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Lecture 5: Convex Analysis and Properties Of A Solution to the Linear Programming

1 Recap

- X is a convex set if for any $x_1, x_2 \in X$, $\lambda x_1 + (1-\lambda)x_2 \in X$, $\forall \lambda \in [0,1]$
- Intersection of convex sets is convex.
- Union of convex sets need not be convex, although in some cases it can be.
- Epigraph $(f) = \{(x,\mu) | \mu \geq f(x), x \in \mathbb{R}^n\}$
- A function f is convex if its epigraph is convex.
- For two disjoint, non-empty convex subsets of $\mathbb{R}^n$, there is a hyperplane separating the two subsets
- Given vectors $a_1, a_2 ..., a_n$, the convex cone generated by these vectors is a set of all linear combinations of $a_i s$ with non-zero co-efficients
- A convex polyhedron is an intersection of finitely many closed half spaces in $\mathbb{R}^n$
- If a hyperplane touches the polygon at exactly one point, the point is called vertex.

2 Convex Combination

Definition 1. Let $x^1, \ldots, x^k$ be vectors in R^n and let $\lambda_1, \ldots, \lambda_k$ be nonnegative scalars whose sum is unity. The vector $\sum_{i=1}^k \lambda_i x^i$ is said to be a *convex combination* of the vectors $x^1, \ldots, x^k$.

3 Convex Hull

Definition 2. Let $X \subset R^n$ be a set. The *convex hull* of X is the intersection of all convex sets that contain X. Thus it is the smallest convex set containing X, in the sense that any convex set containing X also contains its convex hull.

This is not a very constructive definition. The convex hull can also be described using convex combinations,

Definition 3. Let $x^1, \ldots, x^k$ be vectors in R^n and let $\lambda_1, \ldots, \lambda_k$ be nonnegative scalars whose sum is unity. The *convex hull* of the vectors $x^1, \ldots, x^k$ is the set of all convex combinations of these vectors.

We can prove that both the definitions are equivalent:-
Let C(X) is the intersection of all convex sets that contain X.

$$\mathrm{C}(X) = \bigcap_\alpha C_\alpha$$

Let $\tilde{C}$ is the set of all convex combinations of points in X.

$$\tilde{C} = \{\sum_{i=1}^{m} \lambda_i x^i : m \geq 1, x^1, \ldots, x^m \in X, \lambda_1, \ldots, \lambda_m \geq 0, \sum_{i=1}^{m} \lambda_i = 1\}$$

Proof. First we prove by induction on m that $\tilde{C} \subset C(X)$.

For $m = 1$, $\lambda_1 = 1$, then $x^1 \in C(X)$
For $m = 2$ it follows directly from the convexity of C(X).

Let us assume, as an induction hypothesis, that a convex combination of m elements of a convex set belongs to that set. Consider $m+1$ elements $x^1, \ldots, x^{m+1}$ of a set X and let $\lambda_1, \ldots, \lambda_{m+1}$ be nonnegative scalars that sum to 1.

We assume, without loss of generality, that $\lambda_{m+1} \neq 1$. We then have

$$\sum_{i=1}^{m+1} \lambda_i x^i = \lambda_{m+1} x^{m+1} + (1 - \lambda_{m+1}) \sum_{i=1}^{m} \left(\frac{\lambda_i}{1 - \lambda_{m+1}} \right) x^i \tag{3.1}$$

The coefficients $\lambda_i/(1 - \lambda_{m+1})$, $i = 1, \ldots, m$ are nonnegative and sum to unity.

Using the induction hypotheses, $\sum_{i=1}^{m} \lambda_i x^i/(1 - \lambda_{m+1}) \in C(X)$. Let $x' = \sum_{i=1}^{m} \lambda_i x^i/(1 - \lambda_{m+1})$ then Eq.(3.1) can be written as

$$\sum_{i=1}^{m+1} \lambda_i x^i = \lambda_{m+1} x^{m+1} + (1 - \lambda_{m+1}) x' \tag{3.2}$$

Then, the fact that C(X) is convex and Eq. (3.2) imply that $\sum_{i=1}^{m+1} \lambda_i x^i \in C(X)$.
Hence proved $\tilde{C} \subset C(X)$.

