Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
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Introduction to University Mathematics **MATH40001/MATH40009**

Solutions to Mid-module Test

1. **Total: 20 Marks**

   (a) Show that for all x, y in $\mathbb{N}$, $\nu(x) \cdot y = x \cdot y + y$.

   Proof. We use (P5) and define the set $A = \{y \in \mathbb{N} | \nu(x) \cdot y = x \cdot y + y\}$. 0 is in A because $\nu \cdot 0 = 0$ by definition of multiplication, and $x \cdot 0 + 0 = 0 + 0 = 0$ by definition of addition and multiplication.
   Assume now that $\nu(x) \cdot y = x \cdot y + y$ for all y in $\mathbb{N}$. We have

$$\begin{aligned} \nu(x) \cdot \nu(y) &= \nu(x) \cdot y + \nu(x) = (x \cdot y + y) + \nu(x) = x \cdot y + (y + \nu(x)) \\ &= x \cdot y + \nu(y + x) = x \cdot y + \nu(x + y) = x \cdot y + (x + \nu(y)) \\ &= (x \cdot y + x) + \nu(y) = x \cdot \nu(y) + \nu(y), \end{aligned}$$

   using (in that order): the definition of multiplication, the induction hypothesis, associativity of addition, the definition of addition, commutativity of addition, the definition of addition, associativity of addition, and finally the definition of multiplication. □

   (b) i. Show that for all $x, y, z \in \mathbb{N}$, if $y \leq z$, then $x + y \leq x + z$.

   Proof. Assume that $y \leq z$. By definition of the ordering, this means that $z = y + v$, for some $v \in \mathbb{N}$. But then $x + z = x + (y + v) = (x + y) + v$ by commutativity and associativity of the addition. Consequently $x + y \leq x + z$ by definition of the ordering. □

   ii. Show that for all $x \in \mathbb{N}$, $x < \nu(x)$

   Proof. By the definition of addition we have $\nu(x) = \nu(x+0) = x+\nu(0)$, hence $x \leq \nu(x)$ by the definition of the order. But $x \neq \nu(x)$ by axiom (P3). Indeed assume $x = \nu(x)$, then as before we get $x + 0 = x = \nu(x) = x + \nu(0)$ and by the cancellation property for addition we have $0 = \nu(0)$, which contradicts axiom (P3). Hence $x < \nu(x)$. □

   iii. if $x < y$, then $\nu(x) \leq y$.

   Proof. We have by assumption that $y = x + n$ for some $n \in \mathbb{N}$, which is non-zero since $y \neq n$. Then $n = \nu(m)$ for some m (by existence of predecessor) and hence $y = x + \nu(m) = \nu(x + m) = \nu(m + x) = m + \nu(x) = \nu(x) + m$. □

   iv. if $x \cdot k = y$ and $x \cdot l = \nu(y)$, for some $k, l \in \mathbb{N}$, then $x = 1$.

   Proof. We have $y < \nu(y)$, hence $x \cdot k < x \cdot l$. We claim that $k < l$. If not, either $k = l$, in which case $x \cdot k = x \cdot l$ (a contradiction), else $k > l$, in which case $x \cdot k \geq x \cdot l$, again a contradiction. So $l = k + m$ for some $m \in \mathbb{N}$, non-zero since $l \neq k$. Since m is non-zero, we have $m = \nu(m')$ for some m', and hence $m = \nu(m') \geq \nu(0) = 1$. Then by distributivity, $x \cdot l = (x \cdot k) + (x \cdot m) \geq (x \cdot k) + (x \cdot m) \geq (x \cdot k) + (x \cdot 1) = (x \cdot k) + x$. Thus, $\nu(y) \geq y + x$. Since $y \neq \nu(y)$, we can't have $x = 0$. Thus there is a predecessor of x, write $x = \nu(x')$. Then $\nu(y) \geq y + \nu(x') = \nu(y + x')$. Therefore $\nu(y) = \nu(y + x') + r = \nu(y + x' + r)$ for some $r \in \mathbb{N}$. As a result $y = y + x' + r$, so $x' + r = 0$ by cancellation. Finally, since $x' = 0$, $x = \nu(0) = 1$. □

2. **Total: 20 Marks**

   (a) Prove that there exists a function $g(n) = 2^n = \underbrace{2 \cdot \ldots \cdot 2}_{n \text{ times}}$. **4 Marks**

Proof. This is just the recursively defined function $R(n)$ defined by $R(0) = 1$ and $R(\nu(n)) = 2R(n)$. $\square$

