Part I: Introduction to PID Control Systems

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Outline

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How a Control System Works-I

- To illustrate how in a general sense a computer controlled system works, a room temperature control is used as an example.

- Suppose that in a cold environment, appropriate room temperature is maintained through heating.

- The heating elements are from an oil furnace. The input to control the room temperature is the fuel injection rate to the furnace and the output is room temperature. The control objective is to maintain the room temperature at $19C^o$.

- The assumption is that the measurement and control action are synchronized in terms of a sequence of time intervals. The intervals between two executions are assumed to be the same, and it is called the sampling interval Δt.

How a Control System Works- II

- Take the output measurement. At the time when the closed-loop control system is switched on, assuming as t_0, the temperature sensor reads the room temperature $y(t_0)$. Then this measurement is read by the computer into its memory.

- Compute feedback error. This temperature measurement at time t_0 is compared with the required room temperature. This comparison generates the feedback error $e(t_0)$, which is the error between your desired room temperature and the actual room temperature.

- Compute the controller output. The control algorithm in computer computes a control signal $u(t_0)$ based the feedback error $e(t_0)$ and the history of the feedback error, depending on the complexity of the control algorithm.

How a Control System Works- III

- This digital value $u(t_0)$, through a device called zero-order-hold, is converted into the analog control signal $u(t)$ (fuel injection rate), where $u(t) = u(t_0)$, $t_0 \leq t \leq t_0 + \Delta t$.
- This analog signal is the fuel injection rate for the time interval, $t_0 \leq t \leq t_0 + \Delta t$.
- The clock ticks to the next sample period, $t_1 = t_0 + \Delta t$, the temperature sensor reads the room temperature as $y(t_1)$, and the entire controlling process repeats the steps listed.

Design and Implementation of Control Systems-I

- Understand the plant to be controlled. If the plant is completely new to us, then we may have to consider the configuration of the control system by choosing appropriate input and output signals.
- Build a dynamic model for the plant to be controlled. This is always one of the key steps. How well the control system is going to perform strongly depends on the quality of the dynamic model built for controller design. We may use a physical model based approach to find the dynamics of the system. Or we may choose to conduct some experiments and build a dynamic model based on experimental data from the plant.

Design and Implementation of Control Systems-II

- Control system design, which is based on the dynamic model obtained. In the control system design step, we will find the controller parameters and verify some of the closed-loop characteristics. It is possible for some controllers with one or two parameters to be determined that the controller parameters can be tuned trial-by-error, without using a dynamic model.

Design and Implementation of Control Systems-III

- Simulation of the closed-loop control system. There are soft- ware- in- the- loop simulation and hardware in-the-loop simulation. Simulations are the necessary means used to verify the control system designed with the assumptions, and find the faults and make the necessary corrections. Software in-the-loop-simulation is the cheapest and safest way to make sure that the control system designed will lead to bad consequences. Hardware in-the-loop simulation will utilize some key hardware in the simulation and will bring the simulation steps closer to the actual control system implementation.

- Implementation of the control systems. This may require transferring the computer codes used in simulation platform to the actual computer language used in the real-time control system, which matches the existing system, through either micro-controller, DSP or other computational platforms. Safety jackets will be implemented in the software to ensure the safety operation of the plant.

Outline

Proportional Controller

- With this term **proportional**, the feedback control signal $u(t)$ is computed in proportion to the feedback error $e(t)$ with the formulae,

$$u(t) = K_c e(t) \tag{1}$$

- where K_c is the proportional gain and the feedback error as the difference between the reference signal $r(t)$ and the output signal $y(t)$ ($e(t) = r(t) - y(t)$).
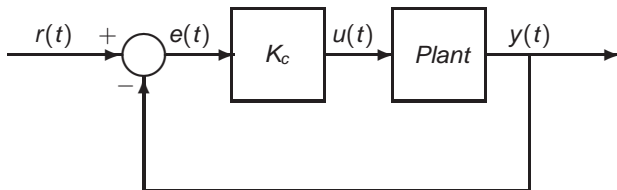

Figure 1: Proportional Feedback Control System

Applications of Proportional Controller

- Because the controller only involves one parameter to be determined, it is possible to choose K_c without detailed information about the plant.
- Because of its simplicity, the proportional controller is often used in the cases when the system dynamics is unknown, the system is unstable and the required control performance in steady-state operation is not demanding.
- Use of proportional controller is common in the design of cascaded control systems.
- The limitation of a simple proportional controller is that the steady-state error of the closed-loop control system will not be completely eliminated.

