Exercises 9

14 March

All questions are unassessed.

1. The Hamiltonian cycle problem HAMCYCLE is: Given a graph G, does G have a Hamiltonian cycle?
(a) Show that HAMCYCLE is in NP.
(b) Show that HAMCYCLE is NP-hard. You may assume that HAMPATH is NP-hard (lecture notes page 89).
[Hint: Define an appropriate reduction by adding nodes and/or arcs to the graph.]

2. The problem COMPOSITE is as follows: Given a natural number n, is n composite (i.e. not a prime number)? Show that COMPOSITE is NP. Define the associated verification problem VER-COMPOSITE.

3. The problem GRAPHISOM is as follows: Given two graphs G_1, G_2, is there an isomorphism between G_1 and G_2? Show that GRAPHISOM is NP. Define the associated verification problem VER-GRAPHISOM.

4. Show that many-one reduction $\leq$ on decision problems (lecture notes pages 93-94) is (a) reflexive and (b) transitive.

5. Let D and D' be decision problems. Show that if D is NP-hard and $D \leq D'$ then D' is also NP-hard.

6. In the lectures we showed that both $\mathrm{TSP(D)}$ (slide 332) and $\mathrm{VRPC(D)}$ (slide 343) are NP-complete. Explain why $\mathrm{TSP(D)} \sim \mathrm{VRPC(D)}$ (slide 325).

7. (a) Show that exponentiation m^n of natural numbers can be carried out with only polynomially many multiplications and divisions. [Hints: Remember that the input size is $\log m + \log n$, not $m+n$. Try writing a recursive program which goes into cases depending on whether the exponent n is even or odd.]
(b) Given that multiplication and division are p-time, does this show that exponentiation is a p-time operation?

8. The exam timetabling problem is: given a set of exams E, sets of candidates L_e (each $e \in E$), and a set of times T, is there a timetable which has no clashes? Here a timetable is a function $f : E \to T$, and a clash is when a candidate has to take two exams which are scheduled at the same time.
Explain why the exam timetabling problem is in NP.

9. [2018 exam] (a) The problem 3COL is defined as follows: given a graph G, is G 3-colourable? Explain why 3COL belongs to the complexity class NP.
(b) The problem 4COL is defined as follows: given a graph G, is G 4-colourable?
Show that if 3COL is NP-complete then 4COL is NP-complete.

① a) we guess a sequence of nodes $x_1, \dots, x_n$ and check in p-time that it is a Hamiltonian cycle for G, that is

- G has n nodes
- all x_i are distinct and nodes of G
- There is an arc in G between x_i and x_{i+1} $(i = 1, \dots, n)$

b) we show HamPath $\leq$ Ham Cycle. Since HamPath is NP-hard we need a reduction function f such that G has a Ham. path. iff $f(G)$ has a Ham Cycle.

Let G have n nodes Let f add a new node x to G. and join every node of G to x by n new arcs.
this is in p-time as we add 1 node and n arcs.

suppose G has Ham path with endpoints y, z then $f(G)$ has a Ham cycle where we add the arcs (y, x), (x, z)

Conversely suppose G has Ham cycle wlog can assume this cycle starts at x (and ends at x) such that $x, y, \dots, z, x$ so, $y \to z$ must be a Ham path for G.

Hence Hampath $\leq$ Ham Cycle then Ham Cycle is NP-hard.

② given n we can guess non-trivial factorisation m_1, m_2 such that $m_1 \times m_2 = n$. (multiplication is p-time) guesses are bound by the input size Hence COMPOSITE $\in$ NP

VER-COMPOSITE $(n, m_1, m_2) \longleftrightarrow n = m_1 * m_2$ and $m_1, m_2 > 1$

Then COMPOSITE $(n) \longleftrightarrow \exists m_1, m_2$ VER-COMPOSITE (n, m_1, m_2)

(3) Given G_1 G_2 we can guess the potential isomorphism which is a pair of mapping (f, g) for nodes and arcs. we can check in p-time that (f, g) is a isomorphism between 2 graph f, g a bijection on nodes and arcs; and can be p-bounded so that $f \subseteq nodes(G_1) \times nodes(G_2)$ $g \subseteq arcs(G_1) \times arcs(G_2)$

hence GraphIsom $\in NP$

VER-GRAPH ISOM $(G_1, G_2, f, g) \leftrightarrow (f, g)$ is an isomorphism between G_1 and G_2.

(4)

a) Let D be a decision problem Then $D \leq D$ with the identity function as reduction function:

$D(x) \leftrightarrow D(f(x))$ where $f(x) = x$ as identity is p-time computable.

b) D, D', D'' where $D \leq D' \leq D''$ assume p-time computable f, g

such that:

$$D(x) \leftrightarrow D'(f(x))$$
$$D'(y) \leftrightarrow D''(g(x))$$

Clearly $D(x) \leftrightarrow D''(g \circ f(x))$ $g \circ f$ is p-time computable by proposition on composition of p-time computable functions

Hence $D \leq D''$ as required.

(5) $D < D'$ and $D \in$ NP-hard

let D'' be any problem in NP. Since D is NP-hard we have $D'' \leq D$ But $D'' \leq D'$ by transitivity of $\leq$; Hence for any problem D'' we have $D'' \leq D'$ $\rightarrow$ therefore D' is NP-hard.

⑥ TSP(D) and VRPC(D) are NP-complete, both ∈ NP and NP-hard

so for all D ∈ NP we have D ≤ TSP(D) and D ≤ VRPC(D)

so VRPC(D) ≤ TSP(D) and TSP(D) ≤ VRPC(D).

Hence VRPC(D) ~ TSP(D)

Works in general for any 2 NP-complete problems

⑦ a) showExp m n

| n==0 = 1

| otherwise = m * showExp (m, n-1)

n multiplication is exponential in the input size log m + log n

showExp m n

| n== 0 = 1

| n%2==0 = showExp (m*m, $\underline{n/2}$)

| otherwise = showExp(m*m, $\underline{(n-1)/2}$)

for exponent n we do log n multiplications because of

b) Exponentiation cannot be p-time since the size of the output log (m^n) = n log m which is exponential in the input size. We can do it in p-time w/ modular arithmetic

⑧ f: E → T ...

⑨ a) guess an assignment of 3 colours to nodes and check in p time that adjacent nodes have different colours.

$$3COL(G) \Longleftrightarrow \underbrace{\exists c.\ VER\text{-}3COL(G, c)}_{\text{must be } \in P}$$

b) suppose 3COL is NP-complete then show 4COL ∈ NP similar to a) this means 3COL ≤ 4COL

Let G have n nodes, let f be a function that adds a node x to G and join x to all nodes of G by an arc. This is in p-time

Claim: G is 3 colourable iff $f(G)$ is 4 colourable

($\Rightarrow$): We extend 3 col. of G to a 4-col of $f(G)$ by colouring x with a new colour

($\Leftarrow$): Suppose $f(G)$ has a 4 colouring c. Then $c(x) \neq c(y)$ for all $y \in nodes(G)$ so c restricted to $nodes(G)$ is a 3 colouring of G.