Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2020-2021

Introduction to University Mathematics **MATH40001/MATH40009**

Final Exam

Instructions: The **neatness, completeness and clarity of the answers** will contribute to the final mark. You must turn in handwritten solutions written on paper and scanned. You should upload your answers to this test as a single PDF via the Turnitin Assignment called *Whole exam dropbox* which you will find in the *Exam Paper and Dropbox(es)* folder.

IMPORTANT – Use the following naming convention for your file **AND** for the Title of your Turnitin submission: `[CID]_[ModuleCode]_full_submission.pdf`. Your first page should be the "Maths Coversheet for Submission" (which can be download from blackboard) and be completed.

Maths students must attempt all three questions, JMC students will only attempt the first two questions. For this module, the module code is **MATH40001 for Maths students** and **MATH40009 for JMC students**.

In this exam, you may assume any results proved in the course notes or videos unless explicitly asked to prove them.

1. **Total: 20 Marks**

   (a) Give the definitions of what it means for a binary relation R on a set X to be

   i. reflexive;
   ii. symmetric;
   iii. antisymmetric;
   iv. transitive;
   v. a partial order;
   vi. an equivalence relation. 3 Marks

   Now let X and Y be sets, and let $f : X \to Y$ be a function. In this question we will use f to construct a binary relation on one of these sets from a binary relation on the other. In part (b) we'll start with a relation on Y and construct one on X; in part (c) we'll start with a relation on X and construct one on Y.

   (b) Say S is a binary relation on Y. Define a binary relation R on X by $R(x_1, x_2) = S(f(x_1), f(x_2))$. In other words, x_1 is related to x_2 by R if and only if $f(x_1)$ is related to $f(x_2)$ by S.

   i. Prove that if S is an equivalence relation on Y, then the relation R constructed above is an equivalence relation on X. 6 Marks
   ii. Now let Y be the set $\{1\}$, and let S be the binary relation on Y defined by $S(1, 1) =$ `true`. Check that S is a partial order on Y. 2 Marks
   iii. With Y and S as in the previous part, let X be the set $\{2, 3\}$. Define a function $f : X \to Y$ and show that if R is the binary relation defined using this S, f, X and Y as above, then R is *not* a partial order on X. 3 Marks

(c) Back to general X, Y and f. Now say T is a binary relation on X, and let's define a binary relation U on Y by saying $U(y_1, y_2)$ is true if and only if for every x_1 such that $f(x_1) = y_1$ and for every x_2 such that $f(x_2) = y_2$, we have that $T(x_1, x_2)$ is true.

i. Prove that if T is symmetric, then U is symmetric. $\boxed{\text{2 Marks}}$

ii. Give an explicit example of sets X and Y, a function $f : X \to Y$ and a binary relation T on X which is reflexive, such that U is not reflexive. $\boxed{\text{4 Marks}}$

2. $\boxed{\textbf{Total: 20 Marks}}$

(a) i. Define multiplication on the natural numbers $\mathbb{N}$. $\boxed{\text{1 Mark}}$

ii. Assuming only the axioms and the definition of addition and multiplication, show that $0 \cdot n = 0$, for all $n \in \mathbb{N}$. $\boxed{\text{3 Marks}}$

iii. Let $S \subset \mathbb{N}$. If S has a least element, then prove that it is unique. $\boxed{\text{2 Marks}}$

(b) i. Show that if $p \in \mathbb{N}$, $p > 1$ is prime, then the equation $ax \equiv 1 \mod p$ has a solution for any $a \not\equiv 0 \mod p$. $\boxed{\text{2 Marks}}$

ii. Show that for all $x \in \mathbb{R}$, $x > 0$ there exists a natural number n such that $nx > 1$. $\boxed{\text{1 Mark}}$

iii. Let $n \in \mathbb{N} - \{0\}$. Show that if $x \in I_n := [0, \frac{1}{n}]$ for all $n \in \mathbb{N}$, then $x = 0$. $\boxed{\text{3 Marks}}$