Example of Proportional Control

The plant is a first order system with the following transfer function,

$$G(s) = \frac{0.3}{s+1}. \tag{2}$$

with proportional controller K_c ($K_c > 0$). Suppose that the reference signal is a step signal with amplitude 1 and its transfer function is $R(s) = \frac{1}{s}$. Find the steady-state value of the output with respect to the reference signal.

Solution I

The closed-loop control system from the set-point signal to the plant output has the transfer function,

$$\frac{Y(s)}{R(s)} = \frac{K_c G(s)}{1 + K_c G(s)} = \frac{0.3 K_c}{s + 1 + 0.3 K_c} \tag{3}$$

With any positive K_c, the closed-loop system is stable where the closed-loop pole is determined by the solution of the polynomial equation,

$$s + 1 + 0.3 K_c = 0 \tag{4}$$

which is $-1 - 0.3 K_c$.

Solution II

The Laplace transfer function of the output, $Y(s)$, is

$$Y(s) = \frac{0.3K_c}{s + 1 + 0.3K_c} R(s) = \frac{0.3K_c}{s(s + 1 + 0.3K_c)}. \tag{5}$$

where $R(s) = \frac{1}{s}$. Applying the final value theorem to the stable closed-loop system, we calculate

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s \times \frac{0.3K_c}{s(s + 1 + 0.3K_c)} = \frac{0.3K_c}{1 + 0.3K_c}. \tag{6}$$

Solution III

For any value of $K_c < \infty$, $\lim_{t\to\infty} y(t) \neq 1$, *i.e.* not equal to the desired value at the steady state response. Figure 2 shows the closed-loop step response with the proportional controller $K_c = 8$ and $K_c = 80$, respectively. It is seen that with the increase of the proportional gain, the closed-loop response speed increases and the steady-state value becomes closer to the desired value 1.
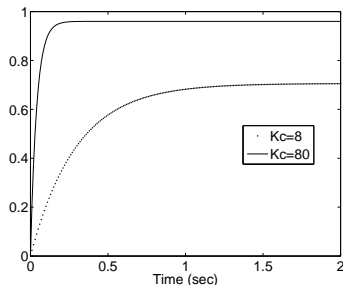

Figure 2: Closed-loop step response of proportional control system

Outline

Proportional Plus Integral Controller

- Proportional plus Integral (PI) Controller is the most widely used controller among PID controllers.
- With the integral action, the steady-state error that had existed with the proportional control system will be completely eliminated.
- The output of the PI controller is calculated using

$$u(t) = K_c e(t) + \frac{K_c}{\tau_I} \int_0^t e(\tau) d\tau \tag{7}$$

where $e(t) = r(t) - y(t)$ is the error signal between the setpoint $r(t)$ and the output $y(t)$, K_c is the proportional gain, and τ_I is the integral time constant.

Signs of K_c and τ_I

- The parameter τ_I is always positive, and its value is inversely proportional to the effect of the integral action taken by the PI controller. A smaller τ_I will result in a stronger effect of the integral term.
- The parameter K_c has the same sign as the steady-state gain of the plant to be controlled. Its value is proportional to the effect of the proportional action taken by the PI controller. A larger K_c will result in a stronger effect of the proportional term.
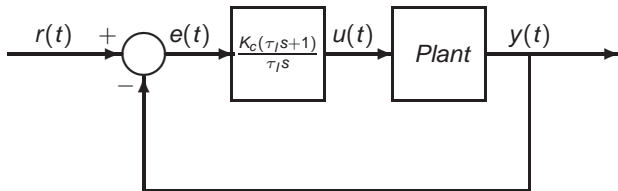- Q: What is steady-state gain of a plant?

Laplace transform of the control signal

The Laplace transform of the controller output is

$$U(s) = K_c E(s) + \frac{K_c}{\tau_I s} E(s) \tag{8}$$

with $E(s)$ being the Laplace transform of the error signal $e(t)$. With this, the Laplace transfer function of the PI controller is expressed as

$$C(s) = \frac{U(s)}{E(s)} = \frac{K_c(\tau_I s + 1)}{\tau_I s} \tag{9}$$

Example

Assume that the plant is a first order system with the transfer function

$$G(s) = \frac{0.3}{s+1} \tag{10}$$

the PI controller has the proportional gain $K_c = 8$, and the integral time constant $\tau_I = 3$ and 0.5 respectively. Examine the locations of the closed-loop poles. With the set-point signal $r(t)$ as a unit step signal, find the steady-state value of the closed-loop output $y(t)$.

