Assignment 2

ADITEYA BARAL (ab12057)

Q1 $X = 6 = (0110)_2$

$Y = 11 = (1011)_2$

- $K(0110, 1011)$ — $1000010_2 = 6 + (15-6-2) \times 2^2 + 2 \times 2^4 = 66_{10}$
  - $K(01, 10)$ — $10_2 = 0 + (1-0-0)2 + 0 = 2_{10}$
    - $K(0,1)$: 0
    - $K(1,1)$: 1
    - $K(1,0)$: 0
  - $K(11, 101)$ — $1111_2 = 1 + (6-2-1) \times 2 + 2 \times 2^2 = 15_{10}$
    - $K(1,10)$: 10
    - $K(10,11)$ — $110_2 = 0 + (2-1-0)2 + 1 \times 2^2 = 6_{10}$
      - $K(1,1)$: 1
      - $K(1,10)$: 10
      - $K(0,1)$: 0
    - $K(1,1)$: 1
  - $K(10, 11)$ — $110_2 = 0 + (2-1-0)2 + 1 \times 2^2 = 6_{10}$
    - $K(1,1)$: 1
    - $K(1,10)$: 10
    - $K(0,1)$: 0

(i)

Q2) At each step, the algorithm divides two n-digit numbers into $n/2$ digit halves each. However, if n is odd, the halves are actually of size $= \lceil n/2 \rceil$. An extra term: $+1$ is used to account for any additional overhead due to a carry bit introduced while splitting the numbers. Since three recursive calls are made for each multiplication, the multiplicative recurrant term is $3 \times T(\lceil n/2 \rceil + 1)$

Alongside multiplication, the algorithm also performs additions and subtractions to compute terms at intermediate steps:

(a) $2 \times$ $n/2$-digit additions
(b) $2 \times$ n-digit subtractions
(c) $1 \times$ 2n-digit Z computation

} All take linear time in n $\Rightarrow 2 \times n/2 + 2n + 2n = 5n$

Thus, a $+5n$ term is used to represent the worst-case number of linear time operations like additions, subtractions and left shifts. A final $O(1)$ term is added to account for fixed overheads such as base cases and other inexpensive but fixed-cost operations like splitting numbers.

$$\Rightarrow \boxed{T(n) = 3T\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) + 5n + O(1)}$$

(ii) $T(n) = 3T\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) + 5n$

We need to show:

$$T(n) \le U(n) = (n+3)^{\lg 3} - Kn \qquad \exists K \geq 0$$

Assume $T(n) \le U(n)$ (ev.) $\forall m < n$

$\Rightarrow T(m) \le (m+3)^{\lg 3} - Km \quad \forall m < n$

By inductive hypothesis, we can bound $T\left(\lceil \frac{n}{2} \rceil + 1\right)$ as,

$$T\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \leq \left(\left\lceil \frac{n}{2} \right\rceil + 1 + 3\right)^{\lg 3} - K\left(\left\lceil \frac{n}{2} \right\rceil + 1\right)$$

We know that $\left\lceil \frac{n}{2} \right\rceil + 1 \leq \frac{n+3}{2}$

$$\Rightarrow T\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \leq \left(\frac{n+3}{2} + 3\right)^{\lg 3} - K\left(\frac{n+3}{2}\right)$$

$$= \left(\frac{n+9}{2}\right)^{\lg 3} - K\left(\frac{n+3}{2}\right)$$

Substituting into $T(n)$,

$$T(n) \leq 3\left[\left(\frac{n+9}{2}\right)^{\lg 3} - K\left(\frac{n+3}{2}\right)\right] + 5n$$

$$= 3\left(\frac{n+9}{2}\right)^{\lg 3} - 3K\left(\frac{n+3}{2}\right) + 5n$$

We need to show $T(n) \leq U(n)$,

$$\Rightarrow 3\left(\frac{n+9}{2}\right)^{\lg 3} - 3K\left(\frac{n+3}{2}\right) + 5n \leq (n+3)^{\lg 3} - Kn$$

$$(n+9)^{\lg 3} - (n+3)^{\lg 3} \leq 3K\left(\frac{n+3}{2}\right) - 5n - Kn$$

As $n \to \infty$, $(n+9)^{\lg 3} \approx (n+3)^{\lg 3}$

$$\Rightarrow \frac{3Kn}{2} + \frac{9K}{2} - 5n - Kn \geq 0$$

$$\left(\frac{K}{2}\right)n + \frac{9K}{2} - 5n \geq 0$$

$$\Rightarrow \frac{Kn}{2} + \frac{9K}{2} - 5n \geq 0$$

$$\frac{9K}{2} + n\left(\frac{K}{2} - 5\right) \geq 0$$

Since $\frac{9K}{2}$ is positive, and as $n \to \infty$, $\left(\frac{K}{2} - 5\right)$ must be ≥ 0 for the inequality to hold,

$$\Rightarrow \frac{K}{2} - 5 \geq 0 \Rightarrow \boxed{K \geq 10}$$

(iii) $V(n)$ is not an honest upper bound because:

(a) The $(n+3)^{\lg 3}$ term follows a smooth power-law growth by assuming a smooth division of 'n'. However due to varying sizes of n (caused by $\lceil n/2 \rceil + 1$), subproblems of varying sizes are created leading to step-wise growth. Thus $V(n)$ is an ideal smooth approximation which is not true always.

(b) The $(n+3)^{\lg 3}$ term is an asymptotic approximation of the dominant term in $T(n)$. It ignores lower order terms and multiplicative constants that are practically significant. The subtraction of Kn is an oversimplified attempt at accounting for these terms. It also does not account for other factors like additional summations or shift operations.

(c) The base case $T(1)$ is not accounted for in $V(n)$, which contributes to overall runtime. Also, the additional constant overhead $O(1)$ term in the recurrence is ignored in $V(n)$. Multiplicative constants such as 3 in $3T(\lceil n/2 \rceil + 1)$ aren't accounted in either.

(d) The D/C approach is thus insufficient at finding an upper bound. We need to explicitly include lower order terms in the bound to acount for all contributions to growth. Base cases need to be handled as well to acount for increased runtime for small n. Ceiling functions $\lceil n/2 \rceil$ and the +1 term must be included to account for non-smooth splitting of n and carry over bits. We can achieve this by opting for a stricter and more expressive upper bound where:

$$\boxed{T(n) \leq C \cdot n^{\lg 3} - Dn + E}$$

where C captures the dominant $n^{\lg 3}$ term, D captures all linear operations such as addition and combining 2. E is used to account for any constant overheads including O(1) time computations. It can be derived as:

$$T(n) = 3T(n/2) + 5n + O(1)$$

$$= 3\left(3T(n/4) + 5\frac{n}{2} + O(1)\right)$$

$$\Rightarrow T(n) = 3^k T\left(\frac{n}{2^k}\right) + 5n\left(1 + \frac{3}{2} + \frac{3^2}{2^2} + \dots + \frac{3^{k-1}}{2^{k-1}}\right) + O(1)\left(1 + 3 + 3^2 + \dots + 3^{k-1}\right)$$

$$5n\left(1 + \frac{3}{2} + \dots + \frac{3^{k-1}}{2^{k-1}}\right) = 5n\,\frac{1-(3/2)^k}{1-3/2} = 10n\left[\left(\frac{3}{2}\right)^k - 1\right]$$

$$O(1)\left(1 + 3 + \dots + 3^{k-1}\right) = O(1) \cdot \frac{3^k - 1}{2}$$

Let $k = \lg n$

$$\Rightarrow T(n) = 3^{\lg n} T(1) + 10n\left(n^{\lg 3/2} - 1\right) + O(1) \cdot \frac{n^{\lg 3} - 1}{2}$$

$$T(n) = n^{\lg 3} T(1) + 10n \cdot n^{\lg 3/2} - 10n + O(n^{\lg 3})$$
$$= n^{\lg 3} T(1) + 10n^{\lg 3} - 10n + O(n^{\lg 3})$$
$$= n^{\lg 3} \left[T(1) + 10 + O(1) \right] - 10n$$

To account for additional overhead, $\lceil n/2 \rceil$ and $a + 1$, add E

$$\boxed{T(n) \leq C \cdot n^{\lg 3} - Dn + E}$$

when n is not even

$C = T(1) + 10 + O(1)$
$D = 10$
$E =$ overhead constant in $O(1)$

This expression is a more realistic upper bound for $T(n)$ since it includes the dominant term for growth, the linear term for lower order operations and multiplicative constants and an additional overhead for handling the splitting of n and other $O(1)$ time operations.

