Calculus

COMP40016

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1. (a) Let S be a non-empty set of real numbers. Prove that the real number α is the supremum of S if and only if both the following conditions are satisfied: [4]

   1. $x \leq \alpha$ for all $x \in S$:
   2. for every $\epsilon > 0$, there is some $x \in S$ such that $\alpha - \epsilon < x' \leq \alpha$.

   Solution: Condition 1 ensures that α is an upped bound of S (1 mark) and condition 2 guarantees that any number $\alpha - \epsilon$ (which is less than α) is not an upper bound (1 mark). Hence α is the least of the upper bounds (1 mark). It follows that if α satisfies 1 and 2 then α is the supremum of S.
   Conversely, if α is the supremum of S, then α must satisfy 1 and 2 (1 mark).

   (b) i. Let $a_n = n^2 + n\cos n\pi$. Show that $a_n \to \infty$ as $n \to \infty$. [2]

   Solution: We observe that

   $$a_n = n^2 + (-1)^n n \geq n^2 - n \geq n^2 - \frac{1}{2}n^2 = \frac{1}{2}n^2 \quad \text{for } (n \geq 2) \qquad \text{(1 mark)}$$

   Write $b_n = \frac{1}{2}n^2$. Then $b_n \to \infty$ and hence $a_n \to \infty$ as $n \to \infty$ (1 mark).

   ii. Let $a_n = \frac{n^2+\sqrt{n}}{n+\cos n}$. Show that $a_n \to \infty$ as $n \to \infty$. [2]

   Solution: For $n > 1$,

   $$a_n > \frac{n^2}{n+\cos n} \geq \frac{n^2}{n+1} > \frac{n^2}{n+n} = \frac{1}{2}n \qquad \text{(1 mark)}$$

   Write $b_n = \frac{1}{2}n$. Since $a_n > b_n (n > 1)$ and $b_n \to \infty$ as $n \to \infty$, we see that $a_n \to \infty$ as $n \to \infty$ (1 mark).

   iii. Let $a_n = \frac{n^2+n+1}{2n^2+1}$. Show that $a_n \to \frac{1}{2}$ as $n \to \infty$. [2]

   Solution: We notice that

   $$a_n - \frac{1}{2} = \frac{n^2+n+1}{2n^2+1} - \frac{1}{2} = \frac{2n+1}{2\left(2n^2+1\right)}.$$

   Hence

   $$\left|a_n - \frac{1}{2}\right| < \frac{2n+1}{2\cdot 2n^2} < \frac{2n+n}{2\cdot 2n^2} = \frac{3}{4n} \qquad \text{for } n > 1 \qquad \text{(1 mark)}$$

   Therefore, given $\epsilon > 0$, $\left|a_n - \frac{1}{2}\right| < \frac{3}{4n} < \epsilon$ for all $n > \max\left\{1, \frac{3}{4\epsilon}\right\}$. Thus if N is chosen to be the smallest integer such that $N \geq \max\left\{1, \frac{3}{4\epsilon}\right\}$, then $\left|a_n - \frac{1}{2}\right| < \epsilon$ for all $n \geq N$ (1 mark).

   (c) i. Evaluate [2]

   $$\lim_{x\to 1} \frac{2x^2 - 3x + 4}{x^3 + 5x + 1}$$

explaining how you arrived at the result.

Solution: Since $\lim_{x\to 1} x = 1$, the product rule gives $\lim\limits_{x\to 1} x^2 = 1$ and $\lim\limits_{x\to 1} x^3 = 1$. Since the constant function $f(x) = k$, for all x, is such that $\lim\limits_{x\to 1} f(x) = k$, the sum and product rules give

$$\lim_{x\to 1}\left(2x^2 - 3x + 4\right) = 3$$

and

$$\lim_{x\to 1}\left(x^2 + 5x + 1\right) = 7$$

(1 mark).
Finally, the quotient rule gives

$$\lim_{x\to 1}\frac{2x^2 - 3x + 4}{x^3 + 5x + 1} = \frac{3}{7}$$

(1 mark).

ii. Show that the function $f(x) = 3x + 7$ is uniformly continuous on $\mathbb{R}$. [2]

Solution: Choose $\epsilon > 0$. Let $\delta = \frac{\epsilon}{3}$. Choose $x_0 \in \mathbb{R}$. Choose $x \in \mathbb{R}$. Assume $|x - x_0| < \delta$ (1 mark).
Then

$$|f(x) - f(x_0)| = |(3x+7) - (3x_0 + 7)| = 3|x - x_0| < 3\delta = \epsilon$$

(1 mark).

