Computational Techniques 2009-2010

1 a) The range space, column space or the range of matrix A, where $A \in \mathbb{R}^{m \times n}$, is defined as the set of all possible linear combinations of the column vectors of matrix A. This is also referred to as the span of A.

The null space, or kernel, of a matrix $A \in \mathbb{R}^{m \times n}$, is defined as the set of solutions to the homogeneous equation:

$$A\bar{v} = 0 \qquad \text{(where } \bar{v} \text{ is a } n \times 1 \text{ vector)}$$

In order to prove that a subset is a subspace, it needs to satisfy the following three properties:

If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following 3 conditions hold:

- the subspace W contains the zero vector
- if $\bar{u}$ and $\bar{v}$ are vectors in W, then $\bar{u}+\bar{v}$ is in W
  → we say that the subspace is closed under addition.
- if $k \in \mathbb{R}$, i.e. it is a real number scalar, and $\bar{u}$ is any vector in W, then $k\bar{u}$ is also in the subspace W.
  → we say that the subspace is closed under scalar multiplication

EXTRA STUFF

$$\underset{m \times n}{A} = \begin{bmatrix} \bar{v}_1, \bar{v}_2, \dots \bar{v}_n \end{bmatrix}$$

m → rows, n → columns

$\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n \in \mathbb{R}^m$

The range of A, $R(A)$, is a subspace of $\mathbb{R}^m$

THEOREM: If A is an $m \times n$ matrix, then $R(A)$ is a subspace of $\mathbb{R}^m$

PROOF: First of all, notice that if $\bar{y}$ is in $R(A)$, then $A\bar{x} = \bar{y}$ for some $\bar{x} \in \mathbb{R}^n$. Since A is $m \times n$ and $\bar{x}$ is $n \times 1$, $A\bar{x} = \bar{y}$ will be $m \times 1$. That is $\bar{y}$ will be in $\mathbb{R}^m$. This shows that $R(A)$ is a subset of $\mathbb{R}^m$.

$$\begin{array}{c c} & n \\ m & \begin{bmatrix} A \end{bmatrix} \end{array} \begin{array}{c c} 1 & \\ \begin{bmatrix} \bar{x} \end{bmatrix} & n \end{array} = \begin{array}{c c} 1 & \\ \begin{bmatrix} \bar{y} \end{bmatrix} & m \end{array}$$

Verifying that the range space, $R(A)$ is a subspace.

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \;A\; \end{bmatrix} \end{array} \begin{array}{cc} 1 & \\ \begin{bmatrix} \bar{x} \end{bmatrix} & n \end{array} = \begin{array}{cc} 1 & \\ \begin{bmatrix} \bar{y} \end{bmatrix} & m \end{array}$$

Let $W = R(A)$

1) Let 0_m denote the zero vector in $\mathbb{R}^m$ and 0_n denote the zero vector in $\mathbb{R}^n$. Notice that $A0_n = 0_m$. Hence $A\bar{x} = 0_m$ is satisfied by at least one $\bar{x} \in \mathbb{R}^n$, namely $\bar{x} = 0_n$. Thus, $0_m \in W$. $\blacksquare$

2) Suppose $\bar{z}, \bar{y} \in R(A)$. Let $\bar{x}_1, \bar{x}_2 \in \mathbb{R}^n$ be such that $A\bar{x}_1 = \bar{y}$ and $A\bar{x}_2 = \bar{z}$. Then $\bar{y} + \bar{z} = A\bar{x}_1 + A\bar{x}_2 = A(\bar{x}_1 + \bar{x}_2) \in R(A)$ $\blacksquare$
[This means that the vector $(\bar{x}_1 + \bar{x}_2)$ is sent by A to $(\bar{y} + \bar{z})$ and we wanted to prove that there is such a vector].

3) Suppose $\bar{y} \in R(A)$. Let $c \in \mathbb{R}$ (c is a real number scalar). Since $\bar{y} \in R(A)$, there exists a vector $\bar{v} \in \mathbb{R}^n$ such that $A\bar{v} = \bar{y}$. Consider $A\bar{x} = c\bar{y}$. Then let $\bar{x} = c\bar{v}$, a vector in $\mathbb{R}^n$. Then $A\bar{x} = A(c\bar{v}) = c(A\bar{v}) = c\bar{y}$
Therefore $c\bar{y}$ is in $R(A)$. $\blacksquare$

Verifying that the null space, $N(A)$ is a subspace.

