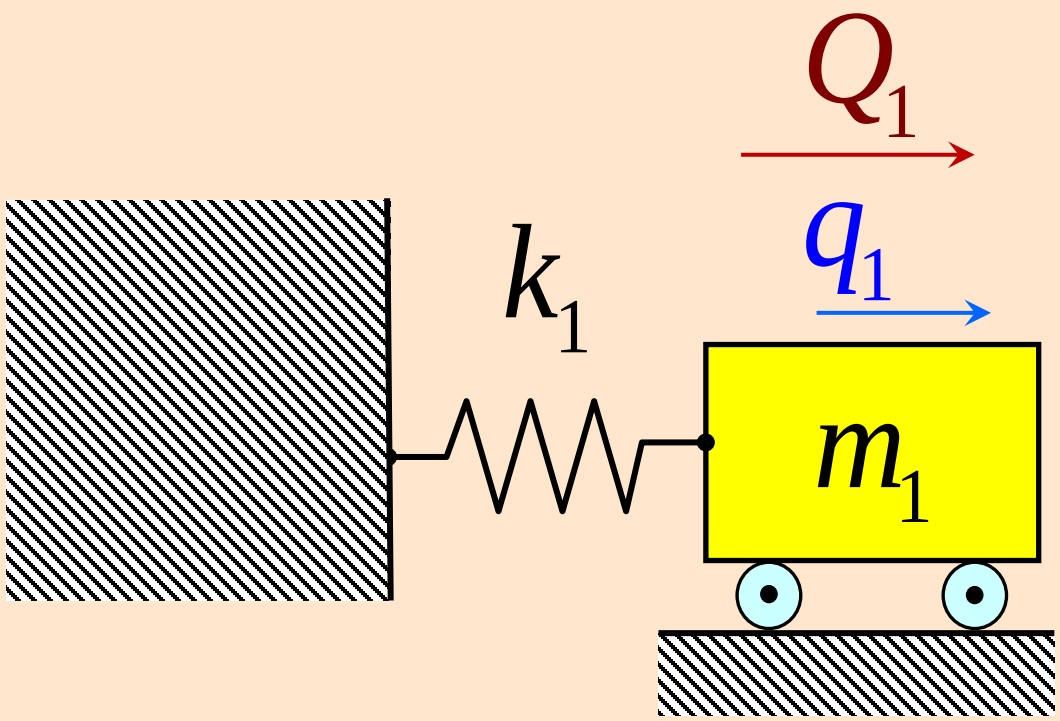
Let us rename notations, returning back to the original notations:

$\tilde{q}_2$	is renamed to	q_1
$\tilde{m}_2$	is renamed to	m_1

SOLUTION: System's GLOBAL stiffness matrix is:

$$[k_1]$$

Equation of Motion for Original Case



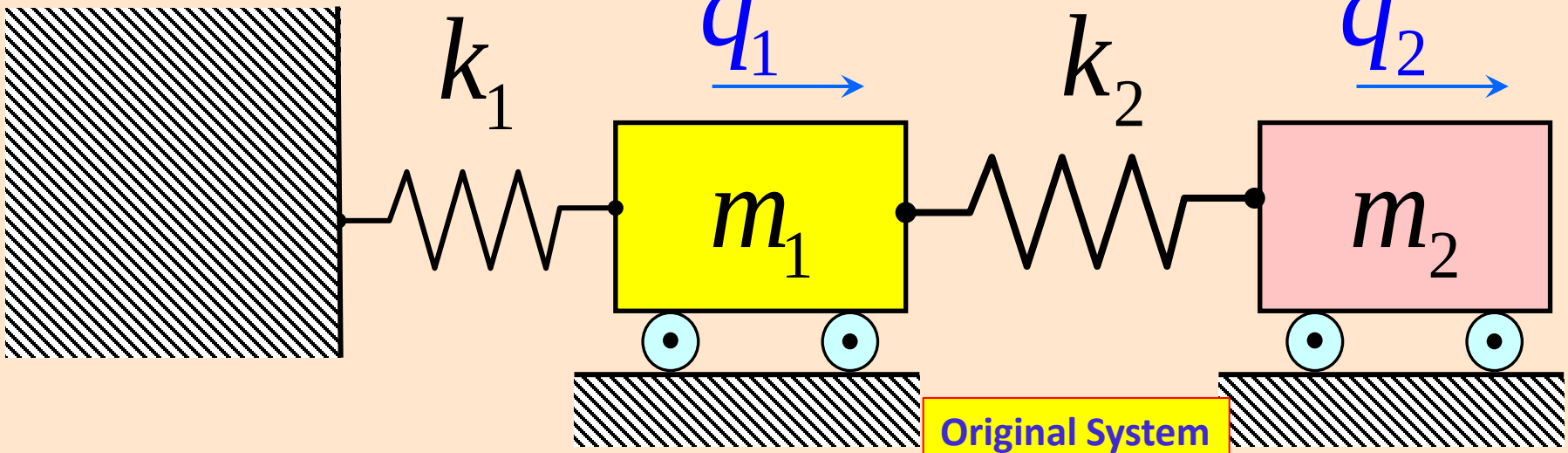
$$m_1 \ddot{q}_1 + k_1 q_1 = Q_1$$

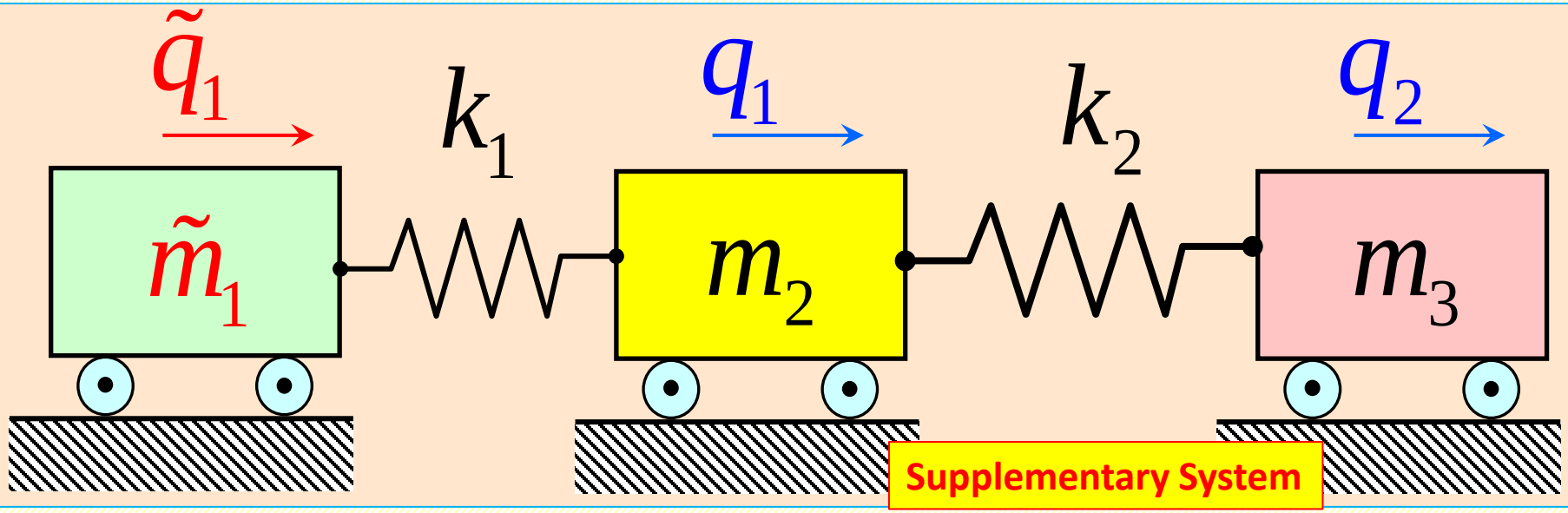
EOM for the Original System, derived by FEM

Introduction into FEM:

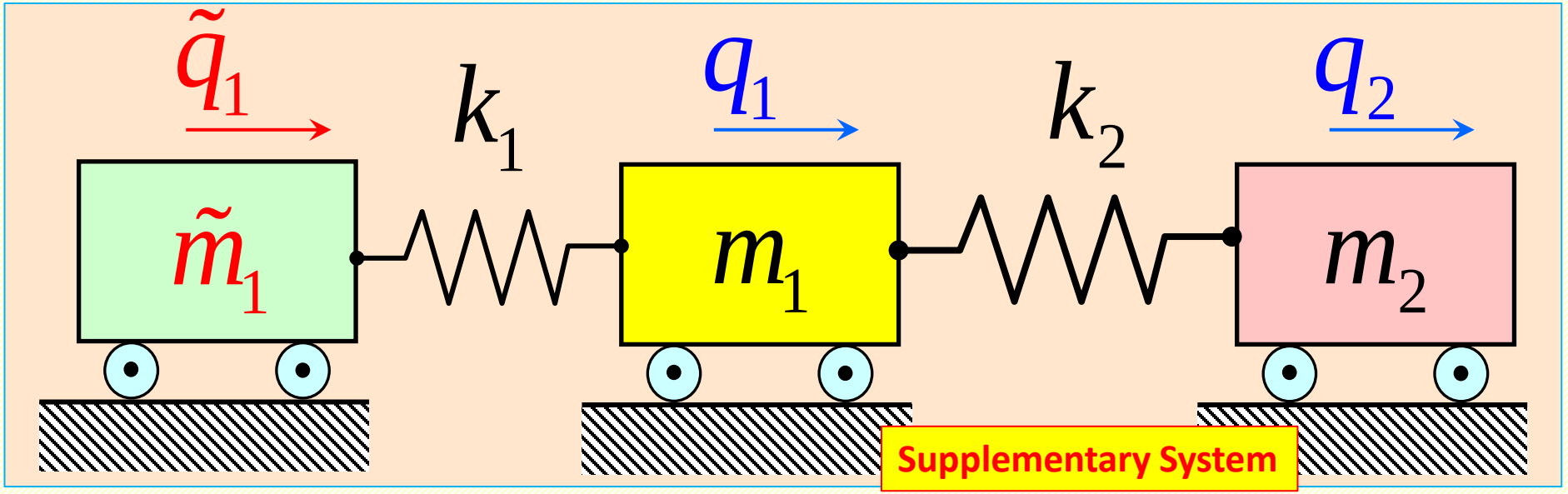
Fixed Bounday Case, 2-masses Example

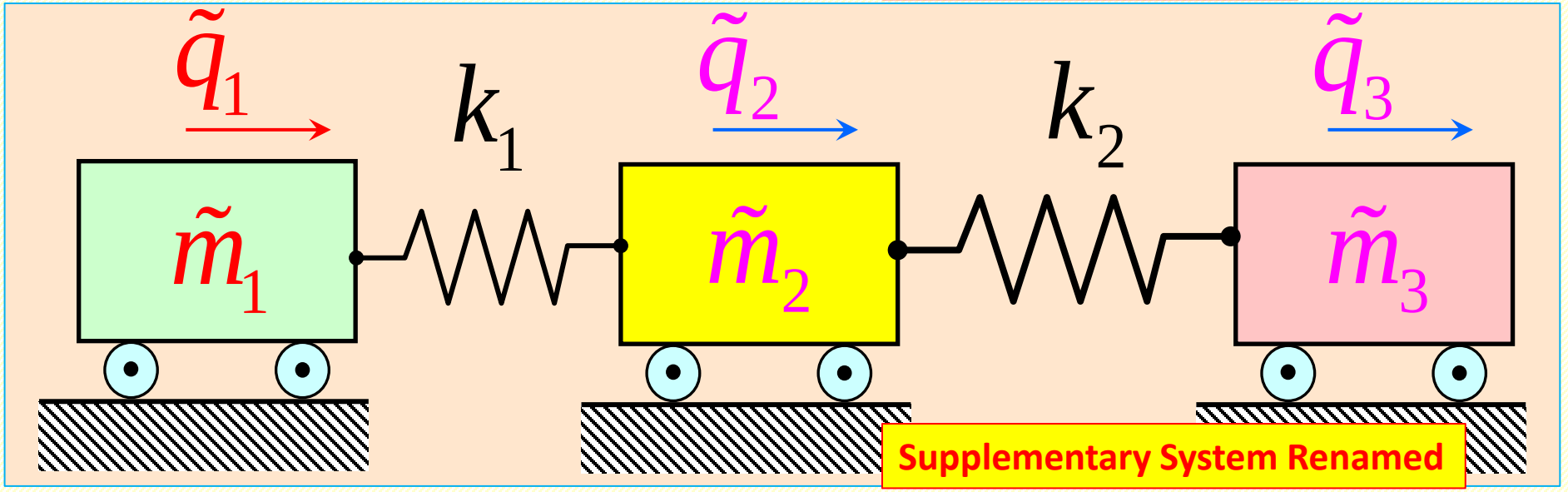
Original & Supplementary Systems



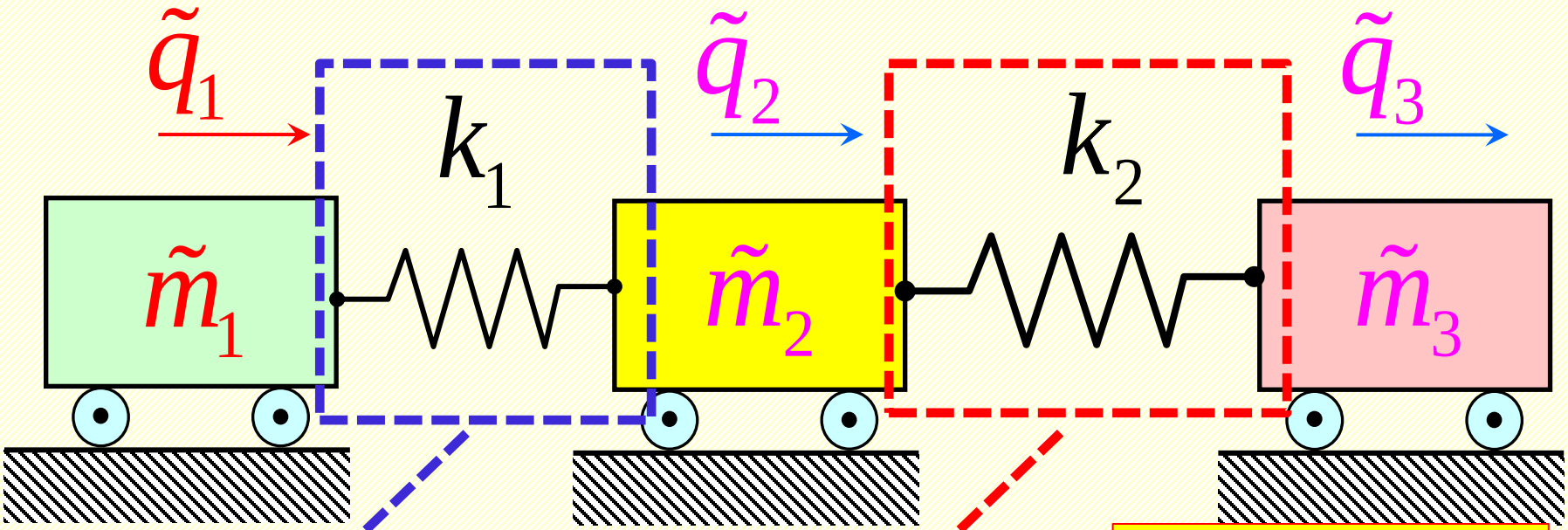


Rename Supplementary Parameters





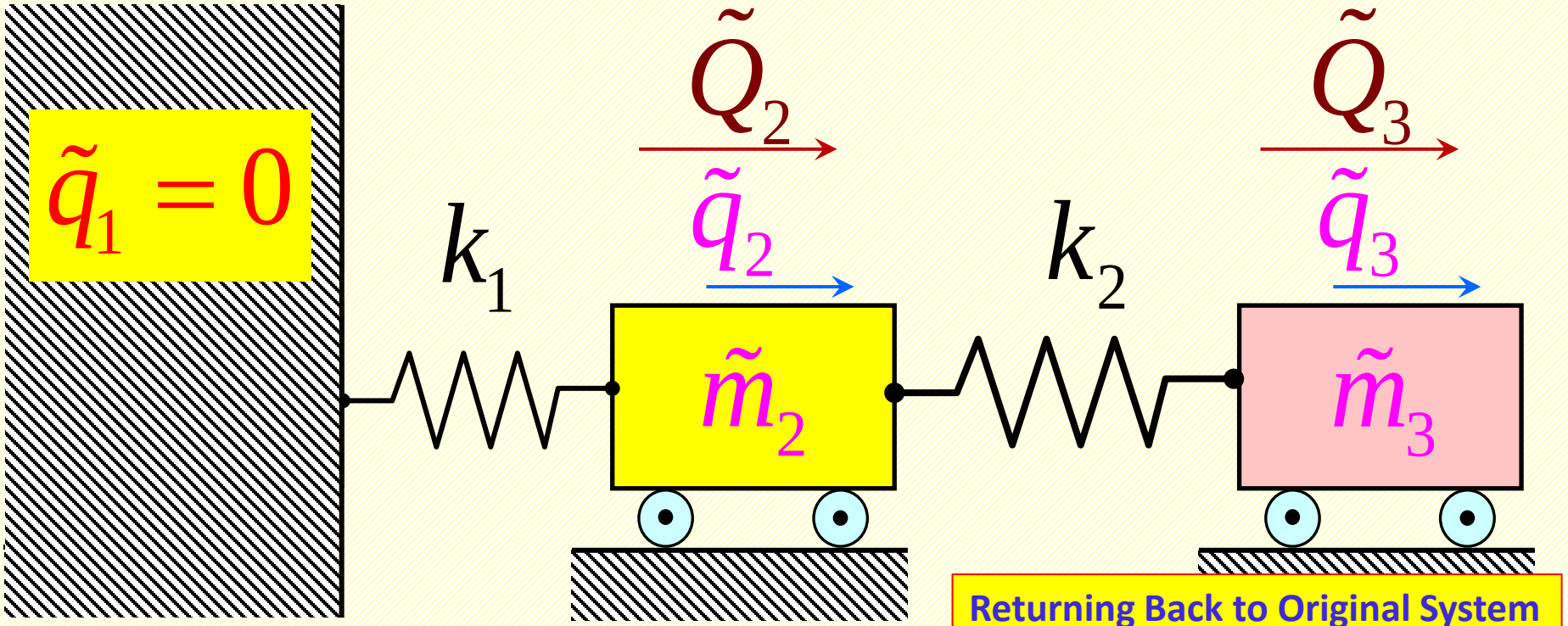
Assembling Stiffness Matrix



Supplementary System

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

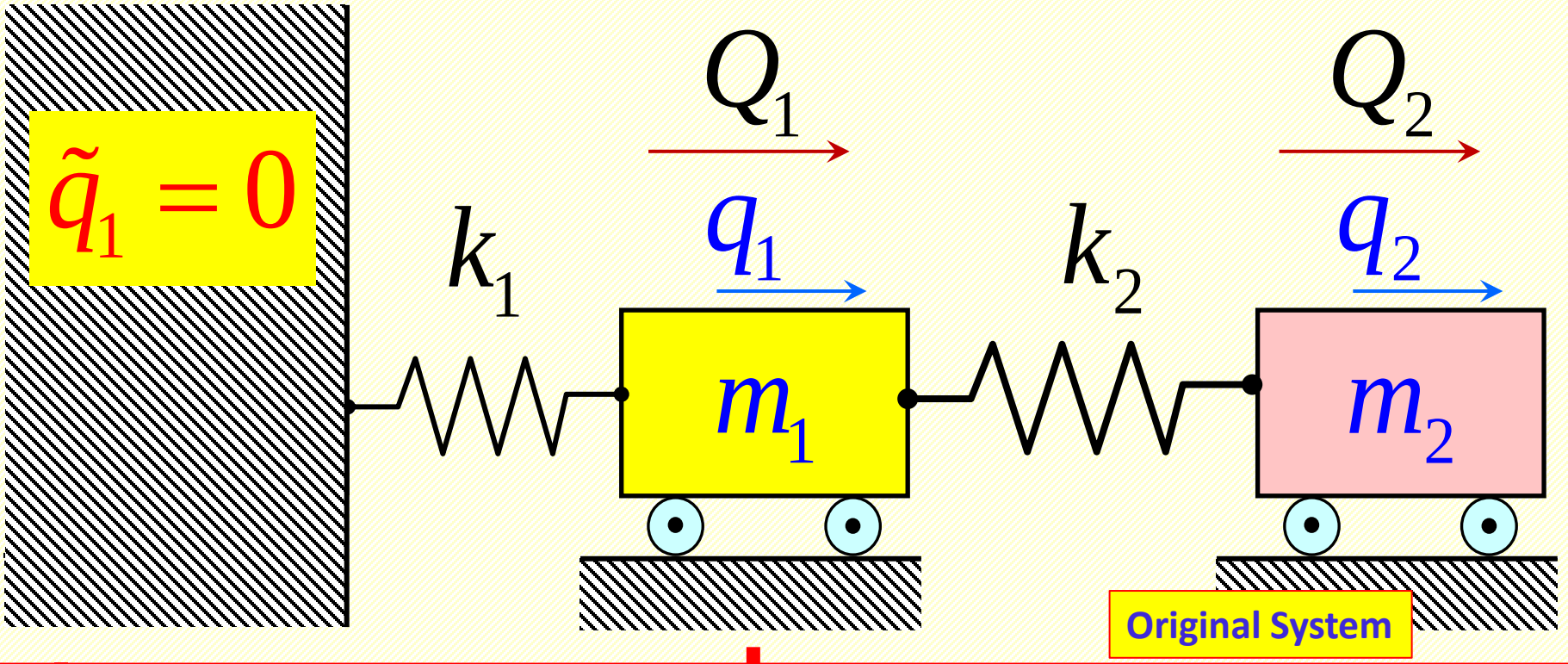
Constraining Supplementary System



Returning Back to Original System

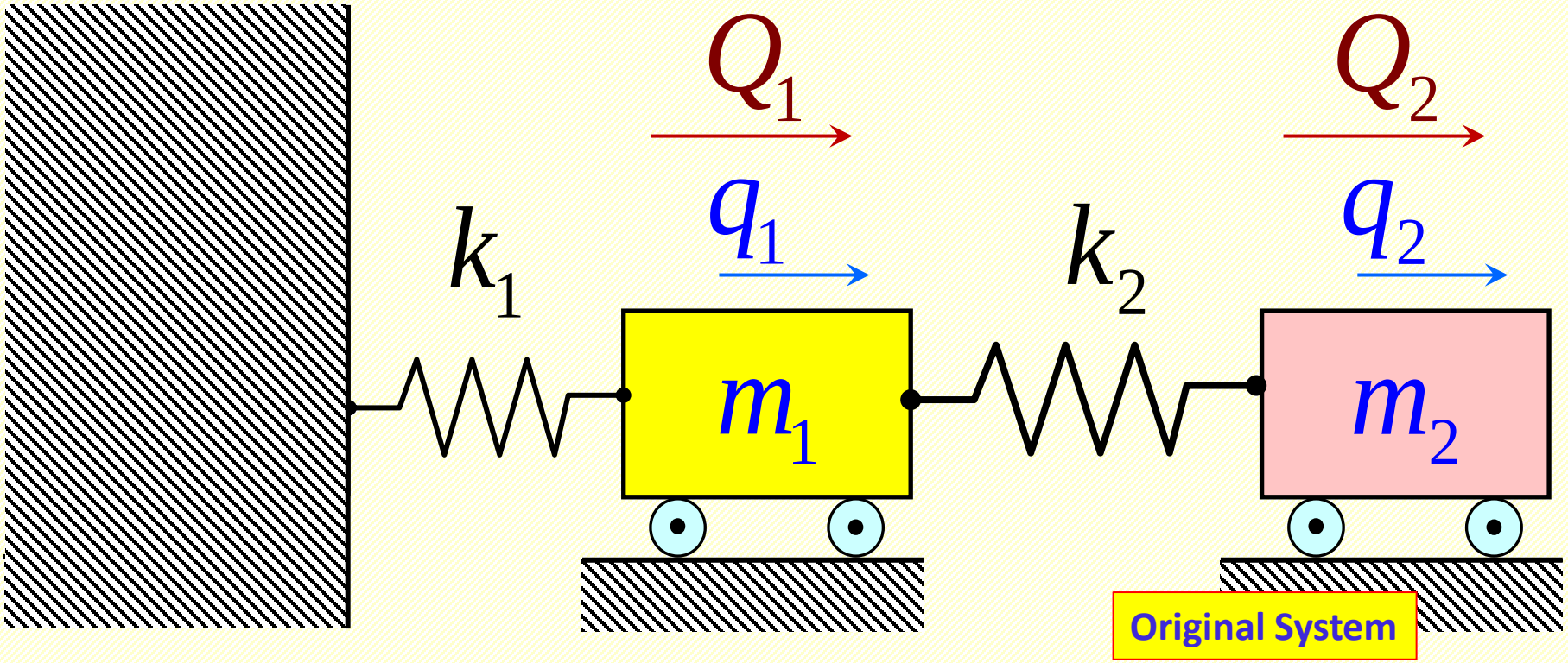
$$\begin{bmatrix} \tilde{m}_1 & 0 & 0 \\ 0 & \tilde{m}_2 & 0 \\ 0 & 0 & \tilde{m}_3 \end{bmatrix} \begin{Bmatrix} \ddot{\tilde{q}}_1 \\ \ddot{\tilde{q}}_2 \\ \ddot{\tilde{q}}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \end{Bmatrix} = \begin{Bmatrix} \tilde{Q}_1 \\ \tilde{Q}_2 \\ \tilde{Q}_3 \end{Bmatrix}$$

