Chapter 4. Elementary Set Theory

This chapter develops the formal mathematical model for events, the theory of sets. We introduce several important notions such as elements, empty sets, intersection, union and disjoint sets.

4.1 Sets, subsets and complements

4.1.1 Sets and notation

A **set** is any collection of distinct objects, and is a fundamental object of mathematics.
$\emptyset = \{\}$, the empty set containing no objects, is included.

The objects in a set can be anything, for example integers, real numbers, or the objects may even themselves be sets.

$\in$	-	"is an element of" (set membership)
$\iff$	-	"if and only if" (equivalence)
$\implies$	-	"implies"
$\exists$	-	"there exists"
$\forall$	-	"for all"
s.t. *or* $\vert$	-	"such that"
wrt	-	"with respect to"

Table 4.1: Notations

4.1.2 Subsets, Complements and Singletons

If a set B contains all of the objects contained in another set A, and possibly some other objects besides, we say A is a **subset** of B and write $A \subseteq B$.

Suppose $A \subseteq B$ for two sets A and B. If we also have $B \subseteq A$ we write $A = B$, whereas if we know $B \nsubseteq A$ we write $A \subset B$.

The **complement** of a set A wrt a universal set Ω (say, of all "possible values") is
$\overline{A} = \{\omega \in \Omega | \omega \notin A\}$.

A **singleton** is a set with exactly one element — $\{\omega\}$ for some $\omega \in \Omega$.

4.2 Set operations

4.2.1 Unions and Intersections

Consider two sets A and B.

The **union** of A and B, $A \cup B = \{\omega \in \Omega | \omega \in A \text{ or } \omega \in B\}$.

The **intersection** of A and B, $A \cap B = \{\omega \in \Omega | \omega \in A \text{ and } \omega \in B\}$.

More generally, for sets $A_1, A_2, \ldots$ we define

$$\bigcup_i A_i = \{\omega \in \Omega | \exists i \text{ s.t. } \omega \in A_i\}$$
$$\bigcap_i A_i = \{\omega \in \Omega | \forall i, \omega \in A_i\}$$

If $A \cap B = \emptyset$, then we say the sets are **disjoint**, that is, the sets have no common element.

The sets $A_1, \ldots, A_k \subseteq \Omega$ form a partition of event $B \subseteq \Omega$ if

(a) $A_i \cap A_j = \emptyset, \quad \text{for } i \neq j,\ i, j = 1, \ldots, k \quad \text{(disjoint)}$

(b) $\bigcup_{i=1}^{k} A_i = B$

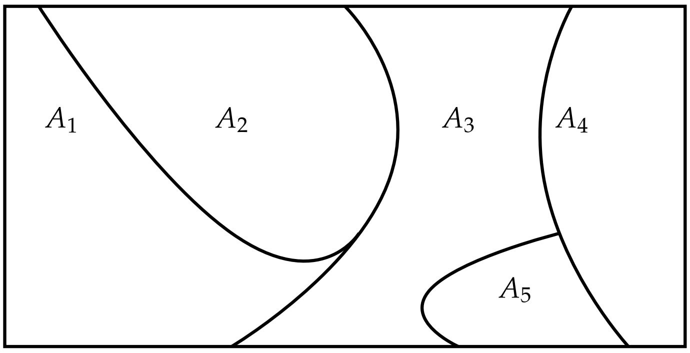

Figure 4.1: Partition of Ω

In Figure 4.1, we have $\Omega = \bigcup_{i=1}^{5} A_i$.

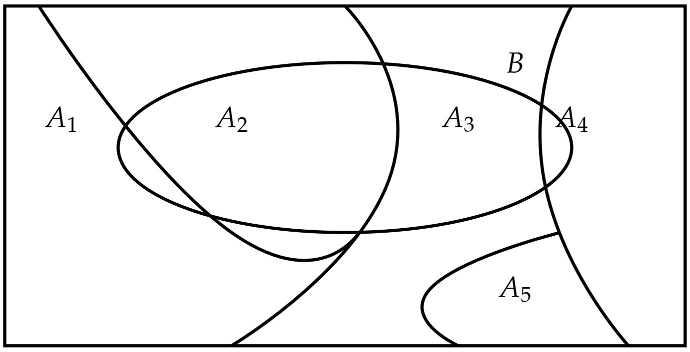

Figure 4.2: Partition of $B \subset \Omega$

In Figure 4.2, we have $B = \bigcup_{i=1}^{5} (A_i \cap B)$, but, for example, $B \cap A_5 = \emptyset$.

Properties of Union and Intersection Operators

Consider the sets $A, B, C \subseteq \Omega$

COMMUTATIVITY	$A \cup B = B \cup A$ $A \cap B = B \cap A$
ASSOCIATIVITY	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
DISTRIBUTIVITY	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
DE MORGAN'S LAWS	$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

The **difference** of A and B is $A \backslash B = A \cap \overline{B} = \{\omega \in \Omega | \omega \in A \text{ and } \omega \notin B\}$.

Example Let $\Omega = \{1,2,3,4,5,6,7,8,9,10\}$ be our universal set and $A = \{1,2,3,4,5,6\}$ and $B = \{5,6,7,8,9\}$ be two sets of elements of Ω.

- $A \cup B = \{1,2,3,4,5,6,7,8,9\}$
- $A \cap B = \{5,6\}$
- $A \backslash B = \{1,2,3,4\}$
- $\overline{(A \cup B)} = \{10\}$
- $\overline{(A \cap B)} = \{1,2,3,4,7,8,9,10\}$
- $\overline{A} = \{7,8,9,10\}, \quad \overline{B} = \{1,2,3,4,10\}$
- $\overline{A} \cap \overline{B} = \{10\}$
- $\overline{A} \cup \overline{B} = \{1,2,3,4,7,8,9,10\}$

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4.2.2 Cartesian Products

For two sets Ω_1, Ω_2, their **Cartesian product** is the set of all ordered pairs of their elements. That is,

$$\Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) | \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$$

More generally, the Cartesian product for sets $\Omega_1, \Omega_2, \ldots$ is written $\prod_i \Omega_i$.

4.3 Cardinality

A useful *measure* of a set is the size, or **cardinality**. The cardinality of a finite set is simply the number of elements it contains. For infinite sets, there are again an *infinite* number of different cardinalities they can take. However, amongst these there is a most important distinction: Between those which are **countable** and those which are not.

A set Ω is countable if $\exists$ a function $f : \mathbb{N} \to \Omega$ s.t. $f(\mathbb{N}) \supseteq \Omega$. That is, the elements of Ω can satisfactorily be written out as a possibly unending list $\{\omega_1, \omega_2, \omega_3, \ldots\}$. Note that all finite sets are countable.

A set is **countably infinite** if it is countable but not finite. Clearly $\mathbb{N}$ is countably infinite. So is $\mathbb{N} \times \mathbb{N}$.

A set which is not countable is **uncountable**. For instance, $\mathbb{R}$ is uncountable.

The empty set $\emptyset$ has zero cardinality,

$$|\emptyset| = 0.$$

For finite sets A and B, if A and B are disjoint (that is $A \cap B = \emptyset$), then

$$|A \cup B| = |A| + |B|$$

otherwise,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Summary of Notation

Ω	universal set
$\emptyset$	empty set
$A \subseteq \Omega$	Subset of Ω
$\overline{A}$	Complement of A
$\lvert A\rvert$	Cardinality (or size) of A
$A \cup B$	union (A or B)
$A \cap B$	intersection (A and B)
$A \subset B$	set inclusion (elements of A are also in B)
A/B	set difference (elements in A that are not in B)