

IMPERIAL COLLEGE LONDON

TIMED REMOTE ASSESSMENTS 2021-2022

BEng Honours Degree in Computing Part I  
MEng Honours Degrees in Computing Part I  
for Internal Students of the Imperial College of Science, Technology and Medicine

*This paper is also taken for the relevant assessments for the  
Associateship of the City and Guilds of London Institute*

PAPER COMP40016

CALCULUS

Monday 16 May 2022, 10:00

Writing time: 80 minutes

Upload time: 25 minutes

*Answer ALL TWO questions*

Open book assessment

This time-limited remote assessment has been designed to be open book. You may use resources which have been identified by the examiner to complete the assessment and are included in the instructions for the examination. You must not use any additional resources when completing this assessment.

The use of the work of another student, past or present, constitutes plagiarism. Giving your work to another student to use constitutes an offence. Collusion is a form of plagiarism and will be treated in a similar manner. This is an individual assessment and thus should be completed solely by you. The College will investigate all instances where an examination or assessment offence is reported or suspected, using plagiarism software, vivas and other tools, and apply appropriate penalties to students. In all examinations we will analyse exam performance against previous performance and against data from previous years and use an evidence-based approach to maintain a fair and robust examination. As with all exams, the best strategy is to read the question carefully and answer as fully as possible, taking account of the time and number of marks available.

Paper contains 2 questions

- 1 a (i) Find the least upper bound for the following set and prove that your answer is correct:

$$S = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots \right\}.$$

- (ii) State the infimum, minimum, and maximum of the above set  $S$ .

- b (i) Using the definition of the limit of a sequence, prove that

$$\lim_{n \rightarrow \infty} \frac{n-1}{n+1} = 1.$$

- (ii) Using the rules for combinations of sequences, determine the limit of the following convergent sequence:

$$\left\{ \frac{3n^2 - 1}{10n + 5n^2} \right\}_{n \geq 1}.$$

- (iii) By using the sandwich theorem or otherwise, prove that

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0.$$

- c (i) Show that the function

$$f(x) = \begin{cases} \frac{2x-6}{x-3}, & \text{when } x \neq 3, \\ 2, & \text{when } x = 3 \end{cases}$$

is continuous at  $x = 3$ .

- (ii) Prove that  $f(x) = mx + b$  is uniformly continuous on  $\mathbb{R}$ .

- d (i) Prove that the series

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

converges. (State clearly the name of any convergence test that you use.)

- (ii) Prove that the series

$$1 + 1 + 0.5 + 0.5 + 0.25 + 0.25 + 0.125 + 0.125 + \dots$$

converges. (State clearly the name of any convergence test that you use.)

- (iii) Does the series

$$\sum_{n=1}^{\infty} \frac{n}{2n+1}$$

converge or diverge? Why?

*The four parts carry, respectively, 20%, 30%, 20%, and 30% of the marks.*

- 2a (i) Recall that the function  $f$  is *differentiable* at  $x$  iff the limit

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

exists and is a real number. In that case, this limit is the *derivative* of  $f$  at  $x$  and is denoted as  $f'(x)$  or, equivalently,  $\frac{dy}{dx}$  when  $y = f(x)$ .

From first principles (i.e., using the above definition) find the derivative of  $\frac{1}{x}$ .

- (ii) Prove that, if  $f$  is twice-differentiable in a neighbourhood of  $x$  and its second derivative is continuous, then

$$f''(x) = \lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

- b (i) Let  $f(x)$  be continuous in  $[a, b]$ . Using the fact that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx$$

or otherwise, evaluate

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n} \right\}.$$

- (ii) Using the properties of the Riemann integral, calculate

$$\int_0^4 |x-1| dx.$$

- c (i) Solve the equation

$$1 - 2x + 4\frac{x^2}{2!} - 8\frac{x^3}{3!} + 16\frac{x^4}{4!} - \dots = 2.$$

- (ii) Taylor's theorem can be used to show that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}e^{\xi}}{(n+1)!}$$

where  $0 < \xi < 1$ . Use this fact to prove that  $e$  is irrational.

- d Suppose that

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{when } x \neq 0, \\ 0, & \text{when } x = 0. \end{cases}$$

The function  $f$  is continuous everywhere, differentiable arbitrarily often everywhere, and 0 is the only solution of  $f(x) = 0$ .

Show that if  $x_0 = 0.0001$ , it takes more than one hundred million iterations of Newton's method (sometimes called the Newton–Raphson method) to get below 0.00005.

- e (i) Show that, if  $f : [0, \infty) \rightarrow [0, \infty)$  is a non-decreasing, concave function (i.e.,  $f(u + v) \leq f(u) + f(v)$  for all non-negative  $u$  and  $v$ ), vanishing only at 0, then  $d(x, y) = f(|x - y|)$  defines a metric on  $\mathbb{R}$ .
- (ii) Use part (i) to show that  $d(x, y) = \sqrt{|x - y|}$  is a metric on  $\mathbb{R}$ .

*The five parts carry equal marks.*