Imperial College London

Department of Mathematics
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Academic Year 2019-2020

Introduction to University Mathematics **MATH40001/MATH40009**

Test 1

1. (a) Let X and Y be sets, and let $f : X \to Y$ be a function.
   i. What does it mean for f to be *injective*?
   ii. What does it mean for f to be *surjective*?
   iii. What does it mean for f to be *bijective*?

   (b) Now say X, Y, Z are sets and $f : X \to Y$ and $g : Y \to Z$ are functions. Define the *composition* $g \circ f$ of f and g. It is a function – what is its domain and its codomain?

   (c) Prove that if f and g are injective, then $g \circ f$ is injective.

   (d) Give an example, with proof, to show that if $g \circ f$ is injective then g might not be injective.

2. (a) Let P and Q be propositions. Prove that the propositions $(P \implies Q)$ and $(\neg Q \implies \neg P)$ are logically equivalent.

   (b) Let X be the empty set, and let R be the proposition $\forall x \in X, x \neq x$. Is R true or false? Justify your answer!

   (c) Let X be a set, and let S and T be subsets of X. Let A be the proposition $\forall x \in X, (x \in S \implies x \in T)$, and let B be the proposition $(\forall x \in X, x \in S) \implies (\forall x \in X, x \in T)$. Are A and B logically equivalent? Either give a proof, or a counterexample.

3. (a) State
   i. the Peano axioms
   ii. the Well-ordering principle.

   (b) Let $A \subseteq \mathbb{N}$ be a set containing 3 and 4. Suppose that for every $a, b \in A$, $a + b \in A$. Prove that A contains $\{8 + 3n | n > 0\}$.

   (c) Let $A \subseteq \mathbb{N}$ be a set such that for every $a \in A$, also $\nu(a) \in A$ and for every nonzero $a \in A$ there exists $a' \in A$ such that $a = \nu(a')$. Prove that either $A = \emptyset$ or $A = \mathbb{N}$.