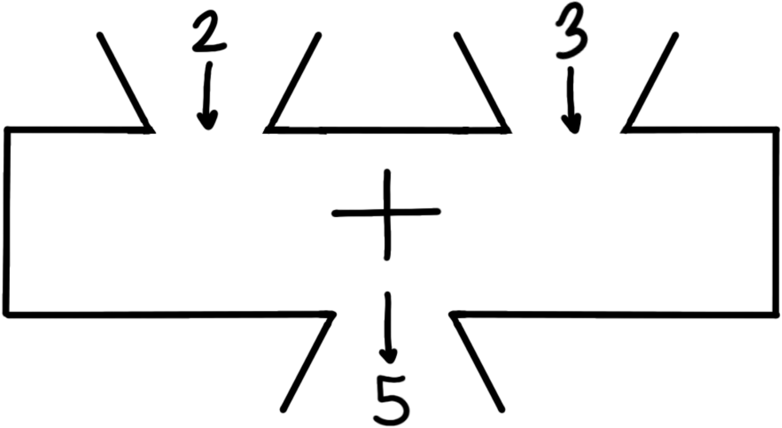
If addition is a function, what is its domain and codomain?



A function is supposed to want to take one input, but addition seems to take two.

In this lecture I will explain *two* ways of fixing this!

First way: products of sets.

Let A and B be sets.

The *product* $A \times B$ is the set whose elements are *ordered pairs* (a, b) with $a \in A$ and $b \in B$.

Picture of the product of $\{a, b, c\}$ and $\{p, q\}$:

q	(a, q)	(b, q)	(c, q)
p	(a, p)	(b, p)	(c, p)
$\times$	a	b	c

The top right 2×3 rectangle is the six elements of the product.

Ordered pairs

The elements of the set $A \times B$ are ordered pairs (a, b).
Equality of ordered pairs works like this: $(a_1, b_1) = (a_2, b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

You have seen this before with coordinates in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Note in particular that $(2, 3) \neq (3, 2) \in \mathbb{R} \times \mathbb{R}$.

Note finally that product of *sets* does not have anything to do with products of *elements* of the sets. It is more to do with the product of the *sizes* of the sets.

Special case of $A \times B$ is when $A = B$; we sometimes use A^2 as notation for $A \times A$.

Example: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Why do we care about products of sets in this course? Two reasons.

1. Graphs of functions.
2. They can be used to make addition into a function.

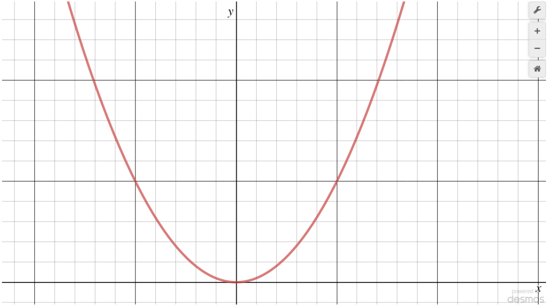
Let's talk about graphs of functions. Say $f : X \to Y$ is a function (think about the case $X = Y = \mathbb{R}$).

The *graph* of f is the subset of $X \times Y$ consisting of the pairs (x, y) such that $y = f(x)$.

If $X = Y = \mathbb{R}$ then we can draw subsets of $X \times Y = \mathbb{R}^2$, by colouring in the points in the subset.

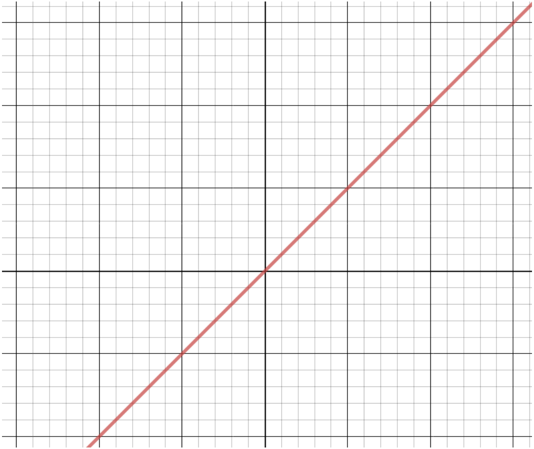
Say $f : \mathbb{R} \to \mathbb{R}$ is the squaring function, $f(x) = x^2$.

