

Problem Sheet 0

Welcome to your first Problem Class at Imperial! This first problem sheet contains a number of problems whose topics have been drawn from the A-levels curriculum. It is meant as an icebreaker to get you started discussing maths with your new colleagues. Try to solve these problems as carefully as possible using mathematical notation and the proper quantifiers when appropriate.

1. Let n be an integer. Prove (carefully) that
 - (a) if 2 divides n , then 2 divides n^2 .
 - (b) if 2 divides n^2 , then 2 divides n .
2. Prove that 3 divides $n^3 - n$ for all integers $n \geq 0$.
3. Can you think of a function which is discontinuous everywhere?
4. What about one which is differentiable everywhere, but with a discontinuous derivative?
5. Can an initial value problem have more than one solution?
6. Consider a box containing radioactive atoms, at any time t , we denote $x(t)$ the number of radioactive atoms remaining in the box. With time, these atoms decay; as each atom has the same chance to decay, the rate of change of atom number is proportional to the number of atoms remaining. Can you write an ordinary differential equation governing the number of atoms in the box? Find a solution to this problem, knowing that there was initially x_0 atoms in the box.
7. Prove the *conjugate root theorem*: for any polynomial $P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ with real coefficients a_0, \dots, a_n , if z is a root of P , then so is \bar{z} .
8. Show that if $|z| = 1$ and $z \neq -1$, then

$$\operatorname{Im} \frac{z}{(z+1)^2} = 0.$$

Is there a nice geometric interpretation of this equation? Find all the points on the complex plane such that $\operatorname{Im} \frac{z}{(z+1)^2} = 0$ — there are more of them than just the ones on the unit circle.

9. Can you find the maximum number of pieces of pizza that can be made with a given number n of straight cuts?
10. For an arithmetic sequence, can you prove that

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}, \text{ for } n \geq 2?$$

What about a geometric sequence with terms a_n all positive, can you write a proof for the following relationship:

$$a_n = \sqrt{a_{n-1}a_{n+1}}, \text{ for } n \geq 2?$$

11. Can you think of any similarities between the *sequences-series* relationship and the *derivative-integral* one? What happens if you integrate and then differentiate? What about if you go from a_n to S_n and then try to make a_n again?

12. Can one take the sine or cosine of complex numbers? If so, what value does $\sin(1 + i)$ take?
13. (a) Find a matrix which describes an anticlockwise rotation of $\frac{3\pi}{4}$ radians about the origin.
- (b) What transformation does the matrix $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ represent?
- (c) What about this matrix: $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$?