Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2021-2022

Introduction to University Mathematics **MATH40001/MATH40009**

Final Exam

Instructions: The **neatness, completeness and clarity of the answers** will contribute to the final mark. You must turn in handwritten solutions written on paper and scanned. You should upload your answers to this test as a single PDF via the Turnitin Assignment called *Whole exam dropbox* which you will find in the *Exam Paper and Dropbox(es)* folder.

IMPORTANT – Use the following naming convention for your file **AND** for the Title of your Turnitin submission: `[CID]_[ModuleCode]_full_submission.pdf`. Your first page should be the "Maths Coversheet for Submission" (which can be download from blackboard) and be completed.

Maths students must attempt all three questions, JMC students will only attempt the first two questions. For this module, the module code is **MATH40001 for Maths students** and **MATH40009 for JMC students**.

> In this test, you may assume any results from the course notes or videos, as long as you state them correctly (unless stated otherwise).

1. **Total: 20 Marks**

   (a) Let T be a set and let R be a binary relation on T. Say a, b, c are three distinct elements of T. Assume that $R(a, b)$ is true, $R(b, c)$ is true and $R(c, a)$ is false. Prove that R is not an equivalence relation on T. 3 Marks

   (b) Let U be a set, and let R be a binary relation on U. Let d, e, f be three distinct elements of U. Assume that $R(d, e)$ is true, $R(e, f)$ is true and $R(f, d)$ is true. Prove that R is not a partial order on U. 3 Marks

   (c) Let X be a set, and let R and S be two binary relations on X.
   Now define two new binary relations A and B on X, by $A(x, y) = R(x, y) \wedge S(x, y)$, and $B(x, y) = R(x, y) \vee S(x, y)$.

      i. Proof or counterexample: If R and S are equivalence relations, then A must be an equivalence relation. 4 Marks

      ii. Proof or counterexample: If R and S are equivalence relations, then B must be an equivalence relation. 4 Marks

      iii. Proof or counterexample: If R and S are antisymmetric, then A must be antisymmetric. 3 Marks

      iv. Proof or counterexample: if R and S are antisymmetric, then B must be antisymmetric. 3 Marks

2. **Total: 20 Marks**

   (a) For this part assume the Peano axioms only!

      i. Show that for all n, m, $\nu(n + m) = \nu(n) + m$. 3 Marks

      ii. Show that for all n, m in $\mathbb{N}$, $n \neq n + \nu(m)$. 4 Marks

   (b) Let p_n be the n-th prime number, so $p_1 = 2$, $p_2 = 3$, $\ldots$.

      i. Show that $p_{n+1} \leq p_1 \cdot \cdots \cdot p_n + 1$, for $n \geq 1$. 4 Marks

ii. Show that $p_n \leq 2^{2^{n-1}}$. $\boxed{\text{3 Marks}}$

(c) Let S be a field and $x \neq 0$ be in S. Show that there exists no element $y \neq 0$ in S such that $x \cdot y = 0$. $\boxed{\text{3 Marks}}$

(d) Prove that, for every positive real number $r \in \mathbb{R}$, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that the set $S := \{f(n) \cdot n^{-1} | n \geq 1, n \in \mathbb{N}\}$ has infimum equal to r. Show that, if r is not a rational number, then the supremum cannot also equal r. $\boxed{\text{3 Marks}}$

3. $\boxed{\textbf{Total: 20 Marks}}$

(a) For each of the following statements, state without proof whether they are TRUE or FALSE:

i. The equation $ax + by + cz + d = 0$ represents a line in space. $\boxed{\text{1 Mark}}$

ii. The cross product of two unit vectors is a unit vector. $\boxed{\text{1 Mark}}$

iii. If $|\mathbf{r}(t)| = 1$ for all t, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t. $\boxed{\text{1 Mark}}$

iv. If $\mathbf{r}(t)$ is a differentiable vector function, then $\frac{d}{dt}|\mathbf{r}(t)| = |\mathbf{r}'(t)|$. $\boxed{\text{1 Mark}}$

(b) Find a parametric equation for the line $\mathcal{L}$ through the points $A(1, 2, 1)$ and $B(2, 1, 2)$. Compute the distance between line $\mathcal{L}$ and the point $C(2, 3, 4)$. $\boxed{\text{4 Marks}}$

