PAPER C142

Discrete Mathematics

Monday 11 May 2020, 11:00
Duration: 80 minutes
Post-processing time: 30 minutes
Answer TWO questions

While this time-limited remote assessment has not been designed to be open book, in the present circumstances it is being run as an open-book examination. We have worked hard to create exams that assesses synthesis of knowledge rather than factual recall. Thus, access to the internet, notes or other sources of factual information in the time provided will not be helpful and may well limit your time to successfully synthesise the answers required.

Where individual questions rely more on factual recall and may therefore be less discriminatory in an open book context, we may compare the performance on these questions to similar style questions in previous years and we may scale or ignore the marks associated with such questions or parts of the questions. In all examinations we will analyse exam performance against previous performance and against data from previous years and use an evidence-based approach to maintain a fair and robust examination.

As with all exams, the best strategy is to read the question carefully and answer as fully as possible, taking account of the time and number of marks available.

Paper contains 2 questions

1 a Prove that, for all sets A, B, and C:

i) if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$.

ii) if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

b Let A, B, and C be finite sets. Using that $|A \cap B| = |A| + |B| - |A \cup B|$, find a similar formula for $|A \cap B \cap C|$; make sure to show all intermediate steps with justification when appropriate.

c i) Let R be a binary relation on A. State the formal properties that R should satisfy in order to be called: *reflexive*; *symmetric*; *transitive*; an *equivalence*.

ii) If $|A| = n$, how many binary relations are there on A? How many of these are symmetric? Justify your answers.

iii) Is the transitive closure of a symmetric relation always reflexive? Justify your answer either through a proof or through a counterexample.

iv) Let R be an equivalence relation on a finite set A of cardinality n such that every equivalence class has the same fixed cardinality m. Express the cardinality of R (*i.e.* the number of pairs in R, not the number of equivalence classes!) in terms of m and n and justify your answer.

d i) Let R be a binary relation on a set A. Specify the formal properties R should satisfy for it to be a *partial order*.

ii) Let $\leq_P$ denote a binary relation on the positive natural numbers, defined by $n \leq_P m \triangleq \exists\, k \in \mathbb{N} \left(n^k = m \right)$. Prove that $\leq_P$ is a partial order.

The four parts carry, respectively, 20%, 20%, 40%, and 20% of the marks.

2 a i) State the properties that a function $f : A \to B$ has to satisfy to be called *injective*, *surjective*, respectively *bijective*.

ii) Let $A = \{1,2,3\}$ and $B = \{a,b,c\}$. How many functions are there from A to B? How many of these functions are *not* onto? How many *not* one-to-one? How many are bijections? How many partial functions are there from A to B? Motivate all your answers.

b i) Let $f : A \to B$ be an arbitrary function. Specify when a function $g : B \to A$ is called, respectively, a *left inverse* of f, a *right inverse* of f, or an *inverse of* f.

ii) Prove that if a function f has a left inverse g, f is injective and if f has a right inverse h, f is surjective.

c Assume that both A and B are countable sets. Prove that $A \times B$ is one as well.

d Show that if V is not countable, and $V \subseteq W$, then neither is W.

e Define $\mathbb{N}^*$ as the set of all infinite sequences built out of elements of $\mathbb{N}$. Show that $\mathbb{N}^*$ is not countable. (State clearly any result in the course that you may use in your proof.)

The five parts carry, respectively, 20%, 25%, 15%, 20%, and 20% of the marks.