

Notes on An Introduction to Quantum Computing

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1 Citation

Phillip Kaye, Raymond Laflamme, Michele Mosca. 2007. *An Introduction to Quantum Computing*.

2 Keywords

Quantum Computing; Computer Architecture; Circuit Design

3 Goals

- Learn the basics of quantum computers
- Learn the basic mathematical background of quantum mechanics
- Learn how to simulate and put together qubits and circuits
- Create condensed and easily referable notes on the book 'An Introduction to Quantum Computing'

1 Introduction and Background

Two measures of complexity of a computation: time and space.

The Church–Turing Thesis says that a computing problem can be solved on any computer that we could hope to build, if and only if it can be solved on a very simple ‘machine’, named a Turing machine. The Turing machine is a mathematical abstraction. The very term *computable* corresponds to what can be computed by a Turing machine.

A probabilistic Turing machine is one capable of making a random binary choice at each step, where the state transition rules are expanded to account for these random bits. There are some important problems that we know how to solve efficiently using a probabilistic Turing machine, but do not know how to solve efficiently using a conventional Turing machine.

(Classical) Strong Church–Turing Thesis: *A probabilistic Turing machine can efficiently simulate any realistic model of computation.*

Quantum Strong Church–Turing Thesis: *A quantum Turing machine can efficiently simulate any realistic model of computation.*

Another useful model of computation is that of a uniform families of reversible circuits. A family of circuits is a set of circuits $\{C_n \mid n \in \mathbb{Z}^+\}$, one circuit for each input size n .

Def 1. *A set of gates is universal for classical computation if, for any positive integers n, m , and function $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$, a circuit can be constructed for computing f using only gates from that set.*

Three measures of complexity of a computation can be used for the circuit model.

- Depth: The total number of time slices in circuit C_n .
- Width: The total number of bits or adjacent wires (analogous to space).
- Total number of gates.

1.1 Linear Algebraic Formulation of the Circuit Model

We can summarize the information in a bit by a 2-dimensional vector of probabilities such that it is in state 0 with probability p_0 and in state 1 with probability p_1 . This description can also be used for deterministic circuits. A wire in a deterministic circuit whose state is 0 could be specified by the probabilities $p_0 = 1$ and $p_1 = 0$.

Similarly, we would be able to represent gates in the circuit by operators that take the form of matrices.

$$\text{NOT} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{NOT} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (1)$$

This implies that we can represent the *NOT* operator as

$$\text{NOT} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{NOT} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} p_1 \\ p_0 \end{pmatrix} \quad (2)$$

To describe the state associated with a given point in a probabilistic circuit having two wires we can use a 4-dimensional vector of probabilities. The four possibilities for the combined state of both wires at the given point are 00,01,10,11 (where the binary string ij indicates that the first wire is in state i and the second wire in state j). The probabilities of each of these four states can be obtained by multiplying the corresponding probabilities: $\text{prob}(ij) = p_i q_j$ and the 4D vector will be:

$$\begin{pmatrix} p_0 q_0 \\ p_0 q_1 \\ p_1 q_0 \\ p_1 q_1 \end{pmatrix} \quad (3)$$

This is the tensor product of the 2D state vectors of the first and second wires separately.

We can also represent gates acting on more than one wire. For example, the controlled-not gate, denoted CNOT. The reversible CNOT gate flips the value of the target bit t if and only if the control bit c has value 1. The CNOT gate can be represented by the matrix:

$$\text{CNOT} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (4)$$

1.2 Reversible Computation

A computation is reversible if it is always possible to uniquely recover the input, given the output. Each gate in a finite family of gates can be made reversible by adding some additional input and output wires if necessary.

Note that the reversible AND gate is a generalization of the CNOT gate (the CNOT gate is reversible), where there are two bits controlling whether the not is applied to the third bit.

In general, given an implementation (not necessarily reversible) of a function f , we can easily describe a reversible implementation of the form:

$$(x, c) \mapsto (x, c \oplus f(x)) \quad (5)$$

1.3 A Preview of Quantum Physics

The quantum mechanical behaviour of microscopic particles be illustrated by a simple experiment.

Simple Beam Splitter: Photon source, beam splitter, two photon detectors. Here the beam splitter acts like a fair coin flipper and the photon arrives at each photon detector 50% of the time.

