Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2022–2023

Introduction to University Mathematics **MATH40001/MATH40009**

Solutions to Mid-module Test

1. **Total: 20 Marks**

   (a) *State Axiom P5 and the principle of mathematical induction.* 4 Marks

   P5: If $X \subseteq \mathbb{N}$, $0 \in X$, and for every $x \in X$, we have $\nu(x) \in X$, then $X = \mathbb{N}$.

   Induction: If $\mathcal{P}(n)$ is a statement for all $n \in \mathbb{N}$, and we know that $\mathcal{P}(0)$ is true, and $\mathcal{P}(n)$ implies $\mathcal{P}(n+1)$, then $\mathcal{P}(m)$ is true for all $m \in \mathbb{N}$.

   (b) *Let $X = \{n \in \mathbb{N} \mid n \geq 5\}$ and let $Y = \{n \in \mathbb{N} \mid n+2 \in X\}$. Suppose that n is a natural number not in Y. Prove that $n \in \{0, 1, 2\}$.* 6 Marks

   Another way to state the desired result is $Y \cup \{0, 1, 2\} = \mathbb{N}$. Let's prove this with P5. Let $Z := Y \cup \{0, 1, 2\}$. Clearly $0 \in Z$, by the definition of union. Now if $n \in Z$, then either $n \in \{0, 1, 2\}$ or else $n \in Y$.

   In the first case, $\nu(n) \in \{1, 2, 3\}$, so we just need to show $3 \in Y$. To see this we need to show that $3 + 2 \in X$. That is we need to show that $3 + 2 \geq 5$. This would be satisfied if we know that $3+2 = 5 = 5+0$. But $3+2 = 3+\nu(\nu(0)) = \nu(3+\nu(0)) = \nu(\nu(3+0)) = \nu(\nu(3)) = 5$. Thus solves the problem.

   In the second case, $n \in Y$ means that $n + 2 \geq 5$. In this case, $n + 2 = m + 5$ for some natural number m. Then $\nu(n) + 2 = \nu(n + 2) = \nu(m + 5) = \nu(m) + 5$, with $\nu(m)$ a natural number. Thus $\nu(n) \in Y \subset X$. This completes the proof.

   (c) *Give the definition of the multiplication on $\mathbb{N}$.* *2 Marks*

   This can be defined recursively. The function $f_a(b) := a \cdot b$ can be defined recursively by $f_a(0) = 0$, and $f_a(\nu(n)) = a + f_a(n)$. Or you can say that it is the unique binary operation $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that

   1) For all $x \in \mathbb{N}$, $x \cdot 0 = 0$,
   2) For all $x, y \in \mathbb{N}$, $x \cdot \nu(y) = x \cdot y + x$.

   (d) *Prove or give a counterexample: if x, y, and z are integers, and $x + z = y + z$, then $x = y$.* *4 Marks*

   This is true. One way to get the result is to add $-z$: $(x+z)+(-z) = (y+z)+(-z)$. We can apply associativity and get $x + (z + (-z)) = y + (z + (-z))$ and apply the inverse property to get $x + 0 = y + 0$, and then finally we get $x = y$.

   (e) *Prove or give a counterexample: if x, y, and z are integers, and $x \cdot z = y \cdot z$, then $x = y$.* *4 Marks*

   This is false. Indeed, for all $x, y \in \mathbb{Z}$, we have $x \cdot 0 = y \cdot 0$, but x need not equal y. A specific example is $1 \cdot 0 = 0 \cdot 0 = 0$.

2. **Total: 20 Marks**

   (a) *Let $x, y, z \in \mathbb{N}$ such that $z \neq 0$ and $x = y \cdot z$. Show that $y \leq x$. You may use the fact that every nonzero natural number has a predecessor.* *4 Marks*

   Since $z \neq 0$, we know that $z = \nu(z')$ for some $z' \in \mathbb{N}$. Now $y \cdot z = y \cdot \nu(z') = (y \cdot z') + y$. By definition of the order, this is at least y.

