Optimization Methods		Date:	*02/02/2018*
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Lecture 8: Linear Programming - Duality

1 Recap

1.1 Initial basic feasible solution

Finding an initial basic feasible solution is not always obvious. Given $A\mathbf{x} = b$ where A is of size mxn, b if of size mx1. We introduce m auxiliary variables, $x_{n+1}, x_{n+2}, .., x_{n+m}$ greater than zero, for each of the m constraints in $A\mathbf{x} = b$. We get a basic feasible solution when all the auxiliary variables become zero. The auxiliary variables are defined as follows:

$$x_{n+i} = b_i - (a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n)$$

Defining a new LP as follows:

$$\text{maximize } -(x_{n+1} + x_{n+2} + \ldots + x_{n+m})$$

$$\text{subject to } Ey = b,\ y_i >= 0$$

$$\text{where } y = (x_1, x_2, ..x_n, x_{n+1}, ...x_{n+m})$$

An initial basic feasible to this LP is

$$(0,\ 0,\ \ldots,\ 0,\ b_1,\ b_2,\ \ldots,\ b_m)$$

Using the above basic feasible solution, we can solve the new LP using simplex which will be the initial basic feasible solution for the original LP

1.2 Rules for pivot selection

1. Largest Coefficient: Choose the variable with the largest coefficient in the objective function
2. Largest Increase: Choose a variable that leads to largest increase in the absolute value of the objective function. This is computationally more expensive than rule 1
3. Random Edge: Choose among improving variables uniformly at random
4. Bland's Rule: Choose the improving variable with the smallest index. This method helps prevent cycling
5. Steepest Edge: Choose a variable that moves the current basic feasible solution in the direction closest to **c**

1.3 Algorithm - Simplex Method

1. Convert the given linear program to its equational form
2. If there is no obvious initial basic feasible solution, use the method discussed above. If the optimal solution to the new LP defined by the constraints $Ey = b$ has a negative solution, the the original system is infeasible.

3. Compute the simplex tableau of the form
$$x_B = \mathbf{p} \ + \ Qx_N$$
$$z = z_0 \ + \ r^T x_N$$
4. If $r <= 0$ return $x^* = (x_B, x_N)$
5. Else, select an entering variable, x_v using one of the pivot rules
6. If the column of the entering variable x_v in the simplex tableau is non negative, then the LP is unbounded
7. Else, select a leaving variable such that:
$$q_{uv} < 0$$
$$-\frac{p_v}{q_{uv}} = min \ \{-\frac{p_i}{q_{iu}} \ : \ q_{iu} < 0, i = 1, 2, \ldots, m\}$$
8. Replace the current feasible basis B with the new basis: $B \backslash \{u\} \cup \{v\}$
9. Update the simplex with the new basis, go to step 4

2 Introduction to Duality

Every constrained linear optimization problem (also known as primal) has a corresponding Dual formulation. Motivation to solve the dual:

1. The dual is solved if there are large number of variables in the primal
2. Sometimes it is easier to solve the dual
3. Dual provides and upper(lower) bound to the corresponding maximization(minimization) Primal, assuming we both primal and dual are feasible
4. Finding an initial basic feasible solution to the dual is sometimes easier
5. In some cases, the dual gives a computational advantage over the primal.

2.1 Example

Let us consider the linear program -
$$\max \ \mathbf{x_1} + \mathbf{x_2}$$
$$\text{subject to}$$
$$\mathbf{x_1} + 2\mathbf{x_2} >= 1 \tag{1}$$
$$2\mathbf{x_1} + 3\mathbf{x_2} <= 6 \tag{2}$$
$$\mathbf{x_1}, \mathbf{x_2} >= 0$$
Dividing (2) by half we get -
$$\mathbf{x_1} + \frac{3}{2}\mathbf{x_2} <= 3$$
Clearly we can see
$$\mathbf{x_1} + \mathbf{x_2} <= \mathbf{x_1} + \frac{3}{2}\mathbf{x_2} <= 3$$
Thus Equation gives one upper bound to the objective function.
Now using (1) and (2):
$$-\mathbf{x_1} - 2\mathbf{x_2} <= -1$$

$$\mathbf{2x_1 + 3x_2 <= 6}$$

On adding the above two equations we get -

$$\mathbf{x_1 + x_2 <= 5}$$

Thus by linear combination of constraints we are getting the upper bound on the objective function. Thus we can combine the two constraints in the linear program with some non-negative coefficients y_1 and y_2.