Solution I

We calculate the closed-loop transfer function between the reference signal

$$\frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)} \tag{11}$$

With $C(s)$ given in (9) and $G(s)$ in (10), we have

$$\frac{Y(s)}{R(s)} = \frac{0.3K_c\tau_I s + 0.3K_c}{\tau_I s^2 + \tau_I(1 + 0.3K_c)s + 0.3K_c} \tag{12}$$

The closed-loop poles of this system are determined by the solutions of the polynomial equation,

$$\tau_I s^2 + \tau_I(1 + 0.3K_c)s + 0.3K_c = 0 \tag{13}$$

$$s_{1,2} = -\frac{1 + 0.3K_c}{2} \pm \frac{1}{2}\sqrt{(1 + 0.3K_c)^2 - \frac{1.2K_c}{\tau_I}} \tag{14}$$

Solution II

If the quantity

$$(1 + 0.3K_c)^2 - \frac{1.2K_c}{\tau_I} = 0$$

then there are two identical real poles located at $s_{1,2} = -\frac{1+0.3K_c}{2}$. If the quantity,

$$(1 + 0.3K_c)^2 - \frac{1.2K_c}{\tau_I} > 0$$

then there are two real poles located at

$$s_{1,2} = -\frac{1 + 0.3K_c}{2} \pm \frac{1}{2}\sqrt{(1 + 0.3K_c)^2 - \frac{1.2K_c}{\tau_I}}$$

If the quantity,

$$(1 + 0.3K_c)^2 - \frac{1.2K_c}{\tau_I} < 0$$

then there are two complex poles located at

$$s_{1,2} = -\frac{1 + 0.3K_c}{2} \pm j\frac{1}{2}\sqrt{\frac{1.2K_c}{\tau_I} - (1 + 0.3K_c)^2}$$

The closed-loop system is stable as long as K_c is positive and $0 < \tau_I < \infty$.

Solution III

Applying the final value theorem, we calculate

$$
\begin{aligned}
\lim_{t \to \infty} y(t) &= \lim_{s \to 0} s \times \frac{0.3 K_c \tau_I s + 0.3 K_c}{\tau_I s^2 + \tau_I (1 + 0.3 K_c) s + 0.3 K_c} \frac{1}{s} \\
&= \frac{0.3 K_c}{0.3 K_c} = 1.
\end{aligned}
\tag{15}
$$

where the steady-state value is equal to the reference signal, and it is independent of the value of integral time constant τ_I.

Solution IV

Figure 4 shows the closed-loop step response with $\tau_I = 3$ and $\tau_I = 0.5$, respectively. It is seen that as τ_I reduces, the closed-loop response speed becomes faster. Nevertheless, the steady-state responses with both τ_I values are equal to one.
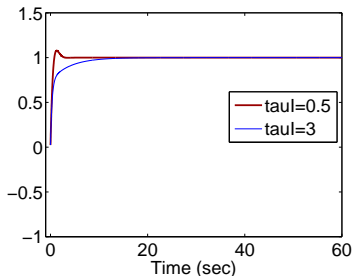

Figure 4: Closed-loop step response of PI control system

Alternative PI Structure

This small change is to put the proportional control on the output signal $y(t)$, instead of the feedback error $e(t) = r(t) - y(t)$.

$$u(t) = -K_c y(t) + \frac{K_c}{\tau_I} \int_0^t (r(\tau) - y(\tau)) d\tau \tag{16}$$

Applying Laplace transform to this equation leads to the Laplace transform of the controller output in relation to the reference signal and the output as

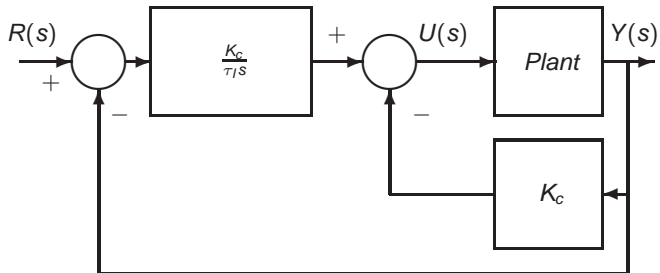$$U(s) = -K_c Y(s) + \frac{K_c}{\tau_I s} (R(s) - Y(s)) \tag{17}$$

