Assignment - 1

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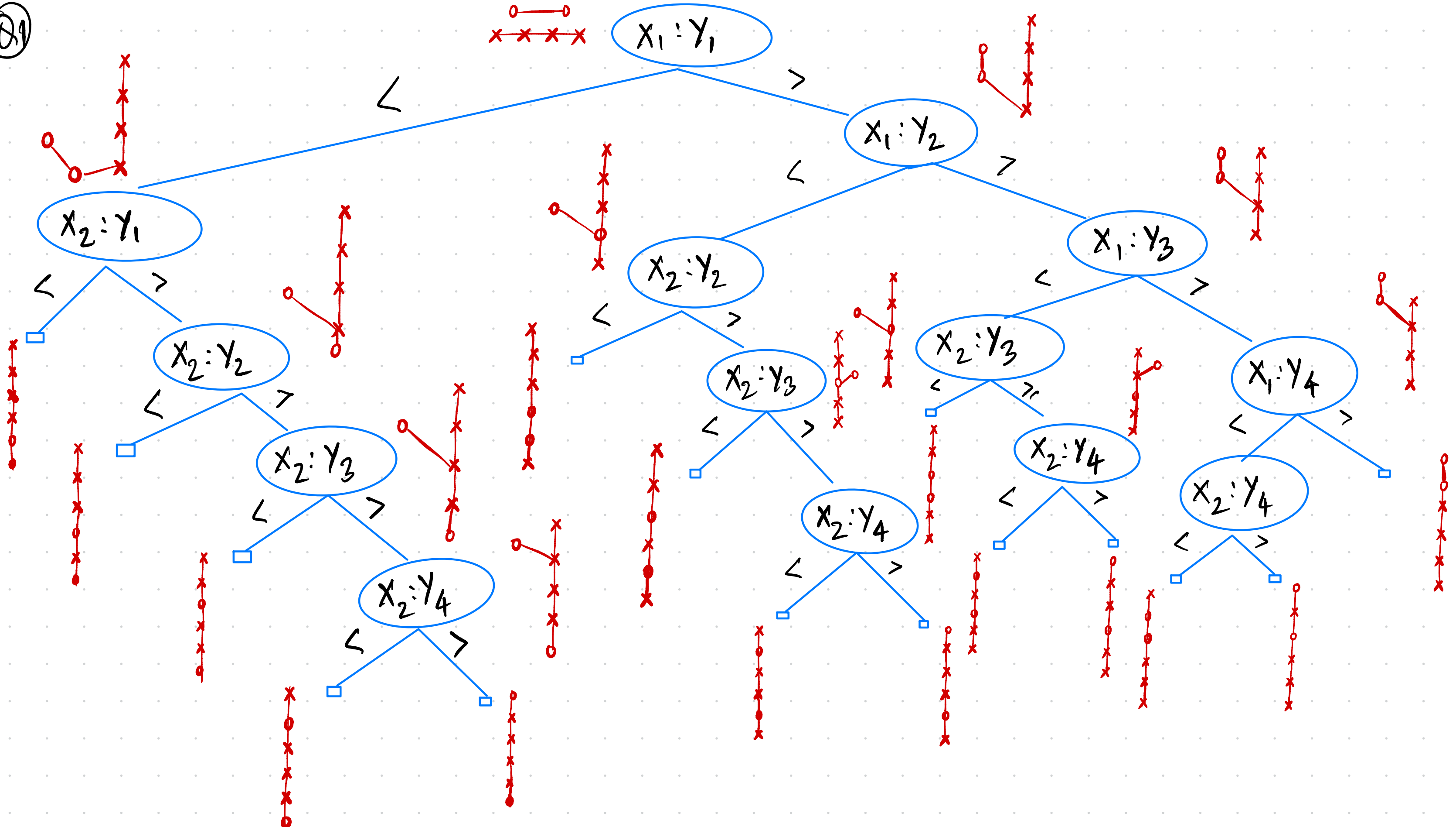
Q1



$x_1 : y_1$
$x_2 : y_1$
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$x_2 : y_4$
$x_2 : y_4$

(Q.2)

Maximum number of comparisons in Merge Sort after accounting for arbitrary n :

$$S(n) = n\lceil \log_2 n\rceil - 2^{\lceil \log n\rceil} + 1$$

$$\Rightarrow S(1000) = 1000\lceil \log_2 n\rceil - 2^{\lceil \log_2 1000\rceil} + 1$$

