Optimization Methods	Date:	*09/02/2018*
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Lecture 9: Primal and Dual Formulation for a Linear Program

Contents

1 Recap of the Last Class

1.1 Homework Problem (Simplex Algorithm)

Solve the following using Largest Coefficient Simplex Algorithm:

$$
\begin{aligned}
\max\ & 9x_1 + 3x_2 + x_3 \\
& \text{s.t} \\
& x_1 \leq 1 \\
& 6x_1 + x_2 \leq 9 \\
& 18x_1 + 6x_2 + x_3 \leq 81 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

The first task here is to convert the problem to the equational form and hence we add x_4, x_5 and

x_6 as slack variables and hence our problem becomes:

$$\begin{aligned}
\max\ & 9x_1 + 3x_2 + x_3 \\
& \text{s.t} \\
& x_1 + x_4 = 1 \\
& 6x_1 + x_2 + x_5 = 9 \\
& 18x_1 + 6x_2 + x_3 + x_6 = 81 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}$$

Now, our basic solution for the above problem is, $\mathbf{X} = (0, 0, 0, 1, 9, 81)$ where we have $\mathbf{X}_B = (x_4, x_5, x_6)$ and $\mathbf{X}_N = (x_1, x_2, x_3)$. Therefore, we can rewrite as:

$$\begin{aligned}
x_4 &= 1 - x_1 \\
x_5 &= 9 - 6x_1 - x_2 \\
x_6 &= 81 - 18x_1 - 6x_2 - x_3 \\
Z &= 9 - 9x_1 + 3x_2 + x_3
\end{aligned}$$

Now, applying largest coefficient algorithm, we see that x_1 has the largest coefficient and thus we replace x_1. We get, $\mathbf{X} = (1, 0, 0, 0, 3, 63)$, $\mathbf{X}_B = (x_1, x_5, x_6)$ and $\mathbf{X}_N = (x_2, x_3, x_4)$. Therefore, simplex tableau will be:

$$\begin{aligned}
x_1 &= 1 - x_4 \\
x_5 &= 3 + 6x_4 - x_2 \\
x_6 &= 63 + 18x_4 - 6x_2 - x_3 \\
Z &= 9 - 9x_4 + 3x_2 + x_3
\end{aligned}$$

Again, applying largest coefficient algorithm, we get that x_2 has the highest coefficient and hence we replace x_2. We get, $\mathbf{X} = (1, 3, 0, 0, 0, 45)$, $\mathbf{X}_B = (x_1, x_2, x_6)$ and $\mathbf{X}_N = (x_3, x_4, x_5)$. Therefore, simplex tableau will be:

$$\begin{aligned}
x_1 &= 1 - x_4 \\
x_2 &= 3 + 6x_4 - x_5 \\
x_6 &= 45 - 18x_4 - 6x_5 - x_3 \\
Z &= 18 + 9x_4 - 3x_5 + x_3
\end{aligned}$$

So, here we see that by using this rule, we traverse the vertices of a convex polyhedral and in worst case we have to visit ${}^{m}C_n$ vertices to get the optimal solution. Therefore, we have to carefully choose which rule to apply in the simplex algorithm for deciding the entering and exiting variable.

The above is an example of **Klee-Minty** problem where the simplex algorithm can take exponential time (Figure Below).

1.2 Primal and Dual Problem Formulation

Primal Formulation: Primal formulation refers to the original linear program problem we are required to solve. A general representation of this formulation can be:

$$\max \quad \mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{1.1}$$

$$s.t \begin{cases} \mathbf{Ax} & \leq \mathbf{b} \\ \mathbf{x} & \geq 0 \end{cases}$$

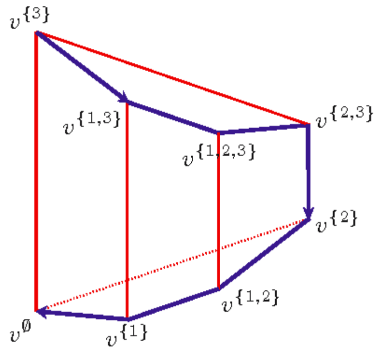

Figure 1: Klee Minty Cube
https://www.semanticscholar.org/topic/Klee-Minty-cube/1329278

