EC451: Advanced Digital Signal Processing

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Sub-Nyquist Sampling and Recovery of Pulse Streams with the Real Parts of Fourier Coefficients

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Research Report

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Abstract

FRI (finite rate of innovation) theory has shown that sampling and recovery of pulse streams of known shape can be achieved at rates much lower than the Nyquist's rate. However, this theory has been found to be ineffective in cases with high rate of innovation or the sampling stage was found to be too complex and redundant. This paper has proposed a method of sampling and recovery using real parts of the Fourier coefficients. This is achieved through modulating the input signal in each channel with a properly chosen cosine signal, followed by filtering with a low-pass filter. Since the modulating process will lead to the signal spectrum aliasing, the authors have proposed a spectrum de-aliasing algorithm to solve this problem, resulting in the real parts of a band of Fourier coefficients from each two channels. A more efficient way to obtain arbitrary frequency bands from the aliased spectrum is designed by combining with the multi-channel sampling structure, which improves the utility of the signal spectrum. By using a sparsity-based recovery algorithm, the time delays and amplitudes of the pulse streams can be recovered from the obtained real parts of the Fourier coefficients.

1 Introduction

The objective of this work is to design a sampling and recovery based system using real parts of the fourier coefficients.

Shannon-Nyquist theorem states that in order to perfectly reconstruct an analog signal from its samples, it must be sampled at the Nyquist rate, i.e., twice its highest frequency. This proposition is valid for bandlimited signals alone. Other prior knowledge on the signal, rather than band limitation, can be exploited in order to reduce the sampling rate. The FRI framework treats sampling and recovery of signals characterized by a finite number of degrees of freedom per unit time. For such models, the goal is to design a sampling scheme operating at the innovation rate, which is the minimal possible rate from which perfect recovery is possible by making use of gaussian, polynomial and exponentially reproducing kernels. However, these methods are unstable for a large number of pulses per unit time

2 Nomenclature

L	Number of pulses
T	Observation time
ρ	The rate of innovation, $\rho = 2L/T$
f_s	Sampling rate
T_s	Sampling period, $T_s = 1/f_s$
ω_{cut}	Cutoff frequency of LPF
$\Delta\omega$	Frequency interval of the obtained real parts of the Fourier Coefficients
M	Quantizing number of bins for frequency period $[0, \omega_{cut})$ with step $\Delta\omega$, $M = \lfloor \frac{\omega_{cut}}{\Delta\omega} \rfloor$
P	Number of channels of the muti-channel sampling system
K	Total number of the obtained real parts of the Fourier Coefficients
N	Quantizing number of bins for time period $[0, T)$
δ	Quantizing step for time period $[0, T)$, $\delta = T/N$

3 Literature Review

Trends and problems in the existing research on FRI based sampling and recovery -

- Gaussian kernels are unstable when the pulse number is larger than 6

- Polynomial and exponential reproducing kernels are unstable when the pulse number is larger than 5

- Sinc sampling kernel, i.e., ideal low-pass filter (LPF) - To obtain the Fourier coefficients
  It was showed that the time delays and amplitudes of the pulse streams can be recovered from a set of the signal's Fourier coefficients. However, using a LPF can only extract a consecutive set of Fourier coefficients, which would lead to a poor recovery performance.
  To extract arbitrary sets of Fourier coefficients, a pre-sampling filtering scheme was introduced. This sampling kernel required multiple pass-bands and extremely high frequency selectivity, which are difficult to satisfy when designing a practical analog filter.

- Multi-channel sampling scheme -
  Discrete Fourier coefficients distributed over a larger part of the signal's spectrum are obtained using

multi-channel mixers and integrators to directly compute and sample the Fourier coefficients. In this method, one channel can only obtain one Fourier coefficient. (complicated hardware support)

In this paper, the authors have proposed a more efficient way to obtain arbitrary frequency bands from the aliased spectrum, which improves the utility of the signal spectrum as well as the recovery performance in the presence of noise. This is achieved through modulating the input signal in each channel with a properly chosen cosine signal and then filtering with a LPF, followed by sampling at twice its cut-off frequency. A spectrum de-aliasing algorithm is used to calculate the real parts of the Fourier coefficients one by one from two staggered and aliased spectrums, which can be obtained from each two channels with close modulation frequencies.Further a sparsity-based recovery algorithm is used to recover the time delays and amplitudes of the pulse streams by using these real parts of the Fourier coefficients.

