CO202 – Software Engineering – Algorithms

Graph Algorithms - Solutions

Exercise 1: Optimal Substructure of Shortest Path

Lemma: Given a weighted, directed graph $G = (V, E)$ with weight function $w: E \to \mathbb{R}$, let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be a shortest path from vertex v_0 to v_k and, for any i and j such that $0 \leq i \leq j \leq k$, let $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$ be the subpath of p from vertex v_i to v_j.

Then, p_{ij} is a shortest path from v_i to v_j.

Exercise 1: Optimal Substructure of Shortest Path

Lemma: Given a weighted, directed graph $G = (V, E)$ with weight function $w: E \to \mathbb{R}$, let $p = \langle v_0, v_1, \dots, v_k \rangle$ be a shortest path from vertex v_0 to v_k and, for any i and j such that $0 \leq i \leq j \leq k$, let $p_{ij} = \langle v_i, v_{i+1}, \dots, v_j \rangle$ be the subpath of p from vertex v_i to v_j.

Then, p_{ij} is a shortest path from v_i to v_j.

Proof: If we decompose path p into $v_0 \overset{p_{0i}}{\rightsquigarrow} v_i \overset{p_{ij}}{\rightsquigarrow} v_j \overset{p_{jk}}{\rightsquigarrow} v_k$, then we have that $w(p) = w(p_{0i}) + w(p_{ij}) + w(p_{jk})$.

Now, assume that there is a path p'_{ij} from v_i to v_j with weight $w(p'_{ij}) < w(p_{ij})$.

Then, $v_0 \overset{p_{0i}}{\rightsquigarrow} v_i \overset{p'_{ij}}{\rightsquigarrow} v_j \overset{p_{jk}}{\rightsquigarrow} v_k$ is a path from v_0 to v_k whose weight $w(p_{0i}) + w(p'_{ij}) + w(p_{jk})$ is less than $w(p)$, which contradicts the assumption that p is a shortest path from v_0 to v_k.

Exercise 2: Good guys, bad guys

There are two types of professional wrestlers: “babyfaces” (good guys) and “heels” (bad guys). Between any pair of professional wrestlers, there may or may not be a rivalry.

Suppose we have n professional wrestlers and we have a list of r pairs of wrestlers for which there are rivalries.

Task: Devise a strategy using graph algorithms that assigns each wrestler to one of the two types such that no rivalry exists between wrestlers of the same type.

Hint: This might not always be possible.

Exercise 2: Good guys, bad guys

Solution:

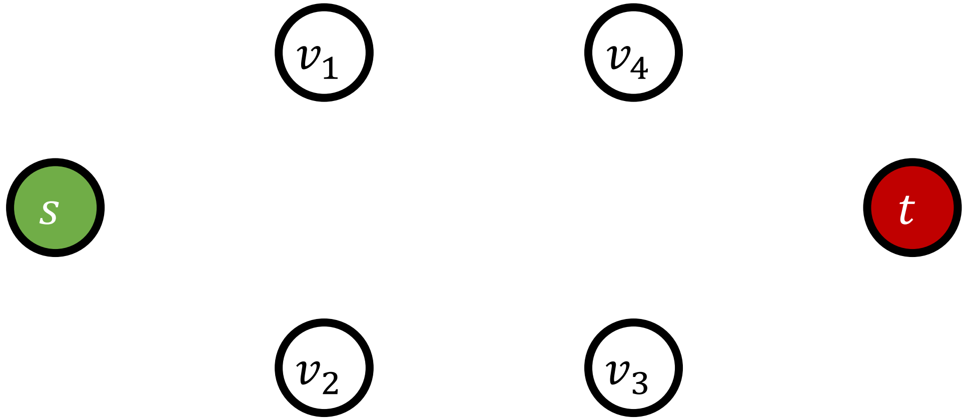
- Create a graph $G = (V, E)$ where each vertex represents a wrestler and each edge represents a rivalry.
- Perform as many BFS's as needed to visit all vertices. Assign all wrestlers whose distance is even to be good guys and all wrestlers whose distance is odd to be bad guys.
- Then check each edge to verify that it goes between a good guy and a bad guy.