   Alternative proof using induction: If $z = 1$ then $x = y \cdot z = y \cdot 1 = y$, so $y \leq x$. Inductively, if $y \leq y \cdot n$, then $y \leq y \cdot n \leq y \cdot n + y = y \cdot \nu(n)$. Therefore by induction, $y \leq y \cdot n$ for all $n \geq 1$. Now $\{n \in \mathbb{N} \mid n \geq 1\} \cup \{0\} = \mathbb{N}$, by P5, since every nonzero number has a successor. Therefore for all nonzero natural n, we have $y \leq y \cdot n$, which proves the assertion.

(b) *Let n be a natural number with $n > 1$. A number $x \in \mathbb{Z}_n \setminus \{[0]\}$ is called a* zero divisor *if there exists a number $y \in \mathbb{Z}_n \setminus \{[0]\}$ such that $x \cdot y = [0]$. Prove that there exists a zero divisor if and only if n is composite. (You may use the previous part.)* $\boxed{\text{5 Marks}}$

Suppose that n is composite. Then there is a factor m of n such that $m \notin \{1, n\}$. As a result of the previous question, $1 < m < n$. Then $n = mb$ for some natural number b, and for the same reason we have $1 < b < n$. Now $[m][b] = [n] = [0]$ shows that $[m]$ and $[b]$ are zero divisors. (3 marks)

Conversely, suppose that $\mathbb{Z}$ has a zero divisor, $[n] = [m][b]$ for some $[m], [b] \in \mathbb{Z}_n$ not equal to $[0]$. Then taking repesentatives in the range $\{0, 1, \ldots, n-1\}$, we get $1 < m, b < n$. Thus $n \mid m \cdot b$. If n were prime this would imply that $n \mid m$ or $n \mid b$, contradicting the fact that $[m] \neq [0], [b] \neq [0]$. This is a contradiction. Since $n > 1$, n is composite. (2 marks)

(c) *Let X be a set whose elements are n for $n \in \mathbb{N}$ and the symbols n' for $n \in \mathbb{N}$. Let $\nu_X : X \to X$ be defined by*

$$\nu_X(n) = \nu(n), \quad \nu_X(n') = \nu(n)', \quad n \in \mathbb{N}.$$

Prove that ν_X does not satisfy Axiom P5, that is, there is some proper subset $Z \subsetneq X$ such that $0 \in Z$ and $z \in Z$ implies $\nu(z) \in Z$. $\boxed{\text{4 Marks}}$

We can let $Z := \mathbb{N} = \{n \mid n \in \mathbb{N}\}$. Then by definition the successor function is the same as for $\mathbb{N}$, so we have $0 \in Z$ and $n \in Z$ implies $\nu(n) \in Z$. But also by definition $Z \neq X$.

(d) *Let X be as in the previous part. Define a partial ordering $<_X$ by:*

$$m <_X n', \forall m, n \in \mathbb{N},$$
$$m <_X n \text{ if } m < n, \quad m' <_X n' \text{ if } m < n, \quad \forall m, n \in \mathbb{N}.$$

(i) *Show that $x < \nu(x)$ for all $x \in X$.* $\boxed{\text{3 Marks}}$

By definition of the partial ordering, this reduces to the statement for $x \in \mathbb{N}$: if $x = n'$ then $n' < \nu(n') = (\nu(n))'$ if and only if $n < \nu(n)$. But we know $n < \nu(n)$ for all $n \in \mathbb{N}$, since $\nu(n) = n + 1$.

(ii) *Prove that $<_X$ satisfies the well ordering principle: for every subset $Y \subseteq X$, there is a least element.* $\boxed{\text{4 Marks}}$

If we have a set $Y \subseteq X$ and $Y \cap \mathbb{N} \neq \emptyset$, then by the well ordering principle for $\mathbb{N}$, there is a least element for $Y \cap \mathbb{N}$. It is less than every element of the form n', hence less than every element of $Y \setminus \mathbb{N}$, hence by definition less than or equal to every element of Y. On the other hand, if $Y \cap \mathbb{N} = \emptyset$, then $Y \subseteq \mathbb{N}' := \{n' \mid n \in \mathbb{N}\}$. In this case, let $Z := \{n \in \mathbb{N} \mid n' \in Y\}$. We have a least element $z \in Z$ by the well ordering principle. By definition of the ordering on X, $z' \in Y$ is a least element for Y: for every element of Y, it is of the form n' for some $n \in \mathbb{N}$, and then $z \leq n$ implies that $z' \leq n'$.

