Optimization Methods		Date:	*23-01-2018*
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Lecture 7: The Simplex Algorithm

1 Recap

- Introduction to the Simplex Method
- Examples covering the Simplex Method.
- *Lemma:* Given a basis there is a unique Simplex Tableau

$$\tau(B) : \frac{x_B = p + Qx_N}{z = z_0 + r^t x_N}$$

- Introduction to some of the problems with the simplex method including, how to compute the initial basic feasible solution, how to pick which non-basis variable should be the entering variable, and how to handle degenerate cases.

2 Concepts

The following topics were covered in the class in this lecture:

1. Examples for which at optimal solution, in simplex tableau the non-basis variables have coefficients 0.
2. Issues with the Simplex Method
   (a) Computing an Initial Basic Feasible Solution
   (b) Pivot Rules
   (c) Unbounded Linear Program
3. The Simplex Algorithm

3 Cases with Coefficients of Non-Basis variables as zero

We now look at some examples corresponding to cases where the coefficient of some of the non-basis variables are zero.

1. Consider the LP,

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{such that} \quad & x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

   To convert this to equatorial form we will introduce a *slack* variable x_3. The LP now becomes,

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{such that} \quad & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

   We will now use the simplex method to solve this LP. The steps are as follows:

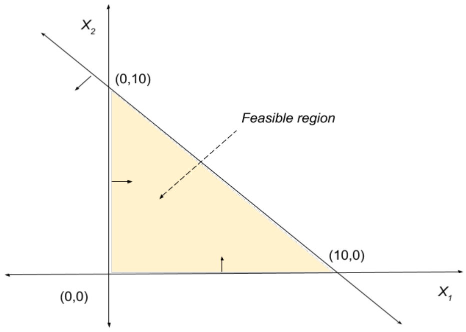

Figure 1: Feasible solution area for the LP 3(1).

(a) Let $(0,0)$ be the initial basic feasible solution, $B = \{3\}$ and $N = \{1,2\}$. Then, the tableu for this would be,

$$\frac{x_3 = 1 - x_1 - x_2}{z = 0 + x_1 + x_2}$$

The optimal value for this solution is 0.

(b) We now pick x_1 as the *entering* variable and x_3 as the corresponding *leaving* variable i.e. $B = \{1\}$ and $N = \{2,3\}$ The tableu for this is,

$$\frac{x_1 = 1 - x_2 - x_3}{z = 1 - x_3}$$

The optimal value for this solution is 1. Notice that in this, x_2's coefficient is 0, hence for this case we have 1 degree of freedom. The solution space in this case, will be localized in 1-dimensional space.

2. Consider another LP,

$$\begin{aligned} &\text{maximize} && x_1 + x_2 \\ &\text{such that} && x_1 + x_2 + x_3 \leq 1 \\ & && x_1, x_2, x_3 \geq 0 \end{aligned}$$

To convert this to equatorial form we will introduce a *slack* variable x_4. The LP now becomes,

$$\begin{aligned} &\text{maximize} && x_1 + x_2 \\ &\text{such that} && x_1 + x_2 + x_3 + x_4 = 1 \\ & && x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

We will again use the simplex method to solve this LP. The steps are as follows:

(a) Let $(0,0,0)$ be the initial basic feasible solution, $B = \{4\}$ and $N = \{1,2,3\}$. Then, the tableu for this would be,

$$\frac{x_4 = 1 - x_1 - x_2 - x_3}{z = 0 + x_1 + x_2}$$

(b) We now pick x_1 as the *entering* variable and x_4 as the corresponding *leaving* variable i.e. $B = \{1\}$ and $N = \{2,3,4\}$. The tableu for this is,