(c) i. Prove that if $s \in \mathbb{R}$, $s > 0$, then $s^{-1} > 0$. You may assume that $0 < 1$. $\boxed{\text{3 Marks}}$

ii. Let x, y be real numbers with $x < y$. Let $s \in \mathbb{R}$, $s > 0$ and $s \notin \mathbb{Q}$ (you can assume such a number exists). Show that there exists a number $r \in \mathbb{Q}$, such that $x < rs < y$. $\boxed{\text{3 Marks}}$

iii. Show that between any two distinct real numbers, there is an irrational number. $\boxed{\text{2 Marks}}$

3. $\boxed{\textbf{Total: 20 Marks}}$

(a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors in $\mathbb{R}^3$, and suppose that $\mathbf{a} \neq \mathbf{0}$. Prove whether the following statements are TRUE or FALSE. $\boxed{\text{3 Marks}}$

i. $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \Rightarrow \mathbf{b} = \mathbf{c}$

ii. $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \Rightarrow \mathbf{b} = \mathbf{c}$

iii. $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \wedge \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \Rightarrow \mathbf{b} = \mathbf{c}$

(b) Consider the space curve parametrized by the following vector function $\mathbf{r}(t) = t^2\hat{\mathbf{i}} + (\sin t - t\cos t)\hat{\mathbf{j}} + (\cos t + t\sin t)\hat{\mathbf{k}}$:

i. Find the length of the curve on the interval $0 \leq t \leq 2$; $\boxed{\text{3 Marks}}$

ii. Find the unit tangent vector $\mathbf{T}(t)$ to the curve; $\boxed{\text{1 Mark}}$

iii. Find the unit normal vector $\mathbf{N}(t)$ to the curve; $\boxed{\text{2 Marks}}$

iv. Find its curvature. $\boxed{\text{2 Marks}}$

(c) A particle starts at the origin with an initial velocity $\mathbf{v}_0 = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and its acceleration is given by $\mathbf{a}(t) = 6t\hat{\mathbf{i}} + 12t^2\hat{\mathbf{j}} - 6t\hat{\mathbf{k}}$. What is the trajectory of the particle? $\boxed{\text{3 Marks}}$

(d) Consider the vector function $\mathbf{r}(t) \in \mathbb{R}^3$. We define $\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$. Show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)]$$

Justify all of your steps. $\boxed{\text{2 Marks}}$

(e) Consider the line defined by $\mathbf{r} \times \mathbf{a} = \mathbf{b}$, where $\mathbf{a} \cdot \mathbf{b} = 0$ and the plane $\mathbf{r} \cdot \mathbf{n} = c$. What are the conditions for this line and this plane to intersect at a unique point? Explain geometrically what this condition means. What are the conditions for the line and the plane to have no intersection point? An infinite number of intersection points? $\boxed{\text{4 Marks}}$

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Solutions to Final Exam

1. $\boxed{\textbf{Total: 20 Marks}}$

   (a) i. R is reflexive if $\forall a \in X, R(a,a)$.
   ii. R is symmetric if $\forall a,b \in X, R(a,b) \implies R(b,a)$.
   iii. R is antisymmetric if $\forall a,b \in X, R(a,b) \wedge R(b,a) \implies a=b$.
   iv. R is transitive if $\forall a,b,c \in X, R(a,b) \wedge R(b,c) \implies R(a,c)$.
   v. R is a partial order if it's reflexive, antisymmetric and transitive.
   vi. R is an equivalence relation if it's reflexive, symmetric and transitive.