Here, $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^n$. Where $m, n \in \mathbb{R}$

Dual Formulation: The dual model of a Linear Programming problem consists of an alternative modeling instance that allows us to recover the information of the original problem. A general representation of this formulation for the above primal problem can be:

$$\min \quad \mathbf{b}^{\mathrm{T}}\mathbf{y} \tag{1.2}$$

$$s.t \begin{cases} \mathbf{A}^T\mathbf{y} & \geq \mathbf{c} \\ \mathbf{y} & \geq 0 \end{cases}$$

Here, $\mathbf{A}$, $\mathbf{c}$, $\mathbf{b}$ are same as in the primal problem and $\mathbf{y} \in \mathbb{R}^m$

Intuition behind Dual formulation: The idea behind converting the primal problem into the dual is to upper bound the objective of the primal. We generate the dual with the linear combination of all the constraints in the primal, in such a manner that the objective of the dual upper bounds the objective in primal. Hence, out task in the dual problem is to minimize the value of the objective.

1.3 Weak Duality

Let $F(P)$ be the set of primal feasible solutions and $F(D)$ be the set of dual feasible solutions. By Weak Duality, for any $\bar{\mathbf{x}} \in F(P)$ and $\mathbf{y} \in F(D)$ we see that

$$\mathbf{c}^{\mathrm{T}}\bar{\mathbf{x}} \leq \mathbf{b}^{\mathrm{T}}\mathbf{y} \tag{1.3}$$

$$\mathbf{c}^{\mathrm{T}}\mathbf{x}^* \leq \mathbf{b}^{\mathrm{T}}\mathbf{y} \quad \forall y \in \mathrm{F(D)} \ \text{, where } \mathbf{x}^* \text{ is the optimal solution of the primal} \tag{1.4}$$

$$\mathbf{c}^{\mathrm{T}}\mathbf{x}^* \leq \mathbf{b}^{\mathrm{T}}\mathbf{y}^* \ \text{, where } \mathbf{y}^* \text{ is the optimal solution of the dual} \tag{1.5}$$

1.4 Strong Duality

We can use this principle in the case when both Primal and Dual problem have feasible solution. If $\mathbf{x}^*$ is the optimal solution for primal and $\mathbf{y}^*$ is the optimal solution for the dual problem then,

$$\mathbf{c}^{\mathrm{T}}\mathbf{x}^* = \mathbf{b}^{\mathrm{T}}\mathbf{y}^* \tag{1.6}$$

1.5 Advantages of Using the Dual Form

1. One of the prime advantage of using the dual form is the reduction in the computational complexity of the problem. Consider the case of the primal problem which has n variables and

m constraints and hence the complexity here is nm^2. When we convert the primal to the dual problem then the complexity of solving is mn^2, therefore in the case when $n << m$, it is better to solve the dual problem.

2. In the cases where we need to find feasible solution to a problem then converting the problem to the dual form may be helpful. For example, when the primal problem is a maximization problem.

2 Understanding Primal and Dual graphically

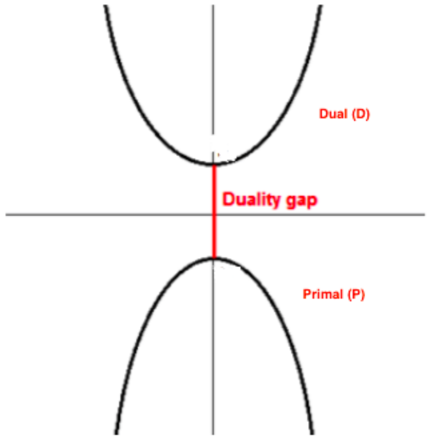

Figure 2: Graphical representation of Primal (P) and Dual (D)

In the schema above, imagine that in our primal problem, we are trying to maximize the function at the bottom of the graph. If we search for a dual function, we could end up with the one at the top of the graph, whose minimum is the point where the primal is maximized . In this case, we clearly see that dual is the upper bound.

