PAPER C145

MATHEMATICS I

Wednesday 13 May 2020, 11:00
Duration: 80 minutes
Post-processing time: 30 minutes
Answer TWO questions

While this time-limited remote assessment has not been designed to be open book, in the present circumstances it is being run as an open-book examination. We have worked hard to create exams that assesses synthesis of knowledge rather than factual recall. Thus, access to the internet, notes or other sources of factual information in the time provided will not be helpful and may well limit your time to successfully synthesise the answers required.
Where individual questions rely more on factual recall and may therefore be less discriminatory in an open book context, we may compare the performance on these questions to similar style questions in previous years and we may scale or ignore the marks associated with such questions or parts of the questions. In all examinations we will analyse exam performance against previous performance and against data from previous years and use an evidence-based approach to maintain a fair and robust examination. As with all exams, the best strategy is to read the question carefully and answer as fully as possible, taking account of the time and number of marks available.

Paper contains 2 questions

Always provide justifications and show any intermediate work for your answers.

1 a Let

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 3 & -7 \\ 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and consider the linear system of equations $\mathbf{Ax} = \mathbf{b}$.

i) Perform elementary row operations on the augmented matrix $[\mathbf{A}|\mathbf{b}]$ to reduce $\mathbf{A}$ to its Reduced Row Echelon Form.

ii) Identify the basic and the free variables.

iii) Find the general solution of $\mathbf{Ax} = \mathbf{b}$.

b Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -2 \end{bmatrix}.$$

i) Find the rank and the nullity of $\mathbf{A}$.

ii) Find a basis for the image space $\mathrm{im}(\mathbf{A})$.

iii) Find a basis for the null space $\mathrm{null}(\mathbf{A})$.

c Let

$$\mathbf{A} = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

i) Show that 6 is an eigenvalue of $\mathbf{A}$ and determine the other eigenvalues.

ii) Prove that $\mathbf{A}$ is invertible (without actually computing its inverse).

ii) Find an eigenvector for each of the eigenvalues of $\mathbf{A}$.

iii) Obtain an invertible $\mathbf{S} \in \mathbb{R}^{3\times 3}$ such that $\mathbf{S}^{-1}\mathbf{AS}$ is diagonal.

iv) Find an invertible $\mathbf{P} \in \mathbb{R}^{3\times 3}$ such that $\mathbf{P}^{-1}\mathbf{A}^{-1}\mathbf{P}$ is diagonal.

d Find the orthogonal projection of the point $(1,0,1)^T$ on the plane $2x_1 - x_2 + x_3 = 0$.

The four parts carry, respectively, 35%, 20%, 25%, and 20% of the marks.

Always provide justifications and show any intermediate work for your answers.

2 a Does the sequence $\left(\frac{(2n^2-1)\cdot(7n+5)}{4n^3+n-1}\right)_{n\geq 1}$ converge?

If so, show why this is the case and compute its limit. Otherwise, demonstrate why this sequence diverges.

b Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \cos(x^4)$.

i) Derive the Maclaurin series of this function $\cos(x^4)$.

[Hint: it may be easier to first compute the Maclaurin series of $\cos(x)$.]

ii) Determine the radius of convergence for this Maclaurin series.

iii) Compute the limit $\lim_{x\mapsto 0} \frac{\cos(x^4)-1+\frac{1}{2}\cdot x^8}{x^{16}}$.

[Hint: you may replace term $\cos(x^4)$ with its Maclaurin series.]

c Consider a variant of **L'Hôpital's Rule** that applies to differentiable functions $f, g\colon (-1,1) \to \mathbb{R}$ that satisfy $\lim_{x\mapsto 0} f(x) = 0 = \lim_{x\mapsto 0} g(x)$, and $g'(x) \neq 0$ for all x in $(-1,1) \setminus \{0\}$.

When these conditions are met, L'Hôpital's Rule says that if $\lim_{x\mapsto 0} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\mapsto 0} \frac{f(x)}{g(x)}$ exists as well and equals $\lim_{x\mapsto 0} \frac{f'(x)}{g'(x)}$.

Now let $f(x) = 5e^x - 5$ and $g(x) = 4x^2 - 15x$ be given:

i) Show that this variant of L'Hôpital's Rule is applicable for this f and g.

ii) Apply this variant of L'Hôpital's Rule to compute $\lim_{x\mapsto 0} \frac{5e^x-5}{4x^2-15x}$.

d Use the Integral Test to show that the series $\sum_{n\geq 1} \frac{1}{\sqrt{n}}$ diverges.

The four parts carry, respectively, 20%, 30%, 30%, and 20% of the marks.