Discrete mathematics, **logic** and reasoning (COMP40018)

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Many thanks to Ian Hodkinson and Robert Craven for some of the original material, and none of the errors.

4.2 Semantics—The meaning

The notion of *situations* has to be extended to give meaning to the new style of predicate logic formulas. We have to specify:

- what a *situation* is for predicate logic,
- how to evaluate predicate logic formulas in a given situation.

We need first to give meaning to all the symbols in a given signature

We then need to define the notions of valuation of terms and evaluation for atomic formulas, and quantifiers.

Functions and relations over a Domain of Discourse

A collection of objects to which a predicate logic might refer is called a *domain of discourse*.

- A *domain of discourse* is a *non-empty* set of objects. We denote it with $\mathbb{D}$.
- For a positive whole number n, the n-ary *Cartesian power* of a set $\mathbb{D}$, written $\mathbb{D}^n$ or $\mathbb{D}\times, \ldots, \times\mathbb{D}$, is the set of all n-tuples that can be constructed from its members.
- A *relation* R of arity n over $\mathbb{D}$ is a subset of $\mathbb{D}^n$, so $R \subseteq \mathbb{D}^n$, with $n \geq 0$.
- A *unary relation* R is a relation of arity 1, so $R \subseteq \mathbb{D}$.
- A *binary relation* R is a relation of arity 2. So $R \subseteq \mathbb{D}^2$.
- A *function of arity n* over $\mathbb{D}$ is a one-to-one or a many-to-one mapping from all of $\mathbb{D}^n$ to $\mathbb{D}$, with $n > 0$.
- A *many-to-one* function means that different n-tuples of objects in $\mathbb{D}$ may be mapped to the same object in $\mathbb{D}$, but each n-tuple is mapped to exactly one object.

Structures (i.e., situations in predicate logic)

Definition 4.4 (L-structure)

Let L be a signature $\langle \mathcal{K}, \mathcal{F}, \mathcal{P} \rangle$. An *L-structure* (or sometimes (loosely) a *model*) M is a pair: $M = \langle \mathbb{D}, \mathbb{I} \rangle$, where

- $\mathbb{D}$ is a *domain of discourse* of M, a non-empty set of objects that M 'knows about'. It's also called *universe* of M, and sometimes written as *dom(M)*.
- $\mathbb{I}$ is an *interpretation* that specifies the meaning of each symbol in L in terms of the objects in $\mathbb{D}$:
  - for each constant c in $\mathcal{K}$, $\mathbb{I}(c) = c_M \in \mathbb{D}$
  - for each n-ary function symbol f in $\mathcal{F}$, $\mathbb{I}(f) = f_M : \mathbb{D}^n \mapsto \mathbb{D}$ for $n > 0$.
  - for each n-ary predicate symbol P in $\mathcal{P}$, $\mathbb{I}(P) = P_M \subseteq \mathbb{D}^n$ for $n \geq 0$.

0-ary predicates are similar to propositional atoms in Propositional Logic. The 0-ary Cartesian power of $\mathbb{D}$, denoted as $\mathbb{D}^0$, is defined to be a singleton set containing the empty set, i.e., $\mathbb{D}^0 = \{\emptyset\}$.

Example of a structure $M = \langle \mathbb{D}, \mathbb{I} \rangle$

For our simple signature L, an L-structure should define the domain and the meaning of the symbols in L.

Let $\mathbb{D}$ be a set of *12 objects*: $\mathbb{D} = \{o_1, o_2, \ldots, o_{12}\}$.

Let $\mathbb{I}$ be the following interpretation:

- $\mathbb{I}(\texttt{tony}) = o_5$, $\mathbb{I}(\texttt{susan}) = o_4$, $\mathbb{I}(\texttt{frank}) = o_3$, $\mathbb{I}(\texttt{clyde}) = o_7$, $\mathbb{I}(c) = o_{11}$, $\mathbb{I}(\texttt{heron}) = o_{10}$.

