142 Discrete Math 2020 Exam Sample Solution

I personally think this paper is inflicting too much pain on us......Imperial students are real fighters

Disclaimer: This is not an official answer key, so please correct any mistake if you find one. There are more than one possible solution for some questions, so keep in mind about that!

1. **a. i)** A possible proof:

$$\begin{aligned} C \subseteq A \wedge C \subseteq B &\triangleq \forall x \in C\ (x \in A) \wedge \forall x \in C\ (x \in B) \\ &\Rightarrow \forall x \in C\ (x \in A \wedge x \in B) \\ &\Rightarrow \forall x \in C\ (x \in A \cup B) \\ &\triangleq C \subseteq A \cap B \end{aligned}$$

ii) A possible proof:

$$\begin{aligned} A \subseteq C \wedge B \subseteq C &\triangleq \forall x \in A\ (x \in C) \wedge \forall x \in B\ (x \in C) \\ &\Rightarrow \forall x \in A\ (x \in C) \vee \forall x \in B\ (x \in C) \qquad (\text{A} \wedge \text{B imples A} \vee \text{B}) \\ &\Rightarrow \forall x \in A \cup B\ (x \in C) \\ &\Rightarrow A \cup B \subseteq C \end{aligned}$$

b. A possible solution:

$$\begin{aligned} |A \cap B \cap C| &= |A \cap (B \cap C)| && (\text{By associativity of } \cap) \\ &= |A| + |B \cap C| - |A \cup (B \cap C)| && (\text{By the fact in question}) \\ &= |A| + (|B| + |C| - |B \cup C|) - |A \cup (B \cap C)| && (\text{By the fact in question}) \\ &= |A| + |B| + |C| - |B \cup C| - |(A \cup B) \cap (A \cup C)| && (\text{By distributivity of } \cup) \\ &= |A| + |B| + |C| - |B \cup C| - (|A \cup B| + |A \cup C| - |(A \cup B) \cup (A \cup C)|) && (\text{By the fact in question}) \\ &= |A| + |B| + |C| - |B \cup C| - |A \cup B| - |A \cup C| + |A \cup B \cup C)| && (\text{By absorption of } \cup) \end{aligned}$$

c. i) $\text{R is reflexitve} \triangleq \forall x \in A\ (x\ R\ x)$

$\text{R is symmetric} \triangleq \forall x, y \in A(x\ R\ y \Rightarrow y\ R\ x)$

$\text{R is transitive} \triangleq \forall x, z \in A(\exists y \in A(x\ R\ y \wedge y\ R\ z) \Rightarrow x\ R\ z)$

$\text{R is an equivalence relation} \triangleq \text{R is reflexive, symmetric, and transitive}$

ii) There are 2^{n^2} binary relations on A since all elements in A cna form n^2 distinct pairs which can be either included or not included in a binary relation. Such inclusion/exclusion decision will have 2^{n^2} differnet combinations with n^2 pairs, hence there are 2^{n^2} differnet binary relations on A.

There are $2^{\frac{n(n+1)}{2}}$ symmetric ones among those binary relations. For each non-reflexive pairs(e.g. <x, y> with $x \neq y$), the opposite pair(e.g. <y, x>) must also appear in a symmetric relation. Those two must appear or not appear together. There are $n^2 - n$ such pairs that have to bond with its opposite pair. We can treat such two pairs as one single element and try to form a new set, so there are $\frac{n^2-n}{2}$ elements of this kind, The reflexive ones(there are n reflexive ones) can either show up or not show up, so the total number of elements in this new set would be

$\frac{n^2-n}{2} + n = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$. All the subsets of this new set of pairs are symmetric, leading to $2^{\frac{n(n+1)}{2}}$ different subsets of this new set, which is also the number of different symmetric relations in total.

iii) The statement is not necessarily true:

Counterexample: Let $A = \{1, 2, 3\}$ and $R = \{<3, 3>\}$. Now, R is symmetric by definition of symmetric relation, but $R^+ = \{<3, 3>\}$ is not reflexive.

iv) There are $m \cdot n$ different pairs in R. Since each equivalence class has fixed cardinality of m, each element in A is paired up to m elements. In other words, each element contributes m pairs in R. Since R is an equivalence relation, it must be reflextive, which means that all elements in A must appear in R since reflexitivity requires the property $\forall x \in A\ (x\ R\ x)$. There are n elements in A, hence $m \cdot n$ pairs in R.

d. i) $\text{R is a partial order} \triangleq \forall x, y \in A\ (x\ R\ y \land y\ R\ x \Rightarrow x =_A y) \land \forall x \in A\ (x\ R\ x)$
$\land \forall x, z \in A\ (\exists y \in A (x\ R\ y \land y\ R\ z) \Rightarrow x\ R\ z)$

ii) We will show that $\leq_p$ is reflextive, transitive, and anti-symmetric.

$\leq_p$ is reflexitve: when $k = 1$, $n^k = n$ for all positive natural numbers.

$\leq_p$ is transitive: let $a^{k_1} = b$ and $b^{k_2} = c$. Then we have $(a^{k_1})^{k_2} = c$ by arithmetic, leading to $a^{k_1 k_2} = c$. Since both k_1 and k_2 are natural numbers, their produce will be natural number as well.

$\leq_p$ is anti-symmetric: let $a^{k_1} = b$ and $b^{k_2} = a$. Then we have $a^{k_1 k_2} = a$ by arithmetic, which leads to $k_1 k_2 = 1$. Since $k_1, k_2 \in \mathbb{N}$, we know that both k_1 and k_2 must be equal to 1. Hence, $a = b$ and $b = a$.