3. $\boxed{\textbf{Total: 20 Marks}}$

(a) *Let $\mathbf{u}, \mathbf{v}$, be two **non-zero** vectors in $\mathbb{R}^2$.*

i. *Define the determinant, det$(\mathbf{u}, \mathbf{v})$, of the two vectors.* $\boxed{\text{2 Marks}}$

Let θ be the orientated angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. Then the determinant of $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\text{det}(\mathbf{u}, \mathbf{v}) = |\mathbf{u}||\mathbf{v}| \sin\theta.$$

ii. *Prove that $(\mathbf{u} \cdot \mathbf{v})^2 + (\text{det}(\mathbf{u}, \mathbf{v}))^2 = |\mathbf{u}|^2|\mathbf{v}|^2$.* $\boxed{\text{2 Marks}}$

$$\begin{aligned}
\text{LHS} &= (\mathbf{u} \cdot \mathbf{v})^2 + (\text{det}(\mathbf{u}, \mathbf{v}))^2 \\
&= |\mathbf{u}|^2|\mathbf{v}|^2 \cos^2\theta + |\mathbf{u}|^2|\mathbf{v}|^2 \sin^2\theta \\
&= |\mathbf{u}|^2|\mathbf{v}|^2(\cos^2\theta + \sin^2\theta) \\
&= |\mathbf{u}|^2|\mathbf{v}|^2 = \text{RHS},
\end{aligned}$$

where we have used the definition of det$(\mathbf{u}, \mathbf{v})$ and the fact that $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ in the second line.

iii. *Verify this identity in the case where $\mathbf{u} = (\sqrt{3}, 1)$ and $\mathbf{v} = (\sqrt{6} - \sqrt{2}, \sqrt{6} + \sqrt{2})$.* *4 Marks*

Using the facts that for vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ we have

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_1 v_1 + u_2 v_2, \\ \text{det}(\mathbf{u}, \mathbf{v}) &= u_1 v_2 - u_2 v_1, \end{aligned}$$

gives

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \sqrt{3}(\sqrt{6} - \sqrt{2}) + \sqrt{6} + \sqrt{2} = 3\sqrt{2} - \sqrt{6} + \sqrt{6} + \sqrt{2} = 4\sqrt{2}, \\ \text{det}(\mathbf{u}, \mathbf{v}) &= \sqrt{3}(\sqrt{6} + \sqrt{2}) - (\sqrt{6} - \sqrt{2}) = 3\sqrt{2} + \sqrt{6} - \sqrt{6} + \sqrt{2} = 4\sqrt{2}, \end{aligned}$$

hence

$$\text{LHS} = (4\sqrt{2})^2 + (4\sqrt{2})^2 = 64.$$

Now on the other hand

$$\begin{aligned} |\mathbf{u}| &= \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2, \\ |\mathbf{v}| &= \sqrt{(\sqrt{6} - \sqrt{2})^2 + (\sqrt{6} + \sqrt{2})^2} = \sqrt{6 + 2 - 2\sqrt{2}\sqrt{6} + 6 + 2 + 2\sqrt{2}\sqrt{6}} = \sqrt{16} = 4, \end{aligned}$$

hence

$$\text{RHS} = (2)^2 \times (4)^2 = 64 = \text{LHS},$$

and so the identity is verified.

(b) *Let $\mathbf{a} = (1, 3, -2)$, $\mathbf{b} = (3, 4, 2)$ and $\mathbf{c} = (0, 6, -4)$ be the position vectors of the points A, B and C in $\mathbb{R}^3$.*

i. *Find the volume of the parallelepiped formed by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.* *3 Marks*

We compute

$$\begin{aligned} \mathbf{b} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 2 \\ 0 & 6 & -4 \end{vmatrix} \\ &= \mathbf{i}(-16 - 12) - \mathbf{j}(-12 - 0) + \mathbf{k}(18 - 0) \\ &= -28\mathbf{i} + 12\mathbf{j} + 18\mathbf{k}. \end{aligned}$$

Next we calculate

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (1, 3, -2) \cdot (-28, 12, 18) \\ &= -28 + 36 - 36 \\ &= -28. \end{aligned}$$

The volume of the parallelepiped formed by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is then

