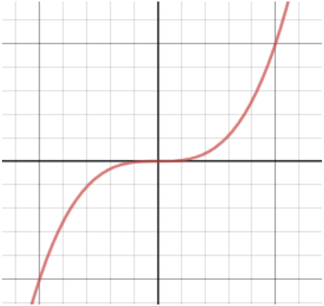
Bijections

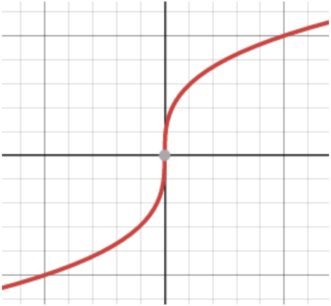
A bijection $f : X \to Y$ is a function with the property that the pre-image of any $y \in Y$ is a single element of X.

This means that it is possible to define a function g from Y to X, sending $y \in Y$ to the unique element of X such that $f(x) = y$.

Another way to think about it: if f is bijective, then it passes the horizontal line test, so meets so its reflection satisfies the vertical line test and is also a function.

The inverse of $x \mapsto x^3$ is $x \mapsto x^{1/3}$.





The graph of the inverse is the reflection of the graph of the function.

Inverse functions

Inverse functions are more confusing than they should be, because you have seen things which *look* like examples, but which are not.

Here are some confusing things.
1) If x is a real number, is $\sin^{-1}(\sin(x))$ always equal to x?
2) If y is a real number, is $\sin(\sin^{-1}(y))$ always equal to y?

Pause the video and think about these questions.

1) If x is a real number, is $\sin^{-1}(\sin(x))$ always equal to x? This cannot be true.

Why not? Because $\sin$ isn't injective. If $x_1 = \theta$ and $x_2 = \theta + 2\pi$ then $x_1 \neq x_2$ but $\sin(x_1) = \sin(x_2)$.

If $\sin(x_1) = \sin(x_2) = y$, then $\sin^{-1}(y)$ cannot be both x_1 and x_2.

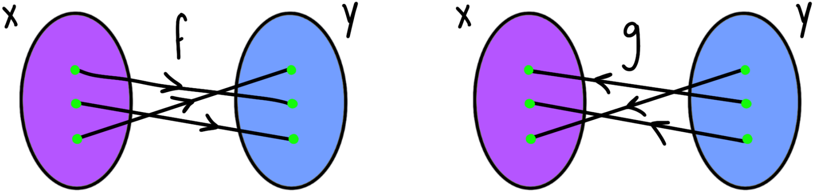
2) If y is a real number, is $\sin(\sin^{-1}(y))$ always equal to y? This cannot be true either.

Because if $y = 37$, then y isn't the sine of any real number! $\sin : \mathbb{R} \to \mathbb{R}$ *is not surjective*.

Say $f : X \to Y$ is a function.

New definition: We say that a function $g : Y \to X$ is a *two-sided inverse of f* if it satisfies *both* of the following conditions:

1) $g \circ f = id_X$ (equivalently, for all $x \in X$, $g(f(x)) = x$).
2) $f \circ g = id_Y$ (equivalently, for all $y \in Y$, $f(g(y)) = y$).



Non-example: $\sin : \mathbb{R} \to \mathbb{R}$ is a function, and $\sin^{-1}$ is *not* a two-sided inverse – indeed both conditions fail.

But we can fix this up. Let X be the closed interval $[-\pi/2, \pi/2]$ and let Y be $[-1, 1]$.

Define $f : X \to Y$ by $f(x) = \sin(x)$.

Now f *does* have a two-sided inverse, and it's $\sin^{-1}$.

$\sin : \mathbb{R} \to \mathbb{R}$ is not injective or surjective.

$\sin : [-\pi/2, \pi/2] \to [-1, 1]$ is injective and surjective.

We will finish this video by showing the relationship between being bijective and having a two-sided inverse.

Let $f : X \to Y$ be bijective.

Then for every $y \in Y$, the pre-image of y in X is exactly one element. Let's define $g(y)$ to be that element.

Then by definition, $g(y)$ is the unique element x of X such that $f(x) = y$.

I claim that g is a two-sided inverse for f. This involves checking two things.

Setup: $f : X \to Y$ a bijection, and $g(y)$ is the unique element $x \in X$ such that $f(x) = y$.

We want to prove:
1) For all $x \in X$, $g(f(x)) = x$;
2) For all $y \in Y$, $f(g(y)) = y$.

Pause the video and try and convince yourselves that these statements are true. Can you write down a proof?

Setup: $f : X \to Y$ a bijection, and $g(y)$ is the unique element $x \in X$ such that $f(x) = y$.

2) How to prove that for all $y \in Y$, $f(g(y)) = y$?

This one is not too hard! By definition, $g(y)$ satisfies $f(g(y)) = y$, which is what we want.

1) How to prove that for all $x \in X$, $g(f(x)) = x$?

We know that $g(y)$ is the unique element z of X such that $f(z) = y$.

So we know that $g(f(x))$ is the unique element z of X such that $f(z) = f(x)$.

By uniqueness, z must be x. So $g(f(x)) = x$.

Rather nicely, the converse is also true.

Theorem

Say $f : X \to Y$ is a function and g is a two-sided inverse for f. Then f is a bijection.

What do we have to do to prove this theorem?
Well, we know that g is a two-sided inverse for f, so we know

$$\forall x, g(f(x)) = x$$

and

$$\forall y, f(g(y)) = y.$$

We have to prove f is injective and surjective. So the proof is in two parts.

The proof of injectivity of f will only use $\forall x, g(f(x)) = x$.
The proof of surjectivity of g will only use $\forall y, f(g(y)) = y$.

Recall the set-up: $f : X \to Y$ and $g : Y \to X$ is a two-sided inverse of f.
Let's prove that $\forall a, b \in X, f(a) = f(b) \implies a = b$.

How do we prove a "for all a and b" statement? We let a and b be arbitrary elements of X, and we now need to prove $f(a) = f(b) \implies a = b$.

How do we prove an implies statement? We assume $f(a) = f(b)$, and let's try to deduce $a = b$.

Well, we know $f(a) = f(b)$, so we can apply g and deduce $g(f(a)) = g(f(b))$.

But because g is a two-sided inverse for f, $g(f(a)) = a$ and $g(f(b)) = b$.

Hence $a = b$ and we're done.

Recall the set-up: $f : X \to Y$ and $g : Y \to X$ is a two-sided inverse of f.
Finally, let's prove that $\forall y \in Y, \exists x \in X, f(x) = y$.

So let y be an arbitrary element of Y.

Our goal is now to prove that there exists $x \in X$ such that $f(x) = y$.

We have an element y of Y – how are we going to build an element x of X?

$$f : X \to Y$$
$$g : Y \to X$$
$$y \in Y$$

This is the data we have, and we also know that g is a two-sided inverse for f.

We are trying to prove that there exists some $x \in X$ (which could depend on f and g and y), such that $f(x) = y$.

Let's try setting $x = g(y)$.

Then we have to prove $f(g(y)) = y$.

But this follows because g is a two-sided inverse for f. So QED.

In the Lean video I will go through a proof that a function has a two-sided inverse if and only if it is a bijection.