Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2019-2020

Introduction to University Mathematics **MATH40001/MATH40009**

Final Exam

There are four questions in this exam. Attempt *three* of them. If you attempt all four, your mark will be based on the *first three questions* that you answer, and we will not mark your fourth solution. You may assume, without proof, any results from the lecture notes, unless you are explicitly asked to prove them.

1. **Total: 20 Marks**

   (a) Let X be a set, and let R be a binary relation on X, so that if $x, y \in X$ are arbitrary, then $R(x, y)$ is a true/false statement. What does it mean for R to be

   i. reflexive; 1 Mark

   ii. symmetric; 1 Mark

   iii. transitive; 1 Mark

   iv. an equivalence relation? 1 Mark

   (b) Explain why the binary relation $\leq$ on the natural numbers is *not* an equivalence relation (you do not have to rigorously justify any explicit inequalities which you are claiming, as long as they are true!). 1 Mark

   (c) Now let Y be a set and let $\sim$ be an equivalence relation on Y. If $y \in Y$, define the *equivalence class* $cl(y)$ of y. 1 Mark

   (d) Prove that if $s, t \in Y$ are two elements with $s \sim t$, then $cl(t) \subseteq cl(s)$. Then prove that if $s \sim t$, then $cl(s) = cl(t)$. 6 Marks

   (e) Now let Z be the set $\{1, 2\}$.

   i. How many binary relations are there on the set Z? Justify your answer. 2 Marks

   ii. How many of these binary relations are equivalence relations? Justify your answer. 5 Marks

   iii. For each of the binary relations which are equivalence relations, give a complete list of the equivalence classes (no proof required). 1 Mark

2. **Total: 20 Marks**

   (a) i. Give the definition of the addition on $\mathbb{N}$. 1 Mark

   ii. Give the definition of the multiplication on $\mathbb{N}$. 1 Mark

   iii. Assuming only the axioms and the definition of addition and multiplication, show that $0 + x = x$ and $x \cdot 1 = x$ for any $x \in \mathbb{N}$. 4 Marks

   (b) i. State the axiom of completeness for the reals. 1 Mark

   ii. Assuming this axiom, deduce that any non-empty set S of real numbers which has a lower bound l has an infimum. 3 Marks

   iii. Using the axioms only, prove that for any $r, x, y \in \mathbb{R}$, if $r + x = 0$ and $r + y = 0$ then $x = y$. 3 Marks

   (c) Let a, n be positive integers such that $\gcd(a, n) = 1$, and let b be an integer.

   i. How many congruence classes modulo n are there? Briefly explain your answer (give just the idea, not the entire proof!). 2 Marks

   ii. Show that the equation $ax \equiv b \mod n$ has a solution $x_0 \in \mathbb{Z}$. 3 Marks

   iii. Show that any other solution to $ax \equiv b \mod n$ is congruent to x_0 modulo n. 2 Marks

3. **Total: 20 Marks**

(a) For each of the following statements, state without proof whether it is TRUE or FALSE 6 Marks

i. If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ as expressed in component form in the standard basis, then $\mathbf{u} \cdot \mathbf{v} = (u_1 v_1, u_2 v_2)$.

ii. For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$.

iii. For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

iv. For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and $\lambda, \mu \in \mathbb{R}$, $\mathbf{w} = \lambda \mathbf{u} + \mu \mathbf{v}$ implies that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0$.

v. For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$, $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w}$.

vi. If $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = 0$, then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

(b) State what it means for the dot product in $\mathbb{R}^n$ to be symmetric and bilinear. Prove it. 4 Marks

(c) Let $\mathbf{u} = (3, 4)$ and $\mathbf{v} = (-2, 1)$ be two vectors in the plane expressed in Cartesian coordinates. Calculate a measure of the oriented angle $\widehat{(\mathbf{u}, \mathbf{v})}$. 5 Marks

(d) Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vectors to show that the diagonals of the quadrilateral are perpendicular. 5 Marks

4. **Total: 20 Marks**

(a) Show that the distance from a point in $\mathbb{R}^2$ with Cartesian coordinates $P(x_0, y_0)$ to the line of Cartesian equation $ax + by + c = 0$ is

