Risk and Decisions (II)

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Introduction to Artificial Intelligence

The lectures

- The agent and the world (**Knowledge Representation**)
  - Actions and knowledge
  - Inference
- Good decisions (**Risk and Decisions**)
  - Chance
  - Gains
- Good decisions in time (**Markov Decision Processes**)
  - Chance and gains in time
  - Patience
  - Finding the best strategy
- Learning from experience (**Reinforcement Learning**)
  - Finding a reasonable strategy

Today

- Utility and expected utility
- Risky moves

Risk and Decisions

Lotteries (and how to win them)

The book

Stuart Russell and Peter Norvig
Artificial Intelligence: a modern approach
Chapters 16-17

If you snooze you lose?

I set the alarm clock(s) to wake up on time for the lectures.

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Let action S_t = snooze the alarm clock t times

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Potential problems, e.g.:

1. planned engineering works
2. my phone dies
3. my mum forgets to call me (very unlikely)

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e.g., missing class vs. sleeping

Chances + preferences

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Decision theory = utility theory + probability theory

Rewards

Sensors Breeze, Glitter, Smell

Actuators Up, Down, Left, Right, Grab, Release, Shoot, Climb

Rewards 1000 escaping with gold, -1000 dying, -10 using arrow, -1 walking

Environment
- Squares adjacent to Wumpus are smelly
- Squares adjacent to pit are breezy
- Glitter iff gold is in the same square
- Shooting kills Wumpus if you are facing it
- Shooting uses up the only arrow
- Grabbing picks up gold if in same square
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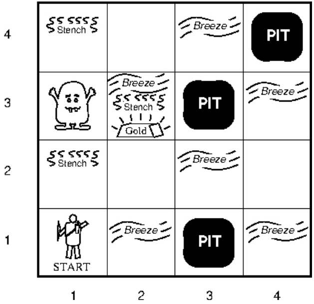

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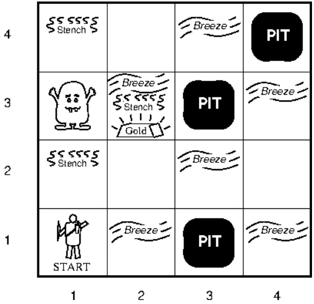

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- The set of states is our sample space

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L is the set of lotteries over S.

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e.g.,
$L_1 = [1, s_1; 0, s_2; \ldots 0, s_n]$

We get s_1 with probability 1, and the rest with probability 0.

Compound lotteries

Consider now the set L of lotteries over S.
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\end{aligned}$$

Compound lotteries can be reduced to simple lotteries

Comparing lotteries: the plan

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Here is the plan:

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  Notice: I said numbers, I haven't said money.

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When we don't have numbers, we can often make them up.

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- $A \sim B = (A \succeq B \text{ and } B \succeq A)$ means that lottery A the same as lottery B value-wise (indifference).

Rational preferences

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Orderability $(A \succ B) \vee (B \sim A) \vee (B \succ A)$

Transitivity $(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$

Continuity $A \succ B \succ C \Rightarrow \exists p \; [p, A; \; 1 - p, C] \sim B$

Substitutability $A \sim B \Rightarrow [p, A; \; 1 - p, C] \sim [p, B; 1 - p, C]$

Monotonicity $A \succ B \Rightarrow (p \geq q \Leftrightarrow [p, A; \; 1 - p, B] \succsim [q, A; \; 1 - q, B])$

Orderability

$$(A \succ B) \vee (B \succ A) \vee (B \sim A)$$

'Either A over B, or B over A, or I don't care.'

Transitivity

$$(A \succ B) \land (B \succ C) \Rightarrow (A \succ C)$$

'If A is better than B, and B better than C, then A is better than C.'

Continuity

$$A \succ B \succ C \;\Rightarrow\; \exists p \;\; [p, A;\; 1 - p, C] \sim B$$

'A is better than B , that is better than C. But if you give me the right mix of A and C then this would be the same as B.'

Substituability

$$A \sim B \;\Rightarrow\; [p, A;\; 1-p, C] \sim [p, B; 1-p, C]$$

'If I'm indifferent to A and B,
then I also don't care of how likely they are.'

Monotonicity

$$A \succ B \;\Rightarrow\; (p \geq q \;\Leftrightarrow\; [p, A;\; 1-p, B] \succsim [q, A;\; 1-q, B])$$

'If I like A more than B,
then I'd rather have a bit more of A than a bit more of B.'