4 Methodology

4.1 Obtaining Real Parts of the Fourier Coefficients from the Aliased Spectrum

4.1.1 Problem Formulation

FRI signals whose pulse shape is known can be modelled as

$$x(t) = \sum_{l=1}^{L} a_l h(t - t_l), a_l \in C, t_l \in [0, T)$$

where a_l is amplitude, h(t) is a known pulse of same shape and short duration in time domain, L is number of pulses, T is observation time, thus giving degree of freedom 2L, rate of innovation = 2L/T. The unknown parameters are $\{a_l,t_l\}_{l = 1 \text{ to } L}$ which would be determined in the recovery part of the paper. Continuous Time Fourier Transform of the signal gives

$$X(\omega) = \int_{-\infty}^{\infty} [\sum_{l=1}^{L} a_l h(t - t_l)] e^{-j\omega t} dt$$

$$= \sum_{l=1}^{L} a_l \int_{-\infty}^{\infty} h(t - t_l) e^{-j\omega t} dt$$

$$= H(\omega) \sum_{l=1}^{L} a_l e^{-j\omega t_l}$$

Since h(t) and thus H(ω) is known, the unknown parameters $\{a_l,t_l\}_{l = 1 \text{ to } L}$ can be found if we obtain a set of non-zero coefficients of X(kω_0) where H(kω_0) is not zero.

4.1.2 Frequency selection and spectrum aliasing problem

A consecutive Fourier subset is obtained using an LPF, followed by sampling at twice the cutoff frequency and the Discrete Fourier Transform would provide required samples, but coefficients distributed over a large part of the signal's spectrum would enhance recovery performance.

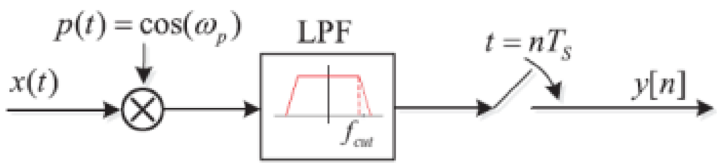

Figure 1: Sampling setup. x(t) is the pulse stream, p(t) is the modulating signal and T_s is the sampling period.

First the input signal is modulated with a cosine signal p(t),

$$p(t) = \cos(\omega_{\mathrm{p}} t)$$

where ω_{p} is the frequency of the cosine signal and can be called modulation frequency. The CTFT of the modulation signal p(t) is,

$$P(\omega) = \pi[\delta(\omega + \omega_{\mathrm{p}}) + \delta(\omega - \omega_{\mathrm{p}})]$$

The modulated signal

$$g(t) = x(t).p(t)G(\omega) = X(\omega) * P(\omega)$$

where G(w) is CTFT of g(t). Substituting the above P(ω) expression in G(ω),

$$G(\omega) = 1/2.[X(\omega + \omega_{\mathrm{p}}) + X(\omega - \omega_{\mathrm{p}})]$$

Filtering it with cutoff frequency ω_{cut} , we get

$$Y(\omega) = rect(\frac{\omega}{\omega_{\mathrm{cut}}})G(\omega)$$

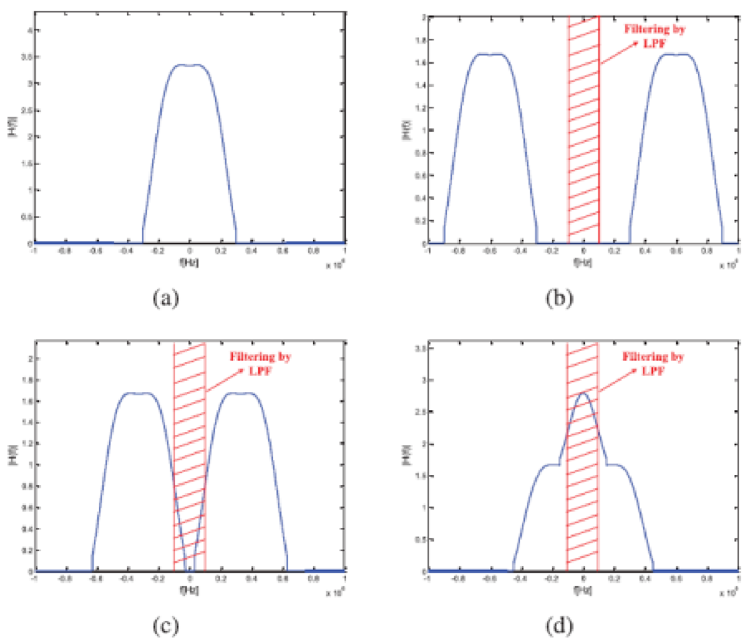

FIGURE The modulating results: (a) Original signal spectrum; (b) Case 1, $\omega_p > \omega_{max} + \omega_{cut}$; (c) Case 2, $\omega_{max} < \omega_p \leq \omega_{max} + \omega_{cut}$; and (d) Case 3, $\omega_p \leq \omega_{max}$.

