Discrete Mathematics
SMA2307
Propositional Logic I

Chapter 2_Part 1

Chapter 2: Logic
Summary

- **Propositional Logic**
  - The Language of Propositions
  - Applications
  - Logical Equivalences
- **Predicate Logic**
  - Predicates
  - The Language of Quantifiers
  - Nested Quantifiers
- **Rules of Inference**
- **GCD**
- **Modular Arithmetic**
- **Cryptography**

Propositional Logic
Section Summary

- **Propositions**
- **Compound propositions**
  - Negation
  - Connectives
    - Conjunction
    - Disjunction
    - Conditional statement/ Implication
    - New conditional statement: Contrapositive, Inverse, Converse
    - Biconditionals
- **Truth Tables for Compound Propositions**

Propositions

Definition: A ***proposition*** (denoted p, q, r, …) is simply:

- a *statement* (*i.e.*, a declarative sentence) *with some definite meaning*, (not vague or ambiguous)

- having a *truth value* that's either *true* (**T**) or *false* (**F**)
  it is **never** both, neither, or somewhere "in between!"

- However, you might not *know* the actual truth value,
  and, the truth value might *depend* on the situation or context.

Examples for Proposition

- It is raining. (In a given situation)
- Beijing is the capital of China.(T)
- 2 + 2 = 5.(F)
- 1 + 2 = 3.(T)

A fact-based declaration is a proposition, even if no one knows whether it is true.

- 11213 is prime.
- There exists an odd perfect number.

Examples for Non-Proposition

The following are **NOT** propositions:

- Who's there? (interrogative, question)
- Just do it! (imperative, command)
- La la la la la. (meaningless interjection)
- does not explain or express things clearly.(vague)
- 1 + 2 (expression with a non-true/false value)
- x + 2 = 5 (declaration about semantic tokens of non-constant value)

Propositions

- Constructing Propositions
  - Propositional Variables: $p, q, r, s, \ldots$
  - The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.

Compound Propositions

- An *operator* or *connective* combines one or more *operand* expressions into a larger expression.
- (*e.g.*, "+" in numeric expressions.)

- ***Unary*** operators take *one* operand (*e.g.*, -3);

- ***Binary*** operators take *two* operands (*e.g.* 3×4).

- ***Propositional*** or ***Boolean operators*** operate on propositions (or their truth values) instead of on numbers.

- The ***Boolean domain*** is the set {T, F}. Either of its elements is called a ***Boolean value***.

Compound Propositions

- Compound Propositions; constructed from logical connectives and other propositions

Formal Name	**Nickname**	**Arity**	**Symbol**
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

Negation

- The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	**F**
F	**T**

- **Example**: If p denotes "The earth is round.", then $\neg p$ *denotes*, "The earth is not round."

Connectives

Conjunction

- The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \wedge q$ denotes,
- "I am at home and it is raining."

Disjunction

- The *disjunction* of propositions p or q is denoted by $p \lor q$ and has this truth table:

p	q	$p \lor q$
T	T	T
T	F	T
F	T	T
F	F	F

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \lor q$ denotes
- "I am at home or it is raining."

The Connective Or in disjunction

- In English "or" has two distinct meanings.
  - "Inclusive Or" - In the sentence **"Students who have taken CS202 or Math120 may take this class,"** we assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of disjunction. For $p \vee q$ to be true, either one or both of p and q must be true.
  - "Exclusive Or" - When reading the sentence **"Soup or salad comes with this entrée,"** we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of p and q must be true, but not both. The truth table for $\oplus$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statements: Implication

- If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Example**: If p denotes "I am at home." and q denotes "It is raining." then $p \rightarrow q$ denotes
- "If I am at home then it is raining."
- In $p \rightarrow q$, p is the *hypothesis* (*premise*) and q is the *conclusion* (or *consequence*).

Understanding Implication

- In $p \rightarrow q$ there does not need to be any connection between the promise or the conclusion. The "meaning" of $p \rightarrow q$ depends only on the truth values of p and q.

- These implications are perfectly fine, but would not be used in ordinary English.
  - "If the moon is made of green cheese, then I have more money than Bill Gates. "
  - "If the moon is made of green cheese then I'm on welfare."
  - "If 1 + 1 = 3, then your grandma wears combat boots."

