Optimization Methods	Date: *20th Feb,2018*
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Lecture 11: Ellipsoid Update Rule & Examples,Travelling Salesman Problem,Two Player Zero Sum Games

1 Recap

Ellipsoid method is an algorithm for linear programming which always runs in polynomial time in terms of number of variables and number of constraints. An ellipsoid can be described by -

$$E = \{Mx + s : x \in B^n\}$$

where B^n be the n-dimensional ball of unit radius centered at 0.
By manipulating this definition we got an in-equality in the last lecture-

$$E = \left\{ y \in R^n : (y - s)^T Q^{-1} (y - s) \leq 1 \right\}$$

where $Q = MM^T$.
This method generates a sequence of ellipsoids whose volume uniformly decreases at every step, thus enclosing a minimizer of a convex function. In practice, simplex method is much faster than ellipsoid method.

2 Ellipsoid Method

2.1 Update Rule:

Let the sequence of ellipsoids generated be E_0,E_1,E_2....E_t

1. Set $k = 0$ and $E_0 = B(0, R)$.
2. If s_k satisfies all inequalities of the system $Ax <= b$ return s_k and stop.
   Else,we define E_{k+1} as,

   $$s_{k+1} = s_k - \frac{1}{n+1} \frac{Q_k h_k}{\sqrt{s_k^T Q_k h_k}}$$

   $$Q_{k+1} = \frac{n^2}{n^2 - 1} \left(Q_k - \frac{2}{n+1} \frac{Q_k h_k h_k^T Q_k}{h_k^T Q_k h_k} \right)$$

   where h_k is the separating hyper-plane.
3. If the volume of E_{k+1} is smaller than the volume of a ball of radius ϵ , return in-feasible and stop. Otherwise, increase k by 1 and go to Step 2.

2.2 Examples:

1. To find a feasible solution to

$$\begin{aligned} x &<= y \\ y &>= x - 0.5 \\ x + y &>= 1.5 \\ x + y &<= 2 \end{aligned}$$

   In this example, the center of the ellipsoid is moving along the line $x = y$.

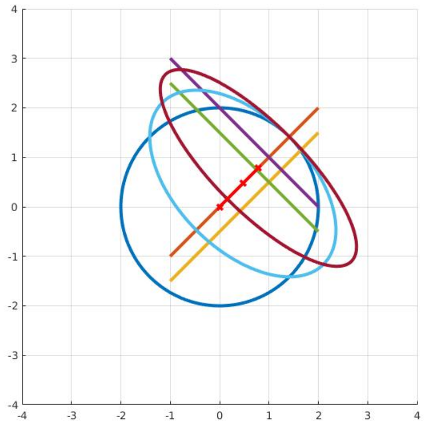

Figure 1: Example1

2. To find a feasible solution to

$$\begin{aligned} x &>= 1 \\ x &<= 1.5 \\ y &>= 1 \\ y &<= 1.5 \end{aligned}$$

   In this case center of the ellipsoid moves along x-axis first and then moves vertically.

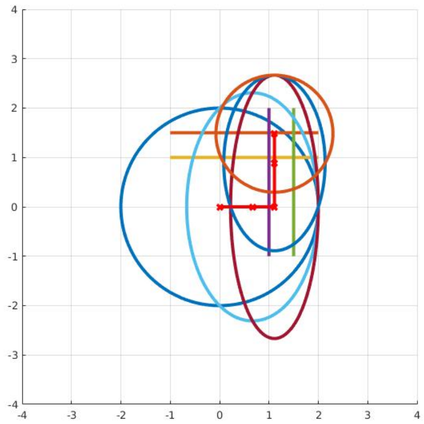

Figure 2: Example2

3. To find a feasible solution to

$$y <= 2x - 0.5$$
$$y >= x - 0.5$$
$$x + y >= 1.5$$
$$x + y <= 2$$

In this case the ellipsoid forms are: dark blue circle, then light blue, brown and then dark blue whose center fall in the feasible region.

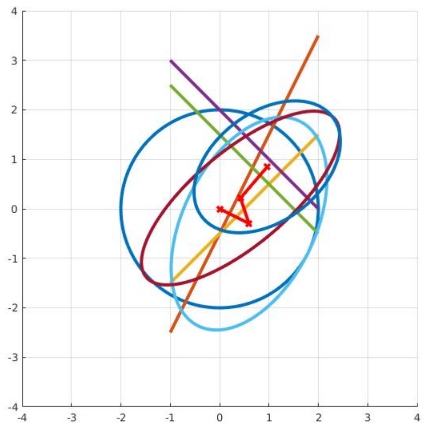

Figure 3: Example3

3 Separation Oracle

- The role of the separation oracle in *Ellipsoid Method* is, given a point p, whether it is in the feasible region or not. If no then give a separating hyper-plane that separates that point and the feasible region.
- Now the problem is given a feasible region, how to find whether a given point p is in it or not.
- For example given below is an LP

$$\begin{aligned} max : c^T x \\ s.t. \\ Ax <= b \\ x >= 0 \end{aligned}$$

  The feasible region will be defined by $Ax <= b$ and $x >= 0$. So one can simply find whether a point p is in the feasible region or not by checking if both $Ap <= b$ and $p >= 0$ holds true.

