Q 1

a) standardising, we get

$$\min\ z = -x_1 - x_2$$

$$\text{st} \begin{cases} -x_1 + x_2 + x_3 = 2 \\ -2x_1 + 3x_2 - x_4 = 1 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases} \quad \text{— ①}$$

Adding an artificial variable ξ_1 to the LHS of ①, we solve the auxiliary LP by minimizing

$$Z = \xi_1 = 1 + 2x_1 - 3x_2 + x_4 .$$

Initial basic representation:

	x_1	x_2	x_3	x_4	ξ_1	RHS	Ratio
Z	-2	3		-1		1	
x_3	-1	1	1			2	2
ξ_1	-2	3		-1	1	1	1/3
Z					-1	0	
x_3	-1/3		1	1/3	-1/3	5/3	
x_2	-2/3	1		-1/3	1/3	1/3	

∴ LP feasible ⇒ ~~start ph~~ start phase 2 using index set $I = \{2, 3\}$

We want to minimise $z = -x_1 - x_2 = -x_1 - \left(\frac{1}{3} + \frac{2}{3}x_1 + \frac{1}{3}x_4\right)$

$$= -\frac{1}{3} - \frac{5}{3}x_1 - \frac{1}{3}x_4$$

	x_1	x_2	x_3	x_4	RHS	Ratio
z	5/3			1/3	-1/3	
x_3	-1/3		1	1/3	5/3	
x_2	-2/3	1		-1/3	1/3	

Now the coefficient of x_1 in the objective row is $5/3 > 0$, but its coefficient in the x_3 & x_2 rows are negative

⇒ by the simplex algorithm, we know that the LP is unbounded.

∴ No optimal solution exists for the LP.

(max of y tends to ∞).

b) Let $x_a = \sum_{k=1}^{N} \alpha_k x^{(k)}$ be an arbitrary weighted average of the N optimal basic feasible solutions (BFS).

$\boxed{\text{Feasibility}}$

Every $x^{(k)}$ satisfies $x^{(k)} \geq 0$ & $Ax^{(k)} = b$ $\forall k \in \{1, \dots, N\}$

$$\Rightarrow \quad Ax_a = A \sum_{k=1}^{N} \alpha_k x^{(k)} = \sum_{k=1}^{N} \alpha_k \left[A x^{(k)}\right]$$

$$\cancel{\alpha_k \sum} \quad = \sum_{k=1}^{N} \alpha_k b$$

$$= b \quad \text{since } \left(\sum \alpha_k = 1\right).$$

Also, $x^{(k)} \geq 0 \Rightarrow \alpha_k x^{(k)} \geq 0$ $\forall k \in \{1, \dots, N\}$ since $\alpha_k \geq 0$.

$$\Rightarrow x_a = \sum_{k=1}^{N} \alpha_k x^{(k)} \geq 0$$

∴ x_a is in the feasible set of the LP.

$\boxed{\text{Optimality}}$

We know the optimal objective value is $c^T x^{(1)} = c^T x^{(2)} = \dots = c^T x^{(N)}$.

Now $c^T x_a = c^T \sum_{k=1}^{N} \alpha_k x^{(k)} = \sum_{k=1}^{N} \alpha_k \left[c^T x^{(k)}\right]$

$$\cancel{\alpha_k \sum c^T x^{(k)}} \quad = \sum_{k=1}^{N} \alpha_k \left[c^T x^{(1)}\right]$$

$$= c^T x^{(1)} \quad \left(\text{since } \sum \alpha_k = 1\right)$$

∴ objective value corresponding to x_a is the same as optimal objective value $\Rightarrow$ x_a is an optimal solution!

∴ Shown that ~~any~~ an arbitrary ~~[illegible]~~ x_a is both feasible & optimal $\Rightarrow$ every such x_a is feasible & optimal!

c) i) True. This is demonstrated by the Klee-Minty problem of n variables, where the ~~[illegible]~~ number of iterations ~~of that~~ the simplex algorithm required in the worst case = number of 'vertices' of the Klee-Minty cube $= 2^n$,

$\Rightarrow$ worst case complexity exponential in n.

ii) ~~True Let x_a = [illegible]~~

ii) True. Let ~~the~~ an index set be I, decision variables $= x_1, \dots, x_n$,
reduced cost of NBV $x_j = r_j$ $(j \notin I)$, objective function $= z$.