(c) The Frenet–Serret formulas are given by

$$\begin{cases} d\mathbf{T}/ds = \kappa\mathbf{N} \\ d\mathbf{N}/ds = -\kappa\mathbf{T} + \tau\mathbf{B} \\ d\mathbf{B}/ds = -\tau\mathbf{N} \end{cases}$$

where $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are respectively the tangent, normal and binormal vectors to a curve parametrized by a vector function $\mathbf{r}$. In writing these formulas, we introduced two important quantities: κ the curvature and τ the torsion of the curve. In this question, we will provide an explicit expression for the torsion. You can use the Frenet–Serret formulas without further proof.

i. First, show that

$$\mathbf{r}'' = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$$

where the primes denote derivatives with respect to t and the notation $(s')^n$ means $(ds/dt)^n$. $\boxed{\text{2 Marks}}$

ii. Then, show that

$$\mathbf{r}' \times \mathbf{r}'' = \kappa(s')^3\mathbf{B}$$

$\boxed{\text{2 Marks}}$

iii. Further, show that

$$\mathbf{r}''' = \left[s''' - \kappa^2(s')^3\right]\mathbf{T} + \left[3\kappa s's'' + \kappa'(s')^2\right]\mathbf{N} + \kappa\tau(s')^3\mathbf{B}$$

$\boxed{\text{2 Marks}}$

iv. Finally, deduce that the torsion of a curve parametrized by vector function $\mathbf{r}$ can be explicitly expressed as

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$$

$\boxed{\text{2 Marks}}$

v. The DNA molecule has the structure of a double helix. One of the helices can be parametrized by the vector function $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$, where a and b are positive real constants. Show that this curve has constant curvature and constant torsion. $\boxed{\text{4 Marks}}$

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Solutions to Final Exam

1. **Total: 20 Marks**

(a) We prove it by contradiction. Assume R is an equivalence relation. Then $R(a,b)$ is true and $R(b,c)$ is true. By transitivity we deduce that $R(a,c)$ is true. By symmetry we deduce that $R(c,a)$ is true. But $R(c,a)$ is false, a contradiction. Hence R is not an equivalence relation. I would also be happy if people argue with equivalence classes ("$R(a,b)$ is true so $cl(a) = cl(b)$" etc etc).

(b) Again by contradiction. Assume R is a partial order. We know $R(d,e)$ is true and $R(e,f)$ is true, so by transitivity $R(d,f)$ is true. We know $R(f,d)$ is true as well, so by antisymmetry we deduce that $d = f$. But d and f are assumed to be distinct, a contradiction. Hence R is not a partial order.

(c) i. This is true. First, R and S are reflexive, so A is reflexive, because for all $x \in X$ we have $A(x,x) = R(x,x) \wedge S(x,x) = \texttt{true} \wedge \texttt{true} = \texttt{true}$. Next, R and S are symmetric, so A is symmetric, because if $x, y \in X$ and $A(x,y)$ is true, then $R(x,y)$ and $S(x,y)$ are true, so $R(y,x)$ and $S(y,x)$ are true by symmetry of R and S, so $A(y,x)$ is true. Finally, R and S are transitive, so A is transitive, becaue if $x, y, z \in X$ with $A(x,y)$ and $A(y,z)$, then $R(x,y)$ and $S(x,y)$ and $R(y,z)$ and $S(y,z)$ are all true, so by transitivity of R and S we deduce $R(x,z)$ and $S(x,z)$ are true, and thus $A(x,z)$ is true.

ii. This is false, although a counterexample is slightly tricky to find. Let $X = \{1,2,3\}$. Recall from the course videos that we can make an equivalence relation on X by partitioning it into nonempty disjoint subsets. So let's let R be the equivalence relation corresponding to the partition of X into $\{1,2\}$ and $\{3\}$, and let S be the equivalence relation corresponding to the partition $\{1\}$ and $\{2,3\}$ (the point of doing it this way is that we don't need to check the axioms for an equivalence relation, we know R and S will satisfy them).
Because 1 and 2 are in the same partition for R, we have that $R(1,2)$ is true, but because 1 and 3 are in different partitions for R, we have that $R(1,3)$ is false. Similarly $S(2,3)$ is true, but $S(1,3)$ is false.
Hence $B(1,2)$ is true, because $R(1,2)$ is true, and $B(2,3)$ is true, because $S(2,3)$ is true, but $B(1,3)$ is not true, because $R(1,3)$ and $S(1,3)$ are both false.
Thus B is not transitive, and hence not an equivalence relation.

iii. This is true. Assume $x, y \in X$ and $A(x,y)$ and $A(y,x)$ are both true; we want to prove that $x = y$. However $A(x,y)$ implies $R(x,y)$ and $A(y,x)$ implies $R(y,x)$, so antisymmetry of R tells us that $x = y$, which is what we wanted.

