145 Mathematics I 2020 Exam Sample Solution

Disclaimer: This is not an official answer key, so please correct any mistake if you find one. There are more than one possible solution for some questions, so keep in mind about that!

1. **a. i)** A possible Gaussian elimination.

$$[A \mid b] = \left[\begin{array}{cccc|c} 3 & 6 & 3 & -7 & 1 \\ 1 & 2 & -1 & 3 & 1 \\ 2 & 4 & 1 & -2 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 3 & 6 & 3 & -7 & 1 \\ 0 & 0 & -2 & \frac{16}{3} & \frac{2}{3} \\ 0 & 0 & -1 & \frac{8}{3} & \frac{1}{3} \end{array}\right] = \left[\begin{array}{cccc|c} 3 & 6 & 3 & -7 & 1 \\ 0 & 0 & 1 & -\frac{8}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{8}{3} & -\frac{1}{3} \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 3 & 6 & 0 & 1 & 2 \\ 0 & 0 & 1 & -\frac{8}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 2 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{8}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

ii) From the Gaussian elimination in i) we know that:

$$\begin{aligned} x_1 + 2x_2 + \frac{1}{3}x_4 = \frac{2}{3} &\Rightarrow x_1 = -2x_2 - \frac{1}{3}x_4 + \frac{2}{3} \\ x_3 - \frac{8}{3}x_4 = \frac{1}{3} &\Rightarrow x_3 = \frac{8}{3}x_4 - \frac{1}{3} \end{aligned}$$

Hence the basic variables are x_1 and x_3, the free variables are x_2 and x_4.

(Notice you can exchange the position of x_3 and x_4 to make x_3 free instead, etc.)

iii) From ii) we know that the solution set is:

$$x = \begin{bmatrix} -2x_2 - \frac{1}{3}x_4 + \frac{2}{3} \\ x_2 \\ \frac{8}{3}x_4 - \frac{1}{3} \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{8}{3} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{2}{3} \\ 0 \\ -\frac{1}{3} \\ 0 \end{bmatrix}$$

b. i) To determine the rank of A, we need to know the number of linearly independent column/row vectors in A. Notice that since we know column rank equals to row rank for any matrix, we can perform Gaussian elimination by rows or by columns.

A possible Gaussian elimination by rows:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A possible Gaussian elimination by column(better and quicker):

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

We have found that there are 2 rows/columns that are linearly independent. Hence $\mathrm{rk(A){=}2}$.

Since there are 3 columns in A and 2 linearly independent columns in A, A must be a matrix that represents a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$. Hence we know that $f_A:\mathbb{R}^3\rightarrow\mathbb{R}^2$According to the rank-nullity equation, the nullity of A is $\dim(\mathbb{R}^3)-\dim(\mathrm{Im(A)})=3-2=1$. We can verify this by solving the solution space of $A\mathbf{x}=\mathbf{0}$

ii) To compute the basis of $\mathrm{Im(A)}$, we need to solve the solution space of $A\mathbf{x}=\mathbf{b}$.

$$\begin{bmatrix}1&2&-1\\0&1&0\\-1&2&1\\2&3&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\Rightarrow\begin{bmatrix}x_1+2x_2-x_3\\x_2\\x_3-x_1+2x_2\\2x_1+3x_2-2x_3\end{bmatrix}=x_1\begin{bmatrix}1\\0\\-1\\2\end{bmatrix}+x_2\begin{bmatrix}2\\1\\2\\3\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\1\\-2\end{bmatrix}$$

Since the first and second columns are linearly independent, $\mathrm{Im(A){=}2}$. A possible basis for $\mathrm{Im(a)}$ could be $\left(\begin{bmatrix}1\\0\\-1\\2\end{bmatrix},\begin{bmatrix}2\\1\\2\\3\end{bmatrix}\right)$.

iii) Basis for $\mathrm{ker(A)}$ could be $\begin{bmatrix}1\\0\\1\end{bmatrix}$ by solving for the solution space of $A\mathbf{x}=\mathbf{0}$

$$\begin{bmatrix}0&0&0\\0&1&0\\-1&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\mathbf{0}\Rightarrow\quad\begin{matrix}x_2=0\\-x_1+x_3=0\Rightarrow x_1=x_3\end{matrix}\quad\Rightarrow\mathbf{x}=x_1\begin{bmatrix}1\\0\\1\end{bmatrix}$$

c. i) To determine the eigenvalues of A, we need to find the value of λ that satisfies $(A-\lambda I_3)\mathbf{x}=\mathbf{0}$. This is equivalent as to solve $\det(A-\lambda I_3)=0$ because we want to solve $(A-\lambda I_3)\mathbf{x}=\mathbf{0}$ nontrivially ($\mathbf{x}\neq\mathbf{0}$).

