# 140 Logic 2020 Exam Sample Solution

*Disclaimer: This is not an official answer key, so please correct any mistake if you find one. There are more than one possible solution for some questions, so keep in mind about that!*

1. **a.** $\neg(\neg(p \wedge (q \to \neg s)) \vee s \wedge p)$

   **b.** This is not necessarily true. Take arbitary atoms $p$ and $q$. If $p$ holds, then $p \vee q$ holds as well. However, if $\neg p$ holds, we can not tell whether $p \vee q$ holds or not since $q$ could be true or false. Hence, let $A \triangleq p$ and $B \triangleq p \vee q$, this configuration would satisfy the condition that $A \models B$ and $\neg A \not\models B$. But clearly, $p \not\equiv p \vee q$. Hence, the assertion is not necessairly true.

   **c. i)** $h \triangleq$ `David's plant is healthy`

   $y \triangleq$ `David's plant is yellow`

   $b \triangleq$ `David's plant is blooming`

   $w \triangleq$ `David's plant is overwatered`

   $1. h \to (\neg b \to y) \qquad 2. h \to b \qquad 3. \neg w \vee \neg y \vee \neg h \qquad 4. \neg h$

   **ii)** Premises: $1, 2, 3$ Conclusion: $4$

   It is not propositionally valid as David's plant can be not overwatered and healthy at the same time, which could still satisfy the premises.

   **d.**

$$
\begin{aligned}
& (p \to \neg q) \vee \neg(s \leftrightarrow r) & \\
\equiv\ & (\neg p \vee \neg q) \vee \neg(s \leftrightarrow r) & (A \to B \equiv \neg A \vee B) \\
\equiv\ & (\neg p \vee \neg q) \vee \neg[(s \wedge r) \vee (\neg s \wedge \neg r)] & (A \leftrightarrow B \equiv (A \vee B) \vee (\neg A \vee \neg B)) \\
\equiv\ & (\neg p \vee \neg q) \vee (\neg(s \wedge r) \wedge \neg(\neg s \wedge \neg r)) & (\text{DeMorgan's Law}) \\
\equiv\ & (\neg p \vee \neg q) \vee ((\neg s \vee \neg r) \wedge (s \vee r)) & (\text{DeMorgan's Law}) \\
\equiv\ & (\neg p \vee \neg q \vee \neg s \vee \neg r) \wedge (\neg p \vee \neg q \vee s \vee r) & (\text{By distributivity of } \vee)
\end{aligned}
$$

   **e.**

| | | | | | |
|---|---|---|---|---|---|
| 1 | $\neg A$ | ass | | | |
| | $\dots$ | | | | |
| 2 | $B$ | | | | |
| 3 | $\neg A \to B$ | $\to I(1,2)$ | | | |
| 4 | $A \vee \neg A$ | Lemma | | | |
| 5 | $A$ | ass | 7 | $\neg A$ | ass |
| | | | 8 | $B$ | $\to E(3,7)$ |
| 6 | $A \vee B$ | $\vee I(5)$ | 9 | $A \vee B$ | $\vee I(8)$ |
| 10 | $A \vee B$ | $\vee E(4, 5-6, 7-9)$ | | | |

**f.** The proof system $\vdash^*$ is not sound. Soundness of $\vdash^*$ means that if $A \vdash^* B$ then $A \models B$. From the rule $\to^* E$ we know that $\neg B$ is provable from $A \to B$ and $\neg A$. However, $A \to B, \neg A \models \neg B$ does not hold since according to the truth table of $\to$, when $A$ is false(or does not hold), $B$ could be true or false(could hold or not hold). Hence, this violates the definition of soundness and $\vdash^*$ is not sound.

