COMP 245 - Assessed Coursework – Solutions

S1) (a) In the following S is the sample space, $\mathcal{F}$ is a σ-algebra on S and P is a probability measure on $(S, \mathcal{F})$. We need to show the following conditions hold:

i.

$$\begin{aligned} P(S|F) &= \frac{P(S \cap F)}{P(F)} \quad \text{by def. of conditional prob.} \\ &= \frac{P(F)}{P(F)} \\ &= 1 \end{aligned}$$

$\boxed{\text{1 marks}}$

ii. Let $A_1, A_2, \dots$ be a disjoint collection on $\mathcal{F}$ so that $A_i \cap A_j = \emptyset$ for all pairs $i \neq j$. Then

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i \middle| F\right) &= P(A_1 \cup A_2 \cup A_3 \dots |F) \\ &= \frac{P((A_1 \cup A_2 \cup A_3 \dots) \cap F)}{P(F)} \quad \text{by def. of conditional prob.} \\ &= \frac{P((A_1 \cap F) \cup (A_2 \cap F) \dots)}{P(F)} \quad \text{distributivity} \\ &= \frac{\sum_{i=1}^{\infty} P(A_i \cap F)}{P(F)} \quad \text{as } A_i \cap F \text{ form a partition and } P \text{ is a prob. measure} \\ &= \sum_{i=1}^{\infty} P(A_i|F) \quad \text{by def. of conditional prob.} \end{aligned}$$

$\boxed{\text{2 marks}}$

iii. We also need to show that for every event $E \subseteq S$ $P(E|F) \geq 0$. For an event $E \subseteq S$, we have

$$P(E|F) = \frac{P(\mathrm{E} \cap F)}{P(F)} \quad \text{by def. of conditional prob.}$$

Since $P(E \cap F) \geq 0$ since P is a probability measure and we are given $P(F) > 0$, we have $P(E|F) \geq 0$. $\boxed{\text{1 marks}}$

(b) To show $\overline{E}$ and $\overline{F}$ are independent events, start with

$$\begin{aligned} P(\overline{E} \cap \overline{F}) &= P(\overline{E \cup F}) \quad \text{by De Morgans Law} \\ &= 1 - P(E \cup F) \\ &= 1 - P(E) - P(F) + P(E \cap F) \quad \text{using } P(E \cup F) = P(E) + P(F) - P(E \cap F) \\ &= 1 - P(E) - P(F) + P(E)P(F) \quad \text{as } P(E \cap F) = P(E)P(F) \text{ by independence} \\ &= (1 - P(E))(1 - P(F)) \\ &= P(\overline{E})P(\overline{F}) \end{aligned}$$

$\boxed{\text{2 marks}}$

(c) First, note that $S = \bigcup_{n=0}^{\infty} E_n$ (the E_n form a disjoint partition of the sample space).

[*2 marks for noticing the partition of the sample space*] $\boxed{\text{2 marks}}$

Thus

$$\begin{aligned}
1 = P(S) &= \sum_{n=0}^{\infty} P(E_n) \\
&= P(E_0) + P(E_1) + P(E_2) + \dots \\
&= P(E_0) + 0.3P(E_0) + 0.3^2 P(E_0) + \\
&= P(E_0)\left(\sum_{j=0}^{\infty} 0.3^j\right) \\
&= P(E_0)\left(\frac{1}{1-0.3}\right) \quad \text{using result for geometric progression.} \\
\implies P(E_0) &= 0.7.
\end{aligned}$$

[*2 for correct manipulations, 1 for correct* $P(E_0)$] $\boxed{\text{3 marks}}$

Then the probability that the individual makes more than one visit to the website is

$$P(n > 1) = 1 - P(n = 0) - P(n = 1) = 1 - 0.7 - (0.3)(0.7) = 0.09$$

$\boxed{\text{1 marks}}$

[**Total for question 1: 12 marks**]