When $\lambda=6$,

$$\det(A-6I_3)=\begin{vmatrix}-2-6&-4&2\\-2&1-6&2\\4&2&5-6\end{vmatrix}=\begin{vmatrix}-8&-4&2\\-2&-5&2\\4&2&-1\end{vmatrix}=\begin{vmatrix}-8&-4&2\\-2&-5&2\\0&0&0\end{vmatrix}=0.$$

Hence 6 is one of the eigenvalues.

To find the rest of the eigenvalues, we need to find the characteristic polynomial:

$$p(\lambda)=\begin{vmatrix}-2-\lambda&-4&2\\-2&1-\lambda&2\\4&2&5-\lambda\end{vmatrix}=-(\lambda+5)(\lambda-3)(\lambda-6)$$

The eigenvalues are $-5,3,6$.

ii) We can determine the singularity of A by checking whether $\mathrm{det(A)}=0$

$$
\begin{aligned}
\det(\mathrm{A}) &= \begin{vmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} + 4\begin{vmatrix} -2 & 2 \\ 4 & 5 \end{vmatrix} + 2\begin{vmatrix} -2 & 1 \\ 4 & 2 \end{vmatrix} \\
&= -2 + 4\cdot(-18) + 2\cdot(-8) \\
&= -90 \neq 0
\end{aligned}
$$

Hence A is not singular and thus has its inverse.

iii) When $\lambda = -5$:

$$
A + 5I_3 = \begin{bmatrix} 3 & -4 & 2 \\ -2 & 6 & 2 \\ 4 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned} \Rightarrow \mathbf{x} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}
$$

A vector in the space of E_{-5} therefore could be $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

When $\lambda = 3$:

$$
A - 3I_3 = \begin{bmatrix} -5 & -4 & 2 \\ -2 & -2 & 2 \\ 4 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned} \Rightarrow \mathbf{x} = x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}
$$

A vector in the space of E_3 therefore could be $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$.

When $\lambda = 6$:

$$
A - 6I_3 = \begin{bmatrix} -8 & -4 & 2 \\ -2 & -5 & 2 \\ 4 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{16} \\ 0 & 1 & -\frac{8}{3} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 - \frac{1}{16}x_3 &= 0 \\ x_2 - \frac{8}{3}x_3 &= 0 \end{aligned} \Rightarrow \mathbf{x} = x_3 \begin{bmatrix} \frac{1}{16} \\ \frac{8}{3} \\ 1 \end{bmatrix}
$$

A vector in the space of E_6 therefore could be $\begin{bmatrix} \frac{1}{16} \\ \frac{8}{3} \\ 1 \end{bmatrix}$.

iv) An invertible S could be the combination of the eigenvectors of A, therefore from iii) we know that $S = \begin{bmatrix} -2 & -2 & \frac{1}{16} \\ -1 & 3 & \frac{8}{3} \\ 1 & 1 & 1 \end{bmatrix}$. Notice that the order of the columns does not matter since the exchange of columns would still produce a diagonal matrix whose elements on the diagonal are exchanged.

v) From Caylaey-Hamilton Theorem we know that $p(A) = 0$.

$$
\begin{aligned}
-(A+5I)(A-3I)(A-6I) &= 0 \\
(A^2+2A-15I)(A-6I) &= 0 \\
A^3-4A^2-27A+90I &= 0 \\
A(A^2-4A-27I) &= -90I \\
A^{-1} &= -\frac{A^2-4A-27I}{90}
\end{aligned}
$$

Hence:

$$
\begin{aligned}
P^{-1}A^{-1}P &= -\frac{1}{90}(P^{-1}A^2P-4P^{-1}AP-27P^{-1}IP) \\
&= -\frac{1}{90}(P^{-1}AP\cdot P^{-1}AP-4P^{-1}AP-27I)
\end{aligned}
$$

We know from the fact that $S^{-1}AS$ in iv) is diagonal, and any sum or matrix multiplication of two diagonal matrices will still produce a diagonal matrix. We can thereby let $P=S$ and $P^{-1}A^{-1}P$ will produce a diagonal matrix. Notice this can also be explained using the fact that any invertible matrix A will have the same eigenvectors as its inverse, but the proof will be too complicate.

d. The points on the plane satisfies the matrix $x_2\begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}+x_3\begin{bmatrix}-\frac{1}{2}\\0\\1\end{bmatrix}$. Therefore,

$B=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\1&0\\0&1\end{bmatrix}$.

