2.) a)

i.) dual is:

$\min\ z = z_1 - 3z_2$

$z_1 - z_1 \geq 2$

$-4z_1 - 5z_2 \geq 1$

$z_1 \geq 0\ ,\ z_2 \leq 0$

(both primal and dual feasible
feasible solution for primal: $(1,1)$

because of duality, the dual is also feasible)

now $x(1,-1)$ is a feasible solution of the dual
the value of the dual

$c^T x = 4$

primal also feasible because of duality
$\Rightarrow$ by weak duality we know that $c^T x \leq b^T z$

so $y = c^T x \leq 4$ so the value of the primal is upper bounded by 4

2 a)

ii.) $\min\ z = -2x_1 - x_2$

$x_1 - 4x_2 + x_3 = 1$

$x_1 + 5x_2 + x_4 = 3$

$x_1, x_2, x_3, x_4 \geq 0$

2. a)

iii.) $I_1 = \{1,4\}$ $\quad I_2 = \{1,2\}$ feasible

$B_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\quad B_2 = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & -4 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{pmatrix}$

for I_1:

feasibility: $x_B = B_1^{-1} b \geq 0$

$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ $\quad b = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$B_1^{-1} b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow I_1$ feasible

2a iii cont.

optimality: $r = c_N - N^T (B_1^{-1})^T c_B \geq 0$

$$c_N = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad N = \begin{pmatrix} -4 & 1 \\ 5 & 0 \end{pmatrix} \qquad c_B = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$r = \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -4 & 5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 8 \\ -2 \end{pmatrix} = \begin{pmatrix} -9 \\ 2 \end{pmatrix} \not\geq 0$$

I_1 is not optimal

for I_2:

feasibility:

$$x_B = B_2^{-1} b \geq 0$$

$$B_2^{-1} = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5/9 & 4/9 \\ -1/9 & 1/9 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$x_B = \begin{pmatrix} 5/9 & 4/9 \\ -1/9 & 1/9 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1\frac{7}{9} \\ \frac{2}{9} \end{pmatrix} \geq 0 \Rightarrow I_2 \text{ feasible}$$

optimality:

$$r = c_N - N^T (B_2^{-1})^T c_B \geq 0$$

$$N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \qquad c_N = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad c_B = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$r = \underline{0} - \begin{pmatrix} 5/9 & -1/9 \\ 4/9 & 1/9 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \geq 0$$

$\rightarrow$ so I_2 optimal

$\Rightarrow$ We showed that I_1 and I_2 are both feasible index sets, and I_2 is optimal

2. a)

iv.) min $z = -2x_1 - x_2$

$x_1 - 4x_2 + x_3 = 1$

$x_1 + 5x_2 + x_4 = 3$

$x_1, x_2, x_3, x_4 \geq 0$

$A = \begin{pmatrix} 1 & -4 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{pmatrix}$

Draw the tableau:

BV	x_1	x_2	x_3	x_4	RHS	Ratio
z	2	1	0	0	0	–
x_3	1	-4	1	0	1	1
x_4	1	5	0	1	3	3
z	0	9	-2	0	-2	–
x_1	1	-4	1	0	1	–
x_4	0	9	-1	1	2	$\frac{2}{9}$
z	0	0	-1	-1	-4	
x_1	1	0	5/9	4/9	17/9	
x_2	0	1	-1/9	1/9	2/9	

in: x_1

out: x_3

in: x_2

out: x_4

all coeffs in neg. reduced cost negative
→ optimal solution

⇒ The optimal solution is $y^* = 4$ with $x_1 = \frac{17}{9}$ $x_2 = \frac{2}{9}$

$x_3 = 0$ $x_4 = 0$

2. a) v.)

calculate shadow prices:

$$\pi = (B^{-1})^T c_B$$

$$B = \begin{pmatrix} 7 & -4 \\ 1 & 5 \end{pmatrix} \to B^{-1} = \begin{pmatrix} 5/9 & 4/9 \\ -1/9 & 1/9 \end{pmatrix}$$

$$c_B = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$\pi = \begin{pmatrix} 5/9 & -1/9 \\ 4/9 & 1/9 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

check feasibility of $v(p)$

$$x_B = B^{-1} p = \begin{pmatrix} 5/9 & 4/9 \\ -1/9 & 1/9 \end{pmatrix} \begin{pmatrix} 1+p \\ 3 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 5p+17 \\ -p+2 \end{pmatrix}$$

$$x_B \geq 0 \iff 5p + 17 \geq 0$$
$$p \geq \frac{-17}{5}$$

$$-p + 2 \geq 0$$
$$p \leq 2$$

in our case $p = 2$ so feasible

$$\Rightarrow v(2) = v(b) + \pi^T (p - b) = -4 + (-1 \ -1) \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \underline{\underline{-6}}$$

$\to$ so $\boxed{y^{*1} = 6}$ (maximisation problem)

2. a)
vi) for $p = 3$ checking it with shadow prices:

$$v(3) \geq v(b) + \pi^T (p-b) = -4 + (-1 \ -1) \begin{pmatrix} 3 \\ 0 \end{pmatrix} = -7$$

doing min so friend says $v(3) = -9$

$\Rightarrow -9 \not\geq -7$

so they are wrong, $v(p)$ is at most 7

2. b) i.) false

counter example:

primal is unbounded and dual infeasible:

Primal: max x s.t. $-x \leq -1$, $x \geq 0$

Dual: min $-y$ s.t. $-y \geq 1$, $y \geq 0$

2. b) ii.) true

proof by contradiction:

assume P unbounded and dual feasible

→ both feas. so can use weak duality:

$$\overset{P}{c^T x} \leq \overset{D}{b^T y}$$

which would mean that primal is bounded (upper bounded by $b^T y$)
which is a contradiction, so the statement is true

2. b) iii.) false can both be infeasable

countrexample:

Primal:

$$\max z = 2x_1 + x_2$$
$$\text{s.t} \quad -x_1 + x_2 \leq -4$$
$$x_1 - x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Dual:

$$\min z = -4y_1 + 2y_2$$
$$\text{s.t} \quad -y_1 + y_2 \geq 2$$
$$y_1 - y_2 \geq 1$$
$$y_1, y_2 \geq 0$$

2. b) iv.) true

by definition

P : $\max\{c^T x : Ax \leq b, x \geq 0\}$

D : $\min\{b^T y : A^T y \geq c, y \geq 0\}$

applying the definition on dual:

D(D): $\max\{c^T x : Ax \leq b, x \geq 0\}$

we get back to the primal

so the dual of the dual is indeed the primal.