Renaming Supplementary Parameters



$$\begin{bmatrix} \tilde{m}_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{\tilde{q}}_1 \\ \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \tilde{q}_1 \\ q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} \tilde{Q}_1 \\ Q_1 \\ Q_2 \end{Bmatrix}$$

Equation of Motion for Original Case

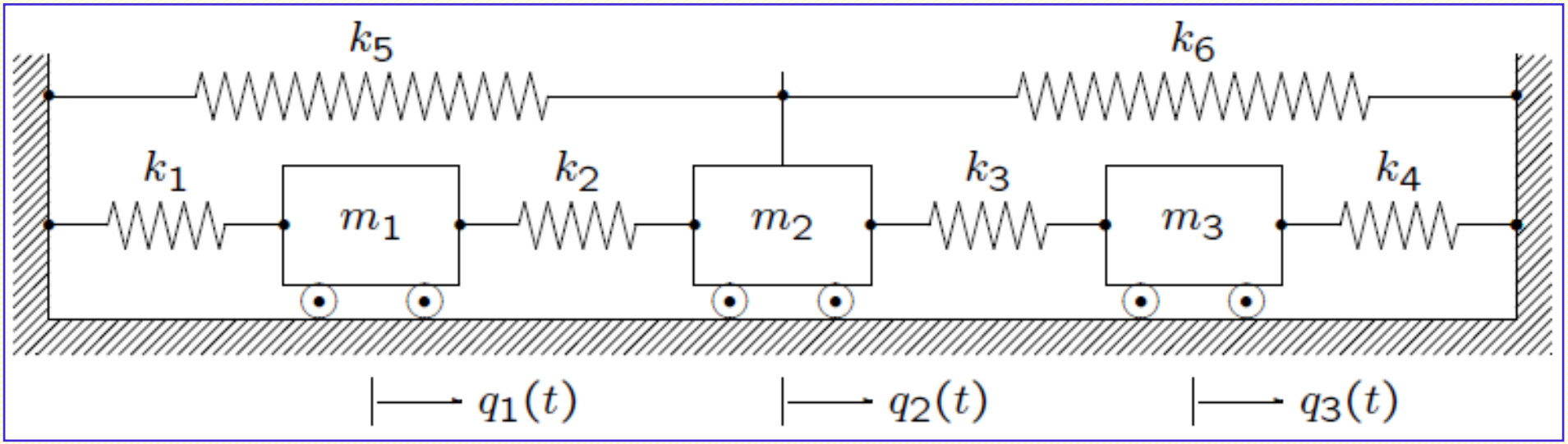


$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

Introduction into FEM:

Homework Examples: try to derive

Case-3 Study: [*m*] & [*k*] matrices



$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix};$$

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 + k_5 + k_6 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}$$

Case-3 Study: EOM with FBDs

$$\begin{cases} m_1\,\ddot{q}_1 &= -k_1\,q_1 - k_2\,(q_1 - q_2), \\ m_2\,\ddot{q}_2 &= -k_5\,q_2 + k_2\,(q_1 - q_2) - k_3\,(q_2 - q_3) - k_6\,q_2, \\ m_3\,\ddot{q}_3 &= -k_4\,q_3 + k_3\,(q_2 - q_3). \end{cases}$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{Bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} +$$

$$+ \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 + k_5 + k_6 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

FEM:

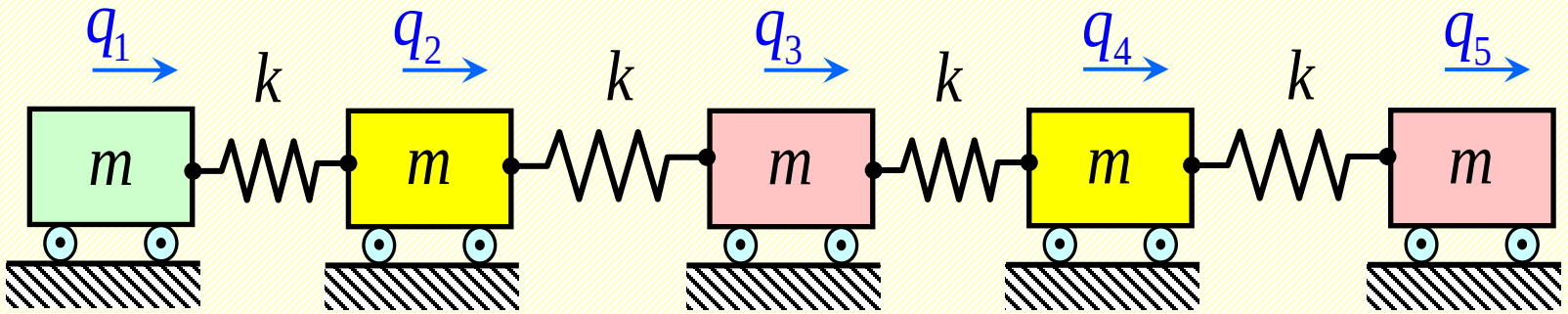
CHAIN MASS-SPRING SYSTEMS

Method-1

Modelling an UNCONSTRAINED Supplementary System First and then Removing Constrained DOFs

A-1: 5-DOF "Mock-Up" System

This is a SUPPLEMENTARY SYSTEM (not exactly as in A1), with NO CONSTRAINTS, where left wall is replaced with a left mass with added Degree-of-Freedom



Creation of the GLOBAL MASS MATRIX of the SUPPEMENTARY system is simple:

$$[M]=\begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & m & 0 \\ 0 & 0 & 0 & 0 & m \end{bmatrix}$$

***N* is a number of FEs;**

In MATLAB: N=4; m=7; M=eye(N+1)*m;

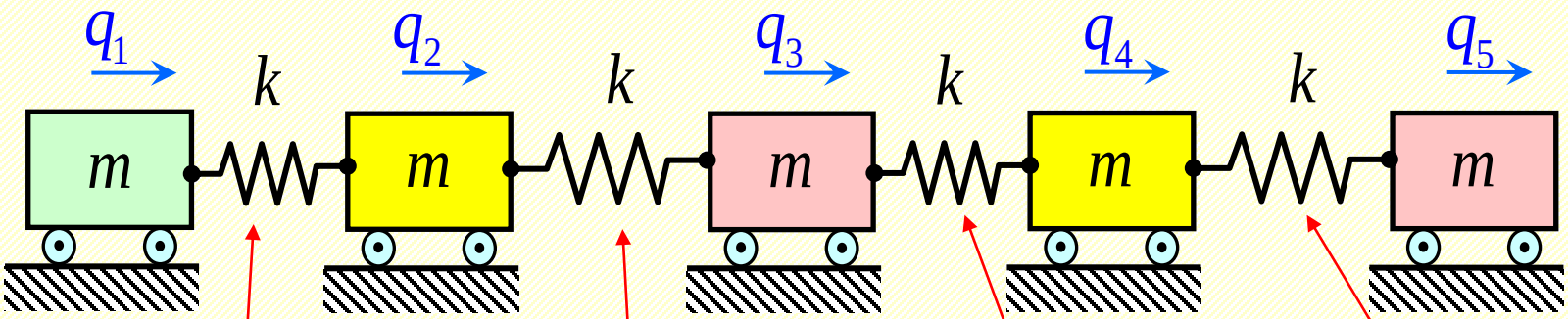
***N*+1 is a number of DOFs**

here N is the number of springs,

hence FINITE ELEMENTS!

A-1: 5-DOF "Mock-Up" System

Creation of the GLOBAL STIFFNESS MATRIX can be done, using FEM:



$$[k_{12}] = \begin{matrix} DOF \\ 1 \\ 2 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{matrix}$$

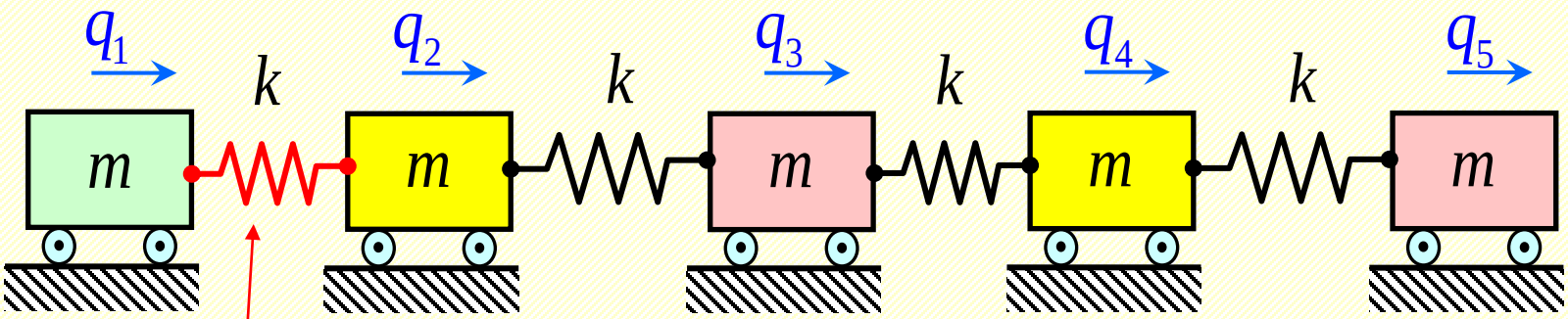
$$[k_{23}] = \begin{matrix} DOF \\ 2 \\ 3 \end{matrix} \begin{matrix} \begin{matrix} 2 & 3 \end{matrix} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{matrix}$$

$$[k_{34}] = \begin{matrix} DOF \\ 3 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 3 & 4 \end{matrix} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{matrix}$$

$$[k_{45}] = \begin{matrix} DOF \\ 4 \\ 5 \end{matrix} \begin{matrix} \begin{matrix} 4 & 5 \end{matrix} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{matrix}$$

Assembling [K]: FE-1 contribution

Creation of the GLOBAL STIFFNESS MATRIX can be done, using FEM:



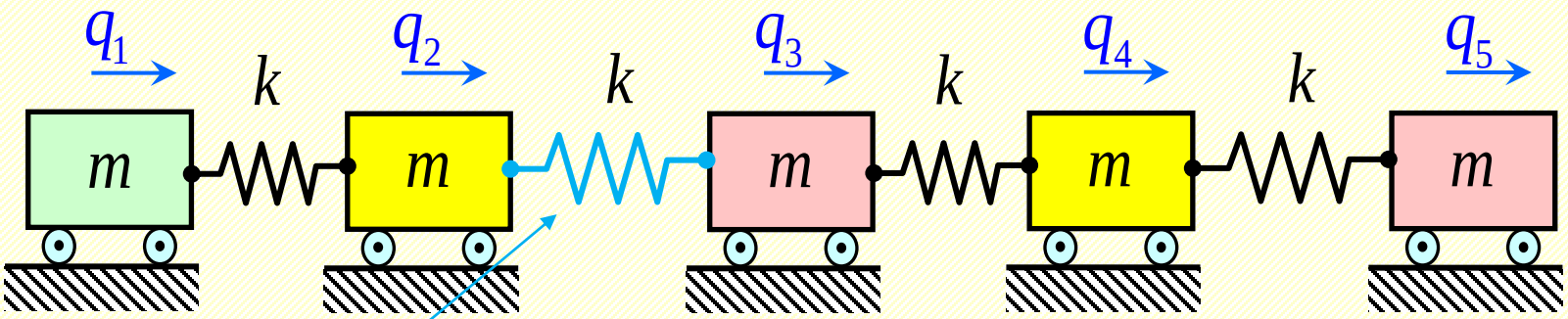
Start assembling GLOBAL STIFFNESS MATRIX: FE #1:

$$[k_{12}] = \begin{array}{c} DOF \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array}$$

$$[K_{12}] = \begin{array}{c|ccccc} DOF & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & k & -k & 0 & 0 & 0 \\ 2 & -k & k & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Assembling [K]: FE-2 contribution

Creation of the GLOBAL STIFFNESS MATRIX can be done, using FEM:



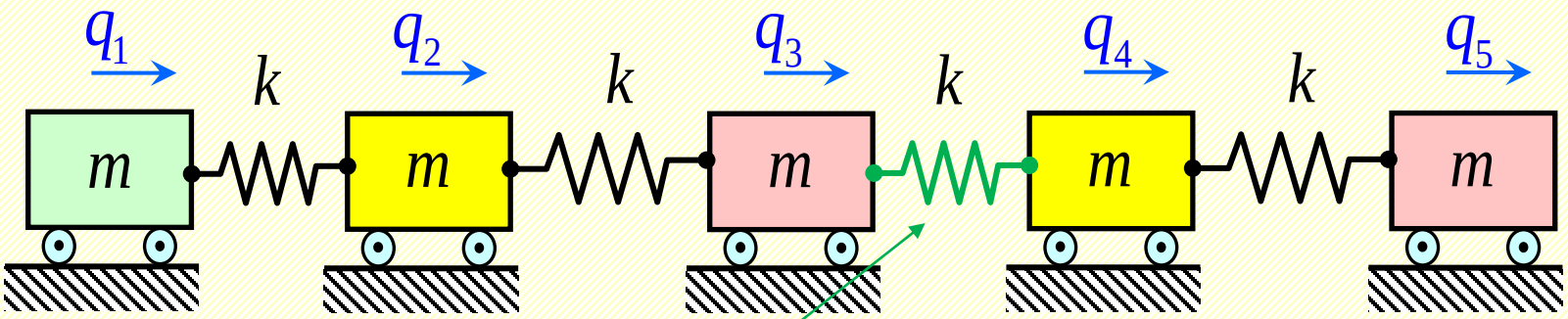
Continue assembling GLOBAL STIFFNESS MATRIX: FE #2:

$$[k_{23}] = \begin{array}{c|cc} DOF & 2 & 3 \\ \hline 2 & k & -k \\ 3 & -k & k \end{array}$$

$$[K_{23}] = \begin{array}{c|ccccc} DOF & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & k & -k & 0 & 0 \\ 3 & 0 & -k & k & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Assembling [K]: FE-3 contribution

Creation of the GLOBAL STIFFNESS MATRIX can be done, using FEM:



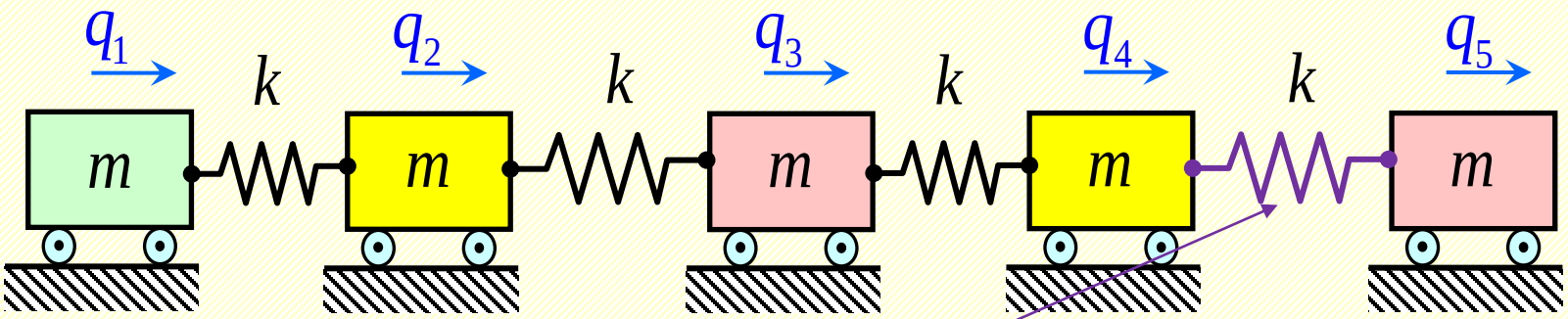
Continue assembling GLOBAL STIFFNESS MATRIX: FE #3:

$$[k_{34}] = \begin{array}{c|cc} DOF & 3 & 4 \\ \hline 3 & k & -k \\ 4 & -k & k \end{array}$$

$$[K_{34}] = \begin{array}{c|ccccc} DOF & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & k & -k & 0 \\ 4 & 0 & 0 & -k & k & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Assembling [K]: FE-4 contribution

Creation of the GLOBAL STIFFNESS MATRIX can be done, using FEM:



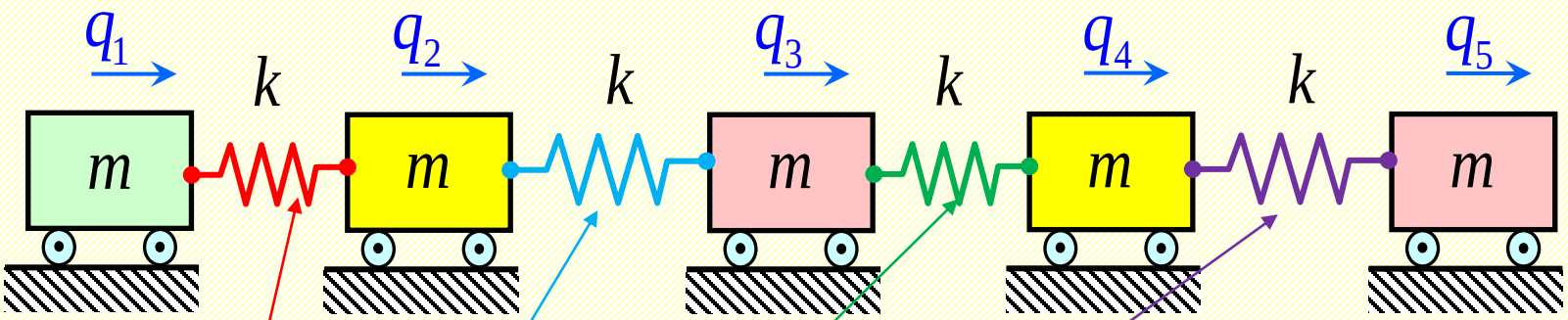
Continue assembling GLOBAL STIFFNESS MATRIX: FE #4:

$$[k_{45}] = \begin{array}{c|cc} DOF & 4 & 5 \\ \hline 4 & k & -k \\ 5 & -k & k \end{array}$$

$$[K_{45}] = \begin{array}{c|ccccc} DOF & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & k & -k \\ 5 & 0 & 0 & 0 & -k & k \end{array}$$

Assembling [K]: FEM-4 contribution

Creation of the GLOBAL STIFFNESS MATRIX can be done, using FEM:

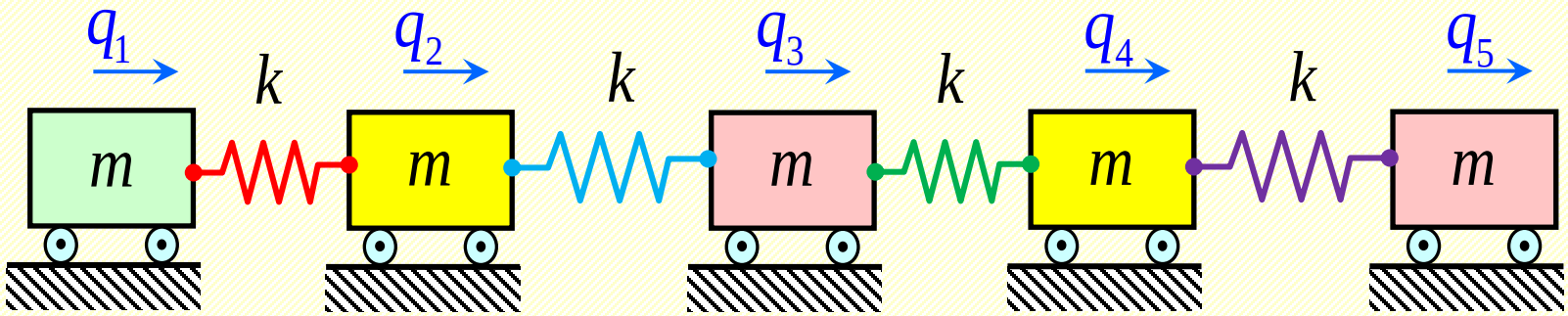


Assembling GLOBAL STIFFNESS MATRIX: ALL FE CONTRIBUTIONS ARE ADDED:

$$[K]=[K_{12}]+[K_{23}]+[K_{34}]+[K_{45}]$$

$$[K]=\begin{bmatrix} k & -k & 0 & 0 & 0 \\ -k & 2k & -k & 0 & 0 \\ 0 & -k & 2k & -k & 0 \\ 0 & 0 & -k & 2k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix}$$

MATLAB Script




```
% SUPPLEMENTARY PROBLEM : ALL MASSES ARE FREE !!!
% Designed by Prof P.M.Trivailo (C)2020
N=4; m=7; k=500;    % N is a number of springs (= number of FEs)

MM=eye(N+1)*m;      % GLOBAL MASS MATRIX - ready
ke=[k -k; -k k];    % ELEMENT STIFFNESS MATRIX
KK=zeros(N+1,N+1); % PROVISION FOR GLOBAL STIFFNESS MATRIX
for ii=1:N
    KK(ii:ii+1, ii:ii+1)= KK(ii:ii+1, ii:ii+1)+ke;
end

[U2,D2]=eig(KK,MM); % SOLVING EIGENVALUE PROBLEM
disp(sprintf('w_1  = %7.4f [rad/s]',sqrt(D2(1,1))))
commandwindow
```

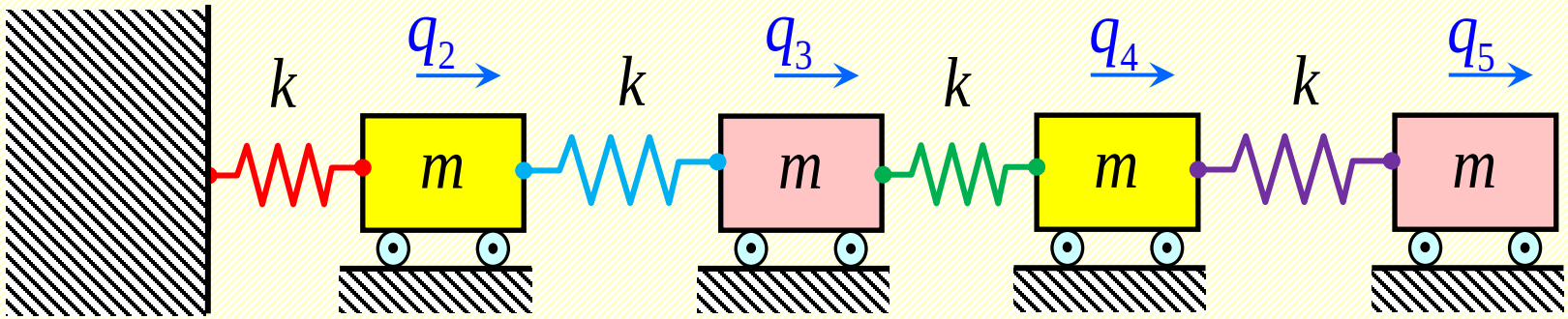
First frequency is ZERO, because system admits rigid body mode

Command Window

New to MATLAB? See resources for Getting Started.

w_1 = 0.0000 [rad/s]

MATLAB Script




```
%% CONSTRAINED SYSTEM
% Designed by Prof P.M.Trivailo (C)2020
N=4; m=7; k=500;

MM=eye(N+1)*m;       % GLOBAL MASS MATRIX - ready
ke=[k -k; -k k];     % ELEMENT STIFFNESS MATRIX
KK=zeros(N+1,N+1); % PROVISION FOR GLOBAL STIFFNESS MATRIX
for ii=1:N
    KK(ii:ii+1, ii:ii+1)= KK(ii:ii+1, ii:ii+1)+ke;
end
KK(1,:)=[]; KK(:,1)=[]; MM(1,:)=[]; MM(:,1)=[]; % ADD CONSTRAINING WALL!

[U,D]=eig(KK,MM); % SOLVING EIGENVALUE PROBLEM
disp(sprintf('w_1   = %7.4f [rad/s]',sqrt(D(1,1))))
commandwindow
```

