3rd method: Checking validity using equivalences

We know that $\top$ is a valid formula.

Showing an argument is valid using equivalences involves:

1. converting the argument into its corresponding implication formula;
2. simplifying the formula to $\top$ (which is a valid formula), always preserving logical equivalence.

Why? Recall this relations ...

argument validity	formula validity	satisfiability	equivalence
$\phi \models \psi$	$\phi \to \psi$ valid	$\phi \land \neg\psi$ unsatisfiable	$(\phi \to \psi) \equiv \top$

Things to note when using equivalences!

1. Name the equivalence law applied when rewriting, or reference its the overall logical form;

2. Apply one equivalence law at a time;

3. You may combine consecutive applications of the same law in one step (e.g., consecutive applications of associativity of $\vee$);

4. Reference the equivalence law next to the *result*;

5. Reference the correct equivalence law,
   e.g., $\top \wedge \phi \equiv \phi$ is different from $\phi \wedge \top \equiv \phi$;

6. Don't forget referencing associativity and commutativity laws.

Writing DNF equivalences from truth tables

Given a formula ϕ, we can construct a semantically equivalent formula in DNF from its truth table.

The basic principle is based on the fact that every truth assignment (row) can be encoded as a propositional formula which is true just on that assignment and false everywhere else.

To construct this formula, take the conjunction of:

1. all propositional atoms that have the value tt in that truth assignment and
2. the negations of all propositional atoms that have the value ff in that truth assignment.

p	q	conjunctive
tt	tt	$p \wedge q$
tt	ff	$p \wedge \neg q$
ff	tt	$\neg p \wedge q$
ff	ff	$\neg p \wedge \neg q$

Writing DNF equivalences from truth tables ctd.

Now to rewrite a formula ϕ in DNF, we:

1. construct the truth table for ϕ;
2. write a conjunctive formula for each assignment in ϕ's truth table in *which* ϕ *evaluates to* tt;
3. take the disjunction of these conjunctive formulas.

Rewrite $(p \to \neg q) \wedge q$ *in DNF.*

p	q	$\neg q$	$p \to \neg q$	$(p \to \neg q) \wedge q$	conjunctive
tt	tt	ff	ff	ff	
tt	ff	tt	tt	ff	
ff	tt	ff	tt	tt	$\neg p \wedge q$
ff	ff	tt	tt	ff	

Therefore, the equivalent DNF formula is $\neg p \wedge q$.

Writing CNF equivalences from truth tables

Similarly, given a formula ϕ, we can construct a semantically equivalent formula in CNF from its truth table.

The basic principle here instead is that every truth assignment (row) can be encoded as a clause (completion) which is false just on that assignment and true everywhere else.

To construct this clause, take the disjunction of

1. all propositional atoms that have the value ff in that truth assignment;
2. the negations of all propositional atoms that have the value tt in that truth assignment.

p	q	completion
tt	tt	$\neg p \lor \neg q$
tt	ff	$\neg p \lor q$
ff	tt	$p \lor \neg q$
ff	ff	$p \lor q$

Writing CNF equivalences from truth tables ctd.

Now to rewrite a formula ϕ in CNF, we follow these steps:

1. Construct the truth table for ϕ;
2. Write a clause (completion) for each assignment in ϕ's truth table in which ϕ has value ff;
3. Take the conjunction of these clauses.

Rewrite $(p \to \neg q) \wedge q$ in CNF.

p	q	$\neg q$	$p \to \neg q$	$(p \to \neg q) \wedge q$	completion
tt	tt	ff	ff	ff	$\neg p \vee \neg q$
tt	ff	tt	tt	ff	$\neg p \vee q$
ff	tt	ff	tt	tt	
ff	ff	tt	tt	ff	$p \vee q$

Therefore, the equivalent CNF formula is

$$(\neg p \vee \neg q) \wedge (\neg p \vee q) \wedge (p \vee q)$$

4th method Proof Systems: Natural Deduction

We should be able to establish validity of argument by breaking it into smaller arguments and showing the validity of these intermediate ones, i.e., construct a *proof*.

A *proof system* is a way of showing formulas to be valid by using purely *syntactic rules*—not using semantics at all. A computer algorithm should be able to apply the rules.

There are many proof systems. We will focus on *natural deduction* (ND), a system invented by F. B. Fitch (and related to work of Gentzen); it is a major topic and will take some time.

Pandora:
https://www.doc.ic.ac.uk/pandora/newpandora/

Pandora is a lab program that allows you to do natural deduction proofs as a kind of computer game.

It was originally done by Dan Ehrlich as a 4th-year individual project, c. 1998. It has evolved since then, always developed by students, supervised by *Krysia Broda* and (in the past) Gabrielle Sinnadurai.

We will use Pandora next week during the lecture. In the meantime, please try it.

What is natural deduction?

A formalisation of 'direct argument'.

Starting perhaps from some given facts—some formulas $\phi_1, \ldots, \phi_n$—we use the rules of the system to reason towards a formula ψ.

(In practice, we reason forwards, backwards, and especially inwards.)

If we succeed, we can write

$$\phi_1, \ldots, \phi_n \vdash \psi$$

$\phi_1, \ldots, \phi_n$ are called *premises*.

ψ is called a *conclusion*.

$\phi_1, \ldots, \phi_n \vdash \psi$ is called a *sequent*.

Assumptions in natural deduction

Proofs in Fitch's system are not based on axioms expressed in logical form, as in Hilbert's axiomatic proof theory.

They sometimes involve making "temporary" assumptions to prove a conclusion.

"Assume for the sake of argument ..."

An *assumption* is just a formula, but it's used in a special way.

We imagine a situation in which a formula is true. Then we derive some additional formulas that help us make progress towards the conclusion.

We need to be careful about how we use these assumption. We will see examples of its use soon.

How natural deduction works

Deduction works by writing down intermediate formulas using *inference rules*.

- State the set of premises $\phi_1, \ldots, \phi_n$, and the conclusion ψ.
- Intermediate formulas form the *proof* of ψ from the givens $\phi_1, \ldots, \phi_n$.
- Once established, they may be usable later.
- Each step of the proof is a *valid argument*.

Natural deduction inference rules

Mostly, there are two rules for each connective:

- one for *introducing* it (bringing in a formula whose principal connective is the connective it introduces),
- one for *using* it (using a formula already present in a step of the proof, whose principal connective is the connective it wants to use).

The rules are based on the semantics for the connectives given earlier.

As we will see later, all rules from propositional logic carry over to predicate logic.

Natural deduction for predicate logic deals also with equality and the two quantifiers.

Rules for conjunction

$\wedge$-introduction, or $\wedge I$: To introduce a formula of the form $\phi \wedge \psi$, you have to have already introduced ϕ and ψ.

1	ϕ	we proved this
2	$\vdots$	(other stuff)
3	ψ	and this
4	$\phi \wedge \psi$	$\wedge I(1,3)$

The line numbers are essential for clarity.

ϕ and ψ in the above need not be atomic.

Rules for conjunction—ctd.

($\wedge$-elimination, or $\wedge E$): If you have managed to write down $\phi \wedge \psi$, you can go on to write down ϕ and/or ψ.

1	$\phi \wedge \psi$		we have this somehow
2	ϕ	$\wedge E(1)$	
3	ψ	$\wedge E(1)$	

Rules for conjunction—example.

Prove that the sequent $p \wedge q, r \vdash q \wedge r$ is valid.

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge E(1)$
4	$q \wedge r$	$\wedge I(3,2)$

Rules for implication

$\to$-introduction, or $\to I$: To introduce a formula of the form $\phi \to \psi$, you *assume* ϕ and then prove ψ.

During the proof, you can use ϕ as well as anything already established.

But you can't use ϕ or anything from the proof of ψ from ϕ later on (because it's based on an extra assumption).

So we isolate the proof of ψ from ϕ, in a box:

1	ϕ	asm
2	$\ldots$	$\langle$the proof$\rangle$ hard struggle
3	ψ	we made it!
4	$\phi \to \psi$	$\to I(1,3)$

Boxes in natural deduction

Boxes are used when making additional assumptions.

The first line should always be labelled ‘asm’ (assumption).

The line immediately following the closed box must match the pattern of the conclusion of the rule that uses the box.

Nothing inside the box can be used later.

Rules for implication—ctd.

$\rightarrow$-elimination, or $\rightarrow E$: If you have managed to write down ϕ and $\phi \rightarrow \psi$, in either order, you can go on to write down ψ.
(This is *modus ponens*.)

1	$\phi \rightarrow \psi$		we got this somehow
2	$\vdots$		(other stuff)
3	ϕ		and this too
4	ψ	$\rightarrow E(1,3)$	

Rules for implication—ctd.

Prove that the sequent $p, p \to q, p \to (q \to r) \vdash r$ *is valid.*

1	p	premise
2	$p \to q$	premise
3	$p \to (q \to r)$	premise
4	$q \to r$	$\to E(3, 1)$
5	q	$\to E(2, 1)$
6	r	$\to E(4, 5)$

Rules for implication—ctd.

Prove that the sequent $\vdash p \to p$ *is valid.*

$$\boxed{\begin{array}{lll} 1 & p & \text{asm} \\ 2 & p & \checkmark(1) \end{array}}$$

$$\begin{array}{lll} 3 & p \to p & \to I(1,2) \end{array}$$

you don't need this last step

Using natural deduction to prove equivalence

We say two formulas ϕ and ψ are provably equivalent in natural deduction (written as $\phi \dashv\vdash \psi$) iff $\phi \vdash \psi$ and $\psi \vdash \phi$.

Prove that $(p \land q) \to r \dashv\vdash p \to (q \to r)$.

We first show $(p \land q) \to r \vdash p \to (q \to r)$.

1	$(p \land q) \to r$	premise
2	p	asm
3	q	asm
4	$p \land q$	$\land I(2,3)$
5	r	$\to E(1,4)$
6	$q \to r$	$\to I(3,5)$
7	$p \to (q \to r)$	$\to I(2,6)$

Using natural deduction to prove equivalence

Now we prove $p \to (q \to r) \vdash p \wedge q \to r$.

1	$p \to (q \to r)$	premise
2	$p \wedge q$	asm
3	p	$\wedge E(2)$
4	$q \to r$	$\to E(1,3)$
5	q	$\wedge E(2)$
6	r	$\to E(4,5)$
7	$p \wedge q \to r$	$\to I(2,6)$

$p \to (q \to r)$ and $p \wedge q \to r$ are provably equivalent formulas.

Rules for disjunction

$\vee$-introduction, or $\vee I$: To prove $\phi \vee \psi$, prove ϕ, or (if you prefer) prove ψ.

$$\begin{array}{lll} \vdots & & \\ 3 & \phi & \text{proved this somehow} \\ 4 & \phi \vee \psi & \vee I(3) \end{array}$$

ψ can be any formula at all!

$$\begin{array}{lll} \vdots & & \\ 3 & \psi & \text{proved this somehow} \\ 4 & \phi \vee \psi & \vee I(3) \end{array}$$

Similarly ϕ can be any formula at all!

Rules for disjunction—ctd.

($\vee$-Elimination, or $\vee E$): To prove something from $\phi \vee \psi$, you have to prove it by assuming ϕ, AND prove it by assuming ψ.

This is arguing by cases.

$$
\begin{array}{lll}
1 & \phi \vee \psi & \text{we got this somehow} \\
\boxed{\begin{array}{lll} 2 & \phi & \text{ass} \\ & \vdots & \\ 6 & \rho & \end{array}} & \boxed{\begin{array}{lll} 7 & \psi & \text{ass} \\ & \vdots & \\ 9 & \rho & \end{array}} & \text{we got } \rho \text{ from both proofs} \\
10 & \rho & \vee E(1, 2-6, 7-9)
\end{array}
$$

The assumptions ϕ, ψ are not usable later, so are put in (side-by-side) boxes.

Both boxes must end with the same ρ; ρ can be any formula.

Nothing inside the boxes can be used later, or in the other box.

Rules for disjunction—ctd.

Prove that the sequent $q \to r \vdash p \vee q \to p \vee r$ is valid.

1	$q \to r$	premise
2	$p \vee q$	asm
3	p	asm
4	$p \vee r$	$\vee I(3)$
5	q	asm
6	r	$\to E(1,5)$
7	$p \vee r$	$\vee I(6)$
8	$p \vee r$	$\vee E(2, 3-4, 5-7)$
9	$p \vee q \to p \vee r$	$\to I(2,8)$

Rules for single negation

These rules involve the notion of *contradiction*. They treat $\neg\phi$ like $\phi \to \bot$.

The formula $\bot$ stands for the contradiction. Examples of contradiction are $p \wedge \neg p$ and $(p \vee q) \wedge \neg(p \vee q)$.

As far as truth is concerned, these are all provably equivalent, for instance

$$p \wedge \neg p \vdash (p \vee q) \wedge \neg(p \vee q) \quad \text{and} \quad (p \vee q) \wedge \neg(p \vee q) \vdash p \wedge \neg p$$

Rules for single negation ctd.

¬-introduction, or $\neg I$: To prove $\neg\phi$, you assume ϕ and prove $\bot$.
$\bot$ stands for the contradiction.

As usual, you can't then use ϕ later on, so enclose the proof of $\bot$ from assumption ϕ in a box:

1	ϕ	asm	
2	$\vdots$		more hard work, oh no
3	$\bot$		we got it!
4	$\neg\phi$	$\neg I(1,3)$	

Idea: If one can prove a contradiction from an assumption ϕ, then one can conclude that 'It is not the case that ϕ'.

Rules for single negation ctd.

($\neg$-elimination, $\neg E$): From ϕ and $\neg\phi$, deduce $\perp$.

1	$\neg\phi$		proved this somehow
2	$\vdots$		
3	ϕ		and this
4	$\perp$	$\neg E(3,1)$	

Note: $\neg E$ allows one to introduce a formula ψ and discharge all assumptions of $\neg\psi$ even if there aren't any. We will see this later.

Rules for single negation—Example

Show that $p \to \neg p \vdash \neg p$ *is a valid sequent.*

1	$p \to \neg p$	premise
2	p	asm
3	$\neg p$	$\to E(1,2)$
4	$\perp$	$\neg E(2,3)$
5	$\neg p$	$\neg I(2,4)$