(b) Show that $2^n > 2$ for $n \geq 2$. $\boxed{\text{3 Marks}}$

Proof. Let $A \subseteq \mathbb{N}$ be the subset of all n such that $2^n > 2$. Notice that $2 \in A$ since $2^2 = 2 \cdot 2 = 4 \geq 2$. Suppose that $n \in A$. Then $2^n > 2$. Then $2^{\nu(n)} = 2 \cdot 2^n = 2^n + 2^n > 2 + 2 > 2$, so that $\nu(n) \in A$ as well. We conclude that the set $B := A \cup \{0, 1\}$ has the property that $B \in A$ and $n \in B$ implies $\nu(n) \in B$ (since if $n \in B$, either $n \in A$, or $n \in \{0, 1\}$, and both $1 = \nu(0)$ and $2 = \nu(1)$ are in B). So by P5, $B = \mathbb{N}$. As $n \geq 2$ implies $n \notin \{0, 1\}$, this implies $n \in A$. Therefore $2^n > 2$ for $n \geq 2$. $\square$

(c) Let $R_{f,x} : \mathbb{N} \to \mathbb{N}$ denote a recursively defined function such that $R_{f,x}(0) = x$ and $R_{f,x}(\nu(n)) = f(R_{f,x}(n))$. Prove that $R_{f,x}(2n) = R_{f \circ f,x}(n)$. $\boxed{\text{5 Marks}}$

Proof. Let $A \subseteq \mathbb{N}$ be the subset of all n such that $R_{f,x}(2n) = R_{f \circ f,x}(n)$. Note that $0 \in A$ since $R_{f,x}(2 \cdot 0) = R_{f,x}(0) = x = R_{f \circ f,x}(0)$. Now suppose that $n \in A$. Then $R_{f,x}(2 \cdot n) = R_{f \circ f,x}(n)$. Applying $f \circ f$ to both sides, we get $R_{f,x}(2 \cdot n + 2) = R_{f,x}(2(n + 1)) = R_{f,x}(2\nu(n)) = R_{f \circ f,x}(\nu(n))$. By P5 we get that $A = \mathbb{N}$, as desired. $\square$

(d) Using previous parts, show that $4^n = 2^{2n}$, where here $4^n = \underbrace{4 \cdot \ldots \cdot 4}_{n \text{ times}}$, defined as in the first part. $\boxed{\text{4 Marks}}$

Proof. As in the first part, we have $4^n = R_{h,1}(n)$ for $h(x) = 4 \cdot x$, whereas $2^n = R_{f,1}(n)$ for $f(x) = 2 \cdot x$. Then $h(x) = f(f(x))$, by the associativity property: $2 \cdot (2 \cdot x) = (2 \cdot 2) \cdot x = 4 \cdot x$. Thus $4^n = R_{f \circ f,1}(n) = R_{f,1}(2n) = 2^{2n}$. $\square$

(e) Prove using the well ordering property that the set $S := \{3m + 4n \mid m, n \in \mathbb{N}\}$ contains all natural numbers other than $1, 2$, and 5.

Proof. We see that $0, 3, 4 \in S$, since $3 \cdot 0 + 4 \cdot 0 = 0$, and $3 \cdot 1 + 4 \cdot 0 = 3$, and $3 \cdot 0 + 4 \cdot 1 = 4$. Now let A be the set of all $n > 5$ such that $n \notin S$. We need to show that A is empty. If it is not, let $x \in A$ be the least element. Since $3 + 3 = 6 \in A$, $3 + 4 = 7 \in A$, $4 + 4 = 8 \in A$, and $3 \cdot 3 = 9 \in A$, it follows that $x \geq 10$. Then $x - 3 \geq 7$ is not in A, that is, $\pi(\pi(\pi(x))) \in S$. Writing $\pi(\pi(\pi(x))) = 3m + 4n$, we get $x = \nu(\nu(\nu(\pi(\pi(\pi(x)))))) = \nu(\nu(\nu(3m + 4n))) = (3m + 4n) + 3 = (3m + 3) + 4n = 3(m + 1) + 4n$. $\square$

3. $\boxed{\textbf{Total: 20 Marks}}$

(a) Consider $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, suppose that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \sqrt{3}$. We find that

i. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \sqrt{3}$.

ii. $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v}) = -\sqrt{3}$ as $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$ and we just obtained that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \sqrt{3}$.

iii. $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}) = -\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = -\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -\sqrt{3}$.

iv. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ as $\mathbf{u} \times \mathbf{v}$ is a vector orthogonal to $\mathbf{u}$.

