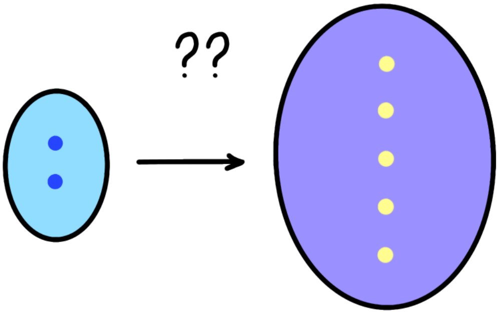
Counting

If X and Y are finite sets, then instead of fixing a function $f : X \to Y$ we could instead consider *all* functions from X to Y.

Question: how many functions are there from a set of size 2 to a set of size 5? Pause the video and try and figure it out.



Remember, functions are extensional – they are determined by their values.

So if $X = \{a, b\}$ and $Y = \{p, q, r, s, t\}$ then a function f from X to Y is determined by $f(a)$ and $f(b)$.

5 possibilities for $f(a) \in Y$ and 5 for $f(b) \in Y$.

So 25 possibilities for choosing both at once.

So there are $25 = 5^2$ possibilities for functions from a set of size 2 to a set of size 5.

How many functions are there from a set of size m to a set of size p? Pause the video and figure it out.

For each of the m elements of domain, we need to make one of p choices. So we have to make $p \times p \times \cdots \times p = p^m$ choices.

Here's another counting question. If α is a finite set with n elements, how many subsets does it have?

Here's one way of thinking about it.

A subset of α is the same information as a predicate on α, which is a function $\alpha \to$`Prop`.

Every proposition is either true or false, and mathematicians believe that propositions are determined by their truth values, so we need to count the functions from α to $\{\texttt{true}, \texttt{false}\}$.

The number of functions from a set of size n to a set of size 2 is 2^n. So a set of size n has 2^n subsets.

We end this video with some notation which we won't be using in this course, but which some people use.

Notation you'll never see again in this course: Let Y^X denote the set of functions from X to Y.

Let's write $|X|$ for the size of a set X. The counting result we proved in this video says that $|Y^X| = |Y|^{|X|}$, which is kind of cute.

Furthermore, the “currying” procedure we learnt about for addition says that a function $X \times Y \to Z$ can be thought of as a function from X to $\mathrm{Hom}(Y, Z)$.

So using this notation we see that there’s some kind of correspondence between $Z^{X \times Y}$ and $(Z^Y)^X$.

Does that remind you of any equation involving numbers?

For more on this, see the first example at Categorification (Wikipedia).