The corresponding subset of $\mathbb{R}^2$ is the points (x, y) with $y = x^2$. Let's draw it!



If f is the identity function from X to X, then the graph of f is the "diagonal" subset of $X \times X$.

It consists of all ordered pairs of the form (x, x).

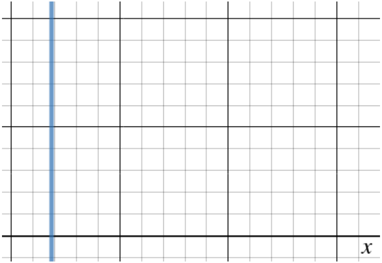


The Vertical Line Test

Did you learn the "vertical line test" at school? Here's how it looks in this generality.

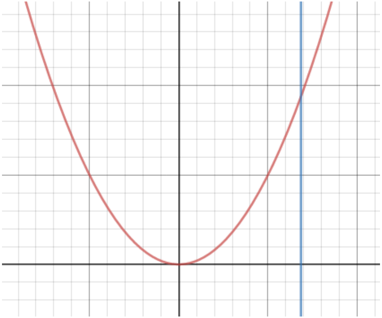
If X and Y are sets, and $x \in X$, let's define the "vertical line through x" to be the set $\{x\} \times Y$.

In other words, it is the set $\{(a, b) \in X \times Y \mid a = x\}$.



Theorem (Vertical line test). A subset $G \subseteq X \times Y$ is the graph of a function from X to Y if, and only if, the following condition holds:

For all $x \in X$, the intersection of G with the "vertical line" $\{x\} \times Y$ consists of a single point.



Proof.

First I claim that the graph of a function meets every vertical line in a single point.

Indeed, if f is a function and G is its graph, then $(x, y) \in G$ if and only if $y = f(x)$, so the intersection of G with $\{x\} \times Y$ is just $(x, f(x))$.

Secondly, I claim that a subset G meeting every vertical line in a single point is the graph of a function.

Indeed, if G meets $\{x\} \times Y$ in the point (x, a), then define $f(x)$ to be a. It is easily checked that G is the graph of f.

$2 + 3 = 5$.

It looks like $+$ takes in two real numbers and outputs a real number.

But of course we can regard $+$ as a function from $\mathbb{R}^2$ to $\mathbb{R}$.

Exercise for you: what does the graph of $+$ look like? How about multiplication? These are subsets of $\mathbb{R}^3$.

Interesting question: does $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R} = \mathbb{R} \times (\mathbb{R} \times \mathbb{R})$?

Curry

The last two slides are non-examinable.

Here is a different way of regarding addition as a function.

Let's imagine the domain of addition is $\mathbb{R}$.

Let's take a number, like 37, and feed it into addition. We get $37+$.

Does $37+$ make any sense?

We can think of $37+$ as a *function*, which takes as input a real number y, and outputs the real number $37 + y$.

So addition can be thought of as a function from $\mathbb{R}$ to $\mathrm{Hom}(\mathbb{R}, \mathbb{R})$.

In Lean, $\mathrm{Hom}(X, Y)$ is just denoted $X \to Y$.

So addition is a function $\mathbb{R} \to (\mathbb{R} \to \mathbb{R})$.

And because of conventions about associativity of $\to$, in Lean we can write addition as a term of type $\mathbb{R} \to \mathbb{R} \to \mathbb{R}$. But don't write this in your maths exams, because mathematicians don't use this convention. For more information, see Currying (Wikipedia).