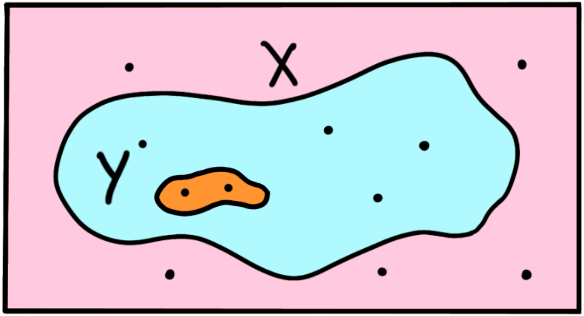
Subsets

In this video, we're going to learn about what we can do with subsets.

Formal definition. A set Y is a *subset* of a set X, if every element of Y is also an element of X.

Set notation: $Y \subseteq X$, pronounced "Y is a subset of X".



Examples of subsets.

Let's do some examples, and also introduce some new notation for subsets.

1) The prime numbers are a subset of the natural numbers. Notation for the primes:

$$\{x \in \mathbb{N} \mid x \text{ is prime } \}$$

2) The real numbers less than 37 are a subset of the real numbers. Notation:

$$\{a \in \mathbb{R} \mid a < 37\}$$

3) The set of counterexamples to the Riemann Hypothesis is a subset of the complex numbers. This is a well-defined set, but nobody knows how many elements it has.

Predicates

Let us now do the general case. Let α be a “base set”, which everything will be a subset of.

Recall that a *predicate* on α is a *family* of true-false statements $P(a)$ which depend on a parameter a in α.

For example if α is the natural numbers, $P(a)$ could be the statement “a is prime”.

We can think of this as infinitely many true-false statements: “0 is prime”, “1 is prime”, “2 is prime”, etc.

Construction of subsets

Given a base set α and a predicate $P(a)$ on α, we can make the subset of things in α where the predicate is true.

Set notation:

$$X = \{a \in \alpha \mid P(a) \text{ is true}\},$$

or just

$$X = \{a \in \alpha \mid P(a)\},$$

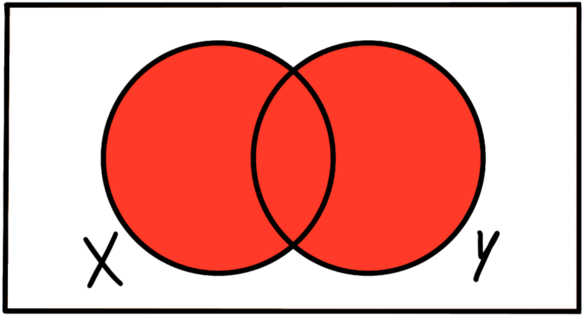
for short, means the elements of α which satisfy P (e.g. the natural numbers which are prime).

Unions

If X and Y are both subsets of a big set α, then $X \cup Y$ is set whose elements are the terms which are either in X, or in Y (or both).

Set notation: $X \cup Y := \{a \in \alpha \,|\, (a \in X) \vee (a \in Y)\}$.

Picture:

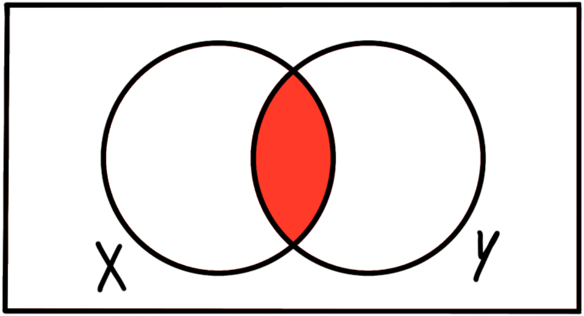


Intersections

If X and Y are both subsets of a big set α, then $X \cap Y$ is set whose elements are the terms which are X and Y.

Set notation: $X \cap Y := \{a \in \alpha \,|\, (a \in X) \wedge (a \in Y)\}$.

Picture:



Do you think distributivity of $\cup$ over $\cap$ is true? Remember what that means? I'll tell you.

It means: do you think that if X, Y and Z are all subsets of α, then

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)?$$

Maybe we could prove this by drawing some "Venn Diagrams" (i.e. pictures of sets).

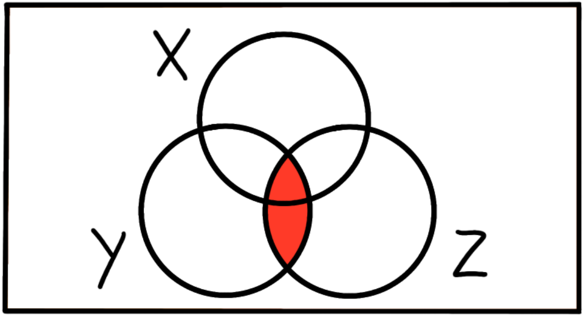
Theorem:

if X, Y and Z are all subsets of α, then

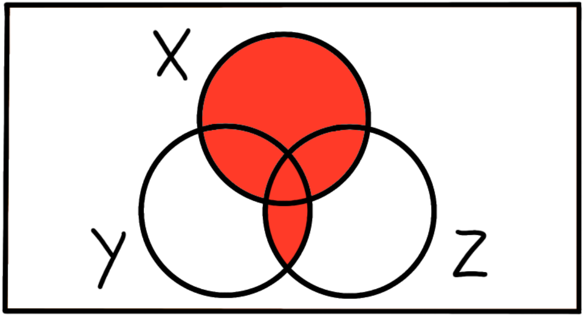
$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z).$$

Proof. First let's draw a picture of the left hand side, and then let's draw a picture of the right hand side.