2. **a.** To prove argument's validity, we need to show $\forall x[P(x) \land \exists y Q(x,y)] \to \exists x \neg P(x) \to \forall x \exists y Q(x,y)$ holds true.

$$
\begin{aligned}
& \forall x[P(x) \land \exists y Q(x,y)] \to \exists x \neg P(x) \to \forall x \exists y Q(x,y) & \\
\equiv\ & \forall x P(x) \land \forall x \exists y Q(x,y) \to \exists x \neg P(x) \to \forall x \exists y Q(x,y) & (\forall x(A \land B) \equiv \forall A \land \forall B) \\
\equiv\ & \forall x P(x) \land \forall x \exists y Q(x,y) \to \neg \exists x \neg P(x) \lor \forall x \exists y Q(x,y) & (A \to B \equiv \neg A \lor B) \\
\equiv\ & \forall x P(x) \land \forall x \exists y Q(x,y) \to \forall x \neg\neg P(x) \lor \forall x \exists y Q(x,y) & (\neg \exists x A \equiv \forall x \neg A) \\
\equiv\ & \forall x P(x) \land \forall x \exists y Q(x,y) \to \forall x P(x) \lor \forall x \exists y Q(x,y) & (\neg\neg A \equiv A) \\
\equiv\ & \neg(\forall x P(x) \land \forall x \exists y Q(x,y)) \lor (\forall x P(x) \lor \forall x \exists y Q(x,y)) & (A \to B \equiv \neg A \lor B) \\
\equiv\ & (\neg \forall x P(x) \lor \neg \forall x \exists y Q(x,y)) \lor (\forall x P(x) \lor \forall x \exists y Q(x,y)) & (\text{DeMorgan's Law}) \\
\equiv\ & (\neg \forall x P(x) \lor \forall x P(x)) \lor (\neg \forall x \exists y Q(x,y) \lor \forall x \exists y Q(x,y)) & (\text{By commutativity of } \lor) \\
\equiv\ & \top \lor \top & (\neg A \lor A \equiv \top) \\
\equiv\ & \top & (\top \lor \top \equiv \top)
\end{aligned}
$$

Alternatively, you can go directly from 4th line to the end by the fact that $A \land B \models A \lor B$

**b. i)** when $\mathrm{c}{=}6$

**ii) A)** $\forall x[Q(x,x) \to \exists y[x \neq y \land Q(y,x)]]$

**B)** $\exists x[Q(x,x) \land \forall y[x \neq y \to \neg Q(x,y)]]$, or $\exists x[Q(x,x) \land \neg \exists y[x \neq y \land Q(x,y)]]$

**C)** $\neg \exists x \forall y[x \neq y \to Q(x,y)]$, or $\forall x \exists y[x \neq y \land \neg Q(x,y)]$

**iii) 1.** x could be 1, 2, 4, 5, 6 **2.** x could be 1, 3, 5 **3.** x could be 2 or 4

**iv)** A possible solution: $P(x) \land \exists y \exists z[Q(x,y) \land Q(x,z) \land y \neq z]$

**c.** Below is the natural deduction proof:

| | | |
|---|---|---|
| 1 | $\exists y \forall x(P(x) \leftrightarrow x = y)$ | premise |
| 2 | $\forall x, y(Q(x,y) \to \neg P(y))$ | premise |
| 3 | $\forall x(P(x) \leftrightarrow x = c)$ | ass |
| 4 | $c = c$ | =reflect |
| 5 | $P(c)$ | $\to \forall E(3,4)$ |
| 6 | $a$ | ∀I const |
| 7 | $Q(a,c) \to \neg P(c)$ | $\forall E(2,6)$ |
| 8 | $Q(a,c)$ | ass |
| 9 | $\neg P(c)$ | $\to E(7,8)$ |
| 10 | $\bot$ | $\neg E(5,9)$ |
| 11 | $\neg Q(a,c)$ | PC(8-10) |
| 12 | $\forall y \neg Q(y,c)$ | ∀I(6-11) |
| 13 | $P(c) \land \forall y \neg Q(y,c)$ | $\land$ I(5,12) |
| 14 | $\exists x(P(x) \land \forall y \neg Q(y,x))$ | ∃I(13) |
| 15 | $\exists x(P(x) \land \forall y \neg Q(y,x))$ | ∃E(1,3-14) |