Understanding Implication (cont)

- One way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes."
  - "If you get 100% on the final, then you will get an A."

- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. This corresponds to the case where p is true and q is false.

Different Ways of Expressing $p \to q$

Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \to q$. You will encounter most if not all of the following ways to express this conditional statement:

"if p, then q"	"p implies q"
"if p, q"	"p only if q"
"p is sufficient for q"	"a sufficient condition for q is p"
"q if p"	"q whenever p"
"q when p"	"q is necessary for p"
"a necessary condition for p is q"	"q follows from p"
"q unless $\neg p$"	"q provided that p"

Different Ways of Expressing $p \leftrightarrow q$

$p \leftrightarrow q$ may be interpreted as-

- (If p then q) and (If q then p)
- p if and only if q
- q if and only if p
- (p if q) and (q if p)
- p is necessary and sufficient for q
- q is necessary and sufficient for p
- p and q are necessary and sufficient for each other
- p and q can not exist without each other
- Either p and q both exist or none of them exist
- p and q are equivalent
- $\sim p$ and $\sim q$ are equivalent

New conditional statements: Converse, Contrapositive, and Inverse

- From $p \to q$ we can form new conditional statements .
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$

Example: Find the converse, inverse, and contrapositive of 'If it's raining, then I won't go to town."

Solution:

converse: ?

inverse: ?

contrapositive: ?

Converse, Contrapositive, and Inverse

- From $p \rightarrow q$ we can form new conditional statements .
  - $q \rightarrow p$ is the **converse** of $p \rightarrow q$
  - $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$
  - $\neg p \rightarrow \neg q$ is the **inverse** of $p \rightarrow q$

Example: Find the converse, inverse, and contrapositive of 'If it's raining, then I won't go to town."

Solution:

converse: If I'm not going to town, then it's raining.

inverse: If it is not raining, then I will go to town.

contrapositive: If I go to town, then it is not raining.

Biconditionals

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q ." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

- Some alternative ways "p if and only if q" is expressed in English:
  - p **is necessary and sufficient for** q
  - **if** p **then** q **, and conversely**
  - p **iff** q

Truth Tables For Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the atomic propositions.
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.

    This includes the atomic propositions

Example Truth Table

- Construct a truth table for $p \vee q \to \neg r$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are e*quivalent* if they always have the same truth value.

- **Example**: Show using a truth table that the conditional is equivalent to the contrapositive.
  - **Solution:**

Equivalent Propositions

- **Two propositions are e*quivalent* if they always have the same truth value.**

- **Example**: Using the truth table show that the conditional$(p \to q)$ is equivalent to the contrapositive.

- **Solution:**

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

- **Example**: Show using truth tables that neither the converse nor inverse of an implication are not equivalent to the implication.
- **Solution:**

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$q \to p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$.

If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.

Exercises

1. Which of these sentences are propositions? What are the truth values of those that are propositions?

a) Miami is the capital of Florida
b) $2 + 3 = 5$
c) $5 + 7 = 10$
d) $X+2=11$
e) What time is it?

2. Write down the negation of each of the following propositions.

a) $2 + 1 = 3$
b) There is no pollution in Kandy
c) 121 is a perfect square.

Exercises(Cont..)

3. Let p and q be the propositions

p : I bought a lottery ticket this week.
q : I won the million dollar jackpot.

Express each of these propositions as an English sentence.

a) $\neg p$
b) $p \vee q$
c) $p \rightarrow q$
d) $p \wedge q$
e) $p \leftrightarrow q$
f) $\neg p \rightarrow \neg q$
g) $\neg p \wedge \neg q$
h) $\neg p \vee (p \wedge q)$

Exercises(Cont..)

4. Let p and q be the propositions

p : It is below freezing.
q : It is snowing.

Write these propositions using p and q and logical connectives (including negations).

a) It is below freezing and snowing.
b) It is below freezing but not snowing.
c) It is not below freezing and it is not snowing.
d) It is either snowing or below freezing (or both).
e) If it is below freezing, it is also snowing.
f) Either it is below freezing or it is snowing, but it is not snowing if it is below freezing.
g) That it is below freezing is necessary and sufficient for it to be snowing.

5. Construct a truth table for the following compound proposition.

$$(p \lor q) \rightarrow (p \land q)$$