Modified Experiment (Two Beam Splitters): Added two mirrors and an additional beam splitter as per figure. Contradictory to classical intuition, the photons arrive at one detector 100% of the time.

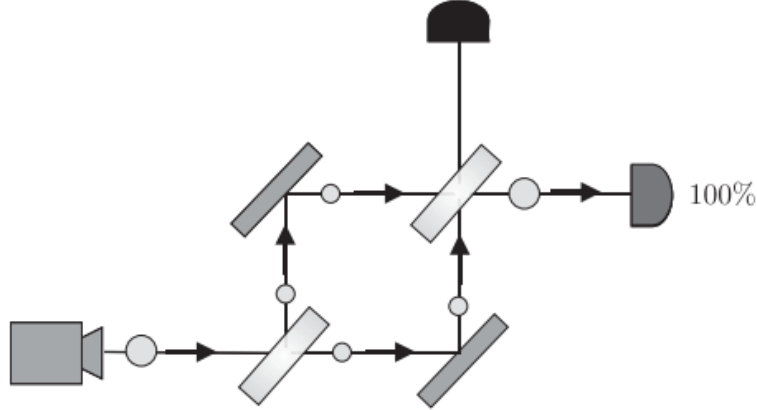


Figure 1: Two beam splitter experiment

To understand this, we will consider the photon in this apparatus as a two state system. These states are represented by the paths they follow in the figures 2 and 3.

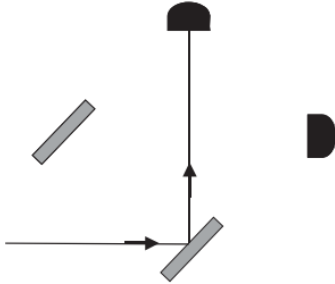


Figure 2: Path 0

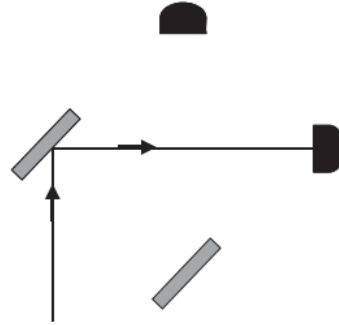


Figure 3: Path 1

According to the quantum mechanical description, the first beam splitter causes the photon to go into a superposition of taking both the '0' and '1' paths. Mathematically, we describe such a superposition by taking a linear combination of the state vectors for the '0' and '1' paths, so the general path state will be described by the vector:

$$\alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \quad (6)$$

If we were to physically measure the photon to see which path it is in, we will find it in path ‘0’ with probability $|\alpha_0|^2$, and in path ‘1’ with probability $|\alpha_1|^2$. Since we should find the photon in exactly one path, we must have $|\alpha_0|^2 + |\alpha_1|^2 = 1$.

When the photon passes through the beam splitter, we multiply its **state vector** by the matrix:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \quad (7)$$

So for the photon starting out in path ‘0’, after passing through the first beam splitter it comes out in state:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{i}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8)$$

If we measure the photon after passing through the first splitter, we find to the path 0 to have a probability of

$$\left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{2},$$

and path ‘1’ with probability:

$$\left| \frac{i}{\sqrt{2}} \right|^2 = \frac{1}{2}.$$

However, if we do not take the measurement, the photon remains in a state of superposition represented as:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad (9)$$

Now if the photon is allowed to pass through the second beam splitter (before making any measurement of the photon’s path), its new state vector is

$$\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \right) = \begin{bmatrix} 0 \\ i \end{bmatrix} \quad (10)$$

Now if we measure the probabilities along paths 0 and 1, which is what the detector is doing, the probability along path 0 will be 0 and the probability along path 1 will be $|i|^2 = 1$. In the language of quantum mechanics, the second beam splitter has caused the two paths (in superposition) to interfere, resulting in cancellation of the ‘0’ path.

2 Linear Algebra and The Dirac Notation

2.1 The Dirac Notation and Hilbert Spaces

In the Dirac notation, the symbol identifying a vector is written inside a ‘ket’, and looks like $|a\rangle$. We denote the dual vector for a (defined later) with a ‘bra’, written as $\langle a|$. Then inner products will be written as ‘bra-kets’ (e.g., $\langle a|b\rangle$).