First frequency is NOT ZERO, because system IS CONSTRAINED

Command Window

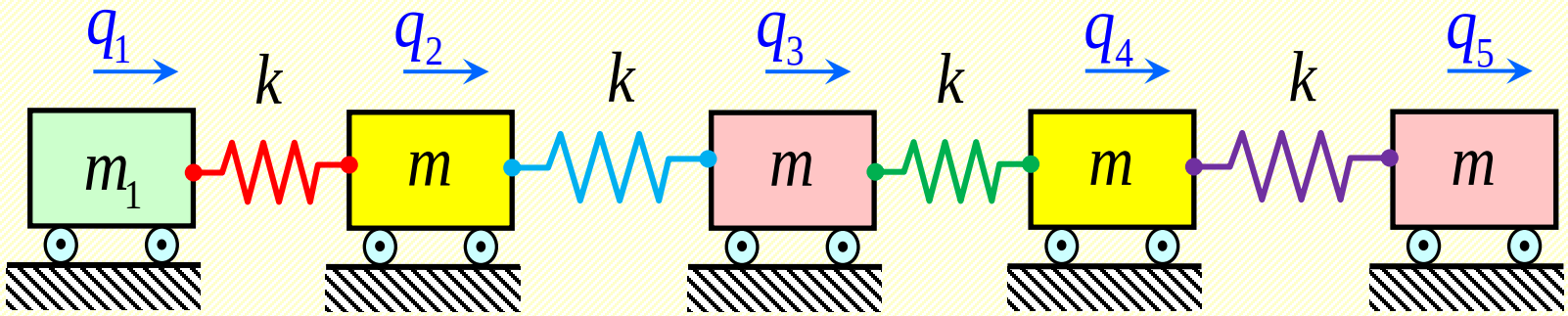
New to MATLAB? See resources for Getting Started.

```
w_1   =   2.9352 [rad/s]
```

Method-2

Modelling of a constraining Wall with a Very Heavy Mass

MATLAB Script




```
% SOLUTION-2 : MODELLING WALL CONSTRAINT AS A HEAVY EXTRA ADDED MASS !!!
% Designed by Prof P.M.Trivailo (C)2020
N=4; k=500;
K=zeros(N+1,N+1); ke=[1 -1; -1 1]*k;

for i=1:N
    K(i:i+1, i:i+1) = K(i:i+1, i:i+1) + ke;
end
M=eye(N+1)*5;

M(1,1)=10^12; % MAKING MASS-1 (m1) VERY HEAVY!!! TO REPLICATE WALL CONSTRAINT

[U,D]=eig(K,M);
disp(sprintf('w_1  = %7.4f [rad/s]',sqrt(D(1,1))))
disp(sprintf('w_2  = %7.4f [rad/s]',sqrt(D(2,2))))
commandwindow
```

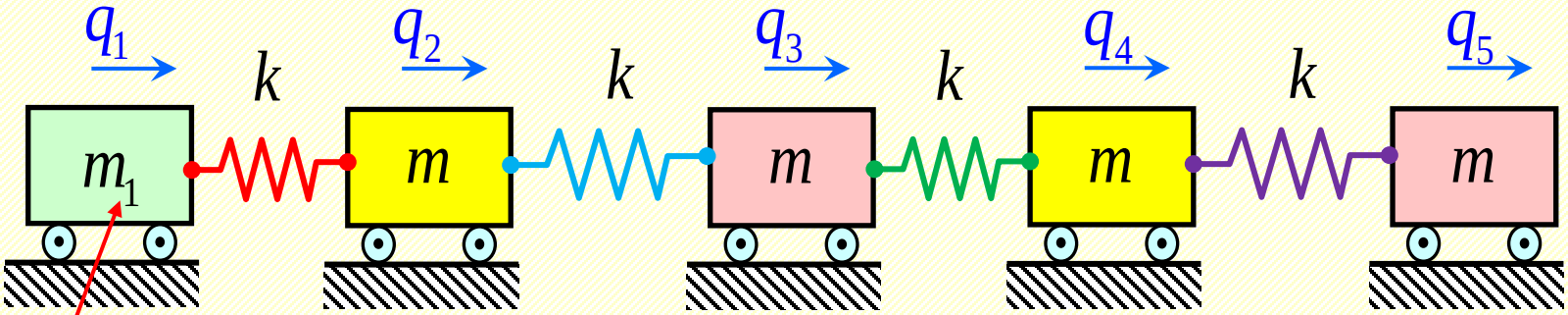
Command Window

New to MATLAB? See resources for Getting Started.

```
w_1  =  0.0000 [rad/s]
w_2  =  2.9352 [rad/s]
```

First frequency is ZERO, because system admits rigid body mode

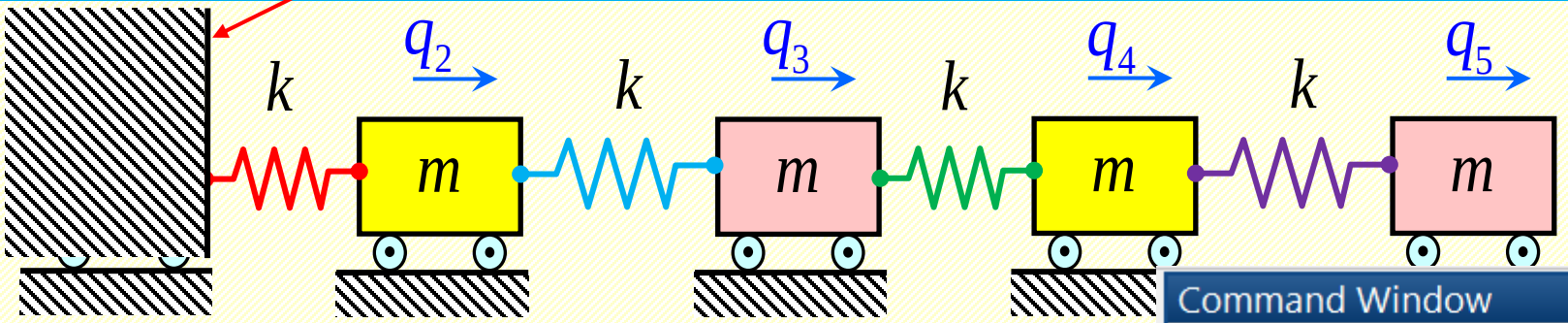
Interpretation of Results



If **mass-1 is very heavy**, it can replicate a WALL CONSTRAINT. However, with this technique, we have **effective reduction by 1** of the DOFs in the unconstrained system.

Hint for A-1:

Therefore, in order to get 1st elastic frequency, we need to take second frequency of the FREE system. In order to get 2nd elastic frequency, we need to take third frequency of the FREE system, etc.



Command Window

New to MATLAB? See resources for Getting Started.

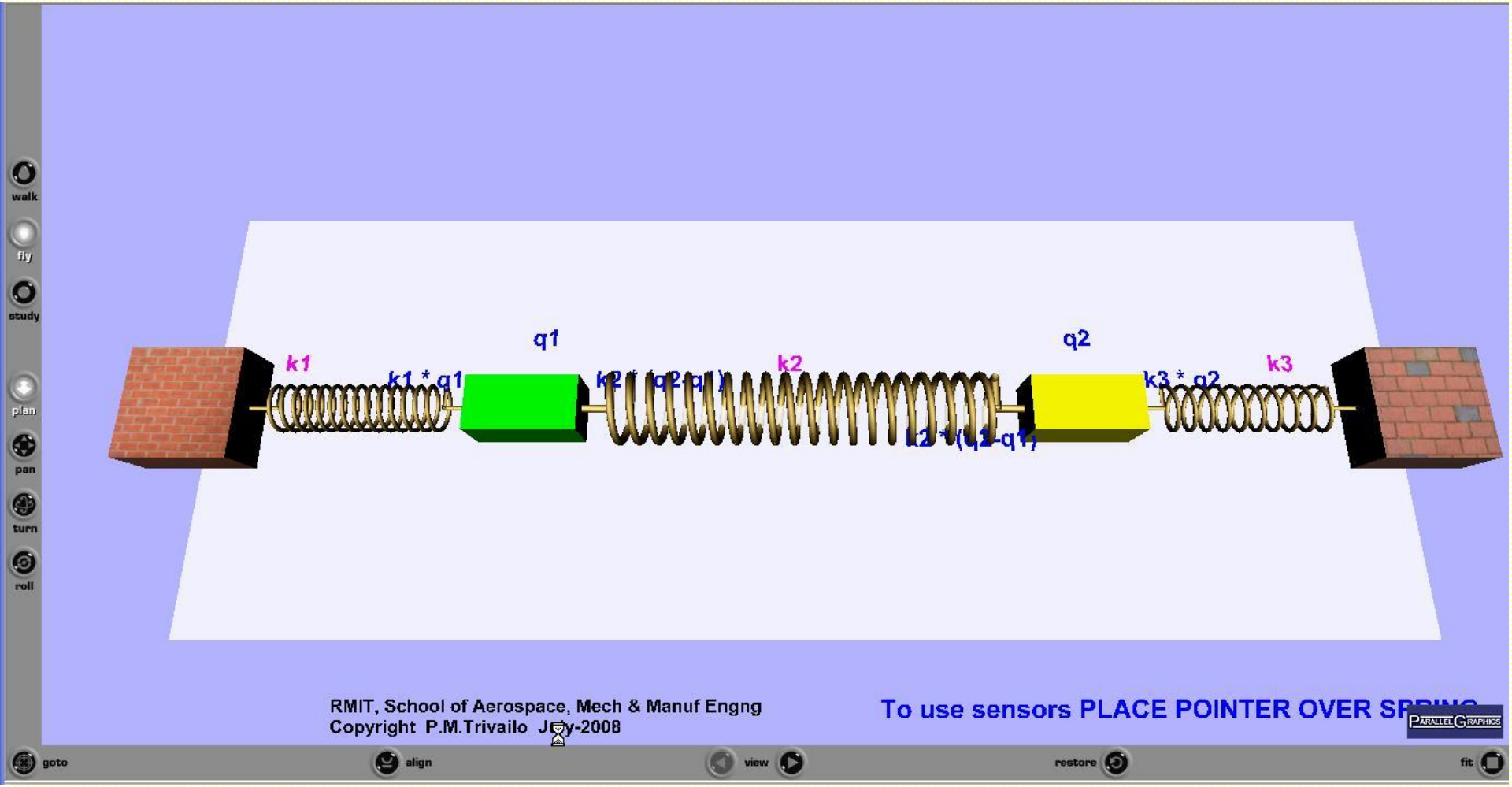
```
w_1   =   0.0000 [rad/s]
w_2   =   2.9352 [rad/s]
```

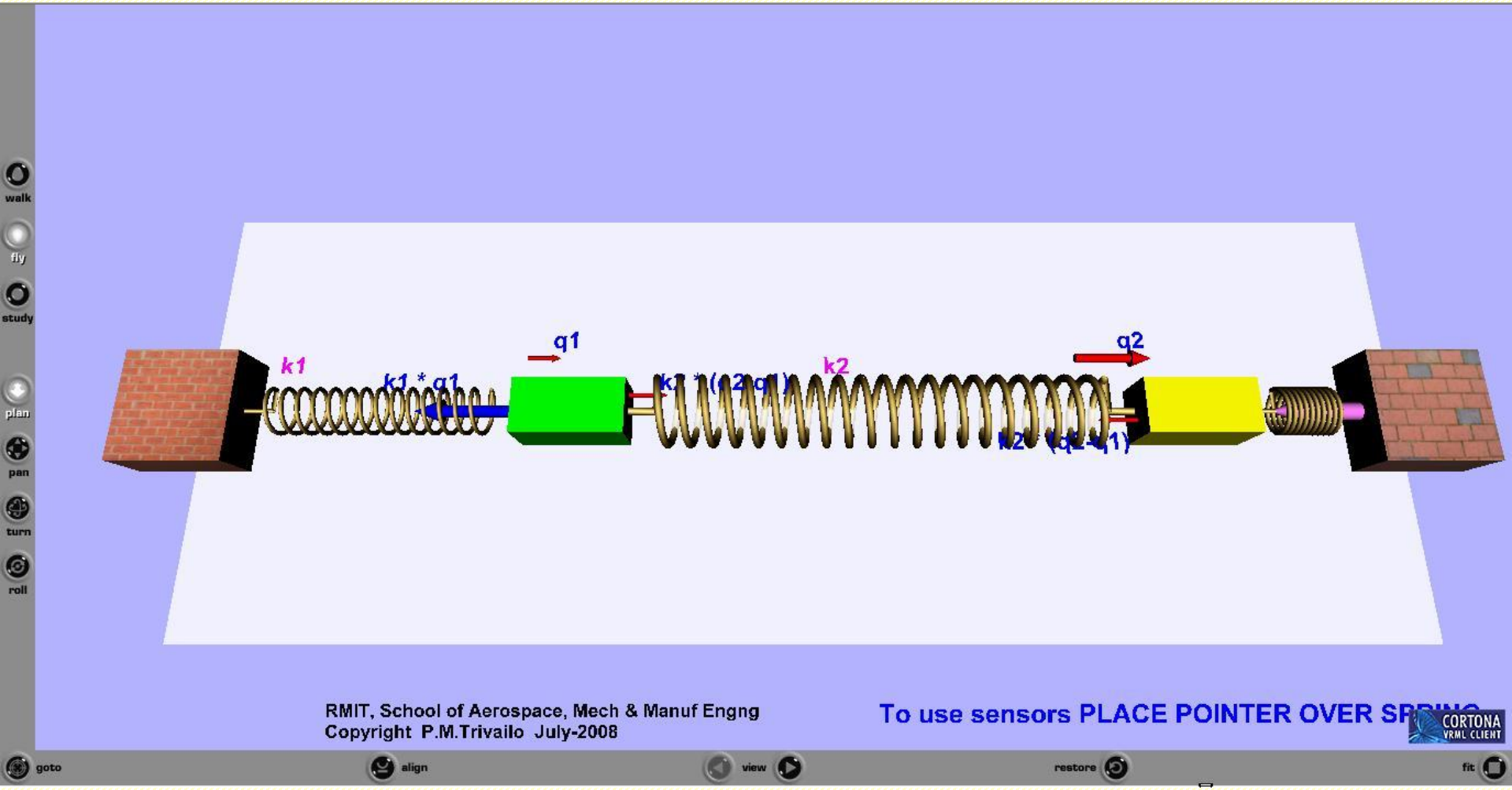
First elastic frequency for the CONSTRAINED system

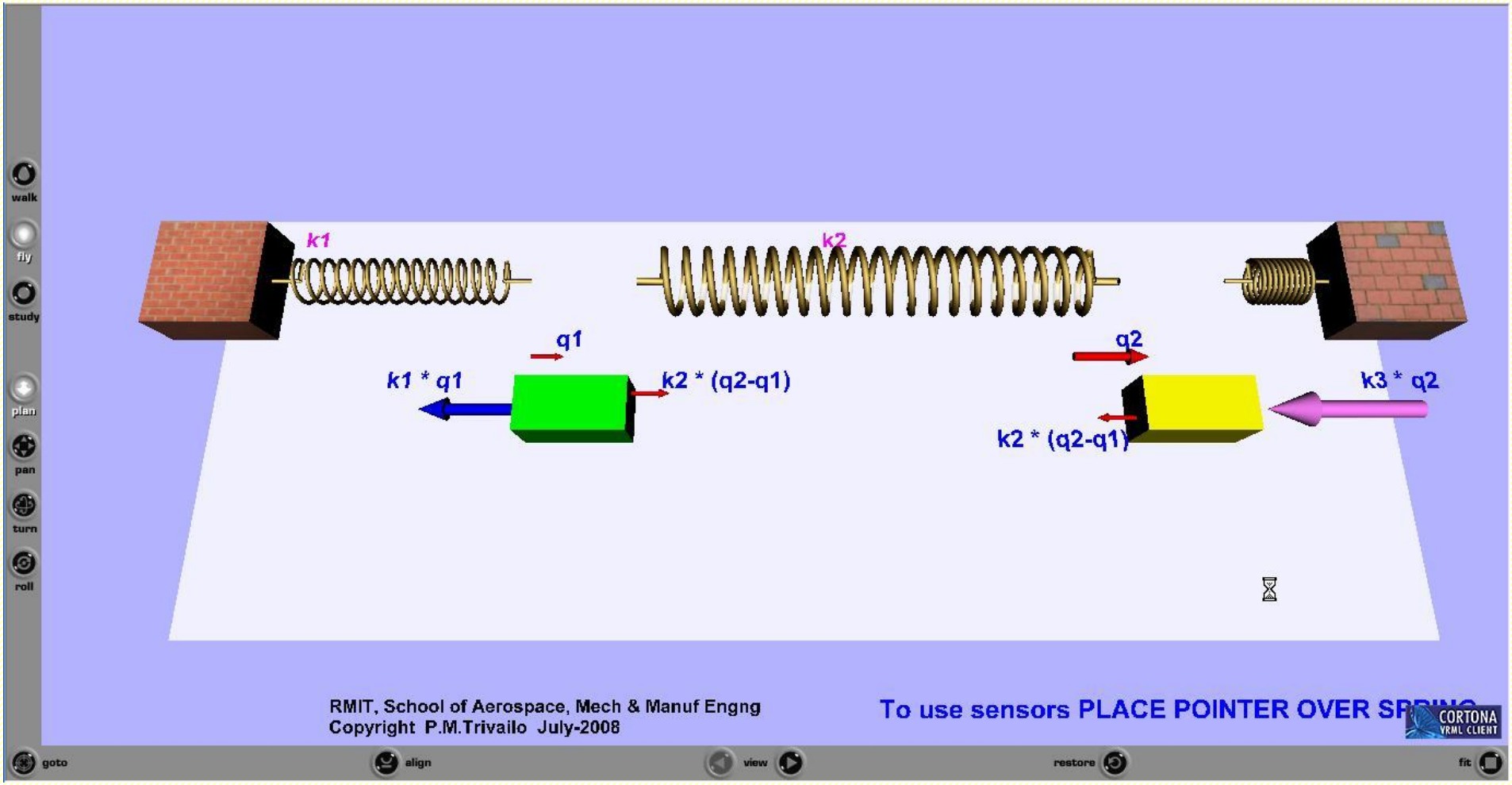
Multi-DOF mass-spring systems:

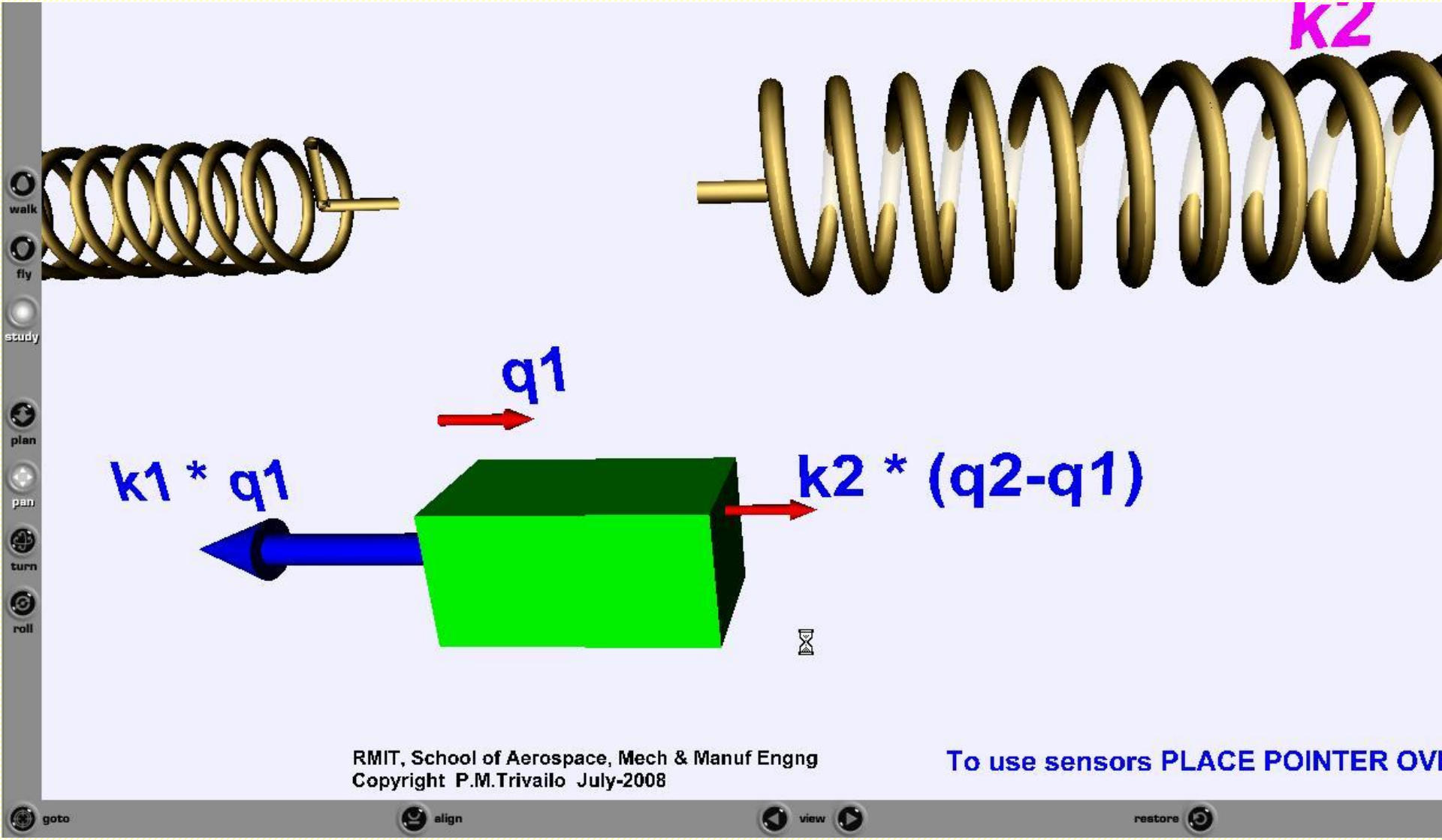
Derivation of EOM using Newton's Second Law & Free-Body Diagrams (FBDs).

Demo in Virtual Reality

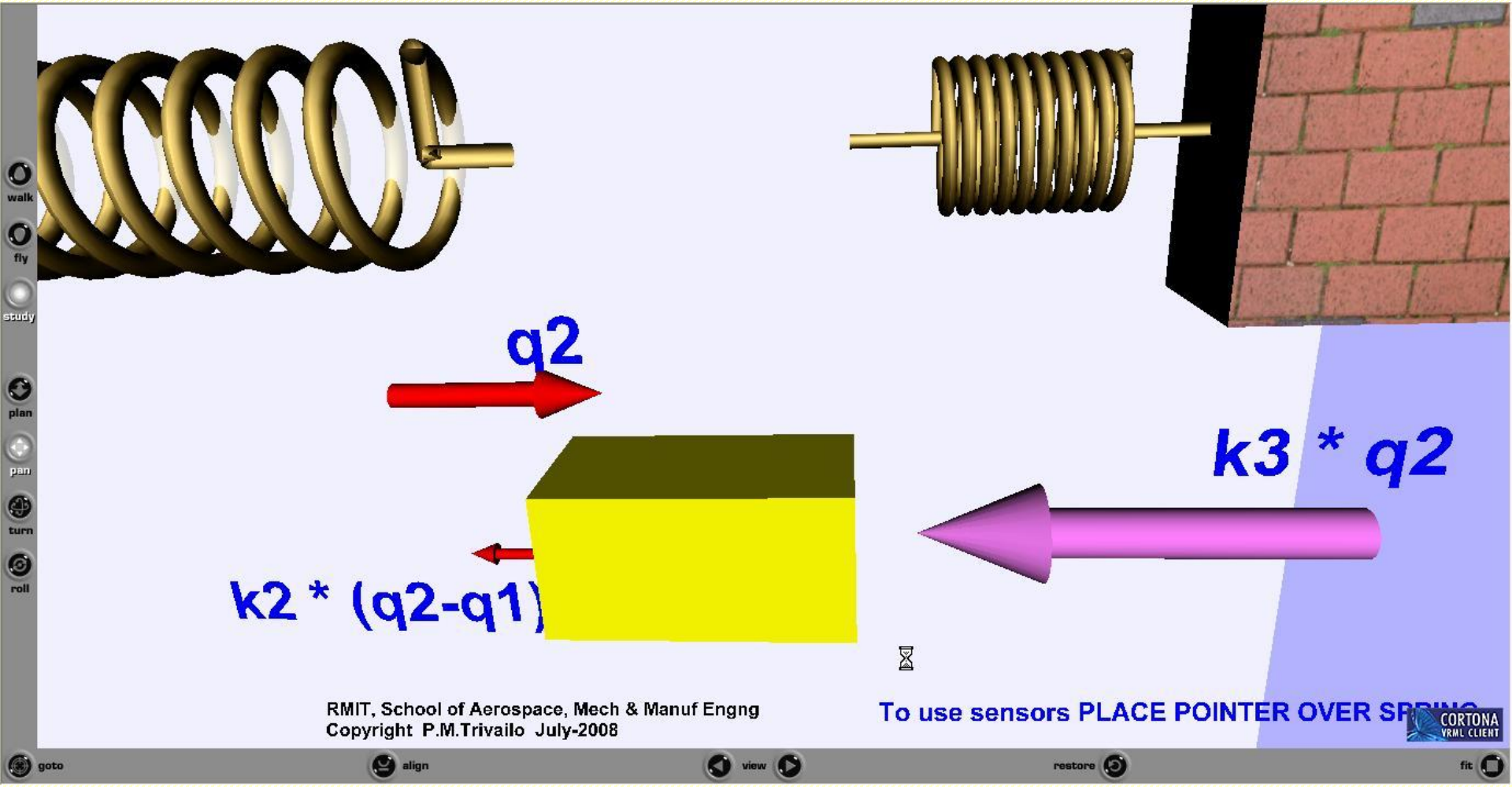



walk
fly
study
plan
pan
turn
roll
k1
k1 * q1
q1
k2
q2
RMIT, School of Aerospace, Mech & Manuf Engng
Copyright P.M.Trivailo July-2008
To use sensors PLACE POINTER OVER SP
CORTONA VRML CLIENT
goto
align
view
restore
fit