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- $\mathbb{I}(father\_of) = f_M : \mathbb{D} \mapsto \mathbb{D}$, given by:
  $\langle o_1 \mapsto o_1, o_2 \mapsto o_1, o_3 \mapsto o_1, o_4 \mapsto o_2, o_5 \mapsto o_2,$
  $o_6 \mapsto o_7, o_7 \mapsto o_8, o_8 \mapsto o_1, o_9 \mapsto o_6,$
  $o_{10} \mapsto o_{11}, o_{11} \mapsto o_{12}, o_{12} \mapsto o_{12} \rangle.$

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  $o_{10} \mapsto o_{11}, o_{11} \mapsto o_{12}, o_{12} \mapsto o_{12} \rangle$.
- $\mathbb{I}(\texttt{Human}) = \{o_1, o_2, o_3, o_4, o_5\}$, $\mathbb{I}(\texttt{Lecturer}) = \{o_2, o_3, o_4, o_5, o_6\}$, $\mathbb{I}(PC) = \{o_{10}, o_{11}, o_{12}\}$.

Example of a structure $M = \langle \mathbb{D}, \mathbb{I} \rangle$

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- $\mathbb{I}(\texttt{Human}) = \{o_1, o_2, o_3, o_4, o_5\}$, $\mathbb{I}(\texttt{Lecturer}) = \{o_2, o_3, o_4, o_5, o_6\}$, $\mathbb{I}(PC) = \{o_{10}, o_{11}, o_{12}\}$.
- $\mathbb{I}(\texttt{Bought}) = \{(o_4, o_{10}), (o_5, o_{10}), (o_4, o_7)\}$

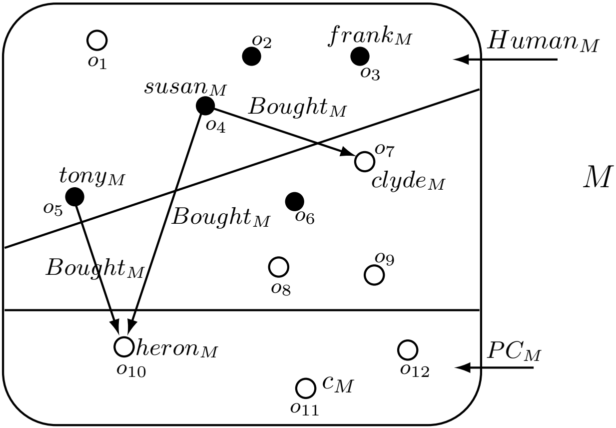
Depicting $M = \langle \mathbb{D}, \mathbb{I} \rangle$ as a diagram

When we have just unary and binary predicates, we can depict a structure as a diagram.

Next slide shows a diagram of our L-structure M.

- The 12 dots are the 12 objects in $dom(M)$.
- Some objects have an additional label (e.g. $frank_M$) to indicate the meaning of the constants in L.
- The interpretations (meanings) of PC, `Human` are drawn as regions. The interpretation of `Lecturer` is indicated by the black dots.
- The interpretation of `Bought` is shown by the arrows between objects.
- We always omit in the diagram the interpretation of the function symbols to avoid confusion with that of the predicate symbols.

The structure $M = \langle \mathbb{D}, \mathbb{I} \rangle$



The structure $M = \langle \mathbb{D}, \mathbb{I} \rangle$

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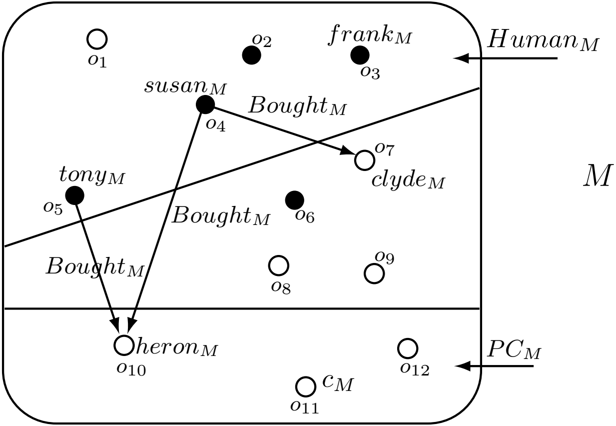

The structure $M = \langle \mathbb{D}, \mathbb{I} \rangle$

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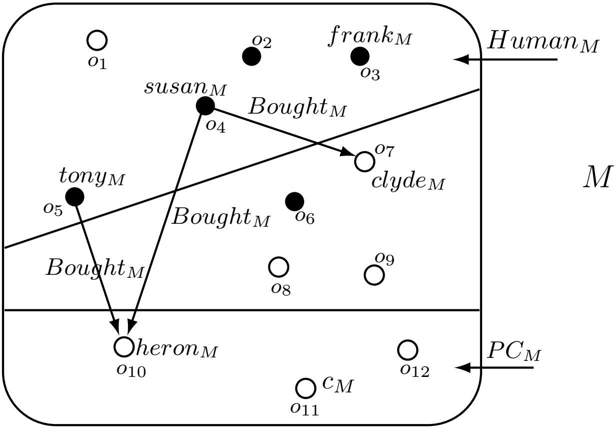