Figure 2

Assuming ω_{max} is the maximum frequency of x(t), the modulating process can be divided into three cases:

1. $\omega_{\text{p}} > \omega_{\text{max}} + \omega_{\text{cut}}$: No sampling values are recorded as modulated spectrum is outside LPF domain
2. $\omega_{\text{max}} \leq \omega_{\text{p}} \leq \omega_{\text{max}} + \omega_{\text{cut}}$: Since $\omega_{\text{cut}} \ll \omega_{\text{max}}$, modulating in this case will extremely reduce the Fourier spectrum utilization
3. $\omega_{\text{p}} < \omega_{\text{max}}$: In this case spectrum would face aliasing problem, with the frequencies aliasing area $[-(\omega_{\text{max}} - \omega_{\text{p}}), (\omega_{\text{max}} - \omega_{\text{p}})]$.

By choosing frequency ω_{p} such that it satisfies case 3, we solve the aliasing problem in the next sub part.

4.1.3 Spectrum De Aliasing

We use two modulation processes from different channels to solve the spectrum aliasing problem, which allows extracting the real parts of an arbitrary band of Fourier coefficients from the aliased spectrum. In this part, the signal x(t) is split into two channels and modulated with two cosine signals, with a difference of frequency $\Delta\delta$ and obtain real parts of coefficients of X(δ). The algorithm used is:

Algorithm 1 Spectrum De-Aliasing Algorithm

Require: Modulation frequency ω_1 and $\Delta\omega$; The corresponding aliased spectrum $Y_1(\omega)$ and $Y_2(\omega)$; Cutoff frequency of LPF ω_{cut}.

