Lemmas about *id* and $\circ$

Lemma. Say X and Y are sets, and $f : X \to Y$ is a function.

Let $id_X : X \to X$ be the identity function.

Then $f \circ id_X = f$.

Before we start the proof, let's check the question makes sense. First, let's check $f \circ id_X$ makes sense.
What does this *mean*? We need to check that the codomain of id_X equals the domain of f, and this is true.

Next, let's check that $f \circ id_X$ and f have the same domain and codomain. Not hard.

OK so this proposition makes sense – let's try to prove it!
Before we start, pause the video and think about how you would prove this yourself.

Proof of the lemma.

Our goal is to prove that two functions are equal.

So it suffices to show that they take the same values on any input.

Say $a \in X$. We need to show $(f \circ id_X)(a) = f(a)$. What does that even mean?

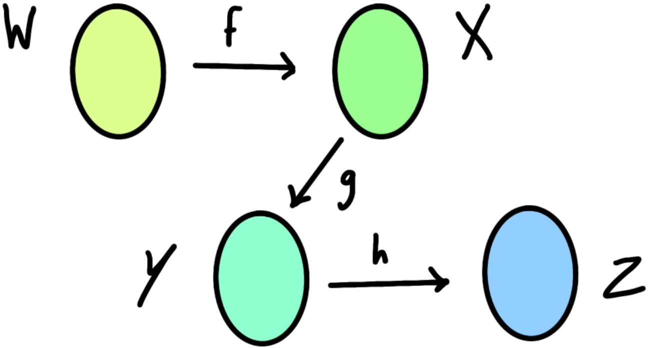
$(f \circ id_X)(a) = f(id_X(a))$ by definition. And $id_X(a) = a$ by definition. So $(f \circ id_X)(a) = f(a)$ by definition. The lemma is proved!

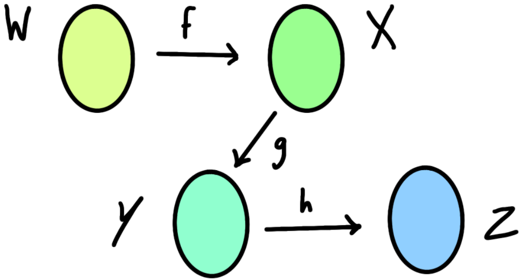
Here's one for you to try:

1. check that $id_Y \circ f$ makes sense;
2. check that $id_Y \circ f = f$ makes sense;
3. Prove that $id_Y \circ f = f$.

We end this video with one last example.

Suppose W, X, Y, Z are sets, and $f : W \to X$, $g : X \to Y$ and $h : Y \to Z$ are functions.





Here are two ways of making a function from W to Z.

1. $h \circ (g \circ f)$;
2. $(h \circ g) \circ f$.

Are these two functions equal?

Remember $5 - (2 - 1)$ isn't equal to $(5 - 2) - 1$.

Theorem. $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof. Let's evaluate both $h \circ (g \circ f)$ and $(h \circ g) \circ f)$ at $w \in W$. First the left hand side.

$(h \circ (g \circ f))(w) = h((g \circ f)(w))$ by definition of $\circ$.
And this equals $h(g(f(w)))$ by definition of $\circ$.

Similarly $(h \circ g) \circ f)(w) = (h \circ g)(f(w))$ and this equals $h(g(f(w)))$ as well.

So the functions are equal. Looking at the picture, this was inevitable. It's like a bus ticket puzzle.

In the next video we will talk about products of sets, and begin to understand how addition is a function.