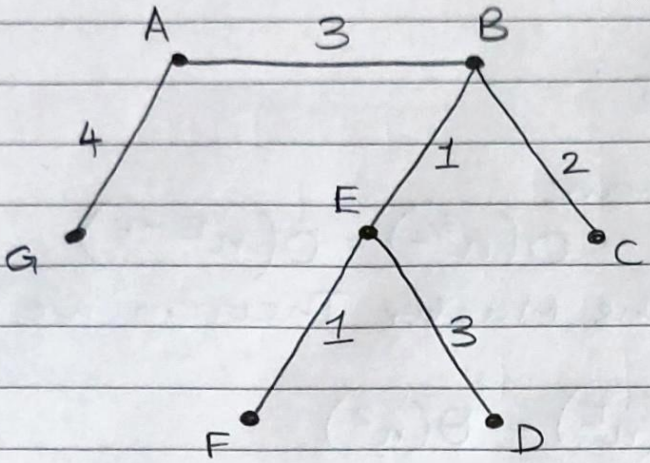
2 a) i) order added: A, B, E, F, C, D, G



ii) 2 distinct MSTs. All arcs of weight 3 or less are used in the above MST. If any of these arcs were to be ~~replaced~~ removed and another arc of the graph that isn't in the MST above were to be included instead, that arc would have a greater weight than the removed arc, thus increasing the weight of the tree (so the result would not be an MST). Thus arcs BE, EF, BC, AB and DE (all arcs of weight 3 or less) must be in <u>any</u> MST.

The only node left to connect after including those mandatory arcs is G. This can be done by including arcs AG, FG or EG. The weight of EG is greater than that of the other two, so either AG or EG must be included. Thus we have only two distinct MSTs of the graph (depending on whether AG or FG is included).

b) i) We have $a = 4$, $b = 2$, $f(n) = 8n$.

Thus $E = \log_b a = \log_2 4 = 2$.

Let $\varepsilon = \frac{1}{2}$.

Then we have

$f(n) = 8n = O(n^{3/2}) = O(n^{E-\varepsilon})$

so by the Master Theorem we have

$$T_1(n) = \Theta(n^E) = \Theta(n^2)$$

ii) We have $a = 8$, $b = 4$, $f(n) = 2n \log n$.

Thus $E = \log_4 8 = \frac{3}{2} = 1.5$

We have $\log n = O(n^m)$ for any $m > 0$,

so $2n \log n = O(n^{1.25})$, since $2n = O(n)$.

Let $\varepsilon = 0.25$

Then we have

$f(n) = 2n \log n = O(n^{1.25})$

$= O(n^{E-\varepsilon})$

so by the Master Theorem,

$$T_2(n) = \Theta(n^E) = \Theta(n^{3/2})$$

c)

No comparisons for the first element.
Since the first $\frac{n}{2}$ elements are sorted,
we have 1 comparison each for the next $\frac{n}{2} - 1$ elements.

Then $\frac{n}{2}$ comparisons for the next element
(i.e. 1 in the given example).

And $\frac{n}{2} + 1$ comparisons for the final $\frac{n}{2} - 1$
elements.

$$\text{So } 0 + 1\left(\frac{n}{2} - 1\right) + \frac{n}{2} + \left(\frac{n}{2} + 1\right)\left(\frac{n}{2} - 1\right)$$

$$= \frac{n}{2} - 1 + \frac{n}{2} + \frac{n^2}{4} - 1$$

$$= \frac{n^2}{4} + n - 2 \text{ comparisons}$$

d) Define another problem VER-INDN as:
given an undirected graph G, a node x
of G, $k \geq 1$ and a set I of nodes of G,
is I an independent set of G of size $\geq k$
containing x?

To determine VER-INDN(G, x, k, I):
1) check size$(I) \geq k$
2) check $x \in I$
3) check no two nodes of I are adjacent.

(1) and (2) can obviously be checked in p-time. (3) can also be checked in p-time by iterating through the adjacency list of G and for each node in I checking that no node that it is adjacent to in G is also a member of I. Thus VER-INDN is in P.

Also clearly (by definition of VER-INDN), we have

$$\text{INDN}(G, x, k) \text{ iff } \exists I.\ \text{VER-INDN}(G, x, k, I)$$

and also clearly there exists a polynomial p such that $|I| \leq p(|G, x, k|)$ since we have $I \subseteq G$.

Thus INDN is in NP.

ii) We show that for all decision problems $D \in NP$, $D \leq$ INDN.

We have that IND is NP-complete, so for all $D \in NP$, $D \leq$ IND.

So it suffices to show IND $\leq$ INDN since by transitivity of $\leq$, if $D \leq$ IND and IND $\leq$ INDN, $D \leq$ INDN.

To show IND $\leq$ INDN we show that for some p-time computable function f,

$$\text{IND}(G, k) \text{ iff } \text{INDN}(f(G, k)).$$

We define f as

$$f(G, k) = \cancel{(G', k', x)}\ (G', x, k')$$

where

$$\text{nodes}(G') = \text{nodes}(G) \cup \{x\}$$
$$k' = k + 1$$
x is a new node

So f adds a node to G that is not adjacent to any other node, and adds 1 to k. Clearly f is p-time computable.

Assume IND(G, k). Then G has an ~~a~~ independent set of size $\geq k$. Thus if $(G', x, k') = f(G, k)$, then G' also has an independent ~~size of~~ set of size $k' = k + 1$, with the extra node being x, which can always be included in an independent set as it is adjacent to no other nodes. Thus IND(G, k) implies INDN$(f(G, k))$.

Assume INDN$(f(G, k))$. Then $G \cup \{x\}$ ~~G~~ has ~~by simply re~~ ~~remove/have~~ an independent set of size $\geq k+1$ containing x, the node added by f. Thus we ~~it~~ also must have an independent set of size $\geq k$ of ~~the~~ the graph G. Hence we have INDN$(f(G, k))$ implies IND(G, k).

So IND(G, k) iff INDN$(f(G, k))$, so IND $\leq$ INDN, and INDN is NP-complete.