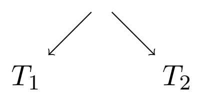
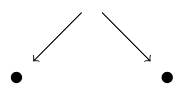
Computation Answers 3: Induction

1. (a) A single `Node` is easy to draw:

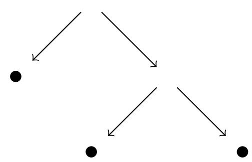
The tree $\texttt{Branch}(T_1, T_2)$ looks like:



$\texttt{Branch}(\texttt{Node}, \texttt{Node})$ looks like:



$\texttt{Branch}(\texttt{Node}, \texttt{Branch}(\texttt{Node}, \texttt{Node}))$ looks like:



(b) We prove by induction on the structure of binary trees that, for any tree T,

$$\texttt{leaves}(T) = \texttt{branches}(T) + 1.$$

Let us refer to this property as $P(T)$. To show that $P(T)$ is true of all binary trees T, the principle of induction says that we must do two things.

Base Case: prove that $P(\texttt{Node})$ is true: that is, that

$$\texttt{leaves}(\texttt{Node}) = \texttt{branches}(\texttt{Node}) + 1.$$

This follows trivially by definition.

Inductive Case: the inductive hypothesis IH is that $P(T_1)$ and $P(T_2)$ are both true, for some T_1 and T_2. So we can assume the IH for T_1 and T_2, namely that

$$\texttt{leaves}(T_1) = \texttt{branches}(T_1) + 1$$

and

$$\texttt{leaves}(T_2) = \texttt{branches}(T_2) + 1$$

From this assumption, we have to derive $P(\texttt{Branch}(T_1, T_2))$: namely that

$$\texttt{leaves}(\texttt{Branch}(T_1, T_2)) = \texttt{branches}(\texttt{Branch}(T_1, T_2)) + 1.$$

By definition,

$$\texttt{leaves}(\texttt{Branch}(T_1, T_2)) = \texttt{leaves}(T_1) + \texttt{leaves}(T_2).$$

By the inductive hypothesis IH,

$$\texttt{leaves}(T_1) = \texttt{branches}(T_1) + 1$$

and

$$\texttt{leaves}(T_2) = \texttt{branches}(T_2) + 1$$

We therefore have

$$\texttt{leaves}(\texttt{Branch}(T_1, T_2)) = \texttt{branches}(T_1) + 1 + \texttt{branches}(T_2) + 1.$$

By the definition of `branches`,

$$\texttt{branches}(\texttt{Branch}(T_1, T_2)) = \texttt{branches}(T_1) + \texttt{branches}(T_2) + 1.$$

It is therefore the case that

$$\texttt{leaves}(\texttt{Branch}(T_1, T_2)) = \texttt{branches}(\texttt{Branch}(T_1, T_2)) + 1.$$

2. (**Totality of $\Downarrow$**) For all expressions E, we wish to establish $P(E)$, which is:

   there is some number n such that $E \Downarrow n$.

   The proof is by induction on the structure of expressions.

   Base Case: We must show $P(n)$ holds for every number n. For any number n, the axiom of the big-step semantics, (B-NUM), gives us that $n \Downarrow n$ so the property is true for every n, as required.

   Inductive Case: We must show $P(E_1 + E_2)$ for arbitrary E_1 and E_2. We assume the inductive hypotheses that $P(E_1)$ and $P(E_2)$ hold.

   We must show that for some n, it is the case that $(E_1 + E_2) \Downarrow n$. By the inductive hypotheses, there are numbers n_1 and n_2 for which $E \Downarrow n_1$ and $E \Downarrow n_2$. We can then apply the rule (B-ADD) to obtain

   $$\frac{E_1 \Downarrow n_1 \quad E_2 \Downarrow n_2}{(E_1 + E_2) \Downarrow n_3}$$

   where $n_3 = n_1 \underline{+} n_2$. So n_3 is the required number which makes $P(E_1 + E_2)$ true.

   Inductive Case: We must show $P(E_1 \times E_2)$ for arbitrary E_1 and E_2, assuming $P(E_1)$ and $P(E_2)$. This case follows a similar pattern to the previous case.

3. We use induction on the structure of E. Let $P(E)$ be the property:

   for all n, $\texttt{den}(E) = n$ if and only if $E \Downarrow n$.

   Base Case: We must prove $P(n)$ for arbitrary number n, that is: $\texttt{den}(n) = m$ if and only if $n \Downarrow m$. Both sides hold exactly when $n = m$ by the definition of `den` and the axiom

   $$\text{(B-NUM)}\,\frac{}{n \Downarrow n}$$

   Inductive Case: $E = E_1 + E_2$. The inductive hypotheses are $P(E_1)$ and $P(E_2)$; that is,

for all n_1, $\texttt{den}(E_1) = n_1$ if and only if $E_1 \Downarrow n_1$
for all n_2, $\texttt{den}(E_2) = n_2$ if and only if $E_2 \Downarrow n_2$

From these assumptions, we must show that $P(E_1 + E_2)$: namely that

for all n, $\texttt{den}(E_1 + E_2) = n$ if and only if $E_1 + E_2 \Downarrow n$

First we show the implication from left to right:

By the definition of $\texttt{den}$, $\texttt{den}(E_1 + E_2) = \texttt{den}(E_1) \underline{+} \texttt{den}(E_2)$. Let n_1 and n_2 be such that $\texttt{den}(E_1) = n_1$ and $\texttt{den}(E_2) = n_2$. By the inductive hypotheses, $E_1 \Downarrow n_1$ and $E_2 \Downarrow n_2$. Using the (B-ADD) rule, we have $(E_1 + E_2) \Downarrow n$ and $n = n_1 \underline{+} n_2 = \texttt{den}(E_1 + E_2)$.

Next we show the implication from right to left:

If $(E_1 + E_2) \Downarrow n$, then this has been derived using the (B-ADD) rule:

$$\text{(B-ADD)}\frac{E_1 \Downarrow n_1 \qquad E_2 \Downarrow n_2}{(E_1 + E_2) \Downarrow n} n = n_1 \underline{+} n_2$$

Hence, we have $E_1 \Downarrow n_1$, $E_2 \Downarrow n_2$ for some n_1, n_2 with $n = n_1 \underline{+} n_2$. By the inductive hypotheses, we have $\texttt{den}(E_1) = n_1$ and $\texttt{den}(E_2) = n_2$, and hence $\texttt{den}(E_1 + E_2) = n_1 \underline{+} n_2 = n$.

Inductive Case: $E = E_1 \times E_2$. Since this case is similar to the previous one, we omit it, concluding our proof.

4. We wish to prove the property $P(E)$ for all E, where

$$P(E) \equiv (\exists n. E = n) \vee (\exists E'. E \rightarrow E')$$

Base Case: We must prove $P(n)$ for arbitrary number n. This holds trivially: there exists the number n.