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

3 Marks

Use this to calculate the distance from the point $(-6, 7)$ to the line $4x - 3y + 5 = 0$. 1 Mark

(b) We denote $A(-3, -1)$, $B(4, 1)$, $C(-2, 3)$ and $\mathbf{u} = (1, 2)$.

i. Determine a Cartesian equation of the line $\mathcal{L}$ going through points A and B. 2 Marks

ii. Determine a parametric representation of the line $\mathcal{L}$. 2 Marks

iii. Determine a Cartesian equation of the line $\mathcal{L}'$ going through point C and parallel to vector $\mathbf{u}$. 2 Marks

iv. Calculate the area of the triangle ABC. 2 Marks

v. Show that the lines $\mathcal{L}$ and $\mathcal{L}'$ intersect and find the intersection point. 2 Marks

vi. Calculate the distance from the point A to the line $\mathcal{L}'$. 2 Marks

(c) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors in $\mathbb{R}^3$.

i. By writing it out in component form, prove that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

3 Marks

ii. Deduce from this result that the vector product is non-associative in general, i.e. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are general vectors in $\mathbb{R}^3$. 1 Mark

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Solutions - Final Exam

1. $\boxed{\textbf{Total: 20 Marks}}$

(a) i. R reflexive means: $\forall x \in X, R(x,x)$. $\boxed{\text{1 Mark}}$

ii. R symmetric means: $\forall x, y \in X, R(x,y) \implies R(y,x)$. $\boxed{\text{1 Mark}}$

iii. R transitive means: $\forall x, y, z \in X, R(x,y) \wedge R(y,z) \implies R(x,z)$. $\boxed{\text{1 Mark}}$

iv. R is an equivalence relation means: R is reflexive, symmetric and transitive. $\boxed{\text{1 Mark}}$

(b) $\leq$ is not symmetric, because $1 \leq 2$ is true but $2 \leq 1$ is false. $\boxed{\text{1 Mark}}$

(c) The equivalence class $cl(y)$ of y is defined to be $\{x \in X : y \sim x\}$ (or equivalently $\{x \in X : x \sim y\}$). $\boxed{\text{1 Mark}}$

(d) First let's prove $s \sim t \implies cl(t) \subseteq cl(s)$. The exact nature of this proof will depend on which definition of equivalence class they gave. With the first one (which was the one I gave in lectures) it comes out nicely.

Proof. Assume $s \sim t$. We want to prove that $cl(t) \subset cl(s)$, which means

$$\forall y \in Y, y \in cl(t) \implies y \in cl(s).$$

So let $y \in Y$ be arbitrary, and assume $y \in cl(t)$. Then by definition we have $t \sim y$. We also know $s \sim t$ by assumption, so by transitivity of $\sim$ we deduce $s \sim y$. This is precisely the assertion that $y \in cl(s)$. $\square$

$\boxed{\text{3 Marks}}$

Now we prove $s \sim t \implies cl(s) = cl(t)$.

Proof. Assume $s \sim t$. By the previous result, $cl(t) \subseteq cl(s)$. However by symmetry of $\sim$ we deduce $t \sim s$, so by the previous result again we deduce $cl(s) \subseteq cl(t)$. Hence $cl(s) = cl(t)$ because we have proved inclusions in both directions. $\square$

$\boxed{\text{3 Marks}}$

(e) Now let Z be the set $\{1, 2\}$.

i. A binary relation is a choice of "T" or "F" for each element of Z^2. Here Z^2 has four elements, we have to make four T/F choices, which we can do in $2^4 = 16$ different ways. Hence there are 16 binary relations on Z. $\boxed{\text{2 Marks}}$

ii. Let R be one of these binary relations. For R to be an equivalence relation we certainly need $R(1,1)$ and $R(2,2)$ to be true, so this only leaves us with $R(1,2)$ and $R(2,1)$. Next note that by symmetry, $R(1,2) \implies R(2,1)$ and $R(2,1) \implies R(1,2)$, which means that $R(1,2) \iff R(2,1)$, and we will need to decide whether $R(1,2)$ is true or false, and then $R(2,1)$ will have the same truth value.