Ensure: A set of real parts of the Fourier coefficients U.

```
X_R(ω_1) = Y_1(0). (Calculate the initial value).
X_R(ω_1 + Δω) = Y_2(0). (Calculate the initial value).
M = ⌊ω_cut / Δω⌋.
if M = 0 then
    U = {X_R(ω_1), X_R(ω_1 + Δω)}. (Obtain 2 real parts of the Fourier coefficients).
else
    for m = 1 to M do
        X_R(ω_1 − mΔω) = 2Y_1R(mΔω) − X_R(ω_1 + mΔω).
        X_R(ω_1 + (m + 1)Δω) = X_R(ω_1 + (m − 1)Δω) − 2Y_1R((m − 1)Δω) + 2Y_2R(mΔω).
    end for
    U = {X_R(ω_1 + bΔω)|b = −M, 1 − M, ··· , M + 1}. (Obtain 2M + 2 real parts of the Fourier coefficients).
end if
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4.1.4 Multi-channel Sampling structure

In this part, we use 2P channels and obtain P bands of Fourier coefficients, satisfying the criteria of $0 < \Delta\omega < \omega_{\text{cut}}$, and $0 < \omega_{\text{i}} < \omega_{\text{max}} - \Delta\omega$, and $|\omega_{\text{i}} - \omega_{\text{j}}| \geq 2\omega_{\text{cut}} + \Delta\delta$ where i,j $\{1, 2, \cdots, \text{P}\}$

We obtain real parts of Fourier coefficients expressed as:

$$U_{\text{i}} = X_{\text{r}}((m_{\text{i}} + b)\Delta\omega) | m_{\text{i}} = \frac{\omega_{\text{i}}}{\Delta\omega}; b = \{-M, 1-M, ..., M+1\}$$

for two channels and a superset,

$$U = \{U_1, U_2, ..., U_{\text{p}}\}$$

for 2P channels.We then use the recovery algorithm to recover the time delay and amplitude parameters using the above coefficients.

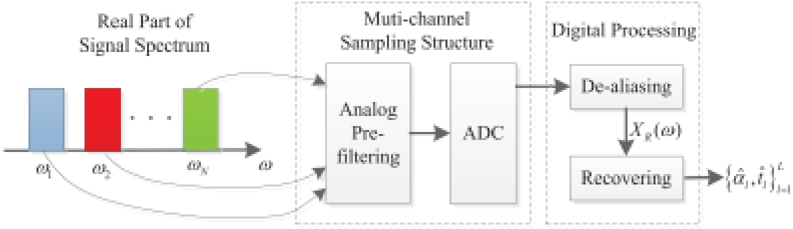

Figure 3: Multi-channel sampling of pulse streams

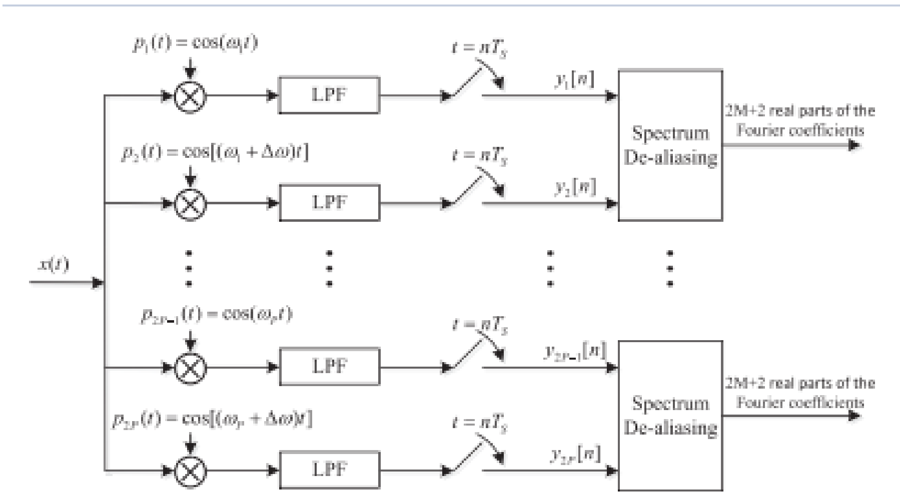

FIGURE Multi-channel sampling structure. Here, each two sampling channels can obtain 2M + 2 ($M = \left\lfloor \frac{\omega_{cut}}{\Delta\omega} \right\rfloor$) real parts of the Fourier coefficients.