We define the difference of optimal objective values as the **Duality gap**. If Duality gap >0 then weak duality holds else if Duality gap = 0 then strong duality holds.

3 Tabular Comparison of Primal and Dual

Parameter	Primal	Parameter	Dual
Variables	$x_1, x_2 \cdots, x_n$	Variables	$y_1, y_2 \cdots, y_m$
Constraints	$\mathbf{A}$	Constraints	$\mathbf{A}^{\mathrm{T}}$
Objective Function	max $\mathbf{c}^{\mathrm{T}}\mathbf{x}$	Objective Function	min $\mathbf{b}^{\mathrm{T}}\mathbf{y}$
i^{th} constraint	$\leq, \geq, =$	i^{th} variables	$\leq 0, \geq 0, \in \mathbb{R}$
i^{th} variables	$\leq 0, \geq 0, \in \mathbb{R}$	i^{th} constraint	$\leq, \geq, =$

4 Example of a Case in Duality

Given, The primal problem:

$$\max \quad x_1 \quad \text{such that,}$$

$$\begin{aligned} x_1 - x_2 &\leq 1 \\ -x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Using the rules from the above table the corresponding dual is

$$\min \quad y_1 + 2y_2 \quad \text{such that,}$$

$$\begin{aligned} y_1 - y_2 &\geq 1 \\ -y_1 + y_2 &\geq 0 \\ y_1, y_2 &\geq 0 \end{aligned}$$

When we analyze the above dual problem then we get that, this formulation is infeasible since $y_1 - y_2 \geq 1$ and $y_1 - y_2 \leq 0$. We also know that the objective value for the dual problem upper bounds that of the primal problem and since the solution for dual is infeasible then we can conclude that the solution for unbounded (since the primal is feasible).

5 Strong Duality

- Neither Primal nor Dual has a feasible solution.
- Primal is unbounded and Dual has no feasilbe solution.
- Primal has no feasible solution and Dual is unbounded.
- Both Primal and Dual have a feasible solution. Then both have an optimal solution, and if $\mathbf{x}^*$ is an optimal solution of Primal and $\mathbf{y}^*$ is an optimal solution of Dual, then

$$\mathbf{c}^{\mathrm{T}}\mathbf{x}^* = \mathbf{b}^{\mathrm{T}}\mathbf{y}^*$$

That is, the maximum of Primal is equal to minimum of Dual.

6 Proving Strong Duality using Simplex method

Let our problem be defined as:

$$\max \quad \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

$$s.t \begin{cases} \mathbf{Ax} & \leq \mathbf{b} \\ \mathbf{x} & \geq 0 \end{cases}$$

Since, the above problem is in standard form, therefore we need to convert it into equational form by adding slack variables $x_{n+1}, \cdots, x_{n+m}$
After, addition of slack variables our new formulation for the problem is:

$$\max \quad \mathbf{c}^{*\mathrm{T}}\mathbf{x}^*$$

$$s.t \begin{cases} \mathbf{A}^*\mathbf{x}^* & \leq \mathbf{b} \\ \mathbf{x}^* & \geq 0 \end{cases}$$

where

$$\mathbf{x}^* = \begin{bmatrix} x_1 & \cdots & x_n & x_{n+1} & \cdots & x_{n+m} \end{bmatrix}$$
$$\mathbf{c}^* = \begin{bmatrix} c_1 & \cdots & c_n & 0 & \cdots & 0 \end{bmatrix}$$
$$\mathbf{A}^* = \begin{bmatrix} \mathbf{A} & | & \mathbf{I_m} \end{bmatrix}$$