Let $W = N(A)$

1) Notice that $A0_n = 0_m$. Hence the equation $A\bar{x} = 0_m$ is satisfied by $\bar{x} = 0_n$
It follows that $0_n \in W$. $\blacksquare$

2) Suppose that $\bar{x}_1, \bar{x}_2 \in W$. This means that $A\bar{x}_1 = 0_m$ and $A\bar{x}_2 = 0_m$.
Let $\bar{x} = \bar{x}_1 + \bar{x}_2$. Then $A\bar{x} = A(\bar{x}_1 + \bar{x}_2) = A\bar{x}_1 + A\bar{x}_2 = 0_m + 0_m = 0_m$
Therefore $\bar{x} = \bar{x}_1 + \bar{x}_2 \in W$. $\blacksquare$

3) Suppose that $x_1 \in W$. Let $c \in \mathbb{R}$ (a real scalar). Since $x_1 \in W$, we have $Ax_1 = 0_m$. Let $\bar{x} = cx_1$. Then: $A\bar{x} = A(c\bar{x}_1) = c(A\bar{x}_1) = c0_m = 0_m$.
Therefore $\bar{x} = cx_1 \in W$ $\blacksquare$

<u>EXTRA STUFF</u>

THEOREM: If A is an $m \times n$ matrix then $N(A)$ is a subspace of $\mathbb{R}^n$

PROOF: First of all notice that if $\bar{x}$ is in $N(A)$, then $A\bar{x} = 0_m$.
Since A is $m \times n$ and Ax is $m \times 1$, it follows that x must be $n \times 1$. That is $x \in \mathbb{R}^n$, Therefore $N(A)$ is a subset of $\mathbb{R}^n$.

b) i) To find the range space of A, perform Gaussian Elimination to achieve echelon form first.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 2 & 1 & -3 \end{bmatrix} \xrightarrow[\substack{R_2 := R_1 \\ R_4 := R_2}]{R_1 := R_4} \begin{bmatrix} 2 & 1 & -3 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_3 := R_2 + R_3} \begin{bmatrix} 2 & 1 & -3 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4 := R_4 + R_3} \begin{bmatrix} 2 & 1 & -3 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 := R_1 - R_2} \begin{bmatrix} 1 & 2 & -3 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 := R_2 - R_1}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 := R_2/3} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 := R_3 - R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 := -1 \times R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Those colums which are linearly independent have leading 1's

$\therefore$ First two columns are linearly independent.

$$\text{range}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

dimension = 2.

$$A^T = \begin{bmatrix} 1 & 0 & -1 & 2 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{bmatrix} \xrightarrow{R_2 := R_2 + R_1} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & -1 & 1 & -3 \end{bmatrix} \xrightarrow{R_3 := R_2 + R_3}$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{null}(A^T) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \quad \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 - x_3 + 2x_4 &= 0 \\ x_2 - x_3 + 3x_4 &= 0 \end{aligned}\right\} \qquad \therefore \text{null}(A^T) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$\left.\begin{aligned} x_1 &= x_3 - 2x_4 \\ x_2 &= x_3 - 3x_4 \end{aligned}\right\}$$

dimension = 2.

ii) range (A) => dimension $= 2$
range (A^T) => dimension $= 2$ $\Big\}$ - always the same
null (A) => dimension $= n - r = 3 - 2 = 1$
null (A^T) => dimension $= m - r = 4 - 2 = 2$

[where m is the row, n is the columns of a $m \times n$ matrix A]

Not exactly sure what it means by "geometric shape".
Attempted answer below:

Suppose I have 2 planes in $\mathbb{R}^3$ and they form a system $A\bar{x} = \bar{b}$.
The nullspace of A represents geometrically the vectors that form the intersection between the 2 planes shifted to the origin.

The range of A represents the span of the normal vectors to the 2nd plane.

iii) null $(A) \perp$ range (A^T)
null $(A^T) \perp$ range (A)

$$\text{range}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{nullspace}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right)$$

Two vectors are orthogonal if their dot product is zero.

