Computation Answers 4: Register Machines

1. (a) The graphical representation looks like:

$$\begin{tikzcd}
& & \mathit{HALT} & \\
\mathit{START} \arrow[r] & R_1^- \arrow[ur] \arrow[d] & & R_3^- \arrow[ll, two heads] \arrow[d, bend left] \\
& R_0^+ \arrow[r] & R_2^- \arrow[r, two heads] \arrow[d] & R_2^+ \arrow[u, bend left] \\
& R_0^+ \arrow[u] & R_3^+ \arrow[l] &
\end{tikzcd}$$

It is very easy to forget to label the start state, but it is *essential* to do so — otherwise how would you know where to begin? ***Always*** **label the start state!**

(b) The computation is: (0,0,2,0,0), (1,0,1,0,0), (2,1,1,0,0), (5,1,1,0,0), (6,1,1,1,0), (0,1,1,1,0), (1,1,0,1,0), (2,2,0,1,0), (3,2,0,0,0), (4,2,0,0,1), (1,3,0,0,1), (2,4,0,0,1), (5,4,0,0,1), (6,4,0,1,1), (5,4,0,1,0), (6,4,0,2,0), (0,4,0,2,0), (7,4,0,2,0).

This register machine computes the sum of the first x odd numbers, this is equivalent to x^2, so:

$$f(x) = \sum_{k=0}^{x-1}(1 + 2k) = x^2$$

Register R_0 is used for the accumulator and final result, R_1 is the input and used for termination of the machine, R_2 is used for the loop that calculates $2k$, and R_3 stores a copy of R_2 whilst it is destructively used by the loop.

States L_1 to L_4 compute $1 + 2k$ and copies R_2 into R_3 whilst R_2 is decremented. L_5 and L_6 moves R_3 back into R_2 and increments the value of R_2 by 1 (this is equivalent to the Σ operation incrementing k for the next addition).

2. (a) i. The following register machine computes f:

$$\begin{aligned}
&L_0 : R_2^- \to L_1, L_2 \\
&L_1 : R_1^- \to L_0, L_4 \\
&L_2 : R_1^- \to L_3, L_4 \\
&L_3 : R_0^+ \to L_2 \\
&L_4 : \mathit{HALT}
\end{aligned}$$

The machine first reduces R_1 (which has initial value x_1) by the amount in R_2 (initially x_2): this is the loop between instructions L_0 and L_1. If it cannot reduce R_1 that far, it means $x_2 > x_1$, and so the machine halts at L_4, with R_0 still at its initial value of 0 (which is $f(x_1, x_2)$). If R_1 can be reduced that far, its contents is then $x_1 - x_2$, which is copied into R_0 by the loop between L_2 and L_3. When this loop exits, the machine will halt with $R_0 = x_1 - x_2 = f(x_1, x_2)$.

ii. Graphically, the register machine looks like:

$$\begin{tikzcd}
START \arrow[r] & R_2^- \arrow[r, bend left] \arrow[d, two heads] & R_1^- \arrow[l, bend left] \arrow[dd, bend left, two heads] \\
& R_1^- \arrow[r, bend left] \arrow[dr, bend right, two heads] & R_0^+ \arrow[l, bend left] \\
& & HALT
\end{tikzcd}$$

(Variations are possible, but this is the simplest that always halts successfully.)

(b) i. Recall what it means for register machine M to compute $g(x_1, x_2)$:

The computation of M starting wtih $R_0 = 0$, $R_1 = x_1$, $R_2 = x_2$ and all other registers set to 0 halts with $R_0 = y$ if and only if $g(x_1, x_2) = y$.