Now, Let $\mathbf{x}^* = \begin{bmatrix} \mathbf{x}^*{}_{\mathrm{B}} & \mathbf{x}^*{}_{\mathrm{N}} \end{bmatrix}$ be the optimal solution. Where $\mathbf{x}^*{}_{\mathrm{B}}$ is the solution to basic variables and $\mathbf{x}*_{\mathrm{N}}$ is the solution to the non-basic variables. Now, at optimal solution we have

$$\mathbf{x}^*{}_N = 0$$

Therefore,

$$\mathbf{x}^*{}_B = \mathbf{A}^{*\text{-}1}_{\mathbf{B}}\mathbf{b}$$

Now,

$$\mathbf{c}^{*\mathrm{T}}\mathbf{x}^* = \begin{bmatrix} \mathbf{c}^*{}_B & | & \mathbf{c}^*{}_N \end{bmatrix} \begin{bmatrix} x^*_B \\ \\ 0 \end{bmatrix}$$
$$= \mathbf{c}^*{}^{\mathrm{T}}_B(\mathbf{A}^*{}^{\text{-}1}_B\mathbf{b})$$
$$= \mathbf{y}^{*\mathrm{T}}\mathbf{b} = \mathbf{b}^{\mathrm{T}}\mathbf{y}^*$$

where

$$\mathbf{y}^{*\mathrm{T}} = \mathbf{c}^*{}^{\mathrm{T}}_B\mathbf{A}^*{}^{\text{-}1}_B \tag{6.1}$$

Now, we need to show that $\mathbf{y}^*$ is a feasible solution to dual.The equality $\mathbf{c}^{*\mathrm{T}}\mathbf{x}^* = \mathbf{b}^{*\mathrm{T}}\mathbf{y}^*$ holds. Now, we need to check the feasibility of **y'** i.e.

$$\mathbf{A}^{\mathrm{T}}\mathbf{y}^* \geq \mathbf{c} \tag{6.2}$$
$$\mathbf{y}^* \geq 0. \tag{6.3}$$

Let us focus on **(6.2)** which can be written as:

$$\mathbf{I_m}\mathbf{y}^* \geq 0 \tag{6.4}$$

Also, using **(6.1)** we get

$$\mathbf{y}^{*\mathrm{T}} = \mathbf{c}^*{}^{\mathrm{T}}_B\mathbf{A}^*{}^{\text{-}1}_B \tag{6.5}$$

Substituting **(6.1)** in **(6.2)** we get,

$$\mathbf{A}^{\mathrm{T}}(\mathbf{c}^*{}^{\mathrm{T}}_B\mathbf{A}^*{}^{\text{-}1}_B) \geq \mathbf{c} \tag{6.6}$$

Let the left hand side be $\mathbf{c_B}$. Therefore,

$$\mathbf{c}^{\mathrm{T}}_B = (\mathbf{c}^*_{\mathbf{B}}{}^{\mathrm{T}}\mathbf{A}^*{}^{\text{-}1}_B\mathbf{A}'_B)$$

Now, for non basic variables.

$$\mathbf{c}^{\mathrm{T}}_N = \mathbf{c}^*{}^{\mathrm{T}}_B\mathbf{A}^*{}^{\text{-}1}_B\mathbf{A}^*{}_N$$

We know that in simplex tableau.

$$\begin{aligned}\mathbf{z} = \mathbf{z}_0 + \mathbf{r^T x_N} \\ \mathbf{r} \leq \mathbf{0}\end{aligned}$$

In other words

$$\mathbf{c}_N^{\mathrm{T}} \leq \mathbf{c^*}_B^{\mathrm{T}} \mathbf{A^*}_B^{\text{-1}} \mathbf{A^*}_N$$

Therefore

$$\mathbf{A}^{\mathrm{T}} \mathbf{y}^* \geq \mathbf{c} \quad as \quad \mathbf{c}_B^{\mathrm{T}} = (\mathbf{c_B^*}^{\mathrm{T}} \mathbf{A^*}_B^{\text{-1}} \mathbf{A^*}_B)$$

Hence it is shown that $\mathbf{y}^*$ is a feasible solution to dual from **(6.1)**.

