Risk and Decisions

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Introduction to Artificial Intelligence

The lectures

- The agent and the world (**Knowledge Representation**)
  - Actions and knowledge
  - Inference
- Good decisions (**Risk and Decisions**)
  - Chance
  - Gains
- Good decisions in time (**Markov Decision Processes**)
  - Chance and gains in time
  - Patience
  - Finding the best strategy
- Learning from experience (**Reinforcement Learning**)
  - Finding a reasonable strategy

Risk and Decisions

Knowing what to expect

Today

- Probabilities: a crash course
- Bayes' rule and conditional independence
- Back to the Wumpus World

The book

Stuart Russell and Peter Norvig
Artificial Intelligence: a modern approach
Chapters 13-14

Holiday

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How worried should you be?

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$w \in \Omega$ is a sample point/possible world/atomic event

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e.g., $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$.

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E.g.,
$P(\text{dice roll} < 4) = P(1) + P(2) + P(3) = 1/6 + 1/6 + 1/6 = 1/2$

Random variables

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Bayes' Rule

Theorem (Bayes' Rule)

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

Bayes' Rule

Useful for assessing causal probability from diagnostic probability:

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E.g., let c be cold, s be sore throat:

$$P(c|s) = \frac{P(s|c)P(c)}{P(s)} = \frac{0.9 \times 0.001}{0.005} = 0.18$$

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$$\mathbf{P}(C|s) = \alpha \langle P(s|c)P(c), P(s|\neg c)P(\neg c) \rangle$$

Bayes' rule with random variables

Theorem (Bayes' rule with random variables)

$$\mathbf{P}(X|Y) = \alpha\,\mathbf{P}(Y|X)\mathbf{P}(X)$$

Holiday solved

You are back from a holiday on an exotic island, and your doctor has bad news and good news. The bad news is that you've been diagnosed a serious disease and the test is 99% accurate. The good news is that the disease is very rare (1 in 10.000 get it).

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Notice:the posterior probability of disease is still very small!

Independence

A and B are independent iff
$\mathbf{P}(A|B) = \mathbf{P}(A)$ or $\mathbf{P}(B|A) = \mathbf{P}(B)$ or $\mathbf{P}(A, B) = \mathbf{P}(A)\mathbf{P}(B)$

$$
\begin{aligned}
& P(cavity) \\
= {} & P(cavity|\text{Weather}) \\
= {} & P(cavity|\text{CristianoRonaldoscores})
\end{aligned}
$$

Combining evidence

Start with the joint distribution:

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	.108	.012	.072	.008
$\neg$ *cavity*	.016	.064	.144	.576

$\mathbf{P}(Cavity|toothache \wedge catch) =$

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$$\mathbf{P}(Cavity|toothache \wedge catch) = \alpha \langle 0.108, 0.016 \rangle = \langle 0.871, 0.129 \rangle$$

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	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	.108	.012	.072	.008
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- It doesn't scale up to a large number of variables
- Can we simplify?

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We can't use absolute independence:

- Toothache and Catch are **not** independent: If the probe catches in the tooth then it is likely the tooth has a cavity, which means that toothache is likely too.
- But they are independent **given** the presence or the absence of cavity! Toothache depends on the state of the nerves in the tooth, catch depends on the dentist's skills, to which toothache is irrelevant

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1. $P(catch|toothache, cavity) = P(catch|cavity)$, the same independence holds if I haven't got a cavity:

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2. $P(catch|toothache, \neg cavity) = P(catch|\neg cavity)$

Catch is conditionally independent of *Toothache* given *Cavity*:

Conditional independence

$$\mathbf{P}(Catch|Toothache, Cavity) = \mathbf{P}(Catch|Cavity)$$

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Equivalent statements:

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Equivalent statements:

$\mathbf{P}(Toothache | Catch, Cavity) = \mathbf{P}(Toothache | Cavity)$
$\mathbf{P}(Toothache, Catch | Cavity) =$
$\mathbf{P}(Toothache | Cavity)\mathbf{P}(Catch | Cavity)$

Conditional independence contd.

Write out full joint distribution using chain rule:

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&= \mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity)
\end{aligned}$$

I.e., $2 + 2 + 1 = 5$ independent numbers (first and second steps remove two). Else 8-1=7. The gain is bigger the more the combinations.

Conditional independence contd.

