Partial orders

A *partial order* on a set X is a binary relation which is all three of

1. reflexive;
2. antisymmetric;
3. transitive.

The standard example is $\leq$ on the natural numbers or the real numbers.
Pause the video and check that you can prove that $\leq$ is a partial order, or at least reduce it to statements which could be assumed without question at school.

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It means that if $X, Y \in \omega$ then $X \subseteq Y$ should be a true-false statement.

Now I claim that $\subseteq$ is a partial order on ω. What does that mean? Let's take a look.

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$$\forall X, X \subseteq X;$$

$$\forall X\ Y, (X \subseteq Y \wedge Y \subseteq X) \implies X = Y;$$

$$\forall X\ Y\ Z, (X \subseteq Y \wedge Y \subseteq Z) \implies X \subseteq Z;$$

Three levels of a computer game.

Here X, Y, Z are running through all subsets of α.

Those statements about sets correspond to the following three statements of logic:

$$P \implies P;$$
$$(P \implies Q \wedge Q \implies P) \implies (P \iff Q);$$
$$(P \implies Q \wedge Q \implies R) \implies (P \implies R).$$

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Or you can prove them constructively if you find it fun.

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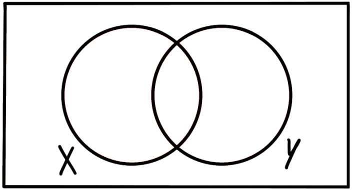
And this is funny because if X and Y are random subsets of α then probably both $X \subseteq Y$ and $Y \subseteq X$ are false!
Antisymmetry doesn't mind that.



If you don't like the possibility of "incomparable objects", you might prefer to study *total orders*. A total order is a partial order (let's call it $\leq$) on a set X satisfying the additional axiom of "totality":

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Can you give an example of α for which $\subseteq$ is a total order on the set of subsets of α?

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I'll finish this video by mentioning an even stronger condition: A *well-ordered set* is a set X equipped with a total order $\leq$ which furthermore satisfies that there are no infinite strictly decreasing sequences $x_1 > x_2 > x_3 > x_4 > \dots$. Here $a > b$ is defined to mean $\neg(a \leq b)$.
Consequence: every non-empty subset has a least element.

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Next time: equivalence relations.