Hence $P_\pi=B(B^TB)^{-1}B^T=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\1&0\\0&1\end{bmatrix}\begin{bmatrix}\frac{5}{6}&\frac{1}{6}\\\frac{1}{6}&\frac{5}{6}\end{bmatrix}\begin{bmatrix}\frac{1}{2}&1&0\\-\frac{1}{2}&0&1\end{bmatrix}=\begin{bmatrix}\frac{1}{3}&\frac{1}{3}&-\frac{1}{3}\\\frac{1}{3}&\frac{5}{6}&\frac{1}{6}\\-\frac{1}{3}&\frac{1}{6}&\frac{5}{6}\end{bmatrix}$.

The projection $p=P_\pi\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}$

2. **a.** This sequence will converge to $\frac{7}{2}$.

$$
\begin{aligned}
\lim_{n\to\infty}\frac{(2n^2-1)(7n+5)}{4n^3+n-1} &= \lim_{n\to\infty}\frac{14n^3+10n^2-7n-5}{4n^3+n-1} \\
&= \lim_{n\to\infty}\frac{14+\frac{10}{n}-\frac{7}{n^2}-\frac{5}{n^3}}{4+\frac{1}{n^2}-\frac{1}{n^3}} \\
&= \frac{14}{4}=\frac{7}{2}
\end{aligned}
$$

b. i) Maclaurin series of $\cos x$:

$$\cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots=\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k}}{(2k)!}$$

Maclaurin series of $\cos(x^4)$ is then:

$$\cos(x^4) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{8k}}{(2k)!} = 1 - \frac{x^8}{2!} + \frac{x^{16}}{4!} - \frac{x^{24}}{6!} + \ldots$$

ii) Radius of convergence:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{8n+8}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{8n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^8}{(2n+2)(2n+1)} \right| \\
&= 0
\end{aligned}$$

This series will converge regardless of the value of x, hence the radius of converngence is ∞.

iii) We can expand $\cos(x^4)$ using the Maclaurin series derived in i):

$$\begin{aligned}
\lim_{x\to\infty} \frac{\cos x^4 - 1 + \frac{x^8}{2}}{x^{16}} &= \lim_{x\to\infty} \frac{(1 - \frac{x^8}{2!} + \frac{x^{16}}{4!} - \frac{x^{24}}{6!} + \ldots) - 1 + \frac{x^8}{2}}{x^{16}} \\
&= \lim_{x\to\infty} \frac{\frac{x^{16}}{4!} - \frac{x^{24}}{6!} + \ldots}{x^{16}} \\
&= \lim_{x\to\infty} \frac{1}{4!} - \frac{x^8}{6!} + \frac{x^{16}}{8!} - \ldots \\
&= \frac{1}{4!} = \frac{1}{24}
\end{aligned}$$

c. i) $f(x)$ and $g(x)$ are both differentiable on $(-1, 1)$. In fact, $f'(x) = 5e^x$ and $g'(x) = 8x - 15$, and $f'(x)$, $g'(x)$ are defined on $(-1, 1)$ as well.

$\lim_{x\to 0} f(x) = \lim_{x\to 0} 5e^x - \lim_{x\to 0} 5 = 5 \cdot 1 - 5 = 0$

$\lim_{x\to 0} g(x) = \lim_{x\to 0} 4x^2 - \lim_{x\to 0} 15x = 0 - 0 = 0$

Also, when $g'(x) = 0$, $x = \frac{15}{8} = 1.875$ which is not in the range $(-1, 1)$.

Hence, $g'(x) \neq 0$ for all $x \in (-1, 1)\backslash\{0\}$

Hence the variant rule is applicable here.

ii) $\lim_{x\to 0} \frac{5e^x - 5}{4x^2 - 15x} = \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{5e^x}{8x - 15} = \frac{5}{-15} = -\frac{1}{3}$

d. Let $a_n = f(n) = \frac{1}{\sqrt{n}}$. We now take the integral of $f(x)$:

$$\int_N^\infty f(x)dx = \int_N^\infty \frac{1}{\sqrt{x}} dx = (2\sqrt{x})\Big|_N^\infty = \infty$$

Hence, $\int_N^\infty f(x)dx$ diverges, and according to the integral test, $\sum_{n\geq 1} \frac{1}{\sqrt{n}}$ diverges as well.