Imperial College
London

CO202 – Software Engineering – Algorithms
Dynamic Programming

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Algorithm Design

How to design an (efficient) algorithm?

Structure the problem!*

Make use of **algorithmic schemes/design paradigms**

- Incremental Approach
- Divide and Conquer (last week)
- **Dynamic Programming** (this week)
- Greedy Algorithms (next week)

*Take some time and think!

Dynamic Programming Principle

- DP solves problems by **combining** the solutions to **subproblems** (similar to Divide and Conquer)
- In D&C subproblems do not overlap: disjoint partitioning
- In DP subproblems **overlap**: subproblems share subsubproblems
- A DP algorithm solves each subproblem just once and **saves its answer in a table**, avoiding recomputations

The term “Programming” refers to a **tabular** method, not to writing computer code. DP is typically applied to **optimisation problems**.

Optimisation Problems

- Such problems can have many possible solutions
- Each solution has a value (e.g. a *cost* or an *energy*)
- Optimisation seeks for a solution with the optimal (minimum or maximum) value: *an** optimal solution

*Note: there could be several solutions with the same optimal value

Dynamic Programming

When developing a DP algorithm, follow four steps:

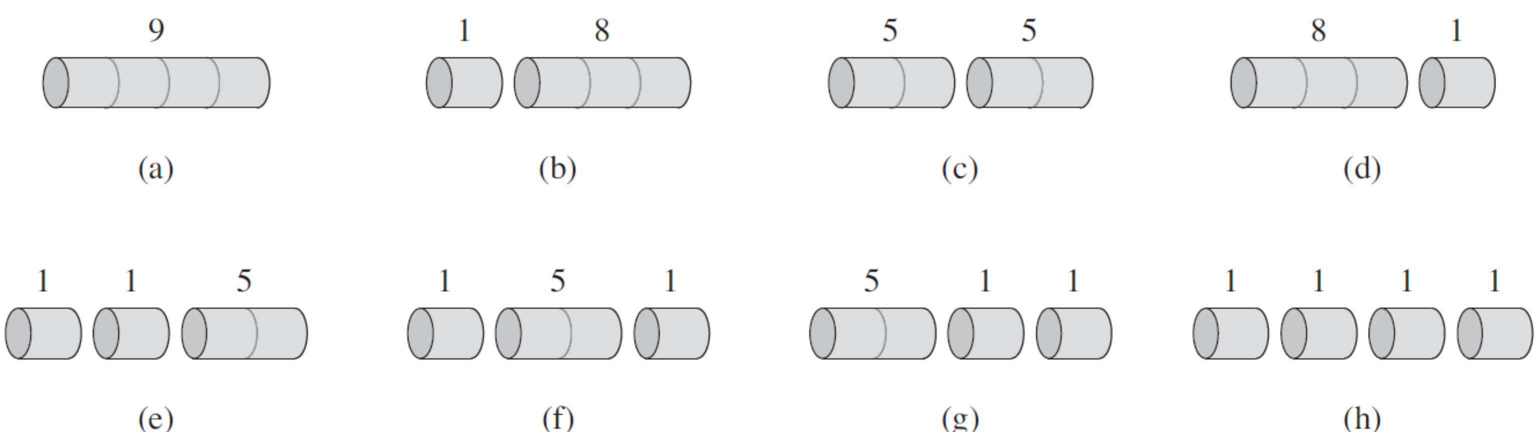
1. Characterize the structure of an optimal solution
2. Recursively define the value of an optimal solution
3. Compute the value of an optimal solution, typically in a bottom-up fashion
4. Construct an optimal solution from computed information

Example: Rod Cutting

Given a rod of length n metres and a table of prices p_i for $i = 1,2,\ldots,n$, determine the maximum revenue r_n obtainable by cutting up the rod and selling the pieces.

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

Possible cuts for rod of length $n = 4$



[Cormen] p.361

Example: Rod Cutting (cont'd)

A rod of length n can be cut up in 2^{n-1} different ways*

*we assume only integer-valued cuts are possible

- An optimal solution cuts the rod into k pieces, for some $1 \leq k \leq n$

- Optimal decomposition is then

$$n = i_1 + i_2 + \cdots + i_k$$

- Providing maximal revenue

$$r_n = p_{i_1} + p_{i_2} + \cdots + p_{i_k}$$

Example: Rod Cutting (cont'd)

Optimal solutions for all $1 \leq i \leq 10$

Max Revenue	Optimal Decomposition
$r_1 = 1$	$1 = 1$ (no cuts)
$r_2 = 5$	$2 = 2$ (no cuts)
$r_3 = 8$	$3 = 3$ (no cuts)
$r_4 = 10$	$4 = 2 + 2$
$r_5 = 13$	$5 = 2 + 3$
$r_6 = 17$	$6 = 6$ (no cuts)
$r_7 = 18$	$7 = 1 + 6$ or $7 = 2 + 2 + 3$
$r_8 = 22$	$8 = 2 + 6$
$r_9 = 25$	$9 = 3 + 6$
$r_{10} = 30$	$10 = 10$ (no cuts)

Recursive Rod Cutting

The maximum revenue r_n can be determined by considering the revenues for shorter rods:

$$r_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \dots, r_{n-1} + r_1)$$

or more compact (with $r_0 = 0$)

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$$

The rod-cutting problem exhibits **optimal substructure**: optimal solutions to a problem incorporate optimal solutions to related subproblems.