The structure $M = \langle \mathbb{D}, \mathbb{I} \rangle$

$\mathbb{I}(father\_of) = f_M : \mathbb{D} \mapsto \mathbb{D}$, given by:
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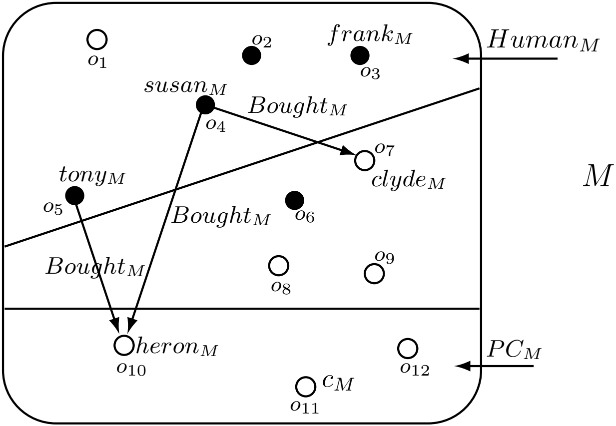
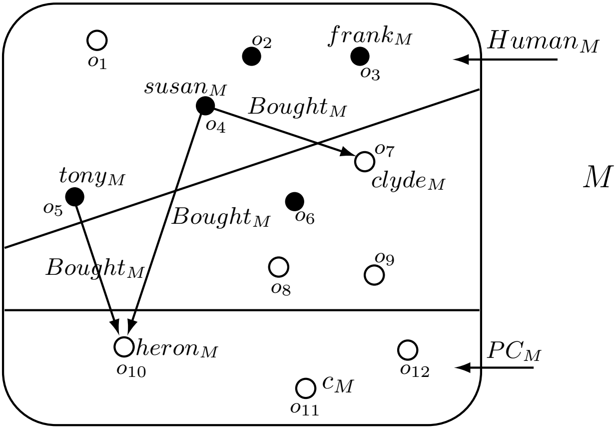


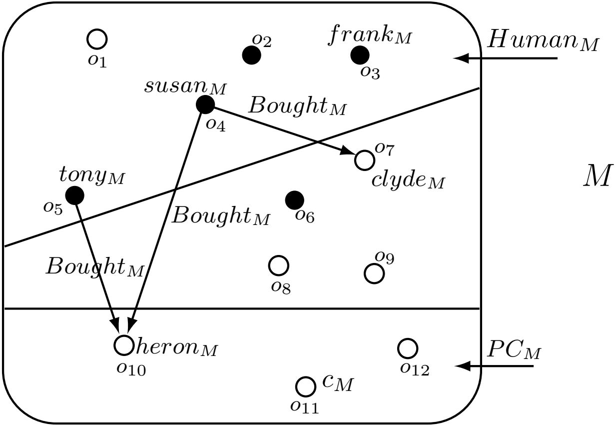
The structure $M = \langle \mathbb{D}, \mathbb{I} \rangle$

$\mathbb{I}(\texttt{Human}) = \{o_1, o_2, o_3, o_4, o_5\}$, $\mathbb{I}(\texttt{Lecturer}) = \{o_2, o_3, o_4, o_5, o_6\}$,
$\mathbb{I}(PC) = \{o_{10}, o_{11}, o_{12}\}$



The structure $M = \langle \mathbb{D}, \mathbb{I} \rangle$

$\mathbb{I}(\texttt{Bought}) = \{(o_4, o_{10}), (o_5, o_{10}), (o_4, o_7)\}$



Drawing other symbols

Our simple signature L has constants, one unary function symbol, unary and binary predicate symbols.

For this L, we drew an L-structure M by

- drawing a collection of objects (the domain of M)
- marking which objects are named by which constants in M
- marking which objects M says satisfy the unary predicate symbols
- drawing arrows between the objects that M says satisfy the binary predicate symbols. The arrow direction matters.

With several binary predicate symbols in L, we'd *really need* to label the arrows.

It is difficult to draw interpretations of 3-ary or higher-arity predicate symbols. We do not draw the interpretation of function symbols, but specify it mathematically as indicated in Slide 176.

0-ary predicate symbols are the same as propositional atoms.

