Assignment 6

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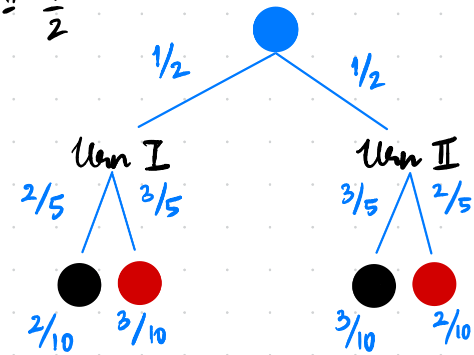
Q1 (a) $P(\text{black}) = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{3}{5} = \frac{1}{2}$



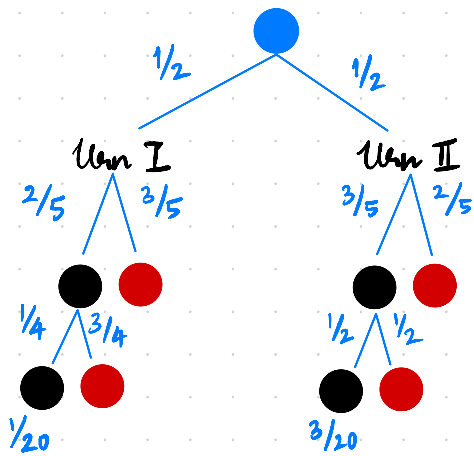
$\Rightarrow \boxed{P(\text{black}) = 1/2}$

(b) $P(B_2 \mid B_1) = \dfrac{P(B_1 \cap B_2)}{P(B_1)}$

$P(B_1) = 1/2$

$P(B_1 \cap B_2) = \frac{1}{2} \times \frac{2}{5} \times \frac{1}{4} + \frac{1}{2} \times \frac{3}{5} \times \frac{1}{2}$

$= \frac{4}{20} = \frac{1}{5}$



$P(B_2 \mid B_1) = \dfrac{1/5}{1/2}$

$\Rightarrow \boxed{P(B_2 \mid B_1) = 2/5}$

(Q2) Total outcomes $= 2^3 = 8$

(a) $\Rightarrow P(\text{each 3 bit number}) = \frac{1}{8}$

We want numbers from 1-6 $\Rightarrow$ 6 outcomes

$\Rightarrow P(1\text{-}6) = \frac{6}{8} = \frac{3}{4}$

$\Rightarrow P(\text{number} \mid \text{number is in } 1\text{-}6) = P(i \mid i = 1\text{-}6)$

$P(i \mid i = 1\text{-}6) = 1/8 \,/\, 6/8 = \frac{1}{6}$

$\Rightarrow \boxed{P(i=1 \mid i = 1\text{-}6) = \frac{1}{6} = P(\text{fair dice})}$

(b) Number of coin tosses per trial $= 3$

$P(i = 1\text{-}6) = \frac{3}{4}$

$\Rightarrow P(\text{repeating a trial}) = 1 - \frac{3}{4} = \frac{1}{4}$

$\Rightarrow$ Expected number tosses $E = 3 + \frac{1}{4} E$

$\Rightarrow \frac{3}{4} E = 3 \Rightarrow E = 4$

$\Rightarrow \boxed{\text{Expected number of coin tosses } E = 4}$

Q3 (a) Let $A = [1,2,3,4,5,6,7,8,9]$

i	r	x	$A \longrightarrow swap(A[i] \leftrightarrow A[x])$
9	0.333389	$\lfloor 9 \times 0.333389 \rfloor + 1 = 4$	$[1,2,3,9,5,6,7,8,4]$
8	0.286361	$\lfloor 8 \times 0.286361 \rfloor + 1 = 3$	$[1,2,8,9,5,6,7,3,4]$
7	0.172394	$\lfloor 7 \times 0.172394 \rfloor + 1 = 2$	$[1,7,8,9,5,6,2,3,4]$
6	0.156509	$\lfloor 6 \times 0.156509 \rfloor + 1 = 1$	$[6,7,8,9,5,1,2,3,4]$
5	0.495376	$\lfloor 5 \times 0.495376 \rfloor + 1 = 3$	$[6,7,5,9,8,1,2,3,4]$
4	0.352533	$\lfloor 4 \times 0.352533 \rfloor + 1 = 2$	$[6,9,5,7,8,1,2,3,4]$
3	0.521332	$\lfloor 3 \times 0.521332 \rfloor + 1 = 2$	$[6,5,9,7,8,1,2,3,4]$
2	0.924328	$\lfloor 2 \times 0.924328 \rfloor + 1 = 2$	(no swap since $i = x$)