Recursive Rod Cutting (cont’d)

Top-down Implementation

```
CUT-ROD(p,n)
if n == 0                           # if no rod
    return 0                        # then no revenue
q = -∞                              # initialise revenue
for i = 1 to n                      # search for maximum
    q = max(q, p[i] + CUT-ROD(p,n-i))
return q
```

What is the running time of the CUT-ROD algorithm?

$$T(n) = 2^n$$

Bad, bad…really bad!

Recursive Rod Cutting (cont'd)

Exponential running time of CUT-ROD

$$T(n) = \begin{cases} 1, & \text{if } n = 0 \\ 1 + \sum_{j=0}^{n-1} T(j), & \text{if } n > 0 \end{cases}$$

Assumption: $T(\mathrm{n}) = 1 + \sum_{j=0}^{n-1} T(j) = 2^n$

Base case: $T(0) = 1 = 2^0$ Holds.

Show for n+1: $T(n+1) = 1 + \sum_{j=0}^{n} T(j)$

$= 1 + \sum_{j=0}^{n-1} T(j) + T(n)$ Extract $T(n)$ from the summation

$= 2T(n)$ Get twice the same term

$= 2(2^n)$ Insert assumption

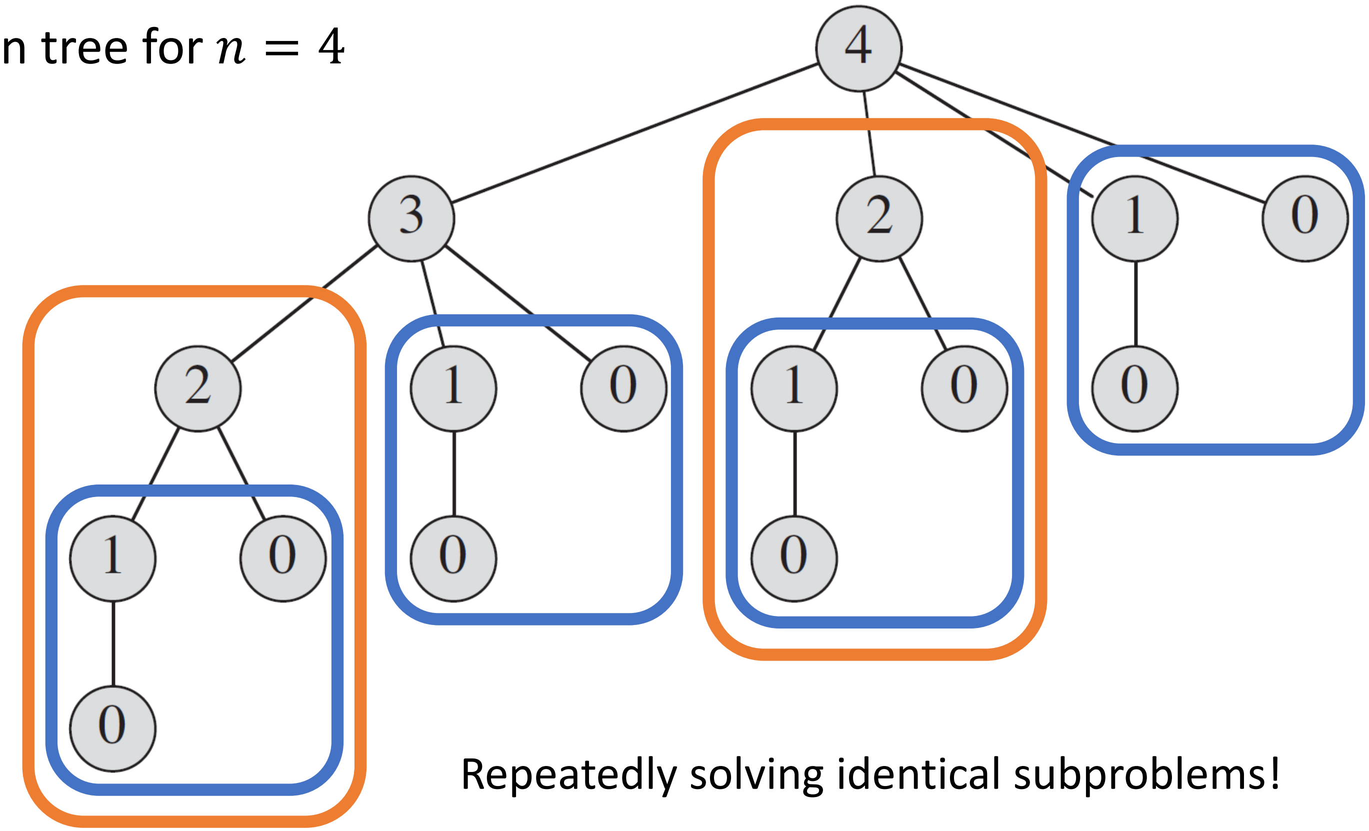
$= 2^{n+1}$ Done.

■

Inefficiency of Recursive Rod Cutting

Why is CUT-ROD so inefficient?

Recursion tree for $n = 4$



Repeatedly solving identical subproblems!

[Cormen] p.364

The Dynamic Programming Approach

- Subproblems are solved **once**, and solutions are **saved**
- If we encounter a subproblem again, just **look up** the solution
- Dynamic programming uses additional memory, often yielding a **time-memory trade-off**
- Two strategies for implementing a DP algorithm:
  1. **Top-down with memoization***
  2. **Bottom-up**

*Note: not a typo, memoization comes from *memo*.