This has cut us down to two binary relations on Z, The first one has $R(1,1) = R(2,2) = T$ and $R(1,2) = R(2,1) = F$, and this one is hence the binary relation of equality on Z. The second one has $R(1,1) = R(2,2) = R(1,2) = R(2,1) = T$ and this is the "always true" binary relation. These are the only possible equivalence relations on Z. But are either of them equivalence relations? Both relations are easily checked to be reflexive and symmetric, the only issue is transitivity. Now equality is certainly transitive – if $a = b$ and $b = c$ then $a = c$, so that gives us one equivalence relation. As for the "always true" relation, it is certainly transitive, because we need to show $R(a,b) \wedge R(b,c) \implies R(a,c)$, but $R(a,c)$ is guaranteed to be true so the implication is certainly true. 5 Marks

iii. For the equality relation, the equivalence classes are $\{1\}$ and $\{2\}$. For the "always true" relation, the one equivalence class is $\{1,2\}$. 1 Mark

2. **Total: 20 Marks**

(a) i. Definition of the addition on $\mathbb{N}$: the unique binary operation $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, i.e. the unique way to assign to each pair (x,y) of natural numbers a natural number $x+y$ such that

$$\forall x \in \mathbb{N}, x + 0 = x. \tag{1}$$
$$\forall x, y \in \mathbb{N},\ x + \nu(y) = \nu(x+y). \tag{2}$$

1 Mark

ii. Definition of the multiplication on $\mathbb{N}$: the unique binary operation $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, i.e. the unique way to assign to each pair (x,y) of natural numbers a natural number $x \cdot y$ such that

$$\forall x \in \mathbb{N}, x \cdot 0 = 0$$
$$\forall x, y \in \mathbb{N}, x \cdot \nu(y) = x \cdot y + x.$$

1 Mark

iii. Define the set

$$A = \{x \in \mathbb{N} | x = 0 + x\}.$$

Obviously $0 \in A$. Assume now that $x \in A$. Then

$$0 + \nu(x) \overset{(2)}{=} \nu(0+x) \overset{x\, in A}{=} \nu(x+0) \overset{(1)}{=} \nu(x).$$

Hence $\nu(x) \in A$, and $A = \mathbb{N}$ by axiom (P5), which means by (1) that $0 + x = x$ for all $x \in \mathbb{N}$.

For the second equality, note that $x \cdot 1 = x \cdot \nu(0) = x \cdot 0 + x = 0 + x = x$, using the definition of 1, of multiplication, and the property of addition by 0. 4 Marks

(b) i. Any nonempty subset $S \in \mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$. 1 Mark

ii. Define $-S := \{-x | x \in S\}$. We claim that $-l$ is an upper bound for $-S$. Indeed every element of $-S$ is of the form $-x$ for some $x \in S$, so $x \geq l$, and $-x \geq -l$, which is what we need. Since $-l$ is an upper bound for $-S$, $-S$ is bounded above and so has a least upper bound L. We claim that $-L$ is the greatest lower bound of S: for any $x \in S$, we have $x \leq -L$ because we have $-x \leq L$, because L is an upper bound of $-S$, which shows that $-L$ is an upper bound of S. Now suppose that C is another lower bound of S. $-C$ will be an upper bound of $-S$ (we have already seen how to show this) so $-C \geq L$ (because L is the least upper bound of $-S$) so $C \leq -L$, which shows that $-L$ is the greatest lower bound of S. 3 Marks

iii. We have

$$\begin{aligned} y &= y + 0 = y + (r + x) \text{ existence of neutral element for } + \text{ and assumption} \\ &= (y + r) + x \text{ associativity} \\ &= (r + y) + x \text{ commutativity} \\ &= 0 + x \text{ assumption} \\ &= x + 0 \text{ commutativity} \\ &= x \text{ existence of neutral element for } + \end{aligned}$$