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Conditional independence is our most basic and robust form of knowledge about uncertain environments.

Bayes' Rule and conditional independence

$\mathbf{P}(Cavity|toothache \land catch)$

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This is an example of a naive Bayes model:

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause)\prod_i \mathbf{P}(Effect_i|Cause)$$

Bayes' Rule and conditional independence

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\begin{aligned}
&\mathbf{P}(Cavity \mid toothache \wedge catch) \\
&\quad = \alpha\, \mathbf{P}(toothache \wedge catch \mid Cavity)\mathbf{P}(Cavity) \\
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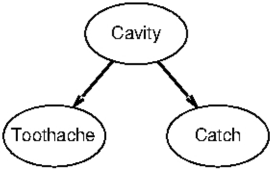

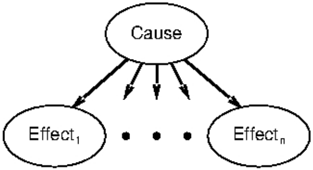

The Wumps World

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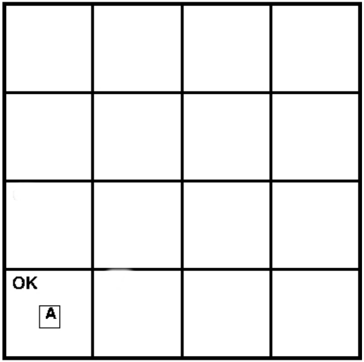

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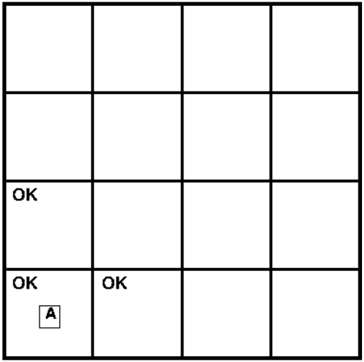

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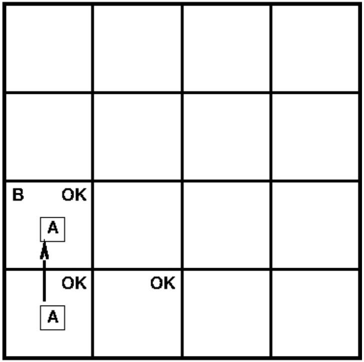

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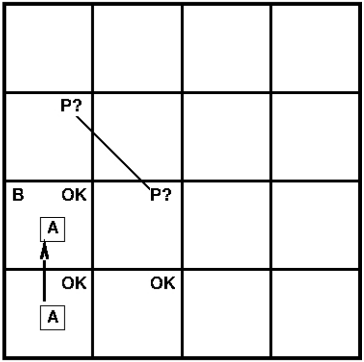

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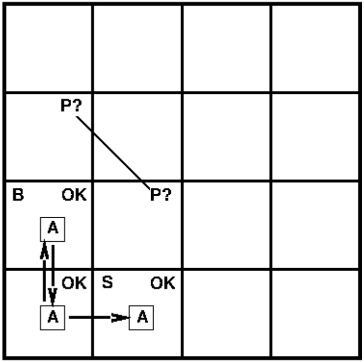

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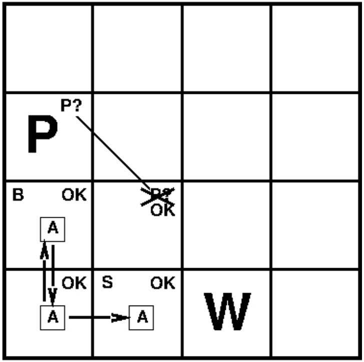

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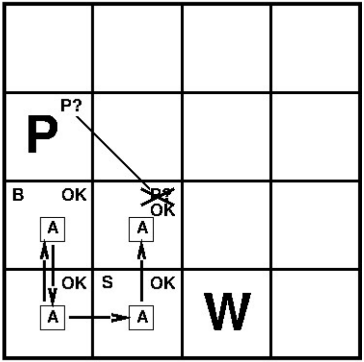

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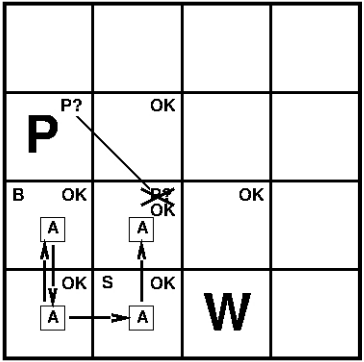