- If yes then return yes and the problem is finished there and then.
- But if no,then either of the following holds true

  –

$$\begin{aligned} \text{if } A = [a_1 a_2 ..a_k ..a_n]^T \\ \exists i \ a_i^T p > b_i \end{aligned}$$

  in which case our separating plane will be a_i

  –

$$\exists i \ p_i < 0$$

  in which case the separating plane will be e_i (a vector with all zero except a one at i^{th} position)

- A naive method can be to check each of the constraints one after other and then return the first inequality that is not satisfied. We can see that this approach will work as long as the number of constraints are polynomial.

3.1 Travelling Salesman Problem

- Formally it is defined as "Given a set of cities and distance between every pair of cities, the problem is to find the shortest possible route that visits every city exactly once and returns to the starting point."
- It can be formulated as a Integer Program as
  - Let c_e is the distance (or cost) for the edge.
  - Let x_e indicate if the edge is in the optimal path.
  - Let edges(v) be the set of all edges directly connected to vertex or vertices in set v.
  - Since the path contains all the vertices, there will be 2 edges connected to every vertex. From here we have our first set constraints.

$$\sum_{e \ \in \ edges(v)} x_e = 2 \quad \forall v \in V$$

- The selected edges will now form only cycles. But there can be multiple disjoint cycles. To avoid this we add another constraint

$$\sum_{e \, \in \, (edges(S) \, \cap \, edges(\overline{S}))} x_e >= 2 \quad \forall S \in P(V)$$

i.e. if we partition the graph into to parts there will always be path edges crossing from one part to other.

- Since a edge can either be present or not present. So

$$x_e \in 0, 1 \quad e \in E$$

- Finally we have

$$min : \sum_{e \, \in \, E} c_e x_e$$
$$\sum_{e \, \in \, edges(v)} x_e = 2 \quad \forall v \in V$$
$$\sum_{e \, \in \, (edges(S) \, \cap \, edges(\overline{S}))} x_e >= 2 \quad \forall S \in P(V)$$
$$x_e \in 0, 1 \quad e \in E$$

- it's approximation as Linear Program will be

$$min : \sum_{e \, \in \, E} c_e x_e$$
$$\sum_{e \, \in \, edges(v)} x_e = 2 \quad \forall v \in V$$
$$\sum_{e \, \in \, (edges(S) \, \cap \, edges(\overline{S}))} x_e >= 2 \quad \forall S \in P(V)$$
$$x_e \in [0, 1] \quad \forall e \in E$$

- We notice here that the number of constraints here are exponential ($|V| + 2^{|V|} + |E|$). So the separation oracle algorithm proposed above will not work as the number of constraints are not polynomial in size.

- One observation is that the second set constraints can be interpreted as that for any cut sum of the cost of the edges crossing it must be at-least 2.

- Hence the min-cut of the graph must also be greater that 2. Min-cut of a graph can be easily found in polynomial time (for example Edmonds-Karp Algorithm).

- We can find the separating hyper-plane via ford fulkerson algorithm if the min-cut value of the graph is greater than 2 or not.

4 Two Player Zero Sum Games

- The simplest type of competitive situations in game theory is *Two-Player-Zero-Sum* Games. These games involve only two players. They are called zero-sum games because gain for one player will be loss for the other.

- Example - ODD or EVEN :

- Consider a simple game called ODD OR EVEN. In this game the two players are given to choose either odd or even. Eventually they show a number by using their finger. After summing up both the numbers, if the result is even, Player1 gets a positive reward and Player2 gets the same negative reward.Let $x = (x_1, x_2)$ be strategy of Player1 and $y = (y_1, y_2)$ be strategy of Player2. For example, if Player1 shows fingers that sum to odd, his strategy is x=(1,0) and if Player2 select even his strategy is y=(0,1)

$$\sum_{j=1}^{2} x_i = 1, x_i >= 0$$
$$\sum_{j=1}^{2} y_i = 1, y_i >= 0$$

- Let a_{ij} be the payoff of Player1 if Player1 performs i^{th} action and Player2 performs j^{th} action. This implies expected payoff for Player1:

$$\sum_{i,y} x_i.y_j.a_{ij} = \sum_{i=1}^{2}\sum_{j=1}^{2} x_i.y_j.a_{ij} = x^T Ay$$