$\Rightarrow \boxed{A = [6,5,9,7,8,1,2,3,4]}$

(b)

$A[i] = i$	r	X	$A \longrightarrow swap(A[i] \leftrightarrow A[X])$
1			[1]
2	0.333389	$\lfloor 2 \times 0.333389 \rfloor + 1 = 1$	[2, 1]
3	0.286361	$\lfloor 3 \times 0.286361 \rfloor + 1 = 1$	[3, 1, 2]
4	0.172394	$\lfloor 4 \times 0.172394 \rfloor + 1 = 1$	[4, 1, 2, 3]
5	0.156509	$\lfloor 5 \times 0.156509 \rfloor + 1 = 1$	[5, 1, 2, 3, 4]
6	0.495376	$\lfloor 6 \times 0.495376 \rfloor + 1 = 3$	[5, 1, 6, 3, 4, 2]
7	0.352533	$\lfloor 7 \times 0.352533 \rfloor + 1 = 3$	[5, 1, 7, 3, 4, 2, 6]
8	0.521332	$\lfloor 8 \times 0.521332 \rfloor + 1 = 5$	[5, 1, 7, 3, 8, 2, 6, 4]
9	0.924328	$\lfloor 9 \times 0.924328 \rfloor + 1 = 9$	[5, 1, 7, 3, 8, 2, 6, 4, 9]

$\Rightarrow \boxed{A = [5, 1, 7, 3, 8, 2, 6, 4, 9]}$

Q4 Minimum needed to play a game = \$2
$P(\text{win}) = 1/3$
$P(\text{lose}) = 2/3$

gain if we win = \$5 − \$2 = \$3
gain if we lose = \$0 − \$2 = −\$2

The lower bound is the number of games we can consecutively lose until we run out of money, i.e \$0

$\Rightarrow \boxed{\text{Lower bound if we lose all games} = \frac{\$10}{\$2} = 5 \text{ games}}$

Expected gain per game $E[g] = P(\text{win}) \times \text{gain}(\text{win}) + P(\text{lose}) \times \text{gain}(\text{lose})$

$\Rightarrow E[g] = \frac{1}{3} \times 3 - \frac{2}{3} \times 2 = -\$\frac{1}{3}$