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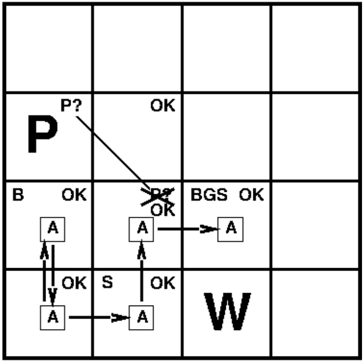

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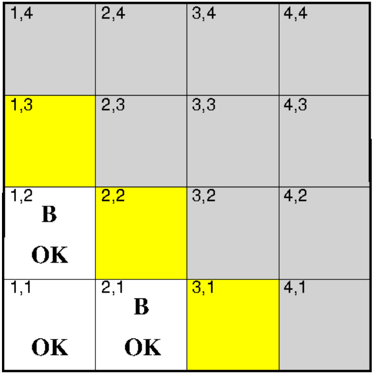

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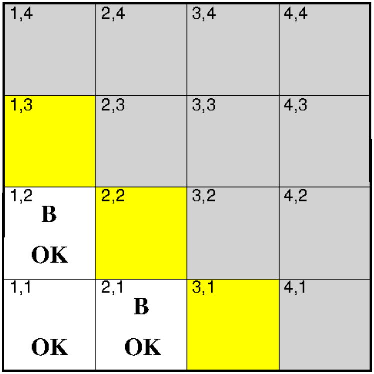

$P_{ij} = true$ iff $[i, j]$ contains a pit
$B_{ij} = true$ iff $[i, j]$ is breezy

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Apply product rule:
$\mathbf{P}(B_{1,1}, B_{1,2}, B_{2,1} \mid P_{1,1}, \ldots, P_{4,4})\mathbf{P}(P_{1,1}, \ldots, P_{4,4})$
(Do it this way to get $P(Effect|Cause)$.)

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$$\mathbf{P}(P_{1,1}, \ldots, P_{4,4}) = \prod_{i,j=1,1}^{4,4} \mathbf{P}(P_{i,j}) = 0.2^n \times 0.8^{16-n}$$

for n pits.

Observations and query

We know the following facts:

$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

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$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$

$explored = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$

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Query is $\mathbf{P}(P_{1,3} | explored, b)$

Define $Unexplored = P_{ij}$s other than $P_{1,3}$ and $Explored$

Complexity

For inference by enumeration, we have

$$\mathbf{P}(P_{1,3}|explored, b) = \alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

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- There are 12 unknown squares
- The summation contains $2^{12} = 4096$ terms

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In general the summation grows exponentiatlly with the number of squares!

Complexity

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$$\mathbf{P}(P_{1,3}|explored, b) = \alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

- There are 12 unknown squares
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And now?

Using conditional independence

Basic insight: observations are conditionally independent of other hidden squares given neighbouring hidden squares

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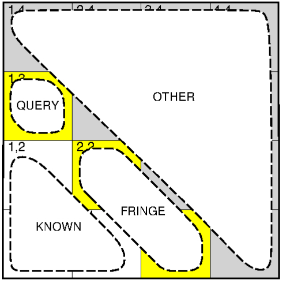

Using conditional independence

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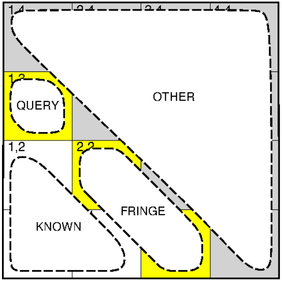

Define *Unexplored* $= Fringe \cup Other$

Using conditional independence

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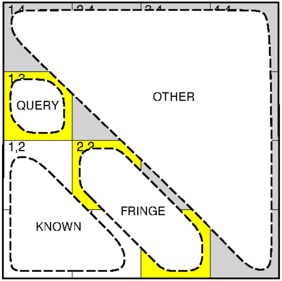

Define $\mathit{Unexplored} = \mathit{Fringe} \cup \mathit{Other}$
$\mathbf{P}(b|P_{1,3}, \mathit{Explored}, \mathit{Unexplored}) = \mathbf{P}(b|P_{1,3}, \mathit{Explored}, \mathit{Fringe})$

