

WELL ORDERING: VIDEO V

Partial and total orders on sets

Reminder: Partial and total order

Things you need to look at if you don't feel comfortable with them:
EQUIVALENCE RELATIONS AND ORDER RELATIONS

Things you need to look at if you don't feel comfortable with them:
EQUIVALENCE RELATIONS AND ORDER RELATIONS

A **partial order relation** R on a set X is a binary relation such that

- $\forall x \in X, R(x, x)$ (Reflexivity)

Things you need to look at if you don't feel comfortable with them:
EQUIVALENCE RELATIONS AND ORDER RELATIONS

A **partial order relation** R on a set X is a binary relation such that

- $\forall x \in X, R(x, x)$ (Reflexivity)
- $\forall x, y \in X, R(x, y) \wedge R(y, x) \Rightarrow x = y$ (Anti-symmetry)

Things you need to look at if you don't feel comfortable with them:
EQUIVALENCE RELATIONS AND ORDER RELATIONS

A **partial order relation** R on a set X is a binary relation such that

- $\forall x \in X, R(x, x)$ (Reflexivity)
- $\forall x, y \in X, R(x, y) \wedge R(y, x) \Rightarrow x = y$ (Anti-symmetry)
- $\forall x, y, z \in X, R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$ (Transitivity)

Reminder: Partial and total order

Things you need to look at if you don't feel comfortable with them:
EQUIVALENCE RELATIONS AND ORDER RELATIONS

A **partial order relation** R on a set X is a binary relation such that

- $\forall x \in X, R(x, x)$ (Reflexivity)
- $\forall x, y \in X, R(x, y) \wedge R(y, x) \Rightarrow x = y$ (Anti-symmetry)
- $\forall x, y, z \in X, R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$ (Transitivity)

R is called a **total order** if additionally $\forall x, y \in X, R(x, y) \vee R(y, x)$

Reminder: Partial and total order

Things you need to look at if you don't feel comfortable with them:
EQUIVALENCE RELATIONS AND ORDER RELATIONS

A **partial order relation** R on a set X is a binary relation such that

- $\forall x \in X, R(x, x)$ (Reflexivity)
- $\forall x, y \in X, R(x, y) \wedge R(y, x) \Rightarrow x = y$ (Anti-symmetry)
- $\forall x, y, z \in X, R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$ (Transitivity)

R is called a **total order** if additionally $\forall x, y \in X, R(x, y) \vee R(y, x)$

Can you think of an partial order which is not total?

Reminder: Partial and total order

Things you need to look at if you don't feel comfortable with them:
EQUIVALENCE RELATIONS AND ORDER RELATIONS

A **partial order relation** R on a set X is a binary relation such that

- $\forall x \in X, R(x, x)$ (Reflexivity)
- $\forall x, y \in X, R(x, y) \wedge R(y, x) \Rightarrow x = y$ (Anti-symmetry)
- $\forall x, y, z \in X, R(x, y) \wedge R(y, z) \Rightarrow R(x, z)$ (Transitivity)

R is called a **total order** if additionally $\forall x, y \in X, R(x, y) \vee R(y, x)$

Can you think of an partial order which is not total?

Example: You have seen that \subseteq is a partial but not a total order on the set of subsets of a set X .

Order on \mathbb{N}

Definition

Let $x, y \in \mathbb{N}$.

We say that x is **smaller than or equal to** y if $\exists v \in \mathbb{N}$ such that $y = x + v$.

Notation: $x \leq y$.

Definition

Let $x, y \in \mathbb{N}$.

We say that x is **smaller than or equal to** y if $\exists v \in \mathbb{N}$ such that $y = x + v$.

Notation: $x \leq y$.

If additionally $v \neq 0$, we say that x is strictly smaller than y and we denote it by $x < y$.

Order on \mathbb{N}

Definition

Let $x, y \in \mathbb{N}$.

We say that x is **smaller than or equal to** y if $\exists v \in \mathbb{N}$ such that $y = x + v$.

Notation: $x \leq y$.

If additionally $v \neq 0$, we say that x is strictly smaller than y and we denote it by $x < y$.

Proposition (\leq is a total order on \mathbb{N})

For all $x, y, z \in \mathbb{N}$, the inequality \leq satisfies the following properties.

- 1) $x \leq x$ (Reflexivity),
- 2) If $x \leq y$ and $y \leq z$, then $x \leq z$ (Transitivity),
- 3) If $x \leq y$ and $y \leq x$, then $x = y$ (Antisymmetry).
- 4) Either $x \leq y$ or $y \leq x$. (Total Order)

Let $x, y \in \mathbb{N}$. Then

- 1) If $x + y = x$, then $y = 0$.
- 2) If $x + y = 0$, then $x = 0$ and $y = 0$

Proof of previous proposition.

