IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2017

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
BEng Honours Degree in Mathematics and Computer Science Part I
MEng Honours Degree in Mathematics and Computer Science Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C140=MC140

LOGIC

Monday 8 May 2017, 14:00
Duration: 80 minutes

Answer ALL TWO questions

Paper contains 2 questions
Calculators not required

1 a Let A be the formula $p \to q \leftrightarrow \neg q \to \neg p$.

i) Draw the formation tree of the formula A.

ii) Use direct argument to show that formula A is valid.

iii) Use propositional equivalences to show that A and $\top$ are logically equivalent.

b Consider the following two formulae: $(p \to q) \wedge \neg q$ and $\neg p$. These are not logically equivalent, but one does logically imply the other. State the direction in which the logical implication holds and give a proof of this using natural deduction.

c For the remaining part of this question, formulae are written using the standard propositional connectives $\wedge$, $\vee$, $\neg$, $\to$, $\top$, and $\perp$ and a new connective Δ, meaning "contrapositive".

Δ has one natural deduction introduction rule and one elimination rule:

$$
\begin{array}{lll}
\hline
\multicolumn{1}{|l}{1} & \neg B & \multicolumn{1}{l|}{\text{ass}} \\
\multicolumn{1}{|l}{} & \vdots & \multicolumn{1}{l|}{} \\
\multicolumn{1}{|l}{2} & \neg A & \multicolumn{1}{l|}{} \\
\hline
3 & A\Delta B & \Delta\text{I}(1,2)
\end{array}
\qquad\qquad
\begin{array}{lll}
1 & A\Delta B & \\
 & \vdots & \\
2 & \neg B & \\
3 & \neg A & \Delta\text{E}(1, 2)
\end{array}
$$

i) Use these rules to show that $p\Delta(q\Delta r) \vdash (p \wedge q)\Delta r$.

ii) Assume the formula $A\Delta B$ to be equal to $A \to B$. Show that ΔI and ΔE are *derived rules* of the usual natural deduction proof system without the new connective Δ.

iii) Assume $\vdash^*$ to be the natural deduction system obtained from the usual natural deduction proof system by replacing the $\to I$ and $\to E$ rules with the ΔI and ΔE rules respectively.

- State what it means to say that $\vdash^*$ is *sound*.
- State whether $\vdash^*$ is sound. Justify your answer.

The three parts carry, respectively, 30%, 20%, and 50% of the marks.

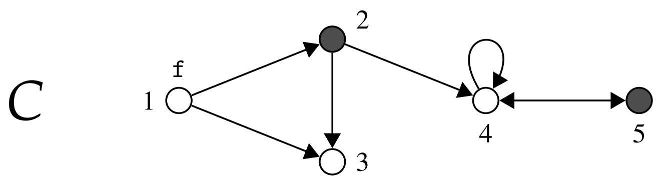
2 The signature L (used to describe company employees) consists of:

- a unary relation symbol P, where $P(a)$ means that *'a is a programmer'*,
- a binary relation symbol M, where $M(a,b)$ means that *'a manages b'*, or *'a is a manager of b'*,
- a constant $\mathtt{f}$ denoting the *founder* of the company.

a Translate each of the following into a logic L-sentence.

i) The founder has no manager.
ii) The founder manages the programmers.
iii) No programmer manages more than one person.
iv) Every programmer who manages a programmer is managed by the founder.
v) No two programmers have the same managers.

b The L-structure C shown below represents the employees of a company. The black circles are the programmers, the arrows show the interpretation of M, and the constant $\mathtt{f}$ is interpreted as person no. 1. For example, $C \models P(2) \wedge M(\mathtt{f}, 3)$.



i) For each of the following L-sentences, state whether it is true in C or false in C. You do not need to justify your answers.

A) $\forall x \exists y (P(y) \wedge \neg M(x,y))$
B) $\exists x \forall y (\neg \exists z M(z,y) \leftrightarrow (y = x \vee y = \mathtt{f}))$

ii) At bonus time, every employee in C predicts: 'none of my managers will get a bonus'. It turned out that *each employee got a bonus if and only if their prediction was correct.* Who got a bonus? Explain your answer.

c Use equivalences to show that the following sentences are logically equivalent:

$$\forall y (\exists x M(x,y) \to P(y)),$$
$$\forall x \neg \exists y (M(x,y) \wedge \neg P(y)).$$

In each step, use only one equivalence, and state its general form.

d Prove by natural deduction that

$$\forall x P(x), \quad \exists x \forall y (P(y) \to y = x) \quad \vdash \quad \exists x \forall y (y = x).$$

The four parts carry equal marks.