Part (a)

Subpart (i)

Clearly we assume $x \neq 0$. By direct substitution with the definition, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}, \\ &= \lim_{h\to 0} \frac{\frac{x-x-h}{x^2+hx}}{h}, \\ &= \lim_{h\to 0} \frac{-h}{hx^2+h^2x}, \\ &= \lim_{h\to 0} \frac{-1}{x^2+hx}. \end{aligned}$$

Note the numerator -1 is constant and $\lim_{h\to 0}(x^2+hx) = x^2$ because $hx \to 0$ by simple addition of limits. As $x^2 \neq 0$, we have that

$$f'(x) = \lim_{h\to 0} \frac{-1}{x^2+hx} = \frac{-1}{x^2},$$

due to the algebra of limits.

Subpart (ii)

finish

Part (b)

Subpart (i)

Using the given fact, we get

$$\begin{aligned} &\lim_{n\to\infty} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right\}, \\ &= \lim_{n\to\infty} \left\{ \frac{1}{n} \left(\frac{1}{1+1/n} + \frac{1}{1+2/n} + \cdots + \frac{1}{1+1} \right) \right\}, \\ &= \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} \right), \\ \text{Using the fact:} \quad &= \int_0^1 \frac{1}{1+x} \mathrm{d}x, \\ &= \ln 1 + x|_0^1, \\ &= \ln 2. \end{aligned}$$

Subpart (ii)

Note that

$$|x-1| = \begin{cases} x-1 & x \geq 1, \\ 1-x & x < 1. \end{cases}$$

Thus

$$\begin{aligned} \int_0^4 |x-1| \mathrm{d}x &= \int_0^1 |x-1| \mathrm{d}x + \int_1^4 |x-1| \mathrm{d}x, \\ &= \int_0^1 1-x \, \mathrm{d}x + \int_1^4 x-1 \, \mathrm{d}x, \\ &= (x - x^2/2)|_0^1 + (x^2/2 - x)|_1^4, \\ &= 1 - 1/2 + 16/2 - 4 - 1/2 + 1, \\ &= 5. \end{aligned}$$

Part (c)

Subpart (i)

Recall that e^x can be represented by its power series $1 + x + x^2/2 + \cdots$. Therefore we have that the equation simplifies into

$$e^{-2x} = 2 \implies x = \frac{-1}{2}\ln 2.$$

Subpart (ii)

Finish

Part (d)

Finish

Part (e)

Subpart (i)

We go property-by-property

Positive: This is trivially true by definition of f as its domain is wholly positive.

Symmetric: Let $x, y \in \mathbb{R}$:

$$d(x,y) = f(|x-y|) = f(|y-x|) = d(x,y).$$

Distance from point to itself is zero: Let $x \in \mathbb{R}$:

$$d(x,x) = f(|x-x|) = f(0) = 0,$$

as by definition of f.

Triangle Inequality: Let $x, y, z \in \mathbb{R}$. Then

$$d(x,y) + d(y,z) = f(|x-y|) + f(|y-z|) \geq f(|x-y| + |y-z|).$$

As f is monotone increasing, we have that f preserves ordering of the reals (i.e. $a \leq b \implies f(a) \leq f(b)$). As $\delta(a,b) := |a-b|$ is a metric on $\mathbb{R}$, we have

$$|x-y| + |y-z| \geq |x-y+y-z| = |x-z|.$$

Hence

$$d(x,y) + d(y,z) \geq f(|x-y| + |y-z|) \geq f(|x-z|) = d(x,z).$$

With all these properties, d is a metric on $\mathbb{R}$.

Subpart (ii)

From part (i), it suffices to show that $x \mapsto \sqrt{x}$ is concave. Its domain already fits the function f in part (i) as we are dealing with positive reals. Therefore we want to show only that $\forall x, y \in [0, \infty)$:

$$\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}.$$

Note that $x, y \in [0, \infty) \implies 2\sqrt{xy} \geq 0$. Then

$$x + y + 2\sqrt{xy} \geq x + y,$$
$$(\sqrt{x} + \sqrt{y})^2 \geq x + y.$$

As $\sqrt{\cdot}$ is monotone increasing function, we have

$$(\sqrt{x} + \sqrt{y})^2 \geq x + y \implies \sqrt{x} + \sqrt{y} \geq \sqrt{x+y}.$$

Thus we can use part (i) and we are done.