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} = (1 \times -2) + (0 \times -3) + (-1 \times 0) + (2 \times 1) = -2 + 0 + 0 + 2 = 0$$

(can also do for other combinations)

c i) $A x = b$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\left.\begin{array}{l} x_1 - x_2 = 2 \\ x_2 - x_3 = 1 \\ -x_1 + x_3 = 0 \\ 2x_1 + x_2 - 3x_3 = 2 \end{array}\right\} \quad \left.\begin{array}{l} x_1 = 2 + x_2 \\ x_2 = 1 + x_3 \\ x_1 = x_3 \end{array}\right\} \quad \left.\begin{array}{l} x_1 = 2 + 1 + x_3 \\ x_1 = x_3 \end{array}\right\} \quad \left\{\begin{array}{l} x_1 = 3 + x_3 \\ x_1 = x_3 \end{array}\right\}$$

inconsistent equations $\therefore$ no solution

ii) Notes on page 73/138 or 70 of the background notes go some way to explaining what to do.

This is a least squares problem, so

$\| Ax - b \|_2$ is minimised when $A^T(b - Ax) = 0$

which in turn is equivalent to :

x minimises $\| Ax - b\|_2$ if and only if $Ax = b_R$ and $b - Ax = b_N$

the range of A

the null space of A

(Need some confirmation on this)

2 a i) trace (A) is the sum of the diagonal entries of a matrix A.

trace (A) = 7 + 6 + 5 = 18

$$\det(A) = 7 \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} - -2 \begin{vmatrix} -2 & -2 \\ 0 & 5 \end{vmatrix} + 0 \begin{vmatrix} -2 & 6 \\ 0 & -2 \end{vmatrix}$$

$$= 7(30 - 4) - -2(-10 - 0) + 0(4 - 0)$$

$$= 7(26) + 2(-10) + 0$$

$$= 182 - 20$$

$$= 162$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a n by n matrix A.

Then:

$\det(A) = \lambda_1 \times \lambda_2 \times \ldots \times \lambda_n$

$\mathrm{trace}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$

ii) Characteristic equation $\det(A-\lambda I)=0$

$$\left|\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right| = 0$$

$$\begin{vmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{vmatrix} = 0$$

$$7-\lambda \begin{vmatrix} 6-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} - -2 \begin{vmatrix} -2 & -2 \\ 0 & 5-\lambda \end{vmatrix} + 0 \begin{vmatrix} -2 & -2 \\ 0 & 5-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)((6-\lambda)(5-\lambda)-4) + 2(-2(5-\lambda)-0) + 0 = 0$$

$$(7-\lambda)(30 - \cancel{6\lambda} - \cancel{5\lambda} + \cancel{\lambda^2} - 4) + 2(-10+2\lambda) = 0$$

$$(7-\lambda)(\lambda^2 - 11\lambda + 26) + 4\lambda - 20 = 0$$

$$7\lambda^2 - 77\lambda + 182 - \lambda^3 + 11\lambda^2 - 26\lambda + 4\lambda - 20 = 0$$

$$-\lambda^3 + 18\lambda^2 - 99\lambda + 162 = 0$$

$$\begin{array}{r|l} & -\lambda^2 + 9\lambda - 18 \\ \hline \lambda - 9 & -\lambda^3 + 18\lambda^2 - 99\lambda + 162 \\ & -\lambda^3 + 9\lambda^2 \\ \hline & 9\lambda^2 - 99\lambda \\ & 9\lambda^2 - 81\lambda \\ \hline & -18\lambda + 162 \\ & -18\lambda + 162 \\ \hline & 0 \end{array}$$

$$(\lambda-9)(-\lambda^2 + 9\lambda - 18)$$

$$(\lambda-9)(-\lambda+6)(\lambda-3)$$

$\therefore$ eigenvalues $\lambda = 9, 6, 3$

iii) There exists a basis of a orthornormal eigenvectors if and only if the matrix A is symmetric (spectral theorem).