This solution would take $O(|V| + |E|)$ time for the BFS, $O(|V|)$ time to designate each wrestler as a good guy or bad guy, and $O(|E|)$ time to check edges, which is $O(|V| + |E|)$ time overall.

Exercise 3: Max-Flow/Min-Cut

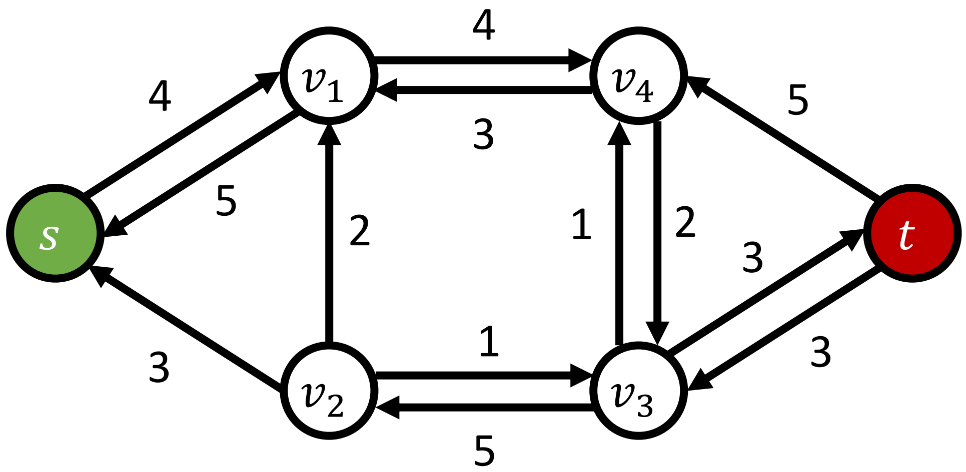
c	s	1	2	3	4	t
s	0	9	3	0	0	0
1	0	0	2	0	7	0
2	0	0	0	6	0	0
3	0	0	0	0	3	6
4	0	0	0	0	0	5
t	0	0	0	0	0	0

f	s	1	2	3	4	t
s	0	5	3	0	0	0
1	-5	0	2	0	3	0
2	-3	-2	0	5	0	0
3	0	0	-5	0	2	3
4	0	-3	0	-2	0	5
t	0	0	0	-3	-5	0



Exercise 3: Max-Flow/Min-Cut

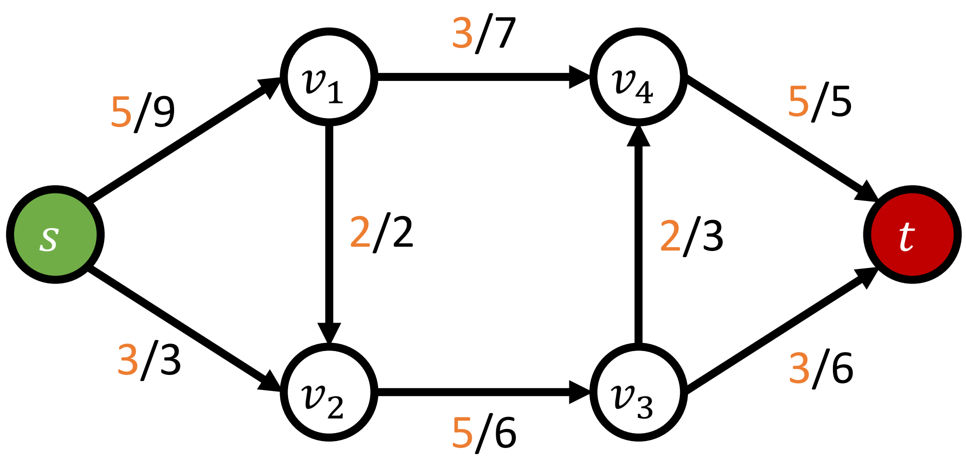
c	s	1	2	3	4	t
s	0	9	3	0	0	0
1	0	0	2	0	7	0
2	0	0	0	6	0	0
3	0	0	0	0	3	6
4	0	0	0	0	0	5
t	0	0	0	0	0	0