$\Rightarrow$ we lose $\$\frac{1}{3}$ per game

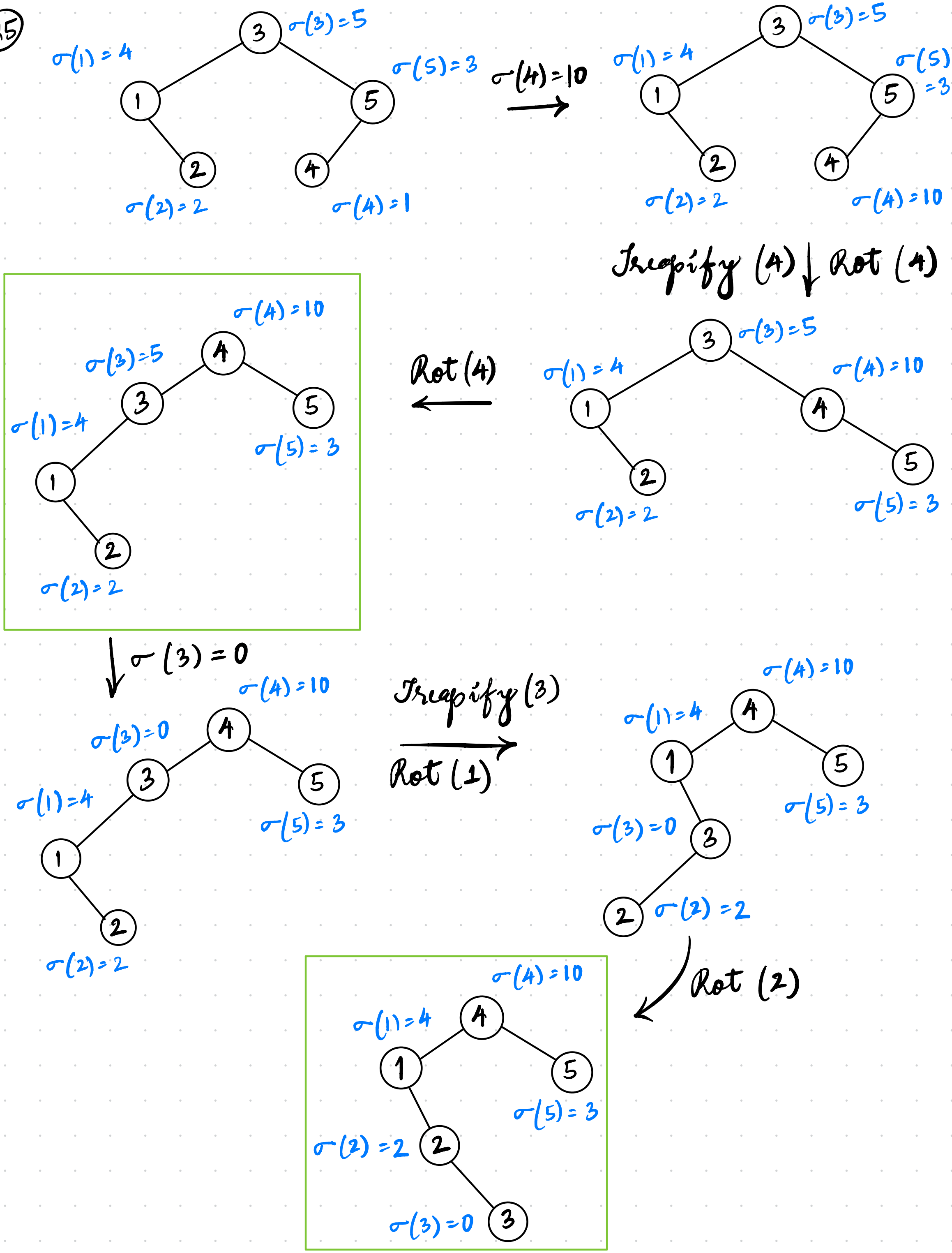
We must stop playing when we have < \$2 remaining

$\Rightarrow 10 - \frac{1}{3} \times E(\text{number of games } n) \geq 2 \Rightarrow n \leq 24$ and $10 - \frac{24}{3} = \$2$

$\Rightarrow$ Expected number of games we can play = 24. This means, on average, we will get to play 24 games, leaving us with \$2. However, since this is an expected value, we may have < \$2 as well. If we play another game with the \$2 $\Rightarrow$ 24+1 = 25 games, we will be left with \$2 − 1/3 = \$1.67 on average and \$0 if we lose $\Rightarrow$ we cannot play further.