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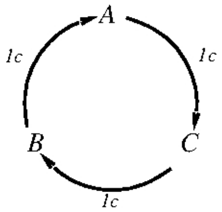

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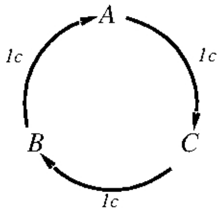

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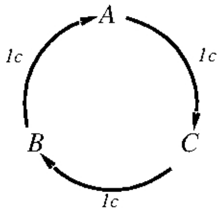

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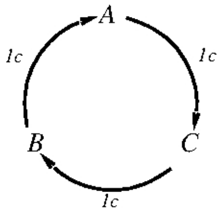

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Theorem (Ramsey, 1931; von Neumann and Morgenstern, 1944)

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[$\Leftarrow$] By contraposition. E.g., pick transitivity and show that if the relation is not transitive there is no way of associating numbers to outcomes.
[$\Rightarrow$] We use the axioms to show that there are infinitely many functions that satisfy them, but they are all "equivalent" to a unique real-valued utility function.

Representation Theorem

Michael Maschler, Eilon Solan and Shmiel Zamir
Game Theory (Ch. 2)
Cambridge University Press, 2013.

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The main message: Give me any order on outcomes that makes sense and I can turn it into a real-valued function!

Utility functions

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$$u : S \to \mathbb{R}$$

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Important:
Utility functions are not the same as money. Utility functions are a representation of happiness, goal satisfaction, fulfilment and the like. They are just a mathematical tool to represent a comparison between outcomes. So altruism, unselfishness, and so fort **can** be modelled using utility functions.

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e.g., rolling a fair six-sided dice, I win $27k$ if 6 comes out, lose $3k$ otherwise. The expected utility is $= \frac{1}{6}27k - \frac{5}{6}3k$

Expected utility

Let $L_1 = [p_1, s_1; p_2, s_2; \ldots p_n, s_n]$ be a lottery.
The expected utility of A is

$$u(L_1) = \sum_{p_i, s_i} p_i \times u(s_i)$$

e.g., rolling a fair six-sided dice, I win $27k$ if 6 comes out, lose $3k$ otherwise. The expected utility is $= \frac{1}{6}27k - \frac{5}{6}3k = 2k.$

Humans and expected utility

'rolling a fair six-sided dice,
you win $27k$ if 6 comes out,
lose $3k$ otherwise'

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Let's change the setup a little bit...

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Modifying utilities and probabilities we can find the indifference point, passed which we change our mind.

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What would you do?

Let's change the setup a little bit...

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Humans and expected utility

Tverski and Kahneman's Prospect Theory:

- Humans have complex utility estimates
- Risk aversion, satisfaction level

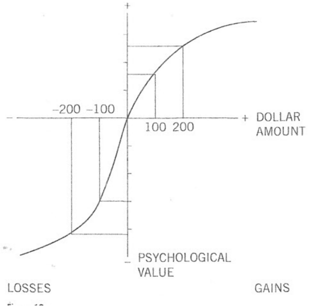

Figure: Typical empirical data

Humans and expected utility

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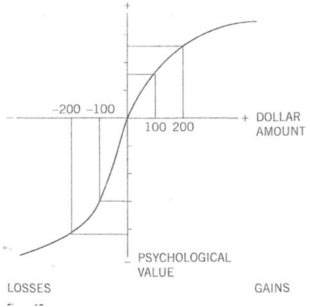
Warning! controversial statement:



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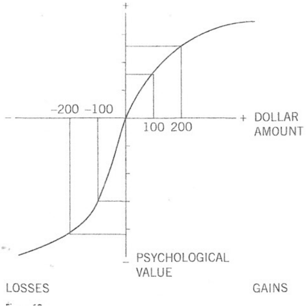

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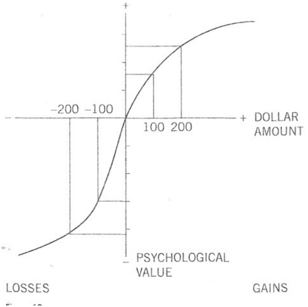
We can incorporate risk aversion and satisfaction as properties of outcomes.