(b) In $\mathbb{R}^3$, let $\mathbf{a}$ be an arbitrary vector and $\mathbf{n}$ be a unit vector in some fixed direction. We notice that an expansion of the triple vector product $\mathbf{n} \times (\mathbf{a} \times \mathbf{n})$ gives the following expression

$$\mathbf{n} \times (\mathbf{a} \times \mathbf{n}) = \mathbf{a}(\mathbf{n} \cdot \mathbf{n}) - \mathbf{n}(\mathbf{a} \cdot \mathbf{n})$$

As $\mathbf{n}$ is a unit vector, we obtain that $\mathbf{a}(\mathbf{n} \cdot \mathbf{n}) = \mathbf{a}$; thus, we can write

$$\mathbf{a} = \mathbf{n}(\mathbf{a} \cdot \mathbf{n}) + \mathbf{n} \times (\mathbf{a} \times \mathbf{n})$$

The first term $\mathbf{n}(\mathbf{a} \cdot \mathbf{n})$ is the component of $\mathbf{a}$ in the direction $\mathbf{n}$, while the second term in the above expression $\mathbf{n} \times (\mathbf{a} \times \mathbf{n})$ is the component of $\mathbf{a}$ that is orthogonal to $\mathbf{n}$.

(c) Consider the two lines with Cartesian equations

$$\begin{cases} \mathcal{L}_1 : 3x + 4y + 3 = 0 \\ \mathcal{L}_2 : 12x - 5y + 4 = 0 \end{cases}$$

We want to find the Cartesian equations of the angle bisectors of these two lines which we will call respectively $\mathcal{B}_1$ and $\mathcal{B}_2$. First, we notice that $(3, 4)$ and $(12, -5)$ are not collinear, and so indeed the lines are not parallel, they intersect in a single point and it makes sense to talk about their angle bisectors. Without loss of generality, we can place the intersection point of the two lines at the origin O. Let M be a point on one of the bisectors, say $\mathcal{B}_1$. Simple trigonometry tells us that

$$\sin(\alpha_1) = \frac{d_1}{|\mathbf{OM}|} \quad \text{and} \quad \sin(\alpha_2) = \frac{d_2}{|\mathbf{OM}|}$$

with α_1 the angle between $\mathcal{B}_1$ and $\mathcal{L}_1$, α_2 the angle between $\mathcal{B}_1$ and $\mathcal{L}_2$, d_1 the distance between M and the line $\mathcal{L}_1$ and d_2 the distance between M and the line $\mathcal{L}_2$.
Thus, we have that

$$d_1 = d_2 \iff \sin\alpha_1 = \sin\alpha_2 \iff \alpha_1 = \alpha_2 \quad \text{with} \quad \alpha_1, \alpha_2 \in [0, \pi/2]$$

So we can equivalently define the bisectors as the lines made out of the point which are equidistant to lines $\mathcal{L}_1$ and $\mathcal{L}_2$ (as this is not a result from the module, you need to prove it).
So we can use distances rather than angles to obtain equations of our bisectors. Let M be a point of the plane. We have that

$$\begin{aligned} M(x, y) \text{ is on one of the bisectors} &\iff M \in \mathcal{B}_1 \cup \mathcal{B}_2 \\ &\iff d(M, \mathcal{L}_1) = d(M, \mathcal{L}_2) \\ &\iff \frac{|3x + 4y + 3|}{\sqrt{9 + 16}} = \frac{|12x - 5y + 4|}{\sqrt{144 + 25}} \\ &\iff \frac{|3x + 4y + 3|}{5} = \frac{|12x - 5y + 4|}{13} \\ &\iff 13|3x + 4y + 3| = 5|12x - 5y + 4| \end{aligned}$$

Here, we need to be a little careful with the absolute values to maintain our equivalences!

$$\begin{aligned} M \in \mathcal{B}_1 \cup \mathcal{B}_2 &\iff \Big(13(3x + 4y + 3) = 5(12x - 5y + 4)\Big) \vee \Big(13(3x + 4y + 3) = -5(12x - 5y + 4)\Big) \\ &\iff \Big((39 - 60)x + (52 + 25)y + 39 - 20 = 0\Big) \\ &\qquad\quad \vee \Big((39 + 60)x + (52 - 25)y + 39 + 20 = 0\Big) \\ &\iff \Big(-21x + 77y + 19 = 0\Big) \vee \Big(99x + 27y + 59 = 0\Big) \end{aligned}$$

We can conclude that the Cartesian equations of the bisectors of $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by

$$\begin{cases} \mathcal{B}_1 : -21x + 77y + 19 = 0 \\ \mathcal{B}_2 : 99x + 27y + 59 = 0 \end{cases}$$

These two bisectors should by definition be orthogonal and indeed, we can check that if we define $\mathbf{n}_1 = (-21, 77)$ a vector orthogonal to $\mathcal{B}_1$ and $\mathbf{n}_2 = (99, 27)$ a vector orthogonal to $\mathcal{B}_2$, we have

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = -21 \times 99 + 77 \times 27 = -7 \times 3 \times 9 \times 11 + 7 \times 11 \times 3 \times 9 = 0$$

and the bisectors are thus orthogonal to each other.

