40018 Logic
Logic exercises 5: Semantics of First-order Logic
Translating English to Logic

There will be no **PMT** submission for this sheet.

1. Situation: All members of DoC (staff and students). The binary relation symbol $\mathit{mtt}(x, y)$ is read as 'x is much taller than y'. If you could adjust the heights of all members of the department, could you adjust them to make each sentence (a)–(d) true? How? And to make them all false? How?

   For example, to make $\forall x \forall y(student(x) \wedge \mathit{staff}(y) \to \mathit{mtt}(x, y))$ true, you'd have to arrange that the students were all much taller than the staff. If they're not, it's false.

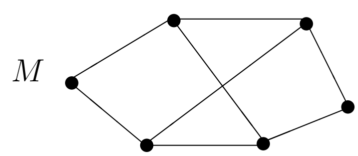
   (a) $\exists u[\mathit{student}(u) \wedge \forall v[\mathit{staff}(v) \to \mathit{mtt}(v, u)]]$

   (b) $\forall y \forall x[\mathit{student}(x) \wedge \mathit{student}(y) \to (\mathit{mtt}(x, y) \leftrightarrow \mathit{mtt}(y, x))]$

   (c) $\forall x[\mathit{student}(x) \to \exists y[\mathit{student}(y) \wedge (\mathit{mtt}(x, y) \vee \mathit{mtt}(y, x))]]$

   (d) $\forall v(\mathit{staff}(v) \to \exists u[\mathit{staff}(u) \wedge \mathit{mtt}(u, v)])$

2. Let L be a signature consisting of a single binary relation symbol E. Consider the following L-structure M. $dom(M)$ has 6 objects, as shown. For objects a, b in $dom(M)$, $E(a, b)$ is true in M just when there's a direct straight line (an 'edge') between a and b. E.g., the top two objects are E-related; the leftmost and rightmost objects are not.



   Which of the following are true in M? Give a brief explanation for each answer.

   (a) $\forall x \neg E(x, x)$

   (b) $\forall x \forall y(E(x, y) \to E(y, x))$

   (c) $\forall x \forall y(E(x, y) \to \forall z(E(y, z) \to E(x, z)))$

   (d) $\exists x \exists y \exists z(E(x, y) \wedge E(y, z) \wedge E(x, z))$

   (e) $\forall x \exists y \exists z(E(x, y) \wedge E(x, z) \wedge y \neq z)$. Here, $y \neq z$ abbreviates $\neg(y = z)$.

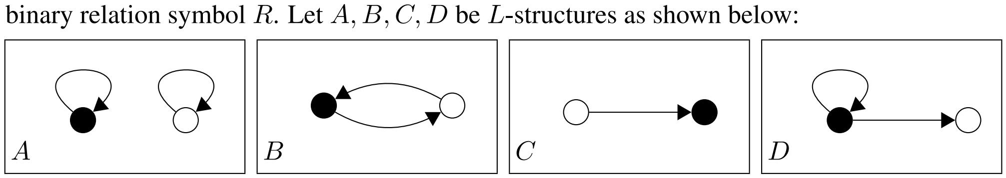
   (f) $\forall x \exists y \exists z \exists v(E(x, y) \wedge E(x, z) \wedge E(x, v) \wedge y \neq z \wedge y \neq v \wedge z \neq v)$.

   Now let P be a new unary relation symbol.

   (g) Define an interpretation of P in M that makes the following sentence true:

   $$\forall x \forall y(E(x, y) \to (P(x) \leftrightarrow \neg P(y))).$$

3. (from exam 2013) Let L be the first-order signature consisting of a unary relation symbol P and a binary relation symbol R. Let A, B, C, D be L-structures as shown below:



   An arrow from a circle a to a circle b means that $R(a, b)$ is true. The objects satisfying P are the black circles.

   For each of the following four L-sentences, state in which of the four structures it is true and in which it is false. You do not need to justify your answers.

(a) $\forall x(P(x) \to \exists y R(x,y))$

(b) $\forall x \forall y(R(x,y) \vee R(y,x))$

(c) $\exists x \forall y(R(x,y) \to P(y))$

(d) $\exists x \forall y \exists z(R(x,z) \wedge \neg(z = y))$

For each of the structures A, B, C, D in turn, write down an L-sentence that is true in that structure and false in the other three.

4. (a) Find a structure with at least two elements in its domain that makes one of $\forall x[F(x) \to G(x)]$ and $\exists x F(x) \to \exists x G(x)$ true, and the other false. Justify your answer.

   (b) Repeat for $\exists v \forall u R(u,v)$ and $\forall x \exists y R(x,y)$.

   (c) In fact, it is only possible to do *one* of the following:

   - find a structure that makes the first sentence of (a) true and the second false, or
   - find a structure that makes the second sentence of (a) true and the first false.

   You cannot do both. Why?