$$\mathbf{y_1(-x_1 - 2x_2) + y_2(2x_1 + 3x_2) <= -y_1 + 6y_2}$$

$$\mathbf{x_1(-y_1 + 2y_2) + x_2(-2y_1 + 3y_2) <= -y_1 + 6y_2}$$

For the above inequality to be an upper bound to the objective function, the coefficient of x1 and x2 should be greater than one.

$$\mathbf{x_1 + x_2 <= x_1(-y_1 + 2y_2) + x_2(-2y_1 + 3y_2) <= -y_1 + 6y_2}$$

For above inequality to be true -

$$(-\mathbf{y_1} + 2\mathbf{y_2}) >= 1$$

$$(-2\mathbf{y_1} + 3\mathbf{y_2}) >= 1$$

And the upper bound on the objective function will be

$$-\mathbf{y_1} + 6\mathbf{y_2}$$

We need to find the least upper bound.Thus the original linear program transforms into -

$$\begin{aligned} \min \quad & -\mathbf{y_1} + 6\mathbf{y_2} \\ \text{subject to} \quad & -\mathbf{y_1} + 2\mathbf{y_2} >= 1 \\ & -2\mathbf{y_1} + 3\mathbf{y_2} >= 1 \\ & \mathbf{y_1}, \mathbf{y_2} >= 0 \end{aligned}$$

This is the dual form of the given linear program.

2.2 Formulation

$$\text{Primal(P):}\ \max c^T\mathbf{x}$$

$$\text{subject to: } A\mathbf{x} = b$$

$$\mathbf{x} >= 0, \mathbf{x} \in \mathbb{R}^n$$

The corresponding Dual(D) becomes:

$$\min b^T\mathbf{y}$$

$$\text{subject to: } A^T\mathbf{y} = c$$

$$\mathbf{y} >= 0, \mathbf{y} \in \mathbb{R}^m$$

Explanation for the formulation is as follows:

$$c^T\mathrm{x} = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\text{re-write } A\mathbf{x} <= b \text{ as:}$$

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n <= b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n <= b_2$$

$$\vdots \vdots \ \vdots \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n <= b_m$$

Multiply each inequality above by some non negative multipliers y_i on both sides:

$$y_1(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n <= b_1)$$

$$y_2(a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n <= b_2)$$

$$\vdots \; \vdots \quad \vdots \; \vdots$$

$$y_m(a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n <= b_m)$$

In each of the above inequalities, each multiplier y_i should be greater than or equal to zero. Otherwise the inequalities would reverse their sign and become >= Adding the new inequalities, we get:

$$\Sigma_{i=1}^{n}(y_1a_{1i} + y_2a_{2i} + \ldots + y_ma_{mi})x_i \ <= y_1b_1 \ + \ y_2b_2 \ + \ \ldots \ + \ y_nb_n \tag{3}$$

The primal will have a clear upper bound if each coefficient of x_i in the LHS of (3) is greater than c_i

$$y_1a_{11} + y_2a_{21} + \ldots + y_ma_{m1} \ >= c_1$$

$$y_1a_{12} + y_2a_{22} + \ldots + y_ma_{m2} \ >= c_2$$

$$\vdots$$

$$y_1a_{n1} + y_2a_{n1} + \ldots + y_ma_{mn} >= \ c_n$$

The above equations can be written as:

$$A^T\mathbf{y} >= c$$

The value of the upper bound for the primal will be the least value (1) can take. That is,

$$\text{minimize: } b^Ty$$

This completes the mathematical explanation for the formulation of the dual. If the primal was a minimization problem, it can be converted to the standard form, where it becomes a maximization problem.