3 Marks

(c) i. There are n congruences classes modulo n. The argument is the following: For any integer a you can consider the Euclidean division by n, namely $a = nq + r$, where $0 \leq r < n$, and hence a is always congruent to at most a number in the set $S := \{0, 1 \ldots n-1\}$ and is therefore in one of the classes $[0]_n, [1]_n, \ldots [n-1]_n$. But all this classes are distinct since if $[b]_n = [c]_n$ for some integers b and c in S, $n|b-c$ which is impossible since $b - c \leq n-1$ unless $b = c$. $\boxed{\text{2 Marks}}$

ii. Let a, n be integers such that $\gcd(a, n) = 1$. Since $\gcd(a, n) = 1$ there exists integers s and t such that $as + nt = 1$. Multiplying by b this yields $abs + nbt = b$. In other words $abs - b = nbt$ and $abs \equiv b \mod n$. Hence $bs =: x_0$ is a solution. $\boxed{\text{3 Marks}}$

iii. Assume x_1 is another solution. Then we have $ax_0 - ax_1 \equiv 0 \mod n$ or by definition of congruences $n|a(x_0 - x_1)$. But by a result proved in test 2, since $\gcd(n, a) = 1$, we must have $n|(x_0 - x_1)$, and therefore $x_0 \equiv x_1 \mod n$. $\boxed{\text{2 Marks}}$

3. $\boxed{\textbf{Total: 20 Marks}}$

(a) i. FALSE $\boxed{\text{1 Mark}}$

ii. FALSE $\boxed{\text{1 Mark}}$

iii. FALSE $\boxed{\text{1 Mark}}$

iv. TRUE $\boxed{\text{1 Mark}}$

v. FALSE $\boxed{\text{1 Mark}}$

vi. TRUE $\boxed{\text{1 Mark}}$

(b) The dot product is a symmetric bilinear form. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$, we have the following properties:

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (symmetry)

(b) $(\lambda\mathbf{u} + \mu\mathbf{w}) \cdot \mathbf{v} = \lambda\mathbf{u} \cdot \mathbf{v} + \mu\mathbf{w} \cdot \mathbf{v}$ (linearity in the first argument)

(c) $\mathbf{u} \cdot (\lambda\mathbf{v} + \mu\mathbf{w}) = \lambda\mathbf{u} \cdot \mathbf{v} + \mu\mathbf{u} \cdot \mathbf{w}$ (linearity in the second argument)

$\boxed{\text{2 Marks}}$

Proof. Let $\mathbf{u} = (u_1, u_2, \cdots, u_n)$, $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ and $\mathbf{w} = (w_1, w_2, \cdots, w_n)$ be vectors in $\mathbb{R}^n$.

☐ **Symmetry**

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= u_1v_1 + u_2v_2 + \cdots + u_nv_n \\
&= v_1u_1 + v_2u_2 + \cdots + v_nu_n \\
&= \mathbf{v} \cdot \mathbf{u}
\end{aligned}$$

☐ **Linearity in the first argument**

$$\begin{aligned}
(\lambda\mathbf{u} + \mu\mathbf{w}) \cdot \mathbf{v} &= \sum_{i=1}^{n} (\lambda\mathbf{u} + \mu\mathbf{w})_i v_i \\
&= \sum_{i=1}^{n} (\lambda u_i + \mu w_i) v_i \\
&= \sum_{i=1}^{n} \lambda u_i v_i + \mu w_i v_i \\
&= \lambda \sum_{i=1}^{n} u_i v_i + \mu \sum_{i=1}^{n} w_i v_i \\
&= \lambda\mathbf{u} \cdot \mathbf{v} + \mu\mathbf{w} \cdot \mathbf{v}
\end{aligned}$$

☐ **Linearity in the second argument** — this is done similarly to the proof of the linearity in the first argument.

