Composition of functions

What makes functions so much fun is that you can *compose* them. Sometimes.

Example: $\sin : \mathbb{R} \to \mathbb{R}$ and $\cos : \mathbb{R} \to \mathbb{R}$.

Two functions. How can we "put them together"?

Examples:

- $a(x) = \sin(x) + \cos(x)$
- $b(x) = \sin(x)\cos(x)$
- $c(x) = \sin(\cos(x))$.

Which of those ideas "works in the greatest generality"? Pause the video and see if you can make any sense of that question.

- $a(x) = \sin(x) + \cos(x)$
- $b(x) = \sin(x)\cos(x) = \sin(x) \times \cos(x)$
- $c(x) = \sin(\cos(x))$.

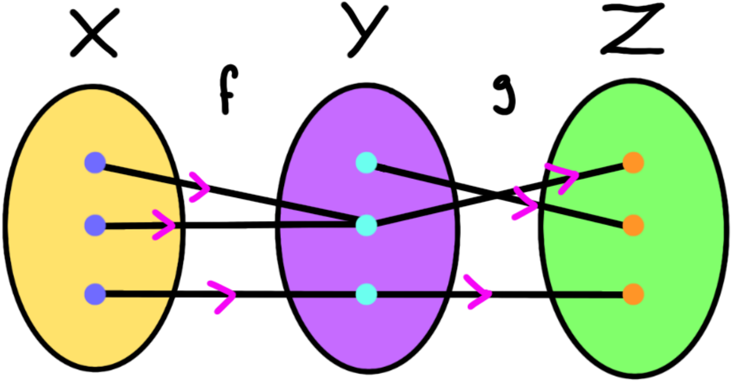
The thing about a is that it relies on the fact that we can add real numbers.

The thing about b is that it uses multiplication (even though we drop the notation).

c doesn't use any new concepts – it "chains" the functions together.

When we move to general sets X and Y, it is "chaining", otherwise known as "composition of functions", which is guaranteed to work.

Say X and Y and Z are sets, and $f : X \to Y$ and $g : Y \to Z$ are functions.



How can we make a function from X to Z?

Say X and Y and Z are sets, and $f : X \to Y$ and $g : Y \to Z$ are functions.
We define a new function from X to Z, by sending x to $g(f(x))$.

Check this makes sense.

Annoying thing: we "do f first, then g".

But the answer is $g(f(x))$.

Unfortunate but standard notation: $g \circ f$ means "do f, then g".

$$(g \circ f)(x) := g(f(x)).$$

Say X and Y and Z are sets, and $f : X \to Y$ and $g : Y \to Z$ are functions.

The composition of f and g is the function $(g \circ f) : X \to Z$.

Important: Note that this *only makes sense* if the codomain of f is equal to the domain of g.

Interlude: matrices

If M is a 2×3 matrix and $v \in \mathbb{R}^3$, then multiplication Mv makes sense, and gives a vector in $\mathbb{R}^2$. Thus M may be thought of as a function from $\mathbb{R}^3$ to $\mathbb{R}^2$.

If P is a 4×2 matrix then $P : \mathbb{R}^2 \to \mathbb{R}^4$.

The reason that the product matrix PM makes sense is because we can compose the functions. The codomain of M equals the domain of P.

The reason MP doesn't make sense is because we can't compose the functions.

The reason the product of matrices is so weird is to make function composition work.

Let X be a set.

The *identity* function $id_X : X \to X$ is defined by $f(a) = a$.

This function is often abbreviated id.

id and $\circ$ interact in a pretty straightforward way. In the next video we'll write it down.