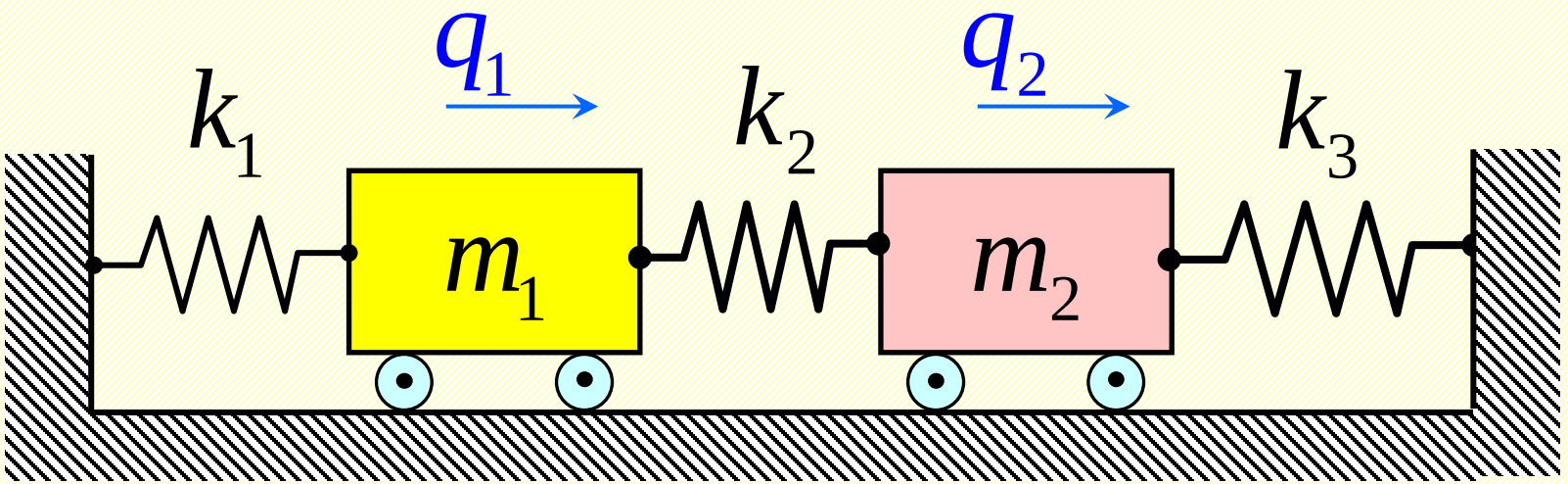

walk
fly
study
plan
pan
turn
roll
k1
k2
q1
k1 * q1
k2 * (q2-q1)
q2
k3 * q2
k2 * (q2-q1)
RMIT, School of Aerospace, Mech & Manuf Engng
Copyright P.M.Trivailo July-2008
To use sensors PLACE POINTER OVER SP
CORTONA VRML CLIENT
goto
align
view
restore
fit



FBD-1: $m_1 \ddot{q}_1 = -k_1 q_1 + k_2 (q_2 - q_1)$



FBD-2: $m_2 \ddot{q}_2 = -k_2(q_2 - q_1) - k_3 q_2$



$$\begin{cases} m_1\ddot{q}_1 + (k_1 + k_2)q_1 - k_2 q_2 = 0 \\ m_2\ddot{q}_2 - k_2 q_1 + (k_2 + k_3)q_2 = 0 \end{cases}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}; \quad [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

EQUATIONS IN STATE-SPACE FORM

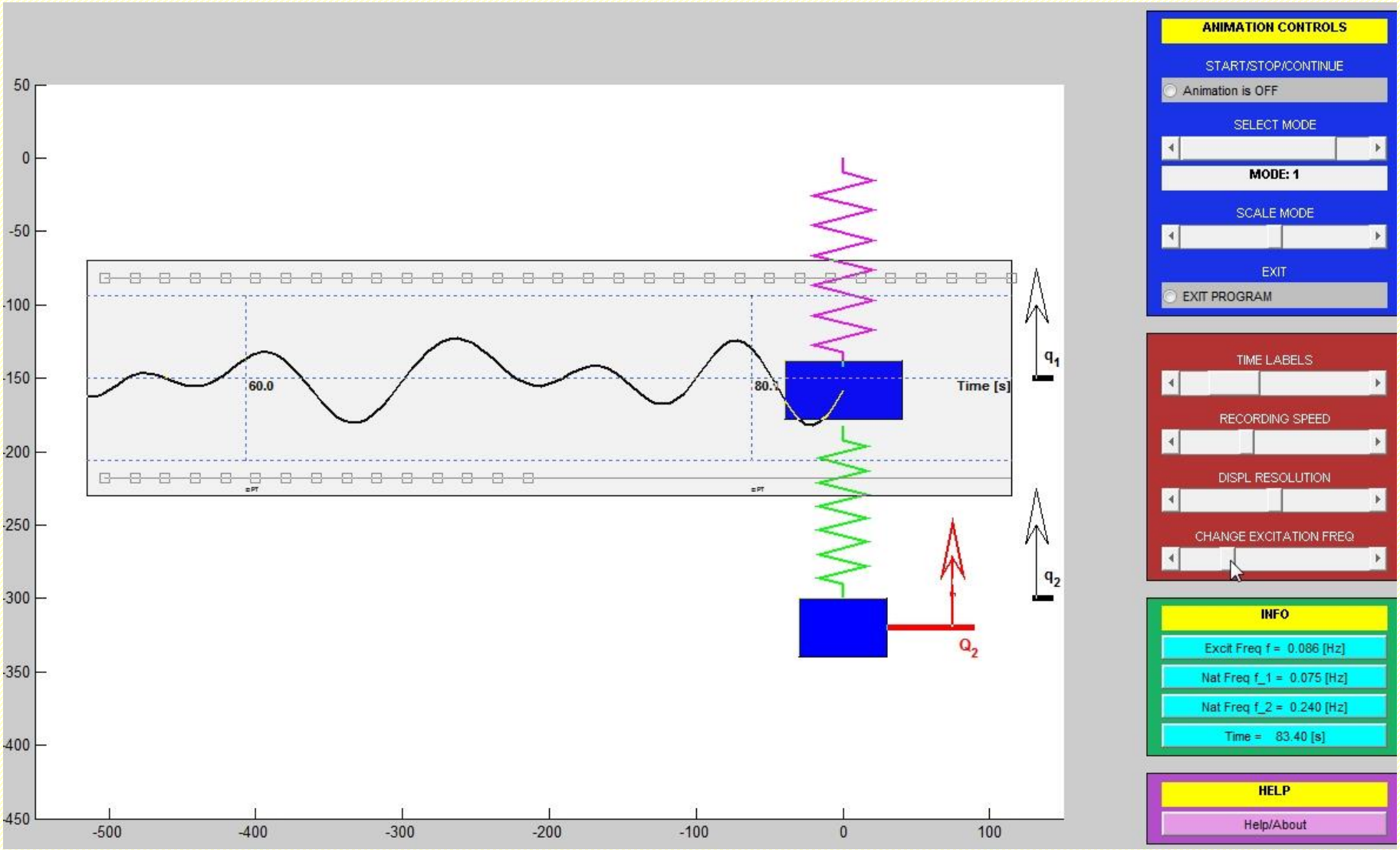
$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

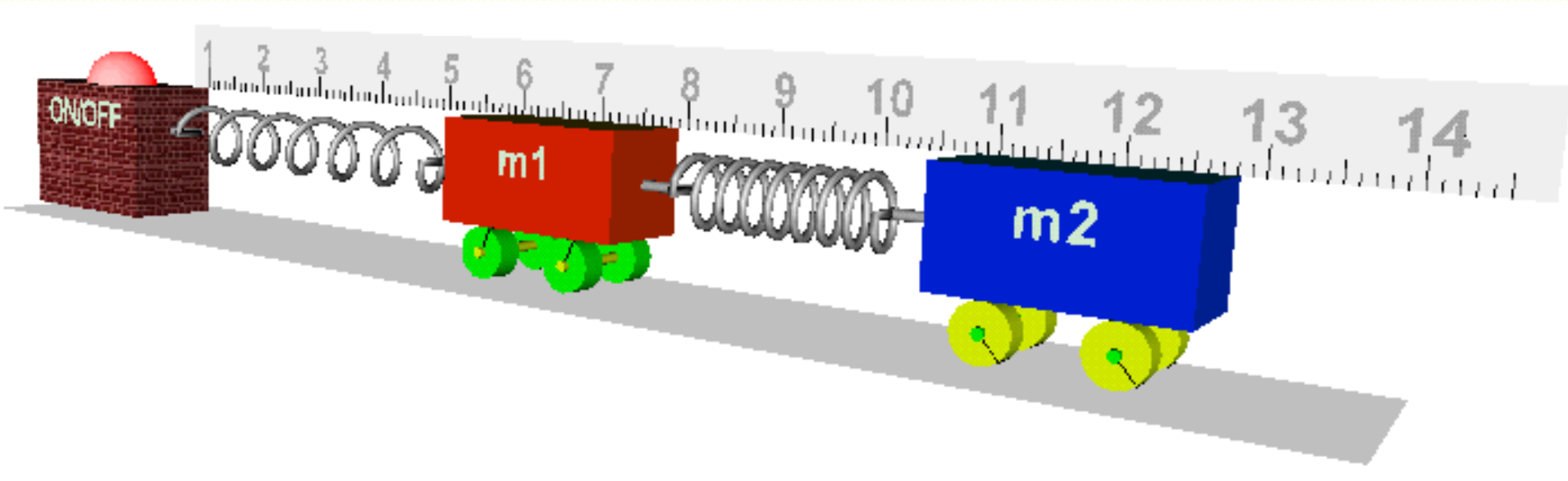
$$A = \left[\begin{array}{c|c} [0] & [1] \\ \hline -[m]^{-1} * [k] & -[m]^{-1} * [c] \end{array}\right]; \quad B = \left[\begin{array}{c} [0] \\ \hline [m]^{-1} \end{array}\right]$$

SOLVING RESPONSES FOR MULTI-DOF SYSTEMS:

MATLAB and ODExxx


ANIMATION CONTROLS
START/STOP/CONTINUE
Animation is OFF
SELECT MODE
MODE: 1
SCALE MODE
EXIT
EXIT PROGRAM
TIME LABELS
RECORDING SPEED
DISPL RESOLUTION
CHANGE EXCITATION FREQ
INFO
Excit Freq f = 0.086 [Hz]
Nat Freq f_1 = 0.075 [Hz]
Nat Freq f_2 = 0.240 [Hz]
Time = 83.40 [s]
HELP
Help/About
60.0
Time [s]
q1
q2
Q2
50
0
-50
-100
-150
-200
-250
-300
-350
-400
-450
-500
-400
-300
-200
-100
0
100

Virtual Laboratory



Virtual Laboratory

EIGENVALUE PROBLEM FOR MULTI-DOF SYSTEMS:

SOLVING EP, USING MATLAB

EIGENVALUE PROBLEM

Matrix Equations of Motion:

Equation of Motion of a Damped n-DOF System

$$[m]\{\ddot{q}(t)\} + [c]\{\dot{q}(t)\} + [k]\{q(t)\} = \{Q(t)\}$$

Equation of Free Motion of an Undamped n-DOF System

$$[m]\{\ddot{q}(t)\} + [k]\{q(t)\} = \{0\}$$

EIGENVALUE PROBLEM FORMULATION

$$[m]\{\ddot{q}(t)\} + [k]\{q(t)\} = \{0\}$$

$$\{q(t)\} = e^{i\omega t}\{u\}$$

Eigenvalue Problem for an n-DOF System:

$$[k]\{u\} = \omega^2[m]\{u\}$$

or

$$\left([k] - \omega^2[m]\right)\{u\} = \{0\}. \qquad (7)$$

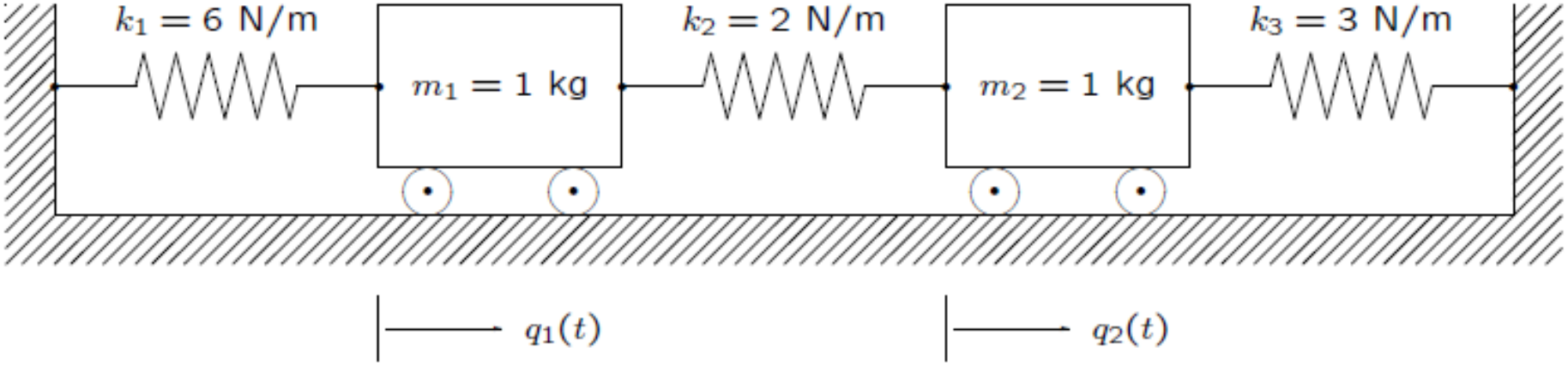
CHARACTERISTIC EQUATION

The problem of determining the values of the parameter ω^2 for which (7) possesses nontrivial solutions is known as eigenvalue problem.

Characteristic Equation for a n-DOF System:

$$\Delta(\omega^2) = |[k] - \omega^2[m]| = 0. \tag{8}$$

Example: 2-DOF MASS-SPRING SYSTEM



$$[m] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad [k] = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}; \quad \{q(t)\} = e^{i\omega t}\{u\}$$

$$-\omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$\begin{bmatrix} 8-\omega^2 & -2 \\ -2 & 5-\omega^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$det\left(\begin{bmatrix} 8-\omega^2 & -2 \\ -2 & 5-\omega^2 \end{bmatrix}\right) = 0.$$

$$\left(\omega^2\right)^2 - 13\left(\omega^2\right) + 36 = 0.$$

$$\lambda_1 = [\omega^2]_1 = 4 \ \left(\frac{rad}{s}\right)^2; \qquad \lambda_2 = [\omega^2]_2 = 9 \ \left(\frac{rad}{s}\right)^2$$

$$\lambda_1 = \omega_1^2 = 4 \left(\frac{rad}{s}\right)^2.$$

$$\begin{bmatrix} 8-4 & -2 \\ -2 & 5-4 \end{bmatrix} \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$4u_{11} - 2u_{21} = 0$$

$$\begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \begin{Bmatrix} 3 \\ 6 \end{Bmatrix}, \ldots$$

$$\lambda_2 = \omega_2^2 = 9 \left(\frac{rad}{s}\right)^2.$$

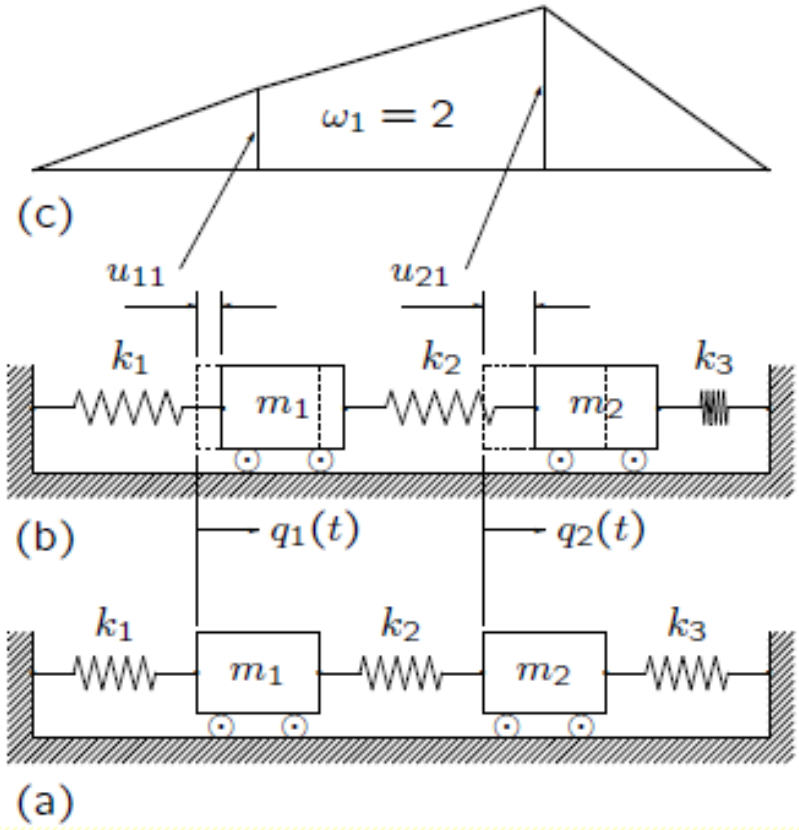
$$\begin{bmatrix} 8-9 & -2 \\ -2 & 5-9 \end{bmatrix} \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

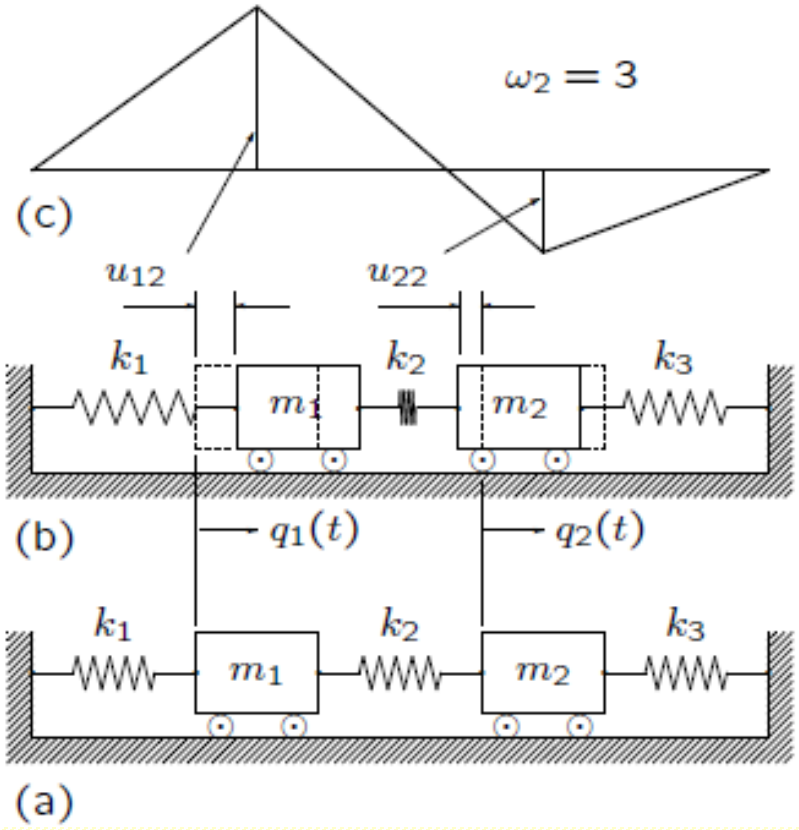
$$-u_{12} - 2u_{22} = 0$$

$$\begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}, \begin{Bmatrix} 4 \\ -2 \end{Bmatrix}, \ldots$$

$$\begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \begin{Bmatrix} 2 \\ 4 \end{Bmatrix}, \ldots$$



$$\begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} = \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}, \begin{Bmatrix} 4 \\ -2 \end{Bmatrix}, \ldots$$



Solve Eigenvalue Problem using MATLAB

MATLAB IN VIBRATION ANALYSIS

MATLAB Command to Calculate

Eigenvalues and Eigenvectors of the System:

$$[U, D] = eig(\ inv(m)\ *\ k\ )$$

or

$$[U, D] = eig(\ k\ ,\ m\ )$$

Normalization Scheme for the Natural Modes:

$$\{u\}_r^T[m]\{u\}_r = 1, \qquad (r = 1, 2, \ldots, n).$$

Consequence from Normalization of the Natural Modes:

$$\{u\}_r^T[k]\{u\}_r = \omega_r^2, \qquad (r = 1, 2, \ldots, n).$$

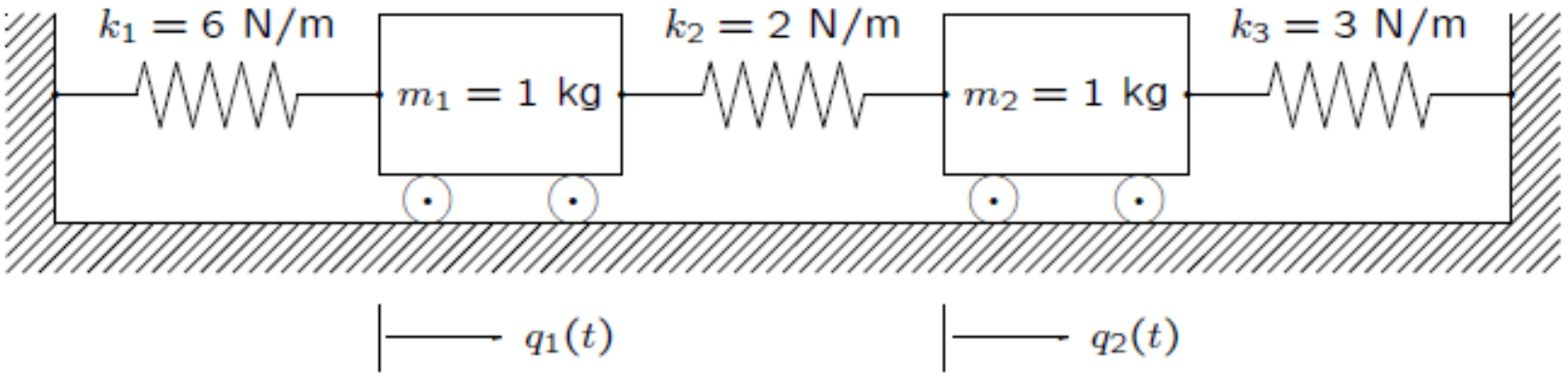
NORMALISATION OF EIGENVECTORS USING MATLAB

To calculate the Normalised Modal Matrix

use the following MATLAB command:

$$Un = U/sqrt(U' * m * U).$$

ORTHOGONALITY: 2-DOFS EXAMPLE



$$[m] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad [k] = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix};$$

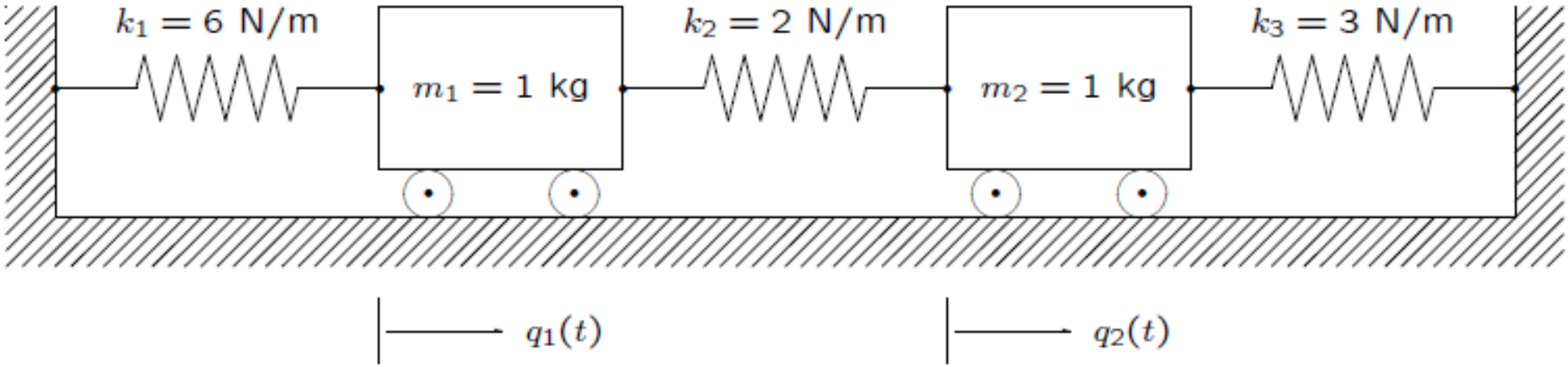
$$\{u\}_1 = \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}, \qquad \{u\}_2 = \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} = \begin{Bmatrix} 2 \\ -1 \end{Bmatrix},$$

$$\{u\}_1^T \; [m] \; \{u\}_2 =$$

$$= \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix}^T \; [m] \; \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} =$$

$$= \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 2 \\ -1 \end{Bmatrix} = 0$$