(d) In the Euclidean space $\mathbb{R}^3$, we define $\mathcal{L}_1$ as the line through the points $(2,-2,4)$ and $(1,1,2)$ and $\mathcal{L}_2$ as the line through the points $(-1,2,0)$ and $(2,1,2)$.

i. The direction vectors associated with both lines are $\mathbf{u} = (-1,3,-2)$ and $\mathbf{v} = (3,-1,2)$. We have $\mathbf{u} \cdot \mathbf{v} = -3-3-4 = -10 \neq 0$ so the vectors are not perpendicular and by extension, the lines are not perpendicular either.

ii. In the previous question, we wrote down the direction vectors for both lines. We can then easily write parametric equations for both of these lines as

$$\begin{cases} \mathcal{L}_1 : \mathbf{r}(\lambda) = \mathbf{m} + \lambda \mathbf{u} = (1,1,2) + \lambda(-1,3,-2) \text{ with } \lambda \in \mathbb{R} \\ \mathcal{L}_2 : \mathbf{p}(\mu) = \mathbf{n} + \mu \mathbf{v} = (2,1,2) + \mu(3,-1,2) \text{ with } \mu \in \mathbb{R} \end{cases}$$

iii. We have defined the following vectors $\mathbf{m} = (1,1,2)$, $\mathbf{n} = (2,1,2)$, $\mathbf{u} = (-1,3,-2)$ and $\mathbf{v} = (3,-1,2)$. We are trying to find the shortest distance between the lines defined by

$$\mathbf{r}(\lambda) = \mathbf{m} + \lambda \mathbf{u} \ (\mathcal{L}_1) \quad \text{and} \quad \mathbf{p}(\mu) = \mathbf{n} + \mu \mathbf{v} \ (\mathcal{L}_2)$$

We know that the shortest distance from a point to a line is obtained for the orthogonal projection of the point on the line. Here, we are thus searching for points such that they are orthogonal to the lines $\mathcal{L}_1$ and $\mathcal{L}_2$. If R is the point (from line $\mathcal{L}_1$) at the shortest distance of $\mathcal{L}_2$ and we define P its orthogonal projection on $\mathcal{L}_2$; then P is the point at the shortest distance of $\mathcal{L}_1$ and P is its orthogonal projection on line $\mathcal{L}_1$. So finding the shortest distance between the lines is finding the distance between R and P with

$$(\mathbf{r}(\lambda) - \mathbf{p}(\mu)) \cdot (-1,3,-2) = 0 \quad \text{and} \quad (\mathbf{r}(\lambda) - \mathbf{p}(\mu)) \cdot (3,-1,2) = 0$$

The system of equation is equivalent to:

$$\begin{cases} (\mathbf{m} - \mathbf{n}) \cdot \mathbf{u} + \lambda \mathbf{u} \cdot \mathbf{u} - \mu \mathbf{v} \cdot \mathbf{u} = 0 \\ (\mathbf{m} - \mathbf{n}) \cdot \mathbf{v} + \lambda \mathbf{u} \cdot \mathbf{v} - \mu \mathbf{v} \cdot \mathbf{v} = 0 \end{cases}$$

In particular, we have the following: $\mathbf{u} \cdot \mathbf{u} = 14$, $\mathbf{u} \cdot \mathbf{v} = -10$ and $\mathbf{v} \cdot \mathbf{v} = 14$ as well as $\mathbf{m} - \mathbf{n} = (-1,0,0)$, $(\mathbf{m} - \mathbf{n}) \cdot \mathbf{u} = 1$ and $(\mathbf{m} - \mathbf{n}) \cdot \mathbf{v} = -3$. So we are trying to solve the following system of equations

$$\begin{cases} 14\lambda + 10\mu = -1 \\ -10\lambda - 14\mu = 3 \end{cases}$$

which admits a single solution $\lambda = 1/6$ and $\mu = -1/3$. Reinjecting this in the equations for $\mathbf{r}(\lambda)$ and $\mathbf{p}(\mu)$ to obtain $r(1/6) = (5/6, 3/2, 5/3)$ and $p(-1/3) = (1, 4/3, 4/3)$. We find that

$$d(\mathcal{L}_1, \mathcal{L}_2) = |\mathbf{r}(1/6) - \mathbf{p}(-1/3)| = |(-1/6, 1/6, 1/3)| = 1/\sqrt{6}$$