Hilbert spaces are finite-dimensional and over complex numbers denoted by \mathcal{H} . Since \mathcal{H} is finite-dimensional, we can choose a basis and alternatively represent vectors (kets) in this basis as finite column vectors, and represent operators with finite matrices.

The Hilbert spaces of interest for quantum computing will typically have dimension 2^n . The standard way to associate column vectors corresponding to these basis vectors is as follows:

$$\begin{aligned} |00\dots 00\rangle &\Longleftrightarrow \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Big\}^{2^n}, & |00\dots 01\rangle &\Longleftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, & \dots \\ \dots, & |11\dots 10\rangle &\Longleftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, & |11\dots 11\rangle &\Longleftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \end{aligned} \quad (11)$$

Each entry in one of the column vectors signify the probability amplitude of the system of qubits to be in a specific basis state. An arbitrary vector in can be written either as a weighted sum of the basis vectors in the Dirac notation, or as a single column matrix.

2.2 Dual Vectors

The inner product of a vector \mathbf{v} with \mathbf{w} is denoted as $\langle \mathbf{v}, \mathbf{w} \rangle$. Dot products are a type of inner product. An inner product is such a function having the following properties:

- Linearity in the second argument
- Conjugate commutativity
- Non-negativity

While the normal interpretation of dot products do not involve conjugates, the dot product of complex vectors are (with $*$ used to denote complex conjugates):

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^n v_i^* w_i \quad (12)$$

Def 2. Let \mathcal{H} be a Hilbert space. The Hilbert space \mathcal{H}^* is defined as a set of linear maps $\mathcal{H} \rightarrow \mathbb{C}$. We denote the elements of \mathcal{H}^* by $\langle\chi|$, where the action of $\langle\chi|$ is

$$\langle\chi| : |\psi\rangle \mapsto \langle\chi|\psi\rangle \in \mathbb{C} \quad (13)$$

where $\langle\chi|\psi\rangle$ is the inner product of $|\chi\rangle \in \mathcal{H}$ with the vector $|\psi\rangle \in \mathcal{H}$.

The set of maps \mathcal{H}^* is a complex vector space in itself and is called the dual vector space associated with \mathcal{H} . The vector $\langle\chi|$ is called the dual of $|\chi\rangle$. $\langle\chi|$ is obtained from $|\chi\rangle$ by taking the corresponding row matrix and then taking the complex conjugate of every element or the 'Hermitean conjugate' of the column matrix for $|\chi\rangle$.

Two vectors are said to be orthogonal if their inner product is zero. The norm or eucledian norm of a vector $|\psi\rangle$, denoted by $||\psi\rangle||$ is the square root of the inner product of $|\psi\rangle$ with itself:

$$||\psi\rangle|| \equiv \sqrt{\langle\psi|\psi\rangle} \quad (14)$$

A vector is called a unit vector if it has norm 1. A set of unit vectors that are mutually orthogonal is called an orthonormal set. The Kronecker delta function $\delta_{i,j}$ is simply defined to be 1 if $i = j$ and 0 otherwise.

Def 3. Consider a Hilbert space \mathcal{H} of dimension 2^n . A set of 2^n vectors $B = \{|b_m\rangle\} \subseteq \mathcal{H}$ is called the orthogonal basis of \mathcal{H} if

$$\langle b_n | b_m \rangle = \delta_{n,m} \quad \forall b_m, b_n \in B \quad (15)$$

and every $|\psi\rangle \in \mathcal{H}$ can be written as

$$|\psi\rangle = \sum_{b_n \in B} \psi_n |b_n\rangle, \text{ for some } \psi_n \in \mathbb{C} \quad (16)$$

The values ψ_n satisfy $\psi_n = \langle b_n | \psi \rangle$, and are called the coefficients of $|\psi\rangle$ with respect to the basis $\{|b_n\rangle\}$.

Hadamard Basis: The Hadamard basis is an alternative orthonormal basis for a 2-dimensional Hilbert space consisting of the states $|+\rangle$ and $|-\rangle$.

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ |-\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned} \quad (17)$$

Both of these basis vectors are normalized and orthogonal.