- For a given strategy of Player2 , Player1 must choose an action such that,

$$\alpha(y) = \max_x x^T Ay$$

- The utility of Player2 is of opposite sign as it is a zero sum game:

$$\beta(x) = \max_y(-x^T Ay) = \min_y x^T Ay$$

4.1 Nash Equilibrium

- A Nash equilibrium is an action profile where nobody has an incentive to deviate from his prescribed action when fixing the others actions. A Nash equilibrium can be of pure-strategies or mixed strategies.
- In a pure strategy Nash equilibrium, each player plays one action in equilibrium.
- In a mixed-strategy Nash equilibrium, a player can play a subset of his available actions according to a certain probability distribution.
- If every player in a game plays his dominant pure strategy, then the outcome will be a Nash equilibrium. The Prisoners' Dilemma is an excellent example of this.
  - In this game, both players get chance to either confess or defect.For example, if both players confess they need to serve six year in prison, If one confess and other defect the one who confess is set free and other get to serve ten years in prison and if both defect each other they both need to serve one years in prison.
  - In this game, both players know that serving one year in prison is better than three years,They both choose to defect and get a to serve one year in prison.This outcome of defect D(-1,-1) is Nash equilibrium.

		Player 2	
		confess	don't confess
Player 1	confess	(-6, -6)	(0, -10)
	don't confess	(-10, 0)	(-1, -1)

Figure 4: Prisoners Dilemma

- (x^*, y^*) is Nash Equilibrium if x^* is such that it achieves $\alpha(y)$ and y^* is such that it achieves $\beta(x)$, i.e.,
  - $\alpha(y^*) = \max_x x^T Ay^* = x^{*T} Ay^*$
    $\beta(x^*) = \min_y x^{*T} Ay = x^{*T} Ay^*$
  - If (x^*, y^*) is Nash Equilibrium implies that x^* and y^* are worst case optimal strategies.
  - If $\beta(x^*) = \alpha(y^*)$, then x^*, y^* is Nash Equilibrium.

4.2 Worst Case Optimal Strategies

- Worst case optimal strategies are such that Player1 assume that Player2 choose a strategy such that payoff for the Player1 is minimized.
- $\min_y(x^T Ay)$, This is the outcome of Player2 who tries to minimize the payoff of the opponent.
- Worst case optimal strategy is to get maximum payoff of Player1 from the minimized payoff set choosen by the Player2.
- x^* is the worst case optimal for Player1 if x^* is s.t,
  - $\beta(x^*) = \max_x \beta(x) = \max_x \min_y x^T Ay$.
- Similarly,y^* is the worst case optimal for Player2 then,
  - $\alpha(y^*) = \min_y \alpha(y) = \min_y \max_x x^T Ay$
- Note that Player1 utilities are negative of Player2 utilities. Hence *maxmin* for Player1's problem becomes *minmax* in column Player2's problem.As we always try to force our opponent to make a bad move.
  - For Player1 Worst case strategy is:

$$\max_x \min_y x^T Ay$$
$$\sum_i x_i = 1$$
$$x_i \geq 0$$

– For Player2 Worst case strategy is:

$$\min_y \max_x x^T Ay$$
$$\sum_i y_i = 1$$
$$y_i \geq 0$$

- One can prove that the above two LPs are primal/dual forms of each other.
  Expanding equations for Player1 Worst case strategy we get:

$$\max_x \left\{ \min_{j \epsilon \{1,2,3,...n\}} x_1 a_{1j} + x_2 a_{2j} + .. + x_m a_{mj} \right\}$$
$$\sum_i x_i = 1$$
$$x_i \geq 0$$

  We know that in LP the objective value takes optimal values at the vertices of the feasible region. The value of y at vertices will be of form $\mathbf{e_j}$, ie the vector will have all zeros except at j^{th} place. which means that we are making move j. Which will result as the same expression as above.
  Expanding equations for Player2 Worst case strategy we get:

$$\min_y \left\{ \max_{i \epsilon \{1,2,3,...m\}} a_{i1} y_1 + a_{i2} y_2 + .. + a_{in} y_n \right\}$$
$$\sum_i y_i = 1$$
$$y_i \geq 0$$

  On careful observation, we can see that both the equations are dual of each other.

- Optimal values in both cases $x^* Ay^*$.
- MINMAX strategies are used as adversarial algorithms in game theory and are nash equilibrium only in two player zero sum games.