Linear Algebra

COMP40017

17 May 2023

1. Let $A\vec{x} = \vec{b}$ be a system of linear equations where,

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 10 & 5 & 4 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

(a) i. Perform elementary row operations on the augmented matrix $[A \mid \vec{b}]$ to reduce A to its Reduced Row Echelon Form.

ii. State the free variables.

iii. Find the general solution (solution set S) of $A\vec{x} = \vec{b}$ and describe it geometrically.

(b) i. Find the rank and nullity of A.

ii. Find a basis for the image space $\mathrm{im}(A)$.

iii. Find a basis for the kernel $\ker(A)$.

iv. Find the rank and nullity of $A^\top$.

v. Find a 'simple' basis for the image space $\mathrm{im}\left(A^\top\right)$.

vi. Find the kernel $\ker\left(A^\top\right)$.

(c) i. Find the solution of $A\vec{x} = \vec{b}$ closest (lowest squared Euclidean distance) to the origin $\vec{0}$.

ii. Find the projection of the vector $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top$ onto the kernel $\ker(A)$.

2. (a) Let $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$

   i. Compute the eigenvalues and eigenspaces of A.

   ii. Determine a transformation matrix B such that $B^{-1}AB$ is a diagonal matrix and provide this diagonal matrix.

   iii. Compute A^9

   iv. Find the expression for A^{-2} in terms of I, A and A^2 using the Cayley-Hamilton Theorem.

   (b) Let $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4 \in \mathbb{R}^4$ such that,

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{a}_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

   Let $A = (\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4)$

   i. Show that A is an ordered basis of $\mathbb{R}^4$.

   ii. Let $B = \left(\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\right)$ be an orthonormal basis of $\mathbb{R}^4$ such that $\vec{b}_1 = \vec{a}_1$. Find $\vec{b}_2, \vec{b}_3$, and $\vec{b}_4$.

   iii. Consider a vector $\vec{x}_0$ whose coordinates with respect to A are

$$\vec{x}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top_{\text{w.r.t. } A}$$

   Compute the coordinates of $\vec{x}_0$ with respect to B.

   iv. Also, consider a vector $\vec{y}_0$ whose coordinates with respect to B are

$$\vec{y}_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top_{\text{w.r.t } B}$$

   Compute the coordinates of $\vec{y}_0$ with respect to A.

   v. Consider a linear mapping $f : \mathbb{R}^4 \to \mathbb{R}^4$ such that in terms of the standard ordered basis, it is defined as follows:

$$f\left([x_1, x_2, x_3, x_4]^\top\right) = [x_1 + x_2, x_2 + x_3, x_3 + x_4, x_4 + x_1]^\top$$

   Compute the same linear transformation f in terms of different bases:

$$f_{AB} : \mathbb{R}^4_B \to \mathbb{R}^4_A$$