Figure 4

4.2 Recovery Algorithm

4.2.1 Problem Formulation

Post sampling, K real parts of the Fourier coefficients have been obtained from the system, where K = (2M+1)P. The Fourier coefficients of the known pulse H(k$\Delta\omega$) can be expressed as

$$H(k\Delta\omega) = c_k e^{j\varphi_k}$$

where c_k is the amplitude and φ_k is the phase. Substituting this, we have

$$X(k\Delta\omega) = H(k\Delta\omega) \sum_{l=1}^{L} a_l e^{-jk\Delta\omega t_l}$$

$$= \sum_{l=1}^{L} a_l c_k e^{-j(k\Delta\omega t_l - -\varphi_k)}$$

$$= \sum_{l=1}^{L} a_l c_k [\cos(k\Delta\omega t_l - \varphi_k) - j\sin(k\Delta\omega t_l - \varphi_k)]$$

Extracting the real parts

$$X_r(k\Delta\omega) = \sum_{l=1}^{L} a_l c_k \cos(k\Delta\omega t_l - \varphi_k)$$

Given a set of non-zero real parts of the Fourier coefficients $X_R(k\Delta\omega)$, 2L unknown parameters $\{a_l, t_l\}_{l=1 \text{ to } L}$ can be recovered.
The recovery process can be summed up as follows:

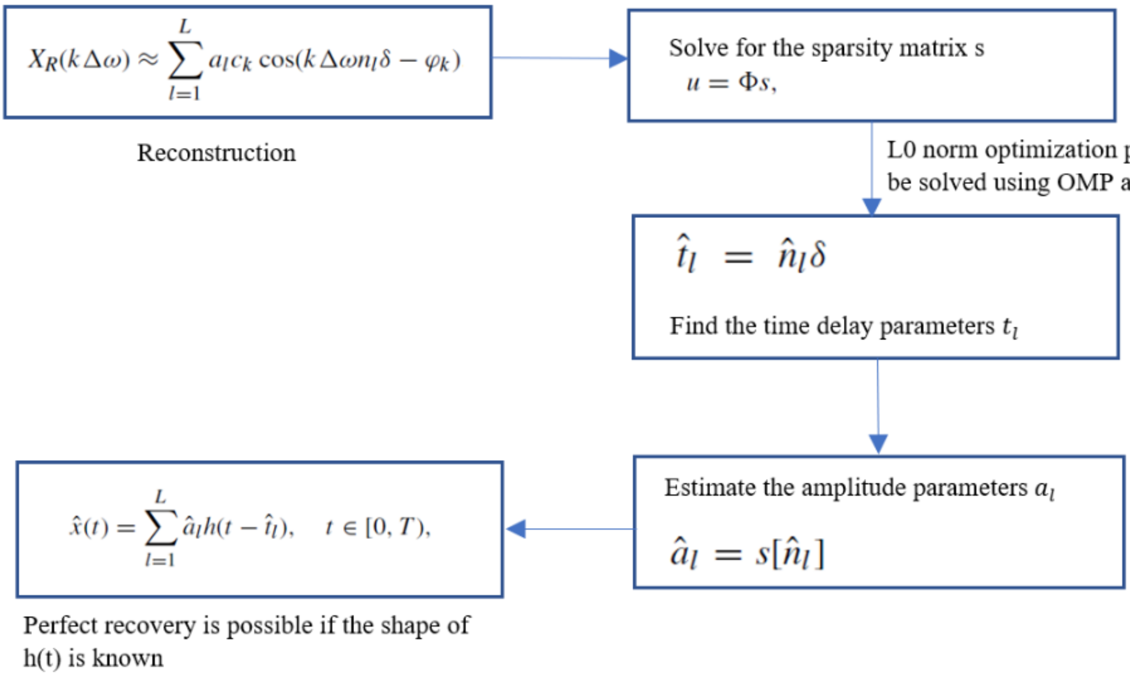

Figure 5: The recovery procedure

4.2.2 Sparsity-based recovery algorithm

The analog time axis is quantized with a resolution step of δ i.e. t = nδ, with n = 0,1,..N-1 and N = T δ. The equation for $X_R(k\Delta\omega)$ can approximated as:

$$X_R(k\Delta\omega) \approx \sum_{l=1}^{L} a_l c_k \cos(k\Delta\omega n_l \delta - \varphi_k)$$

Rewriting the equation in matrix form:

$$\begin{bmatrix} u_{\kappa_1} \\ u_{\kappa_2} \\ \vdots \\ u_{\kappa_K} \end{bmatrix} = \begin{bmatrix} d_{\kappa_1,n_1} & \cdots & d_{\kappa_1,n_L} \\ d_{\kappa_2,n_1} & \cdots & d_{\kappa_2,n_L} \\ \vdots & \ddots & \vdots \\ d_{\kappa_K,n_1} & \cdots & d_{\kappa_K,n_L} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_L \end{bmatrix},$$

