Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2019-2020

Introduction to University Mathematics **MATH40001/MATH40009**

Test 2 - Solutions

1. **Total: 10 Marks**

   (a) i. Two integers a and b are congruent modulo n ($a \equiv b \mod n$) if $n|a-b$.

   1 Mark

   ii. For all integers a and b, $b > 0$ there exist unique integers q and r such that

   $$a = bq + r, \quad 0 \leq r < b.$$

   1 Mark

   (b) $\gcd(486, 160)$=2 and $\text{lcm}(48, 52) = 624$.

   2 Marks

   (c) Let n be a positive integer and $n|ab$, with a, b some integers. Show that if $\gcd(n, a) = 1$, then $n|b$.

   Proof. If $(n, a) = 1$, there exists integers c, d such that $cn + ad = 1$. Because $n \mid ab$, there exists an integer m such that $ab = nm$ or equivalently $a = \frac{nm}{b}$.
   Substituting, we get that $cn + \frac{nmd}{b} = 1$. Multiplying both sides by b, we get that

   $$n \cdot cb + \frac{nmd}{b} \cdot b = n(cb + md) = b,$$

   and $cd + mb$ is an integer because of the closure of $\mathbb{Z}$ under addition and multiplication. Thus, by the definition of divisors, $n \mid b$. $\square$

   4 Marks

   (d) Show that for any integers $a, b, c, d \in \mathbb{Z}$, if If $a \equiv b(\mod n)$ and $c \equiv d(\mod n)$, then $ac \equiv bd(\mod n)$.

   Proof. Write $b = a+nq_1$ and $d = c+nq_2$. Then multiplying, we get $bd = (a+nq_1)(c+nq_2) = ac+naq_2+ncq_1+n^2q_1q_2$. Thus, $bd = ac+n(aq_2+cq_1+nq_1q_2)$, and so $ac \equiv bd \mod n$. $\square$

   2 Marks

2. **Total: 10 Marks**

   (a) Let S be a set.

   i. A non-empty set S is called bounded below if there is a number B (the lower bound) such that $x \geq B$ for all $x \in S$.

   1 Mark

   ii. We call s a least upper bound or supremum of S if

   A. s is an upper bound for S,

   B. if $B < s$ then B is not an upper bound for S.

   1 Mark

   (b) Show directly from the axiom that for given $a, b \in \mathbb{R}$ with $a \neq 0$ there is exactly one x such that $ax = b$.

Proof. Since $a \in \mathbb{R}$ and $a \neq 0$, we know there exists an inverse $y \in \mathbb{R}$ such that $a \cdot y$= 1 (existence of multiplicative inverse). Let $x = yb$. Then,

$$ax = a(yb) = (ay)b = 1b = b.$$

Further, this solution is unique since if x' is another solution, then $ax = b$ and $ax' = b$ implies $ax = ax'$ which in turn implies $x = x'$ using the cancellation rule. □

2 Marks

(c) State the Archimedean property and show that it is equivalent to the fact that the set of natural numbers $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

Proof. Archimedean property: for each $x, y \in \mathbb{R}$, $x > 0$, there exists an $n \in \mathbb{N}$ such that $nx > y$
Suppose that $\mathbb{N}$ is bounded above. This means that there exists a real number $k \in \mathbb{R}$, that is $n = n \cdot 1 \leq k$ for all $n \in \mathbb{N}$. But this contradicts Archimedean property for $x = 1$ and $y = k$.
Conversely if for each $x, y \in \mathbb{R}$, $x > 0$, there exists an $n \in \mathbb{N}$ such that $nx > y$, this means that especially this is true for or $x = 1$ and $y = k$, or in other words for all $k \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ sucht that $n > k$,which is exactly the definition of unbounded for N. □

3 Marks

(d) Let $S = \{x \in \mathbb{R} | x^4 < 16\}$. Find $\inf S$ (justify your answer).

