RATIONALS: VIDEO XIII MOTIVATION AND FIRST DEFINITIONS

The rationals: motivation

- Until now we only have addition, sustraction, multiplication and a notion of divisibility.

The rationals: motivation

- Until now we only have addition, sustraction, multiplication and a notion of divisibility.
- **But we cannot solve for x equations like**

$$4x = 2!$$

We want to have "$x = \frac{2}{4}$"!

The rationals: motivation

- Until now we only have addition, sustraction, multiplication and a notion of divisibility.
- **But we cannot solve for x equations like**

$$4x = 2!$$

We want to have "$x = \frac{2}{4}$"!

Idea: define $\frac{2}{4}$ as the ordered pair $(2, 4) \in \mathbb{Z} \times \mathbb{Z}$?

The rationals: motivation

- Until now we only have addition, sustraction, multiplication and a notion of divisibility.
- **But we cannot solve for x equations like**

$$4x = 2!$$

We want to have "$x = \frac{2}{4}$"!

Idea: define $\frac{2}{4}$ as the ordered pair $(2, 4) \in \mathbb{Z} \times \mathbb{Z}$?
Problem: $\frac{2}{4} = \frac{1}{2} = \frac{-25}{-50} \ldots \ldots \Rightarrow (2,4) = (1,2) = (-25,-50)$ so such a pair is not unique.

The rationals: motivation

- Until now we only have addition, sustraction, multiplication and a notion of divisibility.
- **But we cannot solve for x equations like**

$$4x = 2!$$

We want to have "$x = \frac{2}{4}$"!

Idea: define $\frac{2}{4}$ as the ordered pair $(2,4) \in \mathbb{Z} \times \mathbb{Z}$?
Problem: $\frac{2}{4} = \frac{1}{2} = \frac{-25}{-50} \ldots \ldots \Rightarrow (2,4) = (1,2) = (-25,-50)$ so such a pair is not unique.

★ Note: $\frac{m}{n} = \frac{m'}{n'}$ if and only if $mn' = m'n$★

What would be your next idea....?

Definition of the rationals

Our old Idea:

Define the equivalence relation on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$

$$(m, n) \sim (m', n') \text{ if and only if } mn' = m'n.$$

Definition of the rationals

Our old Idea:

Define the equivalence relation on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$

$$(m, n) \sim (m', n') \text{ if and only if } mn' = m'n.$$

$$\frac{m}{n} := cl((m, n)) = \{(x, y) \in \mathbb{Z} \times (\mathbb{Z} - \{0\}) | (x, y) \sim (m, n)\}$$

Definition of the rationals

Our old Idea:

Define the equivalence relation on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$

$$(m,n) \sim (m',n') \text{ if and only if } mn' = m'n.$$

$$\frac{m}{n} := cl((m,n)) = \{(x,y) \in \mathbb{Z} \times (\mathbb{Z} - \{0\}) | (x,y) \sim (m,n)\}$$

$\mathbb{Q}$ is the set of all equivalence classes $\frac{m}{n}$.

Definition of the rationals

Our old Idea:

Define the equivalence relation on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$

$$(m,n) \sim (m',n') \text{ if and only if } mn' = m'n.$$

$$\frac{m}{n} := cl((m,n)) = \{(x,y) \in \mathbb{Z} \times (\mathbb{Z} - \{0\}) | (x,y) \sim (m,n)\}$$

$\mathbb{Q}$ **is the set of all equivalence classes** $\frac{m}{n}$**.**

Definition of the rationals

Our old Idea:

Define the equivalence relation on $\mathbb{Z} \times (\mathbb{Z} - \{0\})$

$$(m, n) \sim (m', n') \text{ if and only if } mn' = m'n.$$

$$\frac{m}{n} := cl((m,n)) = \{(x,y) \in \mathbb{Z} \times (\mathbb{Z} - \{0\}) | (x,y) \sim (m,n)\}$$

$\mathbb{Q}$ is the set of all equivalence classes $\frac{m}{n}$.

★ Wanted:

$$\frac{m_1}{n_1} + \frac{m_2}{n_2} = \frac{m_1 n_2 + n_1 m_2}{n_1 n_2}, \qquad \frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{m_1 m_2}{n_1 n_2}$$

We define on $\mathbb{Q}$:

Addition: $cl(m_1, n_1) +_{\mathbb{Q}} cl(m_2, n_2) := cl(m_1 n_2 + n_1 m_2, n_1 n_2)$

Multiplication: $cl(m_1, n_1) \cdot_{\mathbb{Q}} cl(m_2, n_2) := cl(m_1 m_2, n_1 n_2)$.

Are you happy with this?

Are you happy with this?

Well if yes you shouldn't...because we haven't checked...

Are you happy with this?

Well if yes you shouldn't...because we haven't checked...

Proposition

Let (m_i, n_i), $(m_i', n_i') \in \mathbb{Z} \times \mathbb{Z}$, $i = 1, 2$. *If* $(m_1, n_1) \sim (m_1', n_1')$ *and* $(m_2, n_2) \sim (m_2', n_2')$, *then*

$$\begin{aligned} cl(m_1, n_1) +_{\mathbb{Q}} cl(m_2, n_2) &= cl(m_1', n_1') +_{\mathbb{Q}} cl(m_2', n_2'), \\ cl(m_1, n_1) \cdot_{\mathbb{Q}} cl(m_2, n_2) &= cl(m_1' n_1') \cdot_{\mathbb{Q}} cl(m_2', n_2') \end{aligned}$$

Are you happy with this?

Well if yes you shouldn't...because we haven't checked...