Inductive Case: Assuming $P(E_1)$ and $P(E_2)$ as the inductive hypotheses, we must prove $P(E_1 + E_2)$. Specifically, the inductive hypotheses are:

$$(\exists n_1. E_1 = n_1) \vee (\exists E_1'. E_1 \rightarrow E_1') \tag{1}$$
$$(\exists n_2. E_2 = n_2) \vee (\exists E_2'. E_2 \rightarrow E_2') \tag{2}$$

Consider the cases for (1):

- $\exists n_1. E_1 = n_1$. Consider the cases for (2):
  - $\exists n_2. E_2 = n_2$. Let $n_3 = n_1 \underline{+} n_2$. Now we have

$$\text{S-ADD}\ \frac{}{n_1 + n_2 \rightarrow n_3}$$

Hence the right side of the disjunction holds by choosing $E' = n_3$, and we are done.

– $\exists E_2'.\, E_2 \to E_2'$. We have

$$\textsc{s-right}\ \frac{E_2 \to E_2'}{n_1 + E_2 \to n_1 + E_2'}$$

Hence the right side of the disjunction holds by choosing $E' = n_1 + E_2'$, and we are done.

- $\exists E_1'.\, E_1 \to E_1'$. We have

$$\textsc{s-left}\ \frac{E_1 \to E_1'}{E_1 + E_2 \to E_1' + E_2}$$

Hence the right side of the disjunction holds by choosing $E' = E_1' + E_2$, and we are done.

Inductive Case: Assuming $P(E_1)$ and $P(E_2)$ as the inductive hypotheses, we must prove $P(E_1 \times E_2)$. This case follows exactly the same pattern as the previous one.

5. (a)

$$\begin{array}{rc} \text{Base Case:} & ops(n) = 0 \\ \text{Inductive Case:} & ops(E_1 + E_2) = ops(E_1) + 1 + ops(E_2) \\ \text{Inductive Case:} & ops(E_1 \times E_2) = ops(E_1) + 1 + ops(E_2) \end{array}$$

(b) We wish to prove the property $P(E)$ for all E, where

$$P(E) \equiv \forall E'.\, E \to E' \implies ops(E) > ops(E')$$

Base Case: We must prove $P(n)$ for arbitrary number n. Assume that E' is such that $n \to E'$. This is a contradiction, because there are no derivation rules that give evaluation steps for numbers. Hence the implication holds trivially.

Inductive Case: Assuming $P(E_1)$ and $P(E_2)$ as the inductive hypotheses, we must prove $P(E_1 + E_2)$. Specifically, the inductive hypotheses are:

$$\forall E_1'.\, E_1 \to E_1' \implies ops(E_1) > ops(E_1')$$
$$\forall E_2'.\, E_2 \to E_2' \implies ops(E_2) > ops(E_2')$$

Assume that E' is such that $E_1 + E_2 \to E'$. There are three rules that could give such a reduction:

- S-LEFT: in this case, $E' = E_1' + E_2$ for some E_1' with $E_1 \to E_1'$. By the inductive hypothesis, $ops(E_1) > ops(E_1')$. Hence, we have

$$ops(E_1 + E_2) = ops(E_1) + 1 + ops(E_2) > ops(E_1') + 1 + ops(E_2) = ops(E')$$

as required.

- S-RIGHT: in this case, $E_1 = n$ and $E' = E_1 + E_2'$ for some n and E_2' with $E_2 \to E_2'$. By the inductive hypothesis, $ops(E_2) > ops(E_2')$. Hence, we have

$$ops(E_1 + E_2) = ops(E_1) + 1 + ops(E_2) > ops(E_1) + 1 + ops(E_2') = ops(E')$$

as required.

- S-ADD: in this case, $E_1 = n_1$, $E_2 = n_2$ and $E' = n_3$ for some n_1, n_2, n_3 with $n_3 = n_1 \underline{+} n_2$. Thus, $ops(E_1{+}E_2) = 1$ and $ops(E') = 0$, so we have $ops(E_1{+}E_2) > ops(E')$ as required.

Inductive Case: Assuming $P(E_1)$ and $P(E_2)$ as the inductive hypotheses, we must prove $P(E_1 \times E_2)$. This case follows exactly the same pattern as the previous case.

(c) Recall that (strong) normalisation requires that there is no infinite sequence of expressions $E_1, E_2, E_3, \ldots$ with $E_i \to E_{i+1}$ for all i. Suppose that we have an evaluation sequence,

$$E_1 \to E_2 \to E_3 \to \ldots$$

Then we also have a sequence of natural numbers

$$ops(E_1) > ops(E_2) > ops(E_3) > \ldots$$

Such a sequence can have no more than $ops(E_1){+}1$ elements, therefore the evaluation sequence must be finite. Hence, $\to$ is (strongly) normalising.

6. We prove each direction of implication separately. We first prove, by induction on E, the property $P(E)$:

$E \Downarrow n$ implies $E \to^* n$.

Base Case: $E = n$ for arbitrary number n. We know that $n \Downarrow n$ and $n \to^0 n$, so the base cases are trivially true.

Inductive Case: $E = E_1 + E_2$ for arbitrary expressions E_1 and E_2. The inductive hypotheses are:

$E_1 \Downarrow n_1$ implies $E_1 \to^* n_1$
$E_2 \Downarrow n_2$ implies $E_2 \to^* n_2$

From these assumptions, we must show that $P(E_1 + E_2)$: namely that

$(E_1 + E_2) \Downarrow n$ implies $(E_1 + E_2) \to^* n$.

Let $(E_1 + E_2) \Downarrow n$. From the rules, this can only happen if $E_1 \Downarrow n_1$ and $E_2 \Downarrow n_2$ for some $n_1 \underline{+} n_2 = n$. By the inductive hypotheses, $E_1 \to^* n_1$ and $E_2 \to^* n_2$. For each step in the reduction

$$E_1 \to E_1' \to E_1'' \to \ldots \to n_1$$

applying the rule for reducing the left argument of an addition gives

$$(E_1 + E_2) \to (E_1' + E_2) \to (E_1'' + E_2) \to \ldots \to (n_1 + E_2).$$

(Formally, we should prove this by induction on the length of evaluation sequences.)

Applying the other rule to the sequence for $E_2 \to^* n_2$ allows us to deduce that

$$(n_1 + E_2) \to^* (n_1 + n_2) \to n_3$$

where $n_3 = n_1 \underline{+} n_2$. (Again, we should prove this by induction on the length of evaluation sequences to be formal.) Hence $(E_1 + E_2) \to^* n_3$. This result shows that if $(E_1 + E_2) \Downarrow n$ then $(E_1 + E_2) \to^* n_3$. Thus, we have shown that $E \Downarrow n$ implies $E \to^* n$. (See below for a more elegant way of proving this.)

Now we must show the other way, that

$E \to^* n$ implies $E \Downarrow n$.

