Imperial College London

Department of Mathematics
Imperial College London
Academic Year 2021-2022

Introduction to University Mathematics **MATH40001/MATH40009**

Mid-module Test

Instructions: The **neatness, completeness and clarity of the answers** will contribute to the final mark. You must turn in handwritten solutions written on paper and scanned. You should upload your answers to this test as a single PDF via the Turnitin Assignment called *Whole exam dropbox* which you will find in the *Exam Paper and Dropbox(es)* folder.

IMPORTANT – Use the following naming convention for your file **AND** for the Title of your Turnitin submission: `[CID]_[ModuleCode]_full_submission.pdf`. Your first page should be the "Maths Coversheet for Submission" (which can be download from blackboard) and be completed.

Maths students must attempt all three questions, JMC students will only attempt the first two questions. For this module, the module code is **MATH40001 for Maths students** and **MATH40009 for JMC students**.

> In this test, you may assume any results from the course notes or videos, as long as you state them correctly (unless stated otherwise).

1. **Total: 20 Marks** In this question, in addition to material in the lecture notes and videos, you may use the existence of a predecessor of a non-zero natural number.

   (a) For this part only, you may not use any properties of multiplication beyond the definition. Show that for all x, y in $\mathbb{N}$, $\nu(x) \cdot y = x \cdot y + y$. 4 Marks

   (b) Show that, for all $x, y, z \in \mathbb{N}$,

      i. if $y \leq z$, then $x + y \leq x + z$. 4 Marks

      ii. $x < \nu(x)$. 4 Marks

      iii. if $x < y$, then $\nu(x) \leq y$. 4 Marks

      iv. if $x \cdot k = y$ and $x \cdot l = \nu(y)$, for some $k, l \in \mathbb{N}$, then $x = 1$. 4 Marks

2. **Total: 20 Marks** In this question you may again use all material in the lecture notes and videos—notably the axiom of recursion and the well-ordering principle. You may also use the existence of a predecessor of a non-zero natural number.

   (a) Prove that there exists a function $g(n) = 2^n = \underbrace{2 \cdot \ldots \cdot 2}_{n \text{ times}}$. 4 Marks

   (b) Show that $2^n > 2$ for $n \geq 2$. 3 Marks

   (c) Let $R_{f,x} : \mathbb{N} \to \mathbb{N}$ denote a recursively defined function such that $R_{f,x}(0) = x$ and $R_{f,x}(\nu(n)) = f(R_{f,x}(n))$. Prove that $R_{f,x}(2n) = R_{f \circ f, x}(n)$. 5 Marks

   (d) Using previous parts, show that $4^n = 2^{2n}$, where here $4^n = \underbrace{4 \cdot \ldots \cdot 4}_{n \text{ times}}$, defined as in the first part. 4 Marks

   (e) Prove using the well ordering property that the set $S := \{3m + 4n \mid m, n \in \mathbb{N}\}$ contains all natural numbers other than $1, 2$, and 5. 4 Marks

3. **Total: 20 Marks**

(a) Consider $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, suppose that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \sqrt{3}$. Find

i. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

ii. $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$

iii. $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w})$

iv. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$

in each case explain your reasoning. 4 Marks

(b) In $\mathbb{R}^3$, let $\mathbf{a}$ be an arbitrary vector and $\mathbf{n}$ be a unit vector in some fixed direction. Show that

$$\mathbf{a} = \mathbf{n}(\mathbf{a} \cdot \mathbf{n}) + \mathbf{n} \times (\mathbf{a} \times \mathbf{n})$$

What is the geometrical interpretation of each of the two terms in the above expression? 4 Marks

(c) Consider the two lines $\mathcal{L}_1$ with Cartesian equation $3x + 4y + 3 = 0$ and $\mathcal{L}_2$ with Cartesian equation $12x - 5y + 4 = 0$. Find the Cartesian equations of their two angle bisectors. 5 Marks

(d) In the Euclidean space $\mathbb{R}^3$, we define $\mathcal{L}_1$ as the line through the points $(2, -2, 4)$ and $(1, 1, 2)$ and $\mathcal{L}_2$ as the line through the points $(-1, 2, 0)$ and $(2, 1, 2)$.

i. Are the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ perpendicular? 1 Mark

ii. Find parametric equations for the lines $\mathcal{L}_1$ and $\mathcal{L}_2$. 2 Marks

iii. What is $d(\mathcal{L}_1, \mathcal{L}_2)$ the shortest distance between lines $\mathcal{L}_1$ and $\mathcal{L}_2$? 4 Marks