$\Rightarrow \boxed{24 \leq E(\text{number of games}) < 25}$

Q5

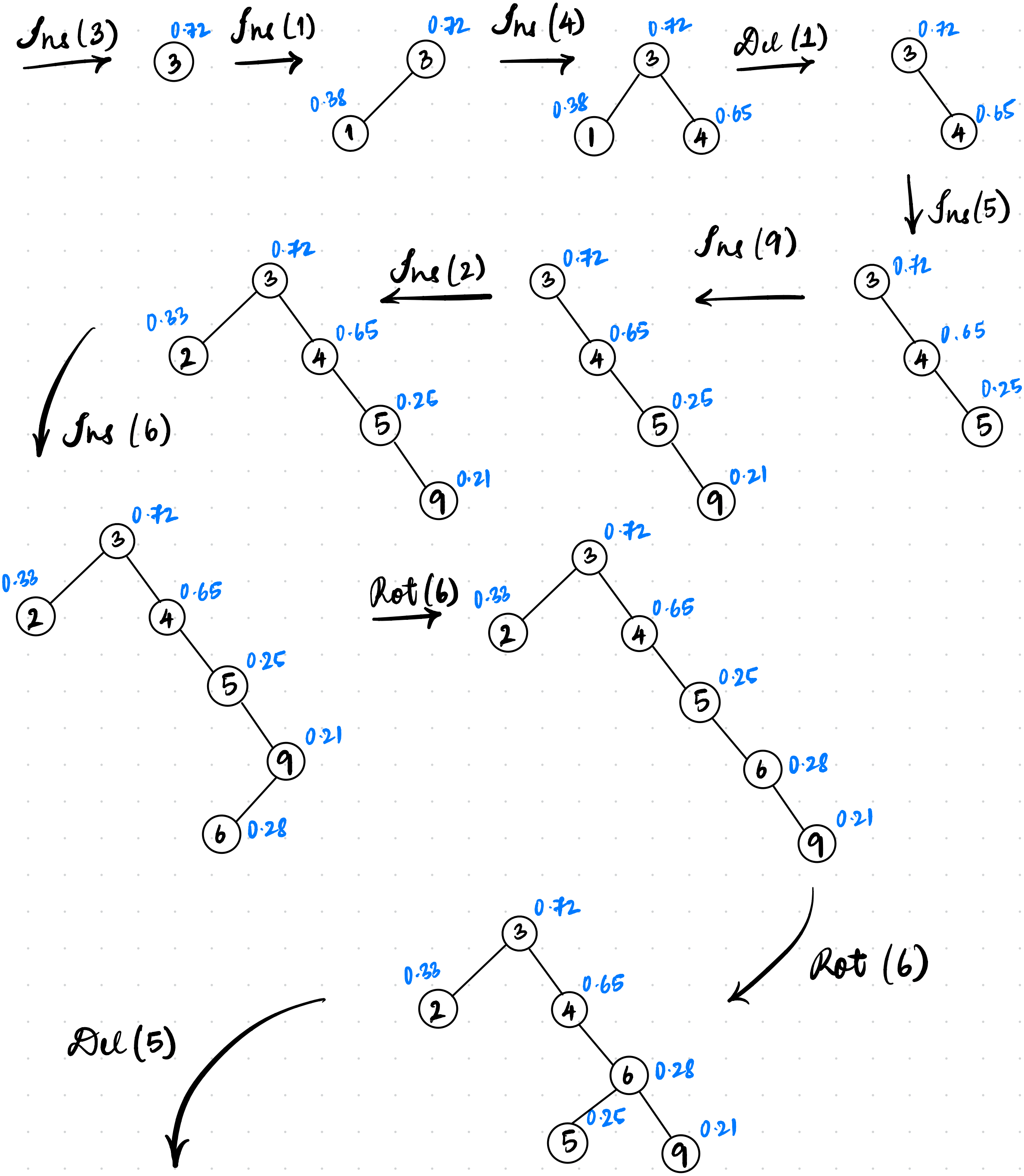


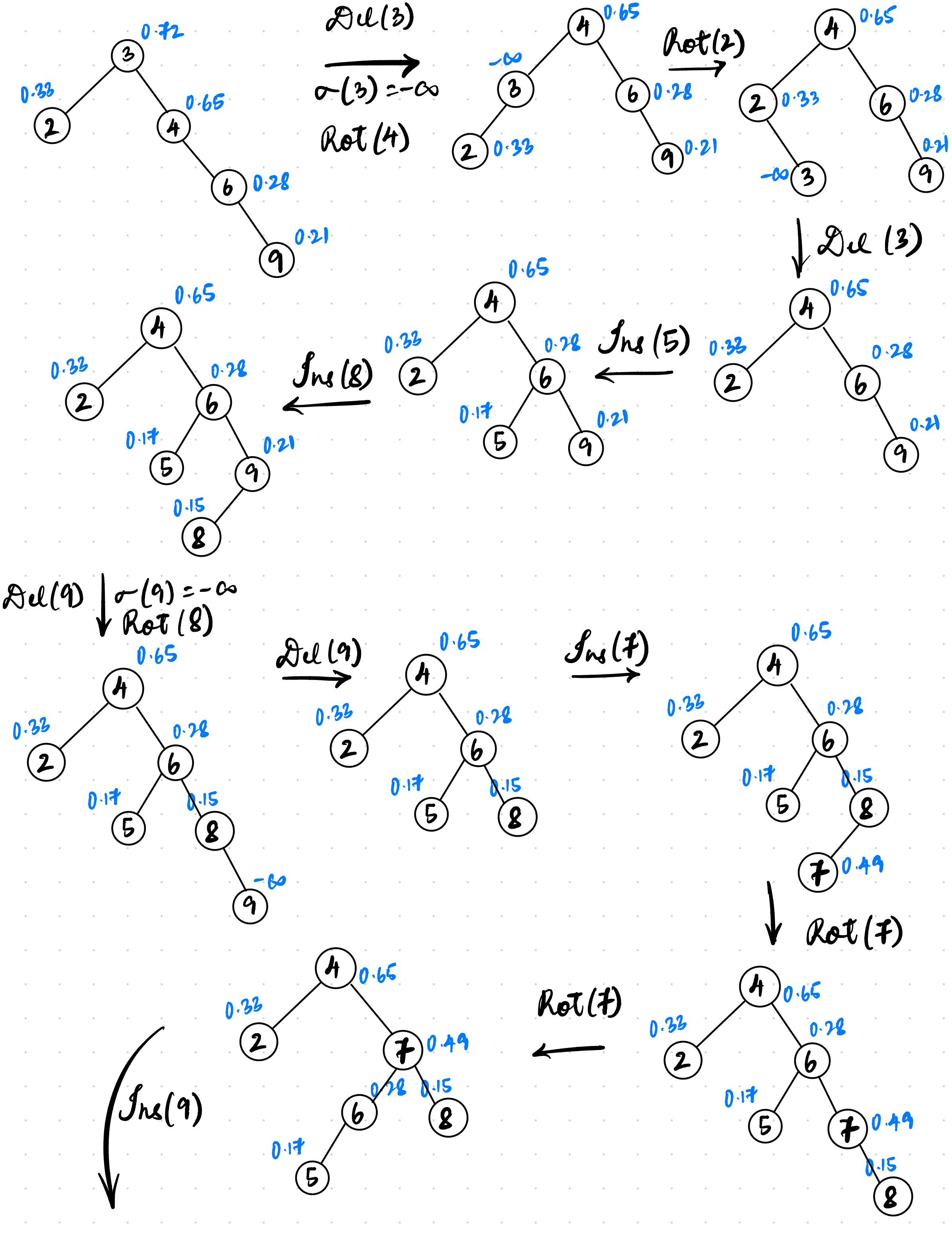
σ(1) = 4
σ(2) = 2
σ(3) = 5
σ(4) = 1
σ(5) = 3
σ(4) = 10
σ(1) = 4
σ(2) = 2
σ(3) = 5
σ(4) = 10
σ(5) = 3
Treapify (4)
Rot (4)
σ(1) = 4
σ(2) = 2
σ(3) = 5
σ(4) = 10
σ(5) = 3
Rot (4)
σ(4) = 10
σ(3) = 5
σ(1) = 4
σ(5) = 3
σ(2) = 2
σ (3) = 0
σ(4) = 10
σ(3) = 0
σ(1) = 4
σ(5) = 3
σ(2) = 2
Treapify (3)
Rot (1)
σ(4) = 10
σ(1) = 4
σ(5) = 3
σ(3) = 0
σ(2) = 2
Rot (2)
σ(4) = 10
σ(1) = 4
σ(5) = 3
σ(2) = 2
σ(3) = 0