Numerical Example

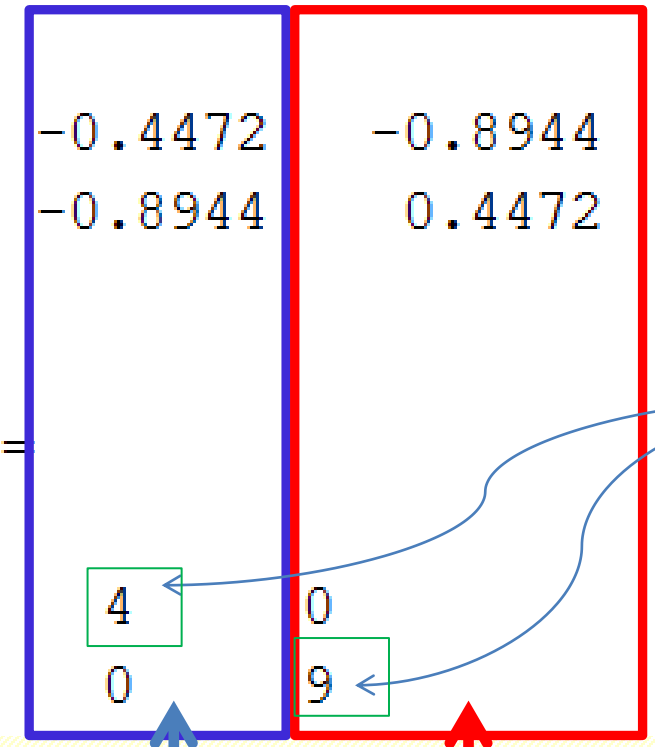


$$[m] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad [k] = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix};$$

```
m1=1; m2=1; k1=6; k2=2; k3=3;
m=[1 0; 0 1];
k=[k1+k2 -k2; -k2 k2+k3];
[U,D] = eig(k,m);
```

Command Window

```
>> m1=1; m2=1; k1=6; k2=2; k3=3;
>> m=[m1 0; 0 m2]; k=[k1+k2 -k2; -k2 k2+k3];
>> [U,D]=eig(k,m)

U =

   -0.4472   -0.8944
   -0.8944    0.4472


D =

    4     0
    0     9
```



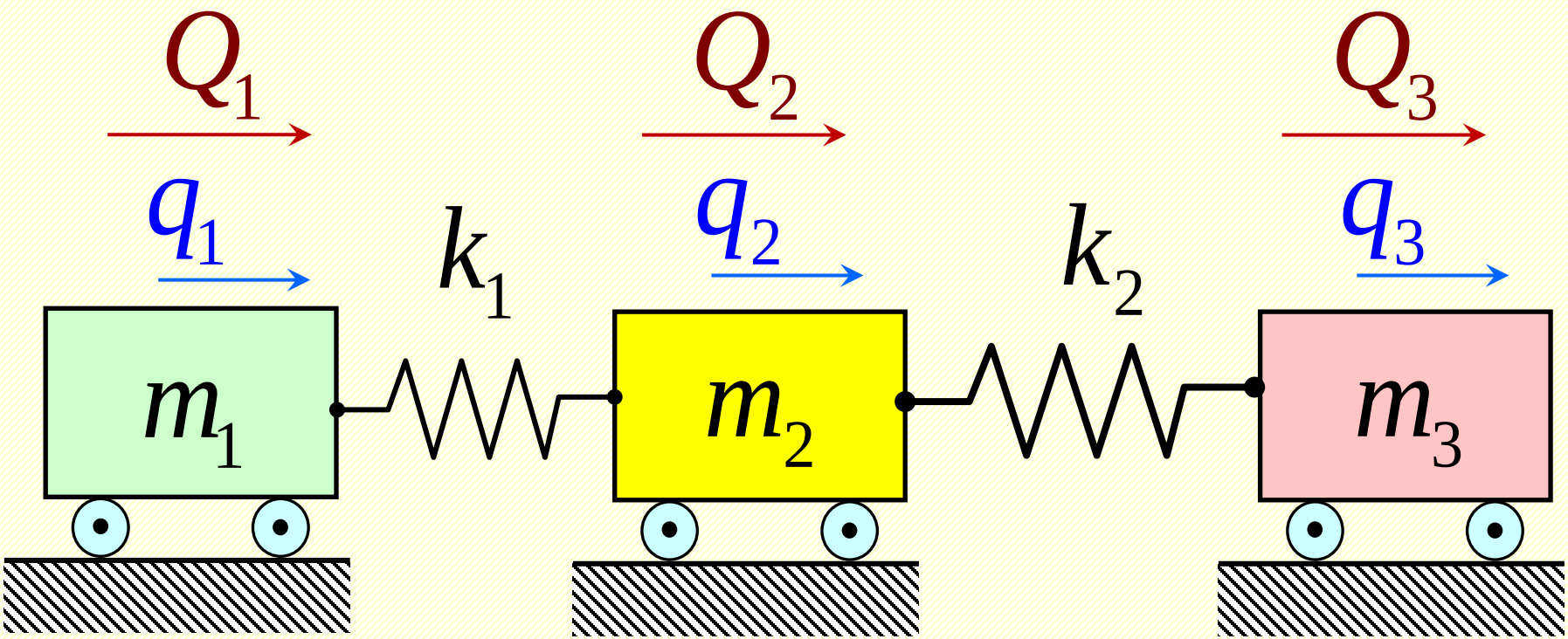
EIGENVECTORS (or 'MODE SHAPES')

EIGENVALUES

MODE-1

MODE-2

Example: 3-DOF System



$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

```
%% MIET2389 Eigenvalue Problem Example
close('all'); clear; clc;
m1=4; m2=9; m3=1;
k1=1000; k2=500;
m=[m1 0 0; 0 m2 0; 0 0 m3];
k=[k1 -k1 0; -k1 k1+k2 -k2; 0 -k2 k2];
[U,D]=eig(k,m);
fprintf('====== Modal Martix [U]: =========\n')
fprintf(' %9.6f %9.6f %9.6f \n',U(1,:));
fprintf(' %9.6f %9.6f %9.6f \n',U(2,:));
fprintf(' %9.6f %9.6f %9.6f \n',U(3,:));
fprintf('====== Natural Frequencies [rad/s] : =========\n')
fprintf('w1= %9.6f [rad/s]; w2= %9.6f [rad/s]; w3= %9.6f [rad/s]\n',sqrt(diag(D)));
fprintf('====== Natural Frequencies [Hz] : ========= \n')
fprintf('f1= %9.6f [Hz];  f2= %9.6f [Hz];  f2= %9.6f [Hz] \n',sqrt(diag(D))/2/pi);
```

Command Window

```
====== Modal Martix [U]: =========
 -0.267261  0.408248  0.109109
 -0.267261 -0.136083 -0.145479
 -0.267261 -0.408248  0.872872
====== Natural Frequencies [rad/s] : =========
w1=  0.000000 [rad/s]; w2= 18.257419 [rad/s]; w3= 24.152295 [rad/s]
====== Natural Frequencies [Hz] : =========
f1=  0.000000 [Hz];  f2=  2.905758 [Hz];  f2=  3.843957 [Hz]
```

Note: **'zero' natural frequency** corresponds to the **'rigid body mode'**

EQUATIONS IN STATE-SPACE FORM

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

EXAMPLES (Class Work)

Exploring “interp1” Command

```
%% MY FIRST EXAMPLE IN MATLAB
% Programmed by Prof P.M.Trivailo
%
%

%% Step-1: Enter the data
t=[0 1 2 3 4 10];
F=[0 0 1 1 0  0];

%% Step-2: Plot the data
plot(t,F);
% alternatively use
% gg1=line('XData',t,'YData', F,'Color',[0 0 1], 'LineWidth',3)
axis
xlabel('Time [s]')
ylabel('Force [N]')

```

```
%% Step-3: Previewing of the Plot and Adjusting the Axes
axis([0 max(t) 0 2])
grid


%% Step-4: Draw a Dot
gg2=line('XData',0,'YData', 0,'Color',[1 0 0], 'MarkerSize',20,'Marker','.')

%% Step-5 & 6: Illustrating the 'interp1' command;
% Illustrating the cells and incremental feature
% Enter increment (for example 0.1) in the '-' window
% for the cells and click '+'
t_interp=1.75;
F_interp = interp1(t,F,t_interp)
set(gg2, 'XData',t_interp,'YData', F_interp);
axis([0 5 0 1.2])
```

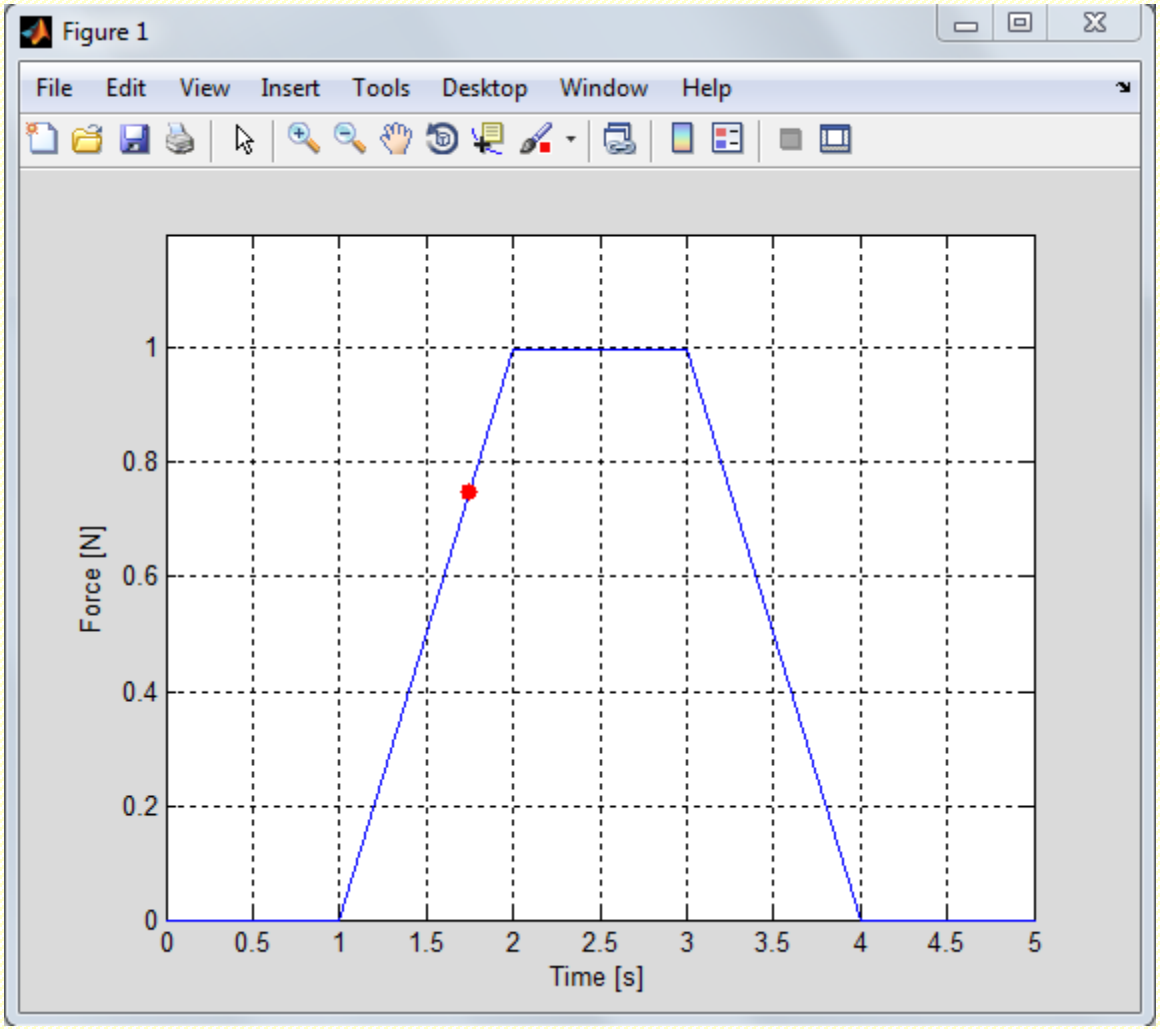
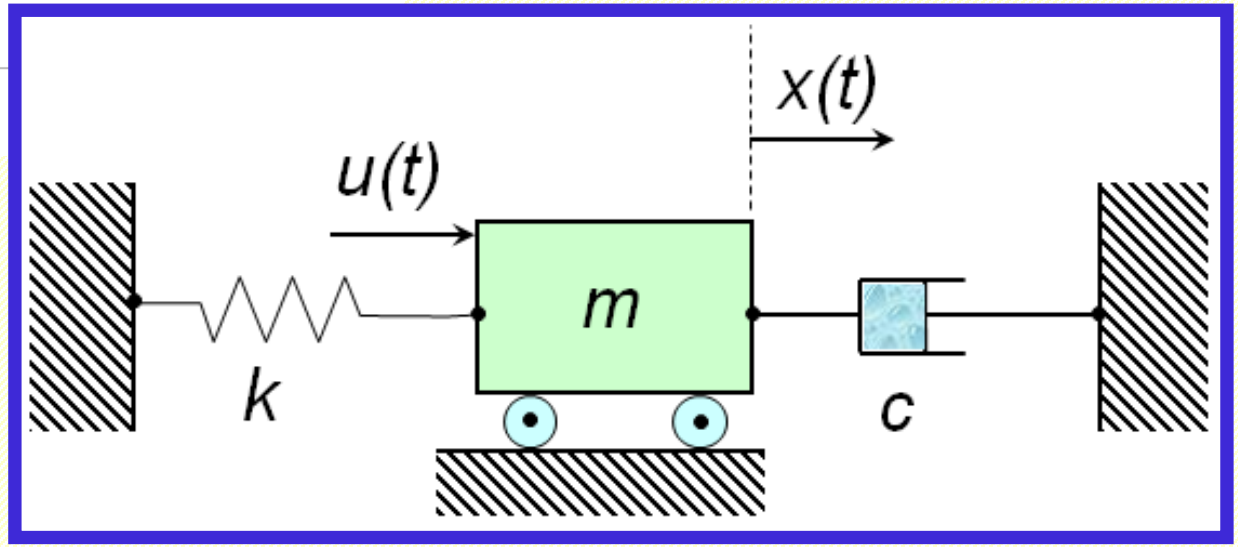


Figure 1
File Edit View Insert Tools Desktop Window Help
Force [N]
Time [s]
0
0.2
0.4
0.6
0.8
1
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5

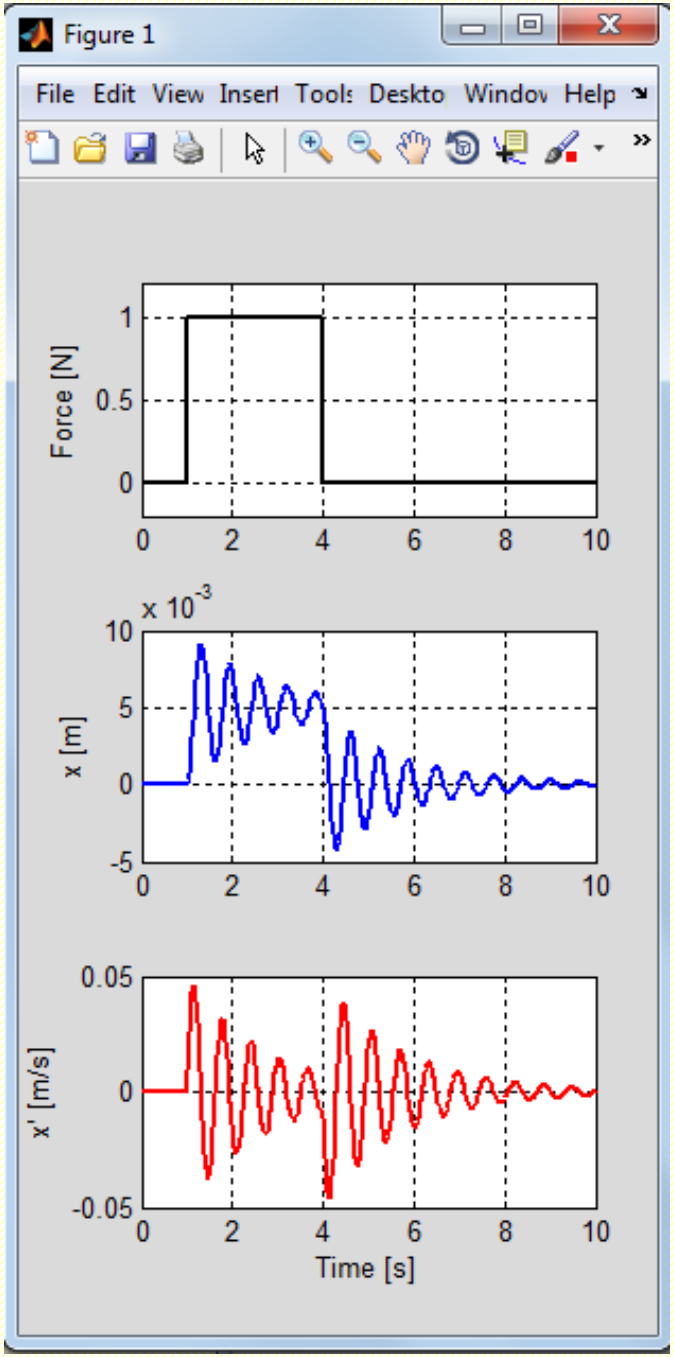
FORCE RESPONSE: 1-DOFS

```
%% 1-DOF SYSTEM: RESPONSE
% Solving Differential Eqs of Motion using MATLAB
% Programmed by Prof P.M.Trivailo
%
%
%
global tt FF A B C D

%% Enter System Characteristics
m = 2; k = 200;
c = 2.3;
%c = 0;

%% Enter Initial Conditions (ICs)
q0 = [0]; q0_dot = [0];

%% Establish the IC vector
x0 = [q0 ; q0_dot];
```

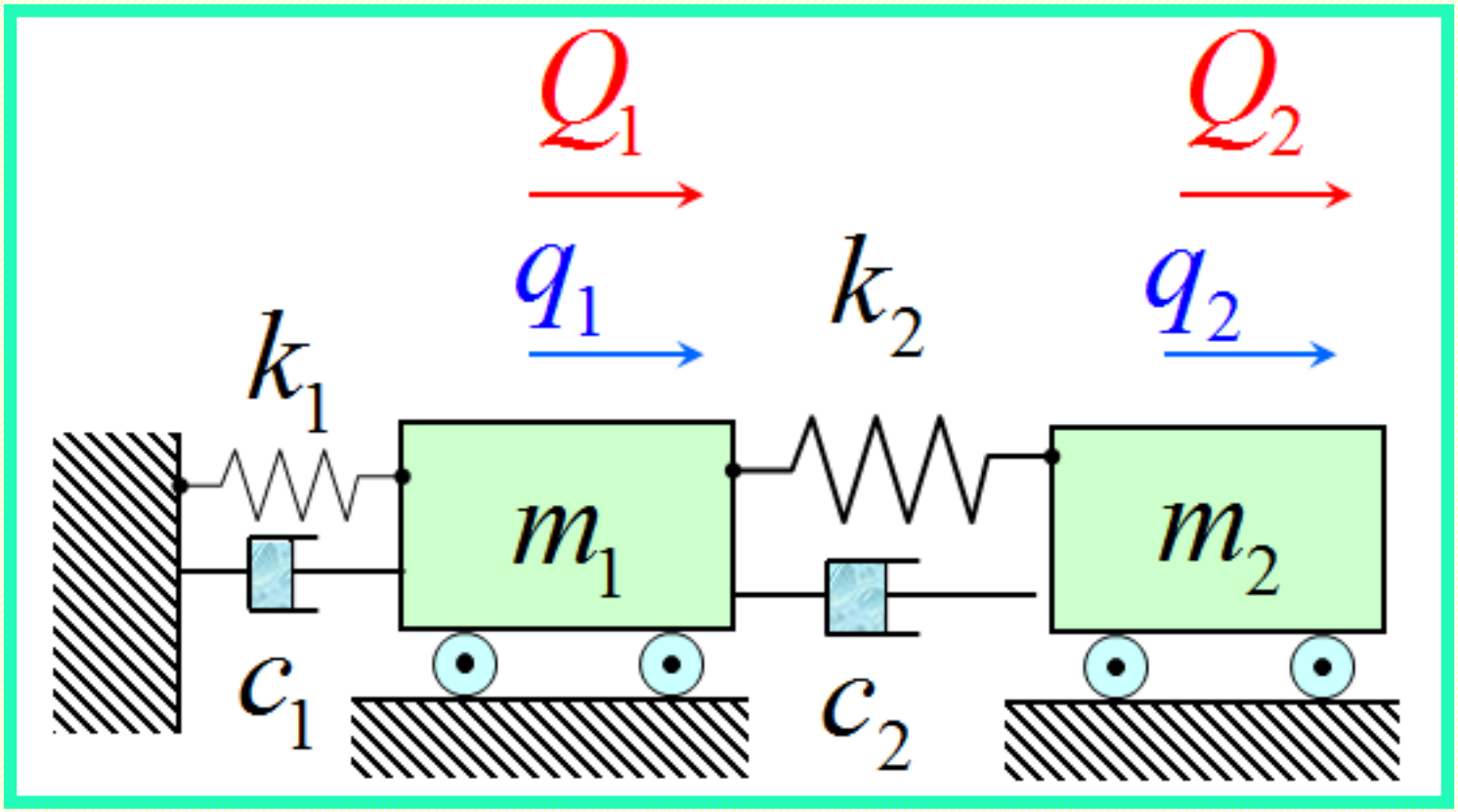


```
19
20      %% Specify the EXTERNAL Excitation Force
21      %tt=[0 1 2 3 4 10];
22 -    tt=[0 1 1.001 3.999 4 10];
23 -    FF=[0 0 1 1 0  0]*1;
24
25      %% Create matrices for state-space formulation
26 -    dof=size(m,1);
27 -    A = [zeros(dof, dof) eye(dof, dof) ; -inv(m)*k -inv(m)*c];
28 -    B = [zeros(dof, dof) ; inv(m)];
29
30      % These matrices will be needed IF SIMULINK is used
31 -    C = eye(2*dof, 2*dof);
32 -    D = zeros(2*dof, dof);
33
34      %% CORE!!! SOLVING DIFFERENTIAL EQS USING ODE23 Integration Procedure
35 -    [t_out,x_out] = ode23('PMTode',[0 max(tt)],x0);
36
```

```
%% PLOTTING Results
% Plotting the excitation function first
subplot(3,1,1)
gg1=plot(tt,FF,'k');
axis([0 max(tt) -0.2  max(FF)+0.2])
%xlabel('Time [s]')
ylabel('Force [N]')
grid

% Plotting Response of the 1-DOF system
subplot(3,1,2)
gg2=plot(t_out,x_out(:,1),'b');
%xlabel('Time [s]')
ylabel('x [m]')
grid

subplot(3,1,3)
gg3=plot(t_out,x_out(:,2),'r');
set([gg1, gg2, gg3],'LineWidth',2)
xlabel('Time [s]')
ylabel('x'' [m/s]')
grid
```



FORCE RESPONSE: 2-DOFS



$$[m]=\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}; \quad [c]=\begin{bmatrix} c_1+c_2 & -c_1 \\ -c_2 & c_2 \end{bmatrix}; \quad [k]=\begin{bmatrix} k_1+k_2 & -k_1 \\ -k_2 & k_2 \end{bmatrix}$$

FORCE RESPONSE: 2-DOFS

```
%% 2-DOF SYSTEM: RESPONSE
% Solving Differential Eqs of Motion for 2DOFS using MATLAB
% Programmed by Prof P.M.Trivailo
%
%
%
global tt1 FF1 tt2 FF2 A B C D

%% Enter System Characteristics
m = 1e2*[1 0 ; 0 2];
k = 1e5*[4 -2 ; -2 2];
c = 1e2*[8 -4 ; -4 4];

%% Enter Initial Conditions (ICs)
q0 = [0; 0];
q0_dot = [0; 0];

%% Establish the IC vector
x0 = [q0 ; q0_dot];
```

```
20
21      %% Specify the EXTERNAL Excitation Force
22      %tt=[0 1 2 3 4 10];
23 -    tt1=[0 1 1.001 3.999 4 10];
24 -    FF1=[0 0 1 1 0  0]*1;
25 -    tt2=[0 5 6 7 10];
26 -    FF2=[0 0 2 0 0];
27
28      %% Create matrices for state-space formulation
29 -    dof=size(m,1);
30 -    A = [zeros(dof, dof) eye(dof, dof) ; -inv(m)*k -inv(m)*c];
31 -    B = [zeros(dof, dof) ; inv(m)];
32
33      % These matrices will be needed IF SIMULINK is used
34 -    C = eye(2*dof, 2*dof);
35 -    D = zeros(2*dof, dof);
36
37      %% CORE!!! SOLVING DIFFERENTIAL EQS USING ODE23 Integration Procedure
38 -    [t_out,x_out] = ode23('PMTode2',[0 max(tt1)],x0);
39
```

```
41      % Plotting the excitation function first
42 -    subplot(3,1,1)
43 -    gg1=plot(tt1,FF1,'b',tt2,FF2,'r--');
44 -    axis([0 max(tt2) -0.2  max(FF2)+0.2])
45 -    ylabel('Force [N]')
46 -    legend('Q_1','Q_2')
47 -    grid
48
49      % Plotting Response of the 2-DOF system
50 -    subplot(3,1,2)
51 -    gg2=plot(t_out,x_out(:,1),'b',t_out,x_out(:,2),'r--');
52 -    ylabel('x [m]')
53 -    legend('q_1','q_2')
54 -    grid
55
56 -    subplot(3,1,3)
57 -    gg3=plot(t_out,x_out(:,3),'b',t_out,x_out(:,4),'r--');
58 -    set([gg1, gg2, gg3],'LineWidth',2)
59 -    xlabel('Time [s]')
60 -    ylabel('x'' [m/s]')
61 -    legend('q''_1','q''_2')
62 -    grid
```

