Mathematics with Propositions

A *proposition* is a true–false statement.

Examples:

- $2 + 2 = 4$
- $2 + 2 = 5$
- If x is any real number, then $x^2 \geq 0$.
- The Riemann Hypothesis (Wikipedia)

It is probably worth giving examples of things which are *not* propositions:

- 2
- +
- =
- $x^2 \geq 0$ if you don't say what x is.

In this video, we will see some operations on propositions.

You will need to know these definitions to pass this module.

All of the definitions are in the course notes.

How do we make new definitions of operations on propositions?

In this part of the course, we will imagine that a proposition is determined by its truth value.

Hence there are only really two propositions: `true` (the true one) or `false` (the false one).

Let P and Q be propositions.

We will define a new proposition called “P and Q”, and use fancy mathematical notation $P \wedge Q$ for it. In words, “P and Q” is true if and only if both P and Q are true.

Here is a formal mathematical definition.

P	Q	$P \wedge Q$
`false`	`false`	`false`
`false`	`true`	`false`
`true`	`false`	`false`
`true`	`true`	`true`

Let’s talk about why is this a definition.

Addition ($+$) is an operation on numbers.

If x and y are numbers, then so is $x + y$.

The "and" operation $\wedge$ is an operation on propositions.

If P and Q are propositions, then so is $P \wedge Q$.

This means that there are certain questions about $+$ which also make sense for $\wedge$.

Here are two facts about addition.

1) “Addition is commutative” : If x and y are any real numbers, then

$$x + y = y + x.$$

2) “Addition is associative” : If x, y and z are any real numbers, then

$$x + (y + z) = (x + y) + z.$$

Aside: these facts are not true for a general mathematical operation. For example, can you show that subtraction $(-)$ is neither commutative nor associative? You will have to give explicit examples of real numbers x, y and z for which the identities fail.

You are probably not in a position to *prove* that addition is commutative on the real numbers. The first step in the proof is to show that addition is commutative on the *natural numbers*, and this is the boss level of addition world in the natural number game.

Let's think instead about how to prove that $\wedge$ is commutative. Let's prove

Theorem

The operator $\wedge$ is commutative. In other words, if P and Q are propositions, then

$$P \wedge Q = Q \wedge P.$$

How might we prove that theorem?

Well, at school you were often given questions for which there was a “method”.

I have bad news for you – at university there is often *no method*, and you have to *think*.

But fortunately, for theorems like $P \wedge Q = Q \wedge P$, there *is* a method. *Check all of the possibilities.*

Theorem

The operator $\wedge$ is commutative. In other words, if P and Q are propositions, then

$$P \wedge Q = Q \wedge P.$$

Proof.

We check all the cases.

P	Q	$P \wedge Q$	$Q \wedge P$
`false`	`false`	`false`	`false`
`false`	`true`	`false`	`false`
`true`	`false`	`false`	`false`
`true`	`true`	`true`	`true`

In every case, we see $P \wedge Q = Q \wedge P$. $\square$

This kind of proof is guaranteed to give you full marks.

One could try to prove that $P \wedge Q = Q \wedge P$ "using words", but this is less likely to give you full marks in an exam.

If you play "advanced proposition world" in the natural number game, you will see other ways of proving theorems like this.

Exercise: prove that $\wedge$ is associative.

Let’s finish this video with the definitions of the other operations we will study on propositions. All of them are in the course notes.

Or

"P or Q", with notation $P \vee Q$, is the proposition which is true exactly when either P or Q, or both, are true.

P	Q	$P \vee Q$
false	false	false
false	true	true
true	false	true
true	true	true

Not

"not P", with notation $\neg P$, is simply the "opposite" proposition to P. It sends `true` to `false`, and `false` to `true`.

P	$\neg P$
`false`	`true`
`true`	`false`

Implication

"P implies Q", with notation $P \implies Q$, is defined to mean: "If P is true, then Q is true". In other words, the only time $P \implies Q$ is false is then P is true and Q is false.

P	Q	$P \implies Q$
`false`	`false`	`true`
`false`	`true`	`true`
`true`	`false`	`false`
`true`	`true`	`true`

Exercises: Is $\implies$ commutative? Is it associative?

Logical equivalence

We say P and Q are *logically equivalent*, written $P \iff Q$, if P implies Q and Q implies P.

P	Q	$P \iff Q$
`false`	`false`	`true`
`false`	`true`	`false`
`true`	`false`	`false`
`true`	`true`	`true`

Exercise: is $\iff$ associative?

In the next video, we will do some worked examples.