2. **a. i)** $\text{f is injective} \triangleq \forall a, b \in A\ (f(a) = f(b) \Rightarrow a = b)$

   $\text{f is surjective} \triangleq \forall b \in B\ \exists a \in A\ (f(a) = b)$

   $\text{f is bijective} \triangleq \text{f is both surjective and injective}$

   ii) $|B^A| = 27$. Since each element in A has to pair up with at least one element in B by function definition, there are 3 different ways of pairing for each element. Hence, there are $3^3 = 27$ different combinations of functions.

   Among those functions, 21 of them are NOT onto. There are 6 functions that are onto: 3 ways to pair element 1 with any of a, b, c; then 2 ways of pairing element 2 and 3 in each cases, leading to total of 6 onto functions. Hence, there are $27 - 6 = 21$ functions that are not onto.

   Among those functions, 21 of them are NOT one-to-one as well. Since $|A| = |B| = 3$, if a function is not onto, then it's not one-to-one as well. Since by pigeon hole principle, at least one of the element in B is mapped by two elements in A provided that f is not onto. This means that f is not one-to-one as well. Hence 21 functions that are not one-to-one.

   As mentioned earlier, there are 6 bijections as $|A| = |B| = 3$.

   There are 64 partial functions as every element in A can now map to 4 different elements: $a, b, c, \perp$. When mapping to $\perp$, it means that the pairs containing the element does not appear in f. This leads to $4^3 = 64$ different partial functions.

b. i) g is left inverse of f $\triangleq \forall a \in A \ (g \circ f(a) = a)$

g is right inverse of f $\triangleq \forall b \in B \ (f \circ g(b) = b)$

g is inverse of f $\triangleq$ g is both left and right inverse of f

ii) Assume that $f(a) = f(a')$. From the definition of left inverse function we know that $g(f(a)) = a$ and $g(f(a')) = a'$. Since g is a function and $f(a) = f(a')$, it follows that $a = a'$. This is because by the definition of function, each element in A must only pair with exactly one element in B. Hence f is injective.

To show that f is surjective, take $b \in B$ arbitarily. From the definition of rigth inverse function we know that $f(h(b)) = b$, meaning that b must be in the image of f. Hence f is surjective(onto).

c. Let's define $f : \mathbb{N} \to A \times B$ as follows:
$f(0) = \langle a_0, b_0 \rangle, f(1) = \langle a_1, b_0 \rangle, f(2) = \langle a_1, b_0 \rangle \ldots$

This is illustrated by the following graph:

$$\begin{matrix} \langle a_0, b_0 \rangle & \langle a_1, b_0 \rangle & \langle a_2, b_0 \rangle & \ldots \\ \langle a_0, b_1 \rangle & \langle a_1, b_1 \rangle & \langle a_2, b_1 \rangle & \ldots \\ \langle a_0, b_2 \rangle & \langle a_1, b_2 \rangle & \langle a_2, b_2 \rangle & \ldots \\ \vdots & \vdots & \vdots & \end{matrix}$$

We can visit each element exactly once via zigzag pattern(both one-to-one and onto), which shows that f is bijective and hence $\mathbb{N} \approx A \times B$, meaning that $A \times B$ is countable.

d. We prove by contradiction. Suppose that W is countable. Then there exists a bijection such that $f : \mathbb{N} \to W$. Let $V \cup P = W$ for some set P, so the bijective function f can also be written as $f : \mathbb{N} \to V \cup P$. However, V is uncountable, meaning that there is no bijection between $\mathbb{N}$ and V. Thus, there should be no bijection $f : \mathbb{N} \to V \cup P$ as well. This leads to a contradiction, so our assumption is incorrect and W is uncountable.

Alternatively: by deifinition of $\subseteq$, we know that $\forall x \in V \ (x \in W)$. Hence, all elements in V must at least stay in W as well, so $|V| \leq |W|$. Since V is uncountable, these common elements in W will be enough to make W uncountable as well.

e. We shall prove that it is not possible to build a bijection $f : \mathbb{N} \to \mathbb{N}^*$ in general.

We build sequences of $\mathbb{N}$ as follows:

$$\begin{aligned} V_0 &= a_0^0 \ a_0^1 \ a_0^2 \ a_0^3 \ \ldots \\ V_1 &= a_1^0 \ a_1^1 \ a_1^2 \ a_1^3 \ \ldots \\ V_2 &= a_2^0 \ a_2^1 \ a_2^2 \ a_2^3 \ \ldots \\ V_3 &= a_3^0 \ a_3^1 \ a_3^2 \ a_3^3 \ \ldots \\ &\vdots \end{aligned}$$

where $a_i^j \in \mathbb{N}$.

Let f be defined as $f(n) = V_n$. We can always find a sequence V such that $V \notin Im(f)$:

$$V = a_0^0 + 1 \ \ a_1^1 + 1 \ \ a_2^2 + 1 \ \ a_3^3 + 1 \ \ \ldots$$

(Notice that as long as V is defined in such way that it differs from any of the sequence above, it will be fine. (e.g. we can define V as $V = 2 * a_0^0 \;\; 2 * a_1^1 \;\; 2 * a_2^2 \;\; 2 * a_3^3 \;\; \ldots$))

Clearly by diagonalization argument, V does not match any of the sequences above it since it differs from V_i by the ith element. Hence V is not in the image of f and there is no way to build a bijection between $\mathbb{N}$ and $\mathbb{N}^*$, hence $\mathbb{N}^*$ is uncountable.