IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2016

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
BEng Honours Degree in Mathematics and Computer Science Part I
MEng Honours Degree in Mathematics and Computer Science Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C140=MC140

LOGIC

Tuesday 3 May 2016, 14:00
Duration: 80 minutes

Answer ALL TWO questions

Paper contains 2 questions
Calculators not required

1 a Draw the formation tree of the formula

$$\forall x P(x) \lor \neg Q(x) \land \exists y \neg R(x,y) \to x = y,$$

and indicate all free occurrences of variables in your tree.

b Use propositional equivalences to show that the two formulas

$$p \lor q \qquad \text{and} \qquad (p \to q) \to q$$

are logically equivalent. In each step of your argument, state the equivalence you use (e.g., $\neg\neg A \equiv A$). Use only one equivalence in each step.

c In natural deduction, the alternative $\lor$-elimination rule $Alt\lor E$ is as follows:

1	$A \lor B$	got this somehow
2	A	ass
3	$\vdots$	hard work
4	B	proved somehow
5	B	$Alt\lor E(1,2,4)$

Let $\vdash$ denote the standard system of natural deduction. Let $\vdash^a$ denote the system of natural deduction obtained from $\vdash$ by adding the rule $Alt\lor E$ and removing the standard rule $\lor E$.

i) Show that $Alt\lor E$ is a derived rule of $\vdash$.

ii) Show that $A \lor B,\ A \to C,\ B \to C \ \vdash^a\ C$.
(You may assume that PC is a derived rule of $\vdash^a$.)

iii) Explain why $\lor E$ is a derived rule of $\vdash^a$, using part c(ii) or otherwise.

iv) What does it mean to say that $\vdash^a$ is *sound and complete*?

v) Is $\vdash^a$ sound and complete? Justify your answer.

The three parts carry, respectively, 10%, 25%, and 65% of the marks.

2a Prove by natural deduction that

$$\exists x P(x), \quad \exists x \forall y (y = x) \quad \vdash \quad \forall x P(x).$$

In the rest of this question, L is the 2-sorted signature with sorts $\mathtt{Nat}$ and $[\mathtt{Nat}]$, constants $\underline{0}, \underline{1}, \underline{2}, \ldots : \mathtt{Nat}$ and $[\,] : [\mathtt{Nat}]$, function symbols $+, -, \times, :, ++, !!, \sharp$, and relation symbols $<, \leq$ and $\mathtt{merge}$, of the appropriate sorts (as in lectures).

Variables i, j, k, m, n, etc., have sort $\mathtt{Nat}$, and xs, ys, zs, us, etc., have sort $[\mathtt{Nat}]$.

The intended semantics is an L-structure M whose domain consists of the natural numbers $0, 1, 2, \ldots$ (sort $\mathtt{Nat}$) and all lists of natural numbers (sort $[\mathtt{Nat}]$). The symbols of L are interpreted in M as in lectures. For example, $M \models \mathtt{merge}(ys, zs, xs)$ if and only if xs is a permutation of $ys{+}{+}zs$ and the relative order of entries in ys and in zs is retained in xs.

You are given an L-formula $\mathtt{in}(n, xs)$ expressing that n is an entry in xs, and an L-formula $\mathtt{count}(n, xs, k)$ expressing that n occurs exactly k times in xs.

b Write down L-formulas expressing the following properties of xs and ys:

i) Some entry in xs is an even number.
ii) There are no repeated entries in xs.
iii) ys is the result of sorting xs in ascending order.
iv) xs has more even entries than odd entries.

c The binary relation $\mathtt{entries}([\mathtt{Nat}], [\mathtt{Nat}])$ is specified informally by:
$M \models \mathtt{entries}(xs, ys)$ if and only if ys lists the entries in xs without repetitions.

Example: $M \models \mathtt{entries}([1,3,1,3,3], ys)$ if and only if ys is $[1,3]$ or $[3,1]$.

Below are three *incorrect* attempts to express $\mathtt{entries}$ by an L-formula:

A1. $\forall n(\mathtt{in}(n, xs) \leftrightarrow \mathtt{in}(n, ys))$
A2. $\forall n(\mathtt{in}(n, xs) \leftrightarrow \mathtt{count}(n, ys, \underline{1}))$
A3. $\exists zs\, \mathtt{merge}(ys, zs, xs) \wedge \forall n(\mathtt{in}(n, xs) \to \mathtt{count}(n, ys, \underline{1}))$

i) For each of the formulas A1–A3, write down lists xs and ys for which the formula is true but $\mathtt{entries}(xs, ys)$ is false, or vice versa.

ii) Write down an L-formula $B(xs, ys)$ that does express $\mathtt{entries}$ (that is, $M \models \mathtt{entries}(xs, ys) \leftrightarrow B(xs, ys)$ for all lists xs, ys). You do not need to justify your answer. You may use $\mathtt{in}$ and $\mathtt{count}$ freely.

The three parts carry, respectively, 35%, 40%, and 25% of the marks.