COMP245: Probability and Statistics 2016 - Problem Sheet 3 Solutions

Probability

S1) **Purpose: Practice finding and using disjoint unions of sets.**

First, note that $E \cup F = (E \cap F) \cup (\overline{E} \cap F) \cup (E \cap \overline{F})$ is a disjoint union. So from the axioms of probability,

$$\mathrm{P}(E \cup F) = \mathrm{P}(E \cap F) + (\overline{E} \cap F) + \mathrm{P}(E \cap \overline{F}). \tag{1}$$

Also, note that $F = (\overline{E} \cap F) \cup (E \cap \overline{F})$ is a disjoint union, and so

$$\mathrm{P}(E \cap \overline{F}) = \mathrm{P}(F) - \mathrm{P}(E \cap F), \tag{2}$$

and similarly

$$\mathrm{P}(\overline{E} \cap F) = \mathrm{P}(E) - \mathrm{P}(E \cap F). \tag{3}$$

Substituting equations (2) and (3) into equation (1) yields the result.

S2) Since E and F are mutually exclusive, $E \cap F = \emptyset$ and so $\mathrm{P}(E \cap F) = 0$.

For independence, we require $0 = \mathrm{P}(E \cap F) = \mathrm{P}(E)\mathrm{P}(F)$, and so that one or both of $\mathrm{P}(E)$ and $\mathrm{P}(F)$ must be zero.

S3) **Purpose: Practice using conditional probability.**

(a) $\frac{3}{6} = \frac{1}{2}$.

(b) If the number is less than 4, it must be one of 1, 2, or 3. Two of these three are odd. Hence the answer is $\frac{2}{3}$.

S4) **Purpose: More practice using conditional probability.**

(a) There are 36 possibilities, only one of which is two 6s. Hence $\frac{1}{36}$.

(b) Two of the 36 possibilities will give a total of 3, these being (1,2) and (2,1). Hence $\frac{2}{36} = \frac{1}{18}$.

S5) **Purpose: Use of independence. Notice again the trick of computing the complement of the event of interest.**

The problem is solved if either one or both of the students solve the problem. Thus

$$\begin{aligned}
\mathrm{P}(\text{solved}) &= 1 - \mathrm{P}(\text{not solved}) = 1 - \mathrm{P}(\text{not solved by either}) \\
&= 1 - \mathrm{P}(\text{A does not solve it and B does not solve it}) \\
&= 1 - \mathrm{P}(\text{A does not solve it}).\mathrm{P}(\text{B does not solve it}) \qquad \text{(by independence)} \\
&= 1 - \left(1 - \frac{2}{5}\right) \times \left(1 - \frac{1}{3}\right) \\
&= 1 - \frac{3}{5} \times \frac{2}{3} \\
&= \frac{3}{5}.
\end{aligned}$$

S6) Let $AX = a$. Then first note that $AX.XB = a(1-a)$. Then $a(1-a) < \frac{3}{16}$ if $a < \frac{1}{4}$ or $a > \frac{3}{4}$. Thus $\mathrm{P}(a(1-a) < \frac{3}{16}) = \mathrm{P}(a < \frac{1}{4}) + \mathrm{P}(a > \frac{3}{4}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

S7) (a) **Purpose: Simple question using equally likely outcomes.** $\frac{18}{37}$.

(b) **Purpose: Practice using the notion of counting in a problem that does not use replacement.**

i. $\frac{x}{x+y}$.

ii.

$$\begin{aligned}
\mathrm{P}(\text{2nd red}) &= \mathrm{P}(\text{2nd red}|\text{first red})\mathrm{P}(\text{first red}) + \mathrm{P}(\text{2nd red}|\text{first green})\mathrm{P}(\text{first green}) \\
&= \frac{x-1}{x+y-1}\frac{x}{x+y} + \frac{x}{x+y-1}\frac{y}{x+y} \\
&= \frac{x}{x+y}
\end{aligned}$$

iii. $\mathrm{P}(\text{first two red}) = \mathrm{P}(\text{2nd red}|\text{first red})\mathrm{P}(\text{first red}) = \frac{x-1}{x+y-1}\frac{x}{x+y}$.

iv. $\frac{x}{x+y}$ by second argument in 7(b)ii.

S8) **Purpose: Practice calculating probabilities for joint events.**

(a) $\frac{1}{4}$.

(b) $0.9 \times 0.7 + 0.8 \times 0.3 = 0.87$.

S9) **Purpose: Practice establishing whether events are independent or not.**

(a) Dependent. If they were independent the chance of rain on two consecutive days would be $0.25 \times 0.25 = 0.0625$, not 0.10.

(b) P(rain tomorrow | rain today) = P(rain tomorrow and rain today) / P(rain today) = 0.1 / 0.25 = 0.40.

