40018 Logic

Exercises 5: Natural Deduction + Syntax and Semantics of First-order Logic

Submit solutions for questions marked **(PMT)** electronically in Scientia by
Mondat 11th December 2023, 7pm GMT.
Questions marked **(PMT)** to be discussed in the PMT session of week 2, in Term 2.

1. (From exam) Let A and B be formulas in propositional logic. Using natural deduction, prove that $(A \wedge B) \vee A \leftrightarrow A \wedge (A \vee B)$ is a theorem.

2. Show that $p \vee q$ is *provably equivalent* to $(p \to q) \to q$.

3. Consider a variant of natural deduction proof system $\vdash^*$ in which the rule for $\to E$ (arrow elimination) is replaced by the elimination rule $\to^* E$ given below.

$$\begin{array}{lll} 1 & A \to B & \\ & \vdots & \\ 2 & \neg A & \\ 3 & \neg B & \to^* E(1,2) \end{array}$$

   Is $\vdash^*$ sound? Justify your answer.

4. Write the following sentences in natural English, using the given intended meanings:

   (a) $\forall x[\texttt{box}(x) \vee \texttt{table}(x)]$

   (b) **(PMT)** $\forall x[(\texttt{table}(x) \to \texttt{red}(x)) \wedge (\texttt{green}(x) \to \texttt{box}(x))]$

   (c) $\neg\exists x[\texttt{red}(x) \wedge \texttt{green}(x)]$

   where $\texttt{box}(x)$ is read as 'x is in the box', $\texttt{red}(x)$ is read as 'x is red', $\texttt{green}(x)$ is read as 'x is green' and $\texttt{table}(x)$ is read as 'x is on the table'.

5. Consider a domain of discourse of people and let L be a signature with predicates $\texttt{male}/1$, $\texttt{father}/2$, $\texttt{mother}/2$, $\texttt{siblings}/2$, and $\texttt{child}/3$, where $\texttt{child}(A, B, C)$ reads as "B is the mother and C is the father of A".

   (a) Express the following statements as predicate logic sentences:

      i. Everybody has a father and a mother.
      ii. **(PMT)** Some people have children.
      iii. A person is the father of another person if and only if he is male and the other person is their child.
      iv. **(PMT)** Siblings have either the same father or the same mother.

   (b) Now complete the following sentence to define the predicate $\texttt{uncle}/2$ according to its intended meaning using the predicates given above:

      i. $\forall x \forall y(\texttt{uncle}(x, y) \leftrightarrow (\ldots))$

   (c) **(PMT)** Now consider a function $\texttt{father\_of}/1$. Express the statement 5(a)iii as a predicate logic sentence using the function $\texttt{father\_of}/1$ instead of the predicate $\texttt{father}/2$.

6. Let L be a signature with constants `Felix`, `Waldo` and binary predicate `chases`/2. Let $M = \langle \mathbb{D}, \mathbb{I} \rangle$ be an L-structure where $\mathbb{D} = \{cat, bird, worm\}$ and $\mathbb{I}$ is defined as follows: $\mathbb{I}(\texttt{Felix}) = cat$, $\mathbb{I}(\texttt{Waldo}) = worm$. The predicate $\texttt{chases}(x, y)$ is interpreted as "x chases y", where cat chases all three, $worm$ is chased by all three and $bird$ only chases $worm$.

   Draw a diagram of M. Which of the following are true in M? Give brief reasons for each answer.

   (a) $\texttt{chases}(\texttt{Felix}, \texttt{Felix}) \vee \texttt{chases}(\texttt{Waldo}, \texttt{Felix})$

   (b) $\exists x\ \texttt{chases}(x, \texttt{Felix})$.

   (c) $\texttt{chases}(\texttt{Felix}, \texttt{Waldo}) \to \texttt{chases}(\texttt{Waldo}, \texttt{Felix})$

   (d) $\forall x(\texttt{chases}(x, x) \to x = \texttt{Felix} \vee x = \texttt{Waldo})$

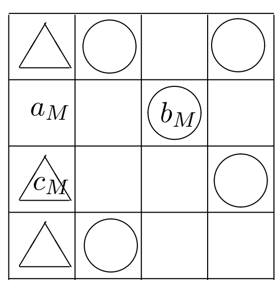
   (e) $\exists x(\texttt{chases}(x, \texttt{Felix}) \wedge \neg(x = \texttt{Felix}))$

   (f) **(PMT)** $\exists u \forall v\ \texttt{chases}(v, u)$

   (g) $\forall y \forall x(\texttt{chases}(x, y) \leftrightarrow \texttt{chases}(y, x))$

   (h) **(PMT)** $\forall v \exists u\ \texttt{chases}(u, v)$

7. Let L is a signature consisting of three constants `a`, `b`, `c`, two unary predicates `triangle`/1, `circle`/1, and two binary predicates `above`/2, `left_of`/2. M is the L-structure whose domain consists of 16 objects, represented by the 16 squares in the diagram. *Note that even an empty square is an object!*



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   The interpretations in M of the symbols of L are as suggested by the diagram. We read $\texttt{circle}(x)$ as 'x is a circle'; $\texttt{tringle}(x)$ means 'x is a triangle'; $\texttt{above}(x, y)$ means 'x is above y' (not necessarily in the same column); and $\texttt{left\_of}(x, y)$ means 'x is to the left of y' (not necessarily in the same row). E.g., $\texttt{circle}(\texttt{b})$, $\neg\texttt{circle}(\texttt{a})$, $\texttt{left\_of}(\texttt{c}, \texttt{b})$, and $\neg\texttt{above}(\texttt{a}, \texttt{b}))$ are all true in M.

   Below are two lists, of English statements and logic sentences.

   (A) Every object is a circle or a triangle.

   (B) Each circle has a circle somewhere below it.

   (C) At least two columns have circles in them.

   (D) Every object in the top row is a circle.

   (E) **(PMT)** All triangles are in the same column.

   (F) a and b are in the same row.

   (G) b is in the rightmost column.

   (H) **(PMT)** Some row has no circles.

   (a) $\exists x \exists y(\texttt{circle}(x) \wedge \texttt{circle}(y) \wedge \texttt{left\_of}(x, y))$

   (b) $\forall x(\texttt{above}(a, x) \leftrightarrow \texttt{above}(b, x))$

   (c) $\exists x \forall y(\texttt{circle}(y) \to \texttt{above}(x, y) \vee \texttt{above}(y, x))$

   (d) $\forall x(\texttt{circle}(x) \vee \texttt{triangle}(x))$

   (e) $\forall x(\neg \exists y\ \texttt{above}(y, x) \to \texttt{circle}(x))$

   (f) $\forall x(\texttt{left\_of}(b, x) \to \exists y\ \texttt{left\_of}(x, y))$

   (g) $\forall x(\texttt{circle}(x) \vee \exists y(\texttt{above}(y, x) \vee \texttt{left\_of}(x, y)))$

   (h) $\forall x(\texttt{circle}(x) \to \exists y(\texttt{circle}(y) \wedge \texttt{above}(x, y)))$

i) *Match up the two lists,* so that each English statement means the same (has the same truth value) as the corresponding logic sentence in any L-structure with the same domain and interpretations of `left_of` and `above` as M (but possibly different interpretations of `circle`, `triangle`, `a`, `b`, `c`). E.g., 'there are no circles' would match $\neg\exists x\ \mathtt{circle}(x)$.

   If you think a logic sentence doesn't match any of the English statements, write your own statement *in plain English* to express it, and vice versa.

ii) Which of the logic sentences are true in the structure M shown above? Explain your answers briefly.