Top-Down with Memoization

```
MEMOIZED-CUT-ROD(p,n)
let r[0..n] be a new array
for i = 0 to n
    r[i] = -∞
return MEMOIZED-CUT-ROD-AUX(p,n,r)
```

```
MEMOIZED-CUT-ROD-AUX(p,n,r)
if r[n] ≥ 0
    return r[n]
if n == 0
    q = 0
else
    q = -∞
    for i = 1 to n
        q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p,n-i,r))
r[n] = q
return q
```

```
CUT-ROD(p,n)
if n == 0
    return 0
q = -∞
for i = 1 to n
    q = max(q, p[i] + CUT-ROD(p,n-i))
return q
```

Bottom-Up Version of Rod Cutting

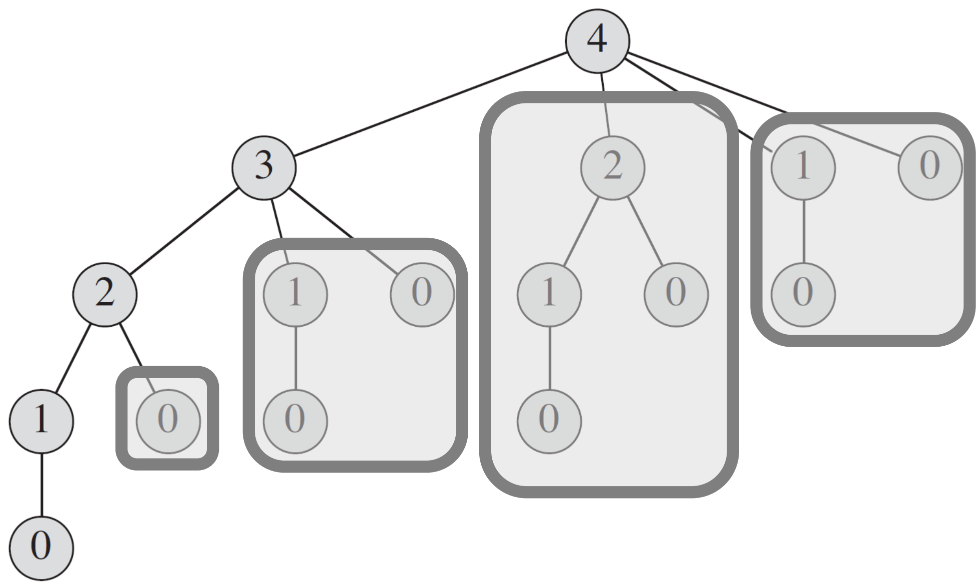
```
BOTTOM-UP-CUT-ROD(p,n)
let r[0..n] be a new array
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        q = max(q,p[i] + r[j-i])
    r[j] = q
return r[n]
```

What is the running time of BOTTOM-UP-CUT-ROD?

$$T(n) = n^2$$

Memoization

- Memoization allows us to make top-down recursive algorithms as efficient as bottom-up approaches
- A memoized recursive algorithm maintains an entry in a table for the solution to each subproblem



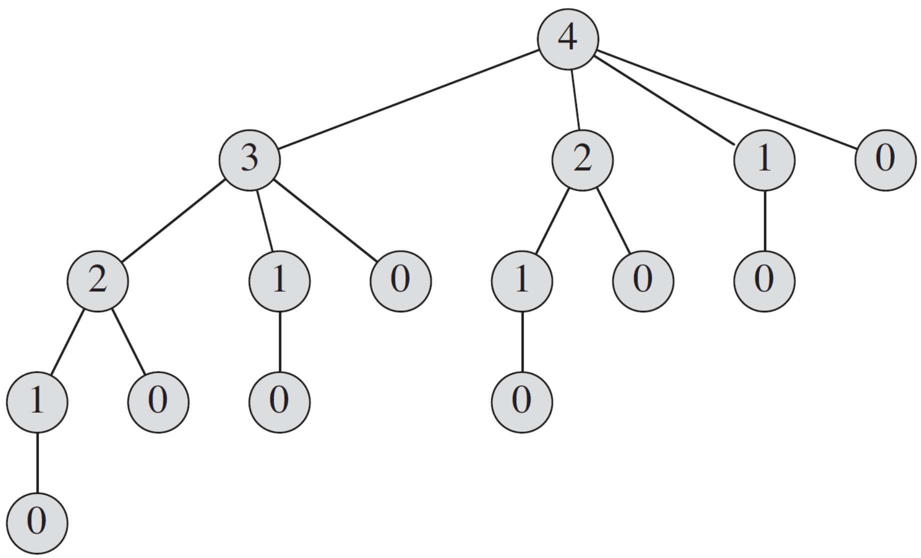
Top-down vs. Bottom-up

Top-down with memoization and bottom-up have the same asymptotic running time

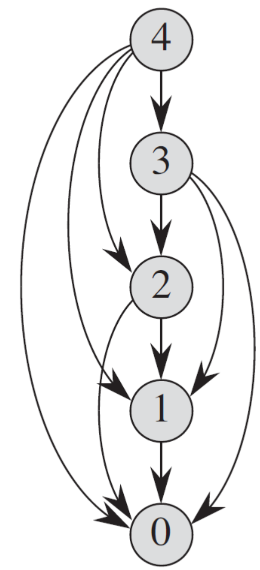
- Bottom-up is usually more efficient by a constant factor because there is no overhead for recursive calls
- Bottom-up might benefit from optimal memory access
- Top-down can avoid to compute solutions of subproblems that are not required
- Top-down implementation is closer to the natural, recursive definition of the problem

