VIDEO XIV
REALS 1: FIELDS

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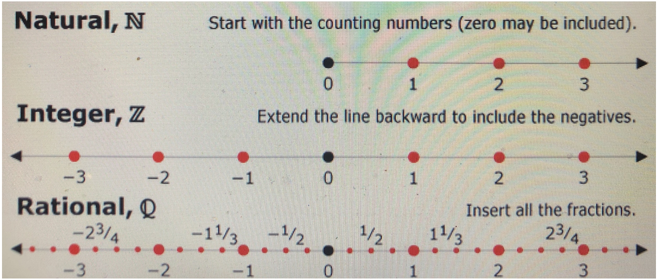

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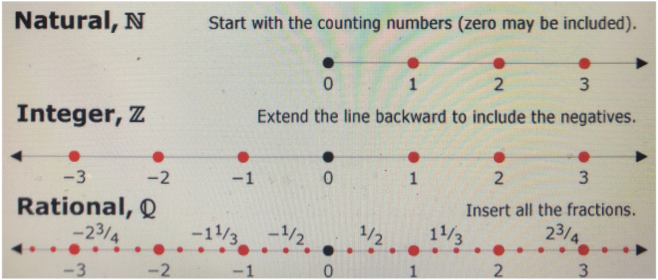

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Example: $\sqrt{2}$ is not rational.

idea: See reals as sequences of rational numbers, i.e. π is the limit of:

$$(3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, ..., 3.14159265358979, ...).$$

Construction of the reals

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This is tidious and beyond the scope of this module...instead we will formulate them as a collection of axioms!!

Fields axioms

We want the reals to be a set $\mathbb{R}$ with an addition $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a multiplication $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $\forall x, y, z \in \mathbb{R}$

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(A) **Axioms for addition**

1. (A1) $x + y = y + x$ (**addition is commutative**)
2. (A2) $(x + y) + z = x + (y + z)$ (**addition is associative**)
3. (A3) $\mathbb{R}$ contains an element 0 such that $0 + x = x$ (**neutral element of** $+$)
4. (A4) $\exists\ -x \in \mathbb{R}$ such that $x + (-x) = 0$. (**additive inverse**)

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(M) **Axioms for multiplication**

1. (M1) $xy = yx$ (**multiplication is commutative**)
2. (M2) $(xy)z = x(yz)$ (**multiplication is associative**)
3. (M3) $\mathbb{R}$ contains an element $1 \neq 0$ such that $1x = x$. (**neutral element of $\cdot$**)
4. (M4) For each $x \neq 0\ \exists\ x^{-1}$, such that $xx^{-1} = 1$ (**multiplicative inverse**)

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A set satisfying all these axioms is called a field

Question

Is it enough to define the reals uniquely?

Example

Let $\mathbb{F}_2 = \{0,1\}$, with $+ : \mathbb{F}_2 \times \mathbb{F}_2 \to \mathbb{F}_2$, $\cdot : \mathbb{F}_2 \times \mathbb{F}_2 \to \mathbb{F}_2$, such that

$$0+0=0,\ 0+1=1+0=1,\ 1+1=0, \qquad 0\cdot 0=0,\ 0\cdot 1=1\cdot 0=0,\ 1\cdot 1=1.$$

Check the axioms!

1. (A1) $x + y = y + x$
2. (A2) $(x + y) + z = x + (y + z)$
3. (A3) $\exists$ an element 0 in $\mathbb{F}_2$ such that $0 + x = x$
4. (A4) $\exists\ -x \in \mathbb{R}$ such that $x + (-x) = 0$.
5. (M1) $x \cdot y = y \cdot x$
6. (M2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
7. (M3) $\exists$ an element $1 \neq 0$ such that $1 \cdot x = x$.
8. (M4) For each $x \neq 0$ $\exists$ x^{-1}, such that $x \cdot x^{-1} = 1$
9. (D) $x \cdot (y + z) = x \cdot y + x \cdot z$.

Computation rules from the axioms

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Answer to our question: No! $\mathbb{Q}$, $\mathbb{F}_2$ satisfies the axioms of fields.

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Nevertheless from these axioms we can prove usual rules for $+$ and $\cdot$

Proposition

Let $x, y, z \in \mathbb{R}$. Then

1) *If $x + z = y + z$, then $x = y$.*
2) *If $x \in \mathbb{R}$, then $x \cdot 0 = 0$*

★: This rules can be derived for an arbitrary field!