7 Optimal and feasible solution of a linear program

Question : Can we find a feasible solution to a linear program faster then finding an optimal solution?
Answer : Turns out to be that both are equally hard to find.

In support of the above statement, consider the following LP formulation

$$\max \ \mathbf{c}^{\mathrm{T}} \mathbf{x} \tag{7.1}$$
$$s.t \begin{cases} \mathbf{Ax} & \leq \mathbf{b} \\ \mathbf{x} & \geq 0 \end{cases}$$

The dual formulation according to the recipe will be be,

$$\min \quad \mathbf{b}^{\mathrm{T}} \mathbf{y} \tag{7.2}$$
$$s.t \begin{cases} \mathbf{A}^T \mathbf{y} & \geq \mathbf{c} \\ \mathbf{y} & \geq 0 \end{cases}$$

Notice that we can always construct a linear program which is as follows

$$\max \quad \mathbf{c}^{\mathrm{T}} \mathbf{x} \tag{7.3}$$
$$s.t \begin{cases} \mathbf{Ax} & \leq \mathbf{b} \\ \mathbf{A}^T \mathbf{y} & \geq \mathbf{c} \\ \mathbf{c}^T \mathbf{x} & \geq \mathbf{b}^T \mathbf{y} \\ \mathbf{x}, \mathbf{y} & \geq 0 \end{cases}$$

This can be rewritten in the standard form as

$$\max \mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{7.4}$$

$$s.t \begin{cases} \mathbf{Ax} & \leq \mathbf{b} \\ -\mathbf{A}^T\mathbf{y} & \leq -\mathbf{c} \\ -\mathbf{c}^T\mathbf{x} + \mathbf{b}^T\mathbf{y} & \leq 0 \\ \mathbf{x}, \mathbf{y} & \geq 0 \end{cases}$$

Here we have increased the complexity of the problem only polynomially, but if we find a feasible solution of this problem, we have already found an optimal solution of the original Primal Problem.

This is because, if $\widetilde{x}$, $\widetilde{y}$ is the feasible solution then we must have

$$\mathbf{c}^T\widetilde{x} \geq \mathbf{b}^T\widetilde{y} \tag{7.5}$$

But from weak duality, we have

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y} \quad \forall\, \mathbf{x}, \mathbf{y} \tag{7.6}$$

Thus

$$\mathbf{c}^T\widetilde{x} = \mathbf{b}^T\widetilde{y} \tag{7.7}$$

which is the strong duality, and thus we can say that $\widetilde{x}$ is the optimal solution of our initially chosen LP.

Thus we can conclude that if, we are able to find even one "feasible" solution here, we are guaranteed to have found an "optimal" solution already for our standard LP. Hence both problems of finding a feasible as well as optimal solution are equally hard.

Ellipsoid method is exactly based on this, where we find one feasible solution for a LP and not the optimal solution.

8 Equation of some standard geometrical curves

- **Circle (2D)**: $x^2 + y^2 = r^2$, r = radius of the circle centered at origin
- **Sphere (3D)**: $x^2 + y^2 + z^2 = r^2$, r = radius of the sphere centered at origin
- **Sphere (in N dimension)**: $x_1^2 + x_2^2 + ... + x_n^2 = r^2$
- **Sphere (in matrix form)**: $B_n(a, r) : (x - a)^T$ (x - a) = Y^2
- **Ellipse (2D)**: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- **Ellipse (in N dimension)**: $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + ... + \frac{x_n^2}{a_n^2} = 1$
- **Ellipse (in matrix form)**: x^T A x = 1, where A is the diagonal matrix having the elements of the following form (where, the off-diagonal elements are 0)

$$A = \begin{bmatrix} \frac{1}{a_1^2} & & \\ & \ddots & \\ & & \frac{1}{a_n^2} \end{bmatrix}$$