Statistics 2010-2011

1 i) $\dfrac{90-\mu}{\sqrt{\sigma^2}} = \Phi^{-1}(0.025)$ $\qquad$ $\dfrac{120-\mu}{\sqrt{\sigma^2}} = \Phi^{-1}(0.005)$

$= -\Phi^{-1}(0.975)$ $\qquad$ $= 2.576$

$= -1.960$

$90-\mu = -1.960\,\sigma$ $\qquad$ $120-\mu = 2.576\,\sigma$

$90+1.960\,\sigma = \mu$ $\qquad$ $120-2.576\,\sigma = \mu$

$90+1.960\,\sigma = 120-2.576\,\sigma$

$90-120 = -2.576\,\sigma - 1.960\,\sigma$

$-30 = -4.536\,\sigma$

$6.6 = \sigma$

$\mu = 90 + 1.960 \times 6.6$

$= 102.9$

$\therefore$ c

ii) since it is discrete

x	1	2	3	4
$P(X=x)$	0.1	0.2	0.3	0.4
$F(x)$	0.1	0.3	0.6	1

$\therefore$ d

iii) $P(E|F) = \dfrac{P(E\cap F)}{P(F)} = \dfrac{P(F|E)\,P(E)}{P(F)}$

$P(E) = P(F)P(E|F) + P(\bar{F})P(E|\bar{F})$ (Law of probability)

$\Rightarrow \quad P(E|F) = \dfrac{P(E) - P(\bar{F})P(E|\bar{F})}{P(E)}$

$= \dfrac{0.5 - 0.5\times 0.6}{0.5} = 0.4$

iv) e the MLE is always asymptotically unbiased & consistent

v) we require $\int_{-\infty}^{\infty} f(x) = 1$

$1 = \int_{-1}^{0} \frac{1}{4}\,dx + \int_{0}^{1} \frac{1}{4} + cx\,dx$

$1 = \left[\frac{1}{4}x\right]_{-1}^{0} + \left[\frac{1}{4}x + \frac{c}{2}x^2\right]_{0}^{1}$

$1 = 0 - \frac{-1}{4} + \frac{1}{4} + \frac{1c}{2} - 0 - 0$

$1 = \frac{1}{2} + \frac{1}{2}c$

$\frac{1}{2} = \frac{1}{2}c \quad \therefore\ c = 1$

2 i)

$P_{XY}(x,y)$		x = 0	1	2	
y	0	$\frac{1}{5}$	$1/10$	$\frac{1}{5}$	$\frac{1}{2}$
	1	$\frac{1}{8}$	$1/5$	$\frac{7}{40}$	$\frac{1}{2}$
		$\frac{13}{40}$	$\frac{3}{10}$	$\frac{3}{8}$	

$$P(X=0 \mid Y=0) = \frac{P(X=0 \cap Y=0)}{P(Y=0)}$$

$$\frac{2}{5} = \frac{P(X=0 \cap Y=0)}{1/2}$$

$$2/5 \times \frac{1}{2} = P(X=0 \cap Y=0)$$

$$\frac{1}{5} =$$

ii) If 2 random variables X and Y are independent, they satisfy the following conditions:

- $P(X \mid Y) = P(X) \quad \forall\, x, y \in \mathbb{R}$
- $P(X \cap Y) = P(X) \times P(Y) \quad \forall\, x, y \in \mathbb{R}$

$\left.\begin{array}{l} P(X=0) = 13/40 \\ P(X=0 \mid Y=0) = 2/5 \end{array}\right\}$ different values therefore not independent.

iii) $E(X) = 0 \times \frac{13}{40} + 1 \times \frac{3}{10} + 2 \times \frac{3}{8} = 1.05 \qquad \left(\sum_0^1 y\, P(Y=y) \right)$

$E(Y) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$

iv) $E(XY) = \sum_x \sum_y xy\, P(x,y)$

$= (0 \times 0 \times 1/5) + (0 \times 1 \times \frac{1}{10}) + (0 \times 2 \times 1/5) +$
$(1 \times 0 \times 1/8) + (1 \times 1 \times 1/5) + (1 \times 2 \times 7/40)$

$= 0.55$

$Cov(X, Y) = E(XY) - E(X)E(Y)$

$= 0.55 - \frac{21}{20} \times \frac{1}{2}$

$= \frac{1}{40}$

Since $Cov(X, Y) > 0$, the variables are positively correlated.