3 Properties of Duality

Theorem 1. *The dual of a dual is primal*

Proof.

$$\text{Primal(P):} \ \ \max \ c^T\mathbf{x}$$

$$\text{subject to: } A\mathbf{x} <= b$$

$$\mathbf{x} >= 0, \mathbf{x} \in \mathbb{R}^n$$

Dual(D) becomes:

$$\min \ b^T\mathbf{y}$$

$$\text{subject to: } A^T\mathbf{y} >= c$$

$$\mathbf{y} >= 0, \mathbf{y} \in \mathbb{R}^m$$

The dual can be written in its standard form as:

$$\text{maximize } \ -b^T\mathbf{y}$$

$$\text{subject to: } \ -A^T\mathbf{y} <= -c$$

$$\mathbf{y} >= 0, \mathbf{y} \in \mathbb{R}^m$$

The dual of the dual now becomes:

$$\min\ -c^T\mathbf{z}$$

$$\text{subject to: } (-A^T)^T\mathbf{z} >= -b$$

$$i.e., -Az >= -b$$

$$\mathbf{z} >= 0, \mathbf{z} \in \mathbb{R}^n$$

Converting the above linear program to its standard form and changing the variable from z to x (because it is just a representation), we get the primal P

$$\max\ c^T\mathbf{x}$$

$$\text{subject to: } A\mathbf{x} <= b$$

$$\mathbf{x} >= 0, \mathbf{x} \in \mathbb{R}^n$$

This completes the proof.

□

3.1 Weak Duality

Let $F(P)$ be the set of feasible solutions for the primal LP, $F(D)$ be the set of feasible solutions for the dual LP. Weak duality states that for any $\mathbf{x} \in F(P)$ and $\mathbf{y} \in F(D)$,

$$c^T\mathbf{x} <= b^T\mathbf{y}$$

If $\mathbf{x}^\star$ is the optimal solution to the primal, and $\mathbf{y}^\star$ is the optimal solution to the dual,

$$c^T\mathbf{x}^\star <= b^T\mathbf{y} <= b^T\mathbf{y}^\star,\ \forall\ \mathbf{y} \in F(D)$$

3.2 Strong Duality

If optimal feasible solutions exist for both the primal(P) and the dual(D) linear programs, then strong duality states the following:

$$c^T\mathbf{x}^\star = b^T\mathbf{y}^\star$$

3.3 Duality theorem of Linear Programming

Let the primal linear program, (P) be defined as follows:

$$\max\ c^T\mathbf{x}$$

$$\text{subject to: } A\mathbf{x} <= b$$

$$\mathbf{x} >= 0, \mathbf{x} \in \mathbb{R}^n$$

and the corresponding dual (D) will be:

$$\min\ b^T\mathbf{y}$$

$$\text{subject to: } A^T\mathbf{y} >= c$$

$$\mathbf{y} >= 0, \mathbf{y} \in \mathbb{R}^m$$

There occurs exactly one of the following four possibilities:

1. Neither (P) nor (D) are feasible
   example:
$$\text{maximize } 2x_1 \ - \ x_2$$
$$\text{subject to:}$$
$$x_1 - x_2 <= 1$$
$$-x_1 + x_2 <= -2,$$
$$x_1, \ x_2 >= 0$$

2. (P) is unbounded and (D) has no feasible solution. If the dual is feasible, from the weak duality theorem, there exists a feasible solution to the dual which is an upper bound to (P), but (P) is unbounded. Hence the dual (D) has no feasible solution
   example:
$$\text{maximize } 5x_1 \ + \ 4x_2$$
$$\text{subject to:}$$
$$x_1 <= 1$$
$$x_1 - x_2 <= 8,$$
$$x_1, \ x_2 >= 0$$

3. (P) has no feasible solution and (D) is unbounded
   example:
$$\text{minimize } x_1 \ + \ x_2$$
$$\text{subject to:}$$
$$x_1 >= 6$$
$$x_2 >= 6,$$
$$-x_1 - x_2 >= -11$$
$$x_1, \ x_2 >= 0$$

4. Both (P) and (D) have a feasible solution and the maximum of (P) equals the minimum of (D), that is,
$$c^T \mathbf{x}^\star = b^T \mathbf{y}^\star$$
   An example for this case is mentioned in section 2.1 of this document, where the optimal value for that example is 3.

4 Dualization Recipe

Table 1: Dualization Recipe

	Primal Linear program	Dual Linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Constraint matrix	$A_{m \times n}$	A^T
RHS of constraints	b	c
Objective Function	max c^Tx	min b^Ty

References

[1] Bernd Gartner and Jir Matousek, *Understanding and Using Linear Programming*, Chapter 6: Duality of Linear Programming

[2] Duality, https://en.wikipedia.org/wiki/Linear_programming#Duality