If we express $|\psi\rangle = \sum_i \alpha_i |\phi_i\rangle$ as a vector in a Hilbert space with respect to an orthogonal basis $|\phi_i\rangle$ then $||\psi\rangle|| = \sum_i |\alpha_i|^2$. The norm squared of the quantum state represents the total probability.

Def 4. The set $\{\langle b_n|\}$ is an orthogonal basis for \mathcal{H}^* called the dual basis.

2.3 Operators

Def 5. A linear operator on a vector space \mathcal{H} is a linear transformation $T : \mathcal{H} \rightarrow \mathcal{H}$ of the vector space to itself.

Outer Products While the inner product of $|\psi\rangle$ and $|\varphi\rangle$ is obtained by multiplying $|\psi\rangle$ on the left by dual vector $\langle\varphi|$, an outer product is obtained by multiplying $|\psi\rangle$ on the right by $\langle\varphi|$ ie. $|\psi\rangle\langle\varphi|$. While the inner product is a complex scalar, the outer product is a linear operator which maps any vector γ in \mathcal{H} as:

$$(|\psi\rangle\langle\varphi|)|\gamma\rangle = |\psi\rangle\langle\varphi|\gamma\rangle \quad (18)$$

Here, $\langle\varphi|\gamma\rangle$ is a scalar which scales the vector $|\psi\rangle$. The outer product can also be represented as matrix.

Orthogonal Projectors The outer product of a vector with itself defines a projection operator: $|\psi\rangle\langle\psi|$. Applying this to another vector φ gives

$$|\psi\rangle\langle\psi|\varphi\rangle \quad (19)$$

This is, the operator given by the outer product projects φ onto a 1D subspace spanned by ψ . This is called an orthogonal projector.

Def 6. Let $B = \{|b_n\rangle\}$ be an orthonormal basis for a vector space \mathcal{H} . Then every linear operator T on \mathcal{H} can be written as:

$$T = \sum_{b_n, b_m \in B} T_{n,m} |b_n\rangle\langle b_m| \quad (20)$$

where $T_{n,m} = \langle b_n|T|b_m\rangle$.

The set of all linear operators on a vector space \mathcal{H} forms a new complex vector space $\mathcal{L}(\mathcal{H})$ where all the vectors in it are the linear operators on \mathcal{H} . As a result of def 6., the basis vectors of $\mathcal{L}(\mathcal{H})$ are all the possible outer products of the basis vectors from \mathcal{H} .

For any orthonormal basis $B = \{|b_n\rangle\}$, the identity operator, called the resolution of the identity in the basis B, can be written as:

$$1 = \sum_{b_n \in B} |b_n\rangle\langle b_n| \quad (21)$$

When an operator T acts on $|\psi\rangle$ belonging to \mathcal{H} , the mapping that arises $|\psi\rangle \mapsto \langle\varphi|(T|\psi\rangle)$ is a linear map from \mathcal{H} to \mathbb{C} , and thus belongs to \mathcal{H}^*

Def 7. Suppose T is an operator on \mathcal{H} . Then the adjoint of T , denoted T^\dagger , is defined as that linear operation on \mathcal{H}^* that satisfies

$$(\langle\psi|T^\dagger|\varphi\rangle)^* = \langle\varphi|T|\psi\rangle, \quad \forall |\psi\rangle, |\varphi\rangle \in \mathcal{H} \quad (22)$$

In the standard matrix representation, the matrix for T^\dagger is the **complex conjugate transpose** (also called the ‘Hermitean conjugate’, or ‘adjoint’) of the matrix for T . Essentially, The adjoint T^\dagger shifts the operator T from the ket to the bra while taking the complex conjugate.

Def 8. An operator U is called unitary if $U^\dagger = U^{-1}$ where U^{-1} is the inverse of U .

This implies that if an operator U is unitary, $U^\dagger U = I$ where I is the identity operator. The unitary operators preserve inner products between vectors, and in particular, preserve the norm of vectors.

Def 9. An operator T in a Hilbert space is called Hermitean or self-adjoint if it equal to its own Hermitean conjugate.

Def 10. A projector on a vector space \mathcal{H} is a linear operator P that satisfies $P^2 = P$. An orthogonal projector is a projector that also satisfies $P^\dagger = P$.