$$\frac{x_1 = 1 - x_2 - x_3 - x_4}{z = 1 - x_3 - x_4}$$

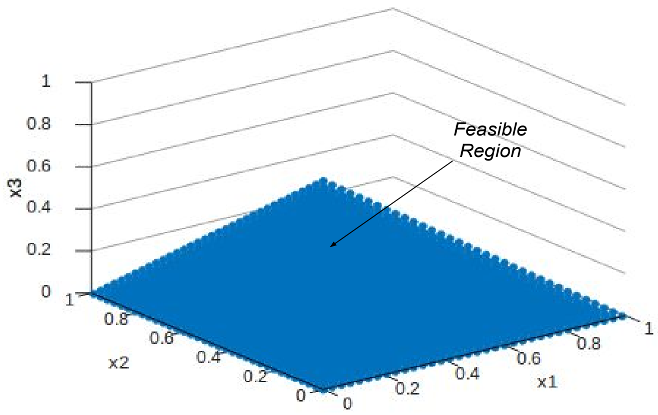

Figure 2: Feasible solution area for the LP 3(2).

In this, similar to the previous example, one of the coefficients of the non-basis variables is 0. The degree of freedom in this case is thus, 1. The solution space in this case, will be localized in 1-dimensional space.

3. Again, consider the LP,

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 + x_3 \\ \text{such that} \quad & x_1 + x_2 + x_3 \le 1 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

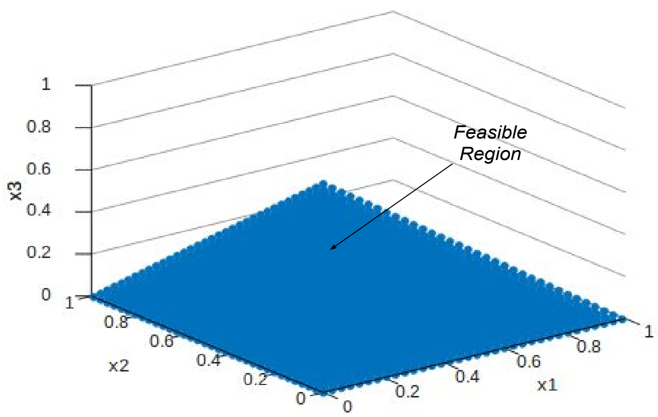

Figure 3: Feasible solution area for the LP 3(3).

To convert this to equatorial form we will introduce a *slack* variable x_4. The LP now becomes,

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 + x_3 \\ \text{such that} & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

We will again use the simplex method to solve this LP. The steps are as follows:

(a) Let $(0, 0, 0)$ be the initial basic feasible solution, $B = \{4\}$ and $N = \{1, 2, 3\}$. Then, the tableu for this would be,

$$\begin{array}{l} x_4 = 1 - x_1 - x_2 - x_3 \\ \hline z = 0 + x_1 + x_2 + x_3 \end{array}$$

(b) We now pick x_1 as the *entering* variable and x_4 as the corresponding *leaving* variable i.e. $B = \{1\}$ and $N = \{2, 3, 4\}$. The tableu for this is,

$$\begin{array}{l} x_1 = 1 - x_2 - x_3 - x_4 \\ \hline z = 1 - x_4 \end{array}$$

However, in here there are 2 coefficients of the non-basis variables which are 0. Hence, the degree of freedom is 2. The solution space in this case, will be localized in 2-dimensional space.

4. Consider another LP,

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{such that} & x_1 + x_2 = 1 \end{array}$$

Here, x_1, x_2 need not be positive. Thus, we introduce auxiliary variables $y_1, y_2 \geq 0$ so that x_1, y_1, x_2 and y_2 are always ≥ 0. The LP now becomes,

$$\begin{array}{ll} \text{maximize} & x_1 - y_1 + x_2 - y_2 \\ \text{such that} & x_1 - y_1 + x_2 - y_2 = 1 \\ & x_1, y_1, x_2, y_2 \geq 0 \end{array}$$

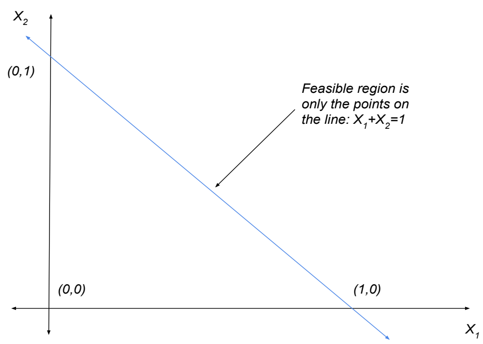

Figure 4: Feasible solution area for the LP 3(4).