Suppose that $E \to^* n$. By the totality of $\Downarrow$ (question 2), we know that $E \Downarrow m$ for some m. By the first half of the result, it follows that $E \to^* m$. By the uniqueness of answers for $\to$ (in the lecture notes), it must be that $m = n$. Therefore $E \Downarrow n$, as required.

It is also possible to prove this result directly, without appealing to normalization and determinacy, by induction on the structure of E.

Formalising the informal arguments. We proved the first part of the implication by an informal argument that we can transform sequences of proof steps. We can formalise this argument as follows:

Lemma

(a) $E_1 \to^r E_1'$ implies $(E_1 + E_2) \to^r (E_1' + E_2)$
(b) $E_2 \to^r E_2'$ implies $(n + E_2) \to^r (n + E_2')$

Proof by induction on r. Let $P(r)$ be the property:

$$\forall E_2. E_1 \to^r E_1' \text{ implies } (E_1 + E_2) \to^r (E_1' + E_2)$$

Base Case: prove that P(0) is true. When r is 0, then it must be the case that $E_1 \to^0 E_1$, and $(E_1 + E_2) \to^0 (E_1 + E_2)$ holds for arbitrary E_2.

Inductive Case: the inductive hypothesis is that $P(k)$ holds: that is,

$$\forall E_2. E_1 \to^k E_1' \text{ implies } (E_1 + E_2) \to^k (E_1' + E_2)$$

From this assumption, we must show that $P(k+1)$: namely that,

$$\forall E_2. E_1 \to^{k+1} E_1' \text{ implies } (E_1 + E_2) \to^{k+1} (E_1' + E_2)$$

So consider $E_1 \to^{k+1} E_1'$ for arbitrary E_1 and E_1', which can be broken down into $E_1 \to^k E_1'' \to E_1'$. By the inductive hypothesis, $(E_1 + E_2) \to^k (E_1'' + E_2)$. Since $E_1'' \to E_1'$, we can apply the rule (S-LEFT) to obtain $(E_1'' + E_2) \to (E_1' + E_2)$, hence the result. The other proof is similar.

Corollary

(a) $E_1 \to^* n_1$ implies $(E_1 + E_2) \to^* (n_1 + E_2)$.
(b) $E_2 \to^* n_2$ implies $(n_1 + E_2) \to^* (n_1 + n_2)$.
(c) $E_1 \to^* n_1$ and $E_2 \to^* n_2$ implies $(E_1 + E_2) \to^* n$ where $n = n_1 \underline{+} n_2$.

7. Fix some arbitrary state s. We wish to show $P(C)$ for all commands C, where

$$P(C) \equiv \forall C_1, C_2, s_1, s_2. \langle C, s\rangle \to_c \langle C_1, s_1\rangle \wedge \langle C, s\rangle \to_c \langle C_2, s_2\rangle \implies \langle C_1, s_1\rangle = \langle C_2, s_2\rangle$$

Proceed by induction on the structure of the command C.

Base Case: We must prove that $P(\texttt{skip})$ holds. This is trivial: since there are no reductions from `skip`, the left-hand-side of the implication cannot hold.

Base Case: We must prove that $P(x := E)$ is true for arbitrary program variable x and arbitrary expression E. Suppose that

$$\langle x := E, s\rangle \to_c \langle C_1, s_1\rangle \tag{3}$$
$$\langle x := E, s\rangle \to_c \langle C_2, s_2\rangle; \tag{4}$$

we must show that $\langle C_1, s_1\rangle = \langle C_2, s_2\rangle$. We consider two cases: whether or not E is a number.

- Suppose that $E = n$, for some number n. It is not the case that $\langle n, s\rangle \to_e \langle E', s'\rangle$ for any E', s', and so the W-ASS.EXP rule does not apply. The only rule that can give a step from $\langle x := n, s\rangle$ is therefore the W-ASS.NUM rule, and so it must be that $\langle C_1, s_1\rangle = \langle \texttt{skip}, s[x \mapsto n]\rangle = \langle C_2, s_2\rangle$. (Note that it is significant that a number expression does not reduce any further: if it did, the semantics would not be deterministic, since both rules could apply!)
- Suppose that E is not a number. The only rule that can apply in this case is the W-ASS.EXP rule, so (3) and (4) must be derived as follows:

$$\text{W-ASS.EXP}\ \frac{\langle E, s\rangle \to_e \langle E_1, s_1\rangle}{\langle x := E, s\rangle \to_c \langle x := E_1, s_1\rangle} \qquad \text{W-ASS.EXP}\ \frac{\langle E, s\rangle \to_e \langle E_2, s_2\rangle}{\langle x := E, s\rangle \to_c \langle x := E_2, s_2\rangle}$$

  where $C_1 = (x := E_1)$ and $C_2 = (x := E_2)$ for some E_1, E_2. Now determinacy for $\to_e$ tells us that $\langle E_1, s_1\rangle = \langle E_2, s_2\rangle$. Therefore, $\langle x := E_1, s_1\rangle = \langle x := E_2, s_2\rangle$.

Inductive Case: We must prove that $P(\texttt{if}\ B\ \texttt{then}\ C_a\ \texttt{else}\ C_b)$ holds for all B, C_a, C_b. For the inductive hypotheses, we assume $P(C_a)$ and $P(C_b)$. (In fact, we will not use either inductive hypothesis.) Suppose that

$$\langle \texttt{if}\ B\ \texttt{then}\ C_a\ \texttt{else}\ C_b, s\rangle \to_c \langle C_1, s_1\rangle$$
$$\langle \texttt{if}\ B\ \texttt{then}\ C_a\ \texttt{else}\ C_b, s\rangle \to_c \langle C_2, s_2\rangle;$$

we must show that $\langle C_1, s_1\rangle = \langle C_2, s_2\rangle$. We consider three cases: where B is `true`, `false`, or something else.

- Suppose that $B = \texttt{true}$. The boolean expression `true` does not reduce any further, so the only applicable rule is w-cond.true. Consequently, $\langle C_1, s_1\rangle = \langle C_a, s\rangle = \langle C_2, s_2\rangle$.
- When $B = \texttt{false}$, by similar argument the only applicable rule is w-cond.false so, $\langle C_1, s_1\rangle = \langle C_b, s\rangle = \langle C_2, s_2\rangle$.
- Otherwise, it must be that $C_1 = \texttt{if}\ B_1\ \texttt{then}\ C_a\ \texttt{else}\ C_b$ where $\langle B, s\rangle \to_b \langle B_1, s_1\rangle$, and $C_2 = \texttt{if}\ B_2\ \texttt{then}\ C_a\ \texttt{else}\ C_b$ where $\langle B, s\rangle \to_b \langle B_2, s_2\rangle$. By the determinacy of $\to_b$, it must be that $\langle B_1, s_1\rangle = \langle B_2, s_2\rangle$, and so $\langle C_1, s_1\rangle = \langle C_2, s_2\rangle$.