   (b) i. Reflexivity: Let $a \in X$ be arbitrary. We must prove $R(a,a)$. But $R(a,a) = S(f(a), f(a))$, and S is reflexive, hence $S(f(a), f(a))$ is true. Thus $R(a,a)$ is true. $\boxed{\text{2 Marks}}$
   Symmetry: Let $a,b \in X$ be arbitrary and let's assume $R(a,b)$ is true. Our goal is to prove $R(b,a)$ is true. Because $R(a,b)$ is true, we know $S(f(a), f(b))$ is true. By symmetry of S, we deduce $S(f(b), f(a))$ is true. Hence $R(b,a)$ is true. Thus R is symmetric. $\boxed{\text{2 Marks}}$
   Transitivity. Let $a,b,c \in X$ be arbitrary and let's assume $R(a,b)$ and $R(b,c)$ are true. Our goal is to prove that $R(a,c)$ is true. Because $R(a,b)$ and $R(b,c)$ are true, we deduce that $S(f(a), f(b))$ and $S(f(b), f(c))$ are true. By transitivity of S, we deduce $S(f(a), f(c))$ is true. Hence $R(a,c)$ is true. Thus R is transitive. $\boxed{\text{2 Marks}}$
   Hence R is an equivalence relation.
   ii. Reflexivity: 1 is the only element of Y, so we just have to check that $S(1,1)$ is true, which it is by definition.
   Antisymmetry: Say $a,b \in Y$ and $S(a,b)$ and $S(b,a)$ are true. We want to deduce $a=b$. But $a,b \in Y$ and hence $a=b=1$, so in particular $a=b$.
   Transitivity: Say $a,b,c \in Y$ and $S(a,b)$ and $S(b,c)$ are true. We want to deduce that $S(a,c)$ is true. But $a,c \in Y$ and hence $a=c=1$, so $S(a,c) = S(1,1)$ which is true, and we are done.
   Hence S is a partial order.
   iii. Let's define $f : X \to Y$ by $f(2) = 1$ and $f(3) = 1$. Then $R(2,3) = S(1,1)$ is true, and $R(3,2) = S(1,1)$ is also true. However $2 \neq 3$. Hence R is not antisymmetric and hence not a partial order.

   (c) i. Say $a,b \in Y$ and $U(a,b)$ is true. Our goal is to prove that $U(b,a)$ is true. We know that for every $x_1, x_2 \in X$ such that $f(x_1) = a$ and $f(x_2) = b$, $T(x_1, x_2)$ is true. By symmetry of T, we deduce that for every such x_1 and x_2 we have $T(x_2, x_1)$ is true. But this is precisely the statement that $U(b,a)$ is true. Hence $U(b,a)$ is true and thus U is symmetric.
   ii. Say $X = \{2,3\}$ and $Y = \{1\}$ and $f(2) = f(3) = 1$, as in part (b). Let T on X be the equality relation: $T(a,b)$ is true iff $a=b$. Then certainly T is reflexive. However $U(1,1)$ is false, because $f(2) = 1$ and $f(3) = 1$ but $T(2,3)$ is false. Thus U is not reflexive.

2. $\boxed{\textbf{Total: 20 Marks}}$

   (a) i. Define multiplication on the natural numbers $\mathbb{N}$. $\boxed{\text{1 Mark}}$
   Multiplication on the natural numbers $\mathbb{N}$ is a binary operation $\cdot : \mathbb{N} \to \mathbb{N}$, such that for all $n \in \mathbb{N}$, $n \cdot 0 = 0$ and for all $n,m \in \mathbb{N}$, $n \cdot \nu(m) = n \cdot m + n$.

ii. Assuming only the axioms and the definition of addition and multiplication, show that $0 \cdot n = 0$, for all $n \in \mathbb{N}$ $\boxed{\text{3 Marks}}$

Proof. Define the set $A = \{n \in \mathbb{N} | 0 \cdot n = 0\}$. By definition of multiplication $0 \in A$. Assume $n \in A$, hence $0 \cdot n = 0$. Again by definition of multiplication $0 \cdot \nu(n) = 0 \cdot n + 0 = 0 \cdot n = 0$ by induction hypothesis and definition of addition. $\square$

iii. Let $S \subset \mathbb{N}$. If S has a least element, then prove that it is unique. $\boxed{\text{2 Marks}}$

Proof. Assume S has two least elements, l and l'. Thus we see that $l \leq l'$ and $l' \leq l$. But by the antisymmetry of the order $l' = l$. $\square$

(b) i. Show that if $p \in \mathbb{N}$, $p > 1$ is prime, then the equation $ax \equiv 1 \mod p$ has a solution for any $a \not\equiv 0 \mod p$. $\boxed{\text{2 Marks}}$

Proof. Assume $a \in \mathbb{Z}$, $a \not\equiv 0 \mod p$. So $\gcd(a, p) = 1$ (otherwise, it would be p since p is prime). By Bézout, $au + pv = 1$ for some integers u, v. Therefore $au \equiv 1 \mod p$ by definition of congruences. $\square$

ii. Show that for all $x \in \mathbb{R}$, $x > 0$ there exists a natural number n such that $nx > 1$. $\boxed{\text{1 Mark}}$

Proof. This is clear by the Archimedean property from lecture with $y = 1$. $\square$

iii. Let $n \in \mathbb{N} - \{0\}$. Show that if $x \in I_n := [0, \frac{1}{n}]$ for all $n \in \mathbb{N}$, then $x = 0$. $\boxed{\text{3 Marks}}$

Proof. Assume $x \in I_n$ for all n, $x \neq 0$. Then since $\mathbb{R}$ is an ordered field, then necessarily $x > 0$. By the previous part, there exists a natural number n such that $nx > 1$. But this implies that $x \notin [0, \frac{1}{n}]$ for that particular n, which is a contradiction. $\square$

(c) i. Prove that if $s \in \mathbb{R}$, $s > 0$, then $s^{-1} > 0$. You may assume that $0 < 1$ $\boxed{\text{3 Marks}}$

Proof. s^{-1} is the inverse of s, hence $ss^{-1} = 1$. Assume s^{-1} is not positive then since $<$ is a total order either $s^{-1} = 0$ or $s^{-1} < 0$. If $s^{-1} = 0$, $ss^{-1} = 0$ by a property showed in the lecture, hence we get a contradiction. If $s^{-1} < 0$, then $1 = s^{-1}s < 0 \cdot s = s \cdot 0 = 0$, by axiom (02) and again by the property mentioned above and commutativity of multiplication. But this is a contradiction. $\square$

ii. Let x, y be real numbers with $x < y$. Let $s \in \mathbb{R}$, $s > 0$ and $s \notin \mathbb{Q}$. Show that there exists a number $r \in \mathbb{Q}$, such that $x < rs < y$. $\boxed{\text{3 Marks}}$

Proof. By the previous part since $s > 0$, $s^{-1} > 0$. By axiom (02) of the reals, since $x < y$, $s^{-1} > 0$ we have that $xs^{-1} < ys^{-1}$. Now by the density of the rationals there exists a rational number r such that $xs^{-1} < r < ys^{-1}$. And again by axiom (02) $x < rs < y$. $\square$

iii. Show that between any two distinct real numbers, there is an irrational number. $\boxed{\text{2 Marks}}$

Proof. Using the previous part, we just have to show that rs is irrational. Assume it is not. then $rs = q \in \mathbb{Q}$. But then $s = qr^{-1}$ and since $r \in \mathbb{Q}$, then $r^{-1} \in \mathbb{Q}$ and consequently $qr^{-1} \in \mathbb{Q}$ which is a contradiction to the fact that s is an irrational number. $\square$

3. $\boxed{\textbf{Total: 20 Marks}}$

(a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors in $\mathbb{R}^3$, we suppose that $\mathbf{a} \neq \mathbf{0}$.

i. This statement is FALSE. Indeed, $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \Rightarrow \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$ so that $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$, which can happen if $\mathbf{b} \neq \mathbf{c}$, e.g. with $\mathbf{a} = (1, 1, 1)$, $\mathbf{b} = (1, 0, 0)$ and $\mathbf{c} = (0, 1, 0)$. $\boxed{\text{1 Mark}}$

ii. This statement is FALSE. Indeed, $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \Rightarrow \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = 0$ so that $\mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{c}$, which can happen if $\mathbf{b} \neq \mathbf{c}$. $\boxed{\text{1 Mark}}$

iii. This statement is TRUE. Indeed, $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c} \wedge \mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \Rightarrow \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times (\mathbf{b} - \mathbf{c}) = 0$ so that $\mathbf{a}$ is both perpendicular and parallel to $\mathbf{b} - \mathbf{c}$. Since $\mathbf{a} \neq 0$, the only possibility is for $\mathbf{b} - \mathbf{c} = \mathbf{0}$ so $\mathbf{b} = \mathbf{c}$. $\boxed{\text{1 Mark}}$

(b) We consider the space curve parametrized by the vector function $\mathbf{r}(t) = t^2\hat{\mathbf{i}} + (\sin t - t\cos t)\hat{\mathbf{j}} + (\cos t + t\sin t)\hat{\mathbf{k}}$:

i. We can easily obtain the derivative of the vector function as

$$\begin{aligned}\mathbf{r}(t) = t^2\hat{\mathbf{i}} &+ (\sin t - t\cos t)\hat{\mathbf{j}} + (\cos t + t\sin t)\hat{\mathbf{k}} \\ &\Rightarrow \mathbf{r}'(t) = 2t\hat{\mathbf{i}} + (\cos t + t\sin t - \cos t)\hat{\mathbf{j}} + (-\sin t + t\cos t + \sin t)\hat{\mathbf{k}} \\ &\Rightarrow \mathbf{r}'(t) = 2t\hat{\mathbf{i}} + t\sin t\hat{\mathbf{j}} + t\cos t\hat{\mathbf{k}} \\ &\Rightarrow |\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\cos^2 t + t^2\sin^2 t} \\ &\Rightarrow |\mathbf{r}'(t)| = \sqrt{5t^2} = \sqrt{5}t \quad \text{(as } t \geq 0\text{)}\end{aligned}$$

Thus, the length of the curve is given by

$$L = \int_0^2 |\mathbf{r}'(t)|dt = \int_0^2 \sqrt{5}t\,dt = \left.\frac{\sqrt{5}t^2}{2}\right|_0^2 = 2\sqrt{5}$$

$\boxed{\text{3 Marks}}$

ii. The unit tangent vector is defined as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{5}t}(2t\hat{\mathbf{i}} + t\sin t\hat{\mathbf{j}} + t\cos t\hat{\mathbf{k}}) = \frac{1}{\sqrt{5}}(2\hat{\mathbf{i}} + \sin t\hat{\mathbf{j}} + \cos t\hat{\mathbf{k}})$$

$\boxed{\text{1 Mark}}$

iii. The unit normal vector is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

We can write that

$$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}(\cos t\hat{\mathbf{j}} - \sin t\hat{\mathbf{k}}) \Rightarrow |\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{0^2 + \cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}$$

Thus,

$$\mathbf{N}(t) = \frac{1/\sqrt{5}}{1/\sqrt{5}}(\cos t\hat{\mathbf{j}} - \sin t\hat{\mathbf{k}}) = \cos t\hat{\mathbf{j}} - \sin t\hat{\mathbf{k}}$$

$\boxed{\text{2 Marks}}$

iv. At this point, the easiest way to obtain the curvature is to remember that

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1/\sqrt{5}}{\sqrt{5}t} = \frac{1}{5t}$$

Otherwise, one could obtain $\mathbf{r}''(t)$ and use the formula involving the cross product of $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$. $\boxed{\text{2 Marks}}$

(c) In this problem, we are provided the acceleration $\mathbf{a}(t)$ of the particle and a set of initial conditions for the velocity and position of the particle. We can integrate the acceleration once to obtain the velocity

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt = \int (6t\hat{\mathbf{i}} + 12t^2\hat{\mathbf{j}} - 6t\hat{\mathbf{k}})dt = 3t^2\hat{\mathbf{i}} + 4t^3\hat{\mathbf{j}} - 3t^2\hat{\mathbf{k}} + \mathbf{c}$$

where $\mathbf{c}$ is a vector constant of integration. We determine $\mathbf{c}$ using the initial condition for the velocity:

$$\mathbf{v}(t=0) = \mathbf{0} + \mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}} \Rightarrow \mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

which means that

$$\mathbf{v}(t) = (3t^2+1)\hat{\mathbf{i}} + (4t^3-1)\hat{\mathbf{j}} + (3-3t^2)\hat{\mathbf{k}}$$

Now to obtain the position of the particle as a function of time, we integrate the velocity

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = \int (3t^2+1)dt\hat{\mathbf{i}} + \int (4t^3-1)dt\hat{\mathbf{j}} + \int (3-3t^2)dt\hat{\mathbf{k}} = (t^3+t)\hat{\mathbf{i}} + (t^4-t)\hat{\mathbf{j}} + (3t-t^3)\hat{\mathbf{k}} + \mathbf{c}$$

where $\mathbf{c}$ is once again a vector constant of integration which we determine using the initial conditions on the particle position

$$\mathbf{r}(t=0) = \mathbf{0} + \mathbf{c} = \mathbf{0} \Rightarrow \mathbf{c} = \mathbf{0}$$

Finally, we conclude that the particle position is given by

$$\mathbf{r}(t) = (t^3+t)\hat{\mathbf{i}} + (t^4-t)\hat{\mathbf{j}} + (3t-t^3)\hat{\mathbf{k}}$$

$\boxed{\text{3 Marks}}$

(d) Let $\mathbf{r}(t) \in \mathbb{R}^3$, we define $\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$. We write that

$$\begin{aligned}
\mathbf{u}'(t) &= \mathbf{r}'(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)] + \mathbf{r}(t) \cdot \frac{d}{dt}[\mathbf{r}'(t) \times \mathbf{r}''(t)] \text{ (by differentiation rule for the scalar product)} \\
&= 0 + \mathbf{r}(t) \cdot \frac{d}{dt}[\mathbf{r}'(t) \times \mathbf{r}''(t)] \text{ (as } \mathbf{r}'(t) \perp \mathbf{r}'(t) \times \mathbf{r}''(t)) \\
&= \mathbf{r}(t) \cdot [\mathbf{r}''(t) \times \mathbf{r}''(t) + \mathbf{r}'(t) \times \mathbf{r}'''(t)] \text{ (by differentiation rule for the cross product)} \\
&= \mathbf{r} \cdot [\mathbf{r}'(t) \times \mathbf{r}'''(t)] \text{ (as } \mathbf{r}''(t) \times \mathbf{r}''(t) = 0)
\end{aligned}$$

$\boxed{\text{3 Marks}}$

(e) The hard part of such a problem is that neither of the equations defining the line and the plane give explicitly $\mathbf{r}$. Thus, we can't simply substitute one equation into the other... However, in lectures we have proven a useful parametric form for the equation of the line. Indeed, we can write that

$$\mathbf{r}(\lambda) = \mathbf{p} + \lambda\mathbf{a} \quad \text{with} \quad \mathbf{p} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2}$$

If we substitute this in the equation of the plane, we find an equation in λ

$$(\mathbf{p} + \lambda\mathbf{a}) \cdot \mathbf{n} = c$$

This determines a unique value of λ (i.e. a unique point of intersection) if and only if $\mathbf{a} \cdot \mathbf{n} \neq 0$. Geometrically, this means that the line is not parallel with the plane. If we had the case where $\mathbf{a} \cdot \mathbf{n} = 0$, then the line can lie in the plane and then we have infinitely many solutions, this is achieved when $\mathbf{p} \cdot \mathbf{n} = c$ or zero intersection point when $\mathbf{p} \cdot \mathbf{n} \neq c$.
$\boxed{\text{3 Marks}}$