(d) i. Show that [2]

$$\sum_{r=1}^{\infty}\frac{1}{r^2 + r} = 1$$

Solution: Since

$$\frac{1}{r^2 + r} = \frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$$

(1 mark). the nth partial sum of the given series can be written as

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Now $S_n \to 1$ as $n \to \infty$, as required (1 mark).

ii. Prove the so-called *vanishing condition*: if $\sum\limits_{r=1}^{\infty} a_r$ is convergent, then $a_n \to 0$ as $n \to \infty$ [1]

Solution: Suppose that (S_n) converges to some limit S. Hence (S_{n-1}) also converges to S. But $a_n = S_n - S_{n-1}$, and so $a_n \to 0$ as $n \to \infty$ (1 mark).

iii. By considering $\sum_{r=1}^{\infty}(\sqrt{r} - \sqrt{r-1})$, show that the converse (reverse implication) of the vanishing condition is false: in other words, show that $\sum_{r=1}^{\infty} a_r$ may still be divergent even if $a_n \to 0$ as $n \to \infty$ [3]

Solution: Consider $\sum_{r=1}^{\infty}(\sqrt{r} - \sqrt{r-1})$. The nth partial sum may be written as

$$S_n = (\sqrt{1} - \sqrt{0}) + (\sqrt{2} - \sqrt{1}) + \ldots + (\sqrt{n} - \sqrt{n-1}) = \sqrt{n}$$

(1 mark). Clearly (S_n) is a divergent sequence and so

$$\sum_{r=1}^{\infty}(\sqrt{r}-\sqrt{r-1})$$

is a divergent series (1 mark).
However,

$$\begin{aligned} a_n &= \sqrt{n}-\sqrt{n-1} \\ &= \frac{(\sqrt{n}-\sqrt{n-1})(\sqrt{n}+\sqrt{n-1})}{\sqrt{n}+\sqrt{n-1}} \\ &= \frac{1}{\sqrt{n}+\sqrt{n-1}} \to 0 \end{aligned}$$

as $n \to \infty$ (1 mark).

2. (a) i. Use the limit definition of differentiability to find the derivative of $f(x) = \frac{1}{x}$ [2]

Solution: From first principles,

$$\begin{aligned} \frac{f(x)-f(c)}{x-c} &= \frac{1/x-1/c}{x-c} \\ &= \frac{c-x}{xc(x-c)} \\ &= -\frac{1}{xc} \end{aligned}$$

for $x \neq c$ (1 mark).
Hence

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = -\frac{1}{c^2}$$

and so f is differentiable at c and $f'(c) = -\frac{1}{c^2}$ (1 mark).

ii. Show that $\lim\limits_{x\to 0}(1+x)^{1/x} = e$. [2]

Solution: We note that

$$\ln\left[\lim_{x\to 0}(1+x)^{1/x}\right] = \lim_{x\to 0}\left[\ln(1+x)^{1/x}\right].$$

By L'Hôpital's rule,

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{1}{1+x} = 1$$

(1 mark). Hence

$$\lim_{x\to 0}(1+x)^{1/x} = \exp\left\{\lim_{x\to 0}\left[\ln(1+x)^{1/x}\right]\right\} = e$$

by the continuity of log and exp (1 mark).

(b) For $x > 0$ define [4]

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$$

Prove that $\Gamma(n+1) = n\,!$ for $n = 0, 1, 2, \ldots$.

Solution: First, for $x \geq 0$,

$$\int_0^b t^x e^{-t} dt = \left[-t^x e^{-t}\right]_0^b + x \int_0^b t^{x-1} e^{-1} dt$$

by integration by parts (1 mark).
Since $\lim_{t\to\infty} t^x e^{-t} = 0$, it follows that $\Gamma(x+1) = x\Gamma(x)$ (1 mark).
By induction on $n, \Gamma(n+1) = n!\Gamma(1)$ (1 mark).
But $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$. Hence $\Gamma(n+1) = n!$ (1 mark).