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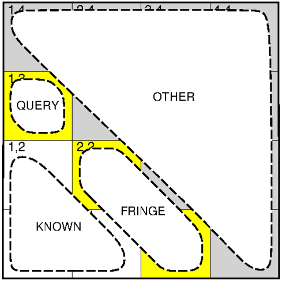
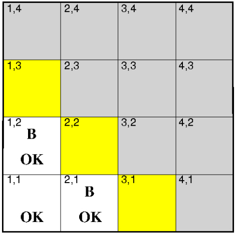


Define $Unexplored = Fringe \cup Other$

$\mathbf{P}(b|P_{1,3}, Explored, Unexplored) = \mathbf{P}(b|P_{1,3}, Explored, Fringe)$

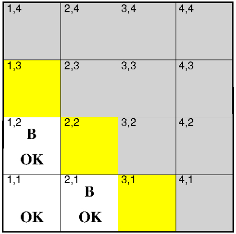
Manipulate query into a form where we can use this!

Using conditional independence



$\mathbf{P}(P_{1,3}|explored, b)$

Using conditional independence



$$\mathbf{P}(P_{1,3}|explored, b) = \alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

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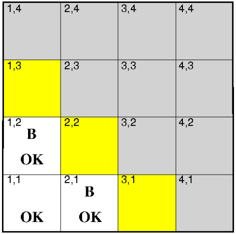

Inference by enumeration

$$\mathbf{P}(P_{1,3}|explored, b) = \alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

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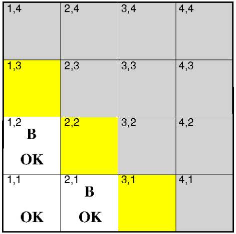

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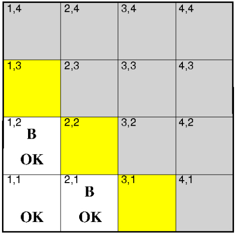
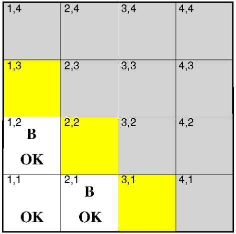


$$\alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

$$= \alpha \sum_{unexplored} \mathbf{P}(b | explored, P_{1,3}, unexplored) \times$$

$$\times \mathbf{P}(P_{1,3}, explored, unexplored)$$

Using conditional independence



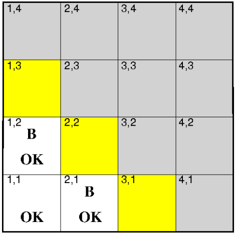
Product rule

$$\alpha \sum_{unexplored} \mathbf{P}(P_{1,3}, unexplored, explored, b)$$

$$= \alpha \sum_{unexplored} \mathbf{P}(b | explored, P_{1,3}, unexplored) \times$$

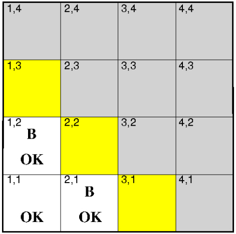
$$\times \mathbf{P}(P_{1,3}, explored, unexplored)$$

Using conditional independence



$$\alpha \sum_{unexplored} \mathbf{P}(b | P_{1,3}, unexplored, explored)\mathbf{P}(P_{1,3}, unexplored, explored)$$

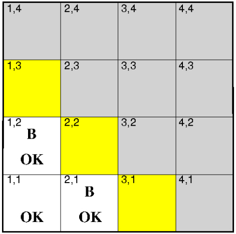
Using conditional independence



$$\alpha \sum_{unexplored} \mathbf{P}(b|P_{1,3}, unexplored, explored)\mathbf{P}(P_{1,3}, unexplored, explored)$$

$$= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe, other) \times$$
$$\times \mathbf{P}(P_{1,3}, explored, fringe, other)$$

Using conditional independence



Distinguishing the unknown

$$\alpha \sum_{unexplored} \mathbf{P}(b|P_{1,3}, unexplored, explored)\mathbf{P}(P_{1,3}, unexplored, explored)$$

$$= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe, other) \times$$
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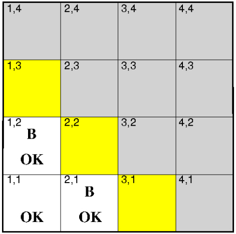
Using conditional independence