iv. This is not true. Let $X = \{1,2\}$ and define a binary relation R on X by $R(1,2)$ is true and $R(x,y)$ is false otherwise. This relation is antisymmetric because if we assume $R(x,y)$ and $R(y,x)$ are true, we must have $x = 1, y = 2$ and $y = 1, x = 2$, but these together prove a contradiction, so we have assumed something false and can deduce $x = y$ as false statements imply anything.
Similarly if S is defined by $S(2,1)$ is true and $S(x,y)$ is false otherwise, then S is also antisymmetric (the same proof as above works, just switch 1 and 2). However $B(1,2)$ and $B(2,1)$ will both be true (because $\texttt{false} \vee \texttt{true} = \texttt{true}$), so B is not antisymmetric because $2 \neq 1$.

2. **Total: 20 Marks**

(a) i. Show that for all n, m, $\nu(n+m) = \nu(n) + m$.

Proof. Define the set

$$A = \{m \in \mathbb{N} | \nu(n+m) = \nu(n) + m\}.$$

0 is obviously in the set by definition of addition. Now assume m is in the set. We want to show $\nu(n+\nu(m)) = \nu(n)+\nu(m)$. But again by definition of addition, induction hypothesis and again definition of addition (in this order) $\nu(n+\nu(m)) = \nu(\nu(n+m)) = \nu(\nu(n)+m) = \nu(n) + \nu(m)$, which shows the statement. □

ii. Show that for all n, m in $\mathbb{N}$, $n \neq n + \nu(m)$.

Proof. Fix $m \in \mathbb{N}$. Define the set

$$A = \{n \in \mathbb{N} | n \neq n + \nu(m)\}.$$

We first show that $0 \in A$, or equivalently $0 \neq 0+\nu(m)$. We have $0+\nu(m) = \nu(m)$, which cannot be 0 by (P3). Now suppose that $n \in A$. Then by (P4), $\nu(n) \neq \nu(n + \nu(m)) = \nu(n) + \nu(m)$. So $\nu(n) \in A$ as well. By (P5), $A = \mathbb{N}$. □

(b) Let p_n be the n-th prime number.

i. Show that $p_{n+1} \leq p_1 \cdot \dots \cdot p_n + 1$, where $n \geq 1$.

Proof. Let $x := p_1 \cdot \dots \cdot p_n + 1$. None of the primes p_i with $0 < i \leq n$ is a factor of x. Indeed assume $q = p_i$ for some i, then $q | p_1 \cdot \dots \cdot p_n$. But then $q | x - p_1 \cdot \dots \cdot p_n = 1$, which is a contradiction. Therefore q has to be a prime numbers which is not in the list $p_1, \dots, p_n$, and therefore $q = p_k$, with $k \geq n+1$ and consequently $p_{n+1} \leq q \leq p_1 \cdot \dots \cdot p_n + 1$. □

ii. Show that $p_n \leq 2^{2^{n-1}}$.

Proof. This is a proof by strong induction. $p_1 = 2$, hence $2 \leq 2^1 = 2^{2^{1-1}}$ and the base case is done.

Assume now that $p_j \leq 2^{2^{j-1}}$ for all $1 \leq j \leq k$. We get using i)

$$p_{k+1} \leq p_1 \cdot \dots \cdot p_n + 1 \leq 2 \cdot 2^{2^1} \cdot 2^{2^2} \dots 2^{2^{k-1}} + 1 \leq 2^{2^k - 1} + 1 \leq 2^{2^k}.$$

This proves the results. □

(c) Let S be a field and $x \neq 0$ be in S. Show that there exists no element $y \neq 0$ in S such that $x \cdot y = 0$.

Proof. Since S is a field and $x \neq 0$, there exists an element x^{-1}, such that $x \cdot x^{-1} = 1$. Assume now that there exists a $y \neq 0$ in S such that $x \cdot y = 0$. We have

$$0 = x^{-1} \cdot 0 = x^{-1} \cdot (x \cdot y) = (x^{-1} \cdot x) \cdot y = 1 \cdot y = y,$$

which is a contradiction. □

(d) Prove that, for every positive real number $r \in \mathbb{R}$, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that the set $S := \{f(n) \cdot n^{-1} | n \geq 1, n \in \mathbb{N}\}$ has infimum equal to r. Show that the supremum cannot also equal r unless r is a rational number.