$\Rightarrow$ Upper bound for $S(1000) = \boxed{8977}$

Minimum number of comparisons = height of comparison tree
Number of possible orderings = $n!$
$\Rightarrow$ Height of comparison tree = $\lg(n!) \equiv$ Minimum number of comparisons

$\Rightarrow$ From Stirling's Approximation, we know that,

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\lg(n!) = \lg\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)$$

$$= \frac{1}{2}\lg(2\pi n) + n\lg(n) - n\lg e$$

$$= \frac{1}{2}\lg 2 + \frac{1}{2}\lg \pi + \frac{1}{2}\lg n + n\lg n - n\lg e$$

$$= \frac{1}{2} + \frac{1}{2}\lg \pi + \left(n+\frac{1}{2}\right)\lg n - \frac{n}{\log_e 2}$$

$\Rightarrow$ Lower bound for $S(1000) = \boxed{8530}$

→ Stirling's approximation is very accurate for the value of n we are dealing with, hence the error due to estimation is very low.

→ Log-properties help make simplification more accurate to mitigate small errors

→ Since Stirling's Approximation is used to find the logarithm of smaller values, there is very marginal error in computation.

→ Thus, using the closed form solution of the recurrence relation we can estimate the upper bound for S(1000). Similarly, we can estimate the lower bound using Stirling's Approximation, which can compute factorials of large numbers very accurately for numbers in that range (1000). This gives us our bounds as:

$$\boxed{8530 \leq S(1000) \leq 8977}$$

Q3) We know that,

$$g = o(f) \rightarrow 0 \le g < C.f, \forall C > 0$$

$$g = w(f) \rightarrow 0 \le f < C.g, \forall C > 0$$

Since $o(f)$ represents functions that strictly grow slower than f and w represents functions that strictly grow faster than f, we cannot have a function that does both at the same time

$\Rightarrow$ $g = o(f)$ and $g = w(f)$ is not possible simultaneously.

$\Rightarrow$ $\theta(f) = o(f) \cap w(f)$ requires $0 \le g < C.f$ **AND** $0 \le f < C.g$ $\forall C > 0$. Since this is not possible, there is no "small-theta" (or, it is an empty set of functions)

$\Rightarrow$ $\theta(f) = o(f) \cap w(f)$ cannot represent any function and thus is an empty set. Hence, no 'θ' notation exists.

Q4 (a) $f = O(g) \Rightarrow g = O(f)$

Let $f(x) = n-1$
$g(x) = n^2 - n + 1$

$f \leq c \cdot g$ $\quad [n-1 \leq c\,(n^2-n+1)]$
$\Rightarrow f = O(g)$

However, $g \nleq c \cdot f$ $\quad [n^2-n+1 \nleq c\,(n-1)]$

$\Rightarrow g \neq O(f)$

$\therefore f = O(g) \not\Rightarrow g = O(f)$ [FALSE]

(b) $\max(f, g) = \Theta(f+g)$

$\max(f,g) = O(f+g)$ and $f+g = O(\max(f,g))$

Let $f(n) = n$, $g(n) = -n+1$

$f(n) + g(n) = n - n + 1 = 1$

$\Rightarrow \Theta(f+g) = \Theta(1)$ [constant growth]

$\max(f,g) = n$

$\Rightarrow \Theta(\max(f,g)) = \Theta(n)$ [grows linearly]

Since $\Theta(\max(f,g)) \neq \Theta(f+g)$

$\Theta(n) \neq \Theta(1)$

$\Rightarrow \max(f,g) \neq \Theta(f+g)$

[FALSE]

(c) If $g > 1$ and $f = O(g)$, then $\ln f = O(\ln g)$

Let $f(x) = 2(1+1/x)$, $g(x) = 1+1/x$

$\Rightarrow f = O(g)$ since $f \leq C g$ where $C = 2$

$\ln f = \ln[2(1+1/x)] = \ln 2 + \ln(1+1/x) \approx \ln 2 + 1/x$

$\ln g = \ln(1+1/x) \approx 1/x$ [as $x \to \infty$]

$\Rightarrow \ln f \not\leq \ln g \Rightarrow \ln f \neq O(\ln g)$

[FALSE]

(d) $f = O(g) \Rightarrow f \circ \log = O(g \circ \log)$

$f(n) \leq C\, g(n) \quad \forall n > n_0$

Let $n = \log m$

$f(\log_b m) \leq C \cdot g(\log_b m) \quad \forall \log_b m > n_0$

$\Rightarrow f(\log_b m) \leq C \cdot g(\log_b m) \quad \forall m > b^{n_0}$

$\Rightarrow f(\log_b m) \leq C \cdot g(\log_b m) \quad \forall m > m_0 \quad [m_0 = b^{n_0}]$