$Y \cap Z$ looks like this:

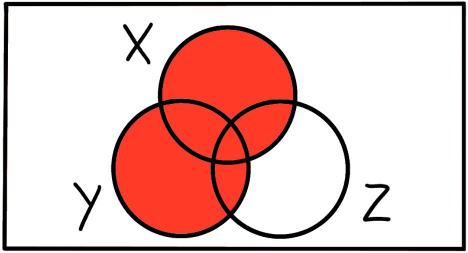


So $X \cup (Y \cap Z)$ looks like this:

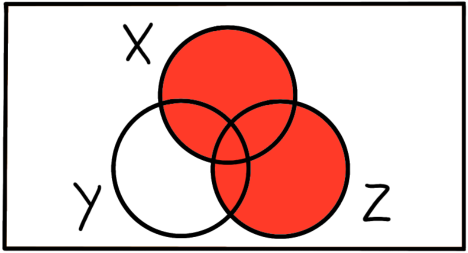


That's the left hand side. Now let's work on the right hand side $(X \cup Y) \cap (X \cup Z)$.

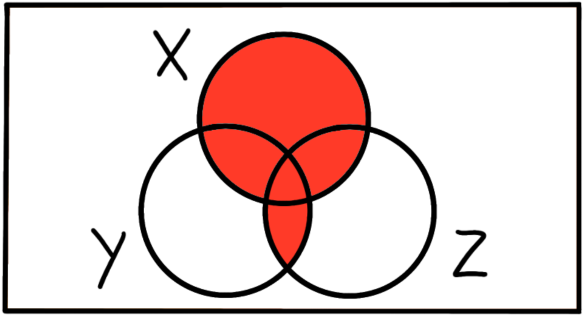
$X \cup Y$ is



and $X \cup Z$ is



so their intersection looks like



which is the same picture as the left hand side.

Do you think that this is an acceptable proof? Does it definitely cover every possible case? Why?

Theorem

If X, Y and Z are all subsets of α, then

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z).$$

Here is a *very* careful proof.

By set extensionality, it suffices to show that for all $a \in \alpha$, we have $a \in X \cup (Y \cap Z)$ if and only if $a \in (X \cup Y) \cap (X \cup Z)$.

By definition of the notation $\cup$ and $\cap$, the question is:

$$(a \in X) \vee (a \in Y \wedge a \in Z) \iff (a \in X \vee a \in Y) \wedge (a \in X \vee a \in Z)$$

If $a \in \alpha$, is it true that

$$(a \in X) \vee (a \in Y \wedge a \in Z) \iff (a \in X \vee a \in Y) \wedge (a \in X \vee a \in Z)$$

is always true? It looks like we lost control of the problem.

But if we define P to be the proposition $a \in X$, let Q be $a \in Y$ and let R be $a \in Z$, then the question becomes

$$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R).$$

Pause the video and tell tell me how we're going to proceed from there.

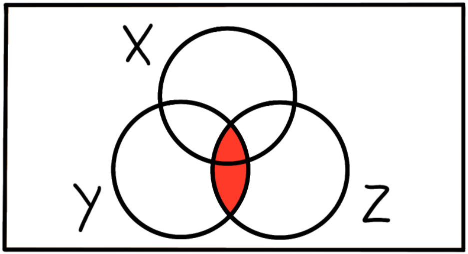
P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
F	F	F	F	F	F	F	F
F	F	T	F	F	F	T	F
F	T	F	F	F	T	F	F
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	T	F	F	T	T	T	T
T	T	T	T	T	T	T	T

We check all eight cases. It's distributivity of $\vee$ over $\wedge$. We write a long boring truth table. QED.

Let's take a look at some of the columns, and some of the pictures.

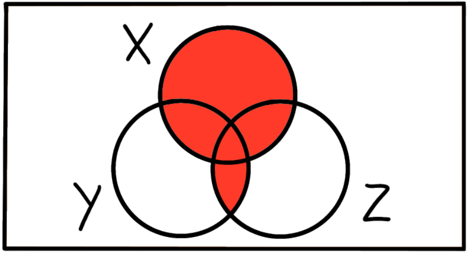
P	Q	R	$Q \wedge R$
F	F	F	F
F	F	T	F
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	F
T	T	F	F
T	T	T	T

$Y \cap Z$



P	Q	R	$P \vee (Q \wedge R)$
F	F	F	F
F	F	T	F
F	T	F	F
F	T	T	T
T	F	F	T
T	F	T	T
T	T	F	T
T	T	T	T

$X \cup (Y \cap Z)$



So we see two interesting things here.

(1) There is some kind of relationship between $\cup$ and $\vee$.
$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ and
$P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$ are related somehow.

If $\cup$ corresponds to $\vee$, what does $\cap$ correspond to? What does the empty set correspond to? What does subset inclusion correspond to? We will talk about these things in the next video.

(2) Drawing Venn Diagrams is just the same as writing out truth tables. It is presenting the same data in a different way.
Conclusion: you *can* prove these things by drawing pictures.