Inductive Case: We must prove that $P(C_a; C_b)$ holds for all C_a, C_b. For the inductive hypotheses, we assume $P(C_a)$ and $P(C_b)$. (In fact, we only use $P(C_a)$.) Suppose that

$$\langle C_a; C_b, s\rangle \to_c \langle C_1, s_1\rangle$$
$$\langle C_a; C_b, s\rangle \to_c \langle C_2, s_2\rangle;$$

we must show that $\langle C_1, s_1\rangle = \langle C_2, s_2\rangle$. We consider two cases: whether or not $C_a = \texttt{skip}$.

- Suppose that $C_a = \texttt{skip}$. `skip` does not make any reductions, and so only the W-SEQ.SKIP rule could make derivations for the above reduction steps. Hence, $\langle C_1, s_1\rangle = \langle C_b, s\rangle = \langle C_2, s_2\rangle$.
- If C_a is not `skip` then for the above reductions to be derivable, it must be that $C_1 = C_{1a}; C_b$ with $\langle C_a, s\rangle \to_c \langle C_{1a}, s_1\rangle$, and $C_2 = C_{2a}; C_b$ with $\langle C_a, s\rangle \to_c \langle C_{2a}, s_2\rangle$. The inductive hypothesis $P(C_a)$ tells us

$$\forall C_{1a}, C_{2a}, s_1, s_2.\ \langle C_a, s\rangle \to_c \langle C_{1a}, s_1\rangle \wedge \langle C_a, s\rangle \to_c \langle C_{2a}, s_2\rangle \implies \langle C_{1a}, s_1\rangle = \langle C_{2a}, s_2\rangle$$

Consequently, we must have $\langle C_{1a}, s_1\rangle = \langle C_{2a}, s_2\rangle$, and hence $\langle C_1, s_1\rangle = \langle C_2, s_2\rangle$. (Note, this is the only truly inductive case.)

Inductive Case: We must prove that $P(\texttt{while } B \texttt{ do } C)$ holds for all B, C. For the inductive hypothesis, we assume $P(C)$. (Again, the inductive hypothesis is not used.) As in previous cases, we suppose that there are reduction steps to $\langle C_1, s_1\rangle$ and $\langle C_2, s_2\rangle$. There is only one rule that applies, which requires that

$$\langle C_1, s_1\rangle = \langle \texttt{if } B \texttt{ then } (C; \texttt{while } B \texttt{ do } C) \texttt{ else skip}, s\rangle = \langle C_2, s_2\rangle$$

8. (a) The rule for multiplication follows the structure of the rule for addition:

$$\textsc{Times}\ \frac{\langle E_1, s\rangle \Downarrow \langle n_1, s\rangle \qquad \langle E_2, s\rangle \Downarrow \langle n_2, s\rangle}{\langle E_1 \times E_2, s\rangle \Downarrow \langle n_3, s\rangle}\ n_3 = n_1 \underline{\times} n_2$$

(b) We wish to show $P(E)$, for all expressions E, numbers n and states s where:

$$P(E) \equiv \langle f(E), s\rangle \Downarrow \langle n, s\rangle \implies \langle E, s\rangle \Downarrow \langle n, s\rangle$$

We proceed by induction on the structure of the expression E.

Base Cases: The base cases are the cases where $E = n$ and where $E = x$. In each of these cases, $f(E) = E$, so $P(E)$ holds trivially.

Inductive Case 1: Consider the case in which $E = E_1 \times E_2$. We assume the inductive hypotheses $P(E_1)$ and $P(E_2)$.

We wish to show

$$\langle f(E_1 \times E_2), s\rangle \Downarrow \langle n, s\rangle \implies \langle E_1 \times E_2, s\rangle \Downarrow \langle n, s\rangle$$

By the definition of f it is sufficient to show:

$$\langle f(E_1) \times f(E_2), s\rangle \Downarrow \langle n, s\rangle \implies \langle E_1 \times E_2, s\rangle \Downarrow \langle n, s\rangle$$

Since the only rule that applies to an expression of the form $E_1 \times E_2$ is the Times rule, it is sufficient to show:

$$\langle f(E_1), s\rangle \Downarrow \langle n_1, s\rangle \wedge \langle f(E_2), s\rangle \Downarrow \langle n_2, s\rangle \implies \langle E_1, s\rangle \Downarrow \langle n_1, s\rangle \wedge \langle E_2, s\rangle \Downarrow \langle n_2, s\rangle$$

which holds trivially from the inductive hypotheses.

Inductive Case 2: Consider the case in which $E = E_1 + E_2$. We assume the inductive hypotheses $P(E_1)$ and $P(E_2)$.

There are two possible cases: The case in which $f(E_1) = f(E_2)$ and the "otherwise" case. The "otherwise" case follows the structure of the case for $E_1 \times E_2$, and so we omit it here, and focus on the case where $f(E_1) = f(E_2)$.

We wish to show

$$\langle f(E_1 + E_2), s\rangle \Downarrow \langle n, s\rangle \implies \langle E_1 + E_2, s\rangle \Downarrow \langle n, s\rangle$$

By the definition of f it is sufficient to show:

$$\langle 2 \times f(E_1), s\rangle \Downarrow \langle n, s\rangle \implies \langle E_1 + E_2, s\rangle \Downarrow \langle n, s\rangle$$

We assume $\langle 2 \times f(E_1), s\rangle \Downarrow \langle n, s\rangle$ and attempt to show $\langle E_1 + E_2, s\rangle \Downarrow \langle n, s\rangle$:

Since $\langle 2 \times f(E_1), s\rangle \Downarrow \langle n, s\rangle$, we know by the Times rule that there exists n_1 such

that $f(E_1) \Downarrow n_1$ and $n = n_1 \underline{+} n_1$.
Since we are showing the case where $f(E_1) = f(E_2)$, we also know that $f(E_2) \Downarrow n_1$.
By the inductive hypotheses, we therefore know that $E_1 \Downarrow n_1$ and $E_2 \Downarrow n_1$.
By the rule ADD, we know that $\langle E_1 + E_2, s\rangle \Downarrow \langle n', s\rangle$ for any n' such that $n' = n_1 \underline{+} n_2$.
Hence $n' = n$, and the proof is concluded.