Truth in a structure (informally)

When is a formula *without quantifiers and variables* true in a structure?

- `PC`(`heron`) is true in M, because the interpretation of the constant `heron`is an object (o_{10}) that is in the interpretation of `PC`, that is $\mathbb{I}(\texttt{heron}) \in \mathbb{I}(\texttt{PC})$ or in other terms $o_{10} \in \{o_{10}, o_{11}, o_{12}\}$.

  We write this as $M \models \texttt{PC}(\texttt{heron})$. Read as '$M$ says `PC`(`heron`)'.

  Warning: This is a quite different use of $\models$ from what seen in the definition of valid argument. '$\models$' is *overloaded* — it's used for two different things.

- `Bought`(`susan`,`susan`) is false in M, because M does not say that the object labeled $susan_M$ (o_4) "bought" itself, i.e.
  $(\mathbb{I}(\texttt{susan}), \mathbb{I}(\texttt{susan})) \notin \mathbb{I}(\texttt{Bought})$, i.e.,
  $(o_4, o_4) \notin \{(o_4, o_{10}), (o_5, o_{10}), (o_4, o_7)\}$.
  We write this as $M \not\models \texttt{Bought}(\texttt{susan}, \texttt{susan})$.

Truth in a structure (informally)

When is a formula *without quantifiers and variables* true in a structure?

- $\texttt{PC}(\texttt{heron})$ is true in M, because the interpretation of the constant **heron**is an object (o_{10}) that is in the interpretation of $\texttt{PC}$, that is $\mathbb{I}(\texttt{heron}) \in \mathbb{I}(\texttt{PC})$ or in other terms $o_{10} \in \{o_{10}, o_{11}, o_{12}\}$.

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  We write this as $M \not\models \texttt{Bought}(\texttt{susan}, \texttt{susan})$.

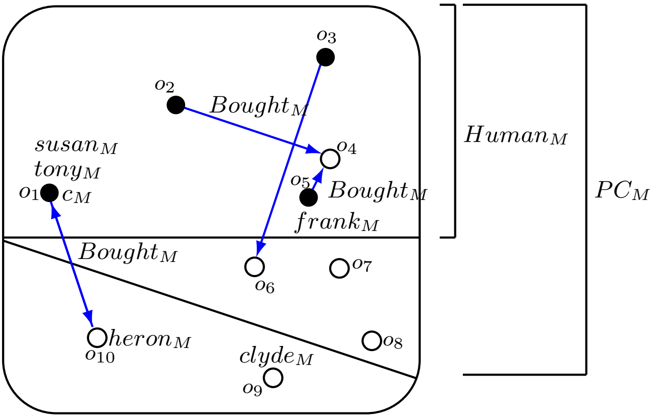
From our knowledge of propositional logic,

- $M \not\models PC(\texttt{tony}) \vee \texttt{Bought}(father\_of(\texttt{frank}), \texttt{clyde})$.

because $o_5 \notin \{o_{10}, o_{11}, o_{12}\}$ and $(o_1, o_7) \notin \{(o_4, o_{10}), (o_5, o_{10}), (o_4, o_7)\}$

Another structure

Here is another L-structure, called M'. $\mathbb{D} = \{o_1, \ldots, o_{10}\}$ and the interpretation $\mathbb{I}(father\_of) = \langle o_1 \mapsto o_2, o_2 \mapsto o_3, o_3 \mapsto o_6,$ $o_4 \mapsto o_5, o_5 \mapsto o_7, o_6 \mapsto o_7, o_7 \mapsto o_8, o_8 \mapsto o_9, o_9 \mapsto o_{10}, o_{10} \mapsto o_{10} \rangle$.



Some statements about M'

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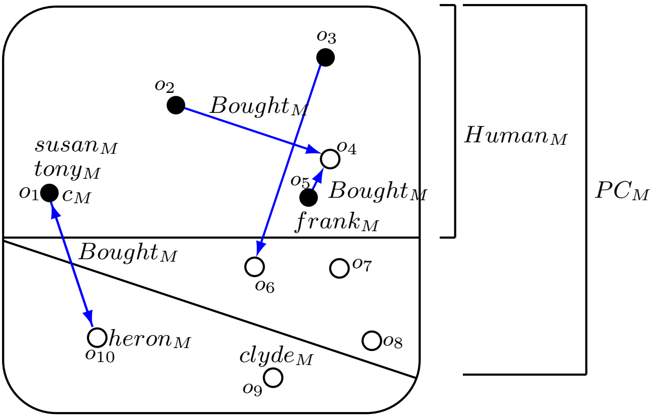

- $M' \not\models \texttt{Bought}(father\_of(\texttt{susan}), \texttt{clyde})$.