$\lambda = 9, 6, 3$

For $\lambda = 9$:

$$\begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} -2x_1 - 2x_2 &= 0 \\ -2x_1 - 3x_2 - 2x_3 &= 0 \\ -2x_2 - 4x_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} -2x_1 &= 2x_2 \\ -2x_2 &= 4x_3 \end{aligned}$$

$$-x_1 = x_2$$

$$-x_2 = 2x_3$$

$$\therefore \bar{v}_1 = \begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$$

For $\lambda = 6$

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} x_1 - 2x_2 &= 0 \\ -2x_1 - 2x_3 &= 0 \\ -2x_2 - x_3 &= 0 \end{aligned}\right\} \begin{aligned} x_1 &= 2x_2 \\ -x_1 &= x_3 \\ -2x_2 &= x_3 \end{aligned} \qquad \therefore \ \bar{V}_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

For $\lambda = 3$

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 4x_1 - 2x_2 &= 0 \\ -2x_1 + 3x_2 - 2x_3 &= 0 \\ -2x_2 + 2x_3 &= 0 \end{aligned}\right\} \begin{aligned} 2x_1 &= x_2 \\ \\ x_3 &= x_2 \end{aligned} \qquad \therefore \ \bar{V}_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

The eigenvectors are orthogonal to each other since A is symmetric.
Normalize each vector to make it 'orthonormal':

$$\| v_1 \| = \sqrt{(-2)^2 + (2)^2 + (1)^2} = 3 \qquad \therefore \ \bar{V}_1 = \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix} \quad \bar{V}_2 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \quad \bar{V}_3 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\| V_2 \| = \sqrt{(2)^2 + (1)^2 + (-2)^2} = 3$$

$$\| V_3 \| = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$$

iv) $Q = \begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ -1 & -2 & 2 \end{pmatrix}$ (combine eigenvectors above)
(we can also check that Q is non-singular by calculating determinant)

$Q^{-1} \Rightarrow$

① find $\det(Q) \Rightarrow$

$$\det(Q) = -2 \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}$$

$$= -2(2 - -4) - 2(4 + 2) + 1(-4 - -1)$$

$$= -2(6) - 2(6) + (-3)$$

$$= -12 - 12 - 3$$

$$= -27$$

$\det(Q) \neq 0$
$\therefore$ non-singular

② $Q^T = \begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}$

③ Find determinant of minor matrices

$$Q^T_{11} = \begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} = 2 - -4 \qquad Q^T_{12} = \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix} = 4 - -2 \qquad Q^T_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1$$

$$Q^T_{21} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} = 4 - -2 \qquad Q^T_{22} = \begin{vmatrix} -2 & -1 \\ 1 & 2 \end{vmatrix} = -4 - -1 \qquad Q^T_{23} = \begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} = -4 - 2$$

$$Q^T_{31} = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -4 - -1 \qquad Q^T_{32} = \begin{vmatrix} -2 & -1 \\ 2 & -2 \end{vmatrix} = 4 - -2 \qquad Q^T_{33} = \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -2 - 4$$

$$\Rightarrow \begin{pmatrix} 6 & 6 & 3 \\ 6 & -3 & -6 \\ -3 & 6 & -6 \end{pmatrix}$$

④ Apply matrix of cofactors.

$$\begin{pmatrix} 6 & 6 & 3 \\ 6 & -3 & -6 \\ -3 & 6 & -6 \end{pmatrix} \times \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} = \begin{pmatrix} 6 & -6 & 3 \\ -6 & -3 & 6 \\ -3 & -6 & -6 \end{pmatrix}$$

$$\therefore \quad Q^{-1} = \frac{1}{-27} \begin{pmatrix} 6 & -6 & 3 \\ -6 & -3 & 6 \\ -3 & -6 & -6 \end{pmatrix}$$

$$= \frac{-3}{-27} \begin{pmatrix} -2 & 2 & -1 \\ -2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}$$

v) This diagonal matrix $\Lambda = Q^{-1} A Q$

$$\Lambda = \frac{1}{9} \begin{pmatrix} -2 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ -1 & -2 & 2 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Which is a matrix containing the eigenvalues corresponding to the eigenvectors in Q.

vi) A positive definite matrix is a symmetric matrix will all positive eigenvalues.

