Linear Algebra

COMP40017

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1. Let $A\vec{x} = \vec{b}$ be a system of linear equations where,

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 10 & 5 & 4 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

(a) i. Perform elementary row operations on the augmented matrix $[A \mid \vec{b}]$ to reduce A to its Reduced Row Echelon Form.

Solution:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 2 & 4 \\ 4 & 3 & 2 & 1 & 5 \\ 10 & 5 & 4 & 1 & 7 \end{array}\right] \xrightarrow{\text{EROS}} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{5} & 0 & -\frac{2}{5} \\ 0 & 1 & \frac{2}{5} & 0 & \frac{11}{5} \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

ii. State the free variables.

Solution: x_3 is a free variable.

iii. Find the general solution (solution set S) of $A\vec{x} = \vec{b}$ and describe it geometrically.

Solution: From the RREF we have the following system of equations:

$$\begin{cases} x_1 + \frac{1}{5}x_3 = \frac{-2}{5} \\ x_2 + \frac{2}{5}x_3 = \frac{11}{5} \\ x_4 = 0 \end{cases}$$

We obtain a line in $\mathbb{R}^4$. The general solution is:

$$\vec{x} = \begin{bmatrix} -\frac{2}{5} \\ \frac{11}{5} \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \\ 1 \\ 0 \end{bmatrix}$$

Since the point and the scaling factor λ are arbitrary, we can find another point on the line (taking $\lambda = 2$) and scaling the direction vector by 5 to get a simpler solution:

$$\vec{x} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \lambda' \begin{bmatrix} 1 \\ 2 \\ 5 \\ 0 \end{bmatrix}$$

(b) i. Find the rank and nullity of A.

Solution: The rank of A is 3 and the nullity is 1.

ii. Find a basis for the image space $\mathrm{im}(A)$.

Solution: We take the columns of A that correspond to the pivot columns of the RREF of A.

$$\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right\}$$

iii. Find a basis for the kernel $\ker(A)$.

Solution: We take the the direction vector of the line in the general solution.

$$\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \\ 0 \end{bmatrix} \right\}$$

iv. Find the rank and nullity of $A^\top$.

Solution:

$$\mathrm{rank}(A) = \mathrm{rank}(A^\top) = 3$$
$$\mathrm{null}(A^\top) = \mathrm{order}(A^\top) - \mathrm{rank}(A^\top) = 0$$

v. Find a 'simple' basis for the image space $\mathrm{im}\left(A^\top\right)$.

Solution: Since $\mathrm{rank}(A^\top) = 3$, we know that all 3 columns of $A^\top$ are linearly independent. Therefore, we can take the columns of $A^\top$ as a basis.
To make it 'simple', we would put the columns of $A^\top$ into a matrix horizontally, and then perform row operations to obtain its RREF.
However, this is identical to the RREF of A, which we have already calculated.

$$\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{1}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{2}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

vi. Find the kernel $\ker\left(A^\top\right)$.

Solution: The nullity is 0, so the kernel is the trivial vector space $\{\vec{0}\}$.

(c) i. Find the solution of $A\vec{x} = \vec{b}$ closest (lowest squared Euclidean distance) to the origin $\vec{0}$.

Solution: A point $\vec{x}$ on the line has coordinates $(\lambda, 3+2\lambda, 5\lambda, 0)$. We want to minimise $\|\vec{x}\|$, which is equivalent to minimising $\|\vec{x}\|^2$, since we know that the norm is always non-negative.

$$\begin{aligned}\|\vec{x}\|^2 &= \lambda^2 + (3+2\lambda)^2 + 5^2 + 0^2 \\ &= 5\lambda^2 + 12\lambda + 34\end{aligned}$$

This is minimised by $\lambda = -\frac{6}{5}$, which gives $\vec{x} = \left(-\frac{6}{5}, \frac{3}{5}, -6, 0\right)$.

ii. Find the projection of the vector $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top$ onto the kernel $\ker(A)$.

Solution: Very similar method to previous question. We minimise

$$\|\vec{v} - \lambda\vec{x}\|^2 = (1-\lambda)^2 + (1-2\lambda)^2 + (1-5\lambda)^2 + 1 = 30\lambda^2 - 16\lambda + 4$$

with $\lambda = \frac{16}{60} = \frac{4}{15}$, giving us the projection $\begin{bmatrix} \frac{4}{15} & \frac{8}{15} & \frac{4}{3} & 0 \end{bmatrix}^\top$

2. (a) Let $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$

i. Compute the eigenvalues and eigenspaces of A.

Solution:

$$\begin{cases} \lambda_1 + \lambda_2 = \operatorname{tr}(A) & = 3 \\ \lambda_1 \times \lambda_2 = \det(A) & = -10 \end{cases}$$

We obtain the eigenvalues $\lambda_1 = -2, \lambda_2 = 5$.
Substituting these into A, we obtain the following matrices:

$$A_{\lambda_1} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad A_{\lambda_2} = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}$$

We can read off the eigenspaces from these matrices directly:

$$x = -2y \quad \text{and} \quad y = 3x$$

$$E_{\lambda_1} = \operatorname{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\} \quad \text{and} \quad E_{\lambda_2} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$$

ii. Determine a transformation matrix B such that $B^{-1}AB$ is a diagonal matrix and provide this diagonal matrix.