☐

2 Marks

(c) To calculate this angle θ, first we compute the dot product between the vectors to obtain:

$$\begin{cases} \mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = \sqrt{3^2+4^2}\sqrt{(-2)^2+1} = 5\sqrt{5}\cos\theta \\ \mathbf{u}\cdot\mathbf{v} = (3,4)\cdot(-2,1) = 3(-2)+4(1) = -2 \end{cases}$$

So we obtain that

$$\cos\theta = -\frac{2}{5\sqrt{5}}$$

We know that at this point, we have determined the angle θ up to a sign, i.e. $\theta = \pm\arccos(-2/5\sqrt{5})$. 2 Marks

Let's now compute the determinant of these two vectors to determine the sign.

$$\det(\mathbf{u},\mathbf{v}) = \begin{vmatrix} 3 & 4 \\ -2 & 1 \end{vmatrix} = 3(1) - (-2)4 = 11 > 0$$

2 Marks

Finally, we obtained that $\cos\theta < 0$ and $\sin\theta > 0$, so the angle we are looking for is located in the second quadrant of trigonometric circle and hence:

$$\boxed{\theta = \arccos\left(-\frac{2}{5\sqrt{5}}\right)}$$

1 Mark

(d) Let the quadrilateral have vertices A, B, C and D. The diagonals can be represented by vectors $\mathbf{AC}$ and $\mathbf{BD}$.

In particular, we can write that $\mathbf{AC} = \mathbf{AB} + \mathbf{BC}$ and $\mathbf{BD} = \mathbf{BC} + \mathbf{CD}$. Since opposite sides and of same length and parallel, we know that $\mathbf{CD} = -\mathbf{AB}$.

Thus, we wrote: $\mathbf{BD} = \mathbf{BC} - \mathbf{AB}$.

2 Marks

To prove that $\mathbf{AC}$ and $\mathbf{BD}$ are orthogonal, we can compute their scalar product:

$$\begin{aligned} \mathbf{AC}\cdot\mathbf{BD} &= (\mathbf{AB}+\mathbf{BC})\cdot(\mathbf{BC}-\mathbf{AB}) \\ &= \mathbf{AB}\cdot(\mathbf{BC}-\mathbf{AB}) + \mathbf{BC}\cdot(\mathbf{BC}-\mathbf{AB}) \text{ linearity of scalarproduct} \\ &= \mathbf{AB}\cdot\mathbf{BC} - |\mathbf{AB}|^2 + |\mathbf{BC}|^2 - \mathbf{AB}\cdot\mathbf{BC} \\ &= |\mathbf{BC}|^2 - |\mathbf{AB}|^2 \end{aligned}$$

As opposite sides have the same length, we just showed that $\mathbf{AC}\cdot\mathbf{BD} = 0$. Since both of these vectors are non-zero, we have shown that the diagonals are orthogonal. 3 Marks

4. **Total: 20 Marks**

(a) There are different ways to show this; we could use scalar projections or show it like in lectures using determinants. In Cartesian coordinates, consider a line $\mathcal{L}$ going through point $A(x_1, y_1)$ and oriented by vector $\mathbf{u} = (\alpha, \beta)$. We denote H the orthogonal projection of point $P(x_0, y_0)$ on the line $\mathcal{L}$; we have that the distance from P to $\mathcal{L}$ is the length of the vector $\mathbf{HP}$. We can decompose the vector $\mathbf{AP} = \mathbf{AH} + \mathbf{HP}$. Further, we know that by bilinearity of the determinant, we have:

$$\det(\mathbf{AP},\mathbf{u}) = \det(\mathbf{AH},\mathbf{u}) + \det(\mathbf{HP},\mathbf{u})$$

As $\mathbf{AH}$ and $\mathbf{u}$ are collinear by definition of H, we then have:

$$\det(\mathbf{AP}, \mathbf{u}) = \det(\mathbf{HP}, \mathbf{u})$$

and as $\mathbf{HP}$ and $\mathbf{u}$ are orthogonal vectors, we know that

$$|\det(\mathbf{HP}, \mathbf{u})| = |\mathbf{HP}||\mathbf{u}|$$

which gives us

$$d(M, \mathcal{L}) = \frac{|\det(\mathbf{AP}, \mathbf{u})|}{|\mathbf{u}|} = \frac{|\beta(x_0 - x_1) - \alpha(y_0 - y_1)|}{\sqrt{\alpha^2 + \beta^2}}$$