9. (a) We wish to show $P(s_2, s_3)$ for all $s_2, s_3 \in S$ with $s_2 \rightarrowtail^* s_3$, where

$$P(s_2, s_3) \equiv \forall s_1.\, s_1 \rightarrowtail^* s_2 \implies s_1 \rightarrowtail^* s_3$$

We proceed by induction on the structure of the derivation of $s_2 \rightarrowtail^* s_3$.
Base Case: The derivation ends with the REFL rule. In this case, the derivation has the form:

$$\text{REFL}\ \frac{}{s_2 \rightarrowtail^* s_2}$$

with $s_3 = s_2$.
Suppose that $s_1 \rightarrowtail^* s_2$. Then clearly $s_1 \rightarrowtail^* s_3$, as required.
Inductive Case: The derivation ends with the STEP rule. In this case, the derivation has the form:

$$\text{STEP}\ \frac{s_2 \rightarrowtail^* s_3' \qquad s_3' \rightarrowtail s_3}{s_2 \rightarrowtail^* s_3}$$

for some s_3'. For the inductive hypothesis, we assume $P(s_2, s_3')$, namely:

$$\forall s_1.\, s_1 \rightarrowtail^* s_2 \implies s_1 \rightarrowtail^* s_3'$$

Suppose that $s_1 \rightarrowtail^* s_2$. By the inductive hypothesis, $s_1 \rightarrowtail^* s_3'$. We know from the derivation of $s_2 \rightarrowtail^* s_3$ that $s_3' \rightarrowtail s_3$. We therefore have the following derivation:

$$\text{STEP}\ \frac{s_1 \rightarrowtail^* s_3' \qquad s_3' \rightarrowtail s_3}{s_1 \rightarrowtail^* s_3}$$

Hence, $s_2 \rightarrowtail^* s_3$, as required.
This completes the proof.

(b) We wish to show that, for all $s_1, s_2 \in S$, $s_1 \rightarrowtail^* s_2$ if and only if $s_1 (\rightarrowtail^*)^* s_2$. Consider each direction of the implication separately.
First, the "only-if" case. Assume $s_1 \rightarrowtail^* s_2$. We have the following derivation:

$$\text{STEP}\ \frac{\text{REFL}\ \dfrac{}{s_1 (\rightarrowtail^*)^* s_1} \qquad s_1 \rightarrowtail^* s_2}{s_1 (\rightarrowtail^*)^* s_2}$$

so $s_1 (\rightarrowtail^*)^* s_2$ as required.
Now, the "if" case. Assume $s_1 (\rightarrowtail^*)^* s_2$. Let us prove $s_1 \rightarrowtail^* s_2$ by induction on the structure of the derivation of $s_1 (\rightarrowtail^*)^* s_2$.
Base Case: The derivation ends with the REFL rule. In this case, the derivation has that form:

$$\text{REFL}\ \frac{}{s_1 (\rightarrowtail^*)^* s_1}$$

with $s_2 = s_1$. We have the following derivation:

$$\text{REFL}\ \frac{}{s_1 \rightarrowtail^* s_1}$$

and so, since $s_1 = s_2$, we have $s_1 \rightarrowtail^* s_2$, as required.

Inductive Case: The derivation ends with the STEP rule. In this case, the derivation has the form:

$$\text{STEP}\ \frac{s_1(\rightarrowtail^*)^* s_2' \qquad s_2' \rightarrowtail^* s_2}{s_1(\rightarrowtail^*)^* s_2}$$

for some s_2'. For the inductive hypothesis, we assume $s_1 \rightarrowtail^* s_2'$.

Since the inductive hypothesis gives $s_1 \rightarrowtail^* s_2'$ and the derivation gives $s_2' \rightarrowtail^* s_2$, we have, by the transitivity of $\rightarrowtail^*$, that $s_1 \rightarrowtail^* s_2$, as required.

This completes the "if" case and the proof.

10. (a) We wish to establish $\forall n.\, P_1(n)$. Proceed by induction on n.

Base Case: $n = 0$. We assume $t \rightsquigarrow^0 t_1$ (and so $t = t_1$) and $t \rightsquigarrow t_2$. Let $t' = t_2$. Since $t_1 = t \rightsquigarrow t_2 = t'$, we have $t_1 \rightsquigarrow t'$. Since $t' = t_2$, we also have $t_2 \rightsquigarrow^* t'$, as required.

Inductive Step: For the inductive hypothesis, assume that $P_1(n)$ holds. We require to show that $P_1(n+1)$ holds.

We assume $t \rightsquigarrow^{n+1} t_1$ and $t \rightsquigarrow t_2$. There must be some t_1' such that $t \rightsquigarrow^n t_1' \rightsquigarrow t_1$. By the inductive hypothesis, either $t_2 \rightsquigarrow^* t_1'$ or there exists t'' such that $t_1' \rightsquigarrow t''$ and $t_2 \rightsquigarrow^* t''$. In the first case, we have $t_2 \rightsquigarrow^* t_1' \rightsquigarrow t_1$, so $t_2 \rightsquigarrow^* t_1$ and we are done. Consider the second case.

Since $t_1' \rightsquigarrow t_1$ and $t_1' \rightsquigarrow t''$, by strong confluence either $t'' \rightsquigarrow^* t_1$ or there exists t' with $t_1 \rightsquigarrow t'$ and $t'' \rightsquigarrow^* t'$. In the first case, we have $t_2 \rightsquigarrow^* t'' \rightsquigarrow^* t_1$, and so $t_2 \rightsquigarrow^* t_1$ and we are done. In the second case, we know t' satisfies the first condition: $t_1 \rightsquigarrow t'$; since $t_2 \rightsquigarrow^* t'' \rightsquigarrow^* t'$, we also have the second condition: $t_2 \rightsquigarrow^* t'$.

(b) We wish to establish $\forall m.\, P_2(m)$. Proceed by induction on m.

Base Case: $m = 0$. We assume $t \rightsquigarrow^* t_1$ and $t \rightsquigarrow^0 t_2$ (and so $t = t_2$). Let $t' = t_1$. We therefore have that $t_1 \rightsquigarrow^* t'$. Since $t_2 = t \rightsquigarrow^* t_1 = t'$, we also have $t_2 \rightsquigarrow^* t'$, as required.

Inductive Step: For the inductive hypothesis, assume that $P_2(m)$ holds. We require to show that $P_2(m+1)$ holds.

We assume $t \rightsquigarrow^* t_1$ and $t \rightsquigarrow^{m+1} t_2$. There must be some t_2' with $t \rightsquigarrow^m t_2' \rightsquigarrow t_2$. By the inductive hypothesis, there exists t'' with $t_1 \rightsquigarrow^* t''$ and $t_2' \rightsquigarrow^* t''$. Now, we have $t_2' \rightsquigarrow^* t''$ (and so $t_2' \rightsquigarrow^n t''$ for some n) and $t_2' \rightsquigarrow t_2$. Thus by the result established in the first part, either $t_2 \rightsquigarrow^* t''$ or there exists t' with $t'' \rightsquigarrow t'$ and $t_2 \rightsquigarrow^* t'$.

In the first case, let $t' = t''$. We have $t_1 \rightsquigarrow^* t'' = t'$, so the first condition is met: $t_1 \rightsquigarrow^* t'$. We also have $t_2 \rightsquigarrow^* t'' = t'$, so the second condition is met: $t_2 \rightsquigarrow^* t'$.

In the second case, we have $t_1 \rightsquigarrow^* t'' \rightsquigarrow t'$, so the first condition is met: $t_1 \rightsquigarrow^* t'$. We also have the second condition: $t_2 \rightsquigarrow^* t'$.

