# **COMP245: Probability and Statistics 2016** - Problem Sheet 6

Solutions

### Continuous Random Variables

S1) Begin with $F(-x) + F(x) = \int_{u=-\infty}^{-x} f(u)du + \int_{v=-\infty}^{x} f(v)dv$. Taking the change of variable $v = -u$ for the first part leads to $F(-x) + F(x) = -\int_{v=\infty}^{x} f(-v)dv + \int_{v=-\infty}^{x} f(v)dv = \int_{v=x}^{\infty} f(v)dv + \int_{u=-\infty}^{x} f(v)dv = \int_{v=-\infty}^{\infty} f(v)dv = 1$.

S2) (a) Since a particle is equally likely to hit anywhere on the plate, for $0 < r < 1$ the probability that it will strike inside a circle of radius r is $\frac{\pi r^2}{\pi 1^2} = r^2$. Hence

$$F(x) = \begin{cases} 0, & x \leq 0 \\ x^2, & 0 < x < 1 \\ 1, & x \geq 1. \end{cases}$$

(b) $\mathrm{P}(r < X < s) = \mathrm{P}(X < s) - \mathrm{P}(X < r) = s^2 - r^2$.

(c) From 2a the cdf of $X$ is given by $F(x) = x^2$ for $0 \leq x \leq 1$. So the pdf, $f(x) = F'(x) = 2x$ for $0 \leq x \leq 1$, and 0 everywhere else.

(d) The expected distance from the origin is

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)dx = \int_{0}^{1} x.2x dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}.$$

S3) A random variable $X \sim \mathrm{Exp}(\lambda)$ has density $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. So

$$\begin{aligned} \mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)dx = \int_{0}^{\infty} x\lambda e^{-\lambda x}dx = \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}dx \\ &= [0-0] - \left[\frac{-e^{-\lambda x}}{\lambda}\right]_0^\infty = \left[0 - \left(-\frac{1}{\lambda}\right)\right] = \frac{1}{\lambda}. \end{aligned}$$

$$\begin{aligned} \mathrm{E}\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{0}^{\infty} x^2\lambda e^{-\lambda x}dx = \left[-x^2e^{-\lambda x}\right]_0^\infty + \int_0^\infty 2xe^{-\lambda x}dx \\ &= [0-0] + \frac{2}{\lambda}\mathrm{E}(X) = \frac{2}{\lambda^2} \\ \Rightarrow \mathrm{Var}(X) &= \mathrm{E}\left(X^2\right) - \{\mathrm{E}\left(X\right)\}^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}. \end{aligned}$$

S4) $X$ has range $[0,1]$ and pdf $f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$

Thus the cdf for $Y = e^X$ is given by

$$F_Y(y) = \mathrm{P}_Y(Y \le y) = \mathrm{P}(e^X \le y) = \mathrm{P}(X \le \log(y)) = \int_{-\infty}^{\log(y)} f_X(x)dx = \int_0^{\log(y)} dx = \log(y),$$

for $0 < \log(y) < 1$; that is, for $1 < y < e$.

For the pdf, differentiating $F_Y(y)$ wrt $y$ gives

$$f_Y(y) = \begin{cases} \frac{1}{y}, & 1 < y < e \\ 0, & \text{otherwise.} \end{cases}$$

S5) Let $g(x) = \dfrac{x-\mu}{\sigma}$, so $Y = g(X)$. First we note that $g$ is clearly a continuous, monotonically increasing function of $x$. Therefore we have

$$f_Y(y) = f_X\{g^{-1}(y)\}|g^{-1\prime}(y)|.$$

Well $g^{-1}(y) = \sigma y + \mu$, so $g^{-1\prime}(y) = \sigma$. Since $X \sim \mathrm{N}(\mu, \sigma^2)$,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

and hence

$$f_Y(y) = f_X(\sigma y + \mu)\sigma = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\sigma y + \mu - \mu)^2}{2\sigma^2}\right\}\sigma = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} \implies Y \sim \mathrm{N}(0,1).$$

S6) (a) $\forall a \neq 0$, $F_Y(y) = \mathrm{P}_Y(Y \le y) = \mathrm{P}(aX + b \le y) = \mathrm{P}(aX \le y - b)$.

If $a > 0$,

$$F_Y(y) = \mathrm{P}\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right),$$

whereas if $a < 0$,

$$F_Y(y) = \mathrm{P}\left(X \ge \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right).$$

(b) The pdf of a continuous random variable is the derivative of the cdf, $f_Y(y) = F_y'(y)$.

So if $a > 0$,

$$f_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{1}{a} F_X'\left(\frac{y-b}{a}\right) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right),$$

whereas if $a < 0$,

$$f_Y(y) = \frac{d}{dy}\left\{1 - F_X\left(\frac{y-b}{a}\right)\right\} = -\frac{1}{a} F_X'\left(\frac{y-b}{a}\right) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right),$$

So either way, we have

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

S7)

(a) $z = 1.16$ or $z =$ -1.16. (b) $z = 1.09$. (c) $z =$ -1.35 or $z =$ -1.69.

S8) (a) 0.3849 (c) 0.6636 (e) 0.8997

(b) 0.2517 (d) 0.1828