WELCOME TO PART II OF THE INTRODUCTORY MODULE!

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PEANO AXIOMS: VIDEO I

The natural numbers: motivation

"Numbers are free creations of the human intellect, they serve as a means of grasping more easily and more sharply the diversity of things."
(Dedekind)

Natural numbers $= 0, 1, 2, 3, \ldots$ **???**

Question: Can we characterize the essential properties of natural numbers without refering to counting, or numbers or...arithmetic?

Things you need to look at if you don't feel comfortable with them: SETS, FUNCTIONS BETWEEN SETS, EQUIVALENCE RELATIONS

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Populating the rest of $\mathbb{N}$ is the purpose of Axiom 2–4...

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$$\heartsuit =: 0, \nu(\heartsuit) =: 1, \nu\nu(\heartsuit) =: 2, \ldots$$

So are we done???.....not quite....can you see why?

Assume we have additional elements ☺ and ☹ in $\mathbb{N}$ such that

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This doesn't hurt axiom 1–4...and we have build a small loop 'aside' from our infinite chain.

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AXIOM 5 (P5): Let $A \subseteq \mathbb{N}$, such that $0 \in A$ and $\nu(n) \in A$ for all $n \in A$, then $A = \mathbb{N}$. (Axiom of Induction)

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What does it mean? It means that $\mathbb{N}$ is the minimal set satisfying axioms 1–4. THIS TIME WE ARE DONE!!!

Now try to understand why **axioms 1–4 are not enough** and why **axiom 5 fixes the problem...Discussion in the next video!!**

PEANO AXIOMS: VIDEO II

Why **axioms 1–4 are not enough** and why **axiom 5 fixes the problem!!**

Summary

[Peano axioms] The natural numbers $\mathbb{N}$ are a set with the following properties

(P1) There exists a distinguished element $0 \in \mathbb{N}$.

(P2) There exists a map $\nu : \mathbb{N} \to \mathbb{N}$ called the successor map.

(P3) There exists no element n such that $\nu(n) = 0$.

(P4) The map ν is injective, i.e. for all n_1, n_2 in $\mathbb{N}$ if $n_1 \neq n_2$, then $\nu(n_1) \neq \nu(n_2)$.

(P5) Let $A \subseteq \mathbb{N}$, such that $0 \in A$ and $\nu(n) \in A$ for all $n \in A$, then $A = \mathbb{N}$.

All other properties of the natural numbers can be derived from Axiom 1–5!

PEANO AXIOMS: VIDEO III

Addition and Multiplication

The axiom of recursion

Axiom **(R) (general)** Let X be a set. Assume that $x \in X$ and let $f : X \to X$ be a function. Then for all $n \in \mathbb{N}$, there exists a unique function $R : \mathbb{N} \to X$, such that:

- $R(0) = x$
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(R) Let $n_0 \in \mathbb{N}$. For all $n \in \mathbb{N}$, there exists a unique function $R : \mathbb{N} \to \mathbb{N}$, such that:

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To go beyond:

- Axiom (R) is actually equivalent to axiom (P5) of induction!
- Try to prove unicity of the function R using induction (see lecture notes!)

Addition on $\mathbb{N}$

Proposition

There exists a unique binary operation $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, *such that*

$$\forall x \in \mathbb{N}, x + 0 = x. \tag{1}$$

$$\forall x, y \in \mathbb{N}, \ x + \nu(y) = \nu(x + y). \tag{2}$$

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Proof.

Immediate from axiom of recursion! □

Multiplication on $\mathbb{N}$

Proposition (Definition of multiplication)

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Convince yourself that this is immediate by axiom (R)!

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Exercise

Compute $1 + 1$, $1 + 2$ *and* $3 \cdot 2$ *with our new definitions...do you get what you expect?*

PEANO AXIOMS: VIDEO IV
Basic properties/ proofs using the axiom of induction

Proposition (Properties of the operations)

Let $x, y, z \in \mathbb{N}$. *Then*

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Proposition (Properties of the operations)

Let $x, y, z \in \mathbb{N}$. *Then*

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(M4) $(x+y) \cdot z = x \cdot z + y \cdot z$ *(Distributivity law)*

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And some more...

Proposition

Let $x, y \in \mathbb{N}$. *Then*

1) *If* $x + y = x$, *then* $y = 0$.
2) *If* $x + y = 0$, *then* $x = 0$ *and* $y = 0$

LET'S PROVE THE TWO FIRST TOGETHER!

Some recipe for this kind of proofs

We first prove (A2): $0 + x = x + 0 = x$ for all $x \in \mathbb{N}$.

- **Define an appropriate set S**
- **Prove that $0 \in S$**
- **Prove that if $n \in S$ for n arbitrary, then $\nu(n) \in S$**
- **Conclude using Axiom (P5) that $S = \mathbb{N}$, hence the property is satisfied for all natural numbers.**

We now prove (A1): $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{N}$.