Diagram

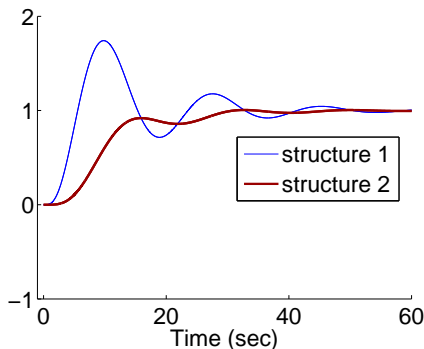
Figure 5: Alternative PI controller structure

Example

Assume that the plant is described by the transfer function

$$G(s) = \frac{1}{s(s+1)^3} \tag{18}$$

and the PI controller has the parameters: $K_c = 0.56$, $\tau_I = 8$. Find the closed-loop transfer function between the reference signal $R(s)$ and the output signal $Y(s)$ for the original PI controller structure (see Figure 3) and the alternative PI controller structure (see Figure 5), and compare their closed-loop step responses.

Results

Detailed solution can be found in Example 1.3 of the lecture notes.

Figure 6: Closed-loop step response of PI control system. Structure 1: the response from the original structure; Structure 2: response from the alternative structure.

Outline

Ideal PID Controllers

A PID controller consists of three terms: proportional (P), integral (I) and derivative (D). The output $u(t)$ of a PID controller is the sum of of the three terms,

$$u(t) = K_c e(t) + \frac{K_c}{\tau_I} \int_0^t e(\tau)d\tau + K_c \tau_D \frac{de(t)}{dt} \tag{19}$$

where $e(t) = r(t) - y(t)$ is the feedback error signal between the reference signal $r(t)$ and the output $y(t)$, and τ_D is the derivative gain. The Laplace transfer function of the PID controller is

$$\frac{U(s)}{E(s)} = K_c(1 + \frac{1}{\tau_I s} + \tau_D s) \tag{20}$$

Practical PID controllers I

- Almost without exception, the derivative term is changed from the original form $K_c \tau_D s$ in the implementations.
- This is because the derivative term $K_c \tau_D (\frac{dr(t)}{dt} - \frac{dy(t)}{dt})$ is not realizable in a practical implementation and the differentiation of the output signal $y(t)$ leading to amplification of measurement noise.
- Q: Why is that?

Practical PID controllers II

- Firstly, in order to avoid the problem caused by a step reference trajectory change, the derivative action is only implemented on the output signal $y(t)$.
- Q: Why differentiation of a step signal causes problem?
- Secondly, in order to avoid amplification of the measurement noise, a derivative filter is in a companionship with the derivative term $K_c \tau_D s$.
- A commonly used derivative filter:

$$F_D(s) = \frac{1}{\beta \tau_D s + 1} \tag{21}$$

where β is typically chosen to be 0.1. β is chosen to be larger if the measurement noise is severe.

Practical PID controllers III

With the derivative filter $F_D(s)$, a typical PID control output is calculated as

$$u(t) = K_c(r(t) - y(t)) + \frac{K_c}{\tau_I} \int_0^t (r(\tau) - y(\tau))d\tau - K_c\tau_D\frac{dy_f(t)}{dt} \qquad (22)$$

where $y_f(t)$ is the filtered output response using (21).

Laplace transform of control signal

In Laplace transform, the control signal is expressed as

$$
\begin{aligned}
U(s) &= K_c(R(s) - Y(s)) + \frac{K_c}{\tau_I s}(R(s) - Y(s)) - \frac{K_c \tau_D s}{\beta \tau_D s + 1} Y(s) \\
&= K_c(1 + \frac{1}{\tau_I s})(R(s) - Y(s)) - \frac{K_c \tau_D s}{\beta \tau_D s + 1} Y(s)
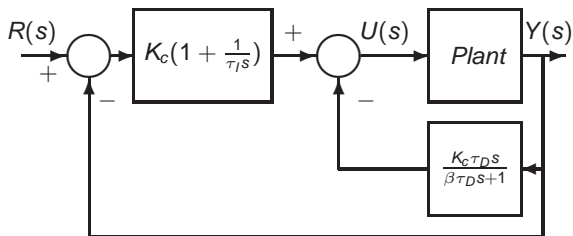\end{aligned}
\tag{23}
$$

Figure 7: PID controller structure

Putting Proportional Control on Output

To avoid the overshoot in output response to a set-point change, the proportional term in the PID controller may also be implemented on the plant output. In this case, the control signal is calculated using

$$u(t) = -K_c y(t) + \frac{K_c}{\tau_I} \int_0^t (r(\tau) - y(\tau))d\tau - K_c \tau_D \frac{dy_f(t)}{dt} \quad (24)$$

