40018 Logic
Logic exercises 5: Semantics of First-order Logic
Translating English to Logic

There will be no **PMT** submission for this sheet.

1. Situation: All members of DoC (staff and students). The binary relation symbol $\mathit{mtt}(x, y)$ is read as 'x is much taller than y'. If you could adjust the heights of all members of the department, could you adjust them to make each sentence (a)–(d) true? How? And to make them all false? How?

   For example, to make $\forall x \forall y(\mathit{student}(x) \wedge \mathit{staff}(y) \to \mathit{mtt}(x, y))$ true, you'd have to arrange that the students were all much taller than the staff. If they're not, it's false.

   (a) $\exists u[\mathit{student}(u) \wedge \forall v[\mathit{staff}(v) \to \mathit{mtt}(v, u)]]$

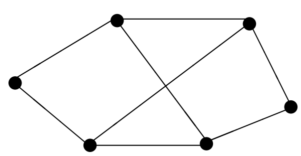
   (b) $\forall y \forall x[\mathit{student}(x) \wedge \mathit{student}(y) \to (\mathit{mtt}(x, y) \leftrightarrow \mathit{mtt}(y, x))]$

   (c) $\forall x[\mathit{student}(x) \to \exists y[\mathit{student}(y) \wedge (\mathit{mtt}(x, y) \vee \mathit{mtt}(y, x))]]$

   (d) $\forall v(\mathit{staff}(v) \to \exists u[\mathit{staff}(u) \wedge \mathit{mtt}(u, v)])$

2. Let L be a signature consisting of a single binary relation symbol E. Consider the following L-structure M. $dom(M)$ has 6 objects, as shown. For objects a, b in $dom(M)$, $E(a, b)$ is true in M just when there's a direct straight line (an 'edge') between a and b. E.g., the top two objects are E-related; the leftmost and rightmost objects are not.

   M

   Which of the following are true in M? Give a brief explanation for each answer.

   (a) $\forall x \neg E(x, x)$

   (b) $\forall x \forall y(E(x, y) \to E(y, x))$

   (c) $\forall x \forall y(E(x, y) \to \forall z(E(y, z) \to E(x, z)))$

   (d) $\exists x \exists y \exists z(E(x, y) \wedge E(y, z) \wedge E(x, z))$

   (e) $\forall x \exists y \exists z(E(x, y) \wedge E(x, z) \wedge y \neq z)$. Here, $y \neq z$ abbreviates $\neg(y = z)$.

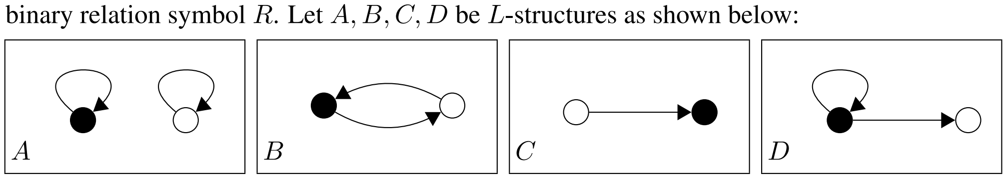
   (f) $\forall x \exists y \exists z \exists v(E(x, y) \wedge E(x, z) \wedge E(x, v) \wedge y \neq z \wedge y \neq v \wedge z \neq v)$.

   Now let P be a new unary relation symbol.

   (g) Define an interpretation of P in M that makes the following sentence true:

   $$\forall x \forall y(E(x, y) \to (P(x) \leftrightarrow \neg P(y))).$$

3. (from exam 2013) Let L be the first-order signature consisting of a unary relation symbol P and a binary relation symbol R. Let A, B, C, D be L-structures as shown below:

   An arrow from a circle a to a circle b means that $R(a, b)$ is true. The objects satisfying P are the black circles.

   For each of the following four L-sentences, state in which of the four structures it is true and in which it is false. You do not need to justify your answers.

(a) $\forall x(P(x) \to \exists y R(x,y))$

(b) $\forall x \forall y(R(x,y) \vee R(y,x))$

(c) $\exists x \forall y(R(x,y) \to P(y))$

(d) $\exists x \forall y \exists z(R(x,z) \wedge \neg(z=y))$

For each of the structures A, B, C, D in turn, write down an L-sentence that is true in that structure and false in the other three.

4. (a) Find a structure with at least two elements in its domain that makes one of $\forall x[F(x) \to G(x)]$ and $\exists x F(x) \to \exists x G(x)$ true, and the other false. Justify your answer.

   (b) Repeat for $\exists v \forall u R(u,v)$ and $\forall x \exists y R(x,y)$.

   (c) In fact, it is only possible to do *one* of the following:

   - find a structure that makes the first sentence of (a) true and the second false, or
   - find a structure that makes the second sentence of (a) true and the first false.

   You cannot do both. Why?

   (d) Repeat part (c) using the sentences in (b) instead of those in (a).

5. Let L be the signature consisting of constants $\underline{1}, \underline{2}, \underline{3}, \ldots$, binary relation symbols $<, >, \leq, \geq$, and binary function symbols $+, \times$. Let N be the structure whose domain consists of the *positive* integers $1, 2, 3, \ldots$, and with the symbols of L interpreted in the natural way.

   The formula $\exists v(x = \underline{2} \times v)$ expresses that x is even. In the same kind of way, write first-order L-formulas expressing that:

   (a) x is divisible by 3 without remainder.

   (b) x is prime.

   (c) Every even number bigger than 2 is the sum of two primes.
   Your answer must be an L-sentence — no free variables please.

   (d) x is a square number.

   (e) x is the sum of two square numbers.

   (f) There are infinitely many prime numbers. (This is 'Euclid's theorem'. $\top$ expresses it, since the theorem is true, but you should write a more direct translation of the meaning of the theorem.)

6. Translate the following sentences into logic as faithfully as possible. Invent your own predicates. Hints: write formulas expressing the subconcepts first, then piece them together into a full solution. E.g., in 6c and 6d, 'x is a chimp', 'y is a prize', 'x won y', 'y was won by a chimp', and 'x won all the prizes' are useful subconcepts. The patterns $\exists x(A \wedge B)$ and $\forall y(A \to B)$ are common — e.g., 'for all y, if y is a prize then x won y' would express 'x won all the prizes'.

   (a) No animal is both a cat and a dog.

   (b) Anyone who admires himself admires someone.

   (c) Every prize was won by a chimpanzee.

   (d) One particular chimpanzee won all the prizes.

   (e) Jack cannot run faster than anyone in the team.

   (f) Jack cannot run faster than everyone in the team.

   (g) All first year students have a PPT tutor.

   (h) No student has the same PMT and PPT tutor.