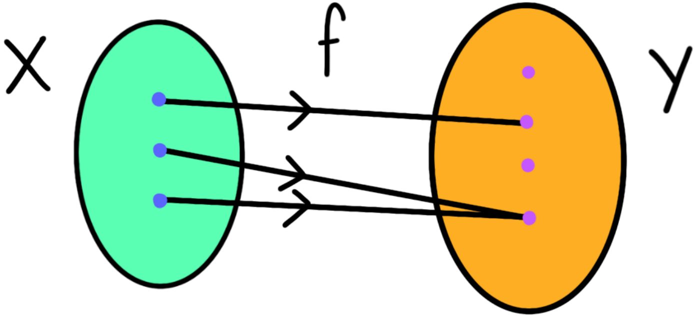
Part I of the intro module is in four pieces:

- Logic;
- Sets;
- Functions;
- Relations.

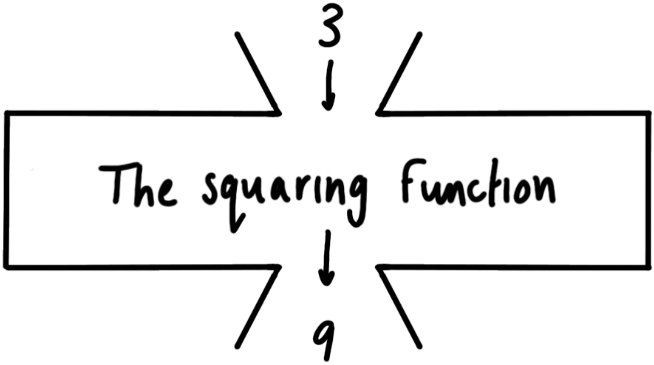
This is the first video about functions.

Let's start with some pictures.

Functions



Functions



Let X and Y be sets.

If f is a *function* from X to Y, then f takes as input an element $x \in X$ and it gives as output an element $f(x) \in Y$.

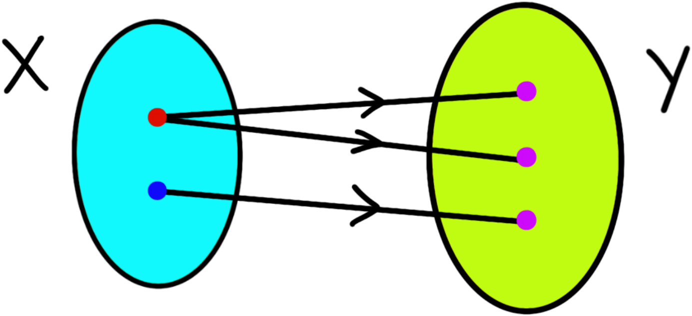
New definitions: X is the *domain* of the function f, and Y is the *codomain* of f.

New notation: $f : X \to Y$ means that f is a function from X to Y.

Basic facts about functions.

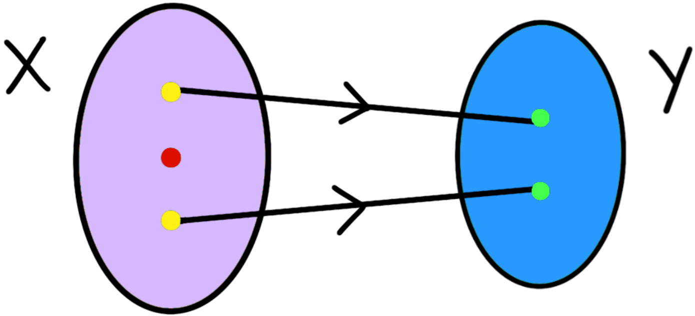
Basic fact 1) A function cannot change its mind.

If $f(x) = 3$ today, then $f(x) = 3$ tomorrow.



That picture does not represent a function.

Basic fact 2) A function must be defined everywhere.



That is not a function either.

Basic facts about functions.

Basic fact 3) A function might not be defined by “a rule which you can type into a calculator”.

Example: $X = \{3, 37, \pi\}$, $Y = \mathbb{R}$, $f : X \to Y$ with $f(3) = 3$, $f(37) = e^{\pi}$, $f(\pi) = -\sqrt{2}$.

Sometimes we do calculations with specific functions. But sometimes we prove theorems about general classes of functions. The function is the variable.

“If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function, then f is continuous”.

In theory, a function is supposed to know its domain and codomain.

In practice, life is not so easy.

$$\sin : \mathbb{R} \to \mathbb{R}$$

$$\sin : \mathbb{Z} \to [-1, 1]$$

(here $[-1, 1]$ means $\{x \in \mathbb{R} \mid -1 \leq x \leq 1\}$).

The set of all functions from X to Y is denoted $\mathrm{Hom}(X, Y)$.

So we could write $f \in \mathrm{Hom}(X, Y)$ instead of $f : X \to Y$.

Non-examinable remark: In type theory, `X → Y` is the type of functions from X to Y, and `f : X → Y` is a term of this type.

When are two functions equal?

Mathematicians treat functions as *extensional* objects, like sets and propositions.

For us mathematicians, a function is determined by *what it does*, not *how it does it*.

Formally, if f and g are functions which have different domains or codomains, we don't usually ask whether f and g are "equal".

But if $f : X \to Y$ and $g : X \to Y$ then

$$f = g \iff \forall x \in X, f(x) = g(x).$$

Two functions are equal if and only if they are the same on all inputs.

Computer science.

Mathematical theorem: all sorting algorithms are equal.

A key difference between mathematics and computer science: computer scientists think of functions as algorithms, but we believe in functional extensionality.

Mathematicians think about specifications, and computer scientists think about implementations.