Proof. For every n, let $f(n)$ be the unique natural number such that $f(n) - 1 < r \cdot n \leq f(n)$: this can be defined as the minimum natural number greater than or equal to $r \cdot n$, and in this case $f(n) - 1 < r \cdot n$ because $r \cdot n \geq 0$. If $n \geq 1$, by division, $f(n) \cdot n^{-1} \geq r$, so that r is indeed a lower bound of the set S. We claim that it is the greatest lower bound. Suppose for a contradiction that r' is another lower bound of S, and $r' > r$. Then, by the Archimedean property, there exists n such that $n(r' - r) \geq 1$. By definition, $f(n) - 1 < r \cdot n \leq f(n)$. Since r' is also a lower bound, we have $f(n) \cdot n^{-1} \geq r'$, i.e., $f(n) \geq r' \cdot n$. Now by assumption, $r' \cdot n \geq r \cdot n + 1$, so that $f(n) \geq r \cdot n + 1$, i.e., $f(n) - 1 \geq r \cdot n$, a contradiction.

For the final question, if r were also the supremum, then it would be both a lower bound and an upper bound, which would imply that $S = \{r\}$. As $S \subseteq \mathbb{Q}$ by definition, this would require that r be rational. (Remark: actually, one can see that r would have to be an integer, since it can be represented as fractions $p \cdot q^{-1}$ and $p' \cdot (q')^{-1}$ with $\gcd(q, q') = 1$.) □

3. **Total: 20 Marks**

(a) Consider the following statements:

i. The equation $ax + by + cz + d = 0$ represents a line in space. **FALSE** 1 Mark

ii. The cross product of two unit vectors is a unit vector. **FALSE** 1 Mark

iii. If $|\mathbf{r}(t)| = 1$ for all t, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t. **TRUE** 1 Mark

iv. If $\mathbf{r}(t)$ is a differentiable vector function, then $\frac{d}{dt}|\mathbf{r}(t)| = |\mathbf{r}'(t)|$. **FALSE** 1 Mark

(b) We consider the line $\mathcal{L}$ going through the points $A = (1, 2, 1)$ and $B = (2, 1, 2)$. The vector $\mathbf{u} = (2-1, 1-2, 2-1) = (1, -1, 1)$ and we can take the reference point $A = (1, 2, 1)$. So a parametric equation for $\mathcal{L}$ is given by

$$\mathbf{r} = A + \lambda \mathbf{u}, \text{with } \lambda \in \mathbb{R}$$

i.e.

$$\begin{cases} x = 1 + \lambda \\ y = 2 - \lambda \\ z = 1 + \lambda \end{cases}$$

2 Marks

The distance between line $\mathcal{L}$ and point C is by definition

$$d(\mathcal{L}, C) = \frac{|\mathbf{AC} \times \mathbf{u}|}{|\mathbf{u}|}$$

We have $|\mathbf{u}| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$. Further, $\mathbf{AC} = (1, 1, 3)$ so that $\mathbf{AC} \times \mathbf{u} = (4, 2, -2)$. Finally, we have

$$d(\mathcal{L}, C) = \frac{\sqrt{4^2 + 2^2 + (-2)^2}}{\sqrt{3}} = \frac{2\sqrt{6}}{\sqrt{3}} = 2\sqrt{2}$$

2 Marks

(c) In this problem, we assume that we can use without further proof Frenet–Serret formulas

$$\begin{cases} \text{(FS1):} \quad d\mathbf{T}/ds = \kappa \mathbf{N} \\ \text{(FS2):} \quad d\mathbf{N}/ds = -\kappa \mathbf{T} + \tau \mathbf{B} \\ \text{(FS3):} \quad d\mathbf{B}/ds = -\tau \mathbf{N} \end{cases}$$

i. The definition of the tangent vector gives us

$$\mathbf{r}'(t) = |\mathbf{r}'(t)|\mathbf{T}$$

but we know that $|\mathbf{r}'(t)| = ds/dt$. Said differently, we have

$$\mathbf{r}' = s'\mathbf{T} \Rightarrow \mathbf{r}'' = s'\mathbf{T}' + s''\mathbf{T} \Rightarrow \mathbf{r}'' = s''\mathbf{T} + (s')^2 \frac{d\mathbf{T}}{ds} \Rightarrow \mathbf{r}'' = s''\mathbf{T} + \kappa(s')^2\mathbf{N} \quad \text{by (FS1).}$$