Subproblem Graphs

Rod cutting with $n = 4$



Recursion Tree



Subproblem Graph

[Cormen] p.364

[Cormen] p.367

Reconstructing a Solution

Compute not only the maximum revenue, but also record the solution (i.e. the cut off pieces)

```
EXTENDED-BOTTOM-UP-CUT-ROD(p,n)
let r[0..n] and s[1..n] be a new array
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        if q < p[i] + r[j-i]
            q = p[i] + r[j-i]
            s[j] = i
    r[j] = q
return r and s
```

i	0	1	2	3	4	5	6	7	8	9	10
r[i]	0	1	5	8	10	13	17	18	22	25	30
s[i]		1	2	3	2	2	6	1	2	3	10

Print a Solution

Print the list of piece sizes for an optimal decomposition of a rod of length n

```
PRINT-CUT-ROD-SOLUTION(p,n)
(r,s) = EXTENDED-BOTTOM-UP-CUT-ROD(p,n)
while n > 0
    print s[n]
    n = n – s[n]
```

Key Elements of Dynamic Programming

When should we consider DP?

- **Optimal substructure**: Optimal solution to a problem contains optimal solutions to subproblems
- **Overlapping subproblems**: The space of subproblems must be “small”. Subproblems are solved over and over, rather then generating new subproblems

Longest Common Subsequence (LCS)

Consider the following two strands of DNA

A strand of DNA is a string over the finite set $\{\mathrm{A}, \mathrm{C}, \mathrm{G}, \mathrm{T}\}$, standing for bases adenine, cytosine, guanine, and thymine.

$$S_1 = \mathrm{ACCGGTCGAGTGCGCGGAAGCCGGCCGAA}$$

$$S_2 = \mathrm{GTCGTTCGGAATGCCGTTGCTCTGTAAA}$$

How similar are these two strands?

One way of defining similarity is to look for bases that appear in same order, but not necessarily consecutively

$$S_3 = \mathrm{GTCGTCGGAAGCCGGCCGAA}$$

Formal Definition of Subsequence

Given a sequence

$$X = \langle x_1, x_2, \dots, x_m \rangle$$

Another sequence

$$Z = \langle z_1, z_2, \dots, z_k \rangle$$

is a **subsequence** of X if there exists a strictly increasing sequence $\langle i_1, i_2, \dots, i_k \rangle$ of indices of X such that for all $j = 1, 2, \dots, k$ we have $x_{i_j} = z_j$.

Example: $Z = \langle B, C, D, B \rangle$ is a subsequence of $X = \langle A, B, C, B, D, A, B \rangle$ with corresponding index sequence $\langle 2,3,5,7 \rangle$.

Common Subsequence

Given two sequences X and Y, then the sequence Z is a **common subsequence** if it is a subsequence of X and Y.

Example: $X = \langle A, B, C, B, D, A, B \rangle$ and $Y = \langle B, D, C, A, B, A \rangle$ then $Z = \langle B, C, A \rangle$ is a common subsequence. What is the LCS?

The LCS Problem

Given two sequences X and Y, find a maximum-length common subsequence. We will solve this using DP!

Reminder: Dynamic Programming

When developing a DP algorithm, follow four steps:

1. Characterize the structure of an optimal solution
2. Recursively define the value of an optimal solution
3. Compute the value of an optimal solution
4. Construct an optimal solution

LCS – Step 1: Characterize the Structure

Brute-force: Enumerate all subsequences of X and check each whether it is also a subsequence of Y. **Impractical!**

Definition of prefix: The ith prefix of X, for $i = 0,1, \dots, m$, is defined as $X_i = \langle x_1, x_2, \dots, x_i \rangle$ with X_0 being the empty sequence

Optimal substructure of LCS (Theorem):

Given $X = \langle x_1, x_2, \dots, x_m \rangle$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$ and let $Z = \langle z_1, z_2, \dots, z_k \rangle$ be any LCS of X and Y

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is LCS of X_{m-1} and Y_{n-1}
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is LCS of X_{m-1} and Y
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$$\boxed{\begin{gathered} X_7 = \langle A, B, C, B, D, A, B \rangle \\ Y_6 = \langle B, D, C, A, B, A \rangle \\ Z_4 = \langle B, C, A, B \rangle \end{gathered}}$$

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$$Y_2 = \langle B, D, C, A, B, A \rangle$$
$$Z_1 = \langle B, C, A, B \rangle$$

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Optimal substructure of LCS (Theorem):

Given $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is LCS of X_{m-1} and Y_{n-1}
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is LCS of X_{m-1} and Y
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is LCS of X and Y_{n-1}

$$X_2 = \langle A, B, C, B, D, A, B \rangle$$
$$Y_2 = \langle B, D, C, A, B, A \rangle$$
$$Z_1 = \langle B, C, A, B \rangle$$

LCS – Step 1: Characterize the Structure

Optimal substructure of LCS (Theorem):

Given $X = \langle x_1, x_2, \dots, x_m \rangle$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$ and let $Z = \langle z_1, z_2, \dots, z_k \rangle$ be any LCS of X and Y

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is LCS of X_{m-1} and Y_{n-1}
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is LCS of X_{m-1} and Y
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is LCS of X and Y_{n-1}

$$\boxed{\begin{gathered} X_2 = \langle A, B, C, B, D, A, B \rangle \\ Y_1 = \langle B, D, C, A, B, A \rangle \\ Z_1 = \langle B, C, A, B \rangle \end{gathered}}$$

LCS – Step 1: Characterize the Structure

Optimal substructure of LCS (Theorem):

Given $X = \langle x_1, x_2, \dots, x_m \rangle$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$ and let $Z = \langle z_1, z_2, \dots, z_k \rangle$ be any LCS of X and Y

