1(a) $V \cap W = \{\{1\}\}$

$V \cup W = \{\{1\}, \{2\}, 1, 2\}$

$$pW = \{ \phi,$$
$$\{\{1\}\}, \{1\}, \{2\},$$
$$\{\{1\}, 1\}, \{\{1\}, 2\}, \{1, 2\},$$
$$\{\{1\}, 1, 2\}$$
$$\}$$

$V \cap pW = \{\{1\}, \{2\}\}$

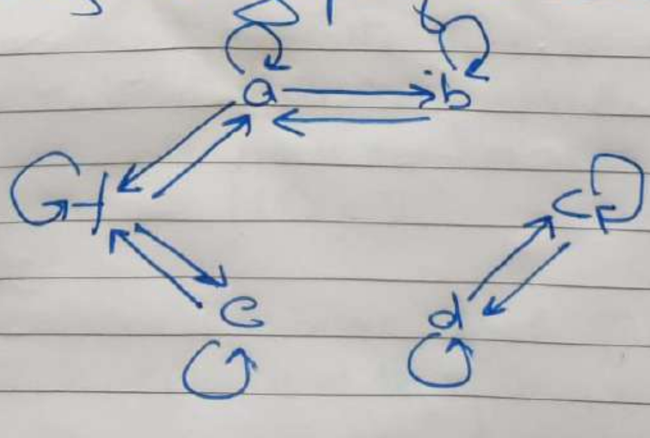
$$V \times W = \{\langle\{1\}, \{1\}\rangle, \langle\{1\}, 1\rangle, \langle\{1\}, 2\rangle,$$
$$\langle\{2\}, \{1\}\rangle, \langle\{2\}, 1\rangle, \langle\{2\}, 2\rangle\}$$

$V \triangle W = \{\{2\}, 1, 2\}$

b Since R is reflexive, $\langle a, a\rangle$, $\langle b, b\rangle$, $\langle c, c\rangle$, $\langle d, d\rangle$, $\langle e, e\rangle$, $\langle f, f\rangle$ must belong to R

Since it is symmetric, $\langle b, a\rangle$, $\langle f, a\rangle$, $\langle c, d\rangle$, $\langle f, e\rangle$ must belong to R.

Drawing the graph of the elements we know $\in R$



So to make R transitive, we must have <e,a>, <e,b>, <f,b>, <a,e>, <b,f>, <b,e>

Notice that symmetry is still maintained

Hence, we need to add the following pairs:
<a,a>, <b,b>, <c,c>, <d,d>, <e,e>, <f,f>, <b,a>,
<f,a>, <c,d>, <f,e>, <e,a>, <e,b>, <f,b>,
<a,e>, <b,f>, <b,e>

(c) Take an arbitrary $x \in (A \cup B) \cap (C \cup D)$.
Then by def. of intersection, we have '$x \in (A \cup B)$ and $x \in (C \cup D)$'. So, we have both '$x \in (A \cup B)$' and '$x \in (C \cup D)$'. By def. of union, we have '$x \in A$ or $x \in B$' and '$x \in C$ or $x \in D$'. We have 4 cases

($x \in A$ and $x \in C$): Using 'and', we get '$x \in A$'. Then, '$x \in A$ or $x \in C$'. By definition of union, '$x \in (A \cup C)$'. Again, by def. of union, '$x \in (A \cup C) \cup (B \cap D)$'

($x \in A$ and $x \in D$): Using 'and', we get '$x \in A$'. Proceeding similarly as above, we get '$x \in (A \cup C) \cup (B \cap D)$'.

($x \in B$ and $x \in C$): Using 'and', we get '$x \in C$'. Then, '$x \in A$ or $x \in C$'. By def. of union, '$x \in (A \cup C)$'. Again, by def. of union, '$x \in (A \cup C) \cup (B \cap D)$'.

($x \in B$ and $x \in D$): By def. of intersection we get '$x \in B \cap D$'. Then '$x \in (A \cup C)$ or $x \in (B \cap D)$'. Hence, we have '$x \in (A \cup C) \cup (B \cap D)$'.

In all the four cases we have '$x \in (A \cup C) \cup (B \cap D)$'. Hence, we conclude that '$x \in (A \cup C) \cup (B \cap D)$'. Further as x was an arbitrary element of $(A \cup B) \cap (C \cup D)$, we have '$(A \cup B) \cap (C \cup D) \subseteq (A \cup C) \cup (B \cap D)$'

(d) To show $\overline{(R \cup S)}^{-1} \subseteq \overline{R^{-1}} \cap \overline{S^{-1}}$

Take arbitrary x, y such that '$x \overline{(R \cup S)}^{-1} y$'. Then by def. of inverse, '$y \overline{(R \cup S)} x$'. So by def. of complement, '$\langle y, x \rangle \notin (R \cup S)$'. By De Morgan's law, we get '$\langle y, x \rangle \notin R$ and $\langle y, x \rangle \notin S$'. ~~By def. of complement, we get~~

Suppose '$\langle x, y \rangle \in R^{-1}$'. Then '$\langle y, x \rangle \in R$' by def. of inverse, which is a contradiction. So we conclude '$\langle x, y \rangle \notin R^{-1}$'. Similarly, '$\langle x, y \rangle \notin S^{-1}$'. Then by def. of complement, we have '$x \overline{R^{-1}} y$ and $x \overline{S^{-1}} y$'. So, by def. of intersection, we have '$x \overline{R^{-1}} \cap \overline{S^{-1}} y$'.