   (d) Repeat part (c) using the sentences in (b) instead of those in (a).

5. Let L be the signature consisting of constants $\underline{1}, \underline{2}, \underline{3}, \ldots$, binary relation symbols $<, >, \leq, \geq$, and binary function symbols $+, \times$. Let N be the structure whose domain consists of the *positive* integers $1, 2, 3, \ldots$, and with the symbols of L interpreted in the natural way.

   The formula $\exists v(x = \underline{2} \times v)$ expresses that x is even. In the same kind of way, write first-order L-formulas expressing that:

   (a) x is divisible by 3 without remainder.

   (b) x is prime.

   (c) Every even number bigger than 2 is the sum of two primes.
   Your answer must be an L-sentence — no free variables please.

   (d) x is a square number.

   (e) x is the sum of two square numbers.

   (f) There are infinitely many prime numbers. (This is 'Euclid's theorem'. $\top$ expresses it, since the theorem is true, but you should write a more direct translation of the meaning of the theorem.)

6. Translate the following sentences into logic as faithfully as possible. Invent your own predicates. Hints: write formulas expressing the subconcepts first, then piece them together into a full solution. E.g., in 6c and 6d, 'x is a chimp', 'y is a prize', 'x won y', 'y was won by a chimp', and 'x won all the prizes' are useful subconcepts. The patterns $\exists x(A \wedge B)$ and $\forall y(A \to B)$ are common — e.g., 'for all y, if y is a prize then x won y' would express 'x won all the prizes'.

   (a) No animal is both a cat and a dog.

   (b) Anyone who admires himself admires someone.

   (c) Every prize was won by a chimpanzee.

   (d) One particular chimpanzee won all the prizes.

   (e) Jack cannot run faster than anyone in the team.

   (f) Jack cannot run faster than everyone in the team.

   (g) All first year students have a PPT tutor.

   (h) No student has the same PMT and PPT tutor.

Logic exercises 6 - solutions

1. (a) The sentence $\exists u[\mathit{student}(u) \wedge \forall v[\mathit{staff}(v) \to \mathit{mtt}(v,u)]]$ says that there is a student that is shorter than all staff members. To make it true, there would have to be at least one such student, who all staff (if any) are much taller than. If there isn't such a student, it's false.

   (b) $\forall y \forall x[\mathit{student}(x) \wedge \mathit{student}(y) \to (\mathit{mtt}(x,y) \leftrightarrow \mathit{mtt}(y,x))]$ is probably not true — being much taller than someone is an asymmetric relationship. Consider two students x and y such that x is much taller than y. Then $\mathit{mtt}(x,y)$ is true and $\mathit{mtt}(y,x)$ false. If we can find such x, y, the sentence is false. Similarly, the sentence is false if we can find students x and y such that y is much taller than x — which we can under the same circumstances as before. If we can't, then it's true. So the sentence is true just when *all students are about the same height.*

   (c) The sentence $S = \forall x[\mathit{student}(x) \to \exists y[\mathit{student}(y) \wedge (\mathit{mtt}(x,y) \vee \mathit{mtt}(y,x))]]$ says that for any given student x, there's some student y who is either much taller or much shorter than x. This explains how to make S *true.* We can also kick out all the students: then S is 'vacuously true', as no x is a student, so $\mathit{student}(x)$ is false for all x — and 'false implies anything' is true.

   S will be *false* just when there is an x making $\mathit{student}(x) \to \exists y[\mathit{student}(y) \wedge (\mathit{mtt}(x,y) \vee \mathit{mtt}(y,x))]$ false. That is, there is some 'utterly average' student x, in the sense that there is *no* student y who's much taller or shorter than x.

   One way that this can happen is if (there is at least one student and) *all students are about the same height.* So this is one way to falsify S. There is another way:[1]

   Here, student 1 is a bit taller than 2, but not much. 2 is a bit taller than 3, but not much. But 1 is much taller than 3. We can take x to be student 2. No student is much taller or shorter than 2, so S is false.

   (d) To make the sentence $\forall v(\mathit{staff}(v) \to \exists u[\mathit{staff}(u) \wedge \mathit{mtt}(u,v)])$ true, you'd have to arrange that every v in DoC satisfies $\mathit{staff}(v) \to \exists u[\mathit{staff}(u) \wedge \mathit{mtt}(u,v)]$. That is, every staff member v satisfies $\exists u[\mathit{staff}(u) \wedge \mathit{mtt}(u,v)]$. But for v the tallest staff member, $\exists u[\mathit{staff}(u) \wedge \mathit{mtt}(u,v)]$ is false, because no staff member u is much taller than v! So the sentence can only be true if DoC manages to employ no staff at all (quite possible), or (less possible) infinitely many staff of larger and larger heights — then there is no tallest staff member.