$\Rightarrow \boxed{f \circ \log = O(g \circ \log)}$ [TRUE]

(e) $f = O(g) \Rightarrow 2^f = O(2^g)$

Let $f(n) = 2n$, $g(n) = n$

$f \leq 2g \Rightarrow f = O(g)$ $[f = O(g) \Rightarrow f \leq C \cdot g]$

$2^f = 2^{2n} = 4^n$

$2^g = 2^n$

$\Rightarrow 2^f \neq O(2^g)$ since $4^n \geq 2^n$

[FALSE]

(f) $f = O(g) \Rightarrow 2^f = O(2^g)$

$f < cg$

Let $c = 1 \Rightarrow f < g$

$\Rightarrow 2^f < 2^g \rightarrow 2^f < c' 2^g$ $[c' = 1]$

$\Rightarrow \boxed{2^f = O(2^g)}$ [TRUE]

(g) $f = O(f^2)$

$f \leq c f^2$ $\exists c > 0$

Let $f = 1/n$

For large values of n, $f(n) = 1/n >> f(n)^2 = 1/n^2$

$\Rightarrow f \nleq c \cdot f^2$

[FALSE]

(h) $f(n) = \theta(f(n/2))$

$f(n) = O(f(n/2))$ and $f(n/2) = O(f(n))$

Let $f(n) = 2^n \Rightarrow f(n/2) = 2^{n/2} = \sqrt{2^n}$

$2^{n/2} \leq c_1 2^n$ but $2^n \nleq c_2 2^{n/2}$

$\Rightarrow f(n/2) = O(f(n))$ but $f(n) \neq O(f(n/2))$ since 2^n grows much faster than $2^{n/2}$

[FALSE]

Q5 $T(n) = 3T(n/2) + n$

$$= 3\left(3T(n/4) + n/2\right) + n$$
$$= 9T(n/4) + 3n/2 + n$$
$$= 9\left[3T(n/8) + n/4\right] + 3n/2 + n$$
$$= 27T(n/8) + 9n/4 + 3n/2 + n$$

$$\Rightarrow T(n) = 3^k T(n/2^k) + n\sum_{i=0}^{k-1}\left(\frac{3}{2}\right)^i$$

$$\sum_{i=0}^{k-1}\left(\frac{3}{2}\right)^i = \frac{(3/2)^k - 1}{(3/2) - 1} = 2\left((3/2)^k - 1\right)$$

$$\Rightarrow T(n) = 3^k T(n/2^k) + 2n\left[(3/2)^k - 1\right]$$

Stopping condition $\rightarrow T(1) \Rightarrow n/2^k = 1 \Rightarrow n = 2^k$ or $k = \log_2 n$

$$\left(\frac{3}{2}\right)^k = \left(\frac{3}{2}\right)^{\log_2 n} = n^{\log_2(3/2)}$$

$$\Rightarrow T(n) = 3^k T(1) + 2n\left[n^{\log_2(3/2)} - 1\right]$$
$$= 3^k T(1) + 2n^{\lg(3/2)+1} - 2n$$

$$3^k = 3^{\lg n} = n^{\lg 3}$$

$$\Rightarrow T(n) = n^{\lg 3} T(1) + 2n^{\lg(3/2)+1} - 2n$$

$$\Rightarrow \boxed{T(n) = \Theta(n^{\log_2 3}) = \Theta(n^{1.585})}$$

Q.6 (a) $f(x) = 4x^2 - 10x$

To prove that $f(x)$ is polynomial, show that

$f(x) \le c\, f(x/2) \quad \forall x > x_0, \ c > 1$

$f(x/2) = \frac{4x^2}{4} - \frac{10x}{2} = x^2 - 5x$

$4x^2 - 10x \le c\,(x^2 - 5x)$

$4x^{\cancel{2}} - 10\cancel{x} \le c\,x^{\cancel{2}} - 5c\cancel{x}$

$x(4-c) \le 10 - 5c$

Let $c = 4$

$0 \le 10 \Rightarrow c$ cannot be 4

Let $c = 5$

$x(4-5) \le 10 - 5(5)$

$x \ge 15$

$\Rightarrow \boxed{c = 5 \quad x_0 = 14}$

(Also note: $f(x)$ is eventually ≥ 0 $[x=0]$ and is an increasing function)

