Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022–2023

Introduction to University Mathematics **MATH40001/MATH40009**

Mid-module Test

Instructions: The **neatness, completeness and clarity of the answers** will contribute to the final mark.

Maths students must attempt all three questions, JMC students will only attempt the first two questions. For this module, the module code is **MATH40001 for Maths students** and **MATH40009 for JMC students**.

> In this test, you may assume any results from the course notes or lectures/videos, as long as you state them correctly (unless stated otherwise).

1. **Total: 20 Marks**

   (a) State Axiom P5 and the principle of mathematical induction. 4 Marks

   (b) Let $X = \{n \in \mathbb{N} \mid n \geq 5\}$ and let $Y = \{n \in \mathbb{N} \mid n + 2 \in X\}$. Suppose that n is a natural number not in Y. Prove that $n \in \{0, 1, 2\}$. 6 Marks

   (c) Give the definition of the multiplication on $\mathbb{N}$. 2 Marks

   (d) Prove or give a counterexample: if x, y, and z are integers, and $x + z = y + z$, then $x = y$. 4 Marks

   (e) Prove or give a counterexample: if x, y, and z are integers, and $x \cdot z = y \cdot z$, then $x = y$. 4 Marks

2. **Total: 20 Marks**

   (a) Let $x, y, z \in \mathbb{N}$ such that $z \neq 0$ and $x = y \cdot z$. Show that $y \leq x$. You may use the fact that every natural number other than zero has a predecessor. 4 Marks

   (b) Let n be a natural number with $n > 1$. A number $x \in \mathbb{Z}_n \setminus \{[0]\}$ is called a *zero divisor* if there exists a number $y \in \mathbb{Z}_n \setminus \{[0]\}$ such that $x \cdot y = [0]$. Prove that there exists a zero divisor if and only if n is composite. (You may use the previous part.) 5 Marks

   (c) Let X be a set whose elements are n for $n \in \mathbb{N}$ and the symbols n' for $n \in \mathbb{N}$. Let $\nu_X : X \to X$ be defined by

   $$\nu_X(n) = \nu(n), \quad \nu_X(n') = \nu(n)', \quad n \in \mathbb{N}.$$

   Prove that ν_X does not satisfy Axiom P5, that is, there is some proper subset $Z \subsetneq X$ such that $0 \in Z$ and $z \in Z$ implies $\nu(z) \in Z$. 4 Marks

   (d) Let X be as in the previous part. Define a partial ordering $<_X$ by:

   $$m <_X n', \forall m, n \in \mathbb{N},$$
   $$m <_X n \text{ if } m < n, \quad m' <_X n' \text{ if } m < n, \quad \forall m, n \in \mathbb{N}.$$

   (i) Show that $x < \nu(x)$ for all $x \in X$. 3 Marks

   (ii) Prove that $<_X$ satisfies the well ordering principle: for every subset $Y \subseteq X$, there is a least element. 4 Marks

3. $\boxed{\textbf{Total: 20 Marks}}$

(a) Let $\mathbf{u}, \mathbf{v}$, be two **non-zero** vectors in $\mathbb{R}^2$.

i. Define the determinant, det$(\mathbf{u}, \mathbf{v})$, of the two vectors. $\boxed{\text{2 Marks}}$

ii. Prove that $(\mathbf{u} \cdot \mathbf{v})^2 + (\text{det}(\mathbf{u}, \mathbf{v}))^2 = |\mathbf{u}|^2|\mathbf{v}|^2$. $\boxed{\text{2 Marks}}$

iii. Verify this identity in the case where $\mathbf{u} = (\sqrt{3}, 1)$ and $\mathbf{v} = (\sqrt{6} - \sqrt{2}, \sqrt{6} + \sqrt{2})$. $\boxed{\text{4 Marks}}$

(b) Let $\mathbf{a} = (1, 3, -2)$, $\mathbf{b} = (3, 4, 2)$ and $\mathbf{c} = (0, 6, -4)$ be the position vectors of the points A, B and C in $\mathbb{R}^3$.

i. Find the volume of the parallelepiped formed by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. $\boxed{\text{3 Marks}}$

ii. Show that A, B and C lie in a unique plane. $\boxed{\text{2 Marks}}$

iii. Find a cartesian equation for this plane. $\boxed{\text{4 Marks}}$

iv. The points A, B, C and D are the four corners of a parallelogram in $\mathbb{R}^3$. Find all possible position vectors of the fourth corner D. $\boxed{\text{3 Marks}}$