IMPERIAL COLLEGE LONDON

TIMED REMOTE ASSESSMENTS 2020-2021

BEng Honours Degree in Computing Part I
MEng Honours Degrees in Computing Part I
for Internal Students of the Imperial College of Science, Technology and Medicine

This paper is also taken for the relevant assessments for the Associateship of the City and Guilds of London Institute

PAPER COMP40002

MATHEMATICS I: FOUNDATIONS

Thursday 6 May 2021, 10:00
Duration: 95 minutes
Includes 15 minutes for access and submission

Answer ALL TWO questions
Open book assessment

Paper contains 2 questions

1 a Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -2 \\ -1 & 2 & 2 & -5 \\ 2 & 3 & 1 & -7 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

and consider the linear system of equations $\mathbf{Ax} = \mathbf{b}$.

i) Perform elementary row operations on the augmented matrix $[\mathbf{A}|\mathbf{b}]$ to reduce $\mathbf{A}$ to Row Echelon Form.

ii) Identify the basic and the free variables.

iii) Find the general solution of $\mathbf{Ax} = \mathbf{b}$ and describe it geometrically.

iv) Find a solution of $\mathbf{Ax} = \mathbf{b}$ with least distance from the origin.

b Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 3 \\ 1 & 5 & -8 & 3 \end{bmatrix}.$$

i) Find the rank and the nullity of $\mathbf{A}$.

ii) Find a basis for the image space $\mathrm{im}(\mathbf{A})$ and also a basis for the null space $\mathrm{null}(\mathbf{A})$.

iii) Find the rank and the nullity of $\mathbf{A}^T$.

c Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

i) Show that 5 is an eigenvalue of $\mathbf{A}$ and determine the other eigenvalues.

ii) Find an eigenvector for each of the eigenvalues of $\mathbf{A}$.

iii) Determine the angle between each pair of the eigenvectors of $\mathbf{A}$.

iv) Obtain an invertible $\mathbf{S} \in \mathbb{R}^{3\times 3}$ such that $\mathbf{S}^{-1}\mathbf{AS}$ is diagonal.

d A linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ acts on an ordered basis $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ as follows:

$$f(\mathbf{v}_1) = 2\mathbf{v}_3 - \mathbf{v}_1 - 2\mathbf{v}_2, \qquad f(\mathbf{v}_2) = 3\mathbf{v}_2 - 5\mathbf{v}_1 - \mathbf{v}_3, \qquad f(\mathbf{v}_3) = \mathbf{v}_2 - \mathbf{v}_1.$$

Find the matrix representation of f with respect to $\mathbf{V}$.

The four parts carry, respectively, 35%, 20%, 30%, and 15% of the marks.

2a i) Find the supremum and the infimum of the following set S:

$$S = \{x \in \mathbb{Q} : x^4 < 5\},$$

where $\mathbb{Q}$ is the set of rational numbers.

ii) Using the $(\epsilon - N)$ definition of limits, prove that the sequence $(a_n)_{n\geq 1}$ with

$$a_n = \frac{1}{n^2 - \frac{1}{2}}$$

converges.

iii) Determine whether or not the following limit exists and obtain the limit if it does exist:

$$\lim_{x\to 1} \frac{x-1}{\cos(\frac{\pi}{2}x)}.$$

b Determine whether or not each of the following series converges and prove your answer.

i) $\sum_{n=0}^{\infty}(\sqrt{n+1} - \sqrt{n})$.

ii) $\sum_{n=0}^{\infty}\left(\frac{5n-3n^3}{7n^3+2}\right)^n$.

iii) $\sum_{n=1}^{\infty}\frac{1}{n^{1.1}}$.

c i) Find the MacLaurin series for the following function $f : \mathbb{R} \to \mathbb{R}$ and determine the radius of convergence of the series.

$$f(x) = \frac{1}{1+2x^2}$$

(**Hint.** Express $f(x)$ as the sum of a geometric series.)

ii) Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has a Maclaurin series with radius of convergence $1/2$, i.e., its Maclaurin series converges for $|x| < 1/2$. Find the first three terms for the Maclaurin series of the function g defined by $g(x) = f(1 - \cos \pi x)$ and determine the radius of convergence of this series.

The three parts carry, respectively, 30%, 40%, and 30% of the marks.