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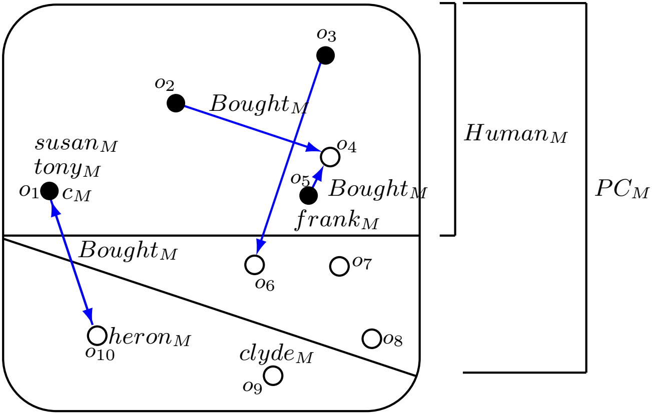

- $M' \models$ `susan` $=$ `tony`?

Some statements about M'

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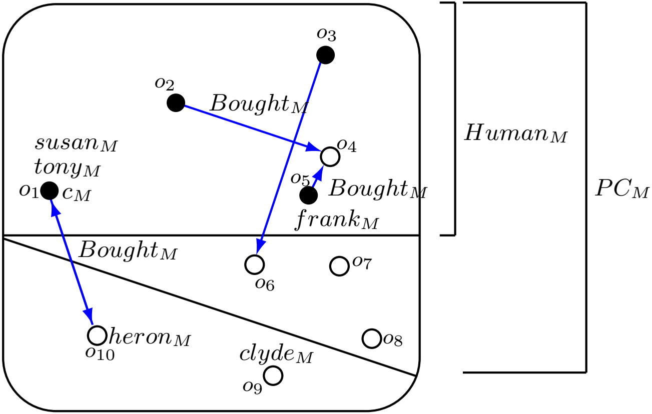

- $M' \models \texttt{Human}(\texttt{tony}) \wedge PC(\texttt{frank})$?

Some statements about M'

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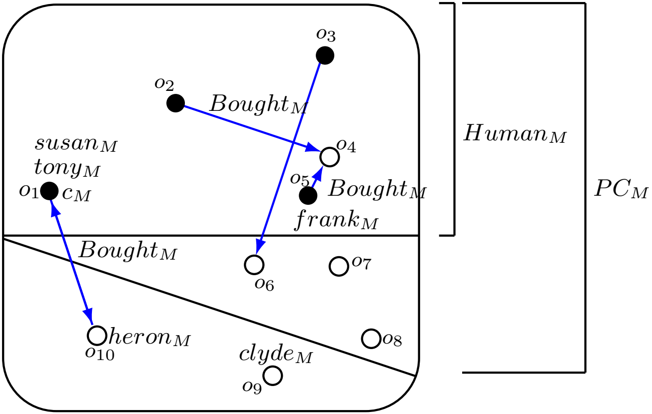

- $M' \models \texttt{Bought}(\texttt{tony}, father\_of(\texttt{heron})) \wedge \texttt{Bought}(Heron, c)$?

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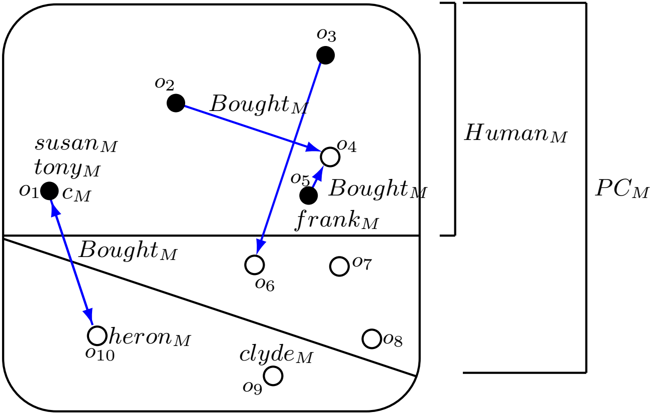

- $M' \models \texttt{Bought}(\texttt{susan}, \texttt{clyde}) \rightarrow \texttt{Human}(\texttt{clyde})$?