Rules for negation—double

$\neg\neg$-introduction, $\neg\neg I$: From ϕ, deduce $\neg\neg\phi$

1	ϕ		proved this somehow
2	$\neg\neg\phi$	$\neg\neg I(1)$	

Intuitively, saying that 'it is *not* true that the party is *not* fun' is just like saying 'the party is fun'

$\neg\neg$-elimination, $\neg\neg E$: From $\neg\neg\phi$, deduce ϕ.

1	$\neg\neg\phi$		proved this somehow
2	ϕ	$\neg\neg E(1)$	

Rules for negation—double ctd.

Show that $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$.

Contradiction in natural deduction

Any formula can be derived from a contradiction... This can be confusing at first.

Suppose

p: Cows can fly.

q: I like going to the theatre.

It is perfectly valid to say

$$p \wedge \neg p \vdash q$$

$\vdash$ tells us what we can infer, given the premises on its left (which may be always false!). The derivation does not care whether these premises make sense.

Rules for bottom

You might find the argument above hard to accept. You will not be alone: it has led some to reject or reformulate classical logic.

I could just say that we are just following proof rules. If the rules allow us to get from $\perp$ to ϕ, then fine.

Better: the English paraphrase says:

SUPPOSE we are in a situation in which $\perp$ is true… then ϕ is true. (And $\neg\phi$ is true too, so ϕ is false,…)

BUT *there is no such situation.* So you can't dispute the crazy things that follow from this supposition.

Rules for bottom–ctd.

$\perp$-elimination, $\perp E$: This encode the fact that a contradiction can prove anything.

1	$\perp$		we got this
2	ϕ	$\perp E(1)$	

$\perp$-introduction, or $\perp I$: To prove $\perp$, you must prove ϕ and $\neg A$ (for any ϕ you like).

1	ϕ	got this somehow	
2	$\vdots$		
3	$\neg\phi$		and this
4	$\perp$		$\perp I(1, 3)$

Rules for bottom—ctd.

The practical point is that there is no situation in which $\perp$ is true, so you can only prove $\perp$ *under (contradictory) assumptions in a box* in the middle of a bigger proof.

You often get such assumptions—e.g., in an argument by cases ($\vee E$).

From $\perp$ you can then deduce anything you want, so the box is no obstacle to the overall goal. So $\perp$ is a useful formula to aim to prove.

Note that $\perp I$ is the same rule as $\neg E$. (There are two names for this rule!)

The rules for $\perp$ can actually be obtained from the other rules already explained.

Rules for bottom—ctd.

Show that $\neg p \vee q \vdash p \rightarrow q$ *is valid.*

1	$\neg p \vee q$	premise
	$p \rightarrow q$	

Rules for top

$\top$-introduction, or $\top I$: You can introduce $\top$ anywhere (for all the good it does you).

$\top$-elimination, $\top E$: You can prove nothing new from $\top$, sorry!

Examples of natural deduction—Tricky!

Show that $\vdash p \vee \neg p$ *is valid.*

Derived rules vs. primitive rules

The deduction we saw in the last slide (for $p \vee \neg p$) is very handy in (other) proofs.

It has a special name: *tertium non datur* or *law of excluded middle* (LEM).

If you haven't got ϕ, you've got $\neg\phi$. This law is true of classical logic, but not of some other logics.

We call it a *derived rule* (DL) because we can infer it from the 'primitive' rules $\vee I$, $\neg I$ and $\neg E$.

Derived rules are NOT necessary, but they do help to speed proofs up.

Some 'purists' only allow for 'primitive rules' (one that are independent of each other) in proofs. We don't here!

There are other common derived rules.

DL: Double Negation Introduction

We have seen the rule for $\neg\neg I$ before. It's also a derived rule.

Show that the sequent $\phi \vdash \neg\neg\phi$ is valid.

1	ϕ	premise
2	$\neg\phi$	asm
3	$\bot$	$\neg E(1, 2)$
4	$\neg\neg\phi$	$\neg I(2, 3)$

DL: Modus Tollens

Modus Tollens (MT): From $\phi \to \psi$ and $\neg\psi$, derive $\neg\phi$.

1	$\phi \to \psi$	proved this somehow
2	$\vdots$	
3	$\neg\psi$	and this
4	$\neg\phi$	MT(1,3)

DL: Proof by Contradiction

Proof by Contradiction (PC): To prove ϕ, assume $\neg\phi$ and prove $\perp$.

The effect of PC is to combine applications of $\neg$I and $\neg\neg$.

$$\begin{array}{lll} \begin{array}{|lll|} \hline 1 & \neg\phi & \text{asm} \\ 2 & \vdots & \\ 3 & \perp & \\ \hline \end{array} \\ 4 & \neg\neg\phi & \neg I(1,3) \\ 5 & \phi & \neg\neg E(4) \end{array} \qquad \text{replaced by:} \qquad \begin{array}{lll} \begin{array}{|lll|} \hline 1 & \neg\phi & \text{asm} \\ 2 & \vdots & \\ 3 & \perp & \\ \hline \end{array} \\ 4 & \phi & PC(1,3) \end{array}$$

Using PC cuts out a line. This is surprisingly helpful.

The proof for ϕ is *indirect*; we achieve it by showing the impossibility of $\neg\phi$.

DL: Proof by Contradiction examples

Show that the sequent $p \wedge \neg q \to r, \neg r, p \vdash q$ *is valid*.

Deduction with Lemmas

A lemma is something you prove that helps in proving what you really wanted.

So in natural deduction proofs, I'll generally allow you to quote LEM $\phi \vee \neg\phi$ as a lemma, without proving it. Justify by *'Lemma'.*

But always *you* have to choose which ϕ to use.

Warning: Pandora calls this particular lemma 'EM' (standing for Excluded Middle). Click on the 'EM' button to use it.

The Pandora 'lemma' button just chops a proof into two halves (try it!). It lets you make your own lemmas. Don't use it if you want $\phi \vee \neg\phi$—use the 'EM' button!

Deduction with Lemmas—ctd.

Using LEM, show that $p \to q \vdash \neg p \vee q$ *is valid.*

DL: Rules for bidirectional implication

Roughly, we treat $\phi \leftrightarrow \psi$ as $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$.

$\leftrightarrow$-introduction, or $\leftrightarrow$I: To prove $\phi \leftrightarrow \psi$, prove both $\phi \rightarrow \psi$ and $\psi \rightarrow \phi$.

1	$\phi \rightarrow \psi$		proved this somehow
	$\vdots$		
2	$\psi \rightarrow \phi$		and this
3	$\phi \leftrightarrow \psi$	$\leftrightarrow I(1,2)$	

DL: Rules for bidirectional implication ctd.

($\leftrightarrow$-elimination, $\leftrightarrow$E): From $\phi \leftrightarrow \psi$ and ϕ, you can prove ψ. From $\phi \leftrightarrow \psi$ and ψ, you can prove ϕ.

1	$\phi \leftrightarrow \psi$		proved this somehow
2	ϕ		and this
3	ψ	$\leftrightarrow E(1,2)$	

or

1	$\phi \leftrightarrow \psi$		proved this somehow
2	ψ		and this
3	ϕ	$\leftrightarrow E(1,2)$	

DL: Rules for bidirectional implication ctd.

Show that $\vdash (p \to q) \leftrightarrow (\neg q \to \neg p)$ *is valid*.

Rules for bidirectional implication—ctd.

Show that $(p \wedge q) \leftrightarrow r, q \leftrightarrow s \vdash r \rightarrow s$ *is valid*.

More on natural deduction

With rules for $\wedge, \rightarrow, \vee$ we can already do quite a lot. (We still need rules for $\neg, \top, \bot$.)

The rules are natural . . . they are simple and self-evident inference rules based upon methods of proof and of reasoning applied in deductive practice.

Though natural deduction rules are *motivated* by the meaning of $\wedge, \vee, \ldots$, they are just syntactic rules.

A computer could do a natural deduction proof without knowing about 'meaning'.

More on natural deduction—ctd.

Definition 2.1 (Natural deduction proof)

Let $\phi_1, \ldots, \phi_n, \psi$ be arbitrary formulas.

$$\phi_1, \ldots, \phi_n \vdash \psi$$

means that there is a (natural deduction) proof of ψ, starting with the formulas $\phi_1, \ldots, \phi_n$ as premises.

You can read $\phi_1, \ldots, \phi_n \vdash \psi$ as 'ψ is provable from $\phi_1 \ldots \phi_n$'.

$\vdash \psi$ means we can prove ψ with no premises at all. We then say that ψ is a *theorem* (of natural deduction).

'$\vdash$' is called 'single turnstile'. Do not confuse it with $\models$.

- $\vdash$ is syntactic and involves proofs.
- $\models$ is semantic and involves situations.

Advice on natural deduction

1. Learn the rules. Practise! Use Pandora.
2. Think of a direct argument to prove what you want. Then translate it into natural deduction.
3. If really stuck in proving ϕ, it can help to:
   - Assume $\neg\phi$ and prove $\bot$. ('Reductio ad absurdum', or 'proof by contradiction', PC.)
     We did this when proving $\vdash \phi \vee \neg\phi$.
   - Use the lemma $\psi \vee \neg\psi$, for suitable ψ. It amounts to arguing by cases.

Nasty-ish example

Let's show

$$\phi \vee \psi, \quad \neg\rho \to \neg\phi, \quad \neg(\psi \wedge \neg\rho) \quad \vdash \quad \rho.$$

is valid.

Well, assume for the sake of argument that you had $\neg\rho$.

Then you'd have $\neg\phi$—but you're given ϕ or ψ, so you get ψ.

Now you've got both ψ and $\neg\rho$, which you're told you don't: contradiction.

SO, you must have ρ.

This is quite easy to translate into ND. (Note the use of the $\vee E$ rule.)

Nasty-ish example—ctd.

1	$\phi \vee \psi$	premise
2	$\neg\rho \to \neg\phi$	premise
3	$\neg(\psi \wedge \neg\rho)$	premise
14	ρ	$PC(4, 13)$

More on box in natural deduction

A box is 'its own little world' with its own assumptions.

A box always starts with an assumption (the only exception is in $\forall I$ in predicate logic).

An assumption can only occur on the first line of a box.

Inside a box, you can use any earlier formulas (except formulas in completed earlier boxes).

The only ways of exporting information from a box are by the rules $\to I$, $\vee E$, $\neg I$, and PC (and also $\exists E$ and $\forall I$ in predicate logic). The first line after a box must be justified by one of these.

No formula inside a box can be used outside, except via the above rule.

You can check these conditions for yourself. If your box doesn't meet them, your proof is wrong.

More on box in natural deduction—ctd.

Show $\neg\phi \vdash \neg\phi$ *(!)*

1st try:

1	$\neg\phi$	premise
2	$\neg\phi$	$\checkmark(1)$ the best proof!

2nd try:

1	$\neg\phi$	premise
2	ϕ	asm
3	$\bot$	$\neg E(2,1)$
4	$\neg\phi$	$\neg I(2,3)$

(lines 2–3 boxed) correct but silly

3rd/4th tries:

1	$\neg\phi$	premise
2	ϕ	asm
3	$\bot$	$\neg E(2,1)$
4	$\neg\phi$	$\bot E(3)$

$\leftarrow$ WRONG $\rightarrow$

1	$\neg\phi$	premise
2	ϕ	asm
3	$\bot$	$\neg E(2,1)$
4	$\neg\phi$	$\bot E(3)$
5	$\neg\phi$	$\checkmark(4)$

Variants of natural deduction

The natural deduction system we've seen can be varied by

– changing the rules (carefully!)
– introducing new connectives and giving rules for them.

From exam 2007: The *IF* connective can be defined as

$$IF(p, q, r) = (p \to q) \wedge (\neg p \to r).$$

Here's an introduction rule for *IF*, based on the rules $\to$I and $\wedge$I:

1	ϕ	asm	2	$\neg\phi$	asm
	$\vdots$			$\vdots$	
2	ψ	got this	4	ρ	and this

5 $\quad IF(\phi, \psi, \rho) \qquad\qquad IF\ I(1-2, 3-4)$

Exercise: what would a good elimination rule (or rules) be?

Semantic validity vs. natural deduction

Our main concern is with validity $\models$.

Recall $\phi_1, \ldots, \phi_n \models \psi$ if ψ is true in all situations in which $\phi_1, \ldots, \phi_n$ are true.

$\vdash$ is useless unless it helps to establish $\models$.

Definition 2.2 (Soundness and completeness)

A proof system is *sound* if every theorem is valid, and *complete* if every valid formula is a theorem.

Recall in natural deduction, a theorem is any formula ϕ such that $\vdash \phi$.

Soundness of natural deduction rules

It can be shown that natural deduction is sound, i.e., they preserve truth values computed by PL semantics.

Theorem 2.3 (Soundness of natural deduction)

Let $\phi_1, \ldots, \phi_n, \psi$ be any propositional formulas. If $\phi_1, \ldots, \phi_n \vdash \psi$, then $\phi_1, \ldots, \phi_n \models \psi$.

In other words,

“Any provable propositional formula is valid”

“Natural deduction never makes mistakes”

The proof is not covered in this module but for those interested, it’s done by mathematical induction on the length of the proof.

Completeness of natural deduction rules

Theorem 2.4 (Completeness)

Let $\phi_1, \ldots, \phi_n, \psi$ be any propositional formulas. If $\phi_1, \ldots, \phi_n \models \psi$, then $\phi_1, \ldots, \phi_n \vdash \psi$.

In other words,

"Any propositional validity can be proved."

"Natural deduction is powerful enough to prove all valid formulas."

Bottom line: *We can use natural deduction to check validity.*

The proof is also not covered in this module but for those interested, it's done by mathematical induction on the height of the formation tree.

Satisfiability vs. consistency

Definition 2.5 (Consistency)

A formula ϕ is said to be *consistent* if $\nvdash \neg\phi$.

A collection $\phi_1, \ldots, \phi_n$ of formulas is said to be *consistent* if $\nvdash \neg \bigwedge_{1 \leq i \leq n} \phi$.

(One can extend this definition to infinite collections of formulas too.)

Recall a propositional formula is *satisfiable* if it is true in at least one situation.

By soundness and completeness (Theorems 2.3 and 2.4), we get:

Theorem 2.6

A formula ϕ is *consistent* if and only if it is *satisfiable*.

Semantic equivalence vs. provable equivalence

Definition 2.7 (Provable equivalence)

Two propositional formulas ϕ and ψ are *provably equivalent* if and only $\phi \vdash \psi$ and $\psi \vdash \phi$, denoted $\phi \dashv\vdash \psi$.

Recall, two propositional formulas ϕ and ψ are *semantically equivalent* if they are true in exactly the same situations.

Roughly speaking: they mean the same.

By soundness and completeness (Theorems 2.3 and 2.4), we get:

Theorem 2.8

Two formulas are *provably equivalent* if and only if they are *semantically equivalent*.