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is LCS of X_{m-1} and Y_{n-1}
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is LCS of X_{m-1} and Y
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is LCS of X and Y_{n-1}

$$X_2 = \langle A, B, C, B, D, A, B \rangle$$
$$Y_1 = \langle B, D, C, A, B, A \rangle$$
$$Z_1 = \langle B, C, A, B \rangle$$

LCS – Step 1: Characterize the Structure

Optimal substructure of LCS (Theorem):

Given $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is LCS of X_{m-1} and Y_{n-1}
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is LCS of X_{m-1} and Y
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is LCS of X and Y_{n-1}

$$X_1 = \langle A, B, C, B, D, A, B \rangle$$
$$Y_0 = \langle B, D, C, A, B, A \rangle$$
$$Z_0 = \langle B, C, A, B \rangle$$

LCS – Step 1: Characterize the Structure

Optimal substructure of LCS (Theorem):

Given $X = \langle x_1, x_2, \dots, x_m \rangle$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$ and let $Z = \langle z_1, z_2, \dots, z_k \rangle$ be any LCS of X and Y

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is LCS of X_{m-1} and Y_{n-1}
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is LCS of X_{m-1} and Y
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is LCS of X and Y_{n-1}

Proof: (Part 1)

1) If $z_k \neq x_m$, then we could append $x_m = y_n$ to Z and get an LCS with $k + 1$, which is a contradiction to Z being LCS of X and Y.

   The prefix Z_{k-1} with length $k - 1$ is LCS of X_{m-1} and Y_{n-1}. Suppose there is LCS W that is greater than $k - 1$, then appending $x_m = y_n$ to W produces an LCS with length $> k$, which is a contradiction.

LCS – Step 1: Characterize the Structure

Optimal substructure of LCS (Theorem):

Given $X = \langle x_1, x_2, \dots, x_m \rangle$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$ and let $Z = \langle z_1, z_2, \dots, z_k \rangle$ be any LCS of X and Y

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is LCS of X_{m-1} and Y_{n-1}
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is LCS of X_{m-1} and Y
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is LCS of X and Y_{n-1}

Proof: (Part 2 & 3)

2) If $z_k \neq x_m$, then Z is a common subsequence of X_{m-1} and Y. If there were a common subsequence W of X_{m-1} and Y with length $> k$, then W would also be a common subsequence of X_m and Y, which is a contradiction with Z being LCS of X and Y.
3) The proof is symmetric to (2).

$\blacksquare$

LCS – Step 2: Recursive Solution

There are two cases:

1. If $x_m = y_n$, find LCS of X_{m-1} and Y_{n-1} and append $x_m = y_n$
2. Otherwise, solve two subproblems
   a) Find LCS of X_{m-1} and Y
   b) Find LCS of X and Y_{n-1}
   Whichever LCS is longer, it is also an LCS of X and Y

Overlapping-subproblems property:

In order to find LCS of X and Y we may need to find LCSs of X and Y_{n-1} and X_{m-1} and Y. Each of these subproblems has the subsubproblem of finding LCS of X_{m-1} and Y_{n-1}.

LCS – Step 2: Recursive Solution (cont'd)

Let $c[i, j]$ be the length of an LCS of sequences X_i and Y_j

$$c[i,j] = \begin{cases} 0, & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1, & \text{if } i, j > 0 \text{ and } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]), & \text{if } i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

LCS – Step 3: Computing a Solution (Bottom-up)

```
LCS-LENGTH(X,Y)
m = X.length
n = Y.length
let b[1..m,1..n] and c[0..m,0..n] be new tables
for i = 1 to m                                # initialize first column
    c[i,0] = 0
for j = 0 to n                                # initialize first row
    c[0,j] = 0
for j = 1 to n
    for i = 1 to m
        if x_i == y_j                         # case 1
            c[i,j] = c[i-1,j-1] + 1
            b[i,j] = ↖
        elseif c[i-1,j] ≥ c[i,j-1]            # case 2a
            c[i,j] = c[i-1,j]
            b[i,j] = ↑
        else                                  # case 2b
            c[i,j] = c[i,j-1]
            b[i,j] = ←
return c and b
```

LCS – Step 4: Construct a Solution

```
PRINT-LCS(b,X,i,j)
if i == 0 or j == 0
    return
if b[i,j] == ↖
    PRINT-LCS(b,X,i-1,j-1)
    print x_i
elseif b[i,j] == ↑
    PRINT-LCS(b,X,i-1,j)
else
    PRINT-LCS(b,X,i,j-1)
```

Approximate String Matching

How similar are `SUNNY` and `SNOWY`?

- LCS has length 3: `SNY`
- **LCS Distance**: How many operations does it need to transform one string into the other? Allowed operations are `DELETE` and `INSERT`

S	U	N	N	-	-	Y
S	-	N	-	O	W	Y

Distance: 4

Levenshtein Distance

Allowed operations are DELETE, INSERT and REPLACE

S	U	N	N	-	Y
S	-	N	O	W	Y

Distance: 3

$$d[i,j] = \begin{cases} \max(i,j)\,, & \text{if } i = 0 \text{ or } j = 0 \\ \min \begin{cases} d[i-1,j] + 1 \\ d[i,j-1] + 1 \\ d[i-1,j-1] + \mathbf{1}_{(x_i \neq y_j)} \end{cases}, & \text{otherwise} \end{cases}$$

Levenshtein Distance (cont’d)

```
LEV-DISTANCE(X,Y)
m = X.length
n = Y.length
let d[0..m,0..n] be new table
for i = 1 to m
    d[i,0] = i
for j = 0 to n
    d[0,j] = j
for j = 1 to n
    for i = 1 to m
        c = x_i == y_j ? 0 : 1
        d[i,j] = min(d[i-1,j] + 1, d[i,j-1] + 1, d[i-1,j-1] + c)
return d[m,n]
```