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Multicriteria decision-making

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Multicriteria decision-making

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- Certain outcomes seem difficult to compare:
    - what factors are more important?
    - have we considered all the relevant ones?
    - do factors interfere with one another?
- In other situations the utility function may be updated because of new incoming information (e.g., evaluating non-terminal positions in a long extensive game like Chess or Go)

Multicriteria decision-making

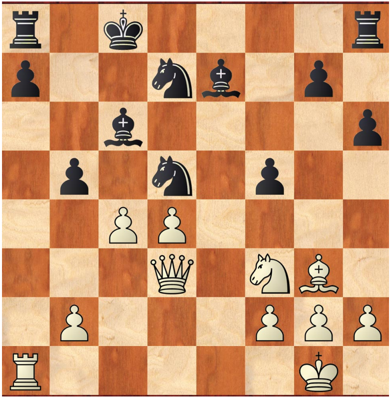

Figure: Deep Blue vs. Kasparov 1996, Final Game. Garry Kasparov (Black) to move: material favours him but the position is hopeless.

Multicriteria decision-making

How do we handle multiple many variables?

Multicriteria decision-making

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e.g., what is
$u(\text{king safety}, \text{material advantage}, \text{control of the centre})$?

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Multicriteria decision-making

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u(king safety, material advantage, control of the centre)?
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- Search methods to avoid multi-criteria altogether: Monte Carlo Tree Search generates random endgames.

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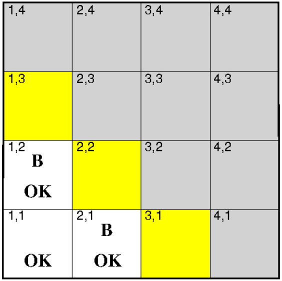
- We need to find ways to compare bundles of factors, might be difficult in general
- Search methods to avoid multi-criteria altogether: Monte Carlo Tree Search generates random endgames.

We assume there is a way of assigning a utility function to bundles of factors and therefore compare them.

Beliefs and expected utility



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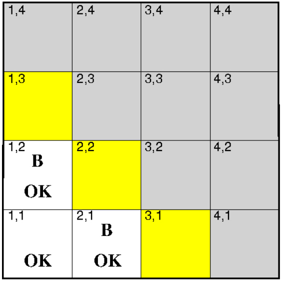

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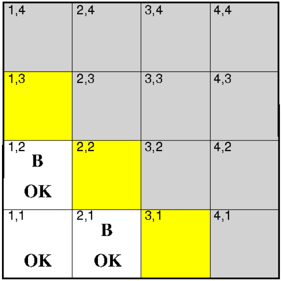

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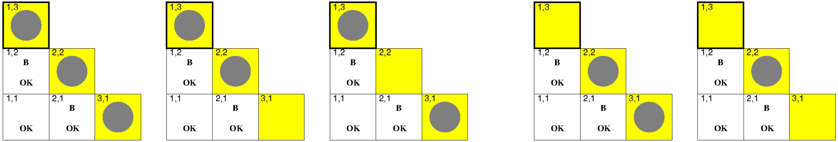
What's the expected utility of going to $[3, 1], [2, 2], [1, 3]$?

Using conditional independence contd.



0.2 x 0.2 = 0.04 | 0.2 x 0.8 = 0.16 | 0.8 x 0.2 = 0.16 | 0.2 x 0.2 = 0.04 | 0.2 x 0.8 = 0.16

$$
\begin{aligned}
\mathbf{P}(P_{1,3}|known, b) &= \alpha' \langle 0.2(0.04+0.16+0.16),\ 0.8(0.04+0.16) \rangle \\
&\approx \langle 0.31, 0.69 \rangle
\end{aligned}
$$

$$
\mathbf{P}(P_{2,2}|known, b) \approx \langle 0.86, 0.14 \rangle
$$

Beliefs and expected utility

The expected utility $u(1,3)$ of the action $(1,3)$ of going to $[1,3]$ from an explored adjacent square is:

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Clearly going to $[2,2]$ from either $[1,2]$ or $[2,1]$ is irrational.
Either going to $[1,3]$ or $[3,1]$ is the rational choice.

Risky moves

Actuators

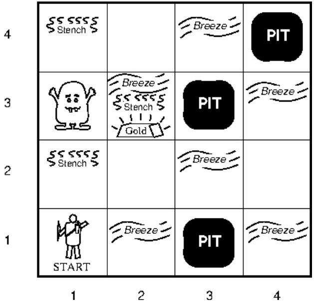
Sensors Breeze, Glitter, Smell

Actuators Turn L/R, Go, Grab, Release, Shoot, Climb

Rewards 1000 escaping with gold, -1000 dying, -10 using arrow, -1 walking

Environment

- Squares adjacent to Wumpus are smelly
- Squares adjacent to pit are breezy
- Glitter iff gold is in the same square
- Shooting kills Wumpus if you are facing it
- Shooting uses up the only arrow
- Grabbing picks up gold if in same square
- Releasing drops the gold in same square