Question

But so how do we get $\mathbb{R}$ then........?

VIDEO XV
REALS 2: ORDERED FIELDS

Ordered fields axioms

Reminder:

A **partial order relation** R on a set X is a binary relation such that

- $\forall x \in X, R(x,x)$ (Reflexivity)
- $\forall x, y \in X,\ R(x,y) \wedge R(y,x) \Rightarrow x = y$ (Anti-symmetry)
- $\forall x, y, z \in X,\ R(x,y) \wedge R(y,z) \Rightarrow R(x,z)$ (Transitivity)

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Ordered fields axioms:
A field $\mathbb{F}$ is called ordered, if there is a total order $\leq$ on $\mathbb{F}$ such that additionally if $x, y, z \in \mathbb{F}$, $x \leq y$ then

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2. (O2) if moreover $z \geq 0$ then $xz \leq yz$

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Question

Is this additional property enough to define the reals uniquely now?

Answer: Still not! $\mathbb{Q}$ satisfies this axioms! Such fields are called ordered fields.

Nevertheless from there we can derive all usual rules for the inequality!

Proposition

For all $x, y, z \in \mathbb{F}$, $\mathbb{F}$ an ordered field,

1. *if $x < y$, then $x + z < y + z$,*
2. *if $x < y$ and $z > 0$, then $xz < yz$*
3. *$x < 0$ if and only if $-x > 0$ and $x > 0$ if and only if $-x < 0$.*
4. *0<1*

VIDEO XVI
REALS 3: THE AXIOM OF COMPLETENESS

Lowest/upper bound–supremum/infimum

Definition

- A set S is called bounded above if there is a number B such that $x \leq B$ for all $x \in S$.
  Any such B is called an upper bound for S.
- A set S is called bounded below if there is a number B such that $x \geq B$ for all $x \in S$.
  Any such B is called an lower bound for S.
- We call s a least upper bound or supremum of S if
  1. s is **an upper bound** for S,
  2. if $B < s$ then B is **not an upper bound** for S.

  Notation: $\sup S$
- We call s a largest lower bound or infimum of S if
  1. s is a **lower bound** for S,
  2. if $B > s$ then B is **not a lower bound** for S.

  Notation: $\inf S$

Now we can finish our axiomatic definition of the reals!

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Definition

The **reals** are the field denoted by $\mathbb{R}$ satisfying **all the 12 axioms** we have seen so far:

- axioms (A1)–(A4), (M1)–(M4) and (D) for addition and multiplication,
- axioms (O1) and (O2) for the total order,
- axiom of completeness (C).

It is called a **complete ordered field**. $\mathbb{R}$ is the only such field.

Corollary (Archimedean property /Axiom of Eudoxos)

$$\forall x, y \in \mathbb{R}, x > 0, \exists n \in \mathbb{N}, \ \textit{such that } nx > y.$$

Proof: Exercise!

Lemma

Let $r \in \mathbb{R}$, $r > 0$. Then there exists a unique $n \in \mathbb{N}$, such that $n - 1 \leq r < n$

proof: Consider the set $S = \{m \in \mathbb{N} | m > r\}$

- By the Archimedean property $S \subseteq \mathbb{N}$ is not empty.
- Therefore by the Well-Ordering Principle there exists a (unique) least element $n \in S$ (i.e. m>n, $\forall m \in S$).

Applications

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- But $n > r$ since $n \in S$
- Case 1: n=1, then 0<r<n=1.
  Case 2: n>1, then $n - 1 \in \mathbb{N}$, but not in S! Hence $n - 1 \leq r < n$.

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Lemma

Let $x \in \mathbb{R}$, $x > 0$. The there exists a unique $n \in \mathbb{N}$, such that $n - 1 \leq x < n$

Proposition (Density of the rationals)

For all $x, y \in \mathbb{R}$, there exists $q \in \mathbb{Q}$, such that $x < q < y$.

Proof: Assume WLoG that $0 < x, y$, hence $y - x > 0$.

- Use Archimedean property:
- Use Above Lemma:
- Find a contradiction

☺ THANKS FOR LISTENING TO ME UNTIL NOW ☺!