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |-28| = 28 \text{ units}^3.$$

ii. *Show that A, B and C lie in a unique plane.* *2 Marks*

Three points in $\mathbb{R}^3$ lie in a unique plane so long as they are not all colinear. Let's check this:

$$\begin{aligned} \mathbf{b} - \mathbf{a} &= (3, 4, 2) - (1, 3, -2) = (2, 1, 4), \\ \mathbf{c} - \mathbf{a} &= (0, 6, -4) - (1, 3, -2) = (-1, 3, -2), \end{aligned}$$

and we see that $\mathbf{b} - \mathbf{a} \neq \lambda(\mathbf{c} - \mathbf{a})$ for any $\lambda \in \mathbb{R}$. Hence the three points are not colinear and must therefore lie in a unique plane.

iii. *Find a cartesian equation for this plane.* $\boxed{\text{4 Marks}}$

The cartesian equation will be of the form $ax + by + cz + d = 0$. But, since we know that A, B and C lie in the plane, then we must have

$$a + 3b - 2c + d = 0, \quad (1)$$
$$3a + 4b - 2c + d = 0, \quad (2)$$
$$6b - 4c + d = 0. \quad (3)$$

Now if we take (2)-3(1) we find

$$-5b + 8c - 2d = 0, \quad (4)$$

and then (4)+2(3) leads to

$$b = 0.$$

Back substituting this into (3) gives

$$c = \frac{d}{4},$$

and then back substituting for both b and c into (1) gives

$$a = -\frac{d}{2}.$$

Hence a cartesian equation for the plane is any equation of the form

$$-\frac{d}{2}x + \frac{d}{4}z + d = 0,$$

for some non-zero $d \in \mathbb{R}$. In particular we can choose, say, $d = -4$ to give

$$2x - z - 4 = 0.$$

iv. *The points A, B, C and D are the four corners of a parallelogram in $\mathbb{R}^3$. Find all possible position vectors of the fourth corner D.* $\boxed{\text{3 Marks}}$

There are 3 possible positions of the final vertex D, these can be seen in figure 1 showing the parallelogram each of them forms in a different colour.

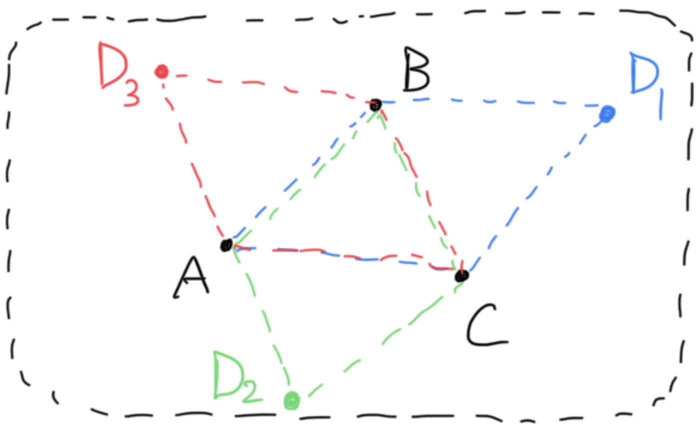

Figure 1: Possible positions of the fourth vertex D.

The position vectors of these points can be found as follows: for example, for D_1, first travel to A, then to C then to D_1. In vector form this gives

$$\mathbf{d}_1 = \mathbf{a} + (\mathbf{c} - \mathbf{a}) + (\mathbf{b} - \mathbf{a}) = \mathbf{b} + \mathbf{c} - \mathbf{a} = (2, 7, 0),$$

where we have used the fact that the vector from C to D_1 is equal to $\mathbf{b} - \mathbf{a}$ by the symmetry of the parallelogram. Similarly the other two position vectors can be found

$$\mathbf{d}_2 = \mathbf{b} + (\mathbf{a} - \mathbf{b}) + (\mathbf{c} - \mathbf{b}) = \mathbf{a} + \mathbf{c} - \mathbf{b} = (-2, 5, -8),$$
$$\mathbf{d}_3 = \mathbf{c} + (\mathbf{a} - \mathbf{c}) + (\mathbf{b} - \mathbf{c}) = \mathbf{a} + \mathbf{b} - \mathbf{c} = (4, 1, 4).$$