where $u_{\kappa_i} = X_R(\kappa_i \Delta\omega)$ and $d_{\kappa_i,n_l} = c_{\kappa_i} \cos(\kappa_i \Delta\omega n_l \delta - \varphi_{\kappa_i})$, with κ_i $(i = 1, 2, \cdots, K)$ the fundamental element of the set κ.

Considering that the time domain of the pulse streams x(t) is limited to [0,T), a complete set of analog time can be obtained as $\eta = \{0, \delta, 2\delta, \ldots, (N-1)\delta\}$ with N = T δ, if the quantization error is ignored. The time delay parameter set is $\gamma = \{n_0\delta, n_1\delta, ..., n_{L-1}\delta\}$, which is a smaller subset of the set η
The solution matrix can be re-written as a sparsity matrix:

$$\begin{bmatrix} u_{\kappa_1} \\ u_{\kappa_2} \\ \vdots \\ u_{\kappa_K} \end{bmatrix} = \begin{bmatrix} d_{\kappa_1,0} & \cdots & d_{\kappa_1,N-1} \\ d_{\kappa_2,0} & \cdots & d_{\kappa_2,N-1} \\ \vdots & \ddots & \vdots \\ d_{\kappa_K,0} & \cdots & d_{\kappa_K,N-1} \end{bmatrix} \cdot \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N-1} \end{bmatrix},$$

where $[s_0, s_1, \cdots, s_{N-1}]^T$ is a $N \times 1$ vector, formed by L amplitude parameters $\{a_l\}_{l=1}^{L}$ and $N - L$ zero values.

The goal is to find the nonzero entries of vector s. For simplicity, the matrix equation may be written as

$$u = \phi s$$

where u is Kx1 vector. The most direct way of solving this equation is by solving a L0 norm optimization problem using Orthogonal Matching pursuit algorithm.
Once the L sparse signal s is solved, the position of the nonzero elements n_l can be known, following which the time delay parameters can be directly calculated as