We shall solve this using the simplex method similar to the previous examples.

(a) Let $(1, 0)$ be the initial basic feasible solution, $B = \{x_1\}$ and $N = \{y_1, x_2, y_2\}$. Then, the tableu for this would be,

$$\begin{array}{l} x_1 = 1 + y_1 - x_2 + y_2 \\ \hline z = 1 \end{array}$$

In this example, all 3 of the non-basis variables have coefficients equal to 0. Thus, the LP has degree of freedom as 3, and the solution is locally 3-dimensional.

5. In general, a LP can have at most $n - m$ degrees of freedom which would be for the case when the coefficient of every non-basis variable is 0.

4 Issues with the Simplex Method

We now look in detail at the 3 issues introduced in the last lecture with respect to the simplex method and their solutions.

4.1 Computing an Initial Basic Feasible Solution

If the given LP has no *obvious* feasible basis, we look for an initial feasible basis by similar to the way we find an optimal solution to the LP. In general, finding any feasible solution of a linear program is equally as difficult as finding an optimal solution. We use the simplex method itself to find the initial basis solution. The method is as follows [1]:

1. Consider a LP L, with constraints of the form $Ax = b$ i.e

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{1m} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

2. Let $Y = \{y_{n+1}, \dots, y_{n+m}\}$ be a set of variables such that,

$$\begin{aligned} y_{n+1} &= b_1 - (a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n) \\ y_{n+2} &= b_2 - (a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n) \\ &\vdots \\ y_{n+m} &= b_m - (a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n) \end{aligned} \tag{1}$$

3. The set of variables $\{y_{n+1}, \dots, y_{n+m}\}$ can take any value on the number line. But, because the set of eq. 1 are in equational form, we can multiply throughout by -1 the equation for which $y_i < 0$, $i \in \{n+1, \dots, n+m\}$. Hence, we can always make certain that $y_i > 0 \; \forall i \in \{n+1, \dots, n+m\}$ because of which we can define a new LP $\bar{L}$, with constraints given by the set of eq. 1.

4. A feasible solution for $\bar{L}$, given by a vector y will be,

$$y = (x_1, x_2, \dots, x_n, y_{n+1}, \dots, y_{n+m}).$$

   It it trivial to see that

$$y = (0, 0, \dots, 0, b_1, b_2, \dots, b_n), \tag{2}$$

   is a satisfying solution to $\bar{L}$. Let this be the *initial basic solution* for $\bar{L}$.

5. Formally, $\bar{L}$ is given by the LP,

$$\begin{aligned} \text{maximize} \qquad & -(y_{n+1} + \dots + y_{n+m}) \\ \text{such that} \qquad & Ey = b \\ & y \geq 0. \end{aligned}$$

   Here, E is of the form $[A|I_m]$, where I_m is an identity matrix of order m.

   Note. If b is < 0, then we will multiply the constraint $Ey = b$ by -1 throughout.

6. We will solve $\bar{L}$ with the simplex method using y given in eq. 2 as the initial basic solution. The optimal solution (for optimal value 0) for $\bar{L}$,

$$y^* = (\bar{x_1}, \ldots, \bar{x_n}, 0, \ldots, 0),$$

will be the initial basic solution for the original LP, L.

Note. If the objective value of $\bar{L}$ does not come out to be 0 then L does not have a feasible solution.