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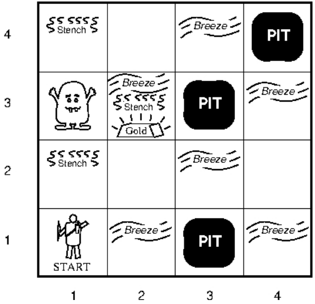
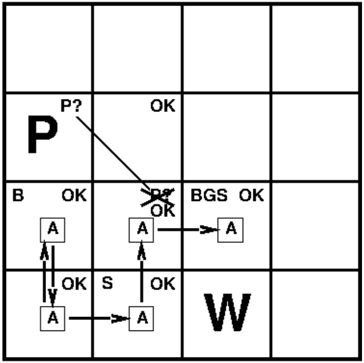


Deterministic actions

Actions in the Wumpus World are **deterministic**



Deterministic actions

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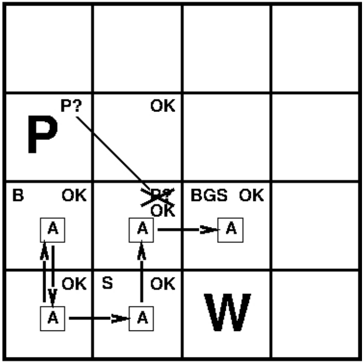
If I want to go from [2, 3] to [2, 2] I just go.



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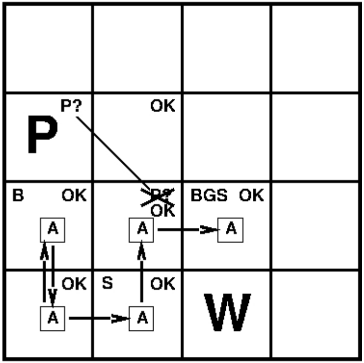
$P([2,2] \mid [2,3],(2,2))$



Deterministic actions

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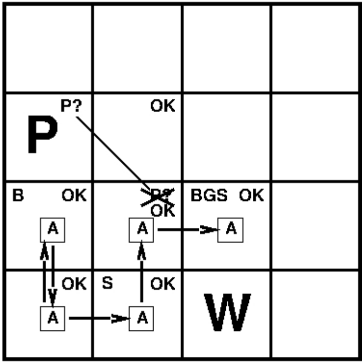
If I want to go from [2, 3] to [2, 2] I just go.

$P([2,2] \mid [2,3],(2,2)) = 1$



Stochastic actions

Stochastic actions 'simulate' lack of control. The agent can try to go to the intended direction but much can work against:

- The environment
- The opponents
- The agent themselves!

Stochastic actions

The result of performing a in state s is a lottery over S, i.e., probability distribution over the set of all possible states.

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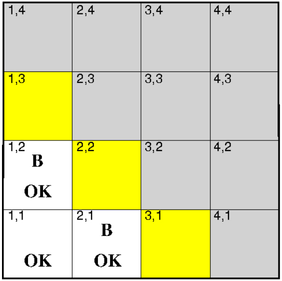
- Bumps their head to the wall and stays in $[2, 1]$ with prob. 0.1
- Goes to any other square with probability 0

Beliefs, expected utility and stochastic actions



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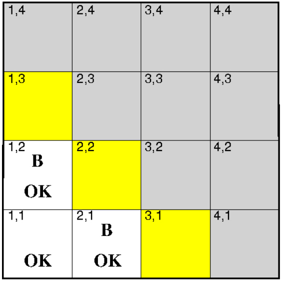

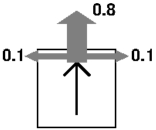

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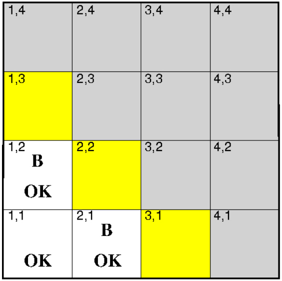

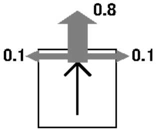

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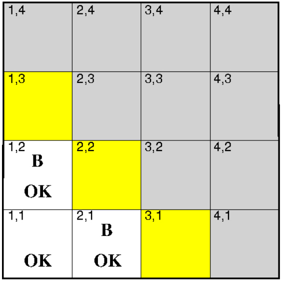

