3rd method: Checking validity using equivalences

We know that $\top$ is a valid formula.

Showing an argument is valid using equivalences involves:

1. converting the argument into its corresponding implication formula;
2. simplifying the formula to $\top$ (which is a valid formula), always preserving logical equivalence.

Why? Recall this relations . . .

argument validity	formula validity	satisfiability	equivalence
$\phi \models \psi$	$\phi \to \psi$ valid	$\phi \wedge \neg\psi$ unsatisfiable	$(\phi \to \psi) \equiv \top$

Equivalences

We do this kind of thing in arithmetic (preserving the value in each step):

$$
\begin{aligned}
3(x+y^2) - 2y(y+4) &= 3x + 3y^2 - 2y^2 - 8y \\
&= 3x + y^2 - 8y.
\end{aligned}
$$

We are going to list some examples of equivalences that have been found useful for this.

There are quite a lot, but they all embody basic logical principles that you should know.

You should check their logical equivalence, using truth tables or direct argument.

Equivalences involving $\wedge$

In the following, ϕ, ψ, ρ will denote arbitrary formulas.

For short, I will often say 'equivalent' rather than 'logically equivalent'.

1. $\phi \wedge \psi$ is logically equivalent to $\psi \wedge \phi$ (*commutativity of* $\wedge$)
2. $\phi \wedge \phi$ is logically equivalent to ϕ (*idempotence of* $\wedge$)
3. $\phi \wedge \top$ and $\top \wedge \phi$ are logically equivalent to ϕ
4. $\bot \wedge \phi$, $\phi \wedge \bot$, $\phi \wedge \neg\phi$, and $\neg\phi \wedge \phi$ are all equivalent to $\bot$
5. $(\phi \wedge \psi) \wedge \rho$ is equivalent to $\phi \wedge (\psi \wedge \rho)$ (*associativity of* $\wedge$)

Equivalences involving $\vee$

6. $\phi \vee \psi$ is equivalent to $\psi \vee \phi$ (*commutativity of* $\vee$)
7. $\phi \vee \phi$ is equivalent to ϕ (*idempotence of* $\vee$)
8. $\top \vee \phi$, $\phi \vee \top$, $\phi \vee \neg\phi$, and $\neg\phi \vee \phi$ are equivalent to $\top$
9. $\phi \vee \bot$ and $\bot \vee \phi$ are equivalent to ϕ
10. $(\phi \vee \psi) \vee \psi$ is equivalent to $\phi \vee (\phi \vee \psi)$ (*associativity of* $\vee$)

Equivalences involving $\neg$ and $\to$

11. $\neg\top$ is equivalent to $\bot$
12. $\neg\bot$ is equivalent to $\top$
13. $\neg\neg\phi$ is equivalent to ϕ

14. $\phi \to \phi$ is equivalent to $\top$
15. $\top \to \phi$ is equivalent to ϕ
16. $\phi \to \top$ is equivalent to $\top$
17. $\bot \to \phi$ is equivalent to $\top$
18. $\phi \to \bot$ is equivalent to $\neg\phi$
19. $\phi \to \psi$ is equivalent to $\neg\phi \vee \psi$, and also to $\neg(\phi \wedge \neg\psi)$
20. $\neg(\phi \to \psi)$ is equivalent to $\phi \wedge \neg\psi$

Equivalences involving $\leftrightarrow$

21. $\phi \leftrightarrow \psi$ is equivalent to
    - $(\phi \to \psi) \wedge (\psi \to \phi)$,
    - $(\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$,
    - $\neg\phi \leftrightarrow \neg\psi$.

22. $\neg(\phi \leftrightarrow \psi)$ is equivalent to
    - $\phi \leftrightarrow \neg\psi$,
    - $\neg\phi \leftrightarrow \psi$,
    - $(\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)$.

    (This is one way to express the *exclusive or* of ϕ, ψ.)

Equivalences—De Morgan laws

Augustus de Morgan (19th-century logician) did not discover these (they are much older).

23. $\neg(\phi \wedge \psi)$ is equivalent to $\neg\phi \vee \neg\psi$
24. $\neg(\phi \vee \psi)$ is equivalent to $\neg\phi \wedge \neg\psi$

An example of (24) in English:

It is not the case that Michelle or Sangita went to the theatre

$$\equiv$$

Michelle did not go to the theatre and Sangita did not go to the theatre

Equivalences—ctd.