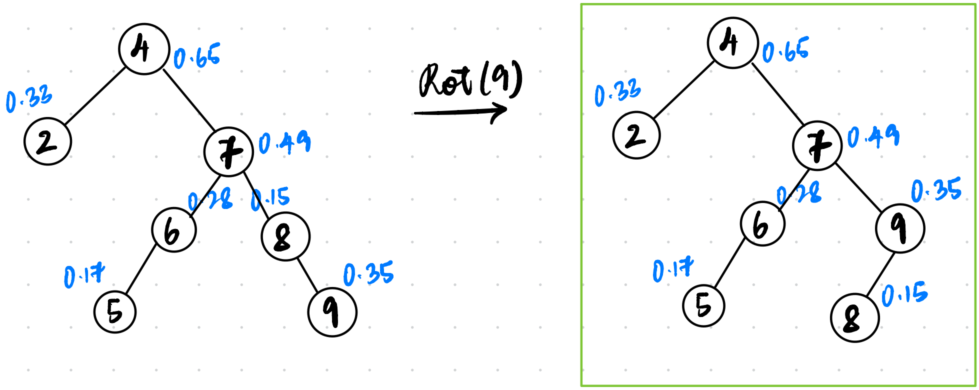
Q6

i	r	σ
3	0.72183	[3]
1	0.381787	[3, 1]
4	0.659242	[3, 4, 1]
1		[3, 4]
5	0.254401	[3, 4, 5]
9	0.210082	[3, 4, 5, 9]
2	0.333389	[3, 4, 2, 5, 9]
6	0.286361	[3, 4, 2, 6, 5, 9]
5		[3, 4, 2, 6, 9]
3		[4, 2, 6, 9]
5	0.172394	[4, 2, 6, 9, 5]
8	0.156509	[4, 2, 6, 9, 5, 8]
9		[4, 2, 6, 5, 8]
7	0.495376	[4, 7, 2, 6, 5, 8]
9	0.352533	[4, 7, 9, 2, 6, 5, 8]

Q6




Del(3)
σ(3) = -∞
Rot(4)
Rot(2)
Del (3)
Ins (5)
Ins (8)
Del(9)
σ(9) = -∞
Rot (8)
Del (9)
Ins (7)
Rot (7)
Rot (7)
Ins(9)



4
0.65
0.33
2
7
0.49
0.28
0.15
6
8
0.17
0.35
5
9
Rot (9)
4
0.65
0.33
2
7
0.49
0.28
0.35
6
9
0.17
5
8
0.15

Q7 Padded Mapping ($\mathbb{Z}_{29}^{5}$) →

fig = $(6, 9, 7, 0, 0)$

plum = $(16, 12, 21, 13, 0)$

kiwi = $(11, 9, 23, 9, 0)$

pear = $(16, 5, 1, 18, 0)$

lime = $(12, 9, 13, 5, 0)$

date = $(4, 1, 20, 5, 0)$

apple = $(1, 16, 16, 12, 5)$

melon = $(13, 5, 12, 15, 14)$

mango = $(13, 1, 14, 7, 1, 5)$

Computing $a = (a_0 \dots a_5)$ →

$a_0 = \lfloor 29 \times 0.721383 \rfloor = 20$

$a_1 = \lfloor 29 \times 0.381787 \rfloor = 11$

$a_2 = \lfloor 29 \times 0.659242 \rfloor = 19$

$a_3 = \lfloor 29 \times 0.254401 \rfloor = 7$

$a_4 = \lfloor 29 \times 0.210082 \rfloor = 6$

$a_5 = \lfloor 29 \times 0.333389 \rfloor = 9$

$$k = a_0 + \sum_{i=1}^{5} a_i x_i$$

$$k_{fig} = 20 + 11\times6 + 19\times9 + 7\times7 + 6\times0 + 9\times0 = 306$$

$$k_{plum} = 20 + 11\times16 + 19\times12 + 7\times21 + 6\times13 + 9\times0 = 649$$

$$k_{kiwi} = 20 + 11\times11 + 19\times9 + 7\times23 + 6\times9 + 9\times0 = 527$$

$$k_{pear} = 20 + 11\times16 + 19\times5 + 7\times1 + 6\times18 + 9\times0 = 406$$

$$k_{lime} = 20 + 11\times12 + 19\times9 + 7\times13 + 6\times5 + 9\times0 = 444$$

$$k_{date} = 20 + 11\times4 + 19\times1 + 7\times20 + 6\times5 + 9\times0 = 253$$

$$k_{apple} = 20 + 11\times1 + 19\times16 + 7\times16 + 6\times12 + 9\times5 = 564$$

$$k_{melon} = 20 + 11\times13 + 19\times5 + 7\times12 + 6\times15 + 9\times14 = 558$$

$$k_{mango} = 20 + 11\times13 + 19\times1 + 7\times14 + 6\times7 + 9\times15 = 457$$