Proposition

Let (m_i, n_i), $(m'_i, n'_i) \in \mathbb{Z} \times \mathbb{Z}$, $i = 1, 2$. *If* $(m_1, n_1) \sim (m'_1, n'_1)$ *and* $(m_2, n_2) \sim (m'_2, n'_2)$, *then*

$$
\begin{aligned}
cl(m_1, n_1) +_{\mathbb{Q}} cl(m_2, n_2) &= cl(m'_1, n'_1) +_{\mathbb{Q}} cl(m'_2, n'_2), \\
cl(m_1, n_1) \cdot_{\mathbb{Q}} cl(m_2, n_2) &= cl(m'_1 n'_1) \cdot_{\mathbb{Q}} cl(m'_2, n'_2)
\end{aligned}
$$

Now you can be happy :)...

Are you happy with this?

Well if yes you shouldn't...because we haven't checked...

Proposition

Let (m_i, n_i), $(m_i', n_i') \in \mathbb{Z} \times \mathbb{Z}$, $i = 1, 2$. *If* $(m_1, n_1) \sim (m_1', n_1')$ *and* $(m_2, n_2) \sim (m_2', n_2')$, *then*

$$
\begin{aligned}
cl(m_1, n_1) +_{\mathbb{Q}} cl(m_2, n_2) &= cl(m_1', n_1') +_{\mathbb{Q}} cl(m_2', n_2'), \\
cl(m_1, n_1) \cdot_{\mathbb{Q}} cl(m_2, n_2) &= cl(m_1' n_1') \cdot_{\mathbb{Q}} cl(m_2', n_2')
\end{aligned}
$$

Now you can be happy :)...

...almost...

Are you happy with this?

Well if yes you shouldn't...because we haven't checked...

Proposition

Let (m_i, n_i), $(m'_i, n'_i) \in \mathbb{Z} \times \mathbb{Z}$, $i = 1, 2$. *If* $(m_1, n_1) \sim (m'_1, n'_1)$ *and* $(m_2, n_2) \sim (m'_2, n'_2)$, *then*

$$\begin{aligned} cl(m_1, n_1) +_{\mathbb{Q}} cl(m_2, n_2) &= cl(m'_1, n'_1) +_{\mathbb{Q}} cl(m'_2, n'_2), \\ cl(m_1, n_1) \cdot_{\mathbb{Q}} cl(m_2, n_2) &= cl(m'_1 n'_1) \cdot_{\mathbb{Q}} cl(m'_2, n'_2) \end{aligned}$$

Now you can be happy :)...

...almost...

You still need to check that $\mathbb{Z} \subseteq \mathbb{Q}$.

Are you happy with this?

Well if yes you shouldn't...because we haven't checked...

Proposition

Let (m_i, n_i), $(m_i', n_i') \in \mathbb{Z} \times \mathbb{Z}$, $i = 1, 2$. If $(m_1, n_1) \sim (m_1', n_1')$ and $(m_2, n_2) \sim (m_2', n_2')$, then

$$\begin{aligned} cl(m_1, n_1) +_{\mathbb{Q}} cl(m_2, n_2) &= cl(m_1', n_1') +_{\mathbb{Q}} cl(m_2', n_2'), \\ cl(m_1, n_1) \cdot_{\mathbb{Q}} cl(m_2, n_2) &= cl(m_1' n_1') \cdot_{\mathbb{Q}} cl(m_2', n_2') \end{aligned}$$

Now you can be happy :)...

...almost...

You still need to check that $\mathbb{Z} \subseteq \mathbb{Q}$. This is done with the proposition

Proposition

The map

$$i : \mathbb{Z} \to \mathbb{Q}, n \mapsto i(n) = \frac{n}{1}.$$

is injective and preserves addition and multiplication.

Ordering on $\mathbb{Z}$ and $\mathbb{Q}$

For the integers, we say $a - b <_{\mathbb{Z}} c - d$ if $a + d < b + c$. This does not depend on the choice of representatives, and extends the ordering on $\mathbb{N}$. We can now define an ordering on $\mathbb{Q}$.

Definition

Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$. Assuming $b, d > 0$, we say that $\frac{a}{b} <_{\mathbb{Q}} \frac{c}{d}$ if $ad < bc$.

Ordering on $\mathbb{Z}$ and $\mathbb{Q}$

For the integers, we say $a - b <_{\mathbb{Z}} c - d$ if $a + d < b + c$. This does not depend on the choice of representatives, and extends the ordering on $\mathbb{N}$. We can now define an ordering on $\mathbb{Q}$.

Definition

Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$. Assuming $b, d > 0$, we say that $\frac{a}{b} <_{\mathbb{Q}} \frac{c}{d}$ if $ad < bc$.

★ We use the symbol $<_{\mathbb{Q}}$ to denote our ordering relation in $\mathbb{Q}$ so as to distinguish it from the orderings $<_{\mathbb{Z}}$ in $\mathbb{Z}$ or $<$ in $\mathbb{N}$.

Ordering on $\mathbb{Z}$ and $\mathbb{Q}$

For the integers, we say $a - b <_{\mathbb{Z}} c - d$ if $a + d < b + c$. This does not depend on the choice of representatives, and extends the ordering on $\mathbb{N}$. We can now define an ordering on $\mathbb{Q}$.

Definition

Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$. Assuming $b, d > 0$, we say that $\frac{a}{b} <_{\mathbb{Q}} \frac{c}{d}$ if $ad < bc$.

★ We use the symbol $<_{\mathbb{Q}}$ to denote our ordering relation in $\mathbb{Q}$ so as to distinguish it from the orderings $<_{\mathbb{Z}}$ in $\mathbb{Z}$ or $<$ in $\mathbb{N}$.

Proposition

$<_{\mathbb{Q}}$ defines a total order on $\mathbb{Q}$.