Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2020-2021

Introduction to University Mathematics **MATH40001/MATH40009**

Mid-module Test

Instructions: The **neatness, completeness and clarity of the answers** will contribute to the final mark. You must turn in handwritten solutions written on paper and scanned. You should upload your answers to this test as a single PDF via the Turnitin Assignment called *Whole exam dropbox* which you will find in the *Exam Paper and Dropbox(es)* folder.

IMPORTANT – Use the following naming convention for your file **AND** for the Title of your Turnitin submission: `[CID]_[ModuleCode]_full_submission.pdf`. Your first page should be the "Maths Coversheet for Submission" (which can be download from blackboard) and be completed.

Maths students must attempt all three questions, JMC students will only attempt the first two questions. For this module, the module code is **MATH40001 for Maths students** and **MATH40009 for JMC students**.
In this test, you may assume any results from the course notes or videos, as long as you state them correctly (unless stated otherwise).

1. **Total: 20 Marks**

   (a) i. Show that for all $n \in \mathbb{N}$, $n \neq \nu(n)$. 4 Marks

   ii. Show that for all $n, m \in \mathbb{N}$, $n \neq \nu(n+m)$. 4 Marks

   iii. Conclude that for all $n, m \in \mathbb{N}$ and $m \neq 0$, $n \neq n+m$. 2 Marks

   (b) Let $n, m \in \mathbb{N}$.

   i. Show that if $m \leq n$, then $\nu(m) \leq \nu(n)$. 2 Marks

   ii. Show that the relation $\leq$ is antisymmetric. 2 Marks

   iii. Show that no two of the properties $n < m$ or $n = m$ or $n > m$ can hold. 3 Marks

   (c) Let S be a nonempty subset of $\mathbb{Z}$. We say that $a \leq b$ on $\mathbb{Z}$ if $a = b + u$ for some $u \in \mathbb{N}$. Show that if there exists an element $m \in \mathbb{Z}$ such that for all $x \in S$, $x > m$, then S has a least element. 3 Marks

2. **Total: 20 Marks**

   (a) Compute $\gcd(10672, 4147)$ (show your work). 4 Marks

   (b) i. Show that if n, m and k are nonzero integers, then $k\gcd(m,n) = \gcd(km,kn)$. 5 Marks

   ii. Show that if $a \equiv b \mod m$, then $ac \equiv bc \mod mc$, for $c > 0$. 5 Marks

   (c) Let p be a prime, $p \mid n_1 n_2 \ldots n_k$, where $n_1, n_2 \ldots, n_k$ are nonzero natural numbers. Then $p \mid n_j$, for some j, such that $1 \leq j \leq k$. 6 Marks

3. **Total: 20 Marks**

   (a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors in $\mathbb{R}^3$.

   i. Prove that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. 5 Marks

   ii. Show that

   $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}.$$

   3 Marks

(b) Find the real numbers x such that the vectors $\mathbf{a} = (3, 2, x)$ and $\mathbf{b} = (2x, 4, x)$ are orthogonal. $\boxed{\text{3 Marks}}$

(c) A tetrahedron is a solid with four vertices P, Q, R and S and four triangular faces.

i. Let $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$ be vectors with lengths equal to the areas of the faces opposite the vertices P, Q, R and S respectively, and direction perpendicular to the respective faces and pointing outwards. Show that

$$\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}.$$

$\boxed{\text{5 Marks}}$

ii. Suppose that the tetrahedron has a trirectangular vertex S (this means that the three angles at S are all right angles). Let A, B and C be the areas of the three faces that meet at S, and let D be the area of the opposite face PQR. Use the result from the previous question to prove the three-dimensional version of the Pythagorean Theorem, that is, show that

$$D^2 = A^2 + B^2 + C^2.$$

$\boxed{\text{4 Marks}}$