Distributivity of $\wedge$, $\vee$

25. $\phi \wedge (\psi \vee \rho)$ is equivalent to $(\phi \wedge \psi) \vee (\phi \wedge \rho)$.
    $(\psi \vee \rho) \wedge \phi$ is equivalent to $(\psi \wedge \phi) \vee (\rho \wedge \phi)$.
26. $\phi \vee (\psi \wedge \rho)$ is equivalent to $(\phi \vee \psi) \wedge (\phi \vee \rho)$
    $(\psi \wedge \rho) \vee \phi$ is equivalent to $(\psi \vee \phi) \wedge (\rho \vee \phi)$

Absorption

27. $\phi \wedge (\phi \vee \psi)$ and $\phi \vee (\phi \wedge \psi)$ are equivalent to ϕ.
    So are $\phi \wedge (\psi \vee \phi)$, $(\phi \wedge \psi) \vee \phi$, etc.

Absorption Law—proof

Show using truth tables that $p \vee (p \wedge q) \equiv p$.

p	q	$p \wedge q$	$p \vee (p \wedge q)$
tt	tt	tt	tt
tt	ff	ff	tt
ff	tt	ff	ff
ff	ff	ff	ff

Absorption Law—proof

Show, using equivalences $(1-26)$, *that* $p \vee (p \wedge q) \equiv p$.

$$
\begin{aligned}
& p \vee (p \wedge q) && \text{[the original formula]} \\
\equiv\ & (p \wedge \top) \vee (p \wedge q) && [\text{by } \phi \wedge \top \equiv \phi] \\
\equiv\ & p \wedge (\top \vee q) && [\text{by distributivity of } \wedge] \\
\equiv\ & p \wedge \top && [\text{by } \top \vee \phi \equiv \top] \\
\equiv\ & p && [\text{by } \phi \wedge \top \equiv \phi]
\end{aligned}
$$

Done!

Equivalences—Normal form

Equivalences can be used to re-write a formula into a *normal form*.

These can improve the efficiency of checking validity/satisfiability of a formula which otherwise takes time exponential in the number of atoms.

We consider two common normal forms:

- Disjunctive Normal Form (DNF)
- Conjunctive Normal Form (CNF)

Equivalences—Disjunctive normal form

Definition 2.1 (Normal form DNF)

A formula ϕ is in *disjunctive normal form* if it is a disjunction of conjunctions of literals, and is not further simplifiable by equivalences without leaving this form. (See Def. 1.3 for literals.)

Examples in DNF:

$p \vee q \vee \neg r$

$(p \wedge \neg q) \vee r \vee (\neg p \wedge q \wedge \neg r)$

q

Examples not in DNF:

$(p \wedge p) \vee (q \wedge \top \wedge \neg q)$

$\neg(q \vee r)$

A formula in DNF is unsatisfiable, if and only if each of its conjunctions contains some literal and its negation.

Satisfiability of formulas in DNF can be checked in linear time.

Every formula can be equivalently written as a formula in DNF.

Equivalences—Conjunctive normal form

Definition 2.2 (Normal form CNF)

A formula ϕ is in *conjunctive normal form* if it is a conjunction of disjunctions of literals (that is, a conjunction of clauses), and is not further simplifiable by equivalences without leaving this form.

CNF is sometimes synonymous to *clausal normal form*.

Examples in CNF:	**Examples not in CNF:**
$(p \vee \neg q) \wedge (q \vee r) \wedge (\neg p \vee q)$	$q \vee (\neg p \wedge r)$
$p \vee q$	$\neg(r \vee s)$
q	

A formula in CNF is valid, if and only if each of its disjunctions contains some literal and its negation.

Validity of formulas in CNF can be checked in linear time.

Every formula can be equivalently written as a formula in CNF.

Rewriting a formula in normal form—ctd.

1. Remove all occurrences of $\to$ and $\leftrightarrow$ by
   - replacing all subformulas $\phi \to \psi$ by $\neg\phi \vee \psi$
   - replacing all subformulas $\phi \leftrightarrow \psi$ by $(\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$

   It's faster to replace $\neg(\phi \to \psi)$ by $\phi \wedge \neg\psi$, and $\neg(\phi \leftrightarrow \psi)$ by $(\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)$.

2. Use the De Morgan laws to push negations down next to atoms. Delete all double negations (replace $\neg\neg\phi$ by ϕ).

3. Rearrange using distributivity to get the desired normal form.

4. Simplify:
   - replacing subformulas $p \wedge \neg p$ by $\bot$, and $p \vee \neg p$ by $\top$.
   - replacing subformulas $\top \vee p$ by $\top$, $\top \wedge p$ by p, $\bot \vee p$ by p, and $\bot \wedge p$ by $\bot$.
   - absorption (equivalence 27) is often useful too.
   - repeat till no further progress.

