

# Sets: making life more interesting.

In the videos on logic, our entire mathematical world only had two elements, `true` and `false`.

Sets are a way of making our world bigger.

The *natural numbers* are a set. Unfortunately people use this phrase to describe *two different sets*!

I was taught that the natural numbers are  $\{1, 2, 3, \dots\}$ . If you are a computer scientist, or a French mathematician, you might think they are  $\{0, 1, 2, \dots\}$ .

We will learn more about this set in Part II of this course, when we do Peano's axioms. (or you could learn more about them now if you play the natural number game).

The exciting thing about the natural numbers: it is definitely *infinite*.

So we can perhaps imagine *infinitely many* true-false statements, indexed by the natural numbers.

$$P(0), P(1), P(2), P(3), \dots$$

Let's look at some examples.

# Examples

$P(n)$  could be the statement “ $n$  is even”.

The truth-values of the propositions  $P(0)$ ,  $P(1)$ ,  $P(2)$ ,  $P(3)$  ... would then be true, false, true, false, ....

Or  $P(n)$  could be “ $n$  is prime”.

Then  $P(n)$  is usually false, but is true infinitely often.

Or  $P(n)$  could be the statement “ $0 + 1 + 2 + \dots + n = \frac{n(n+1)}{2}$ ”.

Then  $P(n)$  is always true.

I will not talk about strategies to *prove* things like “ $P(n)$  is always true, for all  $n$ ”. Dr Lawn will do that in Part II of this module.

This video is about *notation* – that is, a way to *state* that they are all true.

**New notation:**  $\forall n \in \mathbb{N}, P(n)$ . Pronounced “for all  $n$ ,  $P(n)$ ”. This is a true-false statement.

For example, if  $P(n)$  is the statement that  $n$  is prime, then  $\forall n \in \mathbb{N}, P(n)$  is false (why? What is the *proof*? Pause the video and think of a proof.)

But if  $P(n)$  is the statement that  $0 + 1 + \cdots + n = \frac{n(n+1)}{2}$ , then the statement  $\forall n \in \mathbb{N}, P(n)$  is true.

This idea generalises.

Let  $X$  be *any* set.

**Definition.** A *predicate* ([Wikipedia](#)) on  $X$  is a true-false statement  $P(a)$  for every element  $a \in X$ .

For example,  $X = \mathbb{R}$  could be the real numbers and, for  $a \in \mathbb{R}$ ,  $P(a)$  could be  $a < 37$ . Or  $P(a)$  could be the statement that  $2a = a + a$ .

**Definition** (of  $\forall$ ): The true-false statement

$$\forall a \in X, P(a)$$

is the assertion that for every  $a \in X$ , the statement  $P(a)$  is true.

In the next lecture, we will learn about  $\exists$  (“there exists”), a companion for  $\forall$ .