11. We wish to show that

$$E \to E_1 \wedge E \to E_2 \implies E_1 = E_2 \vee \exists E'.\, E_1 \to E' \wedge E_2 \to E'$$

We do so by induction on the structure of the derivation, call it d, of $E \to E_1$.

Base Case:

$$d = \text{(S-ADD)}\ \frac{}{(n_1 + n_2) \to n_3}\ n_3 = \underline{n}_1 + n_2 \qquad E = (n_1 + n_2) \qquad E_1 = n_3$$

We assume that $E \to E_2$. Since n_1 and n_2 are numbers and hence irreducible, the only rule that can derive $E \to E_2$ is S-ADD, which requires that $E_2 = n_3$. Thus we have $E_1 = E_2$, as required.

Inductive Case 1:

$$d = \text{(S-LEFT)}\ \frac{d' = \dfrac{\vdots}{E_l \to E'_l}}{(E_l + E_r) \to (E'_l + E_r)} \qquad E = (E_l + E_r) \qquad E_1 = (E'_l + E_r)$$

For the inductive hypothesis, we assume the premiss for the subderivation d'. That is, we assume (for all E''_l)

$$E_l \to E''_l \implies E'_l = E''_l \lor \exists E'''_l .\, E'_l \to E'''_l \land E''_l \to E'''_l$$

We assume that $E \to E_2$. Consider the cases for the last rule used in the derivation of $E \to E_2$.

- S-ADD: This is impossible, since it would require E_l to be a number, which contradicts the fact that $E_l \to E'_l$.
- S-LEFT: In this case, $E_2 = E''_l + E_r$ for some E''_l, and we have a derivation

$$\text{(S-LEFT)}\ \frac{\dfrac{\vdots}{E_l \to E''_l}}{(E_l + E_r) \to (E''_l + E_r)}$$

  By the inductive hypothesis, since $E_l \to E''_l$, either:
  - $E'_l = E''_l$, in which case $E_1 = (E'_l + E_r) = E_2$, as required;
  - or there is some E'''_l with $E''_l \to E'''_l$ and $E''_l \to E'''_l$. Let $E' = (E'''_l + E_r)$. We then have

$$\text{(S-LEFT)}\ \frac{E'_l \to E'''_l}{E_1 \to E'} \qquad \text{(S-LEFT)}\ \frac{E''_l \to E'''_l}{E_2 \to E'}$$

    as required.
- S-RIGHT': In this case, $E_2 = E_l + E_r$ for some E'_r, and we have a derivation

$$\text{(S-RIGHT')}\ \frac{\dfrac{\vdots}{E_r \to E'_r}}{(E_l + E_r) \to (E_l + E'_r)}$$

  Let $E' = (E'_l + E'_r)$. We then have

$$\text{(S-RIGHT')}\ \frac{E_r \to E'_r}{E_1 \to E'} \qquad \text{(S-LEFT)}\ \frac{E_l \to E'_l}{E_2 \to E'}$$

  as required.

Inductive Case 2: For the second inductive case, the derivation of d concludes with the S-RIGHT' rule. Since this rule is symmetrical to the S-LEFT rule, the proof for this case is similar to the S-LEFT case, so I omit it here.

12. It is sufficient for us tor prove that, for all n, all statements C_1, C_2, C_3, all states s, s',

$$\langle C_1; (C_2; C_3), s\rangle \to^n \langle \texttt{skip}, s'\rangle \quad \text{if and only if} \quad \langle (C_1; C_2); C_3, s\rangle \to^n \langle \texttt{skip}, s'\rangle.$$

The proof is by induction on the number of reduction steps, n.

Base Case: $n = 0$. Neither $C_1; (C_2; C_3)$ nor $(C_1; C_2); C_3$ is $\texttt{skip}$, so this case holds trivially.

Inductive Case: $n = k + 1$.
Inductive hypothesis: for all statements C'_1, C'_2, C'_3, all states r, r',

$$\langle C'_1; (C'_2; C'_3), r\rangle \to^k \langle \texttt{skip}, r'\rangle \quad \text{if and only if} \quad \langle (C'_1; C'_2); C'_3, r\rangle \to^k \langle \texttt{skip}, r'\rangle.$$

In the left-to-right direction, consider the first step of the reduction sequence, which must have the form $\langle C_1; (C_2; C_3), s\rangle \to \langle C, s''\rangle$ for some statement C and state s'' for which $\langle C, s''\rangle \to^k \langle \texttt{skip}, s'\rangle$. There are only two derivation rules that could give such a conclusion, namely the rules for sequential composition.

Consider the first of these:

$$\frac{}{\langle \texttt{skip}; (C_2; C_3), s\rangle \to \langle C_2; C_3, s\rangle}$$

In this case, it must be that $C_1 = \texttt{skip}$, $C = C_2; C_3$ and $s'' = s$. We can derive the following:

$$\frac{\frac{}{\langle \texttt{skip}; C_2, s\rangle \to \langle C_2, s\rangle}}{\langle (\texttt{skip}; C_2); C_3, s\rangle \to \langle C_2; C_3, s\rangle}$$

Since we have established that $\langle (\texttt{skip}; C_2); C_3, s\rangle \to \langle C_2; C_3, s\rangle$ and we know that $\langle C_2; C_3, s\rangle \to^k \langle \texttt{skip}, s'\rangle$, it must be that $\langle (\texttt{skip}; C_2); C_3, s\rangle \to^n \langle \texttt{skip}, s'\rangle$, as required.

Consider the second case:

$$\frac{\langle C_1, s\rangle \to \langle C'_1, s''\rangle}{\langle C_1; (C_2; C_3), s\rangle \to \langle C'_1; (C_2; C_3), s''\rangle}$$

In this case, it must be that $C = C'_1; (C_2; C_3)$ and $\langle C_1, s\rangle \to \langle C'_1, s''\rangle$. We can derive the following:

$$\frac{\frac{\langle C_1, s\rangle \to \langle C'_1, s''\rangle}{\langle C_1; C_2, s\rangle \to \langle C'_1; C_2, s''\rangle}}{\langle (C_1; C_2); C_3, s\rangle \to \langle (C'_1; C_2); C_3, s''\rangle}$$

We know that $\langle C'_1; (C_2; C_3), s''\rangle \to^k \langle \texttt{skip}, s'\rangle$, and so, by the inductive hypothesis, we also know that $\langle (C'_1; C_2); C_3, s''\rangle \to^k \langle \texttt{skip}, s'\rangle$. Together with the fact that $\langle (C_1; C_2); C_3, s\rangle \to \langle (C'_1; C_2); C_3, s''\rangle$, this implies that $\langle (C_1; C_2); C_3, s\rangle \to^n \langle \texttt{skip}, s'\rangle$, as required.

