COMP245: Probability and Statistics 2016 - Problem Sheet 4 Solutions

Further Probability

S1)

$$\begin{aligned}
\mathrm{P}\{A \cap (B \cup C)\} &= \mathrm{P}\{(A \cap B) \cup (A \cap C)\} \\
&= \mathrm{P}(A \cap B) + \mathrm{P}(A \cap C) - \mathrm{P}(A \cap B \cap C) \\
&= \mathrm{P}(A)\mathrm{P}(B) + \mathrm{P}(A)\mathrm{P}(C) - \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C) \\
&= \mathrm{P}(A)\{\mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(B \cap C)\} \\
&= \mathrm{P}(A)\mathrm{P}(B \cup C),
\end{aligned}$$

so A and $B \cup C$ are independent.

S2) (a) $\mathrm{P}(A) = \mathrm{P}(\{a\}) + \mathrm{P}(\{b\}) = 0.2 + 0.3 = 0.5$.

(b) $\mathrm{P}(B) = \mathrm{P}(\{b\}) + \mathrm{P}(\{c\}) + \mathrm{P}(\{d\}) = 0.3 + 0.4 + 0.1 = 0.8$.

(c) $\mathrm{P}(\overline{A}) = \mathrm{P}(\{c, d\}) = \mathrm{P}(\{c\}) + \mathrm{P}(\{d\}) = 0.4 + 0.1 = 0.5$.

(d) $\mathrm{P}(A \cup B) = \mathrm{P}(\{a, b, c, d\}) = \mathrm{P}(S) = 1$.

(e) $\mathrm{P}(A \cap B) = \mathrm{P}(\{b\}) = 0.3$.

S3) Let $A =$ 'Part came from factory 1', $B =$ 'Part is defective'. We want $\mathrm{P}(A|B)$.

We know $\mathrm{P}(B|A) = 100/1000$, $\mathrm{P}(B|\overline{A}) = 150/2000$. Also, since a part is 'selected at random', there is a 1000/(1000+2000) = 1/3 chance that it comes from factory 1. That is, we know $\mathrm{P}(A) = 1/3$ and $\mathrm{P}(\overline{A}) = 2/3$.

The overall probability that a selected part will be defective, $\mathrm{P}(B) = (100{+}150)/(1000{+}2000) = 250/3000 = 1/12$. Hence

$$\begin{aligned}
\mathrm{P}(A|B) &= \mathrm{P}(B|A) \times \mathrm{P}(A)/\mathrm{P}(B) \\
&= (1/10 \times 1/3)/(1/12) \\
&= 0.4.
\end{aligned}$$

S4) Sample space:

(1,1) (1,2) (1,3) (1,4) (1,6) (1,6)

(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

There are 36 equally likely possible outcomes altogether. In 18 of these, the first die is odd, and in 18 the second die is odd. Hence $\mathrm{P}(A) = \mathrm{P}(B) = \frac{1}{2}$. Likewise, from the table we see that in 18 of the 36 the sum is odd, so that $\mathrm{P}(C) = \frac{1}{2}$.

Also from the table we can see that $\mathrm{P}(A \cap B) = \mathrm{P}(A \cap C) = \mathrm{P}(B \cap C) = \frac{9}{36} = \frac{1}{4}$.

However, $\mathrm{P}(A).\mathrm{P}(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ which is equal to $\mathrm{P}(A \cap B)$, so that A and B are independent. Similarly for B and C and for A and C.

Thus A, B, and C are pairwise independent.

Since the sum of two odd numbers is even, we have $\mathrm{P}(A \cap B \cap C) = 0$, which is not equal to $\mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C) = \frac{1}{8}$, the three events A, B, and C are not independent.

S5) Let D be the event that the phone is defective, and let M_i be the event that the phone is manufactured by plant i ($i = 1, 2, 3$).

