Introduction to University Mathematics   **MATH40001/MATH40009**

Test 2

1. Total: 10 Marks

   (a)   i. What does it mean for two integers a and b to be congruent modulo n?   1 Mark
         ii. State the quotient-remainder theorem.   1 Mark

   (b) Compute $\gcd(486, 160)$ and $\mathrm{lcm}(48, 52)$.   2 Marks

   (c) Let n be a positive integer and say $n|ab$, with a, b integers. Show that if $\gcd(n, a) = 1$, then $n|b$.   4 Marks

   (d) Show that for any integers $a, b, c, d \in \mathbb{Z}$, if $a \equiv b \mod n$ and $c \equiv d \mod n$, then $ac \equiv bd \mod n$.   2 Marks

2. Total: 10 Marks

   (a) Let S be a set. Give the definition of
        i. a lower bound of S.   1 Mark
       ii. the supremum of S.   1 Mark

   (b) Show directly from the axioms that for given $a, b \in \mathbb{R}$ with $a \neq 0$ there is exactly one x such that $ax = b$ (you can assume the cancelation rule for multiplication).   2 Marks

   (c) State the Archimedean property and show that it is equivalent to the fact that the set of natural numbers $\mathbb{N}$ is not bounded above in $\mathbb{R}$.   3 Marks

   (d) Let $S = \{x \in \mathbb{R}|x^4 < 16\}$. Find $\inf S$ (justify your answer).   3 Marks

3. Total: 10 Marks

   (a) Consider $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ vectors in $\mathbb{R}^n$. For each of these expression, state whether this operation is allowed and explain your reasoning.   2 Marks
         i. $\mathbf{u} + (\mathbf{v} \cdot \mathbf{w})$
        ii. $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$
       iii. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$
        iv. $(\mathbf{u} + \mathbf{v})|\mathbf{w}|$
         v. $|\mathbf{u} + \mathbf{v}||\mathbf{w}|$

   (b) Show that the vectors $\mathbf{u} = (1, -2, 2)$ and $\mathbf{v} = (2, 3, 2)$ are perpendicular in $\mathbb{R}^3$. Verify directly Pythagoras' Theorem for the right triangles with vertices $\mathbf{0}, \mathbf{u}$ and $\mathbf{v}$ and $\mathbf{0}, \mathbf{u}, \mathbf{u} + \mathbf{v}$.   3 Marks

   (c)   i. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Show that if $\mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}^n$ then $\mathbf{u} = \mathbf{v}$.   2 Marks
         ii. Show that for $\mathbf{u}$ and $\mathbf{v}$ two vectors of $\mathbb{R}^n$, $|\mathbf{u} + \mathbf{v}| \geq \big||\mathbf{u}| - |\mathbf{v}|\big|$. When do we get equality (state without proof)?   3 Marks