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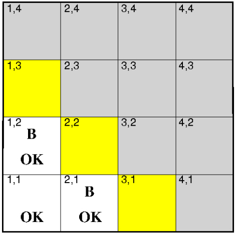

$$\alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe, other) \times$$
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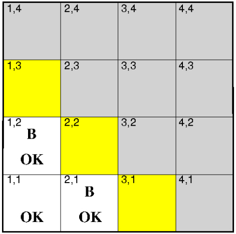

Conditional Independence

$$\alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe, other) \times$$
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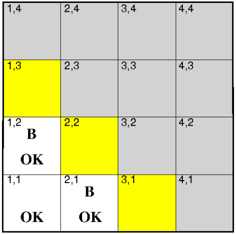
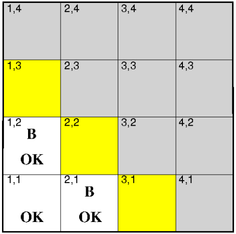


$$\alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe)\times$$
$$\times \mathbf{P}(P_{1,3}, explored, fringe, other)$$

Using conditional independence



$$\alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
$$\times \mathbf{P}(P_{1,3}, explored, fringe, other)$$

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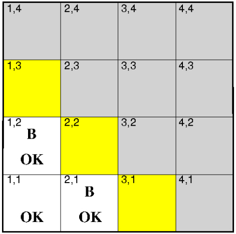
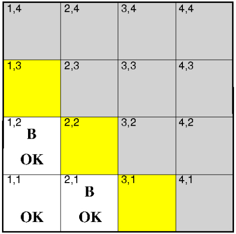


Pushing the sums inwards

$$\alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
$$\times \mathbf{P}(P_{1,3}, explored, fringe, other)$$

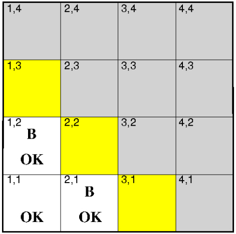
$$= \alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
$$\times \sum_{other} \mathbf{P}(P_{1,3}, explored, fringe, other)$$

Using conditional independence



$$\alpha \sum_{fringe} \mathbf{P}(b | explored, P_{1,3}, fringe) \times$$
$$\sum_{other} \mathbf{P}(P_{1,3}, explored, fringe, other)$$

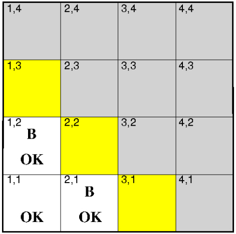
Using conditional independence



$$\alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times \sum_{other} \mathbf{P}(P_{1,3}, explored, fringe, other)$$

$$= \alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times \sum_{other} \mathbf{P}(P_{1,3}) P(explored) P(fringe) P(other)$$

Using conditional independence

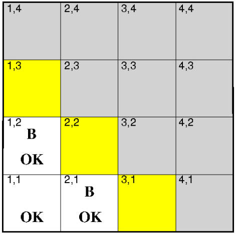


Independence

$$\alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times \sum_{other} \mathbf{P}(P_{1,3}, explored, fringe, other)$$

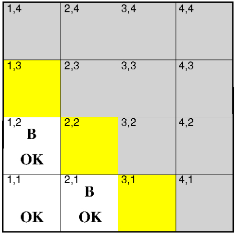
$$= \alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times \sum_{other} \mathbf{P}(P_{1,3}) P(explored) P(fringe) P(other)$$

Using conditional independence



$$\alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
$$\times \sum_{other} \mathbf{P}(P_{1,3}) P(explored) P(fringe) P(other)$$

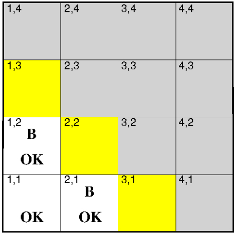
Using conditional independence



$$\alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
$$\times \sum_{other} \mathbf{P}(P_{1,3}) P(explored) P(fringe) P(other)$$

$$= \alpha P(explored) \mathbf{P}(P_{1,3}) \times$$
$$\times \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) P(fringe) \sum_{other} P(other)$$

Using conditional independence

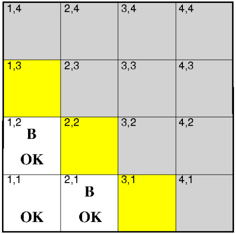


Reordering
and pushing sums inwards

$$\alpha \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) \times$$
$$\times \sum_{other} \mathbf{P}(P_{1,3}) P(explored) P(fringe) P(other)$$