Q3

(a) $T(n) = 4T(n/2) + \dfrac{n^2}{\lg^2 n}$

Let $k = \lg n \Rightarrow n = 2^k$

$$T(2^k) = 4T(2^{k-1}) + \frac{2^{2k}}{\lg^2 2^k}$$

Let $s(k) = T(2^k)$

$$\Rightarrow s(k) = 4s(k-1) + \frac{4^k}{k^2}$$

Let $t(k) = \dfrac{s(k)}{4^k} \Rightarrow s(k) = 4^k t(k)$

$$\Rightarrow 4^k t(k) = 4 \cdot 4^{k-1} t(k-1) + \frac{4^k}{k^2}$$

$$\boxed{t(k) = t(k-1) + \frac{1}{k^2}}$$

$t(k) = t(0) + \dfrac{1}{k^2}$ [Telescoping the recurrence]

$\Rightarrow t(k) = \displaystyle\sum_{i=1}^{k} \frac{1}{i^2} = H_k^{(-2)}$ [Assume $t(0) = 0$]

$$\Rightarrow s(k) = 4^k t(k) = 4^k H_k^{(-2)}$$

$$\Rightarrow T(n) = s(\lg n) = 4^{\lg n} H_{\lg n}^{(-2)}$$

$$\Rightarrow \boxed{T(n) = n^2 H_{\lg n}^{(-2)} = n^2 \sum_{i=1}^{\lg n} \frac{1}{i^2}}$$

If n is a power of 2 $(n = 2^m)$

$$T(n) = n^2 H^{(-2)}_{\lg 2^m} = n^2 H^{(-2)}_{m}$$

As $m \to \infty$, $H^{(-2)}_{m} = \frac{\pi^2}{6} = \Theta(1)$ [constant value]

$$\Rightarrow \boxed{T(n) = n^2 H^{(-2)}_{\lg n} = \Theta(n^2)}$$

For general $n \geq 2$

$$\boxed{T(n) = n^2 H^{(-2)}_{\lfloor \lg n \rfloor} = \Theta(n^2)}$$

At $n < 2 \Rightarrow T(n/2)$ is invalid and $\lg n$ is undefined

DIC $\Rightarrow \boxed{T(n) = 0 \ \forall\, n < 2}$ [Since no subproblems of size $n/2$ exist at $n < 2$, we set $T(n) = 0$]

(b) $T(n) = 4T(n/2) + \frac{n^2}{\sqrt{\lg n}}$

Let $k = \lg n \Rightarrow n = 2^k$

$$\Rightarrow T(2^k) = 4T(2^{k-1}) + \frac{(2^k)^2}{\sqrt{\lg 2^k}}$$

Let $s(k) = T(2^k)$

$$\Rightarrow s(k) = 4s(k-1) + \frac{4^k}{\sqrt{k}}$$

Let $4^k t(k) = s(k)$

$$\Rightarrow \boxed{t(k) = t(k-1) + 1/\sqrt{k}}$$

$$t(k) = t(0) + \sum_{i=1}^{k} \frac{1}{\sqrt{i}} \quad \text{[Telescope the recurrence]}$$

Assume $t(0) = 0$ $[T(1) = 0]$

$$\Rightarrow t(k) = \sum_{i=1}^{k} \frac{1}{\sqrt{i}} = H_k^{(-1/2)}$$

$$\Rightarrow T(n) = n^2 H_{\lg n}^{(-1/2)} = n^2 \sum_{i=1}^{\lg n} \frac{1}{\sqrt{i}}$$

$$T(n) = n^2 \left(2\sqrt{\lg n}\right) \qquad \left[\sum_{i=1}^{\lg n} \frac{1}{\sqrt{i}} \approx 2\sqrt{\lg n} \text{ as } \lg n \to \infty\right]$$

If n is a power of 2:

$$\boxed{T(n) = n^2 H_{\lg n}^{(-1/2)} = 2n^2\sqrt{\lg n} = \Theta\left(n^2\sqrt{\lg n}\right)}$$

For $n \geq 2$, $\boxed{T(n) = n^2 H_{\lfloor \lg n \rfloor}^{(-1/2)} = 2n^2\sqrt{\lfloor \lg n \rfloor} = \Theta\left(n^2\sqrt{\lg n}\right)}$

$$\boxed{\text{DIC} \Rightarrow T(n) = 0 \quad \forall n < 2}$$

Q4 $E_1 = (2^n)^{1/\lg n} = 2^{n/\lg n}$ [exponentially increasing]

$E_2 = (2^n)^{\lg 1/n} = (2^n)^{\lg 1 - \lg n} = 2^{-n \lg n} = n^{-n} = \frac{1}{n^n} = \left(\frac{1}{n}\right)^n$