Conclusions

- DP is a powerful algorithm design technique that can sometimes reduce exponential running time to polynomial (e.g. rod cutting, LCS, …)

- DP is employed in many applications
  - Bioinformatics (computational genomics, DNA matching)
  - Spell checker (via approximate string matching)
  - Time series analysis (speech recognition)
  - Economics

References

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- http://algs4.cs.princeton.edu/lectures/
- https://www.coursera.org/courses?query=algorithms

CO202 – Software Engineering – Algorithms

Dynamic Programming - Exercises

Exercise 1: Compute LCS

	j	0	1	2	3	4	5	6
i		y_j	**B**	**D**	**C**	**A**	**B**	**A**
0	x_i							
1	**A**							
2	**B**							
3	**C**							
4	**B**							
5	**D**							
6	**A**							
7	**B**							

Exercise 1: Compute LCS

	j	0	1	2	3	4	5	6
i		y_j	**B**	**D**	**C**	**A**	**B**	**A**
0	x_i	0	0	0	0	0	0	0
1	**A**	0						
2	**B**	0						
3	**C**	0						
4	**B**	0						
5	**D**	0						
6	**A**	0						
7	**B**	0						

```
for j = 1 to n
    for i = 1 to m
        if x_i == y_j
            c[i,j] = c[i-1,j-1] + 1
            b[i,j] = ↖
        elseif c[i-1,j] ≥ c[i,j-1]
            c[i,j] = c[i-1,j]
            b[i,j] = ↑
        else
            c[i,j] = c[i,j-1]
            b[i,j] = ←
```

Exercise 1: Compute LCS

	j	0	1	2	3	4	5	6
i		y_j	**B**	**D**	**C**	**A**	**B**	**A**
0	x_i	0	0	0	0	0	0	0
1	**A**	0	0					
2	**B**	0						
3	**C**	0						
4	**B**	0						
5	**D**	0						
6	**A**	0						
7	**B**	0						

```
for j = 1 to n
    for i = 1 to m
        if xi == yj
            c[i,j] = c[i-1,j-1] + 1
            b[i,j] = ↖
        elseif c[i-1,j] ≥ c[i,j-1]
            c[i,j] = c[i-1,j]
            b[i,j] = ↑
        else
            c[i,j] = c[i,j-1]
            b[i,j] = ←
```

Exercise 1: Compute LCS

	j	0	1	2	3	4	5	6
i		y_j	**B**	**D**	**C**	**A**	**B**	**A**
0	x_i	0	0	0	0	0	0	0
1	**A**	0	↑ 0					
2	**B**	0						
3	**C**	0						
4	**B**	0						
5	**D**	0						
6	**A**	0						
7	**B**	0						

```
for j = 1 to n
    for i = 1 to m
        if xi == yj
            c[i,j] = c[i-1,j-1] + 1
            b[i,j] = ↖
        elseif c[i-1,j] ≥ c[i,j-1]
            c[i,j] = c[i-1,j]
            b[i,j] = ↑
        else
            c[i,j] = c[i,j-1]
            b[i,j] = ←
```

Exercise 1: Compute LCS

	j	0	1	2	3	4	5	6
i		y_j	**B**	**D**	**C**	**A**	**B**	**A**
0	x_i	0	0	0	0	0	0	0
1	**A**	0	↑ 0					
2	**B**	0	1					
3	**C**	0						
4	**B**	0						
5	**D**	0						
6	**A**	0						
7	**B**	0						

```
for j = 1 to n
    for i = 1 to m
        if xi == yj
            c[i,j] = c[i-1,j-1] + 1
            b[i,j] = ↖
        elseif c[i-1,j] ≥ c[i,j-1]
            c[i,j] = c[i-1,j]
            b[i,j] = ↑
        else
            c[i,j] = c[i,j-1]
            b[i,j] = ←
```

Exercise 1: Compute LCS

	j	0	1	2	3	4	5	6
i		y_j	**B**	**D**	**C**	**A**	**B**	**A**
0	x_i	0	0	0	0	0	0	0
1	**A**	0	↑ 0					
2	**B**	0	↖ 1					
3	**C**	0						
4	**B**	0						
5	**D**	0						
6	**A**	0						
7	**B**	0						

```
for j = 1 to n
    for i = 1 to m
        if x_i == y_j
            c[i,j] = c[i-1,j-1] + 1
            b[i,j] = ↖
        elseif c[i-1,j] ≥ c[i,j-1]
            c[i,j] = c[i-1,j]
            b[i,j] = ↑
        else
            c[i,j] = c[i,j-1]
            b[i,j] = ←
```

Exercise 1: Compute LCS

	j	0	1	2	3	4	5	6
i		y_j	**B**	**D**	**C**	**A**	**B**	**A**
0	x_i	0	0	0	0	0	0	0
1	**A**	0	↑ 0					
2	**B**	0	↖ 1					
3	**C**	0	1					
4	**B**	0						
5	**D**	0						
6	**A**	0						
7	**B**	0						

```
for j = 1 to n
    for i = 1 to m
        if x_i == y_j
            c[i,j] = c[i-1,j-1] + 1
            b[i,j] = ↖
        elseif c[i-1,j] ≥ c[i,j-1]
            c[i,j] = c[i-1,j]
            b[i,j] = ↑
        else
            c[i,j] = c[i,j-1]
            b[i,j] = ←
```