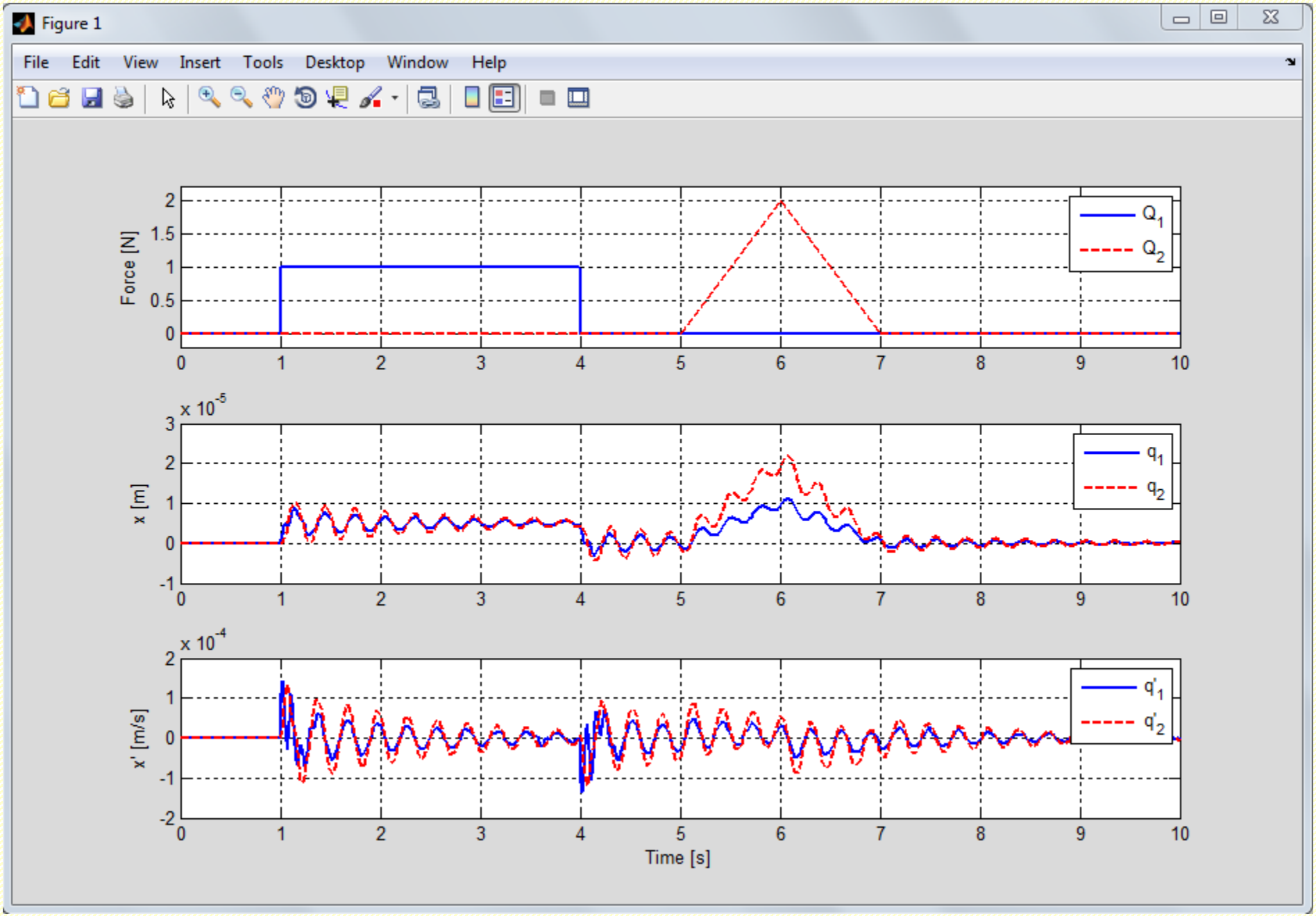


Figure 1
File Edit View Insert Tools Desktop Window Help
Force [N]
Q1
Q2
x 10-5
x [m]
q1
q2
x 10-4
x' [m/s]
q'1
q'2
Time [s]

Non-Linear Systems?
WHY BOTHER ?

Examples

MODELLING:

"CHAIN FOUNTAIN"

EXAMPLE OF A GREAT PRESENTATION

YouTube AU

Courtesy: Steve Mould, https://www.youtube.com/watch?v=_dQJBBklpQQ

Self siphoning beads

1,925,211 views

9.6K 157 SHARE

EXAMPLE OF A GREAT PRESENTATION

Courtesy: Steve Mould, https://www.youtube.com/watch?v=wmFi1xhz9OQ

Investigating the "Mould effect" | Steve Mould | TEDxNewcastle

73,236 views 1.8K 19 SHARE

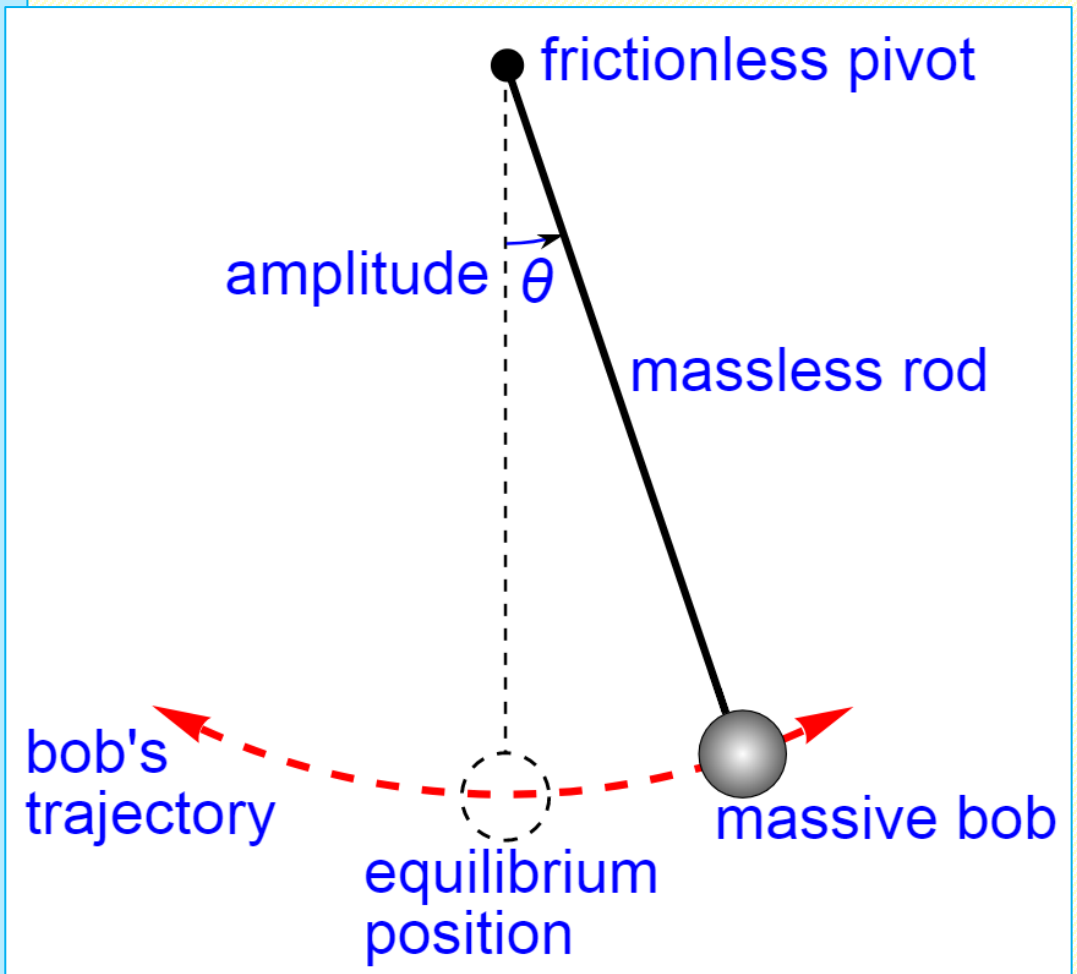
What is Pendulum?

A **pendulum** is a weight suspended from a pivot so that it can swing freely.

When a pendulum is displaced sideways from its resting, equilibrium position, it is subject to a restoring force due to gravity that will accelerate it back toward the equilibrium position.

When released, the restoring force acting on the pendulum's mass causes it to oscillate about the equilibrium position, swinging back and forth.

The time for one complete cycle, a left swing and a right swing, is called the period. The period depends on the length of the pendulum and also to a slight degree on the amplitude, the width of the pendulum's swing.



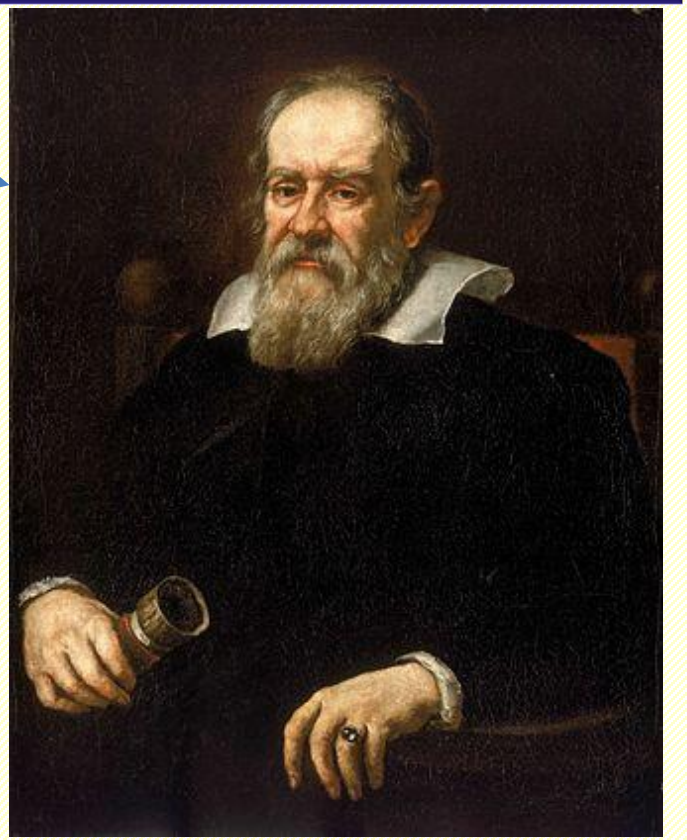
Courtesy: https://en.wikipedia.org/wiki/Pendulum

What is Pendulum?

From the first scientific investigations of the pendulum around 1602 by **Galileo Galilei**, the regular motion of pendulums was used for timekeeping, and was the world's most accurate timekeeping technology until the 1930s.

The pendulum clock invented by **Christian Huygens** in 1658 became the world's standard timekeeper, used in homes and offices for 270 years, and achieved accuracy of about one second per year before it was superseded as a time standard by the quartz clock in the 1930s.

Pendulums are also used in scientific instruments such as accelerometers and seismometers. Historically they were used as gravimeters to measure the acceleration of gravity in geophysical surveys, and even as a standard of length.



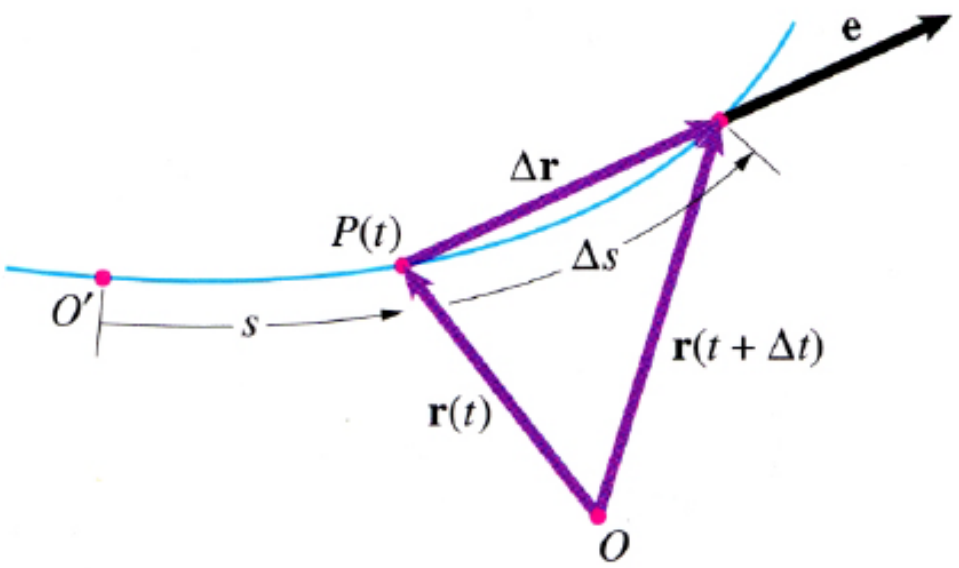
The word "pendulum" is new Latin, from the Latin ***pendulus***, meaning 'hanging'.

Galileo Galilei (15 Feb 1564[3] – 8 Jan 1642) was an Italian polymath. Galileo is a central figure in the transition from natural philosophy to modern science and in the transformation of the scientific Renaissance into a scientific revolution.

Courtesy: https://en.wikipedia.org/wiki/Pendulum https://en.wikipedia.org/wiki/Galileo_Galilei

Normal & Tangential Coordinates
(Slide from the previous Lecture)

Curvilinear Motion: Normal & Tangential Coord



$$\mathbf{v} = v\mathbf{e}_t = \frac{ds}{dt}\mathbf{e}_t$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{e}_t + v\frac{d\mathbf{e}_t}{dt}$$

$$\frac{d\mathbf{e}_t}{dt} = \frac{d\theta}{dt}\mathbf{e}_n$$

$$\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + v\frac{d\theta}{dt}\mathbf{e}_n =$$

$$= \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n$$

Normal acceleration

Tangential acceleration

Derive EOM for Pendulum: Method-1

Newton's Second Law

$$\sum \vec{\mathbf{F}} = m\,\ddot{\vec{\mathbf{r}}}$$

(for translational motion, $m = const$)

is equivalent to

$$\sum F_t = m\frac{dv}{dt}$$ (eq. along tangential direction)

and

$$\sum F_n = m\frac{v^2}{\rho}$$ (eq. along normal direction)

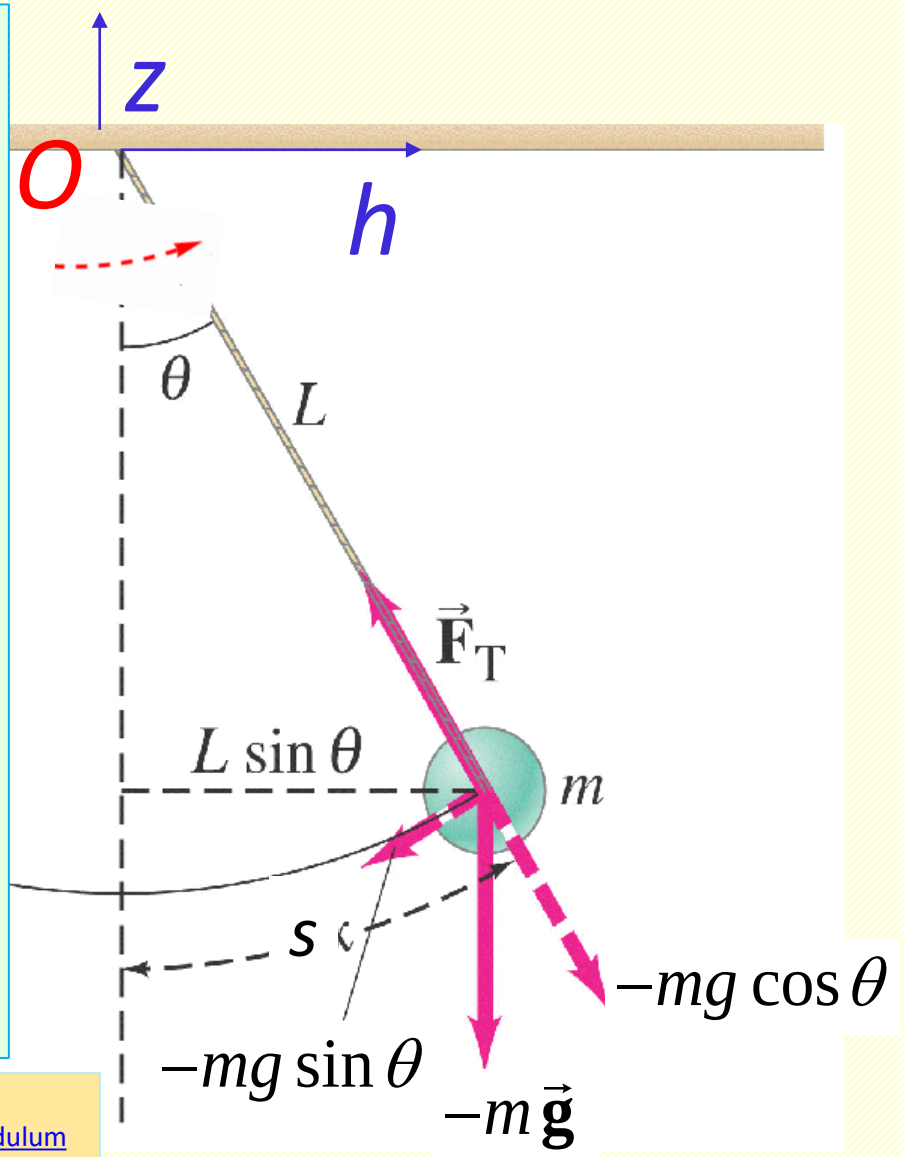


Image (PMT adapted), Courtesy:
https://www.quora.com/How-do-you-calculate-work-done-by-tension-on-a-pendulum

Derive EOM for Pendulum: Method-1

$$\sum F_t = m\frac{dv}{dt}$$

Develop Left Hand Side ($v = \dot{s}; \quad \dot{v} = \ddot{s}$):

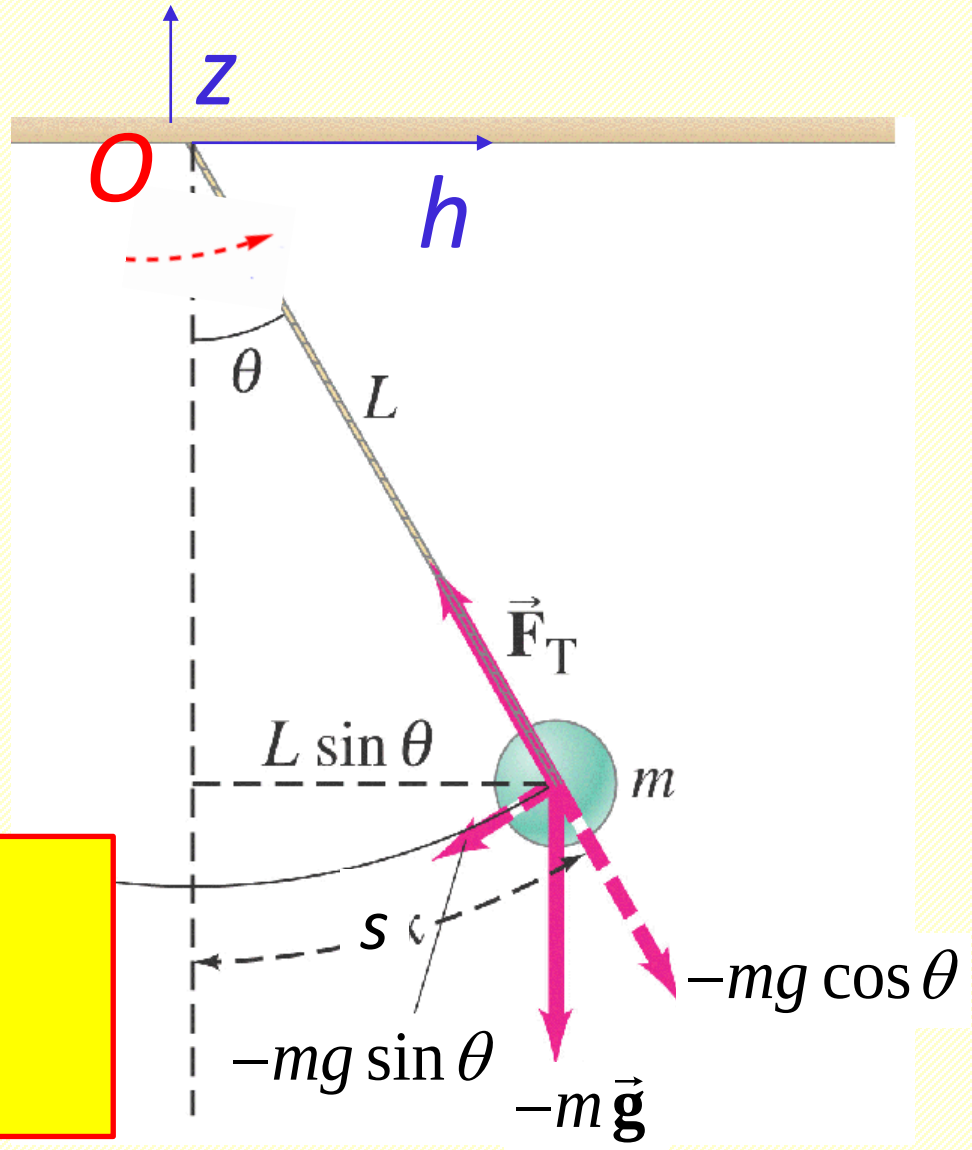
$$\sum F_t = -mg\ \sin\theta$$

Develop Right Hand Side:

$$m\frac{dv}{dt} = m\frac{d^2s}{dt^2} = m\frac{d^2(L\theta)}{dt^2} = L\frac{d^2\theta}{dt^2}$$

LHS=RHS $\Rightarrow$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$



Derive EOM for Pendulum: Method-2

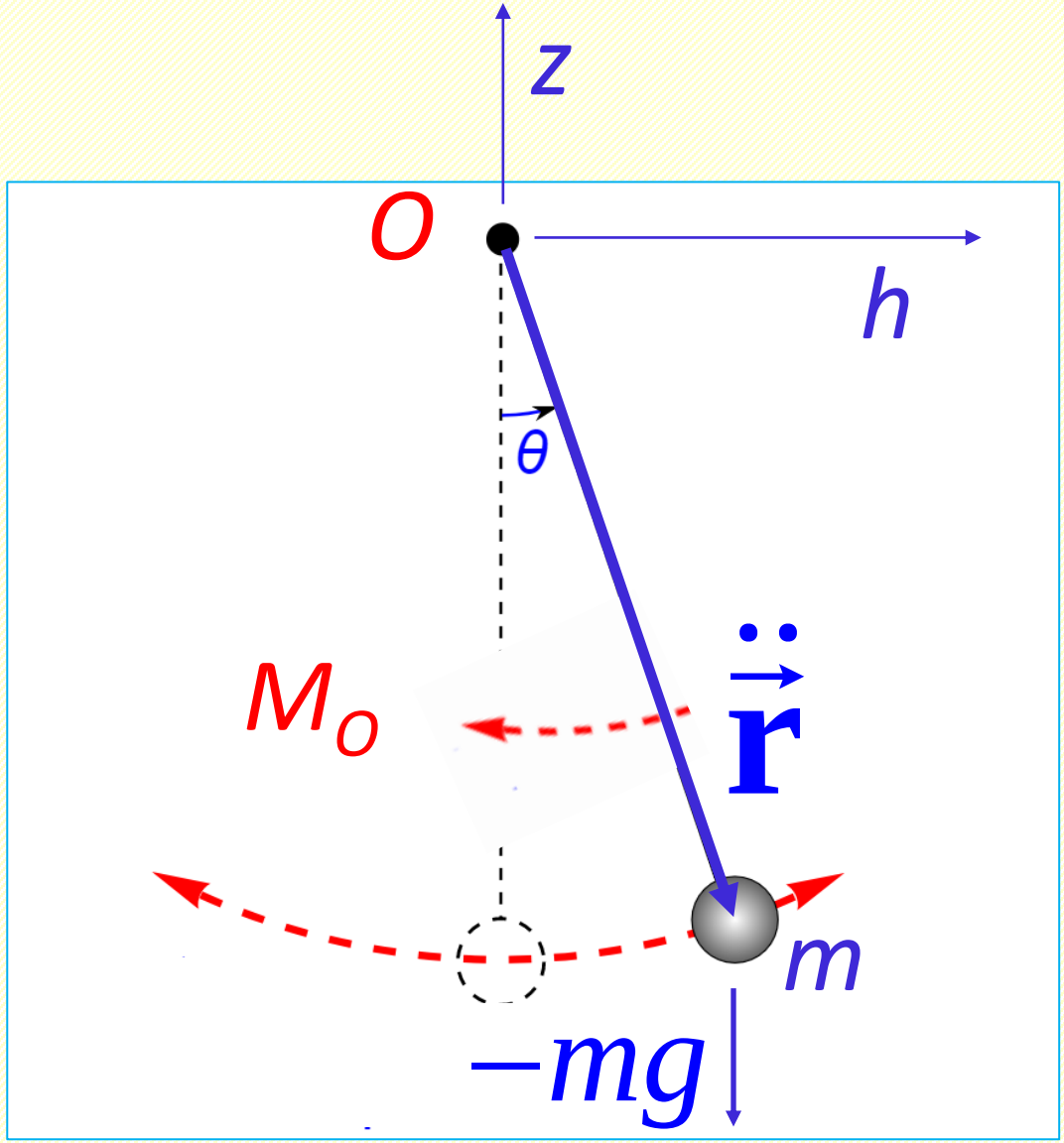
$$\sum \vec{\mathbf{F}} = m\,\ddot{\vec{\mathbf{r}}}$$

