Imperial College London

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
Academic Year 2020-2021

Introduction to University Mathematics **MATH40001/MATH40009**

Solutions to Mid-module Test

1. **Total: 20 Marks**

(a) i) Show that for all $n \in \mathbb{N}$, $n \neq \nu(n)$.
Let $A = \{n \in \mathbb{N} | n \neq \nu(n)\}$. Obviously $0 \in A$, since $0 \neq \nu(0)$, by axiom (P3). Assume $n \in A$, then $n \neq \nu(n)$, but this means by injectivity of ν (axiom (P4)), that $\nu(n) \neq \nu(\nu(n))$, hence $\nu(n) \in A$, and $A = \mathbb{N}$ by axiom (P5).

ii) Show that for all n, $m \in \mathbb{N}$, $n \neq \nu(n+m)$.
Let $A = \{n \in \mathbb{N} | n \neq \nu(n+m), \forall m \in \mathbb{N}\}$. $0 \in A$ as $0 \neq \nu(m)$ for all m in $\mathbb{N}$. Assume $n \in A$, then $n \neq \nu(n+m), \forall n \in \mathbb{N}$. We want to show $\nu(n) \in A$. Assume $\nu(n) = \nu(\nu(n)+m)$ for some m, but then by axiom (P4), $n = \nu(n)+m = m+\nu(n) = \nu(n+m)$, by commutativity of addition and definition of addition. This contradict the induction hypothesis. Therefore $\nu(n) \in A$, and $A = \mathbb{N}$.

iii) Conclude that for all n, $m \in \mathbb{N}$ nonzero, $n \neq n+m$.
Since m is nonzero, $m = \nu(k)$ for some k, by Problem Sheet 1. Now $n+m = n+\nu(k) = \nu(n+k)$. By (ii), we have $n \neq \nu(n+k)$ and hence $n \neq n+m$.

(b) In this problem, don't use any properties of $\leq$ or $<$, just the definition.

i) Show that $(m \leq n) \Rightarrow (\nu(m) \leq \nu(n))$.
Suppose $m \leq n$. Then $n = m+u$ for some $u \in \mathbb{N}$ by definition of $\leq$. By commutativity, $n = u+m$. So $\nu(n) = \nu(u+m) = u+\nu(m)$ by Definition of Addition. By commutativity again, $\nu(n) = \nu(m)+u$. Thus $\nu(m) \leq \nu(n)$.

ii) Show that the relation $\leq$ is antisymmetric.
First we show that if $n \leq m$ and $m \leq n$, then $n = m$ (the antisymmetry property). Indeed, suppose $u, v \in \mathbb{N}$ satisfy $n = m+u$ and $m = n+v$. Then $n = (n+v)+u$. By associativity this implies $n = n+(v+u)$. This implies that $v+u = 0$: for otherwise, $v+u = \nu(k)$ for some $k \in \mathbb{N}$ (as shown in lectures), and then $n = n+\nu(k)$ which contradicts part (ii) above. But now by a result from lectures again we obtain $v = u = 0$. So then $n = m+0 = m$ and $n = m$.

ii) Show that no two of the properties $n < m$ or $n = m$ or $n > m$ can hold.
Now, by definition, $n < m$ implies that $n \neq m$ and similarly $n > m$ implies $n \neq m$. On the other hand, by antisymmetry, $n < m$ and $n > m$ implies $n = m$ but $n \neq m$, a contradiction. So at most one of the three properties above is satisfied.

(c) Let S be a nonempty subset of $\mathbb{Z}$. Show that if there exists an element $m \in \mathbb{Z}$ such that for all $x \in S$, $x > m$, then S has a least element.
By definition, for every $x \in S$, there exists $n \in \mathbb{N}$ such that $x = m+n$. Let $A := \{n \in \mathbb{N} \mid m+n \in S\}$. Since S is nonempty, we have seen that A must also be nonempty. By the well-ordering property, there is a least element $a \in A$. Now we claim that $c := m+a$ is the least element of S. Indeed, for every $x \in S$, $x = m+n$, and we have $n \geq a$ by definition of a. So $n = a+k$ for some $k \in \mathbb{N}$. Finally, $x = m+n = m+(a+k) = (m+a)+k$ where we used the associativity. Hence $x \geq c = m+a$, as desired.

2. **Total: 20 Marks**

(a) $\gcd(10672, 4147) = 29$.