S2) (a) For $X \sim \text{Poisson}(\lambda)$, we have the probability mass function (pmf)

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x \in \mathbb{X} = \{0, 1, 2, \dots\}$$

and $p_X(x) = 0$ for $x \notin \mathbb{X}$. For a valid pmf we require that (i) $\sum_x p_X(x) = 1$ and (ii) $p_X(x) \geq 0$ for all $x \in \mathbb{R}$. First, for (i), we have

$$\begin{aligned}
\sum_{x=0}^{\infty} p_X(x) &= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \\
&= e^{-\lambda} e^{\lambda} = 1,
\end{aligned}$$

where we recognise the Taylor expansion of $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ $\boxed{\text{2 marks}}$

Also, it is clear that $p_X(x) \geq 0$ for all $x \in \mathbb{R}$. *(this needs to be mentioned for the mark.)* $\boxed{\text{1 marks}}$

(b) To find $E(X^2)$ consider the variance $\text{var}(X)$. Since $X \sim \text{Poisson}(\lambda)$ we have $E(X) = \text{var}(X) = \lambda$. Thus

$$\text{var}(X) = E(X^2) - (E(X))^2 \implies \lambda = E(X^2) - \lambda^2$$

A simple rearrangement gives $E(X^2) = \lambda + \lambda^2$. $\boxed{\text{2 marks}}$

Alternatively, one can do this by brute force:

$$\begin{aligned}
E(X^2) &= \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=1}^{\infty} x^2 \frac{\lambda^x}{x!} \\
&= e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^x}{(x-1)!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} (x+1) \frac{\lambda^{x+1}}{x!} \\
&= \lambda e^{-\lambda} \left(\sum_{x=0}^{\infty} x \frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \right) \\
&= \lambda \left(\sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} + e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} \right) \\
&= \lambda \left(E(X) + 1 \right) \\
&= \lambda^2 + \lambda
\end{aligned}$$

as $E(X) = \lambda$.

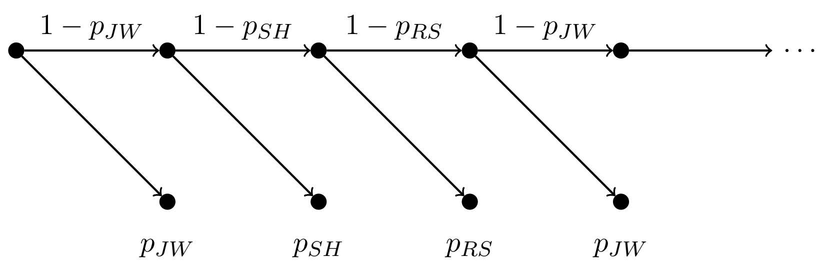
(c) For a valid pmf, we require that $\sum_z p_Z(z) = 1$, so $\boxed{\text{1 marks}}$

$$\begin{aligned}
1 = \sum_{z=0}^{\infty} p_Z(z) &= k \sum_{z \neq 16} p_X(z) + 0.2017 \\
&= k \left(1 - p_X(16)\right) + 0.2017 \\
\implies k &= \frac{0.7983}{1 - p_X(16)} = \frac{0.7983}{1 - \frac{e^{-\lambda}\lambda^{16}}{16!}}
\end{aligned}$$

[*either final form for k is acceptable, however the solution should be simplified as far as possible. 2 for manipulation, 2 for correct final answer.*] $\boxed{\text{4 marks}}$

[**Total for question 2: 10 marks**]

S3) The outcomes of this game can be depicted as follows



We then have that

$$\begin{aligned}
P(\text{SH winning}) =& (1-p_{JW})p_{SH} \\
&+ (1-p_{JW})^2(1-p_{SH})(1-p_{RS})p_{SH} \\
&+ (1-p_{JW})^3(1-p_{SH})(1-p_{RS})p_{SH} \\
&+ \dots \\
=& (1-p_{JW})p_{SH}\sum_{j=0}^{\infty}[(1-p_{JW})(1-p_{SH})(1-p_{RS})]^j \\
=& \frac{(1-p_{JW})p_{SH}}{1-(1-p_{JW})(1-p_{SH})(1-p_{RS})}
\end{aligned}$$

