Calculus

COMP40016

15 May 2023

1. (a) Let S be a non-empty set of real numbers. Prove that the real number α is the supremum of S if and only if both the following conditions are satisfied: [4]

    1. $x \le \alpha$ for all $x \in S$:
    2. for every $\epsilon > 0$, there is some $x \in S$ such that $\alpha - \epsilon < x' \le \alpha$.

   (b) i. Let $a_n = n^2 + n \cos n\pi$. Show that $a_n \to \infty$ as $n \to \infty$. [2]

   ii. Let $a_n = \frac{n^2+\sqrt{n}}{n+\cos n}$. Show that $a_n \to \infty$ as $n \to \infty$. [2]

   iii. Let $a_n = \frac{n^2+n+1}{2n^2+1}$. Show that $a_n \to \frac{1}{2}$ as $n \to \infty$. [2]

   (c) i. Evaluate [2]

$$\lim_{x \to 1} \frac{2x^2 - 3x + 4}{x^3 + 5x + 1}$$

   explaining how you arrived at the result.

   ii. Show that the function $f(x) = 3x + 7$ is uniformly continuous on $\mathbb{R}$. [2]

   (d) i. Show that [2]

$$\sum_{r=1}^{\infty} \frac{1}{r^2 + r} = 1$$

   ii. Prove the so-called *vanishing condition*: if $\sum_{r=1}^{\infty} a_r$ is convergent, then $a_n \to 0$ as $n \to \infty$ [1]

   iii. By considering $\sum_{r=1}^{\infty}(\sqrt{r} - \sqrt{r-1})$, show that the converse (reverse implication) of the vanishing condition is false: in other words, show that $\sum_{r=1}^{\infty} a_r$ may still be divergent even if $a_n \to 0$ as $n \to \infty$ [3]

2. (a) i. Use the limit definition of differentiability to find the derivative of $f(x) = \frac{1}{x}$ [2]

   ii. Show that $\lim\limits_{x \to 0}(1 + x)^{1/x} = e$. [2]

   (b) For $x > 0$ define [4]

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

   Prove that $\Gamma(n + 1) = n\,!$ for $n = 0, 1, 2, \ldots$.

   (c) i. Find the Taylor series for [2]

$$f(x) = (x + 1)e^x,$$

   $x \in \mathbb{R}$, about the point $x = 1$.

   ii. Evaluate [2]

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{(2n + 1)!}$$

   (d) Consider the problem [4]

$$\frac{dy}{dx} = f(x, y)$$

   with $y = y_0$ at $x = x_0$. The iterative scheme

$$x_{n+1} = x_n + h,$$
$$y_{n+1} = y_n + hf(x_n, y_n)$$

is known as *Euler's method* (here h is the step size).

It can be shown that, for a considerable range of values of h, the error produced by this method is proportional to h. Explain why this is the case.

(e) The space X consists of all the sequences of real numbers $x = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)$. Let $y = (\eta_1, \eta_2, \ldots, \eta_n, \ldots) \in X$ be another such sequence. Then we define [4]

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n - \eta_n|}{1 + |\xi_n - \eta_n|}.$$

Prove that (X, ρ) is a metric space.