In the right-to-left direction, consider the first step of the reduction sequence, which must have the form $\langle (C_1; C_2); C_3, s\rangle \to \langle C, s''\rangle$ for some statement C and state s'' for which $\langle C, s''\rangle \to^k \langle \texttt{skip}, s'\rangle$. The only derivation rule with such a conclusion is:

$$\frac{\langle C_1; C_2, s\rangle \to \langle C', s''\rangle}{\langle (C_1; C_2); C_3, s\rangle \to \langle C'; C_3, s''\rangle}$$

for some C' with $C = C'; C_3$. There are two rules that could give derivations of $\langle C_1; C_2, s\rangle \to \langle C', s''\rangle$, namely the rules for sequential composition.

Consider the first case:

$$\frac{}{\langle \texttt{skip}; C_2, s\rangle \to \langle C_2, s\rangle}$$

In this case, it must be that $C_1 = \texttt{skip}$, $C' = C_2$ (so $C = C_2; C_3$), and $s'' = s$. We can derive the following:

$$\frac{}{\langle \texttt{skip}; (C_2; C_3), s\rangle \to \langle C_2; C_3, s\rangle}$$

Since we know that $\langle \texttt{skip}; (C_2; C_3), s\rangle \to \langle C_2; C_3, s\rangle$ and that $\langle C_2; C_3, s\rangle \to^k \langle \texttt{skip}, s'\rangle$, it must be that $\langle \texttt{skip}; (C_2; C_3), s\rangle \to^n \langle \texttt{skip}, s'\rangle$, as required.

Consider the second case:

$$\frac{\langle C_1, s\rangle \to \langle C_1', s''\rangle}{\langle C_1; C_2, s\rangle \to \langle C_1'; C_2, s''\rangle}$$

for some C_1'. In this case, it must be that $C' = C_1'; C_2$ (so $C = (C_1'; C_2); C_3$), and $\langle C_1, s\rangle \to \langle C_1', s''\rangle$. We can derive the following:

$$\frac{\langle C_1, s\rangle \to \langle C_1', s''\rangle}{\langle C_1; (C_2; C_3), s\rangle \to \langle C_1'; (C_2; C_3), s''\rangle}$$

We know that $\langle (C_1'; C_2); C_3, s''\rangle \to^k \langle \texttt{skip}, s'\rangle$, and so, by the inductive hypothesis, we also know that $\langle C_1'; (C_2; C_3), s''\rangle \to^k \langle \texttt{skip}, s'\rangle$. Together with the fact that $\langle C_1; (C_2; C_3), s\rangle \to \langle C_1'; (C_2; C_3), s''\rangle$, this implies that $\langle C_2; (C_2; C_3), s\rangle \to^n \langle \texttt{skip}, s'\rangle$, as required.

13. (a) We wish to prove $P(x, x')$ for all x, x' with $x \to^* x'$, where

$$P(x, x') \equiv \forall y, y'.\, y \to^* y' \implies xy \to^* x'y'$$

The proof is by induction on the derivation of $x \to^* x'$.

Base Case: $x = x'$. Assume that $y \to^* y'$. Let us prove that $xy \to^* x'y'$ by induction on the structure of the derivation of $y \to^* y'$.

Base Case: $y = y'$. We have that $xy = x'y'$, and so $xy \to^* x'y'$ as required.

Inductive Case: $y \to^* y''$ and $y'' \to y'$ for some y''. For the inductive hypothesis, we have that $xy \to^* x'y''$. We also have

$$\textsc{ap-r}\ \frac{y'' \to y'}{x'y'' \to x'y'}$$

and so $xy \to^* x'y'$ as required.

Inductive Case: $x \to^* x''$ and $x'' \to x'$ for some x''. For the inductive hypothesis, we have that, for all y, y', if $y \to y'$ then $xy \to^* x''y'$.

Suppose that $y \to^* y'$. By the inductive hypothesis, we know that $xy \to^* x''y'$. We also have

$$\textsc{ap-l}\ \frac{x'' \to x'}{x''y' \to x'y'}$$

and so $xy \to^* x'y'$ as required.

(b) We wish to prove $P(t, t_1)$ for all t, t_1 with $t \to t_1$, where

$$P(t, t_1) \equiv \forall t_2.\, t \to t_2 \implies \exists t'.\, t_1 \to^* t' \wedge t_2 \to^* t'$$

We do so by induction on the structure of the derivation of $t \to t_1$.

Base Case: The last rule in the derivation is RED-I and, for some x, $t = \mathbf{I}x$ and $t_1 = x$. Suppose that $\mathbf{I}x \to t_2$. We wish to show

$$\exists t'.\, t_1 \to^* t' \wedge t_2 \to^* t'$$

We do so by considering cases of the structure of the derivation of $\mathbf{I}x \to t_2$.

- The last rule is RED-I and $t_2 = x$. Let us choose $t' = x$. Since $t_1 = x = t_2$, we have $t_1 \to^* t'$ and $t_2 \to^* t'$, as required.
- The cases where the last rule is RED-K or RED-S do not apply, since $t = \mathbf{I}x$ does not match the required structure.
- The case where the last rule is AP-L does not apply, since there is no reduction for the term $\mathbf{I}$.
- The last rule is AP-R and $x \to x'$ for some x' with $t_2 = \mathbf{I}x'$. Let us choose $t' = x'$. We know that $t_1 = x \to^* x'$. We also have

$$\text{RED-I}\ \frac{}{\mathbf{I}x' \to x'}$$

  so $t_2 \to^* x'$, as required.

This completes the RED-I base case, since we have considered all possible derivations for $\mathbf{I}x \to t_2$.

Base Case: The last rule in the derivation is RED-K and, for some x, y, $t = \mathbf{K}xy$ and $t_1 = x$. Suppose that $\mathbf{K}xy \to t_2$. We wish to show

$$\exists t'.\, t_1 \to^* t' \wedge t_2 \to^* t'$$

We do so by considering cases of the structure of the derivation of $\mathbf{K}xy \to t_2$. There are three applicable cases: the whole term reduces by the RED-K rule or x or y reduces (by a combination of the AP-L and AP-R rules).

- If the derivation is by RED-K then $t_2 = x = t_1$. Choose $t' = x$ and we have $t_1 \to^* t'$ and $t_2 \to^* t'$, as required.
- Suppose the derivation reduces x, so that $t_2 = \mathbf{K}x'y$ for some x' with $x \to x'$. Let $t' = x'$. We have $t_1 = x \to^* x' = t'$. We also have

$$\text{RED-K}\ \frac{}{\mathbf{K}x'y \to x'}$$

  and so $t_2 \to^* x' = t'$, as required.
- Suppose that the derivation reduces y, so that $t_2 = \mathbf{K}xy'$ for some y' with $y \to y'$. Let $t' = x$. We have $t_1 = x \to^* x = t'$. We also have

$$\text{RED-K}\ \frac{}{\mathbf{K}xy' \to x}$$

  and so $t_2 \to^* x = t'$, as required.

