1.4 Arguments, validity

You now know how to read, write, and evaluate propositional formulas. You know a little about translating between English and logic.

But logic is not just about saying things. It allows arguing too. Some say it is the study of valid arguments.

Valid arguments

Here is an old argument (Socrates syllogism):

- Socrates is a man.
- Men are mortal.
- Therefore, Socrates is mortal.

Is this a valid argument?

What does it mean to say that it is or is not a valid argument?

> The Socrates syllogism is a valid argument if for each situation in which Socrates is a man and also men are mortal, Socrates is mortal in this situation.

Valid Arguments ctd.

Definition 1.1 (valid argument)

Given formulas $\phi_1, \phi_2, \ldots, \phi_n, \psi$, an argument

$$\text{`}\phi_1, \phi_2, \ldots, \phi_n, \text{ therefore } \psi\text{'}$$

is *valid* if:

ψ is true in every situation in which $\phi_1, \phi_2, \ldots, \phi_n$ are all true.

If this is so, we write '$\phi_1, \phi_2, \ldots, \phi_n \models \psi$'.

The '$\models$' is called 'double turnstile'. You can read it as 'logically entails', 'logically implies' or 'semantically entails'.

Examples of arguments

Let ϕ, ψ be arbitrary propositional formulas.

– 'ϕ, therefore ϕ' is valid, since in any situation in which ϕ is true, ϕ is true. $\phi \models \phi$.

– '$\phi \wedge \psi$, therefore ϕ' is valid by the semantics of $\wedge$. $\phi \wedge \psi \models \phi$.

– 'ϕ, therefore $\phi \wedge \psi$' is not in general valid: depending on ϕ and ψ, there might be situations in which ϕ is true but $\phi \wedge \psi$ is not. Then, $\phi \not\models \phi \wedge \psi$.

– 'ϕ, $\phi \to \psi$, therefore ψ' is valid. $\phi, \ \phi \to \psi \models \psi$. This argument style has a name: *modus ponens.*

– '$\phi \to \psi$, $\neg\psi$, therefore $\neg\phi$' is also valid: $\phi \to \psi, \neg\psi \models \neg\phi$. This argument is called *modus tollens.*

– '$\phi \to \psi$, ψ, therefore ϕ' is not valid in general, in spite of what lawyers and politicians may say. $\phi \to \psi, \psi \not\models \phi$.

Valid, satisfiable, equivalent formulas

Three important ideas are related to valid arguments.

1. Valid formula;
2. Satisfiable formula;
3. Equivalent formulas.

Valid formula

Definition 1.2 (valid formula)

A propositional formula is *(logically) valid* if it is true in every situation.

Valid *propositional* formulas are often called *tautologies.* A tautology is just another name for a valid propositional formula.

Valid or not valid?

- $\top$ YES
- $\bot$ NO
- p NO
- $p \wedge \neg p$ NO
- $p \rightarrow p$ YES

Satisfiable formula

Definition 1.3 (satisfiable formula)

A propositional formula is *satisfiable* if it is true in at least one situation.

We typically say a formula is a *contradiction* if it is not satisfiable.

Satisfiable or a contradiction?

- $\top$ satisfiable
- $\perp$ contradiction
- p satisfiable
- $p \land \neg p$ contradiction
- $p \to p$ satisfiable

Equivalent formulas

Definition 1.4 (equivalent formulas)

Two propositional formulas ϕ, ψ are *logically equivalent* if they are true in exactly the same situations. Roughly speaking: they mean the same.

Some people write $\phi \equiv \psi$ for this. But $\equiv$ is also used in other ways, so watch out.

Equivalent?	p	$\top$	$\bot$	$p \to q$
$p \wedge p$	yes	no	no	no
$p \wedge \neg p$	yes	yes	no	no
$p \vee \neg p$	no	yes	no	no
$\neg p \vee q$	no	no	no	yes

Relations between the concepts

Valid arguments and valid, satisfiable, and equivalent formulas are all definable in terms of each other:

argument validity	formula validity	satisfiability	equivalence
$\phi \models \psi$	$\phi \to \psi$ valid	$\phi \land \neg\psi$ unsatisfiable	$(\phi \to \psi) \equiv \top$
$\top \models \phi$	ϕ valid	$\neg\phi$ unsatisfiable	$\phi \equiv \top$
$\phi \not\models \bot$	$\neg\phi$ not valid	ϕ satisfiable	$\phi \not\equiv \bot$
$\phi \models \psi$ and $\psi \models \phi$	$\phi \leftrightarrow \psi$ valid	$\phi \leftrightarrow \neg\psi$ unsatisfiable	$\phi \equiv \psi$

All four statements in each line amount to the same thing.

