IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2015

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C142

DISCRETE MATHEMATICS

Friday 1 May 2015, 10:00
Duration: 80 minutes

Answer ALL TWO questions

Paper contains 2 questions
Calculators not required

1a Let $A = \{\{1\},\{2\}\}$, $B = \{1,2,\{2\}\}$ and $C = \{1,\{2\}\}$. Determine $A \triangle B$, $B \triangle C$, $A \cap (B \cup C)$, and $(A \cap B) \setminus C$.

b For each of the following statements decide whether it is true or false. Support your answer by giving a Venn diagram or give a counterexample.

i) $\forall A,B,C\,(\,A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\,)$.

ii) $\forall A,B,C\,(\,A \cap (B \cup C) = (A \cap B) \cap (B \cup C)\,)$.

iii) $\forall X,Y,Z\,(\,X \subseteq Y \cap Z \Leftrightarrow X \subseteq Y \wedge X \subseteq Z\,)$.

iv) Set difference is commutative.

v) $\forall A,B,C\,(\,A \triangle (B \setminus C) = (A \triangle B) \setminus (A \triangle C)\,)$.

c i) Give an example of a relation on $\{\,a,b,c,d\,\}$ which is reflexive but not symmetric. Express your answer as a directed graph.

ii) How many binary relations are there on a set with n elements? Of these, how many are reflexive? Explain briefly.

d Let R, $S \subseteq A^2$. Prove that the following statements are true:

i) If $R \subseteq S$ then $R^{-1} \subseteq S^{-1}$

ii) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$;

iii) $(\overline{R})^{-1} = \overline{R^{-1}}$;

iv) $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.

v) Show that R is symmetric implies $R \circ R$ is symmetric.

e Let $f : A \to B$ and $g : B \to C$ be functions.

i) Prove that if $g \circ f$ is one-to-one then so is f.

ii) Give an example of f and g such that $g \circ f$ is one-to-one but g is not.

iii) Prove that if $g \circ f$ is onto then so is g.

iv) Give a specific example of f and g such that $g \circ f$ is onto but f is not.

The five parts carry, respectively, 10%, 25%, 15%, 30%, and 20% of the marks.

2a Give a decision tree for the Binary Search algorithm applied to ordered lists of length six, with elements indexed from 0 to 5. Assume that the algorithm chooses the element with lower index at any point where there is a choice.

Also state the worst-case number of comparisons.

b i) State the worst-case number of comparisons for the Insertion Sort algorithm applied to a list of n elements.

Give a justification for your answer.

ii) Suppose that a list L with n distinct elements has the property that each element $L[i]$ is no more than two places away from its sorted position. In other words, if L' denotes L after being sorted, then for each i such that $0 \leq i \leq n-1$, there is j such that $i-2 \leq j \leq i+2$ and $0 \leq j \leq n-1$ and $L[i] = L'[j]$.

Calculate the worst-case number of comparisons in terms of n when Insertion Sort is applied to L.

c i) State without proof the recurrence relation for the worst-case number of comparisons $W(n)$ of the MergeSort algorithm on lists of length n.

(Do not solve your recurrence relation.)

ii) Consider the following modified version of MergeSort, called ModMergeSort. Instead of always merging the two sub-lists, before performing a merge check to see whether the largest element of the left-hand sub-list is less than or equal to the smallest element of the right-hand sub-list. If this is the case then do not perform the merge; otherwise perform the merge as usual.

Write down a recurrence relation for the worst-case number of comparisons $W'(n)$ of ModMergeSort when applied to lists which are *already sorted*.

iii) Solve your recurrence relation from part (ii) for n a power of two ($n = 2^k$ for some $k \geq 0$).

The three parts carry, respectively, 20%, 40%, and 40% of the marks.