Then we have :

$$z = a_0 + \sum_{j \notin I} r_j x_j \quad , \quad \text{where } a_0 = \text{some constant}$$

If $r_q = 0 \Rightarrow$ objective cannot change (i.e increase/decrease) by changing x_q!

iii) False

Consider the LP

$$\min \ x_1$$
$$\text{st} \begin{cases} x_1 \leq 1 \\ x_1 \geq 0 \end{cases} .$$

Then standardizing gives

$$\min \ x_1$$
$$\text{st} \begin{cases} x_1 + x_2 = 1 \\ x_1, x_2 \geq 0 \end{cases}$$

where $x_2 =$ slack ~~variable (only slack)~~ variable (the only 1)

Then taking index set $= \{2\}$, we have the basic solution $(0,1)$,
but ~~and~~ the value of the slack here is $\neq 0$!

iv) False.

It can be the case where the objective function of the phase-1 simplex algorithm is 0 with some artificial variables being still basic (refer to Q5 of Tutorial 4)

In this case, we would need to do additional pivots to remove the artificial variables from the basic variables
(this process does not change the phase-1 objective value ~~$\neq$~~ $\neq$ if it was 0 to begin with)

d) $\underbrace{x_1, x_2 \geq 0 \quad \& \quad x_1 + 3x_2 = 1}_{(*)}$ $\Rightarrow$ denominator $x_1 + x_2$ is always positive in the feasible set.

Evidently $(*)$ implies feasible set is bounded as well.

$\Rightarrow$ Introduce y_0, y_1, y_2 st $x_1 = \frac{y_1}{y_0}$, $x_2 = \frac{y_2}{y_0}$ where $y_0 > 0$, $y_1, y_2 \geq 0$

~~Then we have Homogenizing~~ Homogenizing, we get the equivalent problem:

~~Then, Normalizing the denominator, we get the equivalent problem~~:

$$\min \frac{\max\{y_1 + 3y_2,\ y_0 - 2y_2\}}{y_1 + y_2}$$

(using $y_0 > 0$)

$$\text{st} \begin{cases} y_1 + 3y_2 = y_0 \\ y_0 > 0, \quad y_1, y_2 \geq 0 \end{cases}$$

Normalizing, we get the equivalent problem:

$$\min \left(\max\{y_1 + 3y_2,\ y_0 - 2y_2\} \right)$$

$$\begin{aligned} \text{st} \quad & y_1 + y_2 = 1 \\ & y_1 + 3y_2 = y_0 \\ & y_0 \geq 0, \quad y_1, y_2 \geq 0 \end{aligned}$$

noting that we can loosen the constraint on y_0.

This is equivalent to the LP:

$$\min z$$

$$\text{st} \begin{cases} z \geq y_1 + 3y_2 \\ z \geq y_0 - 2y_2 \\ y_1 + y_2 = 1 \\ y_1 + 3y_2 = y_0 \\ y \geq 0, \quad z \text{ free} \end{cases}$$

Standardizing & setting $z = z^+ - z^-$, where $z^+, z^- \geq 0$, we get the equivalent LP:

$$\min z^+ - z^-$$

$$\text{st} \begin{cases} y_1 + y_2 = 1 \\ -y_0 + y_1 + 3y_2 = 0 \\ y_1 + 3y_2 - z^+ + z^- + x_1 = 0 \\ y_0 - 2y_2 - z^+ + z^- + x_2 = 0 \\ x \geq 0, \quad y \geq 0, \quad z^+, z^- \geq 0 \end{cases}$$

e)

Let $\delta_x = \begin{cases} 1 & \text{if } x \text{ is undertaken} \\ 0 & \text{otherwise} \end{cases}$ for $x \in \{A, B, \dots, G\}$.

Let c_x & p_x = capital & profit of investment x respectively, $\forall x \in \{A, \dots, G\}$.

We want to solve:

$$\max \sum_{x \in \{A, \dots, G\}} \delta_x p_x$$

$$\text{s.t.} \begin{cases} \sum_{x \in \{A, \dots, G\}} \delta_x c_x \leq 10 \\ \delta_A + \delta_C \leq 1 \\ \delta_B + \delta_D + \delta_E \leq 1 \\ \delta_A + \delta_C \geq \delta_B + \delta_D . \\ \delta_A \geq \delta_G \\ \delta_x \in \{0, 1\} \quad \forall x \in \{A, \dots, G\} \end{cases}$$

(Q2)

~~Note that~~

a) i) ~~First check that B is invertible:~~

~~$\det B = 9 - 5 \neq 0 \Rightarrow B$ invertible~~

~~Hence $I = \{1,2\}$ is a valid index set, and its associated basis is B~~

~~Now suffices to check optimality: using notation in notes,~~

a) ii) First standardize:

$$\min \; y = -5x_1 - 6x_2$$

$$\text{st} \quad \begin{cases} x_1 + x_2 + x_3 = p \\ 5x_1 + 9x_2 + x_4 = 45 \\ x \geq 0 \end{cases}$$

Note that B corresponds to the index set $\{1,2\} =: I$.