G_f

Flow: 8

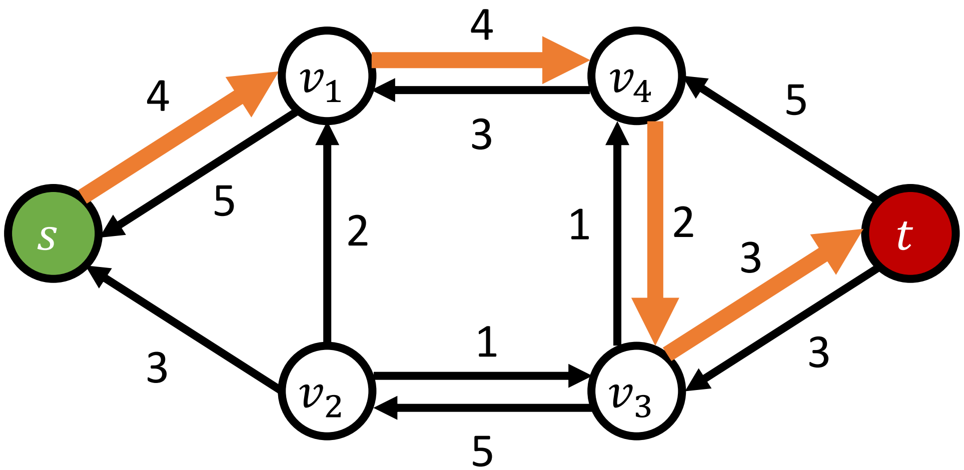
f	s	1	2	3	4	t
s	0	5	3	0	0	0
1	-5	0	2	0	3	0
2	-3	-2	0	5	0	0
3	0	0	-5	0	2	3
4	0	-3	0	-2	0	5
t	0	0	0	-3	-5	0



G

Exercise 3: Max-Flow/Min-Cut

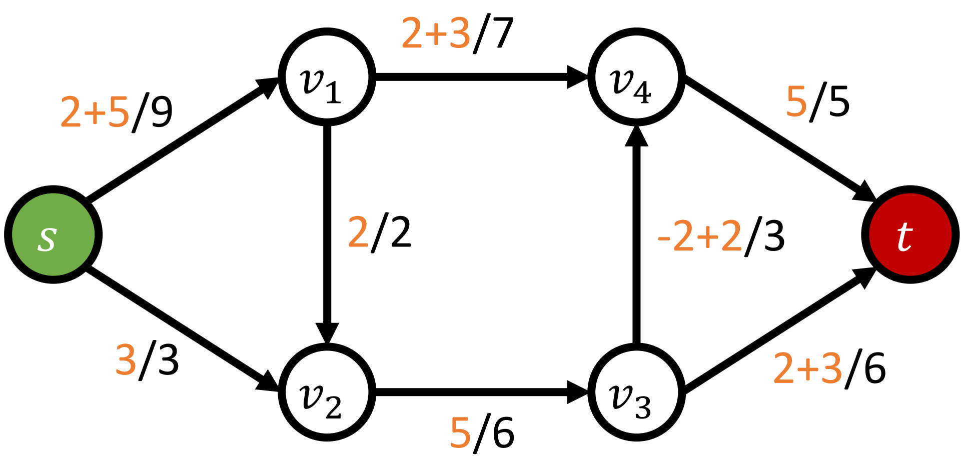
c	s	1	2	3	4	t
s	0	9	3	0	0	0
1	0	0	2	0	7	0
2	0	0	0	6	0	0
3	0	0	0	0	3	6
4	0	0	0	0	0	5
t	0	0	0	0	0	0