A is such a matrix since its eigen values are 9, 6, 3 which are all ≥ 0.

vii) Yes there exists a lower triangular matrix $L \in \mathbb{R}^{3\times 3}$ with $A = LL^T$ since A is symmetric.

b i)

$$A_{mn} = U_{mm} S_{mn} V_{nn}^T$$

S is a diagonal matrix containing the square roots of eigenvalues from U or V in descending order

The columns of U are orthonormal eigenvectors of AA^T

the columns of V are orthonormal eigenvectors of A^TA

Essentially we are computing: $U^T A V = S$

For V

$$\cdot\ A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(A^TA - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda) = 0$$

$$\lambda = 2, 1$$

Finding eigenvectors:

$\lambda = 2$

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-x_2 = 0$$

$$\bar{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ ← any number

$\lambda = 1$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = 0$$

$$\bar{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$ ← any number.

$$\therefore\ \bar{V}_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

ordered by descending value of eigen vectors.

For U

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\det(AA^T - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0 \qquad 0 = 1-\lambda \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 1 & 1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1-\lambda \\ 1 & 0 \end{vmatrix}$$

$$(1-\lambda)(1-\lambda)(1-\lambda) + -(1-\lambda) = 0$$

$$(1-\lambda)(1-\lambda-\lambda+\lambda^2) - 1 + \lambda = 0$$

$$(1-\lambda)(1-2\lambda+\lambda^2) - 1 + \lambda = 0$$

$$\cancel{1} - 2\lambda + \cancel{\lambda^2} - \cancel{\lambda} + 2\lambda^2 - \lambda^3 \cancel{-1} + \cancel{\lambda} = 0$$

$$-2\lambda - \lambda^3 + 3\lambda^2 = 0$$

$$-\lambda^3 + 3\lambda^2 - 2\lambda = 0$$

$$\lambda(-\lambda^2 + 3\lambda - 2) = 0$$

$$\lambda(-\lambda+2)(\lambda-1)$$

Eigenvalues: 0, 1, 2

For $\lambda = 0$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0 \\ x_1 + x_3 &= 0 \end{aligned}\right\} \quad x_1 = -x_3 \qquad \bar{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $\lambda = 1$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} x_3 &= 0 \\ x_1 &= 0 \end{aligned}\right\} \qquad \bar{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 2$

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} -x_1 + x_3 &= 0 \\ -x_2 &= 0 \\ x_1 - x_3 &= 0 \end{aligned}\right\} \qquad \bar{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \bar{U}_I = \cancel{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

ordered by descending value of eigen value.

Finally we have to convert out $\bar{V}_I$ and $\bar{U}_I$ into ~~an~~ orthogonal matrices which we do by applying the Gram-Schmidt orthonormalization process to the column vectors.

For $\bar{V}_I$

① $\bar{U}_1 = \dfrac{\bar{V}_1}{|\bar{V}_1|}$ $\qquad \bar{U}_1 = \dfrac{[\Phi, 0]}{\sqrt{\Phi^2 + 0^2}} = [\Phi, 0]$ ~~(not strictly correct → dividing by 0)~~

② $\bar{\omega}_2 = \bar{V}_2 - \bar{U}_1 \cdot \bar{V}_2 \times \bar{U}_1$

$\bar{u}_2 = \dfrac{\bar{\omega}_2}{|\bar{\omega}|}$

$$\bar{\omega}_2 = \begin{bmatrix} 0 \\ \Phi \end{bmatrix} - \begin{bmatrix} \Phi \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \Phi \end{bmatrix} \times \begin{bmatrix} \Phi \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ \Phi \end{bmatrix} - 0 \times \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ \Phi \end{bmatrix}$$

$$\bar{U}_2 = \frac{[0, \Phi]}{\sqrt{0^2 + \Phi^2}} = [0, \Phi]$$

$$\therefore \bar{V} = \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}$$

For $\bar{U}_I$

① $\bar{U}_1 = \dfrac{\bar{V}_1}{|\bar{V}_1|}$ $\qquad \bar{U}_1 = \dfrac{[1, 0, 1]}{\sqrt{1^2 + 0^2 + 1^2}} = \dfrac{[1, 0, 1]}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