Exercise 1: Compute LCS

	j	0	1	2	3	4	5	6
i		y_j	**B**	**D**	**C**	**A**	**B**	**A**
0	x_i	0	0	0	0	0	0	0
1	**A**	0	↑ 0					
2	**B**	0	↖ 1					
3	**C**	0	↑ 1					
4	**B**	0						
5	**D**	0						
6	**A**	0						
7	**B**	0						

```
for j = 1 to n
    for i = 1 to m
        if x_i == y_j
            c[i,j] = c[i-1,j-1] + 1
            b[i,j] = ↖
        elseif c[i-1,j] ≥ c[i,j-1]
            c[i,j] = c[i-1,j]
            b[i,j] = ↑
        else
            c[i,j] = c[i,j-1]
            b[i,j] = ←
```

Exercise 2: Compute Levenshtein Distance

	j	0	1	2	3	4	5	6	7	8	9
i		y_j	E	M	P	I	R	I	C	A	L
0	x_i										
1	I										
2	M										
3	P										
4	E										
5	R										
6	I										
7	A										
8	L										

Exercise 2: Compute Levenshtein Distance

	j	0	1	2	3	4	5	6	7	8	9
i		y_j	**E**	**M**	**P**	**I**	**R**	**I**	**C**	**A**	**L**
0	x_i	0	1	2	3	4	5	6	7	8	9
1	**I**	1									
2	**M**	2									
3	**P**	3									
4	**E**	4									
5	**R**	5									
6	**I**	6									

```
for j = 1 to n
    for i = 1 to m
        c = x_i == y_j ? 0 : 1
        d[i,j] = min(d[i-1,j] + 1, d[i,j-1] + 1, d[i-1,j-1] + c)
```

Exercise 2: Compute Levenshtein Distance

	j	0	1	2	3	4	5	6	7	8	9
i		y_j	**E**	**M**	**P**	**I**	**R**	**I**	**C**	**A**	**L**
0	x_i	0	1	2	3	4	5	6	7	8	9
1	**I**	1	R 1								
2	**M**	2									
3	**P**	3									
4	**E**	4									
5	**R**	5									
6	**I**	6									

```
for j = 1 to n
    for i = 1 to m
        c = x_i == y_j ? 0 : 1
        d[i,j] = min(d[i-1,j] + 1, d[i,j-1] + 1, d[i-1,j-1] + c)
```

Exercise 2: Compute Levenshtein Distance

	j	0	1	2	3	4	5	6	7	8	9
i		y_j	E	M	P	I	R	I	C	A	L
0	x_i	0	1	2	3	4	5	6	7	8	9
1	I	1	R 1								
2	M	2	R D 2								
3	P	3									
4	E	4									
5	R	5									
6	I	6									

```
for j = 1 to n
    for i = 1 to m
        c = x_i == y_j ? 0 : 1
        d[i,j] = min(d[i-1,j] + 1, d[i,j-1] + 1, d[i-1,j-1] + c)
```

Exercise 2: Compute Levenshtein Distance

	j	0	1	2	3	4	5	6	7	8	9
i		y_j	**E**	**M**	**P**	**I**	**R**	**I**	**C**	**A**	**L**
0	x_i	0	1	2	3	4	5	6	7	8	9
1	**I**	1	R 1								
2	**M**	2	R D 2								
3	**P**	3	R D 3								
4	**E**	4									
5	**R**	5									
6	**I**	6									

```
for j = 1 to n
    for i = 1 to m
        c = x_i == y_j ? 0 : 1
        d[i,j] = min(d[i-1,j] + 1, d[i,j-1] + 1, d[i-1,j-1] + c)
```

Exercise 2: Compute Levenshtein Distance

	j	0	1	2	3	4	5	6	7	8	9
i		y_j	E	M	P	I	R	I	C	A	L
0	x_i	0	1	2	3	4	5	6	7	8	9
1	I	1	R 1								
2	M	2	R D 2								
3	P	3	R D 3								
4	E	4	K 3								
5	R	5									
6	I	6									

```
for j = 1 to n
    for i = 1 to m
        c = x_i == y_j ? 0 : 1
        d[i,j] = min(d[i-1,j] + 1, d[i,j-1] + 1, d[i-1,j-1] + c)
```

