1a i)

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 8 \\ 5 & 6 & 7 & 8 & 4 \\ 9 & 10 & 11 & 12 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 9R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 8 \\ 0 & -4 & -8 & -12 & -36 \\ 0 & -8 & -16 & -24 & -72 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_2 \to -\frac{1}{4} R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 8 \\ 0 & 1 & 2 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 - 2R_2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -10 \\ 0 & 1 & 2 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

ii) $x_3$ and $x_4$ are ~~basic or~~ free variables.

iii) $$\cancel{\vec{x} = \begin{bmatrix} -10 \\ 9 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} -9 \\ 7 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$\vec{x} = \begin{bmatrix} -10 \\ 9 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Plane defined by $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ containing $\begin{bmatrix} -10 \\ 9 \\ 0 \\ 0 \end{bmatrix}$

1ci) $\text{Rank}(A) = 2$

By Rank-Nullity Theorem, $\text{Null}(A) = 4 - 2 = 2$

ii) $\text{Im}(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} \right\}$

iii) $\ker(A)$: $\vec{x}$ for which $A\vec{x} = \vec{0}$

$\ker(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

di) $\text{Rank}(A^T) = \text{Rank}(A) = 2$

By Rank-Nullity Theorem, $\text{Null}(A^T) = 3 - 2 = \underline{1}$

ii) $\begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 4R_1$

$\begin{bmatrix} 1 & 5 & 9 \\ 0 & -4 & -8 \\ 0 & -8 & -16 \\ 0 & -12 & -24 \end{bmatrix}$ $R_2 \to -\frac{1}{4} R_2$, $R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - 3R_2$

$\begin{bmatrix} 1 & 5 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $R_1 \to R_1 - 5R_2$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Columns 1,2 linearly independent

$\text{Im}(A^T) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right\}$

1diii) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$

$x_3$ free

$\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$\ker(A^T) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$