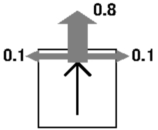

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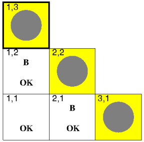
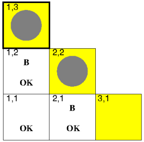
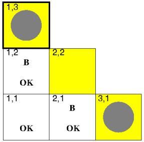
Beliefs, expected utility and stochastic actions



0.2 x 0.2 = 0.04



0.2 x 0.8 = 0.16



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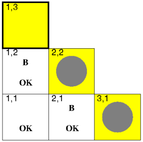

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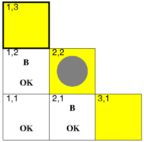

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$$\begin{aligned}\mathbf{P}(P_{1,3}|known, b) &= \alpha' \langle 0.2(0.04+0.16+0.16),\ 0.8(0.04+0.16)\rangle \\ &\approx \langle 0.31, 0.69\rangle\end{aligned}$$

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Then the utility of such action is given by:

$$u(s, a) = \sum_{p_i, L_i} p_i u(L_i) = \sum_{p_i} p_i \sum_{q_i, s_i} q_i u(s_i)$$

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The expected utility of each outcome times the probability of reaching it.

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The expected utility of each outcome times the probability of reaching it.

It is a lottery of lotteries!

Beliefs, expected utility and stochastic actions

$u(1,3) =$

Beliefs, expected utility and stochastic actions

$$u(1,3) = 0.8 \times u[0.31, -1000; 0.69, 0] + 0.1 \times u[1, 0] +$$
$$+ 0.1 \times u[0.86, -1000; 0.14, 0]$$

Beliefs, expected utility and stochastic actions

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We can can get to $[2, 2]$ from two directions, but by symmetry it's the same.

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$u(1,3) = u(3,1)$ (because of symmetry)

Going to $[2,2]$ is still the irrational choice, but not as bad.
The rational choice is either going to $[1,3]$ or $[3,1]$.

Beliefs versus knowledge

- A purely knowledge-based agent has nothing better to do than choosing at random. Which means $\frac{2}{3}u(1,3) + \frac{1}{3}u(2,2)$.
- A belief-based agent can improve the payoff using probabilistic reasoning and going for $u(1,3)$.

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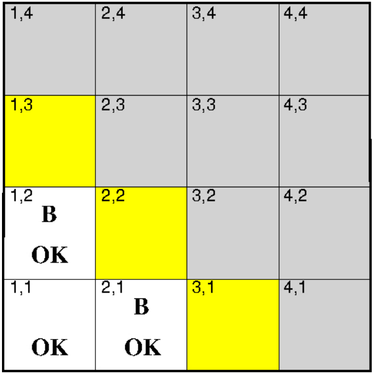
Obviously, the more chaotic the decision system the less the impact of reward difference.

New probability model



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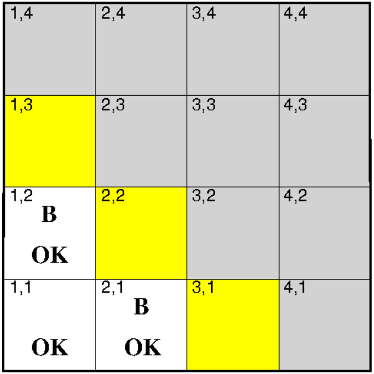
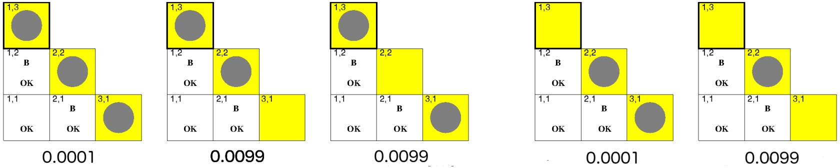


Assume pits can be in a square with probability 0.01

The fringe



Obviously, we can use exactly the same reasoning!

Beliefs, expected utility and stochastic actions

- With deterministic agents, the chance of death is 0.9902 when trying to go to $[2, 2]$.
- With deterministic agents, it tends to 1 with the probability of pit in a square tending to 0;
- The more deterministic the agent, the higher the chance of death.
- Because the way rewards are defined, the expected utility follows the same pattern.

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- The more deterministic the agent, the higher the chance of death.
- Because the way rewards are defined, the expected utility follows the same pattern.

Again, belief-based agents, perform much better than knowledge-based ones

Today's class

- Utility, lotteries and preferences
- Maximisation of expected utility
- Stochastic actions
- Knowledge-based versus belief-based agents

Coming next

- Time
- Risky plans
- What's the best "strategy" to follow?
- Estimating future gains: how patient should we be?

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