Accordingly, the Laplace transform of the control signal is expressed as

$$
\begin{aligned}
U(s) &= -K_c Y(s) + \frac{K_c}{\tau_I s}(R(s) - Y(s)) - \frac{K_c \tau_D s}{\beta \tau_D s + 1} Y(s) \\
&= \frac{K_c}{\tau_I s}(R(s) - Y(s)) - \frac{K_c(\tau_D(\beta + 1)s + 1)}{\beta \tau_D s + 1} Y(s)
\end{aligned}
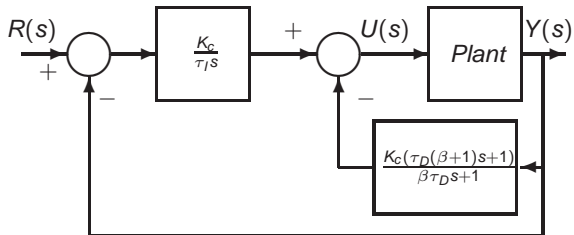\quad (25)
$$

Diagram

Figure 8: Alternative PID controller structure

Example

Suppose that the plant is described by the transfer function

$$G(s) = \frac{1}{s(s+1)^3} \tag{26}$$

and the PI part of the controller has the parameters: $K_c = 0.56$, $\tau_I = 8$. Choosing $\tau_D = 0.1$ and 1, find the closed-loop transfer function for the PID control system shown in Figure 7 and simulate its closed-loop performance. Also, find the closed-loop transfer function of the PID control system shown in Figure 8 and simulate its closed-loop step performance.

Solution I

The control signal from Figure 7 has the Laplace transform given by Equation (23). Substituting $U(s)$ into the following equation,

$$Y(s) = \frac{1}{s(s+1)^3} U(s) \tag{27}$$

re-grouping and re-arranging lead to the closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{K_c(\tau_I s + 1)(\beta \tau_D s + 1)}{\tau_I s^2 (s+1)^3 (\beta \tau_D s + 1) + K_c(\tau_I s + 1)(\beta \tau_D s + 1) + K_c \tau_I \tau_D s^2} \tag{28}$$

There are two zeros in the closed-loop transfer function: $-\frac{1}{\tau_I}$ caused by the integral control, and $-\frac{1}{\beta \tau_D}$ caused by the derivative control. With a small β value, the large effect from the derivative term comes from the term $K_c \tau_I \tau_D s^2$ in the denominator of the closed-loop transfer function.

Solution II

Figure 9 shows the closed-loop step responses for $\tau_D = 0.1$ and $\tau_D = 1$ respectively. With the increase of τ_D, the oscillation in the closed-loop response has been reduced. However, there is a large overshoot in the output response.
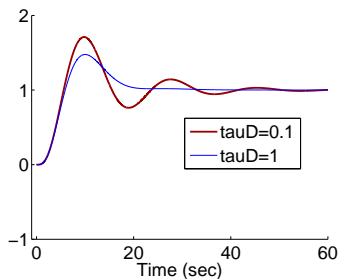
Figure 9: Step responses of PID control system ($K_c = 0.56$, $\tau_I = 8$, $\tau_D = 0.1, 1$)

Solution III

To reduce the overshoot, we calculate the closed-loop transfer function by using the Laplace transform of the controller output using the alternative structure. The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K_c(\beta\tau_D s + 1)}{\tau_I s^2(s+1)^3(\beta\tau_D s + 1) + K_c(\tau_I s + 1)(\beta\tau_D s + 1) + K_c\tau_I\tau_D s^2} \tag{29}$$

With this implementation, the denominator of the closed-loop transfer function is the same, however, there is only one zero at $-\frac{1}{\beta\tau_D}$ caused by the derivative control.

Solution IV

Figure 10 shows the closed-loop step responses with PID controller. In comparison with the responses from the previous case, it is seen that the overshoot in the closed-loop responses have been eliminated.
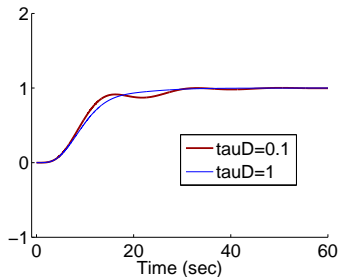
Figure 10: Step responses of PID control system with alternative structure ($K_c = 0.56$, $\tau_I = 8$, $\tau_D = 0.1, 1$)