Now $\det B = 9 - 5 \neq 0 \Rightarrow B$ invertible

$\Rightarrow I$ valid index set & B valid basis.

$\Rightarrow$ suffices to check optimality. Using notation in notes,

$$\text{reduced cost } r = c_N - N^T B^{-T} c_B$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left[\left(\tfrac{1}{4}\right) \begin{pmatrix} 9 & -1 \\ -5 & 1 \end{pmatrix} \right]^T \begin{pmatrix} -5 \\ -6 \end{pmatrix}.$$

$$= -\tfrac{1}{4} \begin{pmatrix} 9 & -5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ -6 \end{pmatrix}$$

$$= -\tfrac{1}{4} \begin{pmatrix} -15 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 15/4 \\ 1/4 \end{pmatrix} \geq 0. \quad \Rightarrow x_B \text{ optimal}$$

Also, when $p = 6$,

$$x_B = \tfrac{1}{4} \begin{pmatrix} 9 & -1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 45 \end{pmatrix}$$

$$= \begin{pmatrix} 9/4 \\ 15/4 \end{pmatrix} \geq 0. \quad \Rightarrow x_B \text{ feasible}$$

$\Rightarrow$ I gives rise to an optimal basic feasible solution

$\Rightarrow$ An optimal basis matrix when $p = 6$ is B.

ii) For $5 \leq p \leq 9$,

$$B^{-1} \begin{pmatrix} p \\ 45 \end{pmatrix} = \tfrac{1}{4} \begin{pmatrix} 9 & -1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} p \\ 45 \end{pmatrix}$$

$$= \tfrac{1}{4} \begin{pmatrix} 9p - 45 \\ 45 - 5p \end{pmatrix} \geq 0$$

Defining $V'_{LP}(p) = \min -5x_1 - 6x_2$

$$\text{st} \begin{cases} x_1 + x_2 \leq p \\ 5x_1 + 9x_2 \leq 45 \\ x \geq 0 \end{cases},$$

we then have (by a result in the notes)

$$V'_{LP}(p) = V'_{LP}(6) + \Pi_1 (p-6), \quad \text{since } B^{-1}\begin{pmatrix} p \\ 45 \end{pmatrix} \geq 0.$$

where Π_1 = first component of the shadow price

$$\Pi = (B^{-1})^T c_B = \frac{1}{4}\begin{pmatrix} 9 & -5 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -5 \\ -6 \end{pmatrix} = \begin{pmatrix} -15/4 \\ -1/4 \end{pmatrix}.$$

$\cancel{\Rightarrow V'_{LP}(p) =}$

Also, $V'_{LP}(6) = -5(9/4) - 6(15/4)$

$$= \frac{-135}{4}.$$

$$\Rightarrow V'_{LP}(p) = -\frac{135}{4} - \frac{15}{4}(p-6)$$

$$= -\frac{1}{4}(15p + 45)$$

Now $V_{LP}(p) = -V'_{LP}(p)$

$$= \frac{1}{4}(15p + 45). \quad // \quad \cancel{\text{for } 5 \leq p \leq 9}$$

b) i) ~~Tr~~ True We know for $5 \leq p \leq 9$, ~~the optimal~~ $\exists$ optimal solution for the LP, by (a).
Assuming the integral version of (1) ~~has~~ is feasible for $5 \leq p \leq 9$, then since in the integral version we are taking a maximum over a smaller set compared to the LP, we have $V_{LP}(p) \geq V_{IP}(p)$. $\forall p \in [5,9]$.

ii) True
Using the formula from (a), if $5 \leq p_1 \leq p_2 \leq 9$

$$\Rightarrow V_{LP}(p_2) = \frac{1}{4}(15p_2 + 45) \geq \frac{1}{4}(15p_1 + 45) = V_{LP}(p_1).$$

iii) True
Taking $p_1 = 6$ & $p_2 = 9$, we know $V_{LP}(p_1) = V_{LP}(6) = \frac{135}{4}$,

and ~~an~~ an element in the feasible set of the ILP when $p_2 = 9$ is $(9,0)$, with objective value 45.