4.2 Pivot Rules

Generally while solving a LP through the simplex method, we encounter cases where there are multiple non-basis variables with positive coefficients i.e., cases where we have to choose between multiple non-basis variables to select the new entering variable. There are multiple method of selection for this, and each method or *rule* is called a *pivot rule.*

Definition 4.1. *A pivot rule is a rule for selecting the entering variable if there are several possibilities, which is usually the case. Sometimes there may also be more than one possibility for choosing the leaving variable, and some pivot rules specify this choice as well. [1]*

Some of the common pivot rules are given below. We refer to an *improving variable* [1] as a nonbasic variable with a positive coefficient in the z-row of the simplex tableau, in simpler words, a candidate for the entering variable.

1. *Largest Coefficient:* This was the first pivot rule suggested by Dantzig, who had proposed the simplex algorithm, which stated to choose an improving variable with the largest co-efficient in the row of the objective function z. This maximized the improvement of z per unit increase of the entering variable. The worst case complexity of this rule is exponential i.e. in the worst case, the simplex algorithm covers every possible basic solution. An example wherein this rule will cover all basic solutions is:

$$\begin{aligned} \text{maximize} \qquad & 9x_1 + 3x_2 + x_3 \\ \text{such that} \qquad & x_1 \le 1 \\ & 6x_1 + x_2 \le 9 \\ & 18x_1 + 6x_2 + x_3 \le 81 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

2. *Largest Increase:* In this rule, we choose an improving variable that leads to the largest absolute improvement in z. This rule is computationally more expensive than the largest coefficient rule, but it locally maximizes the progress. The worst case complexity for this too, is exponential.

3. *Steepest Edge:* In this, we choose an improving variable whose entering into the basis moves the current basic feasible solution in a direction closest to the direction of the vector c i.e., the ratio

$$\frac{c^T(x_{new} - x_{old})}{||x_{new} - x_{old}||}$$

should be maximized, where the x_{old} is the basic feasible solution for the current simplex tableau and x_{new} is the basic feasible solution for the tableau that would be obtained by entering the considered improving variable into the basis.

4. *Bland's Rule:* In this, we choose the improving variable with the smallest index, and if there are several possibilities for the leaving variable, also take the one with the smallest index. This rule has been proven to never run into a *cycle* but it also has an exponential worst case complexity. In practice, we solve the simplex method using one of the other rules, and move to Bland's rule if we happen to run into cycle.

5. *Random Edge:* Here we select the entering variable uniformly at random among all improving variables. This is the simplest example of a randomized pivot rule, where the choice of the entering variable uses random numbers in some way.

4.3 Unbounded linear program

If for a LP, there exists a feasible basis solution but an optimal value does not exist, then it is unbounded i.e at least one variable can be increased to infinity.

Let $B = \{x_1, \ldots, x_B\}$ and $z = z_0 + r_1 x_{B+1} + \ldots$, then this will be unbounded if there exists at least one non-basis variable that can be continued to increase without making any constraint negative i.e all coefficients of that non-basis variable are positive in every constraint z. Figure 5 is an example of such a case.

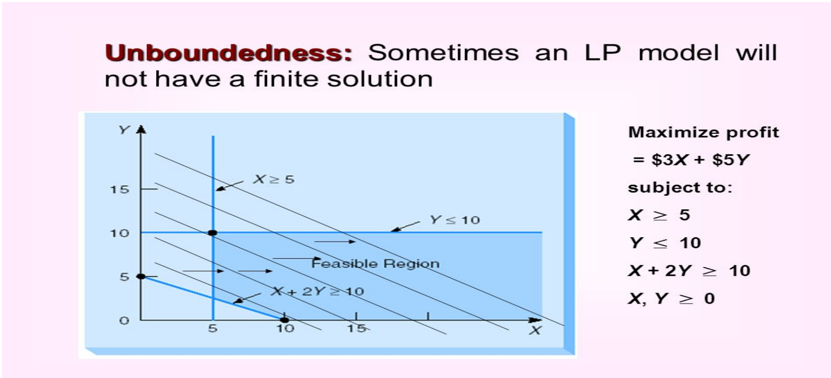

Figure 5: An example of an unbounded LP where basic feasible solution exists. [2]