Q6 (b) $f(x) = 2^x - 10x$

$f(x-k) = 2^{x-k} - 10(x-k)$

Given : $k = 1$

$f(x-1) = 2^{x-1} - 10(x-1)$

Show that $\forall x > x_0 \Rightarrow f(x) > C\, f(x-k)$

$2^x - 10x > C\left[2^{x-1} - 10(x-1)\right]$

$2^x - 10x > C \cdot 2^{x-1} - C \cdot 10x + 10C$

Let $C = 2$

$\cancel{2^x} - 10x > \cancel{2 \cdot 2^{x-1}} - 2 \cdot 10x + 10(2)$

$-10x > -20x + 20$

$10x > 20$

$x > 2$

$\Rightarrow \boxed{\begin{array}{c} C = 2 \\ x_0 = 2 \end{array}}$ (Also note: f is eventually ≥ 0 $[x=6]$ and an increasing function)

Q.6 (c) $f(x) = \frac{2^x}{x!}$

$f(x-k) = \frac{2^{x-k}}{(x-k)!}$

Show that $f(x) < \frac{f(x-k)}{c} \quad \forall x > x_0$

Given, $k=1$

$f(x-1) = \frac{2^{x-1}}{(x-1)!}$

$\frac{2^x}{x!} < \frac{2^{x-1}}{(x-1)!} \times \frac{1}{c}$

$\frac{\cancel{2^x}}{x \times \cancel{(x-1)!}} < \frac{\cancel{2^x}}{2\cancel{(x-1)!}} \times \frac{1}{c}$

$c < \frac{x}{2}$

Let $c = 2$

$2 < \frac{x}{2} \Rightarrow x > 4$

$\Rightarrow \boxed{\begin{array}{l} c = 2 \\ x_0 = 4 \end{array}}$

Q6 (d) Since $f(x)$ and $g(x)$ are polynomial type,

$\Rightarrow f(x) \leq c_1 f(x/2) \quad \forall x > x_{0f}$

$g(x) \leq c_2 g(x/2) \quad \forall x > x_{0g}$

Let $x_0 = \max(x_{0f}, x_{0g})$ such that at $\forall x > x_0$, both $f(x)$ and $g(x)$ are polynomial.

$$
\begin{aligned}
h(x) &= f(x) \cdot g(x) \\
&\leq c_1 f(x/2) \cdot c_2\, g(x/2) \\
&\leq c_1 c_2\, f(x/2)\, g(x/2)
\end{aligned}
$$

$h(x) \leq c\, h(x/2)$, where $c = c_1 c_2$, $h(x/2) = f(x/2) \cdot g(x/2)$

$\Rightarrow h(x) := f(x) \cdot g(x)$ is also polynomial, showing that polynomial functions are closed under multiplication.

Q6 (e) (i) $S_{f(n)} = \sum_{i \geq 1}^{n} i^7 \log^3 i$

Let $f(x) = x^7 \log^3 x \Rightarrow f(x)$ is eventually ≥ 0 $[x=1]$ and increasing

$f(x/2) = \frac{x^7}{2^7} \log^3\left(\frac{x}{2}\right)$

$\cancel{x^7} \log^3 x \leq c \frac{\cancel{x^7}}{2^7} \log^3\left(\frac{x}{2}\right)$

$\log^3 x \leq \frac{c}{2^7} (\log x - \log 2)^3$

At large values of x, $(\log x - \log 2)^3 \approx \log^3 x$

$\Rightarrow \cancel{\log^3 x} \leq \frac{c}{2^7} \cancel{\log^3 x}$

$\Rightarrow c \geq 128 \Rightarrow f(x)$ is polynomial type function

$\Rightarrow S_{f(n)} = \Theta(n \times n^7 \log^3 n)$

$$\boxed{S_{f(n)} = \Theta(n^8 \log^3 n)}$$

Q6 (e) (ii) $S_{f(n)} = \sum_{i \geq 1}^{n} i^2 \log^{-1} i$

Let $f(x) = \dfrac{x^2}{\log x}$ $\Rightarrow$ $f(x)$ is eventually ≥ 0 $[x=2]$ and is an increasing function

