Question

1.

Marks & seen/unseen

Parts

unseen ⇓

seen ⇓

(i) (d).

(ii) (a) 3.463207.

Expected frequencies under the null hypothesis of independence:

	Right handed	Left handed
Right footed	142.492	48.508
Left footed	45.508	15.492

(iii) (a). $(0.1 \times 0.1)/(0.1 \times 0.1 + 0.2 \times 0.25 + 0.7 \times 0.2) = 0.05$.

(iv) (d). $1 - (5/6)^6 = 0.665102$.

(v) (b).

Each 4 marks

Question 2.

Marks & seen/unseen

Parts

seen $\Downarrow$

(i) (a) Denoting the sample as $x_1, \ldots, x_n$ with $n = 12$, respective unbiased estimates for the mean and variance are the sample mean and the bias-corrected sample variance:

$$\bar{x} = 2.509167;$$
$$s^2_{n-1} = 202.48.$$

5 marks

(b) A 95% confidence interval is given by

$$\left[\bar{x} - t_{n-1,0.975}\frac{s_{n-1}}{\sqrt{n}}, \bar{x} + t_{n-1,0.975}\frac{s_{n-1}}{\sqrt{n}}\right] = [-6.531857, 11.550190].$$

5 marks

(c) No, 0 is well within the confidence interval so there is no evidence to reject the hypothesis.

2 marks

(ii) (a) $R = (z_{0.95}, \infty) \approx (1.644854, \infty)$.

4 marks

unseen $\Downarrow$

(b) If $X \sim \mathrm{N}(1, 1)$, then $Z = (X - 1) \sim \mathrm{N}(0, 1)$.

$$\begin{aligned} \mathrm{P}(X \in R) &= \mathrm{P}(X > z_{0.95}) = \mathrm{P}(Z > z_{0.95} - 1) = \Phi(z_{0.95} - 1) \approx \Phi(0.644854) \\ &\approx 0.740489 \\ \implies \beta &\approx 1 - 0.740489 = 0.259511. \end{aligned}$$

4 marks

Question

3.

Marks & seen/unseen

Parts

seen ⇓

(i)

$$\sum_{x=0}^{\infty} p_X(x) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1.$$

3 marks

unseen ⇓

(ii)

$$\frac{p_X(x)}{p_X(x-1)} = \frac{\lambda}{x}.$$

Hence $p_X(x)$ will be greater than or equal to $p_X(x-1)$ whilst $x \le \lambda$. Formally, for $x \in \{1, 2, \ldots, \lfloor \lambda \rfloor\}$. 4 marks

(iii) Since $p_X(x)$ is non-decreasing in x until $x = \lfloor \lambda \rfloor$, and is decreasing thereafter, $\lfloor \lambda \rfloor$ will always provide a maximum of p_X. 3 marks

(iv) The mode is unique when λ is not an integer. 2 marks

When λ is an integer, then both λ and $(\lambda - 1)$ are maxima of p_X, since the ratio of their probability mass function values will be 1. 2 marks

(v) For $z = 0, 1, 2, \ldots$,

$$\begin{aligned}
\mathrm{P}(Z = z) = \sum_{x=0}^{z} p_X(x) p_Y(z-x) &= \sum_{x=0} \frac{\lambda^x e^{-\lambda}}{x!} \frac{\mu^{z-x} e^{-\mu}}{(z-x)!} = e^{-(\lambda+\mu)} \sum_{x=0} \frac{\lambda^x \mu^{z-x}}{x!(z-x)!} \\
&= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda^x \mu^{z-x} \\
&= \frac{e^{-(\lambda+\mu)} (\lambda+\mu)^z}{z!}
\end{aligned}$$

by the binomial theorem. This is the probability mass function of a Poisson$(\lambda + \mu)$ random variable. 6 marks

Question 4.

Marks & seen/unseen

Parts

seen ⇓

(i) For f to be a density function,

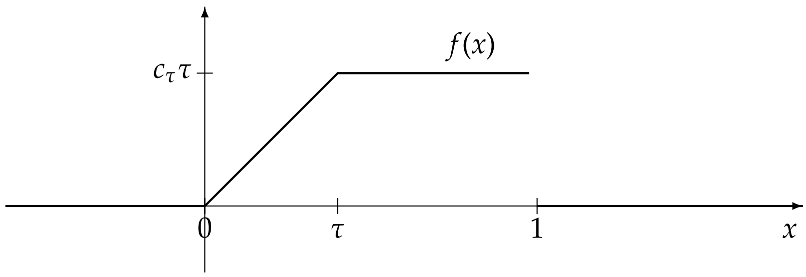
I. $f(x) \geq 0, \forall x \in \mathbb{R}$;

II. $\int_{x=-\infty}^{\infty} f(x)dx = 1$.

3 marks

(ii)

seen sim. ⇓



3 marks

(iii) From the second point, we have

$$1 = \int_{x=-\infty}^{\infty} f(x)dx = c_\tau \left\{ \int_{x=0}^{\tau} xdx + \tau \int_{x=\tau}^{1} dx \right\} = c_\tau \left\{ \frac{\tau^2}{2} + \tau(1-\tau) \right\}$$

$$\implies c_\tau = \frac{2}{2\tau - \tau^2}.$$

5 marks

(iv) Let $g(\tau) = 2\tau - \tau^2$ be the denominator of c_τ. Then the derivative $g'(\tau) = 2(1-\tau)$ is positive for $\tau \in (0, 1)$, implying that g is increasing and hence c_τ is decreasing in τ. 3 marks

However,

$$c_\tau \tau = \frac{2\tau}{2\tau - \tau^2} = \frac{2}{2 - \tau}$$

is clearly increasing in τ. 3 marks

(v) For $\tau \geq x$, we have seen that c_τ is decreasing in τ, so for τ within $(x, 1)$, $f(x)$ is maximised by $\tau = x$. For $\tau \leq x$, we have seen $c_\tau \tau$ is increasing in τ and so is again maximised by $\tau = x$. Hence $\hat{\tau} = x$ is a global maximum and hence the MLE for τ.

unseen ⇓

3 marks