COMP245: Probability and Statistics 2016 - Problem Sheet 1 Solutions

Mathematical Methods Revision

Purpose: The purpose of this problem sheet is to practice basic mathematical methods. These methods will be used throughout the course.

S1) This question uses the results of arithmetic and geometric progression.

(a) $x_n = \frac{1}{4^{n-1}x}$, $S_\infty = \frac{4}{3x}$ for all $x \neq 0$.

(b) $x_n = \frac{1}{x^n}$, $S_\infty = \frac{1}{x-1}$ for all x such that $|x| > 1$.

(c) $x_n = \frac{1}{x^{n-1}}$, $S_\infty = \frac{x}{x-1}$ for all x such that $|x| > 1$.

S2) From 1b) we have

$$\sum_{i=1}^{\infty} x^{-i} = \frac{1}{x-1}.$$

Solving

$$\sum_{i=1}^{\infty} x^{-i} = \frac{1}{x-1} = 1$$

gives $x = 2$.

(Notice that although the sequence is infinite $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ it sums to 1.)

S3) This question uses the formulae for differentiation, namely additivity, the product rules and the chain rules (in this order).

(a) $\sum_{i=1}^{n} i a_i x^{i-1}$;

(b) $\log(x) + 1$;

(c) e^{x+e^x};

S4) This question uses the formulae for integration. The c is the constant of integration.

(a) $\sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1} + c$;

(b) $\frac{x^2}{2}\left(\log(x) - \frac{1}{2}\right) + c$;

(c) $\frac{-e^{-ax}}{a} + c$;

(d) $\dfrac{-e^{-ax}}{a}\left(x+\dfrac{1}{a}\right)+c;$

S5) From 4c) we have that

$$\int_0^\infty e^{-ax}dx = \frac{-e^{-ax}}{a}\Big|_0^\infty = 0-\left(-\frac{1}{a}\right) = \frac{1}{a}.$$

Solving

$$\int_0^\infty e^{-ax}dx = 1$$

implies $a = 1$.

S6) The interior of the quarter-ellipse is given by the region

$$E = \{(x,y)|0 < x < \sqrt{2}, 0 < y < \sqrt{1-x^2/2}\}.$$

The integral of f over E is then

$$\int_E f(x,y)dxdy = \int_{x=0}^{\sqrt{2}} xdx \int_{y=0}^{\sqrt{1-x^2/2}} ydy = \int_{x=0}^{\sqrt{2}} \frac{x}{2}dx \left\{ \left[y^2\right]_{y=0}^{\sqrt{1-x^2/2}} \right\} = \int_{x=0}^{\sqrt{2}} \frac{x}{2} - \frac{x^3}{4}dx$$
$$= \left[\frac{x^2}{4} - \frac{x^4}{16}\right]_{x=0}^{\sqrt{2}} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

By symmetry, the integral of g over the whole ellipse would be equal to 1.

S7) The inverse image is $(-\sqrt{2}, \sqrt{2})$. Hint: if you are unsure why, draw the function $f(x)$ and recall the definition of an inverse image.

S8) (a) At $x = -0.8$, $\hat{y} = 0.7$.

(b) At $x = 1.0$, $\hat{y} = -0.6625$.