Now, we prove that $C(X) \subset \tilde{C}$ by proving $\tilde{C}$ is convex.

Let $y = \sum_{i=1}^{m} \alpha_i x^i$, $z = \sum_{i=1}^{m} \beta_i x^i$ be two elements of $\tilde{C}$, where $\alpha_i \geq 0$, $\beta_i \geq 0$, and $\sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \beta_i = 1$.

Let $\lambda \in [0, 1]$. Then,

$$\lambda y + (1 - \lambda) z = \lambda \sum_{i=1}^{m} \alpha_i x^i + (1 - \lambda) \sum_{i=1}^{m} \beta_i x^i = \sum_{i=1}^{m} (\lambda \alpha_i + (1 - \lambda) \beta_i) x^i \tag{3.3}$$

The coefficients $\lambda \alpha_i + (1 - \lambda) \beta_i$, $i = 1, \ldots, m$, are non-negative and sum to unity. This shows that $\lambda y + (1 - \lambda) z$ is a convex combination of $x^i, \ldots, x^m$ and, therefore, belongs to $\tilde{C}$. Hence $\tilde{C}$ is a convex set, and thus $C(X) \subset \tilde{C}$.

$C(X) \subset \tilde{C}$ and $\tilde{C} \subset C(X)$ will be true if and and only if $C(X) = \tilde{C}$.

□

4 Basic Feasible Solution Of Linear Programming

The general linear-programming problem in equational form is to find a vector $x = (x_1, x_2, \ldots, x_n)$ which maximizes the linear form (i.e the objective function) $c^T x$.

$$c^T x = c_1 x_1 + c_2 x_2 + ... + c_j x_j + ... + c_n x_n \tag{4.1}$$

subject to the linear constraints $Ax = b$.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + ... + a_{1j}x_j + ... + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + ... + a_{2j}x_j + ... + c_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n &= b_i \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + ... + a_{mj}x_j + ... + a_{mn}x_n &= b_m
\end{aligned}
\tag{4.2}
$$

and

$$x_i \geq 0 \qquad i = 1, 2, ..., n \tag{4.3}$$

Definition 4. A *feasible solution* to the linear-programming problem is a vector $x = (x_1, x_2, \ldots, x_n)$ which satisfies the conditions (4.2) and (4.3).

Definition 5. *Basic solution* to (4.2) is a solution obtained by setting $n-m$ variables equal to zero and solving for the remaining m variables, provided that the determinant of the coefficients of these m variables is nonzero. The m variables are called basic variables.

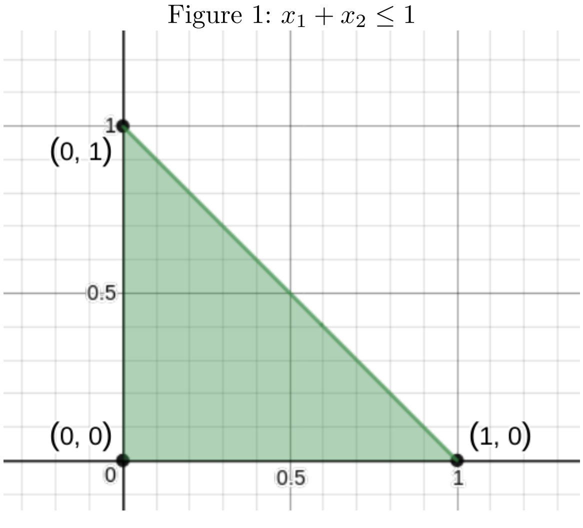
Definition 6. *Basic feasible solution* is a *basic solution* which also satisfies (4.3); that is, all basic variables are nonnegative.

In simple words Basic feasible Solution x of $(Ax = b)$ is a basic solution if the n components of x can be partitioned into m "basic" and $n-m$ "non-basic" variables in such a way that: the m columns of A corresponding to the basic variables form a nonsingular basis and the value of each "non-basic" variable is 0.

Basis is the set B of basic variables. The variables corresponding to B are called basic variables (X_b) and remaining are non-basic variables (X_n). $B \subseteq \{1, 2, 3, ...n\}$. The matrix A_B (m columns of A corresponding to the basic variables) is a non singular matrix.