② $\bar{\omega}_2 = \bar{V}_2 - \bar{U}_1 \cdot \bar{V}_2 \times \bar{U}_1$

$\bar{U}_2 = \dfrac{\bar{\omega}_2}{|\bar{\omega}_2|}$

$$\bar{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - 0 \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

③ $\bar{\omega}_3 = \bar{V}_3 - \bar{U}_1 \cdot \bar{V}_3 \times \bar{U}_1 - \bar{U}_2 \cdot \bar{V}_3 \times \bar{U}_2$

$\bar{U}_3 = \dfrac{\bar{\omega}_3}{|\bar{\omega}_3|}$

$$\bar{\omega}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{\omega}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} - 0 \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \times \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\bar{\omega}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} - 0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\bar{U}_3 = \frac{[1, 1, -1]}{\sqrt{1^2+1^2+(-1)^2}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\therefore \; U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{3} \end{pmatrix}$$

ii) The singular values of A are the square roots of the non-zero eigen values which then populate the diagonal.

Dimensions of A are $m \times n$ so $\Rightarrow$ $\sqrt{2}, \sqrt{1}$

corresponding matrix:

$$\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{bmatrix} = S$$

$\leftarrow$ Augmented.

l_2 norm of $\|A\|_2 = \sqrt{1^2+0^2+0^2+1^2+1^2+0^2}$

$= \sqrt{3}$

iii) Not sure about this one.

We have essentially computed $U^T A V = S$

so $A = USV^T$

so $\mathbb{R}^3$ basis $= U = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{3} \end{pmatrix}$

$\mathbb{R}^2$ basis $= V^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Not convinced that all of my answers are correct since.

$U^T A V \neq S$ for some reason :(

3ai) The p-norm is defined for an m-component vector $\bar{x}$ as follows:

$$\|x\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p}$$

The subordinate matrix norm $\|A\|$ is defined as

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

ii) Condition number of a matrix A is defined as.

$$\text{cond}(A) = \|A\| \cdot \|A^{-1}\|$$

By convention $\text{cond}(A) = \infty$ if A singular.

can check that A is non-singular => $\det(A) \neq 0$

$$\|A\|_1 = \max(|1|+|0|+|2|, |0|+|3|+|-1|, |2|+|-1|+|1|)$$
$$= \max(3, 4, 4)$$
$$= 4$$

$$A^{-1} \Rightarrow \det(A) = 1\begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 3 \\ 2 & -1 \end{vmatrix}$$
$$= 3 - 1 + 2(-6)$$
$$= 2 - 12$$
$$= -10$$

$$A^T \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & -1 \\ 2 & -1 & 1 \end{bmatrix} \text{ determinants of minor matrices } \Rightarrow \begin{bmatrix} 2 & 2 & -6 \\ 2 & -3 & -1 \\ -6 & -1 & 3 \end{bmatrix} \Rightarrow \text{ matrix of cofactors}$$

$$A^{-1} = \frac{1}{-10}\begin{bmatrix} 2 & -2 & -6 \\ -2 & -3 & 1 \\ -6 & 1 & 3 \end{bmatrix}$$

$$\|A^{-1}\| = \max\left(\left|\tfrac{2}{-10}\right| + \left|\tfrac{-2}{-10}\right| + \left|\tfrac{-6}{-10}\right|, \left|\tfrac{-2}{-10}\right| + \left|\tfrac{-3}{-10}\right| + \left|\tfrac{1}{-10}\right|, \left|\tfrac{-6}{-10}\right| + \left|\tfrac{1}{-10}\right| + \left|\tfrac{3}{-10}\right| \right)$$
$$= \max(1, \tfrac{1}{2}, 1)$$
$$= 1$$

$\therefore \text{cond}(A) = 4 \times 1 = 4$

b i) A metric space is a non-empty set S of objects (called points) together with a function $d: S \times S \longmapsto \mathbb{R}^1$ (called the metric of the space) satisfying the following four properties for all points $x, y, z \in S$:

($d \to$ distance)

1) $d(x,x) = 0$
2) $d(x,y) > 0$ if $x \neq y$
3) $d(x,y) = d(y,x)$ (symmetry)
4) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

The non-negative number $d(x,y)$ is to be though of as the distance from x to y

Let $S \in \mathbb{R}^k$ and $d(x,y) = \|x-y\|_2 \rightarrow$ called the euclidean metric.