(c) i. Find the Taylor series for [2]

$$f(x) = (x+1)e^x,$$

$x \in \mathbb{R}$, about the point $x = 1$.

Solution: We have

$$f(x+1) = (x+2)e^{x+1} = \sum_{n=0}^{\infty} \frac{e(n+2)}{n!} x^n,$$

$$x \in \mathbb{R} \text{ (1 mark).}$$

Hence

$$f(x) = \sum_{n=0}^{\infty} \frac{e(n+2)}{n!} (x-1)^n,$$

$$x \in \mathbb{R} \text{ (1 mark).}$$

ii. Evaluate [2]

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{(2n+1)!}$$

Solution: For $x \in \mathbb{R}$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n n}{(2n+1)!} x^{2n+1} &= \frac{1}{2} x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\ &= \frac{1}{2}(x \cos x - \sin x) \end{aligned}$$

(1 mark).
Putting $x = 1$, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{(2n+1)!} = \frac{1}{2}(\cos 1 - \sin 1)$$

(1 mark).

(d) Consider the problem [4]

$$\frac{dy}{dx} = f(x, y)$$

with $y = y_0$ at $x = x_0$. The iterative scheme

$$\begin{aligned} x_{n+1} &= x_n + h, \\ y_{n+1} &= y_n + hf(x_n, y_n) \end{aligned}$$

is known as *Euler's method* (here h is the step size).
It can be shown that, for a considerable range of values of h, the error produced by this method is proportional to h. Explain why this is the case.

Solution: The Taylor series expansion for the function $y(x)$ about the point x_n is the infinite series

$$y(x) = y_n + \frac{(x-x_n)}{1!}y_n' + \frac{(x-x_n)^2}{2!}y_n'' + \dots$$

which can be written

$$y(x) = \sum_{i=0}^{m} \frac{(x-x_n)^i}{i!}y_n^{(i)} + \frac{(x-x_n)^{m+1}}{(m+1)!}y^{(m+1)}(\xi),$$

where $x_n \le \xi \le x$ and where the superscript in parenthesis denotes differentiation (1 mark). Euler's method retains only the first two terms in the series, and the final term indicates the magnitude of the local truncation error, i.e. the error incurred for each step along the x-axis (1 mark).

If we replace x by $x_n + h$, then $y(x_n + h)$, which may be written $y(x_{n+1})$ and also y_{n+1}, is given by

$$y(x_n + h) = y_n + hy_n' + \frac{h^2}{2!}y''(\xi) \qquad \text{(1 mark)}$$

The value of ξ at which the remainder term should be evaluated is not known. At best, therefore, we could compute an upper bound for the error by estimating the maximum value of $y''(\xi)$ over the range $x_n \le \xi \le x_{n+1}$. For present purposes it is sufficient to assume, over a small range of values of x, that y'' is approximately constant, and therefore that the local error - the error per integration step - is proportional to h^2 (1 mark).

(Since, for a given range of x over which the integration is to be performed, the number of integration steps is proportional to $1/h$, the total or global error at the end of the range is proportional to h.)

(e) The space X consists of all the sequences of real numbers $x = (\xi_1, \xi_2, \dots, \xi_n, \dots)$. Let $y = (\eta_1, \eta_2, \dots, \eta_n, \dots) \in X$ be another such sequence. Then we define [4]

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n - \eta_n|}{1 + |\xi_n - \eta_n|}.$$

Prove that (X, ρ) is a metric space.

Solution: To prove the triangle inequality, it suffices to show that for any real numbers a, b,

$$\frac{|a+b|}{1+|a+b|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \qquad \text{(1 mark)}$$

Suppose a and b have the same sign, say both are positive. Then

$$\frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} < \frac{a}{1+a} + \frac{b}{1+b} \qquad \text{(1 mark)}$$

Suppose next that a and b have different signs. We may assume that $|a| \ge |b|$, so that $|a+b| \le |a|$. Since the function $x/(1+x)$ is monotone increasing for $x > 0$ (1 mark), we get

$$\frac{|a+b|}{1+|a+b|} \le \frac{|a|}{1+|a|} \le \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|} \qquad \text{(1 mark)}$$

The proofs of the remaining axioms are trivial.