$E_3 = (\lg n)^{2^{1/n}}$

$E_4 = (\lg n)^{1/2^n} = (\lg n)^{2^{-n}}$

$E_5 = \left(\frac{1}{n}\right)^{2^{\lg n}} = (1/n)^n$

$E_6 = \left(\frac{1}{n}\right)^{\lg 2^n} = (1/n)^n$

$E_2 = E_5 = E_6 \Rightarrow$ decreasing exponentially at the same rate

Exponential functions grow faster than log $\Rightarrow E_1 \gg E_3$

As $n \to \infty$,

$E_3 = (\lg n)^{2^{1/n}} = (\lg n)^{2^0} = \lg n$

$E_4 = (\lg n)^{1/2^n} = (\lg n)^0 = 1$

$\Rightarrow$ At ∞, E_3 grows faster than E_4, which reaches a max value ≈ 1. $\Rightarrow E_3 \gg E_4$

Also, $2^{1/n}$ grows faster than $1/2^n$

$$\boxed{E_1 \gg E_3 \gg E_4 \gg E_2 = E_5 = E_6}$$

Q5

$$T(n) = aT(n/b) + cT(n/d) + 1$$

(a) We can determine the polarity of 'w' by analysing the following equation:

$$1 = \frac{a}{b^w} + \frac{c}{d^w}$$

For $w = 0 \Rightarrow b^w = d^w = 1 \Rightarrow \boxed{a + c = 1}$

For $w > 0 \Rightarrow b^w, d^w > 1 \Rightarrow \frac{a}{b^w} + \frac{c}{d^w} < \frac{a}{1} + \frac{c}{1}$

$$\Rightarrow \boxed{a + c > 1}$$

For $w < 0 \Rightarrow b^w, d^w < 1 \Rightarrow \frac{a}{b^w} + \frac{c}{d^w} > \frac{a}{1} + \frac{c}{1}$

$$\Rightarrow \boxed{a + c < 1}$$

(b) We know that: $\underline{a} \le a \le \bar{a}$ and $\underline{b}^w \le b^w < \bar{b}^w$

$$\Rightarrow \underbrace{\frac{\underline{a}}{\bar{b}^w} \le \frac{a}{b^w} \le \frac{\bar{a}}{\underline{b}^w}}_{(1)} \quad \text{and} \quad \underbrace{\frac{\underline{a}}{\bar{b}^w} \le \frac{c}{d^w} \le \frac{\bar{a}}{\underline{b}^w}}_{(2)}$$

(1) + (2)

$$\Rightarrow \frac{2\underline{a}}{\bar{b}^w} \le \frac{a}{b^w} + \frac{c}{d^w} \le \frac{2\bar{a}}{\underline{b}^w} \Rightarrow \frac{2\bar{a}}{\bar{b}^w} \le 1 \le \frac{2\bar{a}}{\underline{b}^w}$$

$$\Rightarrow \boxed{\log_{\bar{b}}(2\underline{a}) \le w \le \log_{\underline{b}}(2\bar{a})}$$

(c) $\underline{a} = 2, \ \bar{a} = 3, \ \underline{b} = 2, \ \bar{b} = 3$

$$\log_{\bar{b}}(2\underline{a}) = \log_3 4 \approx 1.26$$

$$\log_{\underline{b}}(2\bar{a}) = \log_2 6 \approx 2.58$$

$$\Rightarrow \quad \boxed{1.26 \leq w \leq 2.58}$$

(d) Approx (a, b, c, d, n):

lower $\leftarrow \log_{\bar{b}} \underline{a}$

upper $\leftarrow \log_{\underline{b}} \bar{a}$

mid = (upp + lower) / 2

while (upper − lower $> 10^{-n}$)

  mid $\leftarrow$ (upper + lower) / 2

  value $\leftarrow \dfrac{a}{b^{mid}} + \dfrac{c}{d^{mid}}$

  if (value > 1) then lower $\leftarrow$ mid

  else if (value < 1) then upper $\leftarrow$ mid

  else return mid

return mid

Q6

(a) $T(x) = x + 4T(x/3) + 3T(x/4)$

Show that : $T(x) \le Kx^2$ (ev.)

⇒ Find smallest value of K that holds for sufficiently large 'x'.