So taking: $a = \beta$, $b = -\alpha$ and $c = \alpha y_1 - \beta x_1$, we obtain that the following cartesian equation for this line: $ax + by + c = 0$ and the distance to this line is

$$d(M, \mathcal{L}) = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

$\boxed{\text{3 Marks}}$

In the case of a point with coordinates $(-6, 7)$ and a line with equation $4x - 3y + 5 = 0$, we obtain a distance:

$$\frac{|4(-6) - 3(7) + 5|}{\sqrt{(4)^2 + (-3)^2}} = \frac{|-40|}{\sqrt{16 + 9}} = \frac{40}{5} = 8$$

$\boxed{\text{1 Mark}}$

(b) i. A cartesian equation for a line is of the form

$$ax + by + c = 0$$

We have seen in lectures that the vector (a, b) is a normal vector to the line; thus, the vector $(-b, a)$ is a vector parallel to the line. We are trying to determine the equation of the line going through both A and B. A vector parallel to the line is $\mathbf{AB} = (7, 2)$, so an equation would be written $2x - 7y + c = 0$. As the point A is on the line, we can determine the constant c and we get: $2(-3) - 7(-1) + c = 0 \iff c = -1$. Finally, an equation of $\mathcal{L}$ is given by

$$\boxed{2x - 7y - 1 = 0}$$

$\boxed{\text{2 Marks}}$

ii. A parametric representation of $\mathcal{L}$ is given by the set of points with position vector $\mathbf{r}$ such that $\mathbf{r}(\lambda) = \mathbf{p} + \lambda\mathbf{v}$, with $\mathbf{p}$ the position vector of a point the line is passing through and $\mathbf{v}$ a vector orienting this line. Here we find

$$\boxed{\mathbf{r}(\lambda) = (-3, -1) + \lambda(7, 2) \text{ with } \lambda \in \mathbb{R}}$$

or in component form

$$\boxed{\begin{cases} x = -3 + 7\lambda \\ y = -1 + 2\lambda \end{cases}}$$

$\boxed{\text{2 Marks}}$

iii. $\mathcal{L}'$ is the line parallel to $\mathbf{u}$ going through the point C, similarly to the reasoning in (i), we write that a directing vector for the line is $\mathbf{u} = (1, 2)$, so a vector normal to the line is given by $(-2, 1)$. This leads to equation $2x - y + c = 0$ for which we need to determine the constant c, but we know that the line goes through C so $2(-2) - 3 + c = 0 \iff c = 7$. This leads to the following cartesian equation

$$\boxed{2x - y + 7 = 0}$$

$\boxed{\text{2 Marks}}$

iv. We find the area $\mathcal{A}$ of the triangle ABC using the determinant of two vectors. We have that

$$\mathcal{A} = \frac{1}{2}|\det(\mathbf{AB}, \mathbf{AC})| = \frac{1}{2}\begin{vmatrix} 7 & 2 \\ 1 & 4 \end{vmatrix} = \frac{7(4) - 2(1)}{2} = 13$$

$\boxed{\text{2 Marks}}$

v. We want to show that $\mathcal{L}$ and $\mathcal{L}'$ intersect. We compute the determinant of their directing vectors, i.e. the determinant of the following system

$$(\mathcal{S})\begin{cases} 2x - 7y = 1 \\ 2x - y = -7 \end{cases}$$

The determinant of this system reads

$$\det\mathcal{S} = \begin{vmatrix} 2 & -7 \\ 2 & -1 \end{vmatrix} = 2(-1) - 2(-7) = 12 \neq 0$$

The determinant being non zero means that the vectors normal to the line are not parallel to each other, so the lines intersect in a single point. $\boxed{\text{1 Mark}}$
To determine this point, one needs to find the unique solution to the system of equation $(\mathcal{S})$. We can do that using Cramer's formula or by elimination and substitution, we find with Cramer's formula:

$$x = \frac{\begin{vmatrix} 1 & -7 \\ -7 & -1 \end{vmatrix}}{\det\mathcal{S}} = \frac{1(-1) - (-7)(-7)}{12} = -\frac{25}{6} \quad \text{and} \quad y = \frac{\begin{vmatrix} 2 & 1 \\ 2 & -7 \end{vmatrix}}{\det\mathcal{S}} = \frac{2(-7) - 2(1)}{12} = -\frac{4}{3}$$

