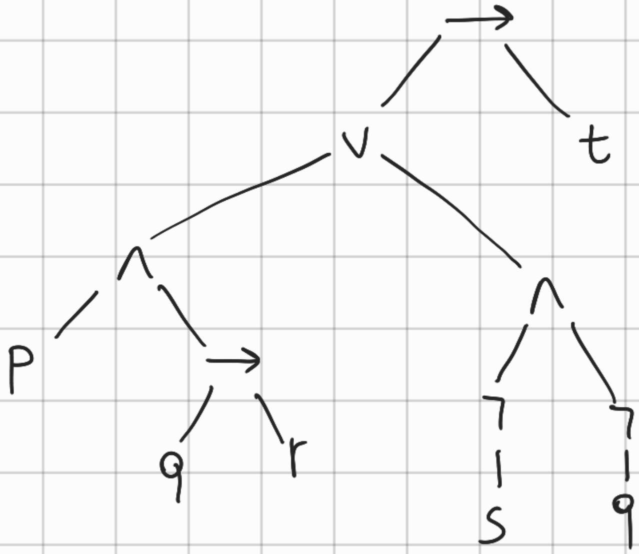
1a $\left(\left(\left(p \wedge (q \to r)\right) \vee \left((\neg s) \wedge (\neg q)\right)\right) \to t\right)$



SUBFORMULAS ARE:

$p, q, r, s, q, t,\ q \to r,\ \neg s,\ \neg q,\ p \wedge (q \to r),\ \neg s \wedge \neg q,$
$(p \wedge (q \to r)) \vee (\neg s \wedge \neg q),\ (p \wedge (q \to r)) \vee (\neg s \wedge \neg q) \to t$

1b
(i) peloton $\wedge$ bike $\to$ sold_out
(ii) $\neg$ soldout $\vee$ (peloton $\wedge$ bike)
(iii) peloton $\wedge$ bike $\to$ sold_out
(iv) bike $\to$ (peloton $\underline{V}$ sold_out)

where $\underline{V}$ is the exclusive or

1c (i) Therefore we have to show $(T \to P) \leftrightarrow (P \leftrightarrow T)$
Take an arbitrary situation.
Then we have two cases:

(1) P is false. Then $T \to P$ is false by semantics of $\to$. But also $P \leftrightarrow T$ is false because T is always true and, by semantics of $\leftrightarrow$, P should be true but it is not. So $(T \to P) \leftrightarrow (P \leftrightarrow T)$ is true because $T \to P$ and $P \leftrightarrow T$ are both false.

(2) P is true. The $T \to P$ is true by semantics of $\to$. Also, $P \leftrightarrow T$ is true because T is always true and P is assumed to be true. But then $(T \to P) \leftrightarrow (P \leftrightarrow T)$ is true

because both $T \to P$ and $P \Leftrightarrow T$ are true.

So the proposition holds in any situation, hence it is valid, as required.

(ii) To show that a proposition is not valid but satisfiable, we need to show that there exist a situation in which the proposition holds and a situation in which it doesn't.

(A) $(\neg P \vee Q) \wedge (P \vee \neg Q)$

- Take $Q = 1$ and $P = 1$:

$(\neg 1 \vee 1) \wedge (1 \vee \neg 1)$

$= 1 \wedge 1$ by $X \vee 1 = 1$ and $1 \vee X = 1$

$= 1$ by $1 \wedge 1 = 1$

- Take $Q = 0$ and $P = 1$:

$(\neg 1 \vee 0) \wedge (1 \wedge \neg 0)$

$= (0 \vee 0) \wedge (1 \wedge \neg 0)$ by $\neg 1 = 0$

$= 0 \wedge (1 \wedge \neg 0)$ by $0 \vee 0 = 0$

$= 0$ by $0 \wedge X = 0$

Hence A is not valid but satisfiable.

(B) $(\neg P \to \neg Q) \to (P \to Q)$ given

- Take $P = 1$ and $Q = 1$:

$(\neg 1 \to \neg 1) \to (1 \to 1)$

$= (\neg 1 \to \neg 1) \to 1$ by $1 \to 1 = 1$

$= 1$ by $x \to 1 = 1$

- Take $P = 1$ and $Q = 0$:

$(\neg 1 \to \neg 0) \to (1 \to 0)$

$= (0 \to 1) \to (1 \to 0)$ by $\neg 1 = 0$ and $\neg 0 = 1$

$= 1 \to (1 \to 0)$ by $0 \to X = 1$

$= 1 \to 0$ by $1 \to 0 = 0$

$= 0$ by $1 \to 0 = 0$

Hence B is not valid but satisfiable.

(1F) $\vdash^*$ is complete because $\vdash^*$ includes all the rules of $\vdash$, so every theorem that can be proved in $\vdash$ can be proved also in $\vdash^*$. Since $\vdash$ is complete, then so is $\vdash^*$.
However, $\vdash^*$ is not sound because using the inference rule you can prove formulas that are not theorems.
An example is the definition of the inference rule itself.
$A \vdash^* A \wedge \neg A$ holds but $A \vDash A \wedge \neg A$ clearly doesn't hold (because $A \wedge \neg A \equiv \bot$ and A is satisfiable, so there exist a situation in which the premise is true but the consequent is not).