Rewriting a formula in normal form—example

Let's write $\neg(p \to q) \lor (r \to p)$ *in DNF*

$$
\begin{aligned}
& \neg(p \to q) \lor (r \to p) && \text{[the original formula]} \\
\equiv\ & \neg(\neg p \lor q) \lor (\neg r \lor p) && [\phi \to \psi \equiv \neg\phi \lor \psi] \\
\equiv\ & (\neg\neg p \land \neg q) \lor (\neg r \lor p) && \text{[by De Morgan laws]} \\
\equiv\ & (p \land \neg q) \lor \neg r \lor p && [\ \neg\neg\phi \equiv \phi] \\
\equiv\ & p \lor (p \land \neg q) \lor \neg r && [\text{by commutativity of } \lor] \\
\equiv\ & (p \lor (p \land \neg q)) \lor \neg r && [\text{by associativity of } \lor] \\
\equiv\ & p \lor \neg r && \text{[by absorption]}
\end{aligned}
$$

Done!

Note that $p \lor \neg r$ is also in CNF.

Rewriting a formula in normal form—example

Let's write $p \wedge q \to \neg(p \leftrightarrow \neg r)$ in DNF

$$
\begin{aligned}
& p \wedge q \to \neg(p \leftrightarrow \neg r) && \text{[the original formula]} \\
\equiv\ & \neg(p \wedge q) \vee \neg(p \leftrightarrow \neg r) && [\phi \to \psi \equiv \neg\phi \vee \psi] \\
\equiv\ & \neg(p \wedge q) \vee ((p \wedge \neg\neg r) \vee (\neg p \wedge \neg r)) && [\neg\phi \leftrightarrow \psi \equiv \\
& && (\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)] \\
\equiv\ & \neg(p \wedge q) \vee ((p \wedge r) \vee (\neg p \wedge \neg r)) && [\neg\neg\phi \equiv \phi] \\
\equiv\ & \neg p \vee \neg q \vee ((p \wedge r) \vee (\neg p \wedge \neg r)) && \text{[by De Morgan law]} \\
\equiv\ & \neg p \vee \neg q \vee (p \wedge r) \vee (\neg p \wedge \neg r) && [\text{by associativity of } \vee] \\
\equiv\ & \neg q \vee (p \wedge r) \vee (\neg p \wedge \neg r) \vee \neg p && [\text{by commutativity of } \vee] \\
\equiv\ & \neg q \vee (p \wedge r) \vee \neg p && \text{[by absorption]}
\end{aligned}
$$

Rewriting a formula in normal form—example ctd.

Consider the last step in the previous slide

$$\equiv \neg q \vee (p \wedge r) \vee \neg p \qquad \text{[by absorption]}$$

We can simplify further if we are willing to leave DNF temporarily:

$$\begin{aligned}
&\equiv \neg q \vee (p \vee \neg p) \wedge (r \vee \neg p) && [\text{by distributivity of } \vee]\\
&\equiv \neg q \vee \top \wedge (r \vee \neg p) && [\phi \vee \neg\phi \equiv \top]\\
&\equiv \neg q \vee (r \vee \neg p) && [\top \wedge \phi \equiv \phi]\\
&\equiv \neg q \vee r \vee \neg p && [\text{by associativity of } \vee]
\end{aligned}$$

Rewriting a formula in normal form—example ctd.

Exercise:
Rewrite $p \wedge q \to \neg(p \leftrightarrow \neg r)$ in CNF.

$p \wedge q \to \neg(p \leftrightarrow \neg r)$ [the original formula]

[Try!]

.....

Showing formula validity using equivalences

Show that $(p \to q) \vee (q \to p)$ *is a valid formula.*

$$
\begin{aligned}
& (p \to q) \vee (q \to p) && \text{[the original formula]} \\
\equiv\ & (\neg p \vee q) \vee (\neg q \vee p) && [\phi \to \psi \equiv \neg\phi \vee \psi] \\
\equiv\ & \neg p \vee (q \vee \neg q \vee p) && \text{[by associativity of } \vee] \\
\equiv\ & (q \vee \neg q \vee p) \vee \neg p && \text{[by commutativity of } \vee] \\
\equiv\ & (q \vee \neg q) \vee (p \vee \neg p) && \text{[by associativity of } \vee] \\
\equiv\ & \top \vee \top && \text{[by } \phi \vee \neg\phi \equiv \top] \\
\equiv\ & \top && \text{[by idempotence]}
\end{aligned}
$$

Done!

Things to note when using equivalences!

1. Name the equivalence law applied when rewriting, or reference its the overall logical form;
2. Apply one equivalence law at a time;
3. You may combine consecutive applications of the same law in one step (e.g., consecutive applications of associativity of $\vee$);
4. Reference the equivalence law next to the *result*;
5. Reference the correct equivalence law,
   e.g., $\top \wedge \phi \equiv \phi$ is different from $\phi \wedge \top \equiv \phi$;
6. Don't forget referencing associativity and commutativity laws.