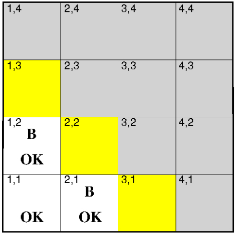
$$= \alpha\, P(explored) \mathbf{P}(P_{1,3}) \times$$
$$\times \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe) P(fringe) \sum_{other} P(other)$$

Using conditional independence



$$\alpha\, P(explored)\mathbf{P}(P_{1,3})\times$$
$$\times \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)P(fringe) \sum_{other} P(other)$$

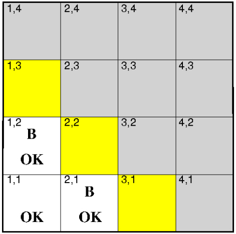
Using conditional independence



$$\alpha\, P(explored)\mathbf{P}(P_{1,3})\times$$
$$\times \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)P(fringe) \sum_{other} P(other)$$

$$= \alpha'\, \mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)P(fringe)$$

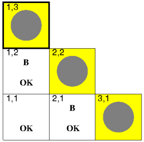
Using conditional independence



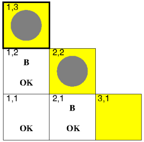
Simplifying

$$\alpha\, P(explored)\mathbf{P}(P_{1,3})\times$$
$$\times \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)P(fringe) \sum_{other} P(other)$$
$$= \alpha'\, \mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b|explored, P_{1,3}, fringe)P(fringe)$$

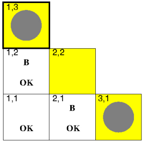
Using conditional independence contd.



0.2 x 0.2 = 0.04



0.2 x 0.8 = 0.16



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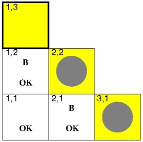

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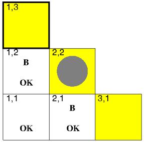
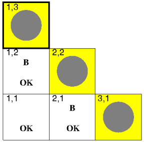


0.2 x 0.8 = 0.16

$\mathbf{P}(b|explored, P_{1,3}, fringe)$

Using conditional independence contd.



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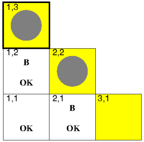

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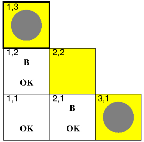

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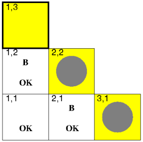

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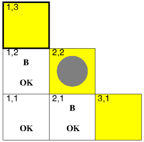
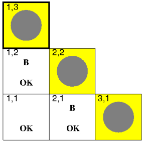


0.2 x 0.8 = 0.16

$\mathbf{P}(b|explored, P_{1,3}, fringe)$

- $= 1$ when the frontier is consistent with the observations
- $= 0$ otherwise

Using conditional independence contd.



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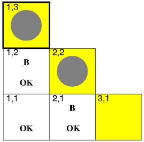

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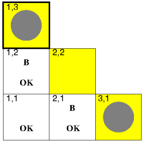

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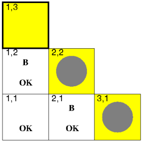

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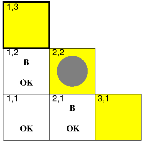


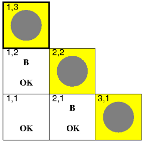
0.2 x 0.8 = 0.16

$\mathbf{P}(b|explored, P_{1,3}, fringe)$

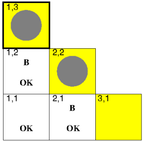
- $= 1$ when the frontier is consistent with the observations
- $= 0$ otherwise

We can sum over the *possible configurations* for the frontier variables that are consistent with the known facts.

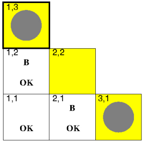
Using conditional independence contd.



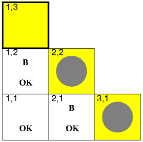
0.2 x 0.2 = 0.04



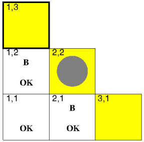
0.2 x 0.8 = 0.16



0.8 x 0.2 = 0.16



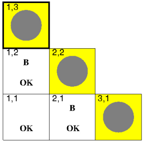
0.2 x 0.2 = 0.04



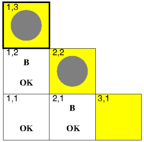
0.2 x 0.8 = 0.16

$$\mathbf{P}(P_{1,3}|explored, b) =$$

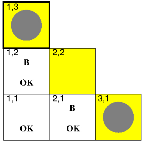
Using conditional independence contd.