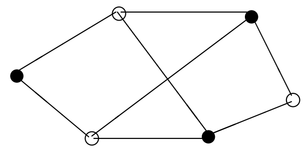
   Therefore, assuming a 'realistic' meaning of *mtt*, you can't make all the sentences true. If (a) is true, there has to be at least one student, say Jane. (Jane is much shorter than all staff but that's not important right now.) If (c) is true, there has to be another student (say Amy) who is much taller or shorter than Jane. Let's say *mtt*(Amy, Jane) is true. Then with a natural reading of *mtt*, *mtt*(Jane, Amy) is false. But this means that (b) is false. (If it were true, then $\mathit{mtt}(x,y) \leftrightarrow \mathit{mtt}(y,x)$ would have to be true for all x, y. But if we take $x =$ Jane and $y =$ Amy, then $\mathit{mtt}(x,y) \leftrightarrow \mathit{mtt}(y,x)$ is false.)

   But you *can* make them all false: e.g., take only the students 1, 2, 3 as pictured above, and take on just one staff member, who is not much taller than any of them.

2. Let L be a signature consisting of a single binary relation symbol E. Consider the following L-structure M. For objects a, b in $dom(M)$, $E(a,b)$ is true in M just when there's a direct straight line (an 'edge') between a and b.

[1]Thanks to Murray Shanahan for pointing this out.

M



(a) $\forall x \neg E(x,x)$ is **true**, because no dot has an edge from itself to itself.

(b) $\forall x \forall y (E(x,y) \to E(y,x))$ is **true**, because if a is connected by an edge to b then b is connected by an edge to a. Edges go both ways.

(c) $\forall x \forall y (E(x,y) \to \forall z (E(y,z) \to E(x,z)))$ is **false** in M, because it's not true that whenever a, b, c are dots and there's an edge from a to b and from b to c, then there's an edge from a to c.

(d) $\exists x \exists y \exists z (E(x,y) \wedge E(y,z) \wedge E(x,z))$ is **false**, because there are no 'triangles' in M.

(e) $\forall x \exists y \exists z (E(x,y) \wedge E(x,z) \wedge y \neq z)$ is **true**, because every dot has two or more different outgoing edges.

(f) $\forall x \exists y \exists z \exists v (E(x,y) \wedge E(x,z) \wedge E(x,v) \wedge y \neq z \wedge y \neq v \wedge z \neq v)$ is **false** in M, because not every dot has three different outgoing edges.

(g) $\forall x \forall y (E(x,y) \to (P(x) \leftrightarrow \neg P(y)))$ is true if we make P true at the black dots only (see picture above), because no edge has endpoints of the same colour (black or white).

3. (a) $\forall x (P(x) \to \exists y R(x,y))$ true in A, B, D.
(It says 'any black object has an arrow coming out'.)

(b) $\forall x \forall y (R(x,y) \vee R(y,x))$ true in none.
(Take $x \neq y$ in A, and $x = y =$ rightmost object in the others.)

(c) $\exists x \forall y (R(x,y) \to P(y))$ true in all.
(It says 'something sees only black objects'; take $x =$ leftmost object in A and $x =$ rightmost object in the others.)

(d) $\exists x \forall y \exists z (R(x,z) \wedge \neg(z = y))$ true in D only.
(It says 'something sees at least two objects' and the leftmost D-object is the only witness.)

Obviously there are many solutions. Here is one:

(a) $\exists x (\neg P(x) \wedge R(x,x))$

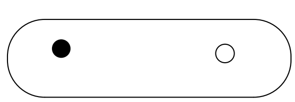
(b) $\exists x \exists y (R(x,y) \wedge R(y,x) \wedge x \neq y)$

(c) $\exists x \exists y (\neg R(x,x) \wedge R(x,y) \wedge \neg R(y,x))$

(d) $\exists x \exists y (R(x,x) \wedge R(x,y) \wedge \neg R(y,x))$

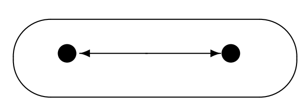
4. Of course there are many solutions. Here's one:

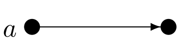
M



F: black dot only
G: white dot only

N



Note: a  b means $R(a,b)$

(a) $M \models \exists x F(x) \to \exists x G(x)$, because $M \models \exists x G(x)$ — take x to be the white dot.
But $M \not\models \forall x [F(x) \to G(x)]$, because if b is the black dot, $M \not\models F(b) \to G(b)$.

(b) $N \models \forall x \exists y R(x,y)$, because for each dot x we can take y to be the other dot.
$N \not\models \exists v \forall u R(u,v)$, because no dot has an arrow (is R-related to) all dots including itself.

(c) For any M, if $M \models \forall x[F(x) \to G(x)]$ then $M \models \exists x F(x) \to \exists x G(x)$.