So the cartesian coordinates of the intersection point are given by $(-25/6, -4/3)$. $\boxed{\text{1 Mark}}$

vi. As we have the coordinate of the point and a cartesian equation for the line we can use the result of question 4(a) to find that

$$\boxed{d(A, \mathcal{L}') = \frac{|2(-3) - (-1) + 7|}{\sqrt{2^2 + (-1)^2}} = \frac{|2|}{\sqrt{5}} = \frac{2}{\sqrt{5}}}$$

$\boxed{\text{2 Marks}}$

(c) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ three vectors in $\mathbb{R}^3$. In the standard basis $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$, we write $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$.

i. We know that in a right-handed orthonormal basis, $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$. We also know that $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$, $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$. As well, we have $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$. First, we can write directly (or rederive) that

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\hat{\mathbf{i}} + (a_3b_1 - a_1b_3)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}$$

So we obtain

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} =& [(a_2b_3 - a_3b_2)\hat{\mathbf{i}} + (a_3b_1 - a_1b_3)\hat{\mathbf{j}} + (a_1b_2 - a_2b_1)\hat{\mathbf{k}}] \times [c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}] \\ =& (a_2b_3 - a_3b_2)(c_2\hat{\mathbf{k}} - c_3\hat{\mathbf{j}}) + (a_3b_1 - a_1b_3)(c_3\hat{\mathbf{i}} - c_1\hat{\mathbf{k}}) + (a_1b_2 - a_2b_1)(c_1\hat{\mathbf{j}} - c_2\hat{\mathbf{i}}) \\ =& (a_3b_1c_3 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2)\hat{\mathbf{i}} + \\ & (a_1b_2c_1 - a_2b_1c_1 - a_2b_3c_3 + a_3b_2c_3)\hat{\mathbf{j}} + \\ & (a_2b_3c_2 - a_3b_2c_2 - a_3b_1c_1 - a_1b_3c_1)\hat{\mathbf{k}}\end{aligned}$$

If we consider the first component of that vector, we can realize that:

$$\begin{aligned}a_3b_1c_3 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2 &= a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3 - a_1b_1c_1 - a_1b_3c_3 - a_1b_2c_2 \\ &= (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (b_1c_1 + b_2c_2 + b_3c_3)a_1 \\ &= (\mathbf{a} \cdot \mathbf{c})b_1 - (\mathbf{b} \cdot \mathbf{c})a_1\end{aligned}$$

We proceed similarly with the other two components to obtain:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times\mathbf{c} =&[(\mathbf{a}\cdot\mathbf{c})b_1-(\mathbf{b}\cdot\mathbf{c})a_1]\hat{\mathbf{i}}+[(\mathbf{a}\cdot\mathbf{c})b_2-(\mathbf{b}\cdot\mathbf{c})a_2]\hat{\mathbf{j}}+[(\mathbf{a}\cdot\mathbf{c})b_3-(\mathbf{b}\cdot\mathbf{c})a_3]\hat{\mathbf{k}}\\ =&(\mathbf{a}\cdot\mathbf{c})(b_1\hat{\mathbf{i}}+b_2\hat{\mathbf{j}}+b_3\hat{\mathbf{k}})-(\mathbf{b}\cdot\mathbf{c})(a_1\hat{\mathbf{i}}+a_2\hat{\mathbf{j}}+a_3\hat{\mathbf{k}})\\ =&(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}\end{aligned}$$

$\boxed{\text{3 Marks}}$

ii. So in particular, we can write using this that on one hand $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$ and on the other hand, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=-(\mathbf{b}\times\mathbf{c})\times\mathbf{a}=-(\mathbf{b}\cdot\mathbf{a})\mathbf{c}+(\mathbf{c}\cdot\mathbf{a})\mathbf{b}$. This means that:

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}-\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}+(\mathbf{b}\cdot\mathbf{a})\mathbf{c}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}\neq\mathbf{0}$$

$\boxed{\text{1 Mark}}$