Reference Sheet for CO233 Computational Methods

Autumn 2017

1 The $\mathbb{R}^n$ and $\mathbb{C}^n$ Vector Spaces

- *Convex combination*: triangle $a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w}$ with vertices $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and $0 \leq a, b, c \leq 1$ and $a + b + c = 1$, for $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$.
- *Inner product*: $\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{3} \boldsymbol{u}_i \boldsymbol{v}_i = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos\theta$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$, where $\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u}_1^2 + \boldsymbol{u}_2^2 + \boldsymbol{u}_3^2}$.
- *Inner product*: $\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \sum_{i=1}^{n} = \boldsymbol{v}_i^* \boldsymbol{w}_i$ for $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{C}^n$.
- *Norm*: $\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}$.

Representation of Linear Maps

- *Linear map* $\mathbb{R}^n \to \mathbb{R}^m$: can be represented by the real matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$:
  - For all $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, $\boldsymbol{A}(\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{A}\boldsymbol{v} + \boldsymbol{A}\boldsymbol{w}$.
  - For all $\boldsymbol{v} \in \mathbb{R}^n, c \in \mathbb{R}$, $\boldsymbol{A}(c\boldsymbol{v}) = c\,(\boldsymbol{A}\boldsymbol{v})$.
- Extends simply to $\mathbb{C}^n$.

2 Norms

2.1 Vector Norms

A vector norm on $\mathbb{R}^n$ is a real-valued map

$$\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$$

which satisfies:

1. For any non-zero vector $\boldsymbol{x} \in \mathbb{R}^n$, $\|\boldsymbol{x}\| > 0$.
2. For any scalar λ and $\boldsymbol{x} \in \mathbb{R}^n$, $\|\lambda \boldsymbol{x}\| = |\lambda| \|\boldsymbol{x}\|$
3. For two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

The l_p Norms

$$\|\boldsymbol{v}\|_p = \left(\sum_{i=1}^{n} |\boldsymbol{v}_i|^p \right)^{1/p}$$

Properties

- For any vector $\boldsymbol{x}$, $\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1$.

2.2 Cauchy-Schwartz Inequality

$$|\langle u, v \rangle| \leq \|\boldsymbol{u}\| \|\boldsymbol{v}\|$$

Proof

- Consider $\lambda \boldsymbol{u} + \boldsymbol{v}$.
- Since the length of any vector is non-negative, $0 \leq (\lambda \boldsymbol{u} + \boldsymbol{v})\,(\lambda \boldsymbol{u} + \boldsymbol{v})$.
- Therefore $\lambda = \frac{\langle u, v \rangle}{\|\boldsymbol{u}\|^2}$.

2.3 Matrix Norms

A matrix norm on $\mathbb{R}^{m \times n}$ is a real-valued map.

3 Linear Independence

For $\boldsymbol{a_i} \in \mathbb{R}^m$, with $i = 1, \ldots, k$, the $\boldsymbol{a_i}$s are linearly independent if whenever $\boldsymbol{x_i} \in \mathbb{R}$, we have $\sum_{i=1}^{k} \boldsymbol{x_i} \boldsymbol{a_i} = 0$, then $\boldsymbol{x_i} = 0$ for $i = 1, \ldots, k$.

Methods for Determining Linear Independence Columns of $\boldsymbol{A}$ are linearly independent if:

- Calculate determinant. Find $\det(\boldsymbol{A}) \neq 0$.
- Solve $\boldsymbol{A}\boldsymbol{x} = 0$. Find $\boldsymbol{x} = 0$.