For, if $M \models \forall x[F(x) \to G(x)]$ and $M \models \exists x F(x)$, then take a in $dom(M)$ with $M \models F(a)$. By $M \models \forall x[F(x) \to G(x)]$, we have $M \models G(a)$. So $M \models \exists x G(x)$.

So we can't find a structure making $\forall x[F(x) \to G(x)]$ true and $\exists x F(x) \to \exists x G(x)$ false, because there isn't one!

(d) For any N, if $N \models \exists v \forall u R(u, v)$ then $N \models \forall x \exists y R(x, y)$.

For, if $N \models \exists v \forall u R(u, v)$ then there is a in $dom(N)$ with $N \models \forall u R(u, a)$. So for any b in $dom(N)$, $N \models R(b, a)$. So $N \models \exists y R(b, y)$. This is true for all b, so $N \models \forall x \exists y R(x, y)$.

5. (a) x is divisible by 3 without remainder: $\exists y(\underline{3} \times y = x)$.

(b) x is prime: by convention, 1 is not prime; a higher number is prime if its only factors are 1 and itself. We can express 'y is a factor of x', by $F(y, x) \; = \exists z(y \times z = x)$, and then express what I said above by

$$Pr(x) \; = x > \underline{1} \wedge \forall y(F(y, x) \to y = \underline{1} \vee y = x).$$

'$\leftrightarrow$' could be used instead of '$\to$' here. Alternatively, you could write

$$Pr'(x) \; = x > \underline{1} \wedge \forall y \forall z(x = y \times z \to y = \underline{1} \vee z = \underline{1}).$$

(c) $\forall x(\exists y(\underline{2} \times y = x) \wedge x > \underline{2} \to \exists y \exists z(Pr(y) \wedge Pr(z) \wedge x = y + z))$.

(d) x is a square number: use $Sq(x) \; = \exists y(y \times y = x)$.

(e) x is the sum of two square numbers: $\exists y \exists z(Sq(y) \wedge Sq(z) \wedge x = y + z)$. (Define $Sq(y), Sq(z)$ as in preceding part.)

(f) There are infinitely many prime numbers is expressed by: $\forall x \exists y(y > x \wedge Pr(y))$, where $Pr(y)$ is as in part 5b. It says that for any number x, there is a prime bigger than x. This is so if and only if there are infinitely many primes (and in fact it's how Euclid's theorem is proved).

6. Please note that there are plenty of answers that are different from mine but which are still correct. This is for several reasons:

- there may have been an ambiguity in the sentence to be translated;
- you have written down an equivalent, but different, sentence;
- one of our answers is not right!
- you have used different relation symbols.

(a) $\forall x[\textit{animal}(x) \to \neg(\textit{cat}(x) \wedge \textit{dog}(x))]$

(b) $\forall x[\textit{admires}(x, x) \to \exists y \; \textit{admires}(x, y)]$

(c) $\forall z[\textit{prize}(z) \to \exists y[\textit{chimp}(y) \wedge \textit{won}(y, z)]]$

(d) $\exists y[\textit{chimp}(y) \wedge \forall x[\textit{prize}(x) \to \textit{won}(y, x)]]$. This read the English sentence as 'some chimp $\cdots$'. If you read it as 'exactly one chimp', use equality to express the extra information that 'any chimp that won all the prizes is the same as y'. Equivalently, use

$$\exists y\big(\textit{chimp}(y) \wedge \forall z(\forall x[\textit{prize}(x) \to \textit{won}(z, x)] \leftrightarrow z = y)\big).$$

This reading might not be what was intended, since the logic allows some prizes to have been won jointly but maybe the English doesn't! If in doubt, ask the author of the English what they meant.

(e) $\forall y[\textit{is-in-team}(y) \to \neg\textit{runs-faster}(\textit{Jack}, y)]$, or alternatively,
$\neg\exists y[\textit{is-in-team}(y) \wedge \textit{runs-faster}(\textit{Jack}, y)]$

(f) $\neg\forall y[\textit{is-in-team}(y) \to \textit{runs-faster}(\textit{Jack}, y)]$

(g) $\forall x[\textit{student}(x) \wedge \textit{first-year}(x) \rightarrow \exists y\ \textit{PPT}(y,x)]$, where $\textit{PPT}(y,x)$ reads as y is a *PPT* for x.

(h) $\neg\exists x[\textit{student}(x) \wedge \exists z[\textit{PPT}(z,x) \wedge \textit{PMT}(z,x)]]$, or alternatively, $\forall x[\textit{student}(x) \rightarrow \forall z[\textit{PPT}(z,x) \rightarrow \neg\textit{PMT}(z,x)]]$. These formulas say that no PPT tutor of a student is also a PMT tutor for that student. However, perhaps unlike the English sentence, they allow a student to have any number of PPT and PMT tutors (as in real life).

Note: we did not need to use =.