If we take x, y, z to be vectors/matrices we can show that the difference is a metric by satisfying the 4 conditions above.

ii)

$$(I - M)G_n = I - M^{n+1}$$

$$IG_n - MG_n = I - M^{n+1}$$

$$I(I + M + M^2 + \ldots + M^n) - M(I + M + M^2 + \ldots + M^n) = I - M^{n+1}$$

$$I + \cancel{M} + \cancel{M^2} + \ldots + \cancel{M^n} - \cancel{M} - \cancel{M^2} - \cancel{M^3} + \ldots + M^{n+1} = I - M^{n+1}$$

$$I - M^{n+1} = I - M^{n+1} \quad \blacksquare$$

iii) If $\|M\| < 1$ then all vectors $\bar{v}$ of M have components, m, such that each component < 1. If $m < 1$, then let $m = \frac{1}{c}$ • M^n means that each component will be multiplied by itself n times, i.e. $(\frac{1}{c})^n$. Hence we can see that as $n \to \infty$, $(\frac{1}{c})^n \to 0$, and hence since this will happen to every component, m, of matrix M, we can see that as $n \to \infty$, $M^n \to 0$.

$$(I - M)G_n = I - M^{n+1}$$

$$(I - M)G_\infty = I - 0$$

as $n \to \infty$

$M^{n+1} \to 0$ since $M^n \to 0$

$$(I - M)G_\infty = I$$

$$G_\infty = \frac{I}{(I - M)}$$

is this a valid deduction?

$$G_\infty = (I - M)^{-1}$$

A a) NOT COVERED

b i) $$\frac{s-1}{(s+1)(s-2)} = \frac{A}{(s+1)} + \frac{B}{(s-2)}$$

$$= \frac{A(s-2) + B(s+1)}{(s+1)(s-2)}$$

$s-1 = A(s-2) + B(s+1)$

$s-1 = As - 2A + Bs + B$

$s-1 = (A+B)s - 2A + B$

Equating coefficients:

$$\left.\begin{aligned} 1 &= A+B \\ -1 &= -2A+B \end{aligned}\right\} \quad \left.\begin{aligned} A &= 1-B \\ A &= \frac{-1-B}{-2} \end{aligned}\right\}$$

$-2(1-B) = -1-B$

$-2 + 2B = -1 - B$

$2B + B = -1 + 2$

$3B = 1$

$B = \frac{1}{3}$ $\quad \therefore A = 1 - \frac{1}{3} = 2/3$

$$\therefore \frac{s-1}{(s+1)(s-2)} = \frac{2}{3(s+1)} + \frac{1}{3(s-2)}$$

ii) $\int \dot{y}_1 = \int 4y_2 + 2e^{-t}$

$\int \dot{y}_1 = 4y_2 t + -2e^{-t} + C$

$y_1 = 4y_2 t - 2e^{-t} + C$

$\int \dot{y}_2 = \int y_1$

$\int \dot{y}_2 = \int 4y_2 t - 2e^{-t} + c$

$\int \dot{y}_2 = 2y_2 t^2 + 2e^{-t} + ct + k$

$y_2 = 2y_2 t^2 + 2e^{-t} + ct + k$

$0 = 4 \times 0 \times 1 - 2e^{-1} + C$

$0 = 0 - 2e^{-1} + C$

$2e^{-1} = C$

$0 = 2.0. -0.5^2 + 2e^{-(0.5)} + \frac{2}{e} \times -0.5 + k$

$0 = 2/e^{\frac{1}{2}} - \frac{1}{e} + k$

$\frac{1}{e} - \frac{2}{e^{\frac{1}{2}}} = k$

$k = \frac{1}{e} - \frac{2e^{\frac{1}{2}}}{e}$

$= \frac{1 - 2e^{\frac{1}{2}}}{e}$

$\therefore y_1 = 4y_2 t - 2e^{-t} + 2/e$

$y_2 = 2y_2 t^2 + 2e^{-t} + 2/e\, t + \frac{1}{e} - \frac{2}{e^{\frac{1}{2}}}$