Collision Chart for $k_i \bmod m \ \forall k_i \in K$

k \ m	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
fig	0	6	9	6	7	14	6	2	0	0	2	6	12	20	7	18	6	20
plum	1	9	0	1	12	7	4	9	3	1	9	9	28	13	5	1	24	25
kiwi	5	7	10	11	7	17	2	15	0	5	15	7	2	23	21	23	2	7
pear	1	6	10	10	3	0	1	6	16	10	7	6	7	10	15	22	6	16
lime	3	6	3	0	2	14	9	12	5	6	9	4	3	4	7	12	19	2
date	1	3	0	1	6	1	13	13	14	1	7	13	1	11	0	13	3	19
apple	6	4	3	0	5	4	9	12	3	6	15	4	24	16	12	12	14	18
melon	0	8	8	6	12	14	3	14	14	0	7	18	15	8	6	6	8	12
mango	7	7	6	1	2	9	7	9	15	7	1	17	19	19	20	1	7	15
	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✗	✓

Total conflict	4	3	2	5	3	2	1	1	5	2	3	2	1	1	1	2	1	$\boxed{0}$

$\Rightarrow$ $\boxed{\text{Smallest } m \text{ for perfect } h_a = 26}$

Q8) $p = 31$

$H_m = \{(ax \bmod 31) \bmod m : a \in \mathbb{Z}_{31}$

$K = \{2, 4, 9, 18, 20, 28, 30\}$

$|K| = 7 \Rightarrow n = 7$

Computing Primary Hash →

To ensure $\frac{(p-1)}{2} = \frac{31-1}{2} = 15$ choices of k satisfy $\sum_{i=0}^{n-1} \binom{b_k^{(i)}}{2} < \frac{n^2}{m} = n$,

we set $m = n = 2$

$\Rightarrow H = (ax \bmod 31) \bmod 7$

Let $a = 2 \Rightarrow H_{a=2} = (2x \bmod 31) \bmod 7$

$B_0 = [\ ] \Rightarrow b_k^0 = 0$
$B_1 = [4, 30] \Rightarrow b_k^1 = 2$
$B_2 = [20] \Rightarrow b_k^2 = 1$
$B_3 = [\ ] \Rightarrow b_k^3 = 0$
$B_4 = [2, 9, 28] \Rightarrow b_k^4 = 3$
$B_5 = [18] \Rightarrow b_k^5 = 1$
$B_6 = [\ ] \Rightarrow b_k^6 = 0$

$$\sum_{i=0}^{n-1} \binom{b_k^i}{2} = \binom{2}{2} + \binom{3}{2} = 1 + 3 = 4 < n$$

$$\Rightarrow \sum_{i=0}^{n-1} \binom{b_k^i}{2} = 4 < 7 \Rightarrow a = 2 \text{ is valid.}$$

$B_1 = [4, 30]$ $\qquad b_1^2 = 2^2 = 4$

$B_2 = [20]$ $\qquad b_2^2 = 1^2 = 1$

$B_4 = [2, 9, 28]$ $\qquad b_4^2 = 3^2 = 9$

$B_5 = [18]$ $\qquad b_5^2 = 1^2 = 1$

Computing Secondary Hash →

$$h_i(x) = (ax \bmod 31) \bmod b_i^2$$

$$= (2x \bmod 31) \bmod b_i^2$$

i	B	b_i^2	$h_i(x)$	Buckets
1	B_1	4	$(2x \bmod 31) \bmod 4$	$[4]_0\ [30]_1\ [\ ]_2\ [\ ]_3$
2	B_2	1	$(2x \bmod 31) \bmod 1$	$[20]_0$
4	B_4	9	$(2x \bmod 31) \bmod 9$	$[9]_0\ [\ ]_1\ [\ ]_2\ [\ ]_3\ [2]_4\ [\ ]_5\ [\ ]_6\ [28]_7\ [\ ]_8$
5	B_5	1	$(2x \bmod 31) \bmod 1$	$[18]_0$