(for translational motion, $m = const$)

is equivalent to

$$\sum M_O = I_O\,\ddot{\theta}$$

(for rotational motion, $I_O = const$)



Derive EOM for Pendulum: Method-2

$$\sum M_O = I_O \frac{d^2\theta}{dt^2}$$

Develop Left Hand Side:

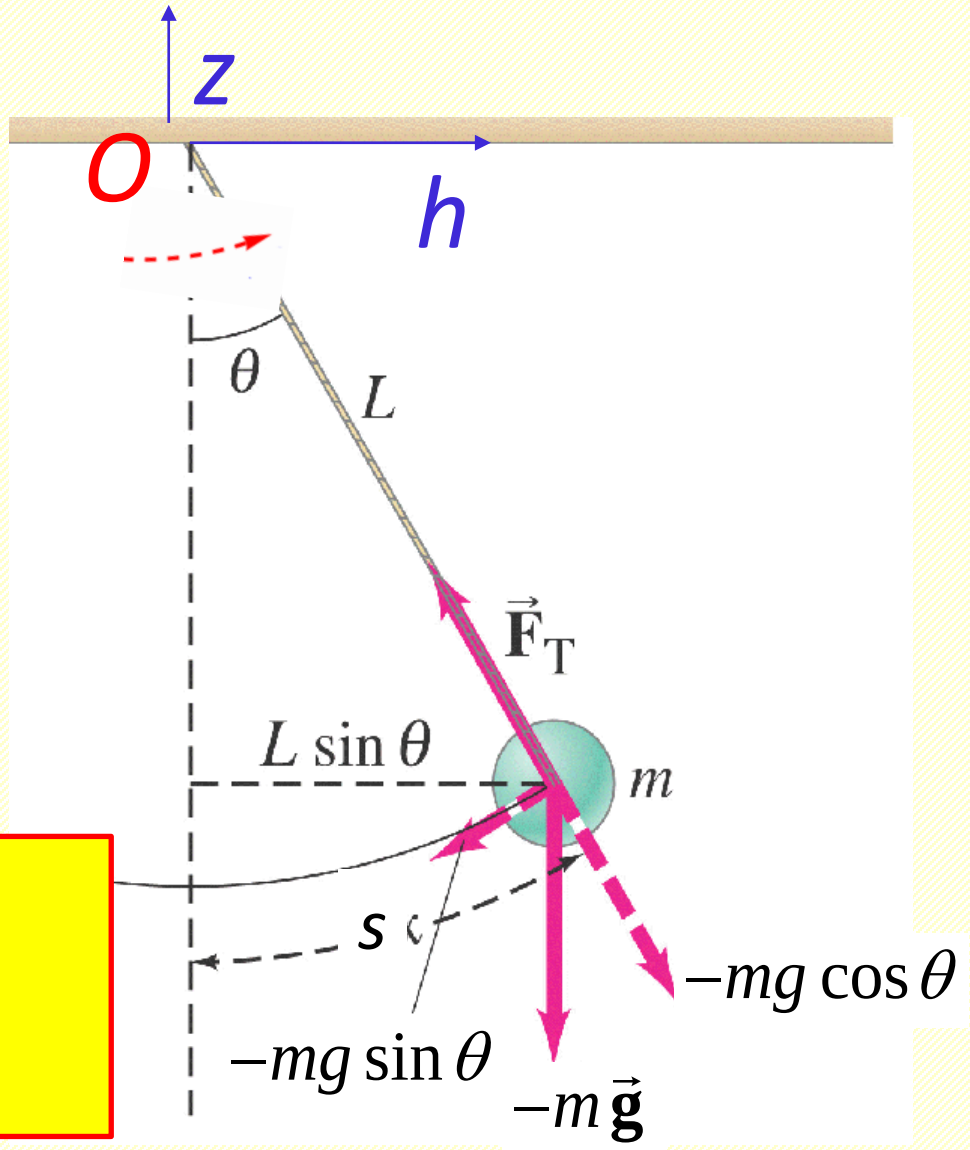
$$\sum M_O = -mg \sin\theta \times L$$

Develop Right Hand Side:

$$I_O \frac{d^2\theta}{dt^2} = mL^2 \frac{d^2\theta}{dt^2}$$

LHS=RHS $\Rightarrow$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$



Why on previous slide $I_O=mL^2$???

On one hand, Kinetic Energy of the System (due to translational considerations):

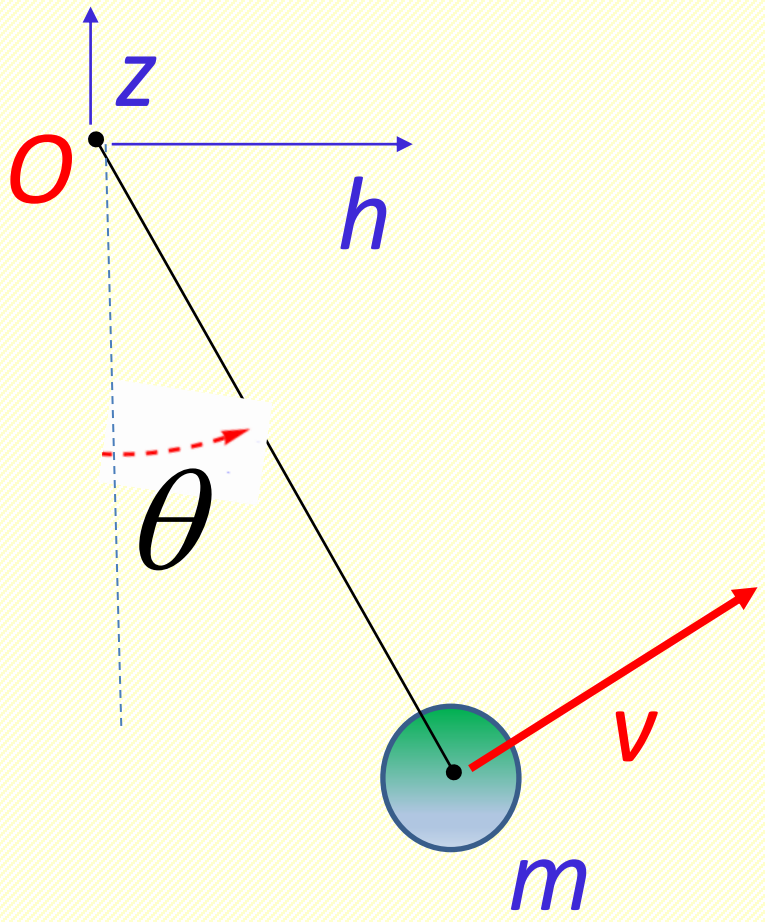
$$T=\frac{1}{2}mv^2=\frac{1}{2}m(\omega L)^2=\frac{1}{2}\boxed{mL^2}\dot{\theta}^2$$

On the other hand, (due to rotational considerations):

$$T=\frac{1}{2}I_O\omega^2=\frac{1}{2}\boxed{I_O}\dot{\theta}^2$$

LHS=RHS $\Rightarrow$

$$I_O=mL^2$$



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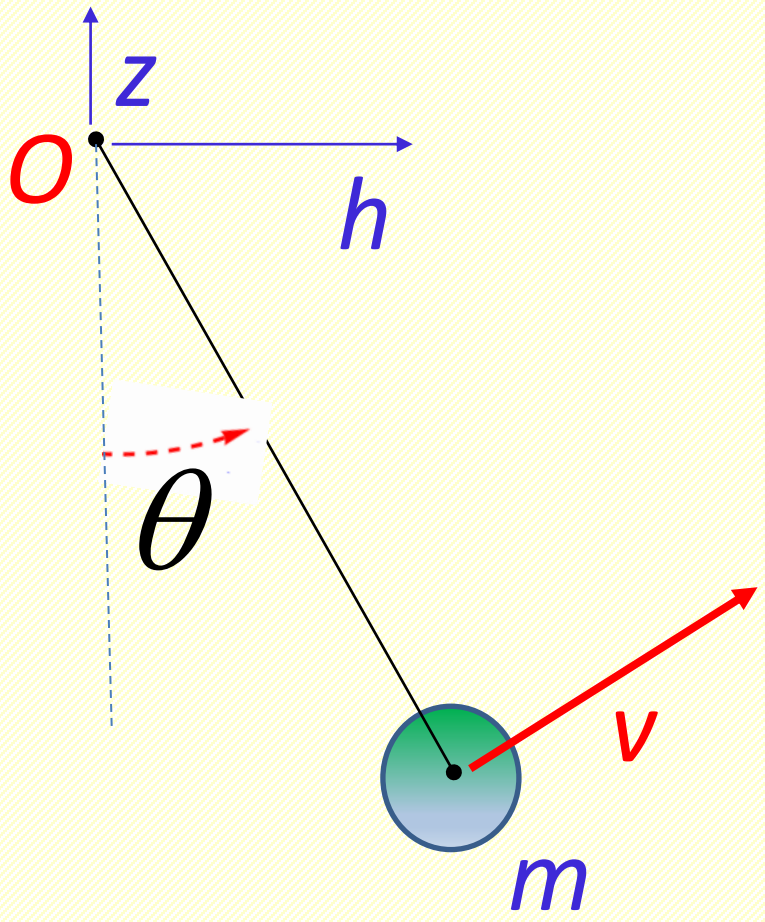
On the other hand, (due to rotational considerations):

$$T=\frac{1}{2}I_O\omega^2=\frac{1}{2}\boxed{I_O}\dot{\theta}^2$$

LHS=RHS $\Rightarrow$

$$I_O=mL^2$$



!!!!!!!!!!!!!

CORE TECHNIQUE

IN THIS COURSE:

NUMERICAL SOLUTION, ILLUSTRATED WITH PENDULUM EXAMPLE

!!!!!!!!!!!!!!

ONE-FILE SOLUTION:

i.e. using Anonymous Function to describe EOM

Re-Write EOM using System's States!

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

$$\begin{cases} x_1 = \theta \\ x_2 = \dot{\theta} \end{cases}$$ therefore $\dot{x}_1 = x_2$

Also, physical EOM can be re-written as

$$\dot{x}_2 = -\frac{g}{L}\sin\theta$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{L}\sin\theta \end{cases}$$

Re-Write EOM using System's States!

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

$$\begin{cases} x_1 = \theta \\ x_2 = \dot{\theta} \end{cases}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{L}\sin\theta \end{cases}$$

SOLUTION: Numerical (!!!) Method

MATLAB SCRIPT (no annotations)

$$\begin{cases} x_1 = \theta \\ x_2 = \dot{\theta} \end{cases}$$

```
%% SIMPLE PENDULUM: NON-LINEAR EXAMPLE
% Designed by Prof P.M.Trivailo (C) 2018
%-------------------------------------
clear; close all; clc;
g=9.81; L=2; tmax=18; thD0=170; th_dot0=0;
th0=thD0*pi/180; t=[0:0.001:1]*tmax;
pend_xdot = @(t, x) ([x(2); -(g/L)*sin(x(1))]);
[tt,zz]=ode45(pend_xdot,t,[th0; th_dot0]);
plot(tt, zz(:,1)*180/pi,'LineWidth',3,'Color',[0 0.5 1]);
grid on;
xl=xlabel('$t$ [s]'); yl=ylabel('$\theta$ [deg]');
set([xl,yl],'Interpreter','LaTeX')
title('\bf Pendulum (No Air Drag): \theta Time History');
set(gca, 'FontSize',18); hold on;
```

SOLUTION: Numerical (!!!) Method

MATLAB SCRIPT (with annotations)

```
%% SIMPLE NON-LINEAR
% Designed by Prof P.M
%
clear; close all; clc;
g=9.81; L=2; tmax=18; thD0=170; th_dot0=0;
th0=thD0*pi/180; t=[0:0.001:1]*tmax;
pend_xdot = @(t, x) ([x(2); -(g/L)*sin(x(1))]);

[tt,zz]=ode45(pend_xdot,t,[th0; th_dot0]);

plot(tt, zz(:,1)*180/pi,'LineWidth',3,'Color',[0 0.5 1]);
grid on;
xl=xlabel('$t$ [s]'); yl=ylabel('$\theta$ [deg]');
set([xl,yl],'Interpreter','LaTeX')
title('\bf Pendu
set(gca, 'Font
```

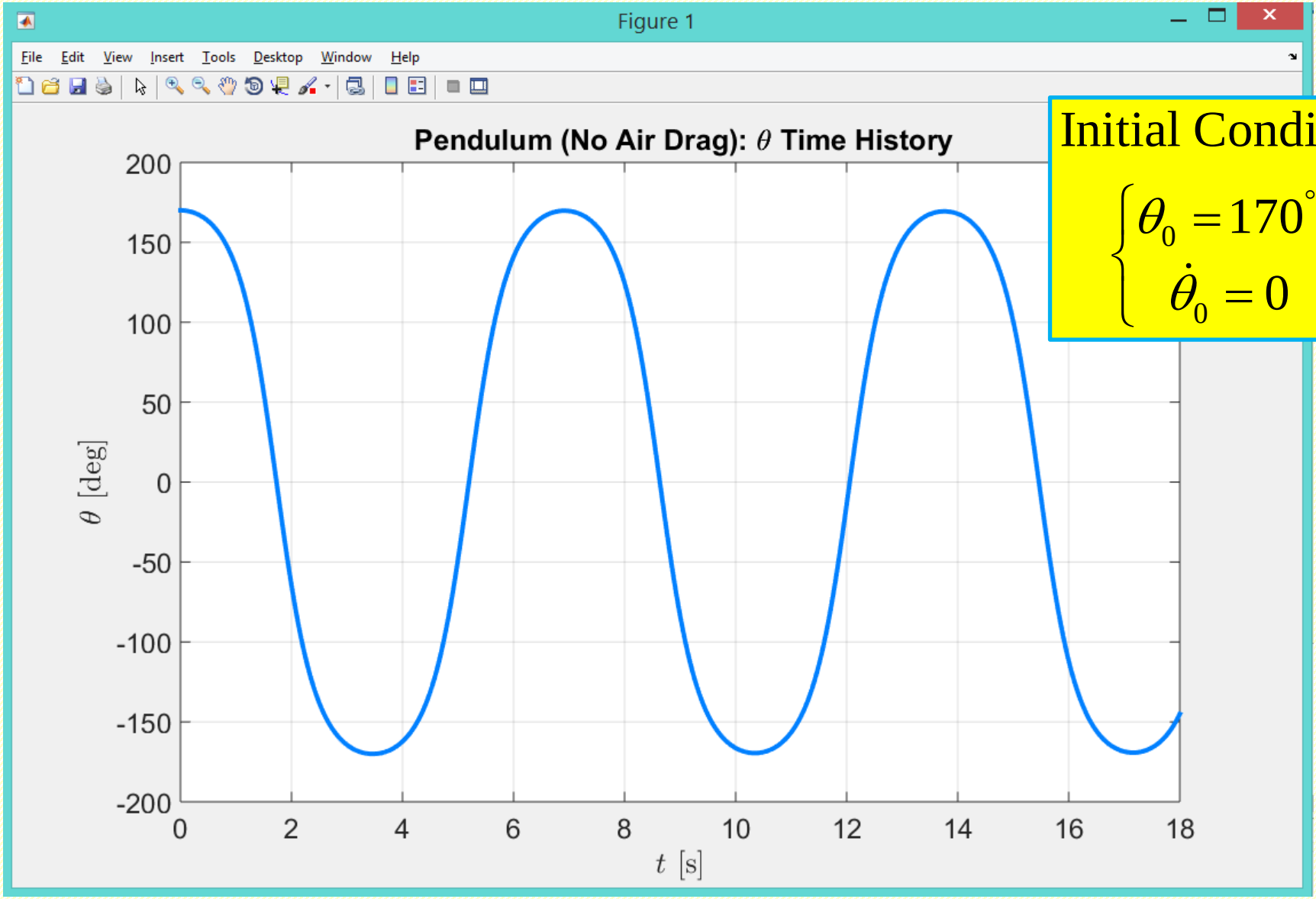
1. Input of the data

2. $$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{L}\sin\theta \end{cases}$$

3. Calling ode45 procedure !

4. Plotting results

Output of the MATLAB script



Initial Conditions

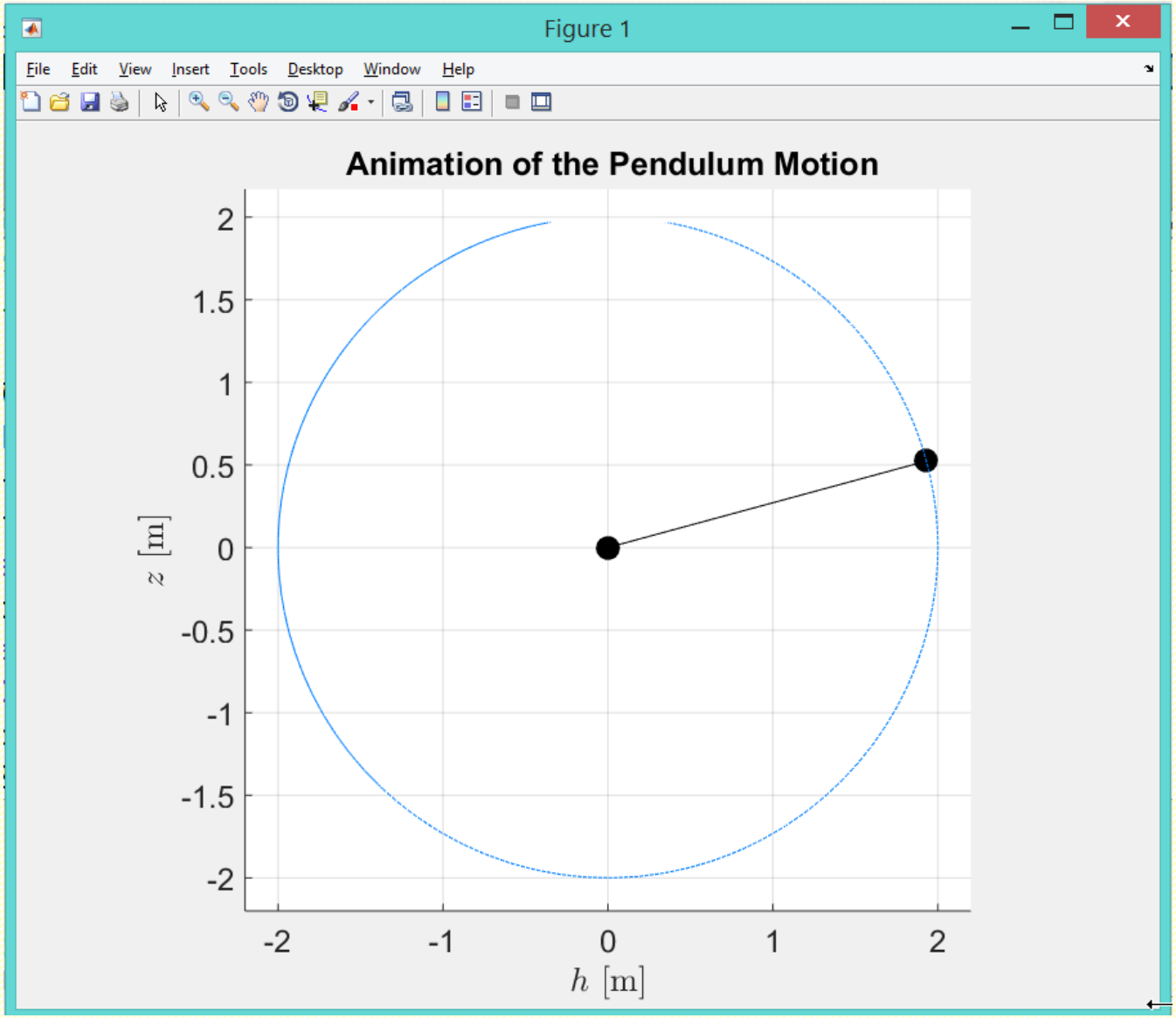
$$\begin{cases} \theta_0 = 170^\circ \\ \dot{\theta}_0 = 0 \end{cases}$$

ANIMATION OF THE SYSTEM (!!!)

MATLAB SCRIPT (no annotations)

```
%% ANIMATION OF SIMPLE PENDULUM: NON-LINEAR EXAMPLE
% Designed by Prof P.M.Trivailo (C) 2018
figure;
add=0.1*L;   % for better visualisation: increase "camera" embrace
hmax=max(L*sin(zz(:,1)))+add;
zmin=-L-add; zmax=max([-L*cos(zz(:,1));  0])+add;
axis([-hmax hmax zmin zmax]); axis equal;
set(gca,'XLimMode','manual'); hold on; grid on;
g=line('XData',[0 L*sin(th0)],…
       'YData',[0 -L*cos(th0)],'Marker','.','MarkerSize',48);
xlabel('$h$ [m]','Interpreter','LaTeX');
ylabel('$z$ [m]','Interpreter','LaTeX');
title('\bf Animation of the Pendulum Motion');
set(gca, 'FontSize',18); hold on;
plot(L*sin(zz(:,1)), -L*cos(zz(:,1)),'LineStyle',':','Color',[0
0.5 1]);
for i=1:length(tt),
   set(g,'XData',[0 L*sin(zz(i,1))],'YData',[0 -L*cos(zz(i,1))]);
   drawnow; pause(0.01);
end
```

Animation: Screen Snap-shot



LINEARISATION of EOMs:

COMPARING LINEAR and NONLINEAR MODELS

Example: 1-DOF Pendulum (Non-Linear)

Example: Response of a 1DOF Pendulum Using Non-Linear AND Linearised EOMs

Let us consider **free vibrations** of a mass m suspended on the inextensible cable of constant length L. Using any of the known methods, and considering a 2D motion of the system, we can derive the non-linear differential equation of motion of the system in terms of the angular displacement θ - the only DOF of the system:

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \qquad \text{(non-linear equation)} \tag{8}$$

Note that for this equation would be very similar to the equation for the 1-DOF inverted pendulum (3) except that for the sign of the second term.

The state-space equivalent of the Eq.(8) is:

$$\begin{cases} \dot{x}_1 = x_2; \\ \dot{x}_2 = -\dfrac{g}{L}\sin\theta \end{cases} \qquad \text{non-linear model} \qquad (9)$$

We can linearise the non-linear equation (8) as fol-

Example: 1-DOF Pendulum (Linearised)

lows:

$$\ddot{\theta} + \frac{g}{L}\theta = 0 \qquad \text{(linearised equation)} \tag{10}$$

The state-space equivalent of the Eq.(10) is:

$$\begin{cases} \dot{x}_1 = x_2; \\ \dot{x}_2 = -\frac{g}{L}x_1 \end{cases} \qquad \text{linearised model} \tag{11}$$

where $x = [\theta; \dot{\theta}]$. This result can be re-written in the

matrix form:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -(g/L) & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \quad \text{linearised model} \quad (12)$$

Note that these linearised equations in the state-space are for the **free-vibrations case** (i.e. the case, when the system is excited via initial conditions only, and via the application of the external force).

Scripts: 1-DOF Pendulum
(Nonlinear Model versus Linear Model: 2-file solution)

MATLAB script (MAIN PROGRAM):

```
%% Simple 1-DOF Pendulum
%% Designed by Prof P.M.Trivailo
%        RMIT University
global g L tt FF A B

%% Enter system's parameters:
g=9.81; % m/s^2
L=2;    % m
x0=[60*pi/180; 0]; % 60deg initial deflection
tmax=14;
tt=[0 tmax]; FF=[0 0];
```

This is where the ARBITRARY EXTERNAL FORCE CAN BE ADDED!