Exercise 2: Compute Levenshtein Distance

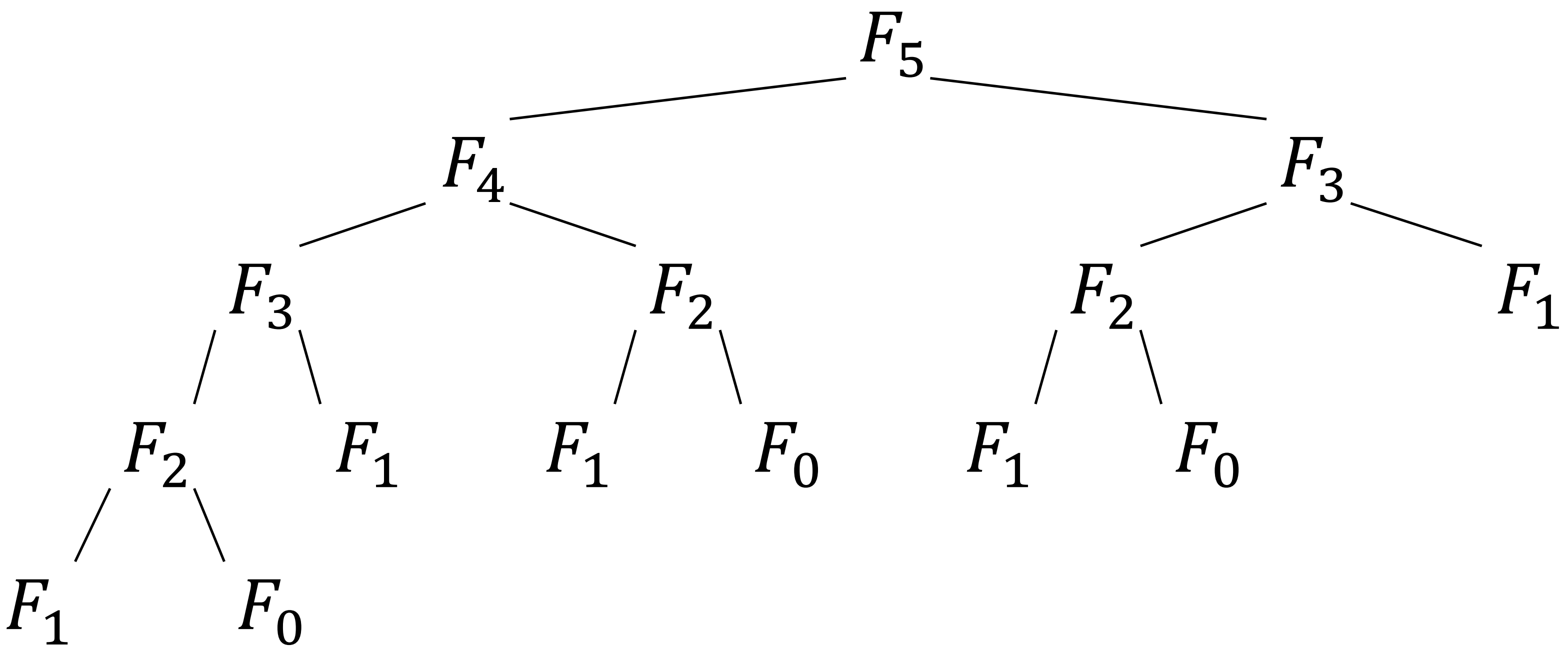
	j	0	1	2	3	4	5	6	7	8	9
i		y_j	E	M	P	I	R	I	C	A	L
0	x_i	0	1	2	3	4	5	6	7	8	9
1	I	1	R 1								
2	M	2	R D 2								
3	P	3	R D 3								
4	E	4	K 3								
5	R	5	D 4								
6	I	6									

```
for j = 1 to n
    for i = 1 to m
        c = x_i == y_j ? 0 : 1
        d[i,j] = min(d[i-1,j] + 1, d[i,j-1] + 1, d[i-1,j-1] + c)
```

Exercise 3: Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

$$F(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ F(n-1) + F(n-2), & \text{otherwise} \end{cases}$$



Exercise 3: Fibonacci Sequence

```
NAÏVE-FIBONACCI(n)
if n == 0
    return 0
if n == 1
    return 1
return NAÏVE-FIBONACCI(n-1) + NAÏVE-FIBONACCI(n-2)
```

Running time of NAÏVE-FIBONACCI:

$$T(n) = O(2^{0.694n})$$

Exercise 3: Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

$$F(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ F(n-1) + F(n-2), & \text{otherwise} \end{cases}$$

```
BOTTOM-UP-FIBONACCI(n)
if n == 0
    return 0
let f[0..n] be a new array
f[0] = 0
f[1] = 1
?
?
?
```

What is the running time of BOTTOM-UP-FIBONACCI?

Exercise 4: Coin Change Problem

Coin change is the problem of finding the least number of coins for a given amount of money.

For example, the UK coin set contains the following coins:

- 1p, 2p, 5p, 10p, 20p, 50p, £1, £2, and £5 (very uncommon).
- For £2.82, the optimal change is £2, 50p, 20p, 10p, 2p.

1. Write a mathematical recurrence equation that determines the least number of coins.

2. Devise a pseudo-code, bottom-up dynamic programming algorithm `coin_change(n,coins)`.

Exercise 5: Fibonacci Challenge

D&C Fibonacci Revisited

Naïve:

$$F(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ F(n-1) + F(n-2), & \text{otherwise} \end{cases}$$

Let's rewrite Fibonacci

$$F(n) = F(2k) = F(k)^2 + 2F(k)F(k-1) \quad \text{for even } n$$

$$F(n) = F(2k-1) = F(k)^2 + F(k-1)^2 \quad \text{for odd } n$$

$$F(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ F(\lceil n/2 \rceil)^2 + 2F(\lceil n/2 \rceil)F(\lceil n/2 \rceil - 1), & \text{if } n \geq 2 \text{ and } n \text{ is even} \\ F(\lceil n/2 \rceil)^2 + F(\lceil n/2 \rceil - 1)^2, & \text{if } n \geq 2 \text{ and } n \text{ is odd} \end{cases}$$

Exercise 5: Fibonacci Challenge

D&C Fibonacci Revisited

$$F(n) = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1 \\ \boldsymbol{F(\lceil n/2 \rceil)}^2 + 2\boldsymbol{F(\lceil n/2 \rceil)}\boldsymbol{F(\lceil n/2 \rceil - 1)}, & \text{if } n \geq 2 \text{ and } n \text{ is even} \\ \boldsymbol{F(\lceil n/2 \rceil)}^2 + \boldsymbol{F(\lceil n/2 \rceil - 1)}^2, & \text{if } n \geq 2 \text{ and } n \text{ is odd} \end{cases}$$

```
DC-FIBONACCI(n)
 1: if n == 0 || n == 1
 2:     return n
 3: else
 4: ?
```

What is the running time?