(a) $\mathrm{P}(D) = \sum_{i=1}^{3} \mathrm{P}(D|M_i)\mathrm{P}(M_i) = 0.02 \times 0.5 + 0.05 \times 0.3 + 0.01 \times 0.2 = 0.027.$

(b) $\mathrm{P}(M_2|D) = \dfrac{\mathrm{P}(D|M_2)\mathrm{P}(M_2)}{\mathrm{P}(D)} = \dfrac{0.05 \times 0.3}{0.027} = 0.556.$

S6) Let, A, B, and C be the event that the player wins, the player wins on the first roll, and the player gains point, respectively. Then $\mathrm{P}(A) = \mathrm{P}(B) + \mathrm{P}(C)$. Now $\mathrm{P}(B) = \mathrm{P}(\text{sum} = 7) + \mathrm{P}(\text{sum} = 11) = 6/36 + 2/36 = 2/9$.

Let A_k be the event that point occurs before 7. Then

$$\mathrm{P}(C) = \sum_{k \in \{4,5,6,8,9,10\}} \mathrm{P}(A_k)\mathrm{P}(\text{point} = k).$$

Well $\mathrm{P}(A_k) = \dfrac{\mathrm{P}(\text{sum} = k)}{\mathrm{P}(\text{sum} = k) + \mathrm{P}(\text{sum} = 7)}$.

We have $\mathrm{P}(\text{sum} = 4) = 3/36$, $\mathrm{P}(\text{sum} = 5) = 4/36$, $\mathrm{P}(\text{sum} = 6) = 5/36$, $\mathrm{P}(\text{sum} = 8) = 5/36$, $\mathrm{P}(\text{sum} = 9) = 4/36$ and $\mathrm{P}(\text{sum} = 10) = 3/36$.

Hence it follows that $\mathrm{P}(A_4) = 1/3$, $\mathrm{P}(A_5) = 2/5$, $\mathrm{P}(A_4) = 5/11$, $\mathrm{P}(A_8) = 5/11$, $\mathrm{P}(A_9) = 2/5$ and $\mathrm{P}(A_10) = 1/3$.

So $\mathrm{P}(A) = \mathrm{P}(B) + \mathrm{P}(C) = 2/9 + 134/495 = 0.49293$.

S7) Here is my R code for the simulations:

```
par(mfrow=c(1,2))
p <- 0.3
n <- 1000
H <- numeric(n) #create vector of length n with all entries 0.
for(i in 1:n){
    if(runif(1)<p) #runif - draws a random number from (0,1)
        H[i] <- 1 #record 1 if heads obtained.
}
plot(1:n,cumsum(H)/(1:n),type="l")#cumsum = cumulative sum
abline(h=p)

p <- 0.03
H2 <- numeric(n)
for(i in 1:n){
    if(runif(1)<p)
        H2[i] <- 1
}
plot(1:n,cumsum(H2)/(1:n),type="l")
abline(h=p)
```

We note that for $p = 0.03$ the plot is more "jumpy" and looks like it takes longer to converge to 0.03. This is because obtaining a head is less likely to appear (i.e. a more extreme event).

S8)

$$P(A) = 1/2, \quad P(B) = 2/3. \quad P(A \cap B) = 1/3 = (1/2)(2/3) = P(A)P(B)$$

Therefore, A and B are independent.

Here is my R code for the simulation

```
n <- 1e4
A <- c(2,4,6)
B <- c(1,2,3,4)
Acount <- 0
Bcount <- 0
ABcount <- 0
for(i in 1:n){
    s <- sample(1:6,1)
```

```
    if(s%in%A)
        Acount <- Acount+1
    if(s%in%B)
        Bcount <- Bcount+1
    if(s%in%A && s%in%B)
        ABcount <- ABcount+1
}
(Acount/n)*(Bcount/n)-ABcount/n
```

I will take $A = \{1,2,3\}$ and $B = \{3,4,5\}$ as two events that are not independent (other dependent events are possible).

```
n <- 1e4
A <- c(1,2,3)
B <- c(3,4,5)

Acount <- 0
Bcount <- 0
ABcount <- 0

for(i in 1:n){
    s <- sample(1:6,1)
    if(s%in%A)
        Acount <- Acount+1
    if(s%in%B)
        Bcount <- Bcount+1
    if(s%in%A && s%in%B)
        ABcount <- ABcount+1
}
(Acount/n) #should be 1/2
(Bcount/n) #should also be 1/2
ABcount/n # should be 1/6
```