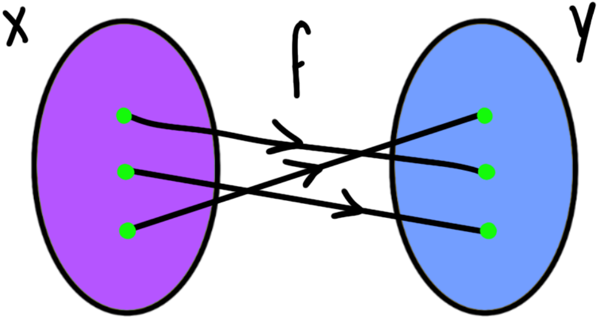
Bijections

This lecture is about a third true-false statement which you can make about a function – you can ask if it is *bijective*.

Let's start with a picture.



A function $f : X \to Y$ is *bijective* if it's injective and surjective.

That's a correct mathematical definition, but it doesn't give much intuition.

$$(\forall w, x \in X, f(w) = f(x) \implies w = x) \wedge (\forall b \in Y, \exists a \in X, f(x) = y).$$

To get some intuition about this idea, let's recall facts about preimages.

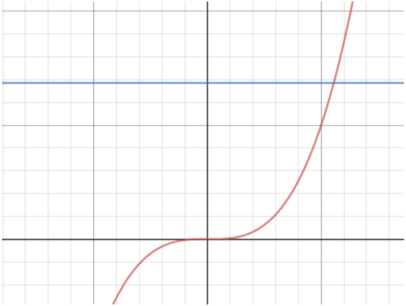
Recall: A function $f : X \to Y$ is injective if and only if forall $y \in Y$, its preimage $f^{-1}(y)$ has *at most one element*.

And f is surjective if and only if for all $y \in Y$, the preimage $f^{-1}(y)$ has *at least one element*.

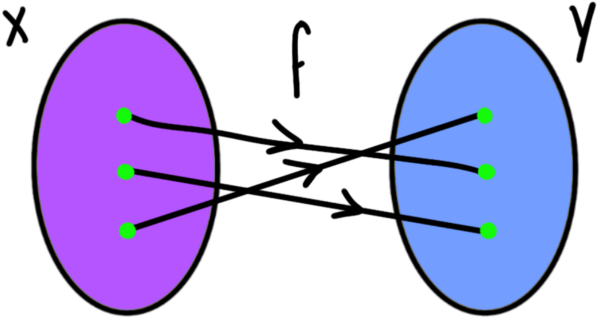
This mean that f is bijective if and only if for all $y \in Y$, the preimage $f^{-1}(y)$ has *exactly one element*.

Another way to say this:
Horizontal line test for bijective functions: f is bijective if and only if it meets every *horizontal* line in exactly one element.

The cubing function on $\mathbb{R}$ is a bijective function (at least if you believe in cube roots):



Let's go back to the abstract picture.



Every element of Y has exactly one arrow going into it.

So a bijection is some kind of correspondence between the elements of X and the elements of Y.

We'll come back to this later. Let's first prove some theorems about bijective functions.

Because we are proving theorems, we will use the recommended definition:

$f : X \to Y$ is bijective if and only if it's injective and surjective.

Is the identity function bijective?

Theorem

Let X be a set. Then the identity function $id_X : X \to X$ is bijective.

Why don't you pause the video and have a go at proving this yourself? Then I will show you my proof.

Proof.

We already proved the identity function was injective.

We already proved the identity function was surjective.

So the identity function is bijective. $\square$

Theorem

Let X, Y and Z be sets, and let $f : X \to Y$ and $g : Y \to Z$ be functions.
If f and g are both bijective, then $g \circ f$ is also bijective.

Proof.

Let's assume that f and g are both bijective. Then they're both injective and both surjective.

Because f and g are injective, so is $g \circ f$.

Because f and g are surjective, so is $g \circ f$.

Hence $g \circ f$ is bijective. $\square$

Counting

Say X and Y are finite sets, and $f : X \to Y$ is an bijective function.

It's injective, so $|X| \leq |Y|$. It's surjective so $|X| \geq |Y|$. Hence $|X| = |Y|$.

This should not surprise you, because the bijection is matching up the elements of X and of Y, so the sets have the same size.

Counting question 1) How many bijections are there from a set of size 2 to a set of size 5?
Counting question 2) How many bijections are there from a set of size 5 to a set of size 5?

Pause the video and have a go at these questions.

Counting question 1) How many bijections are there from a set of size 2 to a set of size 5?

There are none! Indeed, there aren't even any surjections.

Counting question 2) How many bijections are there from a set of size 5 to a set of size 5?

We need to match the elements in the source with the elements in the target.

There are 5 places in Y to send the first element of X, 4 places to send the second,$\ldots$, and 1 place to send the 5th element.

So there are 5! bijections between two sets of size 5.

In the last video I will talk about inverse functions, and how these are related to bijections.