IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

EXAMINATIONS 2023

BEng Honours Degree in Computing Part II
MEng Honours Degrees in Computing Part II
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant examinations for the Associateship of the City and Guilds of London Institute

PAPER COMP50011

COMPUTATIONAL TECHNIQUES

Wednesday 10th May 2023, 10:00
Duration: 90 minutes

Answer ALL TWO questions

CORRECTION:

Q1a ii) $\|A\|_F = \sqrt{\sum_{i=1}^{r} \sigma_i^2}$

Paper contains 2 questions
Calculators required

1 a i) Show that an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$ preserves the ℓ_2 norm of vectors in $\mathbb{R}^n$ and the angle between any two vectors in $\mathbb{R}^n$.

ii) Use the singular value decomposition of $\mathbf{A} \in \mathbb{R}^{m\times n}$ to prove carefully that its Frobenius norm is given by $\|\mathbf{A}\|_F = \sum_{i=1}^{r} \sigma_i$, where σ_i for $1 \le i \le r$ are the singular values of $\mathbf{A}$ and r is the rank of $\mathbf{A}$. State clearly any result you use in your proof.

b i) Find the singular value decomposition of the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

ii) Suppose the matrix $\mathbf{A}$ in part i) represents the linear map $f : \mathbb{R}^2 \to \mathbb{R}^3$ in the standard coordinates system. Determine the image under f of the unit circle. Justify your answer.

c i) Let

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 1 & 3 \\ -1 & -1 \\ 1 & 3 \end{bmatrix}$$

Perform QR decomposition on matrix $\mathbf{A}$ using the Gram-Schmidt process to find orthogonal matrix $\mathbf{Q}$ and upper triangular matrix $\mathbf{R}$ with $\mathbf{A} = \mathbf{QR}$.

ii) Suppose the matrix $\mathbf{A}$ in part i) represents the linear map $f : \mathbb{R}^2 \to \mathbb{R}^4$ in the standard coordinate system. Explain carefully how the matrix representation of f is changed under the QR decomposition.

iii) Find the eigenvalues, with their algebraic and geometric multiplicities, of the projection transformation $P_{\mathbf{u}} = \mathbf{I} - \mathbf{u}\mathbf{u}^T : \mathbb{R}^n \to \mathbb{R}^n$ where $\mathbf{u} \in \mathbb{R}^n$ is a unit vector. Is there an orthonormal basis of $\mathbb{R}^n$ with respect to which $P_{\mathbf{u}}$ is diagonal? If so, find such an orthonormal basis. Otherwise, explain why such an orthonormal basis does not exist.

The three parts carry, respectively, 35%, 30%, and 35% of the marks.

2a i) Using both l_∞ and l_1- norms, calculate the condition number of a given matrix $\mathbf{A}$ with its inverse $\mathbf{A}^{-1}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

ii) Apply Power Method to estimate the dominant eigenvalue and the corresponding eigenvector of the matrix $\mathbf{A}$ introduced in part 2a i). Take $\mathbf{x}^{(0)} = [1,1,1]^T$ as the initial vector and perform 3 iterations. Compute Raleigh quotient on the last iteration.

b i) By applying the Second Derivative Test, find the relative maximum (or minimum) value of:

$$f(x,y) = 2xy + 2x - x^2 - 2y^2$$

ii) Compute 3 iterations of Steepest Ascent (or Descent) method to maximise (or minimise) $f(x,y)$ with an initial guess: $x = -1$ and $y = 1$.

c Solve the following Linear Programming problem using Simplex method.

Maximise:

$$Z = 5x_1 + 10x_2 + 8x_3$$

Subject to constraints:

$$3x_1 + 5x_2 + 2x_3 \leq 60$$
$$4x_1 + 4x_2 + 4x_3 \leq 72$$
$$2x_1 + 4x_2 + 5x_3 \leq 100$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

The three parts carry, respectively, 35%, 35%, and 30% of the marks.