3 i a) bias $(T) = E(T|\mu) - \mu$

$$E(\bar{X}|\mu) = E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{\sum_{i=1}^{n} E(X_i)}{n} = \frac{\sum_{i=1}^{n} \mu}{n} = \frac{n\mu}{n} = \mu.$$

b) $X \sim N(500, 5^2)$ where X is a r.v. which denotes the weight of a bag.

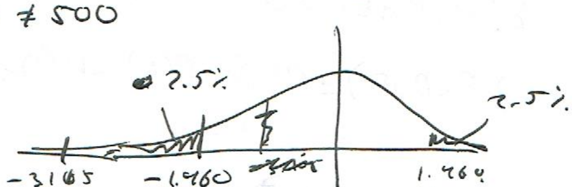
$$\bar{x} = \frac{500.2 + 498.2 + 486.3 + 494 + 502.9 + 503.9 + 487.9 + 496.4 + 483.7 + 497.4}{10}$$

$= 495.09$

c) $\bar{X} \sim N(500, \frac{5^2}{10})$ $\quad H_0 : \mu = 500$ $\quad$ 5% sig lev

$H_1 : \mu \neq 500$

Let $Z = \dfrac{\bar{X} - 500}{\sqrt{2.5}}$



$Z = \dfrac{495.09 - 500}{\sqrt{2.5}}$ $\quad \Phi(1 - 0.025)$

$\Phi(0.975) = 1.960$

$Z = -3.105$

$-3.105 > -1.960$

$\therefore$ reject null hypothesis.

ii a) unbiased estimate of standard deviation =>

$$S_{n-1} = \sqrt{\frac{n}{n-1}\left(\frac{\sum x^2}{n} - (E(X)\,n)^2\right)}$$

$$= \sqrt{\frac{10}{9}\left(\frac{500.2^2 + 498.2^2 + 486.3^2 + \ldots}{10} - (495.09 \times 10)^2\right)}$$

$= 7.0026$ g

b) Since we are using an estimate for the standard deviation, we must use the t-distribution (we can only use $\bar{X}$ distribution if we know σ).

Test statistic $t = \dfrac{\bar{X} - \mu_0}{S\sqrt{n}}$ $\quad$ degrees of freedom $= 10 - 1$

$= 9$

$t = \dfrac{495.09 - 500}{7.003/\sqrt{10}}$

$= -2.2172$

T-distribution with 9 degrees of freedom & @ 0.975 sig. lev. = 2.262

$-2.262 > -2.2172$ $\therefore$ do not reject null hypothesis

c) The differing outcome of the two tests suggests that the mean weights of the bags of grapes may well be 500g, but the variability of the bag weights claimed by the supermarket may be too low.

4 i a) $P(A \cap B) = \emptyset$

b) $P(A|B) = P(A)$

ii a) $P(E|F)$ since E&F independent then $P(E) = 0.4$

b) $P(D \cap E) = 0$

c) $P(D \cup E) = P(D) + P(E) - P(D \cap E)$

$= 0.3 + 0.4 - 0$

$= 0.7$

d) $P(E \cup F) =$

$= P(E) + P(F) - P(E \cap F) = P(E) + P(F) - P(E)P(F)$ by independence of E/F.

$\therefore P(F) = \dfrac{P(E \cup F) - P(E)}{1 - P(E)} = \dfrac{0.8 - 0.4}{1 - 0.4} = \dfrac{0.4}{0.6} = \dfrac{2}{3}$

iii a) $\{X \geq 5\} = F \cap \bar{E}$

b) $P(X \geq 5) = P(F \cap \bar{E})$

By law of probability $P(F) = P(E \cap F) + P(\bar{E} \cap F)$

$P(\bar{E} \cap F) = P(F) - P(E \cap F)$

$= P(F) - P(E)P(F)$

$= 2/3 - 0.4 \times 2/3$

$= 0.4.$