G_f

Flow: 8+2

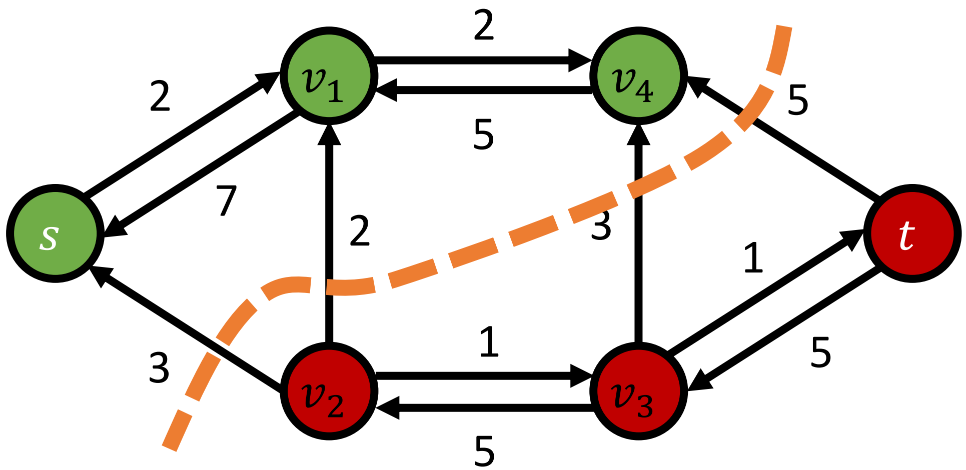
f	s	1	2	3	4	t
s	0	5	3	0	0	0
1	-5	0	2	0	3	0
2	-3	-2	0	5	0	0
3	0	0	-5	0	2	3
4	0	-3	0	-2	0	5
t	0	0	0	-3	-5	0



G

Exercise 3: Max-Flow/Min-Cut

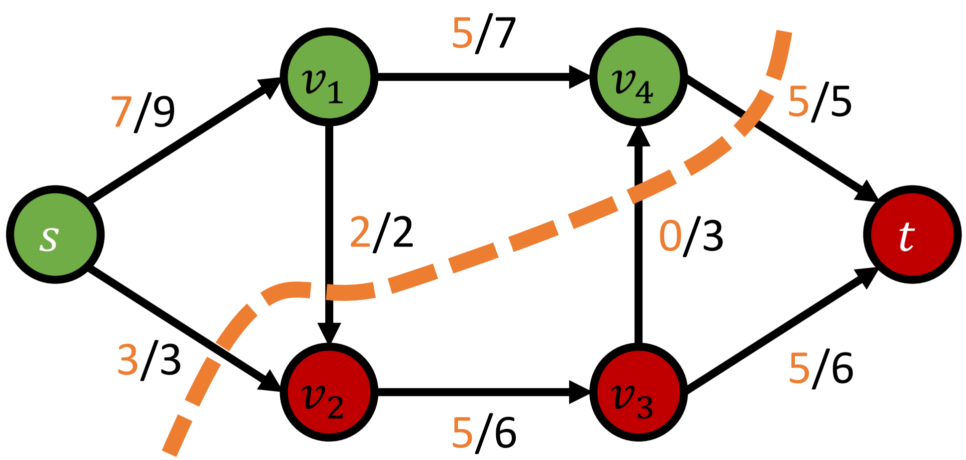
c	s	1	2	3	4	t
s	0	9	3	0	0	0
1	0	0	2	0	7	0
2	0	0	0	6	0	0
3	0	0	0	0	3	6
4	0	0	0	0	0	5
t	0	0	0	0	0	0



G_f

Flow: 10

f	s	1	2	3	4	t
s	0	5	3	0	0	0
1	-5	0	2	0	3	0
2	-3	-2	0	5	0	0
3	0	0	-5	0	2	3
4	0	-3	0	-2	0	5
t	0	0	0	-3	-5	0



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