

IMPERIAL COLLEGE LONDON

TIMED REMOTE ASSESSMENTS 2020-2021

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

*This paper is also taken for the relevant assessments for the
Associateship of the City and Guilds of London Institute*

PAPER COMP40002

MATHEMATICS I: FOUNDATIONS

Thursday 6 May 2021, 10:00

Duration: 95 minutes

Includes 15 minutes for access and submission

Answer ALL TWO questions

Open book assessment

This time-limited remote assessment has been designed to be open book. You may use resources which have been identified by the examiner to complete the assessment and are included in the instructions for the examination. You must not use any additional resources when completing this assessment.

The use of the work of another student, past or present, constitutes plagiarism. Giving your work to another student to use constitutes an offence. Collusion is a form of plagiarism and will be treated in a similar manner. This is an individual assessment and thus should be completed solely by you. The College will investigate all instances where an examination or assessment offence is reported or suspected, using plagiarism software, vivas and other tools, and apply appropriate penalties to students. In all examinations we will analyse exam performance against previous performance and against data from previous years and use an evidence-based approach to maintain a fair and robust examination. As with all exams, the best strategy is to read the question carefully and answer as fully as possible, taking account of the time and number of marks available.

Paper contains 2 questions

1 a Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -2 \\ -1 & 2 & 2 & -5 \\ 2 & 3 & 1 & -7 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

and consider the linear system of equations $\mathbf{Ax} = \mathbf{b}$.

- i) Perform elementary row operations on the augmented matrix $[\mathbf{A}|\mathbf{b}]$ to reduce \mathbf{A} to Row Echelon Form.
- ii) Identify the basic and the free variables.
- iii) Find the general solution of $\mathbf{Ax} = \mathbf{b}$ and describe it geometrically.
- iv) Find a solution of $\mathbf{Ax} = \mathbf{b}$ with least distance from the origin.

b Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 1 & 5 & -8 & 3 \end{bmatrix}.$$

- i) Find the rank and the nullity of \mathbf{A} .
- ii) Find a basis for the image space $\text{im}(\mathbf{A})$ and also a basis for the null space $\text{null}(\mathbf{A})$.
- iii) Find the rank and the nullity of \mathbf{A}^T .

c Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

- i) Show that 5 is an eigenvalue of \mathbf{A} and determine the other eigenvalues.
- ii) Find an eigenvector for each of the eigenvalues of \mathbf{A} .
- iii) Determine the angle between each pair of the eigenvectors of \mathbf{A} .
- iv) Obtain an invertible $\mathbf{S} \in \mathbb{R}^{3 \times 3}$ such that $\mathbf{S}^{-1}\mathbf{AS}$ is diagonal.

d A linear map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ acts on an ordered basis $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ as follows:

$$f(\mathbf{v}_1) = 2\mathbf{v}_3 - \mathbf{v}_1 - 2\mathbf{v}_2, \quad f(\mathbf{v}_2) = 3\mathbf{v}_2 - 5\mathbf{v}_1 - \mathbf{v}_3, \quad f(\mathbf{v}_3) = \mathbf{v}_2 - \mathbf{v}_1.$$

Find the matrix representation of f with respect to \mathbf{V} .

The four parts carry, respectively, 35%, 20%, 30%, and 15% of the marks.

- 2a i) Find the supremum and the infimum of the following set S :

$$S = \{x \in \mathbb{Q} : x^4 < 5\},$$

where \mathbb{Q} is the set of rational numbers.

- ii) Using the $(\epsilon - N)$ definition of limits, prove that the sequence $(a_n)_{n \geq 1}$ with

$$a_n = \frac{1}{n^2 - \frac{1}{2}}$$

converges.

- iii) Determine whether or not the following limit exists and obtain the limit if it does exist:

$$\lim_{x \rightarrow 1} \frac{x - 1}{\cos(\frac{\pi}{2}x)}.$$

- b Determine whether or not each of the following series converges and prove your answer.

i) $\sum_{n=0}^{\infty} (\sqrt{n+1} - \sqrt{n}).$

ii) $\sum_{n=0}^{\infty} \left(\frac{5n-3n^3}{7n^3+2} \right)^n.$

iii) $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}.$

- c i) Find the MacLaurin series for the following function $f : \mathbb{R} \rightarrow \mathbb{R}$ and determine the radius of convergence of the series.

$$f(x) = \frac{1}{1 + 2x^2}$$

(**Hint.** Express $f(x)$ as the sum of a geometric series.)

- ii) Suppose that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ has a Maclaurin series with radius of convergence $1/2$, i.e., its Maclaurin series converges for $|x| < 1/2$. Find the first three terms for the Maclaurin series of the function g defined by $g(x) = f(1 - \cos \pi x)$ and determine the radius of convergence of this series.

The three parts carry, respectively, 30%, 40%, and 30% of the marks.