Example: Consider the LP as described in Figure 5,

$$\begin{aligned} \text{maximize} \quad & 3x + 5y \\ \text{such that} \quad & x \geq 5 \\ & y \leq 10 \\ & x + 2y \geq 10 \\ & x, y \geq 0 \end{aligned}$$

In equational form:

$$\begin{aligned} \text{maximize} \quad & 3x + 5y \\ \text{such that} \quad & -x + z_1 = -5 \\ & y + z_2 = 10 \\ -x - 2y + z_3 = 10 \quad Or \quad & -2y - z_1 + z_3 = 15 \\ & x, y \geq 0 \end{aligned}$$

Using Simplex Method:

1. Let $(x, y, z_1, z_2, z_3) = (0, 0, -5, 10, -10)$ be the initial basic feasible solution, $B = \{3, 4, 5\}$ and $N_B = \{1, 2\}$. Then, the tableu for this would be,

$$\begin{aligned} z_1 &= -5 + x \\ z_2 &= 10 - y \\ z_3 &= 10 + x + 2y \\ \hline z &= 0 + 3x + 5y \end{aligned}$$

   As all the coefficients of a non-basis variable x are positive, it can be increased up to infinity, that means this is an unbounded linear program.

5 The Simplex Algorithm

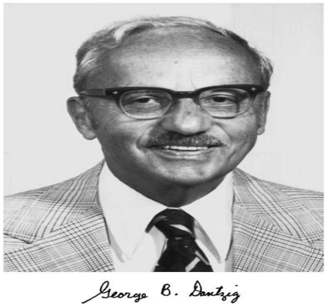

Figure 6:
George Bernard Dantzig

The Simplex Algorithm was proposed by George Bernard Dantzig, who did his Ph.D. from the University of California, Berkeley, in 1946. For the following,

$$Input : A, b, c$$

$$Ouput :$$

(i) $x^* \in \text{argmax}\left\{c^T x | Ax = b, x \geq 0\right\}$

(ii) No fesible solution

(iii) Unbounded Region

the Simplex algorithm, as described in Algorithm 1 takes the above input and outputs either (i),(ii) or (iii).

Algorithm 1 The Simplex Algorithm

Step 1 : Find a basic feasible solution

$$\begin{aligned} \text{maximize} \quad & -\sum_{i=1}^{m} y_{n+i} \\ \text{such that} \quad & Ey = b \\ & y \geq 0 \end{aligned}$$

Step 2 : Let B be basis and construct tableau

$$\tau(B) : \frac{x_B = p + Qx_N}{z = z_0 + r^t x_N}$$

Step 3 : $r \leq 0$ return $x' = (x_B, x_N)$
Step 4 : if some $r_u > 0$ find an entering variable using pivot rule.
Step 5 : if the column of variable x_u in $\tau(B)$ is having all positive coefficients return unbounded.
Step 6 : select a leaving variable

$$\begin{aligned} \text{find} \quad & x_v \\ \text{such that} \quad & q_{vu} < 0 \end{aligned}$$

$$\frac{-p_v}{q_{vu}} = \min\left\{\frac{-p_i}{q_{iu}} : q_{iu} < 0, \forall i = \{1, \dots, m\}\right\}$$

Step 7 : $B \leftarrow (B \setminus \{v\}) \cup \{u\}$ go to Step 2.

6 Takeaways

- Effect on the feasible solution space when the coefficient of a non-basis variable is 0.
- How to deal with issues related to the simplex algorithm such as finding the initial basic feasible solution, handling degenerate cases, optimizing on the value through various pivot rules etc.
- The simplex algorithm.

References

[1] Understanding and Using Linear Programming, Bernd Gartner.

[2] http://slideplayer.com/slide/4938553/