Since $g(x_1, x_2)$ is undefined when $x_2 > x_1$, there is no y with $g(x_1, x_2) = y$. Therefore the machine cannot halt. Instead, it must run forever.

ii. The simplest way to make the machine run forever if $x_2 > x_1$ is to have L_1 loop back on itself when $R_1 = 0$:

$$\begin{aligned}
&L_0 : R_2^- \to L_1, L_2 \\
&L_1 : R_1^- \to L_0, L_1 \\
&L_2 : R_1^- \to L_3, L_4 \\
&L_3 : R_0^+ \to L_2 \\
&L_4 : HALT
\end{aligned}$$

$$\begin{tikzcd}
START \arrow[r] & R_2^- \arrow[r, bend left] \arrow[d, two heads] & R_1^- \arrow[l, bend left] \arrow[loop right, two heads] \\
& R_1^- \arrow[r, bend left] \arrow[dr, bend right, two heads] & R_0^+ \arrow[l, bend left] \\
& & HALT
\end{tikzcd}$$

3. (a) One possible coding is the following:

$$\begin{aligned}
&L_0 : R_1^- \to L_1, L_2 \\
&L_1 : R_1^- \to L_0, L_3 \\
&L_2 : HALT \\
&L_3 : R_0^+ \to L_2
\end{aligned}$$

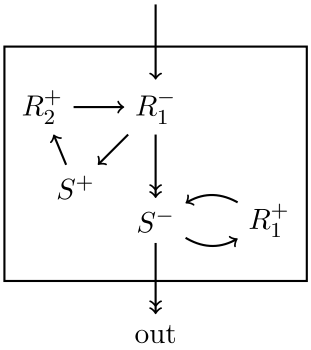
Other codings are possible by renaming L_1, L_2 and L_3 consistently. L_0 cannot be renamed, because it is the start state.

(b) The machine computes the remainder of x divided by 2. That is, the function

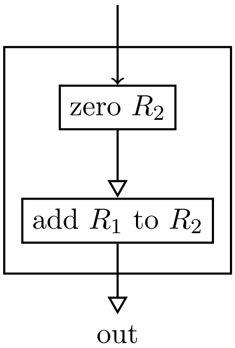
$$f(x) \triangleq \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

To see why, consider that whenever the machine is in state L_0, register R_1 has been decreased by an even amount from its initial value of x. (At the start, R_1 has been decreased by 0, which is even.) If $R_1 = 0$, it must be that x was even, and so the machine halts with $R_0 = 0$. Whenever the machine is in state L_1, register R_1 has been decreased by an odd amount from its initial value. Therefore, if $R_1 = 0$ it must be that x was odd, so by incrementing R_0 then halting the machine halts with $R_0 = 1$. If $R_1 > 0$ in state L_0, it can be decremented by 1 so that it has been decremented an odd number of times when the machine enters state L_1. Similarly, if $R_1 > 0$ in state L_1, it can be decremented by 1 so that it has been decremented an even number of times when the machine enters state L_0. Finally, since R_1 is decreased on every loop, we can conclude that the machine always halts eventually.

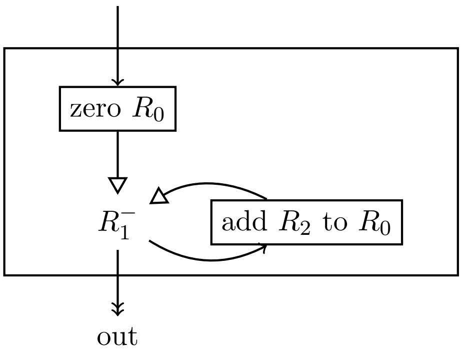
4. (a) add R_1 to R_2:



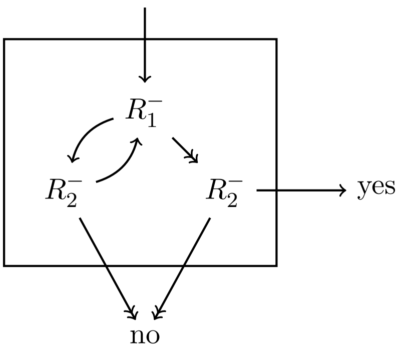
(b) It would be enough for us to add R_1 to R_2 if R_2 was set to 0. So let's zero R_2 first! copy R_1 to R_2:



(c) We can implement multiplication by repeated addition. multiply R_1 by R_2 to R_0:



(d) test $R_1 < R_2$:



(e) The register machine computes the greatest value $f(x)$ such that $(f(x))^2 \leq x$. That is, it computes the floor of the positive square-root of x: $f(x) = \lfloor \sqrt{x} \rfloor$. The machine loops testing whether $(1 + R_0)^2$ is greater than R_1, incrementing R_0 until it is.