The system of functional constraints has n variables and m equations, so this will gives $n-m$ degrees of freedom in solving the system, since any $n-m$ variables can be chosen to be set equal to any arbitrary value in order to solve the m equations in terms of the remaining m variables. This $n-m$ are non basic variables and are considered as zero for finding optimal solution.
The equation $x_1 + x_2 \leq 1$ for $x_1, x_2 \geq 0$ has three corner points and all are basic feasible solution. The corner points are basic feasible solution as there is 1 functional constraint with two variables. So one variable should be zero and point (1,0), (0,1) and (0,0) is satisfying this condition. The equation

Figure 1: $x_1 + x_2 \leq 1$



$x_1 + x_2 \leq 1$ can be written as $x_1 + x_2 + x_3 = 1$ in equational form where $x_1, x2, x3 \geq 0$. The corner points (1,0, 0), (0,1, 0) and (0,0, 0) of the resultant polyhedron is the basic feasible solution and as there is 1 functional constraint with three variables. So 2 variables are zero.
If $x_1 + x2 + x3 = 1$ and $x_1 + x_2 = 1/2$, in this there are two functional constraint with three variables, thus, corner points are (1/2,0,1/2) & (0,1/2,1/2) and both are basic feasible solution. If $x_1 + x2 + x3 = 1$ and $x_3 = 1/2$ and $x_1 = x_2$, here we have three equation with three variables and this constraint will satisfy with no zero at corner point. The corner point satisfying the constraint is (1/4, 1/4, 1/2).
If there are m equations with n variables, then $n - m$ variables should be zero and remaining m values can be non-zero.

Example 1. $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$
For $B = \{1,3\}$*, obtain* $A_B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
A_B *is invertible and* $\{1,3\}$ *is a basis of* A*.*

Example 2. $A = \begin{pmatrix} 1 & 5 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 6 \end{pmatrix}$
For $B = \{2,4\}$ *and* $b^T = [14, 7]$*, obtain* $A_B = \begin{pmatrix} 5 & 4 \\ 1 & 5 \end{pmatrix}$
$x = [0,2,0,1,0]$ *and* A_B *is also non singular. Hence* $\{2,1\}$ *is a basis of* A*.*

Definition 7. A *non-degenerate basic feasible solution* is a basic feasible solution with exactly m positive x_i; that is, all basic variables are positive.

Definition 8. An *optimal basic feasible solution* is a basic solution that satisfies condition (4.1), (4.2) and (4.3).

5 Properties of Feasible basis solution

Proposition 1. *A basic feasible solution is uniquely determined by the set* B*. That is, for every m-element set* $B \subseteq \{1, 2, \ldots, n\}$ *with* A_B *nonsingular there exists at most one feasible solution* $x \in \Re^n$ *with* $x_j = 0$ *for all* $j \notin B$*.*

Proof. For x to be a feasible solution, $Ax = b$ must be true.
Ax can be rewritten to $Ax = A_B x_B + A_N x_N$, where $N = \{1, 2, \ldots, n\} \backslash B$.
For x_N to be a basic feasible solution, x_N must equal 0. Thus,

$$A_B x_B = 0$$

As A_B is a nonsingular square matrix. The system $A_B x_B = b$ has exactly one solution $\tilde{x_B}$.
If any of the component of the $\tilde{x_B}$ is negative then *no* basic feasible solution exist for the considered B. □

Theorem 1. *If a linear programming problem admits of an optimal solution, then the optimal solution will coincide with at least one basic feasible solution of the problem.*

Proof. Let x_O be one of the basic feasible solution of a LP problem.
Let X be defined as, $X = \{\tilde{x} | c^T \tilde{x} \geq c^T x_O\}$

$\tilde{x^*}$ is one of the $\tilde{x}$ which has maximum number of zeros. Only corner points are there in basic feasible solution.
Refer any standard book for the proof of this theorem. □

6 References

1. MOKHTARS S. BAZARAA, JOHN J,JARVIS, HANF D. SHERALI, ”Linear Programming and network flows”.

2. Bernd Grtner, Jir Matouek ”Understanding and Using Linear Programming”.

3. Dimitris Bertsimas, John N.Tsitsiklis ”Introduction to Linear Optimization”.