So we can choose to deal with one of these concepts and get the others for free.

We will for the sake of illustration focus in what follows on valid arguments and valid formulas.

More on argument and formula validity

We will introduce some terminology that we will often use in this section.

Definition 1.5

Let $\phi_1, \ldots, \phi_n \models \psi$ be an argument. We call the formula $\phi_1 \wedge \ldots \wedge \phi_n \to \psi$ its *corresponding implication formula*.

For instance, given an argument

$p \to q, \neg q \models \neg p$

its corresponding implication formula is

$((p \to q) \wedge \neg q) \to \neg p$

More on argument and formula validity—ctd.

We can now express the relationship between valid arguments and valid formulas as follows.

Theorem 1.6

$\phi_1, \ldots, \phi_n \models \psi$ is a valid argument if and only if its corresponding implication formula $\phi_1 \wedge \ldots \wedge \phi_n \rightarrow \psi$ is a valid formula (i.e., tautology).

So we can check the validity of an argument by checking the validity of its corresponding implication formula.

We will allow for arguments to have zero premises and write $\models \psi$, meaning ψ is true in every situation. When such an argument is valid, the formula ψ is said to be a tautology (i.e., a valid formula).

How do we tell whether an *argument* is valid?

In principle, we know how:

To show $\phi_1, \ldots, \phi_n \models \psi$ is a valid argument, check that the conclusion ψ is true in every situation where he premises $\phi_1, \ldots, \phi_n$ are true.

The same methods work for showing a formula ψ is valid. (ψ is a valid formula if and only if $\models \psi$ is a valid argument.)

For *propositional logic*, it can be done, though computationally it's a hard problem.

For *predicate logic*, coming later, it's much harder, because the situations are more complex and numerous (infinitely many).

Ways to check an *argument* is propositionally valid

To check whether an *argument* is propositionally valid, we can:

1. Translate all English sentences into propositional logic;
2. Check whether the resulting *argument* is valid using,

– *Truth tables*—the obvious way. You go through all the possible relevant situations and see if the conclusion is true whenever the premises are true in each.

– *Direct 'mathematical' argument*—the fastest way, once you get used to it.

– *Equivalences*. You make a stock of useful pairs of equivalent formulas. You use them to reduce its corresponding implication formula step by step to $\top$. This takes some art but is quite useful.

– *Various proof systems*, including tableaux, Hilbert systems, natural deduction. We will look at *natural deduction*: it is similar to direct argument. We will cover this intensively.

For valid formulas, equivalence and satisfiability, we proceed in a similar way.

1st method: Checking validity using truth tables

We saw these in specifying the meaning of Boolean connectives.

We use them here to check if an argument is propositionally valid.

Recall the truth table of a propositional formula ϕ summarises its truth values by considering different truth assignments to the propositional atoms appearing in it (which we called *situations*).

When checking if an argument $\phi_1, \ldots, \phi_n \models \psi$ is valid, we check whether the formula ψ is true in every situation where the formula $\phi_1 \wedge \ldots \wedge \phi_n$ is true.

Checking argument validity using truth tables

Is $p \to \neg q, q \models \neg p$ a valid argument?

Recall we write tt for the value true and ff for the value false.

p	q	$\neg q$	$p \to \neg q$	$(p \to \neg q) \wedge q$	$\neg p$
tt	tt	ff	ff	ff	ff
tt	ff	tt	tt	ff	ff
ff	tt	ff	tt	tt	tt
ff	ff	tt	tt	ff	tt

So the argument is valid.

Checking argument validity using truth tables—ctd.

Is $p \to \neg q, q \models \neg p$ a valid argument?

p	q	$\neg q$	$p \to \neg q$	$(p \to \neg q) \wedge q$	$\neg p$	$(p \to \neg q) \wedge q \to \neg p$
tt	tt	ff	ff	ff	ff	tt
tt	ff	tt	tt	ff	ff	tt
ff	tt	ff	tt	tt	tt	tt
ff	ff	tt	tt	ff	tt	tt

Since the argument's corresponding implication formula is valid, hence the argument is valid.

Checking formula validity using truth tables

Show that the formula $(p \to q) \leftrightarrow (\neg p \vee q)$ is valid.

There are two atoms, so 2^2 different truth assignments, hence four rows.

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
tt	tt	tt	ff	tt	tt
tt	ff	ff	ff	ff	tt
ff	tt	tt	tt	tt	tt
ff	ff	tt	tt	tt	tt

The truth value for $(p \to q) \leftrightarrow (\neg p \vee q)$ is tt for every truth value assignment given to p and q. Therefore, it's a valid formula.