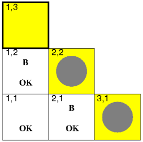
0.2 x 0.2 = 0.04



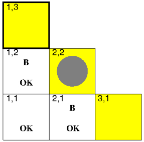
0.2 x 0.8 = 0.16



0.8 x 0.2 = 0.16



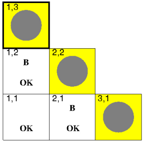
0.2 x 0.2 = 0.04



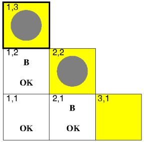
0.2 x 0.8 = 0.16

$$\mathbf{P}(P_{1,3}|explored, b) = \alpha' \langle 0.2(0.04 + 0.16 + 0.16),\ 0.8(0.04 + 0.16)\rangle$$

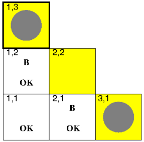
Using conditional independence contd.



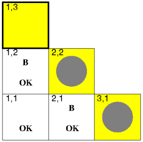
0.2 x 0.2 = 0.04



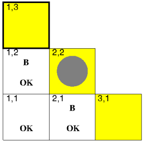
0.2 x 0.8 = 0.16



0.8 x 0.2 = 0.16



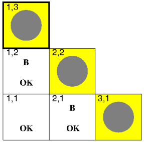
0.2 x 0.2 = 0.04



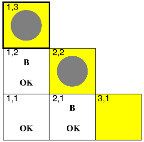
0.2 x 0.8 = 0.16

$$\mathbf{P}(P_{1,3}|explored, b) =$$
$$\alpha' \langle 0.2(0.04 + 0.16 + 0.16),\ \ 0.8(0.04 + 0.16) \rangle$$
$$\approx \langle 0.31, 0.69 \rangle$$

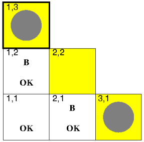
Using conditional independence contd.



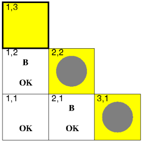
0.2 x 0.2 = 0.04



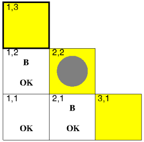
0.2 x 0.8 = 0.16



0.8 x 0.2 = 0.16



0.2 x 0.2 = 0.04

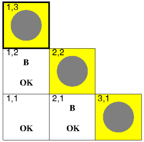


0.2 x 0.8 = 0.16

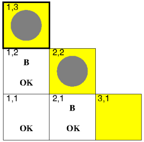
$$\mathbf{P}(P_{1,3}|explored, b) = \alpha' \, \langle 0.2(0.04+0.16+0.16),\ 0.8(0.04+0.16) \rangle \approx \langle 0.31, 0.69 \rangle$$

$$\mathbf{P}(P_{2,2}|explored, b) \approx \langle 0.86, 0.14 \rangle$$

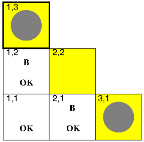
Using conditional independence contd.



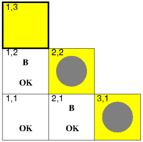
0.2 x 0.2 = 0.04



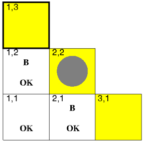
0.2 x 0.8 = 0.16



0.8 x 0.2 = 0.16



0.2 x 0.2 = 0.04



0.2 x 0.8 = 0.16

$$\mathbf{P}(P_{1,3}|explored, b) = \\ \alpha' \langle 0.2(0.04 + 0.16 + 0.16),\ \ 0.8(0.04 + 0.16) \rangle \\ \approx \langle 0.31, 0.69 \rangle$$

$$\mathbf{P}(P_{2,2}|explored, b) \approx \langle 0.86, 0.14 \rangle$$

Today's class

- Probabilities and conditional probabilities
- Independence and conditional independence
- Estimating chances of possible outcomes

Coming next

- Combining chances and rewards
- Maximising the expected reward