$$\Rightarrow V_{IP}(9) \geq 45 > \frac{135}{4} = V_{LP}(6) \Rightarrow \text{True!}$$

c) ~~Using (a), we know~~

$\cancel{x_2 = \frac{15}{4} +}$

Using notation from (a), we know

$$B^{-1}N = \frac{1}{4}\begin{pmatrix} 9 & -1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 9 & -1 \\ -5 & 1 \end{pmatrix}$$

$\Rightarrow$ Since $x_B = B^{-1}b - B^{-1}N x_N$, we know

$$x_2 = \frac{15}{4} + \frac{5}{4}x_3 - \frac{1}{4}x_4$$

$$\Leftrightarrow \quad x_2 - \frac{5}{4}x_3 + \frac{1}{4}x_4 = \frac{15}{4}.$$

$\therefore$ Gomory cut corresponding to x_2 is

$$\left(-\frac{5}{4} - \left\lfloor -\frac{5}{4} \right\rfloor\right)x_3 + \left(\frac{1}{4} - \left\lfloor \frac{1}{4} \right\rfloor\right)x_4 \geq \frac{15}{4} - \left\lfloor \frac{15}{4} \right\rfloor$$

$$\Leftrightarrow \quad \frac{3}{4}x_3 + \frac{1}{4}x_4 \geq \frac{3}{4}$$

$$\Leftrightarrow \quad 3x_3 + x_4 \geq 3$$

Now $x_3 = 6 - x_1 - x_2$ & $x_4 = 45 - 5x_1 - 9x_2$, so substituting:

$$3(6 - x_1 - x_2) + (45 - 5x_1 - 9x_2) \geq 3$$

$$\Leftrightarrow \quad 8x_1 + 12x_2 \leq 60$$

$$\Leftrightarrow \quad 2x_1 + 3x_2 \leq 15.$$

Gomory cut in x_1, x_2.

d) i) It is evident that $x^{LP} = (\frac{1}{2}, \alpha+1)$ satisfies

$$\begin{cases} x_1 \le \frac{1}{2} \\ -2(\alpha+1)x_1 + x_2 \le 0 \\ x_1, x_2 \ge 0 \end{cases}$$

Also, $x_2 \le 2(\alpha+1)x_1 \le 2(\alpha+1)(\frac{1}{2}) = \alpha+1$

$\Rightarrow$ maximum value that the objective can take $= \alpha+1$, which is satisfied by x^{LP}.

$\Rightarrow$ x^{LP} optimal.

Now suppose $x^* = (x_1^*, x_2^*)$ is another optimal solution.

Then objective value corresponding to $x^* = \alpha+1$

$$\Rightarrow x_2^* = \alpha+1$$

$$\Rightarrow -2(\alpha+1)x_1^* + (\alpha+1) \le 0$$

$$\Rightarrow x_1^* \ge \frac{1}{2}.$$

But $x_1^* \le \frac{1}{2} \Rightarrow x_1^* = \frac{1}{2} \Rightarrow x^* = x^{LP}$.

Thus x^{LP} is the unique optimal solution

ii) $x_1 \le \frac{1}{2}$ & $x_1 \in \mathbb{N}$. $\Rightarrow x_1 = 0$.

$-2(\alpha+1)x_1 + x_2 \le 0 \Rightarrow x_2 \le 0$ since $x_1 = 0$.

$\Rightarrow x_2 = 0$ since $x_2 \in \mathbb{N}$.

Hence the only element in the feasible set of the ILP is $(0,0)$.

$\Rightarrow x^{IP} = (0,0)$.

iii) Direct computation: $x_2^{LP} - x_2^{IP} = \alpha+1-0 = \alpha+1$.

iv) No. $\|x^{LP} - x^{IP}\| \cancel{= \sqrt{(\alpha+1)^2+\frac{1}{4}}}$

$$= \sqrt{(\alpha+1)^2 + \frac{1}{4}}$$ using Euclidean metric.

$\approx \alpha+1$ when α is large

Thus when $\alpha = 10000$, distance from x^{LP} to x^{IP} on the plane ≈ 10001, and only increases as $\alpha \to \infty$.

This shows that we cannot deduce anything about

d) ii) This shows that we cannot just search for ~~the~~ an optimal ~~solutions~~ solution of an ILP within ~~~~ a hypersphere (circle in this case) centered about the optimal solution of its LP relaxation.

(As shown in this case, as $\alpha \to \infty$, we need to cut ~~the~~ the radius of this ~~sphere~~ hypersphere $\to \infty$!)

$\Rightarrow$ impractical.