(c) The numbers are the same.

S10) **Purpose: Practice conditioning on multiple events.**

(a) This is the probability that he doesn't leave it in the first three shops but does leave it in the fourth shop given that he didn't leave it in the first three $= \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{27}{256}$.

(b) Let E_i be the event that he left it in the ith shop. We want $\mathrm{P}(E_4|E_1 \cup E_2 \cup E_3 \cup E_4)$. Now

$$\begin{aligned}
\mathrm{P}(E_4|E_1 \cup E_2 \cup E_3 \cup E_4) &= \frac{\mathrm{P}(E_4 \cap (E_1 \cup E_2 \cup E_3 \cup E_4))}{\mathrm{P}(E_1 \cup E_2 \cup E_3 \cup E_4)} \\
&= \frac{\mathrm{P}(E_4)}{\mathrm{P}(E_1 \cup E_2 \cup E_3 \cup E_4)} \\
&= \frac{\mathrm{P}(E_4)}{1 - \mathrm{P}(\overline{E_1 \cup E_2 \cup E_3 \cup E_4})} \\
&= \frac{\mathrm{P}(E_4)}{1 - \mathrm{P}(\overline{E_1} \cap \overline{E_2} \cap \overline{E_3} \cap \overline{E_4})} \\
&= \frac{\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4}}{1 - \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}} \\
&= 0.15.
\end{aligned}$$

(c) This is the same problems as in 10b, but as if there were only three shops. Hence

$$\begin{aligned}
&\frac{\frac{3}{4} \times \frac{3}{4} \times \frac{1}{4}}{1 - \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}} \\
&= 0.24.
\end{aligned}$$

S11) Let S = satisfactory and $U = \overline{S}$ = unsatisfactory; L = low quality, M = medium quality, H = high quality.

(a)

$$\begin{aligned}
\mathrm{P}(S) &= \mathrm{P}(S|L) \times \mathrm{P}(L) + \mathrm{P}(S|M) \times \mathrm{P}(M) + \mathrm{P}(S|H) \times \mathrm{P}(H) \\
&= 0.8 \times 0.4 + 0.9 \times 0.4 + 1 \times 0.2 \\
&= 0.88.
\end{aligned}$$

(b) If two components are tested there can be four outcomes: SS, SU, US, UU. If the probability of an S is p these four possibilities have respective probabilities

$p^2, p(1-p), (1-p)p, (1-p)^2$. Thus the probability of exactly one S $(US \cup SU)$ is $2p(1-p)$. So

$$\begin{aligned}
\mathrm{P}(US \cup SU) &= \mathrm{P}(US \cup SU|L) \times \mathrm{P}(L) + \mathrm{P}(US \cup SU|M) \times \mathrm{P}(M) + \mathrm{P}(US \cup SU|H) \times \mathrm{P}(H) \\
&= 2 \times 0.8 \times 0.2 \times 0.4 + 2 \times 0.9 \times 0.1 \times 0.4 + 2 \times 1 \times 0 \times 0.2 \\
&= 0.2.
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{P}(M|US \cup SU) &= \frac{\mathrm{P}(US \cup SU|M) \times \mathrm{P}(M)}{\mathrm{P}(US \cup SU)} \\
&= \frac{2 \times 0.9 \times 0.1 \times 0.4}{0.2} \\
&= 0.36.
\end{aligned}$$

(d)

$$\begin{aligned}
\mathrm{P}(SS) &= \mathrm{P}(SS|L) \times \mathrm{P}(L) + \mathrm{P}(SS|M) \times \mathrm{P}(M) + \mathrm{P}(SS|H) \times \mathrm{P}(H) \\
&= 0.8 \times 0.8 \times 0.4 + 0.9 \times 0.9 \times 0.4 + 1 \times 1 \times 0.2 \\
&= 0.78. \\
\mathrm{P}(SS \cap H) &= \mathrm{P}(SS|H) \times \mathrm{P}(H) \\
&= 1 \times 1 \times 0.2 \\
&= 0.2. \\
\text{So } \mathrm{P}(H|SS) &= \frac{\mathrm{P}(SS \cap H)}{\mathrm{P}(SS)} \\
&= \frac{0.2}{0.78} \\
&= 0.26.
\end{aligned}$$

S12) **Purpose: Practice using Bayes' Theorem.**

$\mathrm{P}(A) > \mathrm{P}(B) \Rightarrow \dfrac{\mathrm{P}(A)}{\mathrm{P}(B)} > 1$. Then, from Bayes Theorem $\mathrm{P}(A|B) = \mathrm{P}(B|A)\dfrac{\mathrm{P}(A)}{\mathrm{P}(B)} \Rightarrow$ $\mathrm{P}(A|B) > \mathrm{P}(B|A) \times 1$.