A projector maps a vector onto a subspace and leaves it there. Applying it twice has the same effect.

Def 11. A vector $|\psi\rangle$ is called an eigenvector of an operator T if

$$T|\psi\rangle = c|\psi\rangle \quad (23)$$

for some constant c . The constant c is called the eigenvalue of T corresponding to the eigenvector $|\psi\rangle$.

Determinants can be understood as the factor by which an area inside a vector space gets scaled when a transformation is applied to it. Eigenvectors can be understood as the vectors in a space that doesn't get knocked out of it's span by a transformation.

Def 12. If $T = T^\dagger$ and if $T|\psi\rangle = \lambda|\psi\rangle$ then $\lambda \in \mathbb{R}$. In other words, the eigenvalues of a Hermitean operator are real.

Def 13. The trace of an operator A acting on a Hilbert space is:

$$\text{Tr}(A) = \sum_{b_n} \langle b_n | A | b_n \rangle \quad (24)$$

where $\{|b_n\rangle\}$ is any orthonormal basis for the Hilbert space.

Trace of an operator has the cyclic property $\text{Tr}(ABC) = \text{Tr}(BCA)$. The trace of an operator A is independent of the basis, meaning $\text{Tr}(A)$ remains the same after a unitary transformation.

2.4 The Spectral Theorem

Def 14. A normal operator A is a linear operator that satisfies

$$AA^\dagger = A^\dagger A \quad (25)$$

Notice that both unitary and Hermitean operators are normal. Since in this book we are only interested in Hilbert spaces having finite dimensions, we will only consider the spectral theorem in this special case:

Def 15. (Spectral Theorem) For every normal operator T acting on a finite dimensional Hilbert space \mathcal{H} , there is an orthonormal basis of \mathcal{H} consisting of eigenvectors $|T_i\rangle$ of T .

T is diagonal in its own eigenbasis: $T = \sum_i T_i |T_i\rangle \langle T_i|$, where T_i are the eigenvalues corresponding to the eigenvectors $|T_i\rangle$. This is called the **spectral decomposition** of T . The set of eigenvalues of T is called the spectrum of T .

Spectral Theorem tells us that we can always diagonalize normal operators (in finite dimensions). We can restate Spectral Theorem to the more familiar form of diagonalization:

Def 16. For every finite dimensional normal matrix T , there is a unitary matrix P such that $T = P\Lambda P^\dagger$, where Λ is a diagonal matrix. The diagonal entries of Λ are the eigenvalues of T and the columns of P encode the eigenvectors of T .

2.5 Functions of Operators

In quantum mechanics and linear algebra, we often need to compute functions of matrices or operators. Since directly computing this is difficult we use the Spectral theorem and Taylor series analysis. By the Spectral Theorem, we can write every normal operator T in the diagonal form so that it is easy to apply functions to operators.

$$T = \sum_i T_i |T_i\rangle \langle T_i| \quad (26)$$

Additionally, each outer product is a projector and the eigenvectors are orthonormal. Therefore,

$$\begin{aligned} (|T_i\rangle \langle T_i|)^m &= |T_i\rangle \langle T_i| \\ \langle T_i | T_j \rangle &= \delta_{i,j} \end{aligned} \quad (27)$$

This means that computing a power of T (in diagonal form) is equivalent to computing the powers of the diagonal entries of T :

$$\left(\sum_i T_i |T_i\rangle \langle T_i| \right)^m = \sum_i T_i^m |T_i\rangle \langle T_i| \quad (28)$$

Now, the Taylor series of a function f takes the general form:

$$f(x) = \sum_{m=0}^{\infty} a_m x^m \quad (29)$$

The range of values of x for which the Taylor series converges is called the interval of convergence. For any point x in the interval of convergence, the Taylor

series of a function f converges to the value of $f(x)$. In general, the Taylor series for any function f acting on an operator T will have the form:

$$f(T) = \sum_m a_m T^m \quad (30)$$

We can simplify this using Spectral Theorem:

$$\begin{aligned} f(T) &= \sum_m a_m T^m \\ &= \sum_m a_m \left(\sum_i T_i |T_i\rangle \langle T_i| \right)^m \\ &= \sum_m a_m \sum_i T_i^m |T_i\rangle \langle T_i| \\ &= \sum_i \left(\sum_m a_m T_i^m \right) |T_i\rangle \langle T_i| \\ &= \sum_i f(T_i) |T_i\rangle \langle T_i| \end{aligned} \quad (31)$$

Note that the final form of the equation is derived from the general form in Eq. 30. So when T is written in diagonal form, $f(T)$ is computed by applying f separately to the diagonal entries of T .