(b) i. Show that if n, m and k are nonzero integers, then $k\,\mathrm{gcd}(m,n) = \mathrm{gcd}(km, kn)$.
Let $d = gcd(m,n)$, then $d \mid m, n$, hence $dk \mid km, kn$ and therefore $dk \mid \mathrm{gcd}(km, kn)$. Set $\mathrm{gcd}(km, kn) = d'dk$. So $d'dk \mid km, kn$. Hence then $d'd \mid m, n$ and so $d'd \mid \mathrm{gcd}(km, kn) = d$, which implies $d' = 1$. Thus $\mathrm{gcd}(km, kn) = d$.

ii. Show that if $a \equiv b \mod m$, then $ac \equiv bc \mod mc$, for $c > 0$.
If $a \equiv b \mod m$, then there exists an integer k, such that $a = b + km$. Therefore $ac = bc + (km)c = bc + k(mc)$ and by definition of equivalence class modulo mc, $ac \equiv bc \mod mc$.

(c) Let p be a prime, $p \mid n_1 n_2 \ldots n_k$, where n_1, $n_2 \ldots$, n_k are nonzero natural numbers. Then $p \mid n_j$, for some j, such that $1 \leq j \leq k$.
We prove this by induction. For $k = 1$ the result is trivial. Assume now that the result is true for an arbitrary k. Consider $n_1 n_2 ... n_{k+1}$ divisible by p. Notice that either $\mathrm{gcd}(p, n1n2...nk) = 1$ or $\mathrm{gcd}(p, n1n2...nk) = p$ Now if $\mathrm{gcd}(p, n_1 n_2 ... n_k) = 1$ then, $p \mid n_{k+1}$ by Euclid's Lemma. But if $p \mid n_1 n_2 \ldots n_k$, then by the induction hypothesis, there exists an integer i such that $p \mid n_i$.

3. **Total: 20 Marks**

(a) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ three vectors in $\mathbb{R}^3$. In the standard basis $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$, we write $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$.

i. We know that in a right-handed orthonormal basis, $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$. We also know that $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$, $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$ and $\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$. As well, we have $\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$. First, we can write directly (or rederive) that

$$\mathbf{b} \times \mathbf{c} = (b_2c_3 - b_3c_2)\hat{\mathbf{i}} + (b_3c_1 - b_1c_3)\hat{\mathbf{j}} + (b_1c_2 - b_2c_1)\hat{\mathbf{k}}$$

So we obtain

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) =& [a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}] \times [(b_2c_3 - b_3c_2)\hat{\mathbf{i}} + (b_3c_1 - b_1c_3)\hat{\mathbf{j}} + (b_1c_2 - b_2c_1)\hat{\mathbf{k}}] \\
=& (b_2c_3 - b_3c_2)(a_3\hat{\mathbf{j}} - a_2\hat{\mathbf{k}}) + (b_3c_1 - b_1c_3)(a_1\hat{\mathbf{k}} - a_3\hat{\mathbf{i}}) + (b_1c_2 - b_2c_1)(a_2\hat{\mathbf{i}} - a_1\hat{\mathbf{j}}) \\
=& (a_3b_1c_3 - a_3b_3c_1 + a_2b_1c_2 - a_2b_2c_1)\hat{\mathbf{i}} + \\
& (a_3b_2c_3 - a_3b_3c_2 + a_1b_2c_1 - a_1b_1c_2)\hat{\mathbf{j}} + \\
& (a_2b_3c_2 - a_2b_2c_3 + a_1b_3c_1 - a_1b_1c_3)\hat{\mathbf{k}}
\end{aligned}$$

If we consider the first component of that vector, we can realize that:

$$\begin{aligned}
a_3b_1c_3 - a_3b_3c_1 + a_2b_1c_2 - a_2b_2c_1 &= a_1c_1b_1 + a_2c_2b_1 + a_3c_3b_1 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1 \\
&= (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1 \\
&= (\mathbf{a} \cdot \mathbf{c})b_1 - (\mathbf{a} \cdot \mathbf{b})c_1
\end{aligned}$$

We proceed similarly with the other two components to obtain:

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} =& [(\mathbf{a} \cdot \mathbf{c})b_1 - (\mathbf{a} \cdot \mathbf{b})c_1]\hat{\mathbf{i}} + [(\mathbf{a} \cdot \mathbf{c})b_2 - (\mathbf{a} \cdot \mathbf{b})c_2]\hat{\mathbf{j}} + [(\mathbf{a} \cdot \mathbf{c})b_3 - (\mathbf{a} \cdot \mathbf{b})c_3]\hat{\mathbf{k}} \\
=& (\mathbf{a} \cdot \mathbf{c})(b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}) - (\mathbf{a} \cdot \mathbf{b})(c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}) \\
=& (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}
\end{aligned}$$

5 Marks

ii. Using the result from the previous question, we can write

$$\begin{aligned}
&\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \\
&\qquad = [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}] + [(\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}] + [(\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}] \\
&\qquad = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} + (\mathbf{b} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} \\
&\qquad = \mathbf{0}
\end{aligned}$$

where we have used the symmetry of the dot product. 3 Marks

(b) These two vectors are orthogonal if and only if their scalar product is equal to zero:

$(3,2,x)\cdot(2x,4,x) = 0 \iff x^2+6x+8 = 0 \iff (x+2)(x+4) = 0 \iff x = -2$ or $x = -4$

3 Marks

(c) Without loss of generality, we set up a coordinate system such that S is at the origin, $R = (0, y_1, 0)$, $Q = (x_2, y_2, 0)$ and $P = (x_3, y_3, z_3)$.
In this coordinate system, we have

$$\begin{aligned}
\mathbf{SR} &= (0, y_1, 0),\\
\mathbf{SQ} &= (x_2, y_2, 0),\\
\mathbf{SP} &= (x_3, y_3, z_3),\\
\mathbf{QR} &= (-x_2, y_1 - y_2, 0),\\
\mathbf{QP} &= (x_3 - x_2, y_3 - y_2, z_3).
\end{aligned}$$

A vector that is normal to the face opposite vertex S in an outward direction is given by

$$\mathbf{u}_S = \mathbf{QR} \times \mathbf{QP} = (y_1 z_3 - y_2 z_3)\hat{\mathbf{i}} + x_2 z_3 \hat{\mathbf{j}} + (-x_2 y_3 - x_3 y_1 + x_3 y_2 + x_2 y_1)\hat{\mathbf{k}}$$

Similarly, we find that

$$\begin{aligned}
\mathbf{u}_R &= \mathbf{SQ} \times \mathbf{SP} = y_2 z_3 \hat{\mathbf{i}} - x_2 z_3 \hat{\mathbf{j}} + (x_2 y_3 - x_3 y_2)\hat{\mathbf{k}}\\
\mathbf{u}_Q &= \mathbf{SP} \times \mathbf{SR} = -y_1 z_3 \hat{\mathbf{i}} + x_3 y_1 \hat{\mathbf{k}}\\
\mathbf{u}_P &= \mathbf{SR} \times \mathbf{SQ} = -x_2 y_1 \hat{\mathbf{k}}
\end{aligned}$$

We thus conclude that

$$\mathbf{u}_P + \mathbf{u}_Q + \mathbf{u}_R + \mathbf{u}_S = \mathbf{0}$$

By definition of the cross product, we know that $|\mathbf{u}_P|$ is the area of the parallelogram formed by the vectors $\mathbf{SR}$ and $\mathbf{SQ}$; this implies that $|\mathbf{u}_P|$ is twice the area of the triangle QRS. So we conclude that $|\mathbf{u}_P| = 2|\mathbf{u}_1|$, so we get $\mathbf{u}_P = 2\mathbf{u}_1$. We can reason similarly to write that: $\mathbf{u}_Q = 2\mathbf{u}_2$, $\mathbf{u}_R = 2\mathbf{u}_3$ and $\mathbf{u}_S = 2\mathbf{u}_4$. Thus, we conclude that

$$\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}$$

(d) In the previous question, we have defined the vector $\mathbf{u}_1$ to have length equal to the area of the face opposite P, so we can say that $|\mathbf{u}_1| = A$. The direction of $\mathbf{u}_1$ is perpendicular to the face and pointing outward. Similarly, we can write that $|\mathbf{u}_2| = B$, $|\mathbf{u}_3| = C$ and $|\mathbf{u}_4| = D$.
In the previous question, we concluded that $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 = \mathbf{0}$. So

$$\begin{aligned}
\mathbf{u}_4 = -(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) &\Rightarrow |\mathbf{u}_4| = |-(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3)| = |\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3|\\
&\Rightarrow |\mathbf{u}_4|^2 = |\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3|^2\\
&\Rightarrow \mathbf{u}_4 \cdot \mathbf{u}_4 = (\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) \cdot (\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3)\\
&\Rightarrow \mathbf{u}_4 \cdot \mathbf{u}_4 = \sum_{i,j=1}^{3} \mathbf{u}_i \cdot \mathbf{u}_j
\end{aligned}$$

Since the vertex S is trirectangular, we know that the three faces meeting at S are by definition orthogonal to one another, which means that the vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are orthogonal to one another. That means that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for $i \neq j$, with $i, j \in \{1, 2, 3\}$. Thus, we are left with

$$\mathbf{u}_4 \cdot \mathbf{u}_4 = \mathbf{u}_1 \cdot \mathbf{u}_1 + \mathbf{u}_2 \cdot \mathbf{u}_2 + \mathbf{u}_3 \cdot \mathbf{u}_3 \Rightarrow |\mathbf{u}_4|^2 = |\mathbf{u}_1|^2 + |\mathbf{u}_2|^2 + |\mathbf{u}_3|^2 \Rightarrow D^2 = A^2 + B^2 + C^2$$