FINAL

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2018

BEng Honours Degree in Mathematics and Computer Science Part I
MEng Honours Degree in Mathematics and Computer Science Part I
BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER C140=MC140

LOGIC

Friday 4th May 2018, 10:00
Duration: 80 minutes

Answer ALL TWO questions

Paper contains 2 questions
Calculators not required

1 a Determine the truth value of the formula

$$p \vee \neg q \wedge q \leftrightarrow p \rightarrow q$$

in a situation in which p is true and q is false. Show all working.

b Using equivalences, find a formula in disjunctive normal form that is logically equivalent to

$$(p \rightarrow q) \rightarrow p \wedge q.$$

In each step, state the equivalence used (e.g., $\neg\neg A \equiv A$). Use only one equivalence per step.

c Let $\vdash$ be the standard natural deduction system, as in lectures.
Let $\vdash^*$ be the natural deduction system obtained from $\vdash$ by deleting the $\vee E$ rule and adding the following Alternative-$\vee E$ rule:

$$
\begin{array}{l}
1 \quad A \vee B\\
\begin{array}{|ll|ll|}
\hline
2 & A \qquad \text{ass} & 4 & B \qquad \text{ass}\\
 & \vdots & & \vdots\\
3 & C & 5 & D\\
\hline
\end{array}\\
6 \quad C \vee D \quad \text{Alt-}\vee E(1,2,3,4,5)
\end{array}
$$

i) Show that $\bot \vee \bot \rightarrow \bot \quad \vdash^* \quad p \vee q \rightarrow q \vee p$.

ii) Let $\sharp$ be a new binary connective, and let $\vdash^\sharp$ be the natural deduction system obtained from $\vdash$ by adding the following three rules for $\sharp$:

$$
\begin{array}{lll}
1 & A & \\
2 & A \sharp B & \sharp I(1)\\
 & & \\
3 & B & \\
4 & A \sharp B & \sharp I(3)
\end{array}
\qquad\qquad
\begin{array}{l}
5 \quad A \sharp B\\
\begin{array}{|ll|ll|}
\hline
6 & A \qquad \text{ass} & 8 & B \qquad \text{ass}\\
 & \vdots & & \vdots\\
7 & C & 9 & D\\
\hline
\end{array}\\
10 \quad C \sharp D \quad \sharp E(5,6,7,8,9)
\end{array}
$$

Supposing that $\vdash^\sharp$ is sound, what are the possible truth tables for $\sharp$? Justify your answer briefly.

iii) Is $\vdash^*$ complete? Justify your answer briefly.

The three parts carry, respectively, 15%, 25%, and 60% of the marks.

2 In parts a and b, L is the 2-sorted signature with sorts $\texttt{Nat}$ and $[\texttt{Nat}]$, constants $\underline{0}, \underline{1}, \underline{2}, \ldots : \texttt{Nat}$ and $[\,] : [\texttt{Nat}]$, function symbols $+$, $-$, $\times$, $:$, $++$, $!!$, $\sharp$, and relation symbols $<, \leq$ and $\texttt{merge}$, of the appropriate sorts (as in lectures).

Variables i, j, k, m, n, etc., have sort $\texttt{Nat}$, and xs, ys, zs, ts, etc., have sort $[\texttt{Nat}]$.

The L-structure M has domain consisting of the natural numbers $0, 1, 2, \ldots$ (sort $\texttt{Nat}$) and all lists of natural numbers (sort $[\texttt{Nat}]$). The symbols of L are interpreted in M as in lectures. For example, $M \models \texttt{merge}(ys, zs, xs)$ if and only if xs is a permutation of $ys{+}{+}zs$ and the relative order of entries in ys and in zs is retained in xs.

You are given an L-formula $\texttt{in}(n, xs)$ expressing that n is an entry in xs, and an L-formula $\texttt{count}(n, xs, k)$ expressing that n occurs exactly k times in xs.

a Write down L-formulas expressing the following properties of xs:

i) xs is the list $[1, 8]$.
ii) xs is sorted in descending order (e.g., $[3, 3, 2, 1]$).
iii) Every entry in xs occurs an odd number of times.
iv) At least half of the entries in xs are the same (e.g., $[1, 2, 1, 1, 3]$).

b The binary function $\texttt{del} : [\texttt{Nat}] \times [\texttt{Nat}] \to [\texttt{Nat}]$ is specified informally by: $\texttt{del}(xs, ys)$ is the list obtained by deleting from xs all entries that occur in ys.

Example: $M \models \texttt{del}([1, 2, 1, 3], [1, 2, 2, 75]) = [3]$.

Below are three *incorrect* attempts to specify $\texttt{del}$ by an L-formula, where $zs = \texttt{del}(xs, ys)$ in each case:

A1. $\forall x(\texttt{in}(x, zs) \leftrightarrow \texttt{in}(x, xs) \land \neg\texttt{in}(x, ys))$
A2. $\exists ts(\texttt{merge}(zs, ts, xs) \land \exists us\, \texttt{merge}(ts, us, ys))$
A3. $\exists ts\, \texttt{merge}(zs, ts, xs) \land \forall x(\texttt{in}(x, zs) \to \neg\texttt{in}(x, ys))$

i) For each of the formulas A1–A3, write down lists xs, ys, and zs for which the formula is true but $zs = \texttt{del}(xs, ys)$ is false, or vice versa.

ii) Write down an L-formula $D(xs, ys, zs)$ that does express $\texttt{del}$ (that is, $M \models \texttt{del}(xs, ys) = zs \leftrightarrow D(xs, ys, zs)$ for all lists xs, ys, zs). You do not need to justify your answer. You may use $\texttt{in}$ and $\texttt{count}$ if you wish.

c Let f be a unary function symbol. Prove by natural deduction that

$$\exists x \forall y(x = y) \quad \vdash \quad \forall x \exists y(f(y) = x).$$

The three parts carry, respectively, 40%, 25%, and 35% of the marks.