[*3 for the argument, 3 for the result.*] **6 marks**

In a similar fashion, we can get the probabilities of the other players winning:

$$\begin{aligned}
P(\text{JW winning}) &= \frac{p_{JW}}{1-(1-p_{JW})(1-p_{SH})(1-p_{RS})} \\
P(\text{RS winning}) &= \frac{(1-p_{JW})(1-p_{SH})p_{RS}}{1-(1-p_{JW})(1-p_{SH})(1-p_{RS})}
\end{aligned}$$

[*1 mark each for the other probabilities.*] **2 marks**

The game would be fair if $P(\text{SH winning})$, $P(\text{JW winning})$, and $P(\text{RS winning})$ were equal. This could be the case if it is the case that, for example, $p_{JW} = 1/4$, $p_{SH} = 1/3$ and $p_{RS} = 1/2$. So the answer is: **it depends** on the probabilities p_{JW}, p_{SH} and p_{RS}.

[*2 marks for a reasonable argument, 3 for correct conclusion.*] **5 marks**

An alternative solution would be to condition on the first 3 outcomes e.g.

$$\begin{aligned}
P(\textit{JW winning}) &= P(\textit{JW winning in} \leq \textit{3 shots}) + P(\textit{JW winning in} > \textit{3 shots}) \\
&= P(\textit{JW winning on the first shot}) + P(\textit{JW winning})P(> \textit{3 shots}) \\
&= p_{JW} + P(\textit{JW winning})P(\textit{initial 3 shots missed}) \\
&= p_{JW} + P(\textit{JW winning})(1-p_{JW})(1-p_{SH})(1-p_{RS}).
\end{aligned}$$

Rearranging this the same result as before.

[**Total for question 3: 13 marks**]

S4) (a) The number (random variable) of meals required is

$$N = 1 + X_2 + X_3 + X_4 + \cdots + X_k,$$

where X_i is the number of visits required for meal i. **2 marks**

(b) Each X_i has a Geometric distribution with pmf:

$$P(X_i = v) = \frac{k-i+1}{k}\left(\frac{i-1}{k}\right)^{v-1} \qquad v = 1, 2, \dots.$$

That is $X_i \sim \text{Geometric}\left(\frac{k-i+1}{k}\right)$ for $i = 2, 3, \ldots$.

[1 mark for stating the X_i are Geometric, 1 the correct parameters (it is not necessary state the pmf)] $\boxed{\text{2 marks}}$

(c) Hence, $E(X_i) = k/(k-i+1)$. Then by linearity of expectation, we have

$$\begin{aligned} E(N) &= 1 + \frac{k}{k-1} + \frac{k}{k-2} + \frac{k}{k-3} + \ldots \\ &= k \sum_{i=1}^{k} \frac{1}{i}. \end{aligned}$$

[1 expectation of single X_i, 1 for using/stating linearity of expectation, 1 for final result.] $\boxed{\text{3 marks}}$

Further, since the X_i's are independent we have

$$\text{Var}(N) = \sum_{i=1}^{k-1} \text{Var}(X_i).$$

Since $X_i \sim \text{Geometric}((k-i)/k)$, we have $\text{Var}(X_i) = (ik)/(k-i)^2$ and so

$$\text{Var}(N) = \sum_{i=1}^{k-1} \text{Var}(X_i) = k \sum_{i=1}^{k-1} \frac{i}{(k-i)^2}.$$

[1 mark for stating independence, 2 for the final result.] $\boxed{\text{3 marks}}$

[**Total for question 4: 10 marks**]