Base Case: The last rule in the derivation is RED-S and, for some x, y, z, $t = \mathbf{S}xyz$ and $t_1 = xz(yz)$. Suppose that $\mathbf{S}xyz \to t_2$. We wish to show

$$\exists t'.\, t_1 \to^* t' \wedge t_2 \to^* t'$$

We do so by considering cases of the structure of the derivation of $\mathbf{S}xyz \to t_2$. There are four applicable cases: the whole term reduces by the RED-S rule or x, y or z reduces.

- If the derivation is by RED-S then $t_2 = xz(yz) = t_1$. Choose $t' = xz(yz)$ and we have $t_1 \to^* t'$ and $t_2 \to^* t'$, as required.
- Suppose the derivation reduces x, so that $t_2 = \mathbf{S}x'yz$ for some x' with $x \to x'$. We have

$$\textsc{ap-l}\ \dfrac{\textsc{ap-l}\ \dfrac{x \to x'}{xz \to x'z}}{xz(yz) \to x'z(yz)}$$

  By the RED-S rule, we have $t_2 = \mathbf{S}x'yz \to x'z(yz)$. The choice of $t' = x'z(yz)$ therefore fulfils the requirements.
- Suppose the derivation reduces y, so that $t_2 = \mathbf{S}xy'z$ for some y' with $y \to y'$. We have

$$\textsc{ap-r}\ \dfrac{\textsc{ap-l}\ \dfrac{y \to y'}{yz \to y'z}}{xz(yz) \to xz(y'z)}$$

  By the RED-S rule, we have $t_2 = \mathbf{S}xy'z \to xz(y'z)$. The choice of $t' = xz(y'z)$ therefore fulfils the requirements.
- Suppose the derivation reduces z, so that $t_2 = \mathbf{S}xyz'$ for some z' with $z \to z'$. We have

$$\textsc{ap-l}\ \dfrac{\textsc{ap-r}\ \dfrac{z \to z'}{xz \to xz'}}{xz(yz) \to xz'(yz)}$$

  and also

$$\textsc{ap-r}\ \dfrac{\textsc{ap-r}\ \dfrac{z \to z'}{yz \to yz'}}{xz'(yz) \to xz'(yz')}$$

  so $t_1 \to^* xz'(yz')$. By the RED-S rule, we have $t_2 = \mathbf{S}xyz' \to xz'(yz')$. The choice of $t' = xz'(yz')$ therefore fulfils the requirements.

Inductive Case: The last rule in the derivation is AP-L, so $t = xy$ and $t_1 = x_1y$ for some x, y, x_1 with $x \to x_1$. For the inductive hypothesis, we have $P(x, x_1)$, namely

$$\forall x_2.\, x \to x_2 \implies \exists x'.\, x_1 \to^* x' \wedge x_2 \to^* x'$$

Suppose that $xy \to t_2$. We wish to show

$$\exists t'.\, t_1 \to^* t' \wedge t_2 \to^* t'$$

We do so by considering cases of the structure of the derivation of $xy \to t_2$. The RED-I case does not apply because it would require $x = \mathbf{I}$ and no rule could give $\mathbf{I} \to x_1$, which we know. The RED-K case would require $x = \mathbf{K}x_0$ for some x_0, which contradicts the fact that $\mathbf{K}x_0 \to x_1$. Similarly the RED-S case cannot happen as it would require $x = \mathbf{S}x_0x_1$. The cases where the last rule is AP-L or AP-R are therefore the only relevant ones.

- In the AP-L case, $t_2 = x_2y$ for some x_2 with $x \to x_2$. By the inductive hypothesis, there exists x' with $x_1 \to^* x'$ and $x_2 \to^* x'$. Since $y \to^* y$, by (a) we have $x_1y \to^* x'y$ and $x_2y \to^* x'y$. Therefore the choice of $t' = x'y$ fulfils the requirements.

- In the $\textsc{ap-r}$ case, $t_2 = xy'$ for some y' with $y \to y'$. By the $\textsc{ap-r}$ rule, we have $t_1 = x_1 y \to x_1 y'$. By the $\textsc{ap-l}$ rule, we have $t_2 = xy' \to x_1 y'$. The choice of $t' = x_1 y'$ therefore fulfils the requirements.

Inductive Case: The last rule in the derivation is $\textsc{ap-r}$, so $t = xy$ and $t_1 = xy_1$ for some x, y, y_1 with $y \to y_1$. For the inductive hypothesis, we have $P(y, y_1)$, namely

$$\forall y_1.\, y \to y_2 \implies \exists y'.\, y_1 \to^* y' \wedge y_2 \to^* y'$$

Suppose that $xy \to t_2$. We wish to show

$$\exists t'.\, t_1 \to^* t' \wedge t_2 \to^* t'$$

We do so by considering cases of the structure of the derivation of $xy \to t_2$.

- If the last rule in the derivation is $\textsc{red}$-I then $x = \mathbf{I}$ and $t_2 = y$. By the $\textsc{red}$-I rule, we have $t_1 = \mathbf{I}y_1 \to y_1$. We also have $t_2 = y \to y_1$. Therefore the choice of $t' = y_1$ fulfils the requirements.
- If the last rule in the derivation is $\textsc{red}$-K then $x = \mathbf{K}z$ for some z with $t_2 = z$. By the $\textsc{red}$-K rule, we have $t_1 = \mathbf{K}zy_1 \to z$. Therefore the choice of $t' = z$ fulfils the requirements.
- If the last rule in the derivation is $\textsc{red}$-S then $x = \mathbf{S}wz$ for some w, z with $t_2 = wy(zy)$. By the $\textsc{red}$-S rule, we have $t_1 = \mathbf{S}wzy_1 \to wy_1(zy_1)$. We have

$$\textsc{ap-l}\ \dfrac{\textsc{ap-r}\ \dfrac{y \to y_1}{wy \to wy_1}}{wy(zy) \to wy_1(zy)}$$

  and also

$$\textsc{ap-r}\ \dfrac{\textsc{ap-r}\ \dfrac{y \to y_1}{zy \to zy_1}}{wy_1(zy) \to wy_1(zy_1)}$$

  so $t_2 \to^* wy_1(zy_1)$. Hence, the choice of $t' = wy_1(zy_1)$ fulfils the requirements.
- If the last rule in the derivation is $\textsc{ap-l}$, then $t_2 = x'y$ for some x' with $x \to x'$. By the $\textsc{ap-l}$ rule, we have $t_1 = xy_1 \to x'y_1$. By the $\textsc{ap-r}$ rule, we have $t_2 = x'y \to x'y_1$. The choice of $t' = x'y_1$ therefore fulfils the requirements.
- If the last rule in the derivation is $\textsc{ap-r}$, then $t_2 = xy_2$ for some y_2 with $y \to y_2$. By the inductive hypothesis, there exists y' with $y_1 \to^* y'$ and $y_2 \to^* y'$. Since $x \to^* x$, by (a) we have $xy_1 \to^* xy'$ and $xy_2 \to^* xy'$. Therefore the choice of $t' = xy'$ fulfils the requirements.