Question

1.

Marks & seen/unseen

Parts

unseen ⇓

(i) (c). From tables, $\Phi^{-1}(0.025) \approx -1.96$ and $\Phi^{-1}(1-0.005) \approx 2.58$. So we have the equations

$$\begin{aligned} 90 - \mu &\approx -1.96\sigma && (1) \\ 120 - \mu &\approx 2.576\sigma. && (2) \end{aligned}$$

Subtracting (1) from (2) yields

$$\begin{aligned} 4.536\sigma \approx 30 &\implies \sigma \approx 6.614 \\ &\implies \mu \approx 102.963. \end{aligned}$$

(ii) (d). $F(3.4) = \sum_{y \in \mathcal{Y}; y \leq 3.4} p(y) = 0.1 + 0.2 + 0.3 = 0.6$.

seen ⇓

(iii) (a). By the law of total probability, $\mathrm{P}(E) = \mathrm{P}(F)\mathrm{P}(E|F) + \mathrm{P}(\overline{F})\mathrm{P}(E|\overline{F})$. Hence

$$\mathrm{P}(E|F) = \frac{\mathrm{P}(E) - \mathrm{P}(\overline{F})\mathrm{P}(E|\overline{F})}{\mathrm{P}(F)} = \frac{0.5 - 0.5 \times 0.6}{0.5} = \frac{0.2}{0.5} = 0.4.$$

(iv) (e).

(v) (b). We require $\int_{x=-\infty}^{\infty} f(x)dx = 1$, so

$$\begin{aligned} 1 &= \int_{x=-1}^{0} \frac{dx}{4} + c\int_{x=0}^{1} x dx = \frac{1}{2} + c\left[\frac{x^2}{2}\right]_{x=0}^{1} = \frac{1+c}{2} \\ \implies c &= 1. \end{aligned}$$

Each 4 marks

Question 2.

Marks & seen/unseen

Parts

unseen ⇓

(i)

$p_{XY}(x,y)$		$x=0$	$x=1$	$x=2$	
y	0	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$
	1	$\frac{1}{8}$	$\frac{1}{5}$	$\frac{7}{40}$	$\frac{1}{2}$
		$\frac{13}{40}$	$\frac{3}{10}$	$\frac{3}{8}$	1

6 marks

(ii) The random variables X and Y are not independent of one another since, for example, $\mathrm{P}(X=0) = \frac{13}{40}$ but $\mathrm{P}(X=0|Y=0) = \frac{2}{5}$.

seen ⇓

2 marks

(iii)

$$\mathrm{E}(X) = \sum_{x=0}^{2} x\, \mathrm{P}(X=x) = 0 \times \frac{13}{40} + 1 \times \frac{3}{10} + 2 \times \frac{3}{8} = \frac{21}{20} = 1.05.$$

3 marks

$$\mathrm{E}(Y) = \sum_{y=0}^{1} y\, \mathrm{P}(Y=y) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}.$$

2 marks

(iv)

$$\mathrm{E}(XY) = \sum_{x=0}^{2}\sum_{y=0}^{1} x\, y\, p_{XY}(x,y) = \sum_{x=1}^{2} x\, p_{XY}(x,1) = 1 \times \frac{1}{5} + 2 \times \frac{7}{40} = \frac{11}{20} = 0.55.$$

4 marks

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = \frac{11}{20} - \frac{21}{20} \times \frac{1}{2} = \frac{1}{40}.$$

2 marks

Since $\mathrm{Cov}(X,Y) > 0$, the variables are positively correlated.

1 mark

Question 3.

Marks & seen/unseen

Parts

seen ⇓

(i) (a) $\text{bias}(T) = \text{E}[T|\mu] - \mu$. — 1 mark

$$\text{E}(\bar{X}|\mu) = \text{E}\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{\sum_{i=1}^{n} \text{E}(X_i)}{n} = \frac{\sum_{i=1}^{n} \mu}{n} = \frac{n\mu}{n} = \mu.$$

4 marks

(b) $\bar{x} = 495.09$. — 1 mark

(c) Under a null hypothesis $H_0 : \mu = 500\text{g}$, $\bar{X} \sim \text{N}(500, 5^2/10)$. Let $Z = (\bar{X} - 500)/\sqrt{2.5}$, then $Z \sim \Phi$ under H_0 giving a 5% significance level rejection region for the observed statistic z of $(-\infty, -1.96) \cup (1.96, \infty)$. Here $z = -3.1054$, so we reject the null hypothesis. — 5 marks

(ii) (a) The bias-corrected sample standard deviation $s_{n-1} = 7.00261\text{g}$ (considerably larger than the standard deviation claimed by the supermarket). — 2 marks

(b) Under the same null hypothesis, if $T = \sqrt{10}(\bar{X} - 500)/7.00$ then $T \sim t_9$ under H_0 giving a 5% significance level rejection region for the observed statistic t of $(-\infty, -2.26) \cup (2.26, \infty)$. Here $t = -2.2173$, well outside the rejection region and so now there is no evidence to reject the null hypothesis.

5 marks

(c) The differing outcome of the two tests suggests that the mean weights of the bags of grapes may well be 500g, but the variability of the bag weights claimed by the supermarket may be too low. — 2 marks

Question

4.

Marks & seen/unseen

Parts

seen $\Downarrow$

(i) (a) D and E mutually exclusive $\iff D \cap E = \emptyset$.

(b) E and F independent $\iff \mathrm{P}(E \cap F) = \mathrm{P}(E)\mathrm{P}(F)$. — 4 marks

(ii) $\mathrm{P}(E|F) = \mathrm{P}(E) = 0.4$ by independence of E and F. — 2 marks

(iii) $\mathrm{P}(D \cap E) = 0$ since E and F are mutually exlusive. — 1 marks

(iv) $\mathrm{P}(D \cup E) = \mathrm{P}(D) + \mathrm{P}(E) - \mathrm{P}(D \cap E) = \mathrm{P}(D) + \mathrm{P}(E) - 0 = 0.3 + 0.4 - 0 = 0.7$. — 3 marks

(v) $\mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E)\mathrm{P}(F)$ by independence of E and F. Rearranging then gives $\mathrm{P}(F) = \dfrac{\mathrm{P}(E \cup F) - \mathrm{P}(E)}{1 - \mathrm{P}(E)} = \dfrac{0.4}{0.6} = \dfrac{2}{3}$. — 6 marks

unseen $\Downarrow$

(vi) $\{X \geq 5\} \equiv \overline{E} \cap F$. — 1 mark

(vii) $\mathrm{P}(X \geq 5) = \mathrm{P}(\overline{E} \cap F)$. Then by the law of total probability, $\mathrm{P}(F) = \mathrm{P}(E \cap F) + \mathrm{P}(\overline{E} \cap F)$, hence

$$\mathrm{P}(X \geq 5) = \mathrm{P}(F) - \mathrm{P}(E \cap F) = \mathrm{P}(F) - \mathrm{P}(E)\mathrm{P}(F) = \frac{2}{3} \times (1 - 0.4) = 0.4.$$

3 marks