2.6 Tensor Products

The tensor product is a way of combining spaces, vectors, or operators together. Suppose \mathcal{H}_1 and \mathcal{H}_2 are Hilbert spaces of dimension n and m respectively. Then the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a new, larger Hilbert space of dimension $n \times m$. Suppose $\{|b_i\rangle\}_{i \in \{1, \dots, n\}}$ is an orthonormal basis for \mathcal{H}_1 and $\{|c_j\rangle\}_{j \in \{1, \dots, m\}}$ is an orthonormal basis for \mathcal{H}_2 . Then the orthonormal basis for the space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is:

$$\{|b_i\rangle \otimes |c_j\rangle\}_{i \in \{1, \dots, n\}, j \in \{1, \dots, m\}} \quad (32)$$

is an orthonormal basis for the space $\mathcal{H}_1 \otimes \mathcal{H}_2$. The tensor product of two vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ from spaces \mathcal{H}_1 and \mathcal{H}_2 , respectively, is a vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$, and is written $|\psi_1\rangle \otimes |\psi_2\rangle$. The tensor product is characterized by the following axioms:

1. Scalability (analogous to $a(b \times c) = (ab) \times c = a \times (bc)$)
2. Distributivity (Linearity in the First and Second Component)

Suppose A and B are linear operators on \mathcal{H}_1 and \mathcal{H}_2 respectively. Then $A \otimes B$ is the linear operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$ defined by

$$(A \otimes B)(|\psi_1\rangle \otimes |\psi_2\rangle) \equiv A|\psi_1\rangle \otimes B|\psi_2\rangle \quad (33)$$

This definition extends linearly over the elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$(A \otimes B) \left(\sum_{ij} \lambda_{ij} |b_i\rangle \otimes |c_j\rangle \right) \equiv \sum_{ij} \lambda_{ij} A |b_i\rangle \otimes B |c_j\rangle. \quad (34)$$

This is in Dirac notation. In matrix form, the tensor product of a $m \times n$ matrix with a $p \times q$ matrix is a $mp \times nq$ matrix. Refer text for expanded form.

2.7 The Schmidt Decomposition Theorem

Def 17. (Schmidt Decomposition) If $|\psi\rangle$ is a vector in a tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$, then there exists an orthogonal basis $\{|\psi_i^A\rangle\}$ for \mathcal{H}_A and an orthonormal basis $\{|\psi_i^B\rangle\}$ for \mathcal{H}_B and non-negative real numbers $\{p_i\}$ so that:

$$|\psi\rangle = \sum_i \sqrt{p_i} |\psi_i^A\rangle |\psi_i^B\rangle \quad (35)$$

The coefficients $\sqrt{p_i}$ are called *Schmidt Coefficients*.

2.8 Some Comments on Dirac Notation

Some useful identities:

$$\begin{aligned} (|i\rangle |j\rangle)^\dagger |k\rangle |l\rangle &= \langle i|k\rangle \langle j|l\rangle \\ (|i\rangle \otimes |j\rangle)(\langle k| \otimes \langle l|) &= |i\rangle \langle k| \otimes |j\rangle \langle l| \end{aligned} \quad (36)$$

For more general cases:

$$|i\rangle \langle k| = |i\rangle \langle k| \otimes |j\rangle \langle l| \quad (37)$$

3 Qubits and the Framework of Quantum Mechanics

4 Further Reading

Hermitean Operators <https://www.youtube.com/watch?v=xWb97DEq864>

Taylor Series Expansion <https://youtu.be/3d6DsJIBzJ4?si=ny2Xu0IarNYidZVC>

Properties of operators in quantum mechanics https://youtube.com/playlist?list=PL8W2boV7eVfnb10T_COKPozxEYzEKDwns&si=WbWtmNJgW3rWxYX4

5 Bibliography

List papers not cited by “Paper” that inform your understanding of the work.