Scripts: 1-DOF Pendulum
(Nonlinear Model versus Linear Model: 2-file solution)

```
%% Running fully Non-Linear Model first
[tt1,xx1]=ode45('myodeNonLin',[0 tmax],x0);

%% Then running a Linearised Model
A=[0 1; -g/L 0]; B=[0; 0];
[tt2,xx2]=ode45('PMTode',[0 tmax], x0);

%% Plot results on the same plot
plot(tt1, xx1(:,1)*180/pi,'b',tt2,xx2(:,1)*180/pi,'r:');
legend('Non-Linear Model','Linearised Model')
xlabel('Time [s]'); ylabel('\theta [deg]');  grid
```

Scripts: 1-DOF Pendulum
(Nonlinear Model versus Linear Model: 2-file solution)

MATLAB script for the 1-DOF SYSTEM (SEPARATE file ' myodeNonLin.m' with DIFF EOMs):

```
function [x_dot] = myodeNonLin(t,x)
global g L

    x_dot(1,1)= x(2);
    x_dot(2,1)= -(g/L)*sin(x(1));
```

Scripts: 1-DOF Pendulum
(Nonlinear Model versus Linear Model: 2-file solution)

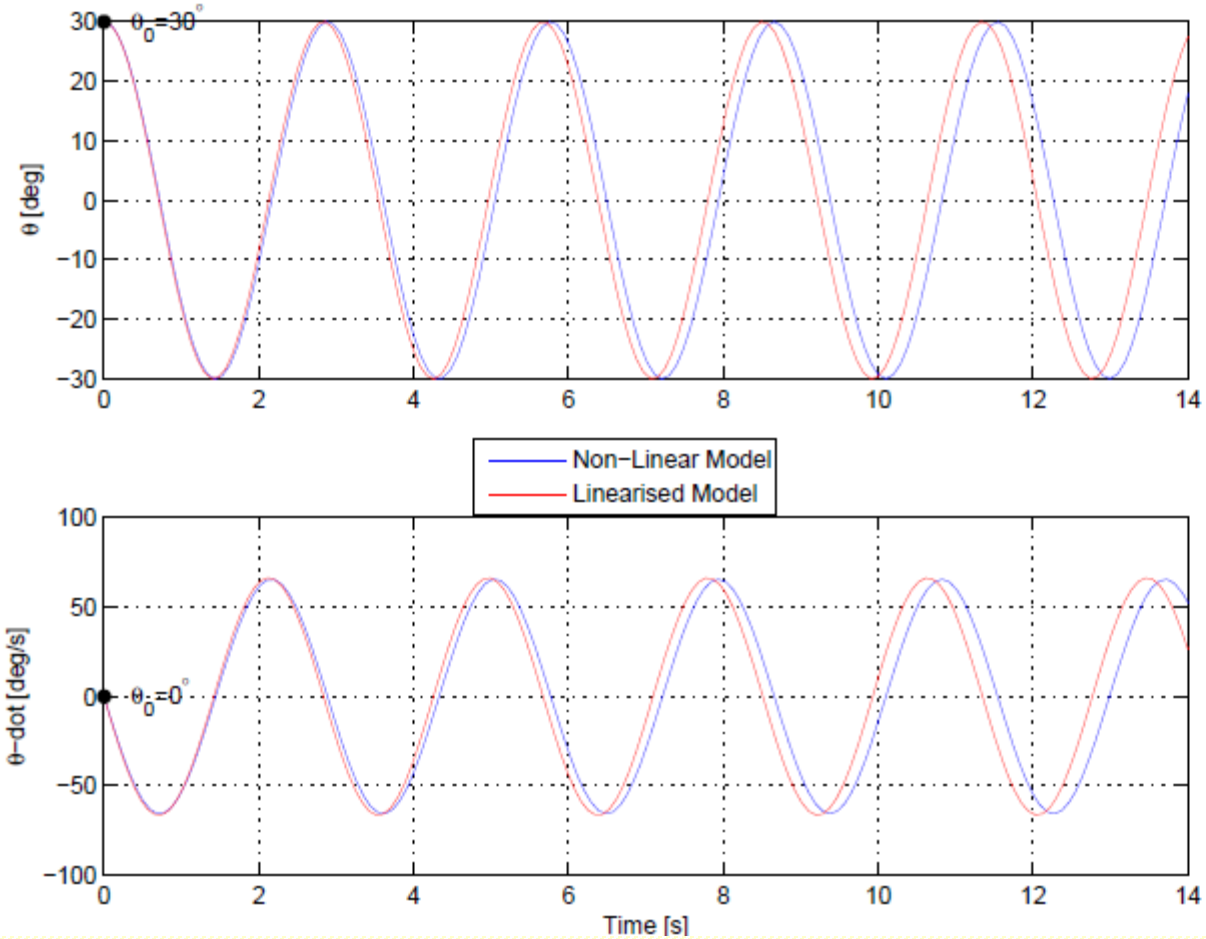
MATLAB script for the 1-DOF SYSTEM

(SEPARATE file ' PMTode.m' with DIFF EOMs)

```
function [x_dot] = PMTode(t,x)
global tt FF A B
Q=interp1(tt,FF,t');
x_dot = A*x+B*Q;;
```

Repsonse to Initial Excitation

Compare Linearised and Non-Linear Model



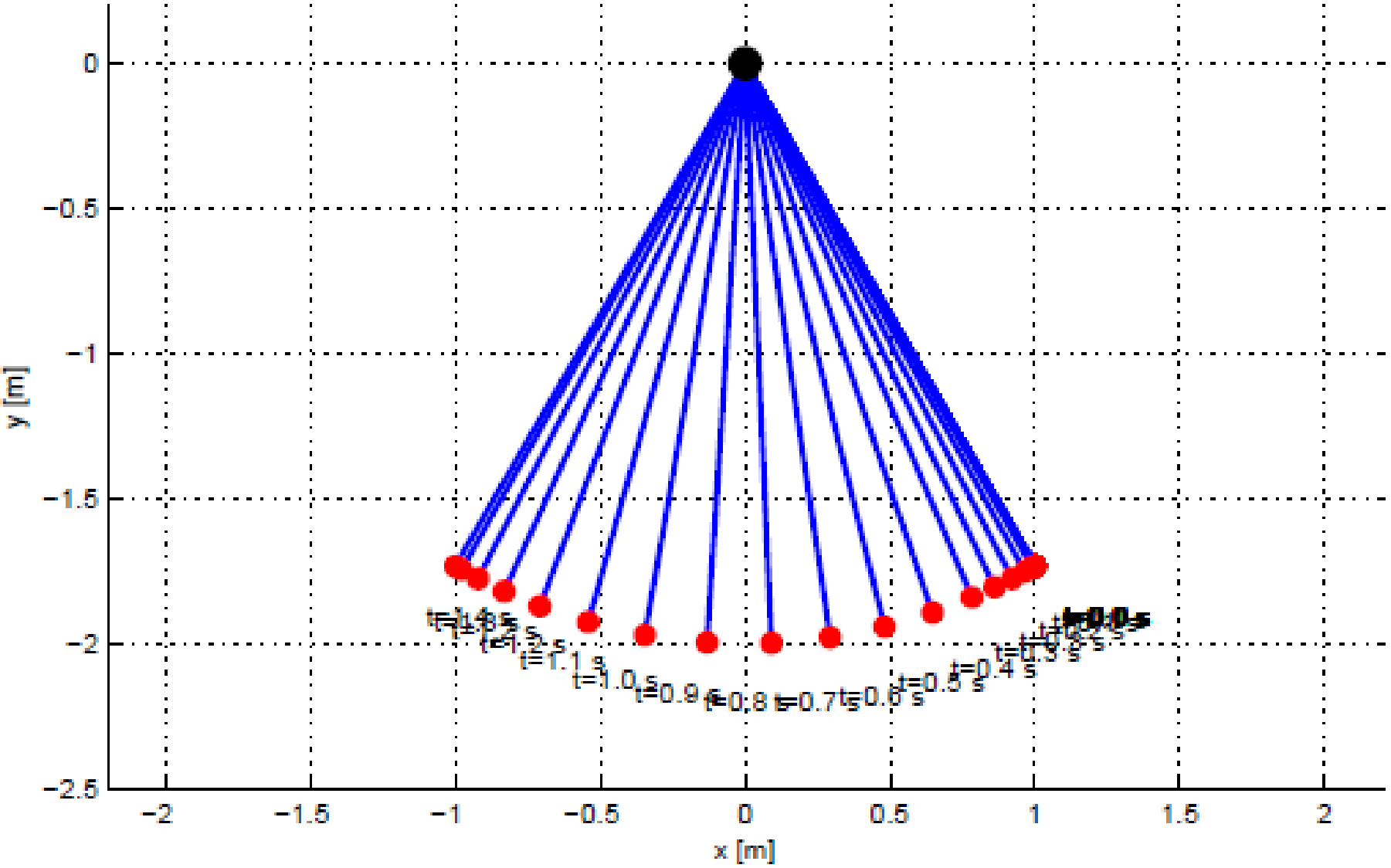
$\theta_0 = 30°$

Results: 1-DOF Pendulum (Nonlinear Model versus Linear Model)

By the way, the PERIOD of oscillations can be calculated for small angles, using

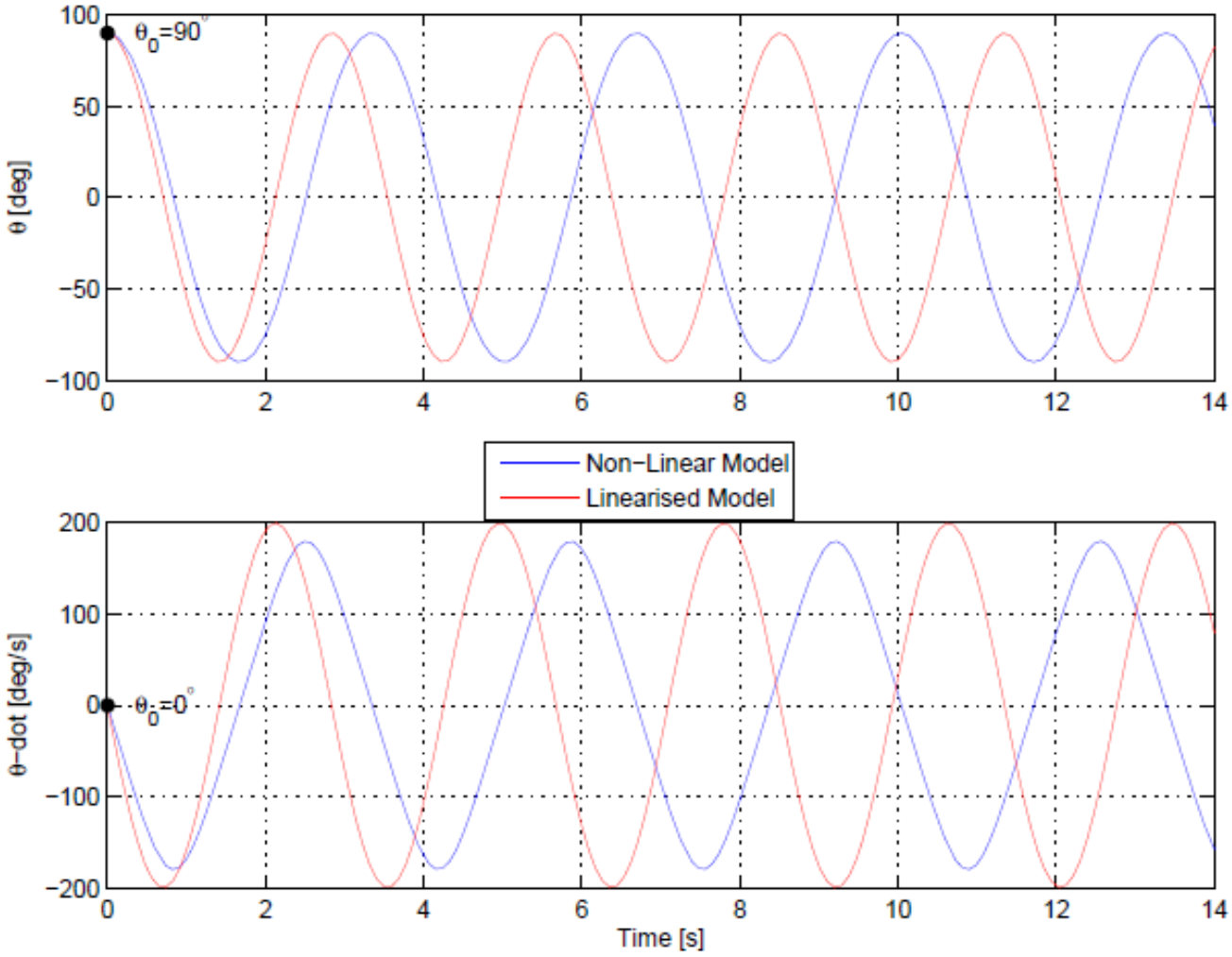
$$T \approx 2\pi \sqrt{\frac{L}{g}}$$

You may wish to check this equation using the simulation data

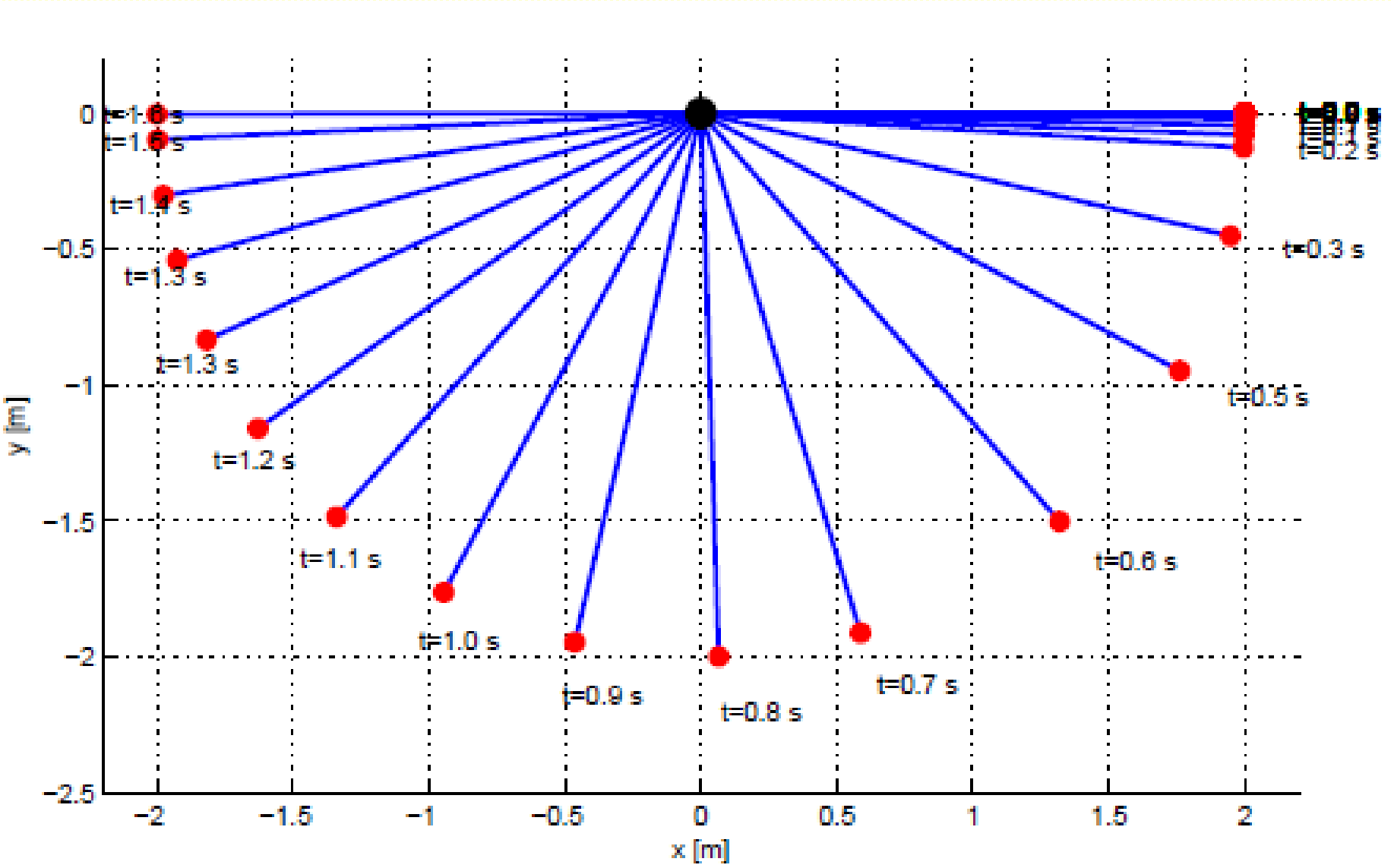

0
−0.5
−1
−1.5
−2
−2.5
y [m]
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
x [m]
t=0.8 s
t=0.7 s
t=0.6 s
t=0.5 s
t=0.4 s

Results: 1-DOF Pendulum (Nonlinear Model versus Linear Model)

Compare Linearised and Non-Linear Model

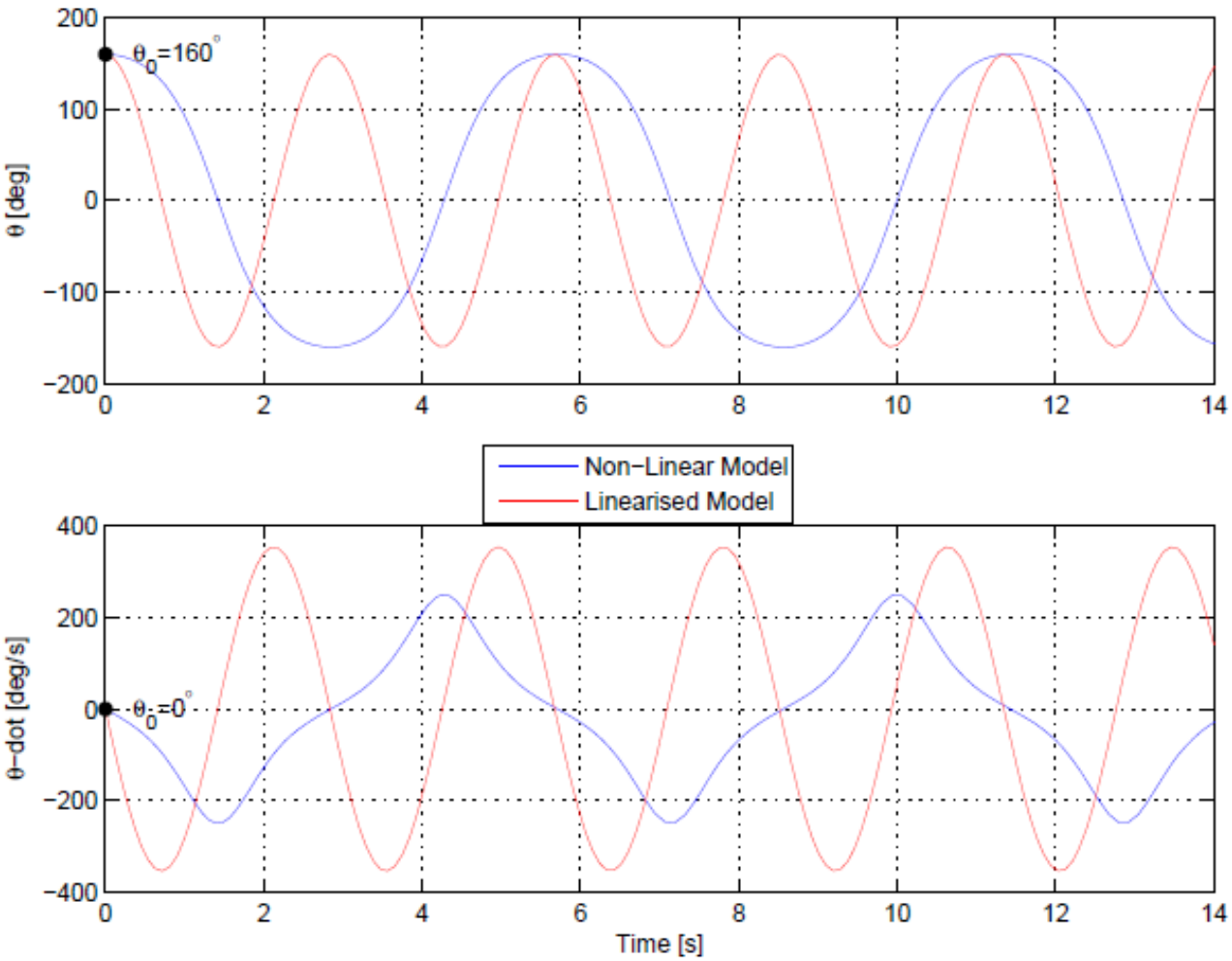


$\theta_0 = 90°$

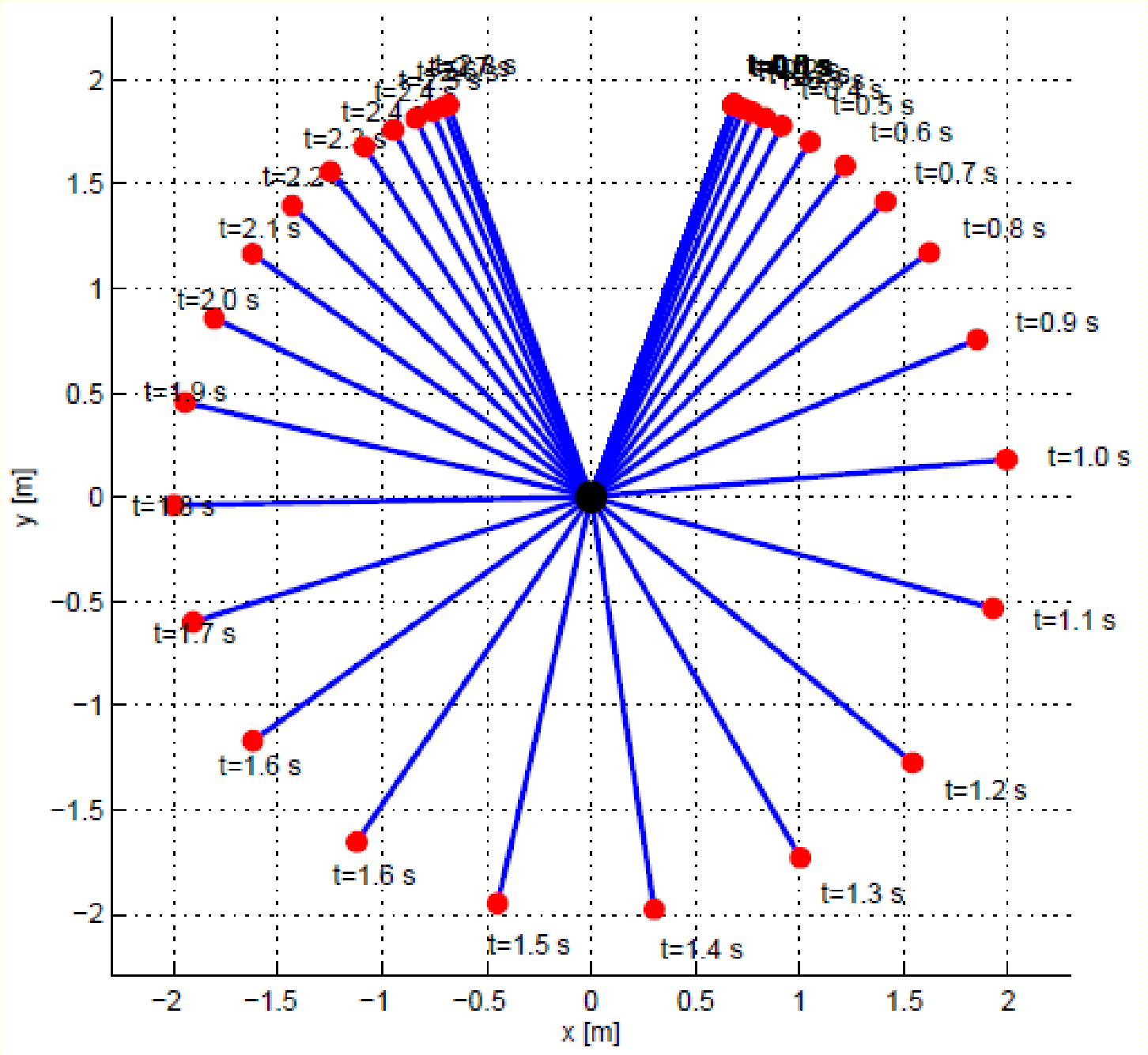

t=1.6 s
t=1.4 s
t=1.3 s
t=1.3 s
t=1.2 s
t=1.1 s
t=1.0 s
t=0.9 s
t=0.8 s
t=0.7 s
t=0.6 s
t=0.5 s
t=0.3 s
t=0.2 s
y [m]
x [m]
0
−0.5
−1
−1.5
−2
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2

Results: 1-DOF Pendulum (Nonlinear Model versus Linear Model)

Compare Linearised and Non-Linear Model



$$\theta_0 = 160°$$


t=0.5 s
t=0.6 s
t=0.7 s
t=0.8 s
t=0.9 s
t=1.0 s
t=1.1 s
t=1.2 s
t=1.3 s
t=1.4 s
t=1.5 s
t=1.6 s
t=1.6 s
t=1.7 s
t=2.0 s
t=2.1 s
t=2.2
t=2.4
y [m]
x [m]
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2

END OF LECTURE-2b SLIDES

APPENDIX

Other Examples