$$\hat{t}_l = \hat{n}_l \delta$$

and the amplitude parameters are estimated as

$$\hat{a}_l = s[\hat{n}_l]$$

As the pulse streams is a delayed and scaled version of the pulse h(t), the original signal x(t) can be recovered:

$$\hat{x}(t) = \sum_{l=1}^{L} \hat{a}_l h(t - \hat{t}_l), t \in [0, T)$$

4.2.3 Effect of Quantization error

In the presence of quantization error σ, the unknown time delay parameters can be re-expressed as

$$t_l = n_l \delta + \sigma_l, \sigma_l \in [0, \delta = \frac{T}{N}]$$

Quantization error would lead to an attenuation of the estimated amplitude parameters during sparsity-based recovery i.e.

$$\hat{a}_l = \mu_l a_l$$

with $\mu_l = \cos(k\Delta\omega\sigma_l) \approx 1$
Time delays can be estimated as $\hat{t}_l \approx n_l \delta = t_l - \sigma_l$ when the number of measurements $K \geq cL\log(N/L)$, with c being a small constant. The reconstruction error of the time delays is denoted as:

$$error_l = \frac{|\hat{t}_l - t_l|}{|t_l|}$$

To guarantee that the reconstruction error $error_l \leq E \in (0, 1)$, the number of quantization bins N should satisfy

$$N \geq \frac{T}{E.min\{t_1, t_2, ..., t_L\}}$$

4.2.4 Remedial measures to improve the robustness of the model

- To improve the temporal resolution and avoid the ambiguity of the time delay parameters, a wider frequency aperture can be obtained by appropriately selecting the modulation frequencies.
- More measurements U can be generated by selecting higher cutoff frequency of LPF, smaller frequency interval of the obtained real parts of the Fourier coefficients and more sampling channels.
- To improve the time delay estimation, a larger quantizing number of bins N can be selected.

