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In this lecture, we will look at four properties which a general binary relation may or may not have:

- Reflexivity;
- Symmetry;
- Antisymmetry;
- Transitivity.

Reflexivity

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Now here's some for you: is $\leq$ a reflexive binary relation on the reals? How about $<$? Pause the video and have a go.

$x \leq x$ is true for every real number x, so $\leq$ is reflexive.

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They proved $\forall x \in X, \neg(x < x)$. They were supposed to prove $\exists x \in X, \neg(x < x)$.

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The thing to write in a test: $\forall a, b \in X, a \star b \implies b \star a$.

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For example, $=$ is a symmetric binary relation on all X, because if $a, b \in X$ and $a = b$, then $b = a$.

Reminder: $\star$ on X is symmetric if (and only if):

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So how about $a = 1$ and $b = 37$? Then $1 \leq 37$ is true, $37 \leq 1$ is false, so $1 \leq 37 \implies 37 \leq 1$ is also false. So $\neg$ it is true!

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So $\leq$ is not symmetric.

What about the relation $P(a, b)$ on $\mathbb{N}$ where $P(a, b)$ is true if and only if a and b were both even or both odd?

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Recall that $=$ on X was symmetric. Is it also antisymmetric? Pause the video and figure out what this question *means*.

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Recall that $=$ on X was symmetric. Is it also antisymmetric? Pause the video and figure out what this question *means*.

It means "Say a and b are elements of X, and $a = b$ and $b = a$. Does $a = b$?"
This is definitely true! So equality is symmetric and antisymmetric.

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Let's make $Q(3, 4)$ and $Q(4, 3)$ both true. Then Q is not antisymmetric.

Now let $Q(1, 3)$ and all the others just be fixed (e.g. random, or all false) and we have a relation which is not symmetric or antisymmetric.

New definition: The binary relation $\star$ on X is *transitive* if for all $a, b, c \in X$ we have $(a \star b \land b \star c) \implies a \star c$.
The thing to write in a test:

$$\forall a, b, c \in X, (a \star b \land b \star c) \implies a \star c$$

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Famous examples of transitivity: $\leq$ and $<$. For example – if $a \leq b$ and $b \leq c$ then $a \leq c$.

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If you've played the natural number game, you'll know how to prove that $\leq$ is transitive on the natural numbers. What the proof of transitivity of $\leq$ looks like on the real numbers depends on what you think the real numbers are.

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$$\forall a, b, c \in X, (a = b \land b = c) \implies a = c$$

.
This is Euclid's first [common notion](#)!

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Nobody knows!

Only known algorithms to compute this take far too long to run on a normal computer. See this link for what is known. Transitivity of a random relation can be hard to check.

So why don't we give it a go! Recall that transitivity is:

$$\forall a, b, c \in X, (a \star b \wedge b \star c) \implies a \star c.$$

Remember the random relation R on $\{A, B, C\}$ defined by $R(A, B)$ and $R(A, C)$ are true, and everything else is false? Is that relation transitive?

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Pause the video and have a go yourself.

Transitivity: $\forall x, y, z \in X, (x \star y \land y \star z) \implies x \star z$.

Consider the random relation R on $\{A, B, C\}$ defined by $R(A, B)$ and $R(A, C)$ are true, and everything else is false: is R transitive?

Here's a proof.

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Is there something wrong with that proof? Pause the video and have a think.

I'll show you how to prove the *R* transitivity question in the Lean video.

In all of the rest of the videos, we will be focussing on relations which are *reflexive* and *transitive*. But symmetry and antisymmetry pull in different directions. We should only assume one. The red pill or the blue pill.