Computation Exercises 4: Register Machines

1. Consider the register machine given by the following code:

$$
\begin{array}{l}
L_0 : R_1^- \to L_1, L_7 \\
L_1 : R_0^+ \to L_2 \\
L_2 : R_2^- \to L_3, L_5 \\
L_3 : R_3^+ \to L_4 \\
L_4 : R_0^+ \to L_1 \\
L_5 : R_2^+ \to L_6 \\
L_6 : R_3^- \to L_5, L_0 \\
L_7 : HALT
\end{array}
$$

   (a) Give the graphical representation of the register machine.

   (b) Give the computation (that is, the sequence of configurations) when the register machine is run from the initial configuration $(0, 0, 2, 0, 0)$.

2. In this question, you will design register machines that implement subtraction.

   (a) Consider the function $f(x_1, x_2)$ defined as

$$
f(x_1, x_2) \triangleq \begin{cases} x_1 - x_2 & \text{if } x_1 \geq x_2 \\ 0 & \text{otherwise} \end{cases}
$$

      i. Define a register machine that computes the function f.

      ii. Draw the graph corresponding to the register machine.

   (b) Consider the partial function $g(x_1, x_2)$ defined as

$$
g(x_1, x_2) \triangleq \begin{cases} x_1 - x_2 & \text{if } x_1 \geq x_2 \\ \text{undefined} & \text{otherwise} \end{cases}
$$

      i. How would a register machine implementing $g(x_1, x_2)$ behave when $x_2 > x_1$? (Hint: consider the definition on Slide 14 of the notes carefully.)

      ii. By adapting your answer to part (a) (or otherwise), define a register machine that computes the partial function g.

3. Consider the register machine represented by the following graph:

$$
\begin{tikzcd}
START \arrow[r] & R_1^- \arrow[r, bend left] \arrow[d, twoheadrightarrow] & R_1^- \arrow[l, bend left] \arrow[d, twoheadrightarrow] \\
& HALT & R_0^+ \arrow[l]
\end{tikzcd}
$$

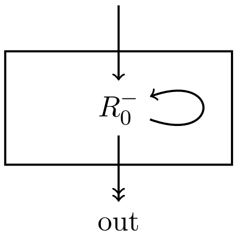
   (a) Give the code of the register machine. (Note, there is more than one way to do this, depending on how you label the states.)

   (b) Describe the function of one argument, $f(x)$, that is computed by the register machine.

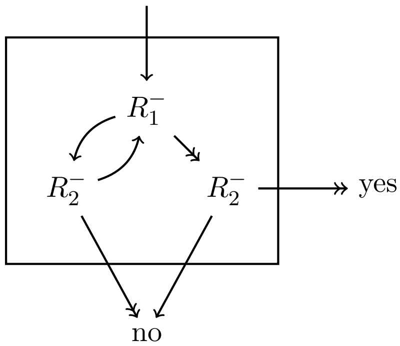
4. In order to construct register machines to perform complex operations, it is useful to build them from smaller components that we'll call *gadgets*, which perform specific operations.

   A gadget will be defined by a (partial) register machine graph that has a designated initial label and one or more designated exit labels (which contain no instructions). The gadget will operate on registers that are specified in the gadget's name, and are used for input and output — we call these the input/output registers. The gadget may use other registers for temporary storage — we call these scratch registers. The gadget may assume the scratch registers are initially set to 0, and *must* ensure that they are set back to 0 when the gadget exits. (Ensuring that the scratch registers are reset to 0 is important so that the gadget may be safely used within loops.)

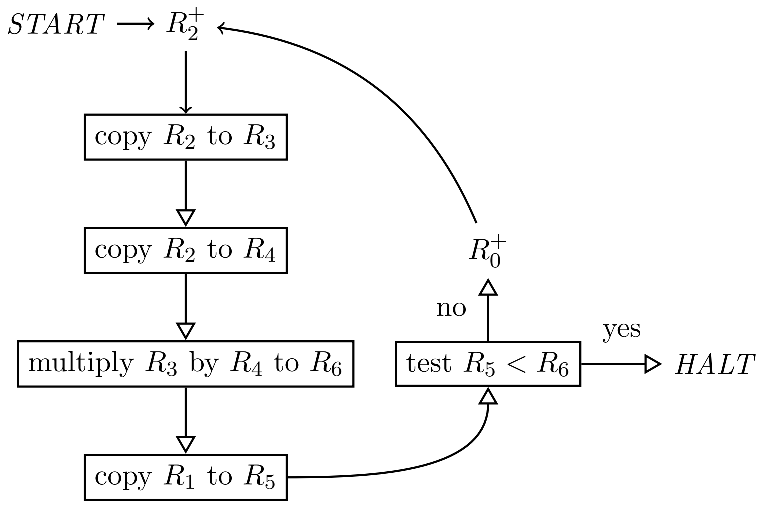
   The gadget "zero R_0" sets register R_0 to be zero, whatever its initial value. It is defined by the graph:



   A slightly more complicated example is the gadget "test $R_1 = R_2$", which determines whether the initial values of R_1 and R_2 are equal, possibly overwriting their values in the process. The gadget has two exit labels: "yes" and "no". The gadget exits to "yes" if the values are equal and "no" if they are unequal. It is defined by the graph:



   We can use gadgets to help us construct other gadgets and complete register machines. For instance we can construct the following register machine, M, using gadgets for copy, multiply and test for $<$:



When we use gadgets this way, we draw the links out of them like —▷. If there is more than one exit point from the gadget, we label these links accordingly. Each of these links can stand for any number of actual edges out of the gadget (for example, with the "test $R_1 = R_2$", the "no" link would correspond to both of the "no" edges).

In constructing such machines, we rename the registers used by each gadget: all of its scratch registers should be renamed to things that do not occur in the rest of the machine, and its input/output registers should be renamed to whichever registers the program requires. As an example, suppose we defined the gadget "copy R_1 to R_2" using a R_3 as a scratch register. To construct the instance "copy R_2 to R_3" that is used in M, we would rename R_3 in the gadget to something fresh, say R_7, and rename R_1 and R_2 to R_2 and R_3 respectively. We would then wire the output edge of the R_2^+ instruction to the input location in the gadget instance, and all of the output edges of the gadget instance to the input instruction of a second instance of the copy gadget corresponding to "copy R_2 to R_4".

(a) Define a gadget "add R_1 to R_2" which adds the initial value of R_1 to register R_2, storing the result in R_2 but restoring R_1 to its initial value. That is, if the initial state is $R_1 = r_1$ and $R_2 = r_2$ then the final state will be $R_1 = r_1$ and $R_2 = r_1 + r_2$. (You will need to use a scratch register to restore R_1 to its initial value. Remember: you can assume the scratch register initially has value 0, but must ensure that it also have value 0 when the gadget exits.)

(b) Define a gadget "copy R_1 to R_2" which copies the value of R_1 into register R_2, leaving R_1 with its initial value. You may construct this gadget from gadgets that have already been defined.

(c) Define a gadget "multiply R_1 by R_2 to R_0" which multiplies R_1 by R_2 and stores the result in R_0, possibly overwriting the initial values of R_1 and R_2. Again, you may use already-defined gadgets in your definition.

(d) Define a gadget "test $R_1 < R_2$" which determines whether the initial value of R_1 is less than that of R_2, possibly overwriting their values in the process. The gadget should exit to "yes" if it is, and "no" otherwise.

(e) Describe the function of one argument $f(x)$ computed by the register machine M defined above.