(1d)

p	q	r	$\neg q$	$\neg r$	$p \to \neg q$	$x = (p \to \neg q) \wedge \neg r$	$x \to q$	$(x \to q) \wedge r$
0	0	0	1	1	1	1	0	0
0	0	1	1	0	1	0	1	1
0	1	0	0	1	1	1	1	0
0	1	1	0	0	1	0	1	1
1	0	0	1	1	1	1	0	0
1	0	1	1	0	1	0	1	1
1	1	0	0	1	0	0	1	0
1	1	1	0	0	0	0	1	1

DNF is the disjuction of conjunctions of literals. So we look at the situation in which $(((p \to \neg q) \wedge \neg r) \to q) \wedge r$ and take the disjuction of them.

$$(((p \to \neg q) \wedge \neg r) \to q) \wedge r$$
$$\equiv$$
$$(\neg p \cdot \neg q \cdot r) \vee (\neg p \cdot q \cdot r) \vee (p \cdot \neg q \cdot r) \vee (p \cdot q \cdot r)$$

(2a) $\forall x \exists y [P(x) \vee Q(y)] \rightarrow \exists y [Q(y)]$ given formula

$\equiv \forall x [\exists y P(x) \vee \exists y Q(y)] \rightarrow \exists y [Q(y)]$ by $\exists x [A \vee B] \equiv \exists x A \vee \exists x B$

$\equiv \forall x [P(x) \vee \exists y Q(y)] \rightarrow \exists y [Q(y)]$ by $\exists y A \equiv A$ if y is not free in A

$\equiv \forall x P(x) \vee \exists y Q(y) \rightarrow \exists y [Q(y)]$ by $\forall x [A \vee B] \equiv \forall x A \vee B$ if x is not free in B

$\equiv \neg(\forall x P(x) \vee \exists y Q(y)) \vee \exists y Q(y)$ by replacing $\rightarrow$

$\equiv (\neg \forall x P(x) \wedge \neg \exists y Q(y)) \vee \exists y Q(y)$ by De Morgan's law

$\equiv (\neg \forall x P(x) \vee \exists y Q(y)) \wedge (\neg \exists y Q(y) \vee \exists y Q(y))$ by distributivity of $\vee$

$\equiv (\neg \forall x P(x) \vee \exists y Q(y)) \wedge T$ by $\neg A \vee A \equiv T$

$\equiv \neg \forall x P(x) \vee \exists y Q(y)$ by $A \wedge T \equiv A$

$\equiv \forall x P(x) \rightarrow \exists y Q(y)$ by $\neg A \vee B \equiv A \rightarrow B$

(2b) (i) $x \in \{2,3,5\}$

(ii) $x \in \{1,4\}$

(iii) $x \in \{5\}$

(iv) $x \in \{1,2,4,5\}$

(2d) (i) $in(n,xs) \wedge in(n,ys)$

(ii) $\exists n{:}Nat\ in(n,xs) \wedge \exists x{:}Nat\ \forall n{:}Nat\ [in(n,xs) \rightarrow n=x]$

(iii) $\neg \exists n{:}Nat\ \exists x{:}Nat\ \exists y{:}Nat\ [in(n,xs) \wedge in(x,xs) \wedge in(y,xs) \wedge \neg(n=x) \wedge \neg(n=y) \wedge \neg(x=y)]$

2e

① $\forall y \neg \exists x [R(y,x) \wedge \neg P(x)]$ premise

② a $\forall I$ const

③ $\exists y\, R(y,a)$ ass

④ $R(b,a)$ ass

⑤ $\neg P(a)$ ass

⑥ $R(b,a) \wedge \neg P(a)$ $\wedge I(4,5)$

⑦ $\exists x [R(b,x) \wedge \neg P(x)]$ $\exists I$ (6)

⑧ $\neg \exists x [R(b,x) \wedge \neg P(x)]$ $\forall E(1)$

⑨ $\perp$ $\perp I(7,8)$

⑩ $\neg\neg P(a)$ $\neg I(5-9)$

⑪ $P(a)$ $\neg\neg E(10)$

⑫ $P(a)$ $\exists E(3,4-11)$

⑬ $\exists y\, R(y,a) \to P(a)$ $\to I(3-12)$

⑭ $\forall x [\exists y\, R(y,x) \to P(x)]$ $\forall I(2-13)$

2c If $\forall x B(x)$ then the statement hold because the RHS of the implication is always true

Take $x=1$: $R(y,x)$ only for $y=1$ but we have $P(1)$, so

$R(y,x) \wedge \neg P(y) \to B(y)$ is true and then

$\forall y (R(y,x) \wedge \neg P(y) \to B(y))$ is true

So $B(x)$ must hold.

Take $x=2$: We have $R(y,x)$ for $y=1 \vee y=3$, but we also have $P(1)$ and $P(3)$, so:

$R(y,x) \wedge \neg P(y) \to B(y)$ is true and then

$\forall y (R(y,x) \wedge \neg P(y) \to B(y))$ is true.

So $B(2)$ must hold.

Take $x=5$: we have $\neg R(y,x)$ for all y, so $\forall y (R(y,x) \wedge \neg P(y) \to B(y))$ is true. Therefore $B(5)$ must hold.

Take $x=4$: We have $R(y,x)$ for $y=4$ or $y=5$. In both cases $\neg P(y)$, so $R(y,x) \wedge \neg P(y)$ holds.

$B(5)$ holds by previous point and:

- $B(4)$: Then $\forall y(R(y,x) \wedge \neg P(y) \rightarrow B(y))$ holds so $B(4)$.
- $\neg B(4)$: Then $\forall y(R(y,x) \wedge \neg P(y) \rightarrow B(y))$ holds so $\neg B(4)$ is coherent.

So $B(4)$ can hold or not.

Take $x=3$: