In the last video we learnt about $\forall$, a way of saying that many statements were all true at once.

In this lecture we are going to think about how $\forall$ interacts with $\neg$ and introduce a new piece of notation – $\exists$.

Say $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the natural numbers (let's assume they start at 0).

Say P is a predicate on $\mathbb{N}$. In other words, we have a true-false statement $P(n)$ for each natural number n.

If we are *very* lucky, all the $P(n)$ are true, and we can say that $\forall n \in \mathbb{N}, P(n)$ is also true.

But what is the *opposite* statement? How can we express the idea that they are *not* all true? Cheat method:

$$\neg(\forall n \in \mathbb{N}, P(n)).$$

But let's instead think about what it *means*, for $\forall n \in \mathbb{N}, P(n)$ to be false.

Consider this statement: $\forall n \in \mathbb{N}, \neg(P(n))$.

By the *opposite* of a true-false statment Q, I just mean $\neg Q$, i.e. the true-false statement which is true if and only if Q is false.

Pause the video, and think about the following question.

If $P(0)$, $P(1)$, $P(2)$, $\ldots$ are infinitely many true/false statements, then is the opposite of $\forall n \in \mathbb{N}, P(n)$ equal to $\forall n \in \mathbb{N}, \neg(P(n))$?

This is not correct. For example imagine $P(n)$ is the statement "n is even".

Then some odd numbers exist (for example 37 is odd), so $\forall n \in \mathbb{N}, P(n)$ is false.

But some even numbers exist too (for example 2 is even), so $\forall n \in \mathbb{N}, \neg(P(n))$ is also false.

Hence these two statements $\forall n \in \mathbb{N}, P(n)$ an $\forall n \in \mathbb{N}, \neg(P(n))$ cannot be opposites of each other.

So what is the opposite of

$$\forall n \in \mathbb{N}, P(n)?$$

How can a collection of true/false statements not all be true?

This happens when *one or more of them are false*.

That's what this means:

$$\exists n \in \mathbb{N}, \neg P(n).$$

If we have a predicate $Q(a)$ on a set X, then

$$\exists a \in X, Q(a)$$

means that there *exists* an element a of X such that $Q(a)$ is true.

Note that that there could be just one element, or 101 such elements, or infinitely many elements with $Q(a)$ true. $\exists$ just means “there exists at least one”.

So the opposite of $\forall x \in X, \ldots$ ("they are all true") is $\exists x \in X, \neg \ldots$. ("There exists at least one false one").

Put it another way: if it's not true for all of them, it's false for at least one of them.

So let's play the opposites game.

The opposite of $\forall n \in \mathbb{N}, P(n)$ is $\exists n \in \mathbb{N}, \neg(P(n))$. It's also $\neg(\forall n \in \mathbb{N}, P(n))$ – these statements are logically equivalent. But the "exists" statement is easier to use or prove things about.

How about this one – what's the opposite of $\exists n \in \mathbb{N}, Q(n)$? Think about what this statement means, pause the video, and see if you can write the opposite statement as a "for all".

The opposite of $\exists n \in \mathbb{N}, Q(n)$ is $\forall n \in \mathbb{N}, \neg(Q(n))$.

Why is this? It's because if we can't find *any* $n \in \mathbb{N}$ such that $Q(n)$ is true, then $Q(n)$ must be false for every n – that's the only way this can happen.

More generally, $\neg\exists a \in X, \ldots$ is logically equivalent to $\forall a \in X, \neg \ldots$.
Let's try a trickier example.

Say $f(x, y)$ is a function of two real variables, for example $f(x, y) = x + y$.

Then $f(37, 42) > 0$ is a true-false statement.

And $\exists y \in \mathbb{R}, f(37, y) > 0$ is also a true-false statement.

So $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, f(x, y) > 0$ is also a true-false statement.

Some questions for you. Pause the video and try them.

1) Is $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0$ true or false?

2) Find a way to write the opposite of $\exists y \in \mathbb{R}, \forall x \in R, x + y > 0$ which doesn't mention $\neg$ at all.

3) Is $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y > 0$ true or false?

Solutions

1) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0$ is a true statement.

What do I have to do to prove it? I need to show that for every $x \in \mathbb{R}$, the claim

$$\exists y \in \mathbb{R}, x + y > 0$$

is true. So let $x \in \mathbb{R}$ be any real number.

To prove $\exists y \in \mathbb{R}, x + y > 0$, we need to find a real number y, which can depend on x, such that $x + y > 0$. How about $y = 37 - x$? Then $x + y = 37 > 0$.

We have hence proved $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0$, so it must be true.

2) Let's figure out the opposite of $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y > 0$.

By definition of $\neg$, the opposite is

$$\neg(\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y > 0).$$

But $\neg\exists$ can be changed to $\forall\neg$, so it's also

$$\forall y \in \mathbb{R}, \neg(\forall x \in \mathbb{R}, x + y > 0).$$

And $\neg\forall$ can be changed to $\exists\neg$, so it's also

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, \neg(x + y > 0).$$

Finally, $\neg t > 0$ is just $t \leq 0$, so the answer is

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x + y \leq 0.$$

Finally Q3. Is $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y > 0$ true or false?

The opposite of that statement is $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x + y \leq 0$. Which one of these two statements is true?

I claim that $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x + y \leq 0$ is true. Because whatever y is, we can always choose x to be $-37 - y$, and then $x + y = -37 \leq 0$.

So the logical opposite of this, which is $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y > 0$, must be false. In the Lean sets video I will show you these proofs again, in Lean.

One last thing to note here. We just showed that

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0$$

was true, and that

$$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, x + y > 0$$

was false.

This shows that $\forall$ and $\exists$ “do not commute”! We can’t just swap them around.