Addition is a function which takes *two* numbers as input.

So if I gave you *five* numbers (e.g. 3, 8, 7, 2, 5) and asked you to add them all up, what would you do?

Because addition is *commutative* ($a + b = b + a$) and *associative* ($(a + b) + c = a + (b + c)$), it doesn't matter which order we add them up in.

e.g. we can do
$((3 + 7) + (8 + 2)) + 5 = (10 + 10) + 5 = 20 + 5 = 25.$

The same is true for finite unions of sets.

In other words, if I have sets A, B, C, D, E then we can unambiguously compute their *union* $A \cup B \cup C \cup D \cup E$.

It doesn't matter which order I compute this in. Why?

Union is also associative and commutative.

But things get more problematic with *infinite* sums.

Maybe you saw the following trick. What is

$$1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots?$$

Well, it's

$$(1 + (-1)) + (1 + (-1)) + (1 + (-1)) + \cdots = 0 + 0 + 0 + \cdots$$

so it's zero. But it's also

$$1 + ((-1) + 1) + ((-1) + 1) + ((-1) + 1) + \cdots = 1 + 0 + 0 + 0 + \cdots$$

so it's also 1.

The theory of infinite sums is subtle. You will learn it later on in the 1st year.

But the theory of infinite *unions* and *intersections* is not so problematic.

Let’s see why.

Say α is our underlying set, and we have infinitely many subsets $X_0, X_1, X_2, \ldots$ of it.

We want to define the *union* $\bigcup_{i \in \mathbb{N}} X_i$.

When should an element a of α be in this infinite union?

a should be in the union if it is in at least one of the X_i.

Can you write down the statement "there exists at least one i such that $a \in X_i$" using some mathematical notation we learnt earlier? Pause the video and have a go.

We can write it as $\exists i \in \mathbb{N}, a \in X_i$.

Note that in the past we had $\exists a \in \alpha$; this time there are two sets involved, namely α and $\mathbb{N}$.

Similarly, we want the infinite intersection $\bigcap_{i \in \mathbb{N}} X_i$ to be the elements which are in *all* of the X_i. Pause the video and fill in the right hand side of this equation yourself:

$$a \in \bigcap_{i \in \mathbb{N}} X_i \iff ???$$

The answer is

$$a \in \bigcap_{i \in \mathbb{N}} X_i \iff \forall i \in \mathbb{N}, a \in X_i.$$

More generally, you could let I be any "index set", with α a "background set" as usual, and you could imagine a subset $X_i \subseteq \alpha$, one for each $i \in I$.

Then

$$a \in \bigcup_{i \in I} X_i \iff \exists i \in I, a \in X_i$$

and

$$a \in \bigcap_{i \in I} X_i \iff \forall i \in I, a \in X_i.$$

These look like big blobs of complicated maths! Make sure you understand them.

Just like $\cup$ corresponded to $\vee$, the arbitrary union $\bigcup_{i \in I} \cdots$ corresponds to $\exists i \in I, \ldots$.

Here is an example. Let $I = \mathbb{Z}$, the integers $\{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$.

Let $\alpha = \mathbb{R}$, the real numbers.

For each $i \in I$, let X_i be the set $\{ a \in \mathbb{R} \mid i \leq a \leq i + 1 \}$. What does this mean?

It means that X_i is the set of real numbers between i and $i + 1$.

What is $\bigcup_{i \in I} X_i$? Pause the video and see if you can figure out what this set is.

The answer is that this union is the whole of $\mathbb{R}$.

A careful proof of this might look like the following. Say $a \in \mathbb{R}$. We need to check that $a \in \bigcup_{i \in \mathbb{Z}} X_i$.

Equivalently, we need to check that $\exists i \in \mathbb{Z}, a \in X_i$.

Substituting in the definition of X_i, we need to find $i \in \mathbb{Z}$ such that $i \leq a \leq i + 1$.

If i is the *floor*, or *integer part* of a, then this i will work (this is a standard fact about real numbers).

Hence an i does exist with $a \in X_i$, and because a was arbitrary we have proved $\mathbb{R} \subseteq \bigcup_{i \in I} X_i$, and the other inclusion is obvious.