The same table shows that $(p \to q)$ and $(\neg p \vee q)$ are equivalent. How? It also shows that $(p \to q) \leftrightarrow (\neg p \vee q)$ is satisfiable. How?

Pros and cons of truth tables

Pros:

– They always work—at least for propositional logic, where only finitely many situations are relevant to a given formula. (This is not true for predicate logic: we need other ways.)

– They can be used to show satisfiability and equivalence.

– They are easy to implement (not recommended).

– They illustrate how hard deciding argument validity, formula validity, satisfiability and equivalence etc., can be—even for 'trivial' propositional logic. (Adding an atom doubles the length of the table.)

Cons:

– They are boring and tedious, dumb, and error-prone.

– No satisfactory way to determine propositional satisfiability is known. The problem is NP-complete.

2nd method: Checking validity using direct argument

It involves showing the truth or falsity of a propositional formula by constructing direct logical (or mathematical) arguments from zero or more premises to one or more conclusions.

Is $p \to (p \vee q)$ a valid formula?

Take an arbitrary situation (truth value assignment to p and q).

To be a valid formula, then any situation must be such that, if p is true in that situation then $p \vee q$ is also true in that situation by semantics of $\to$.

Well, if p evaluates to true, then so does $p \vee q$ (by semantics of $\vee$).

And there we are!

Recall that a formula of the form $\phi \to \psi$ evaluates to false in one situation only: one in which ϕ is true and ψ is false.

Direct argument—Example (1)

Show that $p \wedge (p \to q) \to q$ is a valid formula.

Take any situation. To be a valid formula then the formula must be true in this situation.

For the formula to be true then either $p \wedge (p \to q)$ is false or q is true in that situation (by the semantics of $\to$).

For $p \wedge (p \to q)$ to be true then both p and $(p \to q)$ must be true in that situation (by the semantics of $\wedge$).

For $(p \to q)$ to be true, either p is false, or else p and q are true by semantics of $\to$.

If p is false, then $p \wedge (p \to q)$ is false (by the semantics of $\wedge$), and hence $p \wedge (p \to q) \to q$ is true by the semantics of $\to$.

But if p evaluates to true, then for $(p \to q)$ to be true q must also be true by semantics of $\to$.

Since in any situation where $p \wedge (p \to q)$ is true, q is also true then $p \wedge (p \to q) \to q$ is true that situation by semantics of $\to$.

Direct argument—Example (2)

Show that $(\phi \wedge \psi) \wedge \rho \equiv \phi \wedge (\psi \wedge \rho)$ *for arbitrary formulas* ϕ, ψ, ρ.

For the two formulas to be semantically equivalent, then they must be true in exactly the same situations.

Take any situation. A formula of the form $(\phi \wedge \psi) \wedge \rho$ is true if and only if both $(\phi \wedge \psi)$ and ρ evaluate to true (by the semantic definition of $\wedge$). This is the case if and only if ϕ and ψ evaluate to true, and ρ evaluates to true— that is, they're all true.

This is so if and only if ϕ is true, and also ψ and ρ are true by semantics of $\wedge$.

This is so if and only if ϕ and $\psi \wedge \rho$ are true by semantics of $\wedge$.

This is so if and only if $\phi \wedge (\psi \wedge \rho)$ is true by semantics of $\wedge$.

So $(\phi \wedge \psi) \wedge \rho$ and $\phi \wedge (\psi \wedge \rho)$ have the same truth value in this situation by semantics of $\wedge$. The situation was arbitrary, so they are logically equivalent.

We could have equally shown that a formula of the form $(\phi \wedge \psi) \wedge \rho \leftrightarrow \phi \wedge (\psi \wedge \rho)$ is a valid formula.

Direct argument—Example (3)

Here's a more complex example.

Let's try to show that $((p \to q) \to p) \to p$ (known as 'Peirce's law') is a valid formula.

Take an arbitrary situation.

- *If p is true in this situation*, then $((p \to q) \to p) \to p$ is true, since any formula of the form $\phi \to \psi$ is true when ψ is true. We are done.
- *If not*, then p must be false in this situation.

  So $p \to q$ is true, because $\phi \to \psi$ is true when ϕ is false by semantics of $\to$.

  So $(p \to q) \to p$ is false by semantics of $\to$, because the antecedent is true and the consequent is false: $\phi \to \psi$ is false when ϕ is true and ψ false.

  So $((p \to q) \to p) \to p$ is true by semantics of $\to$, because $\phi \to \psi$ is true when ϕ is false. We are done again, and finished.

This was an *argument by cases:* p true, or p false. They are exhaustive: this is known as the *'law of excluded middle'*.