Solution: We take the eigenvectors as the columns of B.

$$B = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}$$

And the eigenvalues form the diagonal of $B^{-1}AB$.

$$B^{-1}AB = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$$

iii. Compute A^9

Solution: $A^9 = B(B^{-1}AB)^9B^{-1}$. Since $B^{-1}AB$ is diagonal, we can raise each diagonal element to the power of 9.
First, let us calculate B^{-1}.

$$B^{-1} = \frac{1}{\det B}\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$$

Now, we can calculate A^9.

$$A^9 = B\begin{bmatrix} (-2)^9 & 0 \\ 0 & 5^9 \end{bmatrix}B^{-1} = \frac{1}{7}\begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} (-2)^9 & 0 \\ 0 & 5^9 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^9 = \begin{bmatrix} 278579 & 558182 \\ 837273 & 1674034 \end{bmatrix}$$

iv. Find the expression for A^{-2} in terms of I, A and A^2 using the Cayley-Hamilton Theorem.

Solution: First, calculate the characteristic polynomial of A.

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 - 3\lambda - 10$$

According to the Cayley-Hamilton Theorem, A satisfies its own characteristic polynomial, i.e. $p(A) = 0$.

$$\begin{aligned} A^2 &= 3A + 10 \\ A &= 3I + 10A^{-1} \\ A^{-1} &= \frac{1}{10}(A - 3I) \\ A^{-2} &= \frac{1}{100}\left(A^2 - 6A + 9I\right) \end{aligned}$$

(b) Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4 \in \mathbb{R}^4$ such that,

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{a}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Let $A = (\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4)$

i. Show that A is an ordered basis of $\mathbb{R}^4$.

Solution: We can use EROs to reduce A to the identity matrix for $\mathbb{R}^4$. Since the identity matrix is a basis for $\mathbb{R}^4$, A is also a basis for $\mathbb{R}^4$.

ii. Let $B = \left(\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\right)$ be an orthonormal basis of $\mathbb{R}^4$ such that $\vec{b}_1 = \vec{a}_1$. Find $\vec{b}_2, \vec{b}_3$, and $\vec{b}_4$.

Solution: The Gram-Schmidt method would produce an orthonormal basis as required. However, notice that $\vec{b}_1$ is one of the standard basis vectors of $\mathbb{R}^4$. The standard basis is already orthonormal, therefore we can take the remaining standard basis vectors of $\mathbb{R}^4$ as $\vec{b}_2, \vec{b}_3$ and $\vec{b}_4$.

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

iii. Consider a vector $\vec{x}_0$ whose coordinates with respect to A are

$$\vec{x}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^{\top}_{\text{w.r.t.}\,A}$$

Compute the coordinates of $\vec{x}_0$ with respect to B.

Solution: Call the standard basis E. Then

$$\vec{x}_0\ {}_{\text{w.r.t.}\,E} = A \times \vec{x}_0\ {}_{\text{w.r.t.}\,A} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

To go from the standard basis to B, pre-multiply by B^{-1}, which in this case is equivalent to rotating the entries of the vector.

$$\vec{x}_0\ {}_{\text{w.r.t.}\,B} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

iv. Also, consider a vector $\vec{y}_0$ whose coordinates with respect to B are

$$\vec{y}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^{\top}_{\text{w.r.t}\,B}$$

Compute the coordinates of $\vec{y}_0$ with respect to A.

Solution: B is a rearrangement of the standard basis, so this vector remains unchanged on conversion. To go from the standard basis to A, we pre-multiply by A^{-1}. We compute A^{-1} using EROs on the augmented matrix $[\,A \mid I_4\,]$.

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & -1 & -1 & 2 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\vec{y}_{0 \text{ w.r.t.} A} = A^{-1} \times \vec{y}_{0 \text{ w.r.t.} E} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

v. Consider a linear mapping $f : \mathbb{R}^4 \to \mathbb{R}^4$ such that in terms of the standard ordered basis, it is defined as follows:

$$f\left([x_1, x_2, x_3, x_4]^\top\right) = [x_1 + x_2, x_2 + x_3, x_3 + x_4, x_4 + x_1]^\top$$

Compute the same linear transformation f in terms of different bases:

$$f_{AB} : \mathbb{R}^4_B \to \mathbb{R}^4_A$$

Solution: f is a linear map, so we can write it as a matrix F.

$$F_{E \leftarrow E} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Using change of basis matrices, $F_{A \leftarrow B} = I_{A \leftarrow E} F_{E \leftarrow E} I_{E \leftarrow B}$. We know that $A = I_{E \leftarrow A}$ and $B = I_{E \leftarrow B}$, so we will require $I_{A \leftarrow E} = A^{-1}$, which we have already calculated.

$$F_{A \leftarrow B} = A^{-1} F B = \frac{1}{2} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 1 & -1 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

Writing it as a function, we have

$$f_{AB}\left([x_1, x_2, x_3, x_4]^\top\right) = \frac{1}{2} \begin{bmatrix} x_1 + x_2 - 2x_3 - 2x_4 \\ x_1 - x_2 + 2x_4 \\ x_1 + x_2 \\ -x_1 + x_2 + 2x_3 \end{bmatrix}$$