2 Marks

ii. Using the result of part (i), we have $\mathbf{r}'' = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$ and so

$$\begin{aligned} \mathbf{r}' \times \mathbf{r}'' &= (s'\mathbf{T}) \times \left[s''\mathbf{T} + \kappa(s')^2\mathbf{N}\right] \\ &= s's''\cancelto{0}{(\mathbf{T} \times \mathbf{T})} + \kappa(s')^3 (\mathbf{T} \times \mathbf{N}) \\ &= \kappa(s')^3\mathbf{B} \end{aligned}$$

where we have used the linearity of the cross product and the fact that by definition $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. 2 Marks

iii. Using once again the result of part (i), we have $\mathbf{r}'' = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$ which we can differentiate to write

$$\begin{aligned}
\mathbf{r}''' &= \frac{d}{dt}\left[s''\mathbf{T} + \kappa(s')^2\mathbf{N}\right] \\
&= s'''\mathbf{T} + s''\mathbf{T}' + \kappa'(s')^2\mathbf{N} + 2\kappa s's''\mathbf{N} + \kappa(s')^2\mathbf{N}' \\
&= s'''\mathbf{T} + s's''\frac{d\mathbf{T}}{ds} + \kappa'(s')^2\mathbf{N} + 2\kappa s's''\mathbf{N} + \kappa(s')^3\frac{d\mathbf{N}}{ds} \\
&= s'''\mathbf{T} + \kappa s's''\mathbf{N} + \kappa'(s')^2\mathbf{N} + 2\kappa s's''\mathbf{N} + \kappa(s')^3\frac{d\mathbf{N}}{ds} \quad \text{(by FS1)} \\
&= s'''\mathbf{T} + \kappa s's''\mathbf{N} + \kappa'(s')^2\mathbf{N} + 2\kappa s's''\mathbf{N} + \kappa(s')^3\left[-\kappa\mathbf{T} + \tau\mathbf{B}\right] \quad \text{(by FS2)}
\end{aligned}$$

and finally, we obtain as requested

$$\mathbf{r}''' = \left[s''' - \kappa^2(s')^3\right]\mathbf{T} + \left[3\kappa s's'' + \kappa'(s')^2\right]\mathbf{N} + \kappa(s')^3\tau\mathbf{B}$$

2 Marks

iv. Using the results from part (ii) and part (iii), we write

$$(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}''' = \left(\kappa(s')^3\mathbf{B}\right)\cdot\left(\left[s''' - \kappa^2(s')^3\right]\mathbf{T} + \left[3\kappa s's'' + \kappa'(s')^2\right]\mathbf{N} + \kappa(s')^3\tau\mathbf{B}\right)$$

Using the linearity of the scalar product, we can expand this expression. Using the fact that $\mathbf{B}\cdot\mathbf{T} = \mathbf{B}\cdot\mathbf{N} = 0$ as these three vectors are orthogonal to each other, we find that

$$(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}''' = \tau\kappa^2(s')^6\mathbf{B}\cdot\mathbf{B}$$

But we know that by definition $\mathbf{B}$ is a unit vector and so $\mathbf{B}\cdot\mathbf{B} = 1$. Finally, we can isolate τ and write

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{\kappa^2(s')^6}$$

which is also

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$$

according to part (ii). 2 Marks

v. Finally, we want to calculate the curvature and the torsion of the circular helix parametrized by $\mathbf{r} = (a\cos t, a\sin t, bt)$. We know that

$$\text{Curvature:} \quad \kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$$\text{Torsion:} \quad \tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t))\cdot\mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

For this curve, we have

$$\begin{aligned}
\mathbf{r}'(t) &= (-a\sin t, a\cos t, b) \Rightarrow |\mathbf{r}'(t)| = \sqrt{a^2+b^2} \\
\mathbf{r}''(t) &= (-a\cos t, -a\sin t, 0) \\
\mathbf{r}'''(t) &= (a\sin t, -a\cos t, 0)
\end{aligned}$$

which leads to

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = ab\sin t\hat{\mathbf{i}} - ab\cos t\hat{\mathbf{j}} + a^2\hat{\mathbf{k}}$$

leading to

$$|\mathbf{r}' \times \mathbf{r}''| = a\sqrt{a^2+b^2}$$

and finally

$$(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' = a^2 b \sin^2 t + a^2 b \cos^2 t = a^2 b$$

We thus conclude that

$$\text{Curvature:} \quad \kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}$$

$$\text{Torsion:} \quad \tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|} = \frac{a^2 b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}$$

which shows that both only depend on a and b; they are thus constant. $\boxed{\text{4 Marks}}$