Question 1.

Marks & seen/unseen

Parts

seen ⇓

(i) (e).

(ii) (b) 1.0417.

(iii) (a).

unseen ⇓

(iv) (c)

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{1}^{b} xdx = \left[\frac{x^2}{2}\right]_1^b = \frac{b^2-1}{2} \implies b = \sqrt{3}.$$

(v) (f) The probability that they are each of a different colour is $1 \times \frac{6}{8} \times \frac{3}{7} = \frac{9}{28}$.

Each 4 marks

Question 2.

Marks & seen/unseen

Parts

seen ⇓

(i) If $f(x|\mu,\sigma)$ is the density of $\mathrm{N}(\mu,\sigma^2)$ evaluated at x,

$$\ell(\mu,\sigma) = \sum_{i=1}^{n} \log f(x_i|\mu,\sigma) = -\frac{n}{2}\log(2\pi) - n\log(\sigma) - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}.$$

3 marks

(ii)

$$0 = \frac{\partial}{\partial\mu}\ell(\mu,\sigma) = \frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma^2} = \frac{\sum_{i=1}^{n}x_i - n\mu}{\sigma^2} \implies \hat{\mu} = \frac{\sum_{i=1}^{n}x_i}{n} = \bar{x}.$$

4 marks

$$\frac{\partial^2}{\partial\mu^2}\ell(\mu,\sigma) = -\frac{n}{\sigma^2},$$

which is negative everywhere, so $\hat{\mu}$ is the MLE.

2 marks

unseen ⇓

(iii)

$$0 = \frac{\partial}{\partial\sigma}\ell(\hat{\mu},\sigma) = \frac{\partial}{\partial\sigma}\left\{-\frac{n}{2}\log(2\pi) - n\log(\sigma) - \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{2\sigma^2}\right\}$$
$$= -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sigma^3} \implies \hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2.$$

6 marks

(iv) Since $\mathrm{E}(s_{n-1}^2) = \sigma^2$, by linearity of expectation

$$\mathrm{E}(\hat{\sigma^2}) = \frac{n-1}{n}\mathrm{E}(s_{n-1}^2) = \frac{n-1}{n}\sigma^2$$
$$\implies \text{bias}(\hat{\sigma^2}) = \frac{n-1}{n}\sigma^2 - \sigma^2 = \frac{-\sigma^2}{n}.$$

4 marks

No, $\hat{\sigma^2}$ is a biased estimator (although clearly it is asymptotically unbiased).

1 mark

Question 3.

Marks & seen/unseen

Parts

seen ⇓

(i) (a) $X \sim \text{Bin}(25, 0.2)$.
$\text{E}(X) = 25 \times 0.2 = 5$.
$\text{Var}(X) = 25 \times 0.2 \times (1 - 0.2) = 4$. — 5 marks

(b) $\text{P}(X = 5) = 0.196$. — 2 mark

(c) $\text{P}(X \geq 6) = 1 - \sum_{x=0}^{5} \text{P}(X = x) = 0.383$. — 3 marks

(ii) (a) The test should be one-sided, since the change in hard disks is expected to lead to decreased reliability. Let p be the probability that a randomly chosen faulty computer has a hard disk failure. Then

$$H_0 : p = 0.2;$$
$$H_1 : p > 0.2.$$

2 marks

The rejection region $R = \{x | P(X \geq x) < \alpha\} = \{9, 10, 11, \ldots, 25\}$, since $\text{P}(X \geq 8) = 0.109$ and $\text{P}(X \geq 9) = 0.047$. — 3 marks

(b) $6 \notin R$. So there is no significant evidence to reject the null hypothesis. — 2 marks

(c) The p-value is equal to $\text{P}(X \geq 6) = 0.109$ from 3(i)c. — 3 marks

Question 4. | Marks & seen/unseen

Parts | unseen ⇓

(i) When $m_X = m_Y = 2$,

$$1 = p_{XY}(1,1) + p_{XY}(1,2) + p_{XY}(2,1) + p_{XY}(2,2) = 2c + 3c + 3c + 4c = 12c$$
$$\implies c = \frac{1}{12}.$$

2 marks

(ii)

$$1 = \sum_x \sum_y p_{XY}(x,y) = c \sum_{x=1}^{m_X} \sum_{y=1}^{m_Y} (x+y) = c \sum_{x=1}^{m_X} \{m_Y x + m_Y(m_Y+1)/2\}$$
$$= c\{m_Y m_X(m_X+1)/2 + m_X m_Y(m_Y+1)/2\} = c m_X m_Y (m_X + m_Y + 2)/2$$
$$\implies c = 2/\{m_X m_Y (m_X + m_Y + 2)\}.$$

3 marks

(iii)

$$p_X(x) = \sum_y p_{XY}(x,y) = c\{m_Y x + m_Y(m_Y+1)/2\} = \frac{2m_Y x + m_Y(m_Y+1)}{m_X m_Y (m_X + m_Y + 2)}$$
$$= \frac{2x + m_Y + 1}{m_X(m_X + m_Y + 2)}.$$

4 marks

By symmetry,

$$p_Y(y) = \frac{2y + m_X + 1}{m_Y(m_X + m_Y + 2)}.$$

1 mark

If $m_X = m_Y = 2$, $p_X(x) = (2x+3)/12$; similarly $p_Y(y) = (2y+3)/12$.

2 marks

(iv)

$$\mathrm{E}(Y) = \mathrm{E}(X) = \sum_x x p_X(x) = 1.\frac{5}{12} + 2.\frac{7}{12} = \frac{19}{12} \approx 1.583.$$

4 marks

(v)

$$\mathrm{E}(XY) = \sum_x \sum_y xy\ p_{X,Y}(x,y) = \frac{2}{12}.1 + \frac{6}{12}.2 + \frac{4}{12}.4 = \frac{30}{12} = 2.5.$$

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y) = 2.5 - \left(\frac{19}{12}\right)^2$$
$$\approx -0.0069.$$

4 marks