As x was an arbitrary element of $\overline{(R \cup S)}^{-1}$, we conclude $\overline{(R \cup S)}^{-1} \subseteq \overline{R^{-1}} \cap \overline{S^{-1}}$

To show $\overline{R^{-1}} \cap \overline{S^{-1}} \subseteq \overline{(R \cup S)}^{-1}$

Since De Morgan's law is stated in both directions, this follows from the above proof, reading the steps in reverse order.

- To show $\overline{(R \cup S)}^{-1} = \overline{R^{-1}} \cap \overline{S^{-1}}$
  We have '$\overline{(R \cup S)}^{-1} \subseteq \overline{R^{-1}} \cap \overline{S^{-1}}$' and '$\overline{R^{-1}} \cap \overline{S^{-1}} \subseteq \overline{(R \cup S)}^{-1}$'. Hence, by definition of set equality, we have $\overline{(R \cup S)}^{-1} = \overline{R^{-1}} \cap \overline{S^{-1}}$

- To show $\overline{R^{-1}} \cap \overline{S^{-1}} = \overline{(R \cup S)}^{-1}$
  We have '$\overline{R^{-1}} \cap \overline{S^{-1}} \subseteq \overline{(R \cup S)}^{-1}$' and '$\overline{(R \cup S)}^{-1} \subseteq \overline{R^{-1}} \cap \overline{S^{-1}}$'. Hence, by def. of set equality, we have $\overline{R^{-1}} \cap \overline{S^{-1}} = \overline{(R \cup S)}^{-1}$

c To show R is reflexive:
Take any $p, q \in \mathbb{N}$. We have '$p \times 1 = p$' and '$q \times 1 = q$'. But since $1 \in \mathbb{N}$, 'there exists $n, m \in \mathbb{N}$ such that $p \times n = p$ and $q \times m = q$'. Hence, we have '$\langle p, q \rangle R \langle p, q \rangle$'

To show R is transitive:
Assume that '$\langle a, b \rangle R \langle c, d \rangle$' and '$\langle c, d \rangle R \langle e, f \rangle$'. Then by def. of R, there exists $p, q, r, s \in \mathbb{N}$ such that '$a \times p = c$' and '$b \times q = d$' and '$c \times r = e$' and '$d \times s = f$'. Using '$a \times p = c$' and '$c \times r = e$', we have '$a \times (p \times r) = e$'. Using '$b \times q = d$' in '$d \times s = f$', we have '$b \times (q \times s) = f$'. As we have both '$a \times (p \times r) = e$' and '$b \times (q \times s) = f$', and '$p \times r \in \mathbb{N}$' and '$(q \times s) \in \mathbb{N}$', by def. of R, we get '$\langle a, b \rangle R \langle e, f \rangle$'

To show R is anti symmetric
Assume that '$\langle a, b \rangle R \langle c, d \rangle$' and '$\langle c, d \rangle R \langle a, b \rangle$' both hold. Then there exists $p, q, r, s \in \mathbb{N}$ such that '$a \times (p \times r) = a$' and '$b \times (q \times s) = b$'. (Proof is same as case for transitive). As we have $p, q, r, s \in \mathbb{N}$, we conclude $p = q = r = s = 1$. Hence $c = a \times p = a$ and $d = b \times q = b$. So '$\langle a, b \rangle = \langle c, d \rangle$'.

As R is reflexive, transitive and anti symmetric, it follows that R is a partial order.

2a i) Not a function as $\langle 1,1\rangle$ and $\langle 1,-1\rangle$ both belong to the function, so ~~it has~~ 1 has 2 images

ii) This is a function. (It is a total function). It is injective and surjective. $f^{-1}(y) = \left(\frac{1}{y}+8\right)^{1/3}$

iii) This is a total function. Not injective as $\langle -1,1\rangle$ and $\langle 1,1\rangle$ both belong to it. Not surjective as no x gets mapped to a negative number. No inverse.

iv) It is a total function. It is injective. It is not surjective as $\frac{1}{2}$ is not part of the range. No inverse.

b) Given $|A| = m$ and $|B| = n$.

