IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2019

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C145

MATHEMATICAL METHODS

Wednesday 15th May 2019, 14:00
Duration: 80 minutes

Answer ALL TWO questions

Paper contains 2 questions
Calculators required

Always provide justifications and show any intermediate work for your answers. A correct but unsupported answer may not receive any marks. Pay attention to mathematical formalism.

Section A (Use a separate answer book for this Section)

1 a Determine a basis of the intersection $U_1 \cap U_2$ for the vector subspaces

$$U_1 = \left[\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \text{ and } U_2 = \left[\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right] \text{ in } \mathbb{R}^3.$$

b Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & -2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

i) Compute the characteristic polynomial of A and determine the eigenvalues.

ii) Is A diagonalizable? Justify your answer.

c i) Is the mapping $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ defined for $\mathbf{x} = (x_1, x_2, x_3)$ as

$$\Psi(\mathbf{x}) = (3x_1 + x_3 - 2, 2x_2 + 4x_3, 0)$$

linear? Justify your answer.

ii) Prove that $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \to \mathbb{R}$ defined for $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = 5x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$$

is an inner product.

iii) Consider the vector space $M_2(\mathbb{R})$ of real two-dimensional square matrices

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \middle| x_{11}, x_{12}, x_{21}, x_{22} \in \mathbb{R} \right\}.$$

Is the set $S = \{A \in M_2(\mathbb{R}) | \det(A) = 0\}$ a vector subspace of $M_2(\mathbb{R})$? Justify your answer.

The three parts carry, respectively, 30%, 30%, and 40% of the marks.

Section B (Use a separate answer book for this Section)

Always provide justifications and show any intermediate work for your answers. A correct but unsupported answer may not receive any marks. Pay attention to mathematical formalism.

2 a Consider the sequence $(\frac{3n^2}{n^2+1})_{n\geq 1}$.

Guess the limit of this sequence and use the ϵ-definition of convergence to formally prove that this sequence converges to your guessed limit.

b Compute the Maclaurin polynomial of degree 4 for the function $f(x) = \cos(x) \cdot \ln(1-x)$ for $-1 < x < 1$.

[Hint: first compute the Maclaurin series for $\cos(x)$ and for $\ln(1-x)$ and then multiply those series by ignoring resulting terms of degree greater than 4.]

c Consider the following Theorem:

Root Test If there is some x with $0 < x < 1$ and some N such that for all $n \geq N$ we have $0 \leq (a_n)^{1/n} \leq x$, then the infinite series $\sum_{n=1}^{\infty} \mid a_n \mid$ converges.

i) Apply this Root Test to show that the sequence $\sum_{n=1}^{\infty}(\frac{n}{n+1})^{n^2}$ converges.

ii) Show that the Root Test cannot be applied to sequences $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Explain why this means that convergence of $\mid a_n \mid^{1/n}$ to 1 does not tell us anything about the convergence or divergence of $\sum_{n=1}^{\infty} \mid a_n \mid$.

d Consider the function $f(x) = \frac{1}{x^2}$ defined over the interval $[1, \infty]$. We note that f is continuous over that domain of definition.

i) Show that f is positive and decreasing over $[1, \infty]$.

ii) Explain why the Integral Test is applicable in this setting and use that test to show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

The four parts carry, respectively, 20%, 25%, 30%, and 25% of the marks.