→ Assume <u>Real Inductive Hypothesis</u> (RI)

⇒ $\forall y \le x$, $T(y) \le Ky^2$ holds for some $K > 0$

⇒ $T(x/3) \le K\left(\frac{x}{3}\right)^2$ and $T(x/4) \le K\left(\frac{x}{4}\right)^2$

From the recurrence relation :

$$T(x) = x + 4T(x/3) + 3T(x/4)$$

$$\Rightarrow T(x) \le x + 4K\frac{x^2}{9} + 3K\frac{x^2}{16}$$

$$T(x) \le x + Kx^2\left(\frac{4}{9} + \frac{3}{16}\right) = x + Kx^2\left(\frac{91}{144}\right)$$

We need to show that :

$$x + Kx^2\left(\frac{91}{144}\right) \le Kx^2$$

$$Kx^2\left(1 - \frac{91}{144}\right) \ge x$$

$$\frac{K\cancel{x^2}}{\cancel{x^2}}\left(\frac{53}{144}\right) \ge \frac{\cancel{x}}{x^{\cancel{2}}} \qquad [x^2 > 0]$$

$$K\left(\frac{53}{144}\right) > 0 \qquad [\text{ lt } x \to \infty, 1/x \approx 0]$$

$\Rightarrow K > 0$

For the inequality to hold, let $K = \left\lceil \frac{144}{53} \right\rceil$

$\Rightarrow K = 3 \Rightarrow \boxed{T(x) \le 3x^2}$

$\Rightarrow$ $\boxed{\text{Smallest integer value of } K = 3 \text{ for } T(x) \le Kx^2}$

$\Rightarrow$ $\boxed{\text{Partial Real Induction (RI) holds}}$

(b) $T(x) = x + 4T(x/3) + 3T(x/4)$

We can express this as:

$T(x) = G(x, T(g_1(x)), T(g_2(x)))$ where,

$g_1(x) = x/3$
$g_2(x) = x/4$
$f(x) = x^2$

$\Rightarrow$ All these functions are growth functions since they are unbounded as $x \to \infty$, linear, non-decreasing for $x \ge 0$, positive for $x > 0$ and defined for all x.

$G(x) = x + 4t_1 + 3t_2 \rightarrow$ defined for all x, t_1, t_2, non-decreasing and positive for $x > 0$. Since it is a linear combination of x, t_1, t_2, G is also a growth function (x, t_1, t_2 are growth functions)

$\Rightarrow$ All functions are growth functions

We need to show: $\exists\, \delta > 0$ such that $g_i(x) \le x - \delta$

$g_1(x) = x/3$

$\frac{x}{3} \le x - \delta \Rightarrow \delta \le \frac{2x}{3}$

$\Rightarrow$ As $x \to \infty$, this is true if $\delta > 0$

$g_2(x) = x/4$

$\frac{x}{4} \le x - \delta \Rightarrow \delta \le \frac{3x}{4}$

$\Rightarrow$ As $x \to \infty$, this is true if $\delta > 0$

Thus, a gap exists for sufficiently large x ($x \to \infty$) and $g_i(x) \leq x - \delta$ (ev.) for $\delta > 0$ (such as $\delta = 1$)

<u>Showing Real Basis (RB):</u>

We need to show that for sufficiently large K ($K > 0$),

$$G(x, Kf(g_1(x)), Kf(g_2(x))) \leq Kf(x) \quad \text{(ev.)}$$

We know: $f(x) = x^2$, $g_1(x) = x/3$, $g_2(x) = x/4$

$$\Rightarrow G(x, Kf(g_1(x)), Kf(g_2(x))) = x + 4Kf(x/3) + 3Kf(x/4)$$

$$= x + 4K\frac{x^2}{9} + 3K\frac{x^2}{16}$$

$$= x + \frac{91Kx^2}{144}$$

$$\Rightarrow x + \frac{91Kx^2}{144} \leq Kx^2$$

$$x \leq Kx^2\left(1 - \frac{91}{144}\right) = Kx^2\left(\frac{53}{144}\right)$$

$$\frac{1}{x} \leq K\left(\frac{53}{144}\right)$$

$$\Rightarrow K \geq \frac{144}{53x}$$

As $x \to \infty$, the inequality will hold for any $K > 0$ (such as $K = 3$)

$$\Rightarrow \boxed{\text{Real Basis (RB) step holds true.}}$$

Since : 1. $G, f, g_1(x), g_2(x)$ are growth functions,
2. $g_i(x) \leq x - \delta$ (ev.) for $\delta = 1$

3. $G(x, Kf(g_1(x)), Kf(g_2(x))) \leq Kf(x)$ holds for $K = 3$

$\Rightarrow$ $\boxed{\text{RB step follows RI}}$

$$\boxed{T(x) \leq 3x^2 \text{ (ev.)}}$$