$$f(x/2) = \frac{x^2}{4\log(x/2)}$$

$$\frac{\cancel{x^2}}{\log x} \leq C \frac{\cancel{x^2}}{4\log(x/2)}$$

$$C \geq \frac{4\log x/2}{\log x} = \frac{4(\log x - \log 2)}{\log x}$$

At large values of x, $(\log x - \log 2) \approxeq \log x$

$$\Rightarrow C \geq \frac{4\log x}{\log x} \Rightarrow C \geq 4$$

$\Rightarrow$ $f(x)$ is a polynomial type function

$$\Rightarrow \boxed{S_{f(n)} = \Theta\left(\frac{n^3}{\log n}\right)}$$

Q6 (e) (iii) $S_{f(n)} = \sum_{i \geq 1}^{n} \frac{2^i}{i^5 \log i}$

Let $f(x) = \frac{2^x}{x^5 \log x}$

$$f(x-k) = \frac{2^{x-k}}{(x-k)^5 \log (x-k)}$$

$$\frac{2^x}{x^5 \log x} \geqslant C \cdot \frac{2^{x-k}}{(x-k)^5 \log (x-k)}$$

Let $k=1$

$$\frac{2^x}{x^5 \log x} \geqslant C \cdot \frac{2^{x-1}}{(x-1)^5 \log (x-1)}$$

$$\frac{2}{x^5 \log x} \geqslant C \cdot \frac{1}{(x-1)^5 \log (x-1)}$$

$$\frac{2}{\cancel{x^5} \log x} \geqslant \frac{C}{\cancel{x^5} \times (1-\frac{1}{x})^5 [\log x + \log (1-1/x)]}$$

At large values of x, $(1-1/x)^5 \approx 1$ and $\log(1-1/x) \approx 0$

$$\Rightarrow \frac{2}{\log x} \geqslant \frac{C}{\log x} \Rightarrow C \leq 2$$

$\Rightarrow$ For $C=2$, $k=1$, $f(x)$ is exponentially increasing

$$\boxed{S_{f(n)} = \theta\left(\frac{2^n}{n^5 \log n}\right)}$$

Q7 (i) (a) $n^{1/\lg n}$

$\lg n = \log_2 n \Rightarrow 2^{\lg n} = n$

$\Rightarrow (2^{\lg n})^{1/n} = \boxed{2}$

(b) $2^{2^{(\lg \lg n)-1}}$

$= 2^{2^{(\lg \lg n)-1}}$

$= 2^{2^{(\lg \lg n)}/2}$

$= 2^{\lg n/2}$ $\quad [2^{(\lg \lg n)} = \lg n]$

$= (2^{\lg n})^{1/2}$ $\quad [2^{\lg n} = n]$

$= n^{1/2} = \boxed{\sqrt{n}}$

(c) $\sum_{i=0}^{k-1} 2^i$

$= 2^0 + 2^1 + 2^2 + \dots + 2^{k-1}$

$= 1 + 2 + 4 + \dots + 2^{k-1}$

$= \dfrac{2^k - 1}{2-1}$ [Sum of geometric series]

$= \boxed{2^k - 1}$

(ii) (a) $n^{1/\log_c n}$

$n = c^{\log_c n}$

$\Rightarrow (c^{\log_c n})^{1/\log_c n} = \boxed{c}$

(b) $c^{c^{(\log_c \log_c n)-1}}$

$= c^{c^{(\log_c \log_c n)}/c}$

$= c^{\log_c n/c}$

$= (n)^{1/c}$

$= \boxed{\sqrt[c]{n}}$

(c) $\sum_{i=0}^{k-1} c^i$

$= c^0 + c + c^2 + \dots + c^{k-1}$

$= \boxed{\dfrac{c^k - 1}{c-1}}$

(d) $2^{(\lg n)^2}$

$= 2^{\lg n \lg n}$

$= \boxed{n^{\lg n}}$ $\quad [2^{\lg n} = n]$

(e) $4^{\lg n}$

$= (2^2)^{\lg n}$

$= 2^{2 \lg n}$ $\quad [a \lg n = \lg n^a]$

$= 2^{\lg n^2}$

$= \boxed{n^2}$

(f) $(\sqrt{2})^{\lg n}$

$= (2^{1/2})^{\lg n}$

$= 2^{\lg \sqrt{n}}$ $\quad [a \lg n = \lg n^a]$

$= \boxed{\sqrt{n}}$

(d) $c^{(\log_c n)^c}$

$= c^{(\log_c n)^{c-1} (\log_c n)}$

$= \boxed{n^{(\log_c n)^{c-1}}}$

(e) $(c^c)^{\log_c n}$

$= c^{c \log_c n}$

$= c^{\log_c n^c}$

$= \boxed{n^c}$

(f) $(c^{1/c})^{\log_c n}$

$= c^{\log_c n^{1/c}}$

$= \boxed{n^{1/c}}$