- 1) Reflexivity: $x \leq x$
- 2) Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$
- 3) Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$



WELL ORDERING: VIDEO VI

The Well-Ordering Principle

Well-Ordering Principle

Proposition (Well-ordering principle)

Every **non-empty set** of **natural** numbers $A \subseteq \mathbb{N}$ has a **least element**, i.e. there exists a number $a \in A$, such that $a \leq x$, for all $x \in A$.

Well-Ordering Principle

Proposition (Well-ordering principle)

Every **non-empty set** of **natural** numbers $A \subseteq \mathbb{N}$ has a **least element**, i.e. there exists a number $a \in A$, such that $a \leq x$, for all $x \in A$.

Proof by contradiction!

IS THIS RESULT TRIVIAL AND USELESS?

IS THIS RESULT TRIVIAL AND USELESS?

Actually it can be very handy to prove that a statement is true for all $n \in \mathbb{N}$

IS THIS RESULT TRIVIAL AND USELESS?

Actually it can be very handy to prove that a statement is true for all $n \in \mathbb{N}$

Well-ordering proofs for statements of the type $\forall n \in \mathbb{N}, P(n)$:

IS THIS RESULT TRIVIAL AND USELESS?

Actually it can be very handy to prove that a statement is true for all $n \in \mathbb{N}$

Well-ordering proofs for statements of the type $\forall n \in \mathbb{N}, P(n)$:

- Define the set of counterexamples of P $A := \{n \in \mathbb{N} \mid \neg P(n)\}$

IS THIS RESULT TRIVIAL AND USELESS?

Actually it can be very handy to prove that a statement is true for all $n \in \mathbb{N}$

Well-ordering proofs for statements of the type $\forall n \in \mathbb{N}, P(n)$:

- Define the set of counterexamples of P $A := \{n \in \mathbb{N} \mid \neg P(n)\}$
- Assume $A \neq \emptyset$

IS THIS RESULT TRIVIAL AND USELESS?

Actually it can be very handy to prove that a statement is true for all $n \in \mathbb{N}$

Well-ordering proofs for statements of the type $\forall n \in \mathbb{N}, P(n)$:

- Define the set of counterexamples of P $A := \{n \in \mathbb{N} \mid \neg P(n)\}$
- Assume $A \neq \emptyset$
- By the well-ordering principle conclude that there is a least element $l \in A$.

IS THIS RESULT TRIVIAL AND USELESS?

Actually it can be very handy to prove that a statement is true for all $n \in \mathbb{N}$

Well-ordering proofs for statements of the type $\forall n \in \mathbb{N}, P(n)$:

- Define the set of counterexamples of P $A := \{n \in \mathbb{N} \mid \neg P(n)\}$
- Assume $A \neq \emptyset$
- By the well-ordering principle conclude that there is a least element $l \in A$.
- Reach somehow a contradiction (often showing that there is another member $a \in A$, such that $a < l$.

IS THIS RESULT TRIVIAL AND USELESS?

Actually it can be very handy to prove that a statement is true for all $n \in \mathbb{N}$

Well-ordering proofs for statements of the type $\forall n \in \mathbb{N}, P(n)$:

- Define the set of counterexamples of P $A := \{n \in \mathbb{N} \mid \neg P(n)\}$
- Assume $A \neq \emptyset$
- By the well-ordering principle conclude that there is a least element $l \in A$.
- Reach somehow a contradiction (often showing that there is another member $a \in A$, such that $a < l$.
- Conclude that $A = \emptyset$ and therefore $P(n)$ is always true.

IS THIS RESULT TRIVIAL AND USELESS?

Actually it can be very handy to prove that a statement is true for all $n \in \mathbb{N}$

Well-ordering proofs for statements of the type $\forall n \in \mathbb{N}, P(n)$:

- Define the set of counterexamples of P $A := \{n \in \mathbb{N} \mid \neg P(n)\}$
- Assume $A \neq \emptyset$
- By the well-ordering principle conclude that there is a least element $l \in A$.
- Reach somehow a contradiction (often showing that there is another member $a \in A$, such that $a < l$.
- Conclude that $A = \emptyset$ and therefore $P(n)$ is always true.

Example: Show that all $n \in \mathbb{N}$, $n > 1$ can be factored as a product of primes.

- Define the set of counterexamples:
 - Assume $A \neq \emptyset$.
 - By the well-ordering principle, there is a least element $l \in A$.
 - Reach the contradiction:
-
- Conclude that $A = \emptyset$ and therefore...