$|pA| = 2^m$ and $|pB| = 2^n$

considering total functions from pA to pB, each element in pA can choose among $|pB|$ different values for its image. Moreover, there are $|pA|$ elements in pA, so we make this choice $|pA|$ times

So no. of total function $= |pB|^{|pA|} = \cancel{(2^m)^{(2^n)}}\ (2^n)^{(2^m)}$

When considering partial functions, each element in pA has an additional choice to remain undefined. So each element in pA has $(1+p|B|)$ choices.

So no. of partial functions $= |pB+1|^{|pA|} = (2^n+1)^{(2^m)}$

(c) Take any $a \in \mathbb{N}$. Then $\{a\} \in F$. $\{a\}$ is a minimal element of (F, R). Take any $V \in F$ such that $V R \{a\}$. Then '$V = \phi$' or '$V = \{a\}$' since these are the only 2 subsets of $\{a\}$. Moreover, by def. of R, we have '$V \neq \phi$'. So '$V = \phi$' cannot hold and we must have '$V = \{a\}$'.

So we have shown that (since V was arbitrary) '$\forall V \in F. (V R \{a\} \Rightarrow V = \{a\})$'

Note that minimal element is not unique.
$\{1\}, \{2\}, \{3\}, \dots$ are all minimal

No least element exists. Take any minimal element $\{a\}$. Then take '$b = a+1$' and '$c = a+2$', then we have '$\neg(\{a\} R \{b, c\})$'. Hence, '$\forall b \in F (\{a\} R F)$' does not hold for any a.

No maximal element exists as well. Suppose towards a contradiction that '$V = \{a_1, a_2, \dots, a_n\}$' and non-empty (note that V is finite) is a maximal element. Define $W = V \cup \{(\sum_{i=1}^{n} a_i) + 1\}$. Then $V \neq W$. Further, $V \subseteq W$ and $V \neq \phi$. So we have VRW. By def. of V being maximal, we get $V = W$, contradiction to '$V \neq W$'. Hence no maximal element can exist.

By similar reasoning, no greatest element exists as well

(d) Given $A_1 \approx A_2$ and $B_1 \approx B_2$. So there exists bijective functions $f: A_1 \to A_2$ and $g: B_1 \to B_2$

Define $h: (A_1 \times B_1) \to (B_2 \times A_2)$ as follows)
$h(a, b) = \langle g(b), f(a) \rangle$
This function is well defined as $g(b) \in B_2$ and $f(a) \in A_2$, so $\langle g(b), f(a) \rangle \in B_2 \times B_1$. We will show h is bijective.

(h is injective): Assume that '$h(a_1, b_1) = h(a_2, b_2)$'. Then, by def. of h, we have '$\langle g(b_1), f(a_1) \rangle = \langle g(b_2), f(a_2) \rangle$'. Then, we have '$g(b_1) = g(b_2)$' and '$f(a_1) = f(a_2)$'. As f, g are both bijective and hence injective, we get '$b_1 = b_2$' and '$a_1 = a_2$'. So we have $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$. So, h is injective.

(h is surjective): Take any $\langle b_2, a_2 \rangle \in B_2 \times A_2$. Then $b_2 \in B_2$ and $a_2 \in A_2$. As f, g are bijective and hence surjective, there exists $a_1 \in A_1$ such that $f(a_1) = a_2$, and there exists $b_1 \in B_1$ such that $g(b_1) = b_2$. But then $h(a_1, b_1) = \langle g(b_1), f(a_1) \rangle = \langle b_2, a_2 \rangle$. So 'there exists $\langle a, b \rangle \in (A_1 \times B_1)$ such that $h(a_1, b_1) = \langle a_2, b_2 \rangle$'. Hence h is surjective.

Hence h is bijective. But then 'there exists a ~~function~~ bijective function $h: (A_1 \times B_1) \to (B_2 \times A_2)$'. So by def. we have $A_1 \times B_1 \approx B_2 \times A_2$

Hence $A_1 \times B_1 \approx B_2 \times A_2$

(e) i) Let $f : \mathbb{N} \to \{1\}$ be a function. Then, for all $x \in \mathbb{N}$, we have $f(x) = 1$. So, there can only be 1 such function.

So the set of all functions from $\mathbb{N}$ to $\{1\}$ is finite and hence countable by definition.

ii) When considering partial functions, each element can be undefined as well. So we can write any such function as $f : \mathbb{N} \to \{\perp, 1\}$, and the set of such functions as $\{\perp, 1\}^{\mathbb{N}}$.

Using exercise 65, we know that $\{0,1\}^{\mathbb{N}} \approx p(\mathbb{N})$

Suppose towards a contradiction that $\mathbb{N} \approx \{\perp, 1\}^{\mathbb{N}}$. Then by transitivity of $\approx$, we have $\mathbb{N} \approx p(\mathbb{N})$, which by exercise 66, we know is a contradiction. Hence we have $\mathbb{N} \not\approx \{\perp, 1\}^{\mathbb{N}}$.

So, the set of all partial functions from $\mathbb{N}$ to $\{1\}$ is not countable.