5 Results

- Multi-channel modulation and filtering

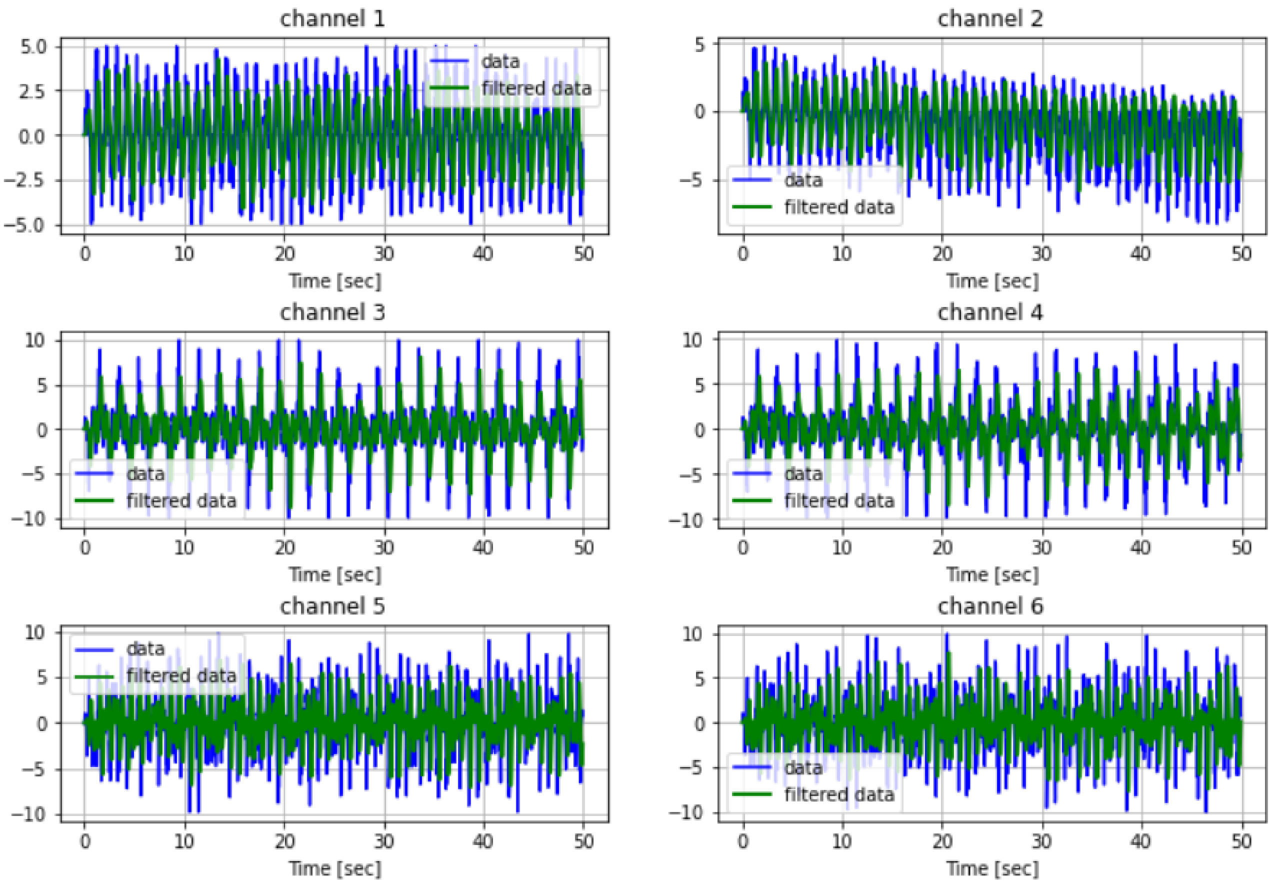

Figure 6: Multi channel system with P = 3

- Recovery of sparse matrix using orthogonal matching pursuit algorithm results

  Orthogonal Matching Pursuit Algorithm (OMP) is a greedy compressed sensing recovery algorithm which selects the best fitting column of the sensing matrix in each iteration. A least squares (LS) optimization is then performed in the subspace spanned by all previously picked columns.

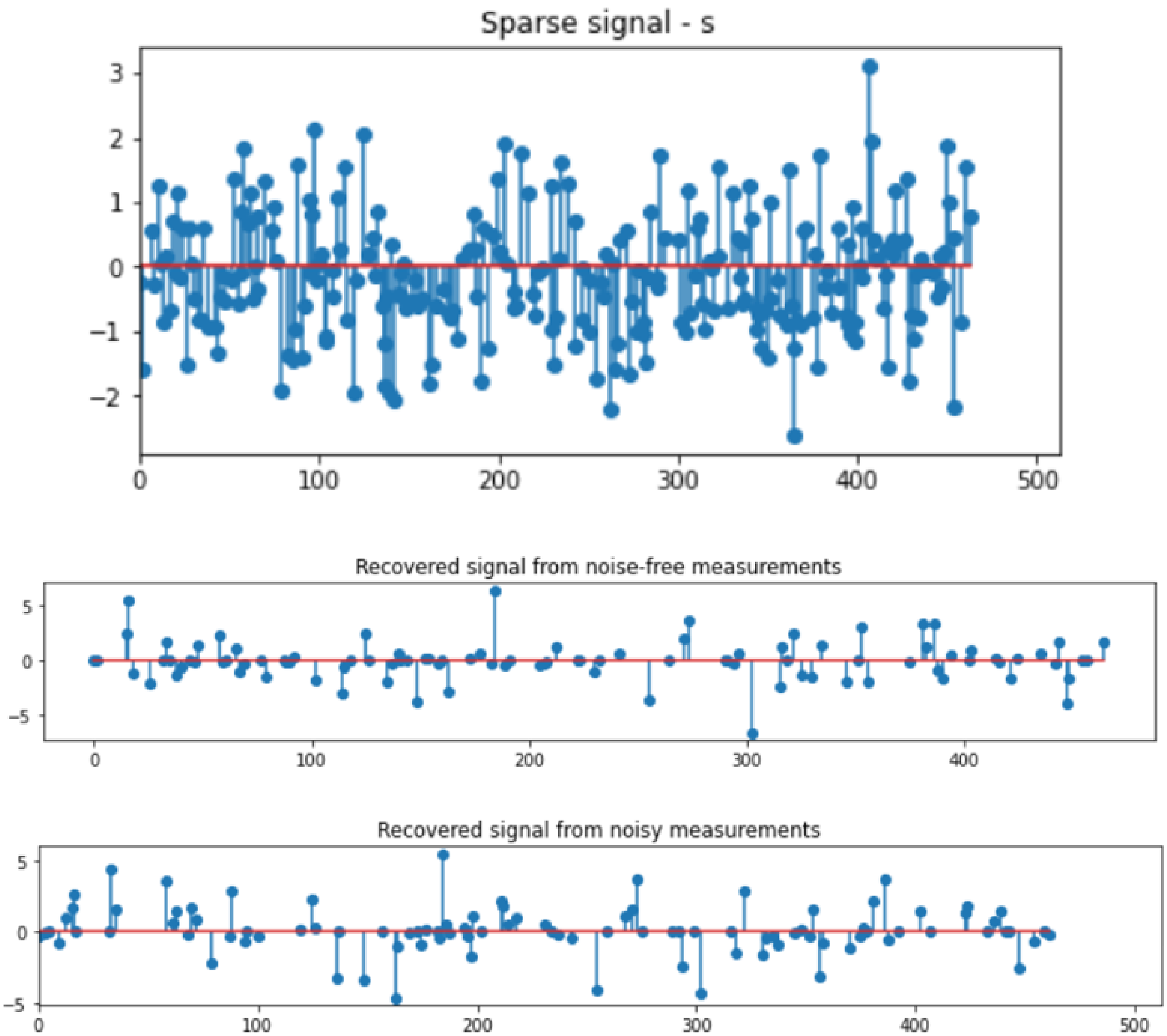

Figure 7: Sparse signal recovery using OMP

- Other Simulation results

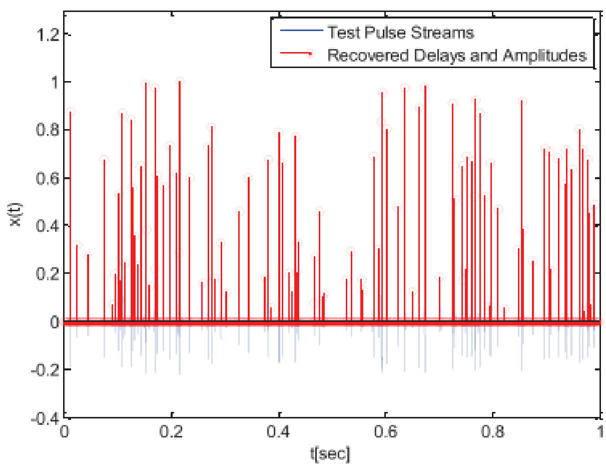

Figure 8: Performance for signal with 100 pulses

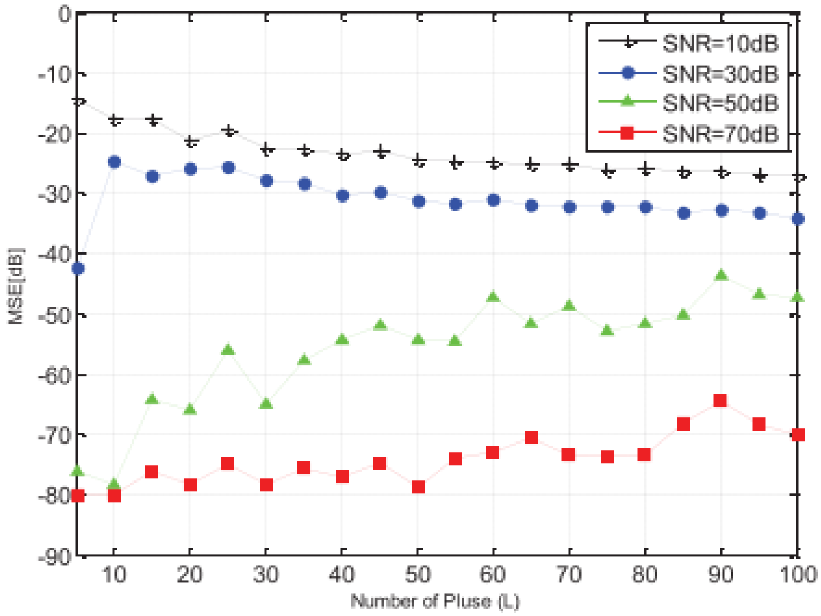

Figure 9: Simulation results with increasing number of pulses

6 Conclusions

The paper proposed a new FRI sampling scheme for pulse streams, which is based on sampling and recovering with the real parts of the Fourier coefficients with a sampling structure mainly consisting of a multiplier, a LPF and a low-rate ADC in each channel. Combined with a spectrum de-aliasing algorithm, it could easily extract the real parts of distinct bands of Fourier coefficients from the aliased signal spectrum. A sparsity-based recovery algorithm was used to recover the pulse delays and amplitudes with the real parts of the Fourier coefficients derived from the sampling system. Based on the simulation tests conducted, the method performs stably even when many pulses are overlapped randomly in time domain, and exhibits good noise robustness.

7 References

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