Introduction to University Mathematics **MATH40001/MATH40009**

Test 1 - Solutions

1. (a) i. f injective: means $\forall x_1, x_2 \in X, f(x_1) = f(x_2) \implies x_1 = x_2$. **1 mark.**
   ii. f surjective: means $\forall y \in Y, \exists x \in X, f(x) = y$. **1 mark.**
   iii. f bijective: means f is injective and surjective. **1 mark.**
   (b) The composition $g \circ f$ is the function with domain X and codomain Z defined by $(g \circ f)(x) = g(f(x))$. **2 marks.**
   (c) Say $x_1, x_2 \in X$ are arbitrary and $g(f(x_1)) = g(f(x_2))$. We want to prove $x_1 = x_2$. By injectivity of g we deduce from $g(f(x_1)) = g(f(x_2))$ that $f(x_1) = f(x_2)$, and then by injectivity of f we deduce that $x_1 = x_2$. **2 marks.**
   (d) We can have $X = \{1\}$, $Y = \{1, 2\}$ and $Z = \{1\}$, and $f(1) = 1$ and $g(1) = g(2) = 1$. Then $g \circ f : X \to Z$ sends 1 to 1 and is hence injective (if x_1 and x_2 are in X then $x_1 = x_2$!), but g is not injective because $g(1) = g(2) = 1$. **3 marks**

2. (a) The cheapest way to do it is to just draw the truth table:

P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$\neg Q \implies \neg P$	$(P \implies Q) \iff (\neg Q \implies \neg P)$
false	false	true	true	true	true	true
false	true	true	false	true	true	true
true	false	false	true	false	false	true
true	true	true	false	false	true	true

   (or just skip the last column and observe that columns 3 and 6 are the same). **2 marks.**
   (b) R is true. Indeed, the logical negation of R is $\exists x \in X, x = x$ and this cannot be true because there is no element of X to be the counterexample. **2 marks.**
   (c) This is not true. For example let X be the set $\{1, 2\}$, let S be the subset $\{1\}$ and let T be the subset $\{2\}$. Then proposition A is false, because if $x = 1$ then $x \in S$ but $x \notin T$. However, proposition B is true, because $\forall x \in X, x \in S$ is false and a false proposition implies anything. **6 marks.**

3. (a) State
   i. The natural numbers $\mathbb{N}$ are a set with the following properties
      (P1) There exists a distinguished element $0 \in \mathbb{N}$.
      (P2) There exists a map $\nu : \mathbb{N} \to \mathbb{N}$ called the successor map.
      (P3) There exists no element n such that $\nu(n) = 0$.
      (P4) The map ν is injective, i.e. for all n_1, n_2 in $\mathbb{N}$ if $n_1 \neq n_2$, then $\nu(n_1) \neq \nu(n_2)$.
      (P5) Let $A \subseteq \mathbb{N}$, such that $0 \in A$ and $\nu(n) \in A$ for all $n \in A$, then $A = \mathbb{N}$. **3 mark**
   ii. Every non-empty set of natural numbers has a least element, i.e. there exists a number $a \in A$, such that $a \leq x$, for all $x \in A$. **1 mark**
   (b) Note first that $8 = 4 + 4 \in A$ by assumption. Now let $B := \{n \in \mathbb{N} \mid 8 + 3n \in A\}$. So $0 \in B$. Suppose $n \in B$. Then $8 + 3n \in A$. Then $8 + 3n + 3 = 8 + 3(n + 1) \in A$ as well. So $\nu(n) \in B$. We deduce that $B = \mathbb{N}$ by P5. Hence $A \supseteq \{8 + 3n \mid n \in \mathbb{N}\}$. **3 mark**
   (c) Assume A is not empty. Let $B = A \setminus \{0\}$, which is nonempty. By the well ordering principle it has a least element $b \in B$. Since $b \neq 0$ and $B \subseteq A$, there exists $b' \in A$ such that $b = \nu(b')$. If $b' \neq 0$ then $b' \in B$ with $b' < b$, a contradiction. So $b' = 0$. We conclude that $0 \in A$. Since $0 \in A$ and $a \in A$ implies $\nu(a) \in A$, by P5, $A = \mathbb{N}$. **3 mark**