Logic exercises 3: Propositional logic Checking validity (Direct argument, equivalences and natural deduction

This exercise sheet focuses on methods for checking validity using direct argument, equivalences and natural deduction.

Questions marked (**TUT**) will be covered during the tutorial session (Thursday 30/11).

Natural deduction proofs can be constructed and printed using Pandora (`https://www.doc.ic.ac.uk/pandora/newpandora/`). You don't have to do this, but please try the program.

1. For each argument below, write its corresponding implication formula, and show whether the corresponding implication formula is valid using truth tables. (Justify your verdict.)

   (a) $p \wedge q \models q$

   (b) $p \leftrightarrow q \models \neg p \leftrightarrow q$

2. Which of the following arguments are valid? In each case, use direct argument to show that the premise semantically entails the conclusion, or (if premise $\not\models$ conclusion) specify a situation in which the premise is true and the conclusion false.

   (a) **TUT**: $p \to q \models q \to p$

   (b) $p \to q \models \neg q \to \neg p$

   (c) $(p \wedge q) \vee (r \wedge s) \models (p \vee r) \wedge (q \vee s)$

   (d) $(p \vee r) \wedge (q \vee s) \models (p \wedge q) \vee (r \wedge s)$

3. Use direct argument to show that the following formulas are logically equivalent.

   (a) $\bot \vee r$ and r

   (b) $\top \vee p$ and $\top$

   (c) $q \wedge \top$ and q

   (d) $\bot \to p$ and $\top$

   (e) **TUT**: $p \vee q$ and $(p \to q) \to q$

   (f) $p \leftrightarrow (q \leftrightarrow r)$ and $(p \leftrightarrow q) \leftrightarrow r$

4. Using equivalences show that

   (a) **(PMT):** $((p \to q) \to p) \to p$ is a valid formula.

(b) $p \wedge q \models q \wedge p$ is a valid argument.

(c) $((p \rightarrow q) \rightarrow q) \leftrightarrow \neg(p \vee q)$ is unsatisfiable.

5. For each of the following state whether the formula is in CNF, DNF, both or neither. Justify your answer.

   (a) $\perp$

   (b) p

   (c) $\neg q$

   (d) **(PMT):** $p \wedge \neg(p \vee r)$

   (e) $p \vee q \vee r$

   (f) $p \vee \neg q \vee \neg r$

   (g) $p \wedge \neg p \vee \neg q \vee r \wedge \neg r$

   (h) $p \wedge \neg p$

   (i) $p \wedge q \vee s$

   (j) **(PMT):**$(p \vee q) \wedge s$

6. Convert the following formula to conjunctive normal form and to disjunctive normal form using equivalences.
$$(p \vee \neg\neg q) \wedge s$$

7. Convert the following formula to conjunctive normal form and to disjunctive normal form using truth tables.
$$(p \rightarrow (p' \rightarrow p'')) \leftrightarrow p$$

8. Using truth tables, rewrite the following formula into an equivalent formula in (i) disjunctive normal form, and (ii) **PMT:** conjunctive normal form. (Do not simplify the formula extracted from the table.)
$$(((p \rightarrow \neg q) \wedge \neg r) \rightarrow q) \wedge r$$

9. Show the following, using natural deduction. *Do not use equivalences to rewrite any formulas.* I advise you always start by thinking up a direct argument to show that *Premises* $\models$ *Conclusion*. Then convert your ideas into a ND proof.

   (a) $p \leftrightarrow q \vdash \neg p \leftrightarrow \neg q$

   (b) $p \wedge q \rightarrow r \vdash p \rightarrow (q \rightarrow r)$

   (c) $p \vee q \vdash (p \rightarrow q) \rightarrow q$

   (d) $(p \rightarrow r) \wedge (q \rightarrow r) \vdash (p \vee q) \rightarrow r$

   (e) $\vdash p \rightarrow (q \rightarrow p)$

   (f) **PMT:** $\vdash p \wedge q \rightarrow p$

   (g) $p \vdash q \rightarrow p$

   (h) $p \vdash q \rightarrow p \wedge q$

   (i) **PMT:** $p \rightarrow (q \rightarrow r) \vdash (p \rightarrow q) \rightarrow (p \rightarrow r)$