IMPERIAL COLLEGE LONDON

TIMED REMOTE ASSESSMENTS 2021-2022

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant assessments for the
Associateship of the City and Guilds of London Institute

PAPER COMP40016

CALCULUS

Monday 16 May 2022, 10:00
Writing time: 80 minutes
Upload time: 25 minutes

Answer ALL TWO questions
Open book assessment

Paper contains 2 questions

1 a (i) Find the least upper bound for the following set and prove that your answer is correct:

$$S = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots \right\}.$$

(ii) State the infimum, minimum, and maximum of the above set S.

b (i) Using the definition of the limit of a sequence, prove that

$$\lim_{n\to\infty} \frac{n-1}{n+1} = 1.$$

(ii) Using the rules for combinations of sequences, determine the limit of the following convergent sequence:

$$\left\{ \frac{3n^2 - 1}{10n + 5n^2} \right\}_{n\geq 1}.$$

(iii) By using the sandwich theorem or otherwise, prove that

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

c (i) Show that the function

$$f(x) = \begin{cases} \frac{2x-6}{x-3}, & \text{when } x \neq 3, \\ 2, & \text{when } x = 3 \end{cases}$$

is continuous at $x = 3$.

(ii) Prove that $f(x) = mx + b$ is uniformly continuous on $\mathbb{R}$.

d (i) Prove that the series

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

converges. (State clearly the name of any convergence test that you use.)

(ii) Prove that the series

$$1 + 1 + 0.5 + 0.5 + 0.25 + 0.25 + 0.125 + 0.125 + \ldots$$

converges. (State clearly the name of any convergence test that you use.)

(iii) Does the series

$$\sum_{n=1}^{\infty} \frac{n}{2n+1}$$

converge or diverge? Why?

The four parts carry, respectively, 20%, 30%, 20%, and 30% of the marks.

2 a (i) Recall that the function f is *differentiable* at x iff the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is a real number. In that case, this limit is the *derivative* of f at x and is denoted as $f'(x)$ or, equivalently, $\frac{dy}{dx}$ when $y = f(x)$.

From first principles (i.e., using the above definition) find the derivative of $\frac{1}{x}$.

(ii) Prove that, if f is twice-differentiable in a neighbourhood of x and its second derivative is continuous, then

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

b (i) Let $f(x)$ be continuous in $[a,b]$. Using the fact that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\, dx$$

or otherwise, evaluate

$$\lim_{n \to \infty} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} \right\}.$$

(ii) Using the properties of the Riemann integral, calculate

$$\int_0^4 |x-1|\, dx.$$

c (i) Solve the equation

$$1 - 2x + 4\frac{x^2}{2!} - 8\frac{x^3}{3!} + 16\frac{x^4}{4!} - \ldots = 2.$$

(ii) Taylor's theorem can be used to show that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1} e^{\xi}}{(n+1)!}$$

where $0 < \xi < 1$. Use this fact to prove that e is irrational.

d Suppose that

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{when } x \neq 0, \\ 0, & \text{when } x = 0. \end{cases}$$

The function f is continuous everywhere, differentiable arbitrarily often everywhere, and 0 is the only solution of $f(x) = 0$.

Show that if $x_0 = 0.0001$, it takes more than one hundred million iterations of Newton's method (sometimes called the Newton–Raphson method) to get below 0.00005.

e (i) Show that, if $f : [0, \infty) \to [0, \infty)$ is a non-decreasing, concave function (i.e., $f(u + v) \leq f(u) + f(v)$ for all non-negative u and v), vanishing only at 0, then $d(x, y) = f(|x - y|)$ defines a metric on $\mathbb{R}$.

(ii) Use part (i) to show that $d(x, y) = \sqrt{|x - y|}$ is a metric on $\mathbb{R}$.

The five parts carry equal marks.