

Mathematics with Propositions

Let's do some examples.

Example 1) Is “or” commutative? In other words, does $P \vee Q$ always have the same truth value as $Q \vee P$? Here is a partially filled in truth table:

P	Q	$P \vee Q$	$Q \vee P$
false	false		
false	true		
true	false		
true	true		

Pause the video and fill in the gaps.

Solution:

P	Q	$P \vee Q$	$Q \vee P$
false	false	false	false
false	true	true	true
true	false	true	true
true	true	true	true

The last two columns are identical, so \vee is indeed commutative.

Example 2) Is \implies associative?

In other words, does $(P \implies Q) \implies R$ always have the same truth value as $P \implies (Q \implies R)$?

How would you go about answering this? Pause the video and have a go.

To prove that $(P \implies Q) \implies R$ always has the same truth value as $P \implies (Q \implies R)$, you would have to check all eight cases.

However, to *disprove* this, you only need to find *one* case where they differ.

Let's write F for false, and T for true.

If P , Q and R are all false, then $(P \implies Q) \implies R$ is $(F \implies F) \implies F$, which equals $T \implies F$, which equals F .

However $P \implies (Q \implies R)$ is $F \implies (F \implies F)$, which is $F \implies T$, which is T .

Hence $(P \implies Q) \implies R$ and $P \implies (Q \implies R)$ are not always the same.

Example 3) If P is a proposition, then prove that $\text{false} \wedge P$ is equivalent to false , and $\text{true} \wedge P$ is equivalent to P .

Pause the video and think about how you would do that.

Solution:

P	$\text{false} \wedge P$	$\text{true} \wedge P$
false	false	false
true	false	true

Just check the two cases P true and P false.

Example 4) You all know that if a , b and c are numbers, then $a \times (b + c) = a \times b + a \times c$. This is just “expanding out the brackets”.

This fancy name for this is *distributivity* of multiplication over addition.

Is \wedge distributive over \vee ? In other words, does $P \wedge (Q \vee R)$ always have the same truth value as $(P \wedge Q) \vee (P \wedge R)$? Pause the video and have a go.

Guaranteed method of finding out:

P	Q	R	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$
F	F	F	F	F
F	F	T	F	F
F	T	F	F	F
F	T	T	F	F
T	F	F	F	F
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T

It is true!

Can we find a less boring method?

The question is whether $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ have the same truth values.

Here is another approach. We know that P is either true or false. Let's consider the two cases separately.

If P is true, then we are being asked to compare $\text{true} \wedge (Q \vee R)$ and $(\text{true} \wedge Q) \vee (\text{true} \wedge R)$. But we just saw that $\text{true} \wedge X$ has the same truth-value as X .

So the question is asking us to compare $Q \vee R$ with $Q \vee R$. These are the same.

If however P is false, then we are being asked to compare $\text{false} \wedge (Q \vee R)$ and $(\text{false} \wedge Q) \vee (\text{false} \wedge R)$. And we just saw that $\text{false} \wedge X$ is always false.

So we are being asked to compare false with $\text{false} \vee \text{false}$. And these are also both the same.

Here's another method. We could set up the question in a computer program called `Lean`; then the question becomes a level of a puzzle game.

Important: There will be *no Lean stuff* in the exam for this course.

But people who play with the ideas of this course in Lean might find they learn more about them.

Here is what the level looks like in Lean:

```
example (P Q R : Prop) :  
  P ∧ (Q ∨ R) ↔ (P ∧ Q) ∨ (P ∧ R) :=  
begin  
  sorry  
end
```

And here is a [link](#) if you want to try the level live.

Here are some solutions. Firstly the start of a first principles solution. If you've played the natural number game, you might be able to solve this puzzle like this.

```
example (P Q R : Prop) :  
  P ∧ (Q ∨ R) ↔ (P ∧ Q) ∨ (P ∧ R) :=  
begin  
  split,  
  -- one direction manually  
  intro hPaQoR,  
  cases hPaQoR with hP hQoR,  
  cases hQoR with hQ hR,  
  left,  
  split,  
  | exact hP,  
  | exact hQ,  
  right,  
  split,  
  | exact hP,  
  | exact hR,  
  -- half way  
  sorry  
end
```

Now a term mode solution, using lambda calculus:

```
example (P Q R : Prop) :  
  P ∧ (Q ∨ R) ↔ (P ∧ Q) ∨ (P ∧ R) :=  
  (λ {hP, hQoR}, or.elim hQoR (λ hQ, or.inl (hP, hQ)) (λ hR, or.inr (hP, hR)),  
   λ h, or.elim h (λ {hP, hQ}, (hP, or.inl hQ)) $ λ {hP, hR}, (hP, or.inr hR))
```

And finally a one-line solution using a high-powered tactic (which checks all the cases):

```
import tactic  
  
example (P Q R : Prop) :  
  P ∧ (Q ∨ R) ↔ (P ∧ Q) ∨ (P ∧ R) :=  
begin  
  tauto!  
end
```

Let me stress again: Lean is a lot of fun, and provides new ways to do some of the questions in the intro module.

But there will be no Lean in the exam.

I run a Lean club; the first meeting of term is 8th October. This term it's on Discord. I'll send an invite around.

I believe that software such as Lean will one day change mathematics.

Example 5) Prove that $(P \implies Q) \iff (\neg Q \implies \neg P)$.

Pause the video and have a go at this.

OK so there's no surprises here, the proof is “draw a truth table”.

But what is useful about this example is that it shows a *practical* way of *changing* how you can think about a problem.

Say we wanted to prove that for every integer n , if n is even then n^2 is even.

This is straightforward – let's do it. If n is even then we can write $n = 2t$. Hence $n^2 = 4t^2 = 2(2t^2)$ is also even.

But how do we prove that if n^2 is even then n is even?

We could write $n^2 = 2t$. And now $n = \sqrt{2t}$ which isn't even obviously a whole number. The “run at the question directly” approach has problems.

We are trying to prove $(n^2 \text{ is even}) \implies (n \text{ is even})$.

But $P \implies Q$ is logically equivalent to $\neg Q \implies \neg P$, so we could try to prove that instead.

$P = "n^2 \text{ is even}"$.

So $\neg P = "n^2 \text{ is odd}"$.

$Q = "n \text{ is even}"$

So $\neg Q = "n \text{ is odd}"$.

And $\neg Q \implies \neg P$ is the statement $"n \text{ is odd} \implies n^2 \text{ is odd}"$.

Pause the video, and check that you understand why these two questions

(1) n^2 is even implies n is even

(2) n is odd implies n^2 is odd

are in some sense *the same question* – they are *logically equivalent questions*.

Also check that you can prove that if n is odd then n^2 is odd.

For why (1) and (2) are the same, I find it helpful to think about when they can fail. For (1) to fail, you need an n such that n^2 is even, but n is odd. What do you need for (2) to fail?

That is the end of the videos for the “logic” part of the course.