Proof. Note that $S = \{x \in \mathbb{R} | -2 < x < 2\}$. So -2 is a lower bound of S. Let M be another lower bound of S. Suppose that $M > -2$. Since the rationals are dense in $\mathbb{R}$, there exists a rational number r such that $M > r > -2$. Then $r \in S$ and $M > r$. But this is a contradiction to the fact that M is a lower bound of S. Thus, $-2 \geq M$ for all lower bounds M so that $\inf S = -2$. Alternatively one can set $r = \frac{M-2}{2}$, the average of M and -2, and proceed in the same way.

□

3 Marks

3. **Total: 10 Marks**

(a)
i. $\mathbf{u} + (\mathbf{v} \cdot \mathbf{w})$ - **not allowed** - sum of a scalar and a vector
ii. $(\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{w}$ - **not allowed** - scalar product returns a scalar
iii. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w}$ - **allowed**
iv. $(\mathbf{u} + \mathbf{v})|\mathbf{w}|$ - **allowed** - scalar multiplication
v. $|\mathbf{u} + \mathbf{v}||\mathbf{w}|$ - **allowed** - multiplication of scalars

2 Marks

(b) Let $\mathbf{u} = (1, -2, 2)$ and $\mathbf{v} = (2, 3, 2)$ be vectors in $\mathbb{R}^3$.
To show that the vectors are perpendicular, one can take their scalar product

$$\mathbf{u} \cdot \mathbf{v} = 1 \times 2 + (-2) \times 3 + 2 \times 2 = 0$$

As, $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u}$ is perpendicular to $\mathbf{v}$.

1 Mark

We can verify Pythagoras theorem for the triangle with vertices $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$. Thus, one needs to check that $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$ We obtain

$$|\mathbf{u} - \mathbf{v}|^2 = (2-1)^2 + (3-(-2))^2 + (2-2)^2 = 26$$
$$|\mathbf{u}|^2 + |\mathbf{v}|^2 = 1^2 + (-2)^2 + 2^2 + 2^2 + 3^2 + 2^2 = 26$$

So, we verified Pythagoras theorem!

1 Mark

We can verify Pythagoras theorem for the triangle with vertices $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$. Thus, one needs to check that $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$ We obtain

$$|\mathbf{u}+\mathbf{v}|^2 = (2+1)^2 + (3+(-2))^2 + (2+2)^2 = 26$$
$$|\mathbf{u}|^2 + |\mathbf{v}|^2 = 1^2 + (-2)^2 + 2^2 + 2^2 + 3^2 + 2^2 = 26$$

So, we verified Pythagoras theorem!

1 Mark

(c) i. Let $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ be vectors in $\mathbb{R}^n$. We assume $\mathbf{u}\cdot\mathbf{w} = \mathbf{v}\cdot\mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}^n$. Thus, we know that it is true in particular for the vectors of the canonical basis. So we can write:

$$\forall i \in [1,n], \mathbf{u}\cdot\hat{\mathbf{e}}_i = \mathbf{v}\cdot\hat{\mathbf{e}}_i \iff \forall i \in [1,n], u_i = v_i \iff \mathbf{u} = \mathbf{v}$$

2 Marks

ii. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors of $\mathbb{R}^n$, the triangle inequality states that

$$|\mathbf{u}+\mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$

with equality if and only if one of the vectors $\mathbf{u}$ or $\mathbf{v}$ is proportional to the other with a nonnegative constant of proportionality.
Substituting in the triangle inequality the following vector $\mathbf{u}+\mathbf{v}$ and $-\mathbf{v}$ for $\mathbf{u}$ and $\mathbf{v}$, we obtain

$$|\mathbf{u}| \leq |\mathbf{u}+\mathbf{v}| + |\mathbf{v}| \iff |\mathbf{u}+\mathbf{v}| \geq |\mathbf{u}| - |\mathbf{v}|$$

By symmetry, we also have the following:

$$|\mathbf{u}+\mathbf{v}| \geq |\mathbf{v}| - |\mathbf{u}|$$

So we conclude that

$$|\mathbf{u}+\mathbf{v}| \geq \big||\mathbf{u}| - |\mathbf{v}|\big|$$

2 Marks

We have equality if and only if one of the vectors $\mathbf{u}$ and $\mathbf{v}$ is proportional to the other with a nonpositive proportionality constant.

1 Mark