Optimization Methods	Date:	*23 Feb 2018*
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Lecture 12: Integer Programming

1 Recap

Considering 2 player zero sum game

$$m : number\ of\ actions\ P1\ can\ take$$

$$n : number\ of\ actions\ P2\ can\ take$$

$$a_{ij} : net\ gain\ to\ P1\ or\ net\ loss\ to\ P2\ if\ P1\ plays\ i\ and\ P2\ plays\ j$$

$$x_i : probability\ of\ P1\ playing\ i^{th}\ action$$

$$y_j : probability\ of\ P2\ playing\ j^{th}\ action$$

- Expected gain for $P1$ can be written as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i a_{ij} y_j$$

$$= x^T Ay$$

- For a given strategy y for $P2$, winning strategy for $P1$

$$\alpha(y) = max_x\ x^T Ay$$

- For a given strategy x for $P1$, winning strategy for $P2$

$$\beta(x) = min_y\ x^T Ay$$

- (x^*, y^*) is Nash Equilibrium if x^* is such that it achieves $\alpha(y^*)$ and y^* is such that it achieves $\beta(x^*)$, i.e.,

$$\alpha(y^*) = max_x\ x^T Ay^* = x^{*T} Ay^* \tag{1}$$

$$\beta(x^*) = min_y\ x^* Ay = x^{*T} Ay^* \tag{2}$$

- If (x^*, y^*) is nash equilibrium strategy then,

$$\alpha(x^*) = \beta(y^*)$$

2 Introduction

An integer programming problem is a mathematical optimization or feasibility program in which some or all of the variables are restricted to be integers.

$$max\ c^T x$$

$$subject\ to\ Ax \leq b$$

$$x \geq 0$$

$$x \in Z^m$$

Here, there is an additional condition from the standard form of LP i.e the variables belong to integer. Integer programming is NP-hard [1].

If we drop the constraint for variables to be integers and add a weaker constraint of interval, it becomes a *LP relaxation* of the 0-1 integer program.

$$max\ c^T x$$

$$subject\ to\ Ax \leq b$$

$$x \geq 0$$

$$x \in [0, 1]$$

Also, if all entries in A are rational, we can safely assume that $b \in Z^m$ as we can cross multiply to change non-integers to integers in all inequalities. In general, it is not possible to solve integer programs in polynomial time except under certain conditions which will be seen in subsequent sections.

So, today we will study one condition on A such that finding solution becomes easier.

2.1 Totally Unimodular Matrices

A matrix A is totally unimodular iff every square submatrix of A has determinant $\in$ {0, 1, -1}

Lemma. *Totally unimodular matrix A should only have entries 0, 1 or -1*

Proof. Every entry of A is a also a submatrix of A of size 1×1. Since, each submatrix should have determinant 0, 1 or -1, every entry should be 0, 1 or -1.

Hence, proved. □

Lemma. *Let A be a totally unimodular matrix and $\bar{A}$ be a matrix obtained by appending a unit vector e as new column to A, then $\bar{A}$ is also totally unimodular (Unit vector e has only one entry as 1 or -1, other entries are 0).*

Proof. Let Q be any square submatrix of $\bar{A}$.

Case 1 Q is a submatrix of A
$Det(Q)$ is 1, 0 or -1 because A is totally unimodular.

Case2 Q contains some entries from e and all these entries are 0.
In this case, $Det(Q)$ is 0.

Case3 Q contains some entries from e and one of these entries is 1 or -1.

Applying Laplace expansion [2] around column c containing entries from e and 1 or -1 is contained in r row of that column then,

$$Det(Q) = 0 \pm 1 \times Det(Q - (c, r))$$

Now, matrix $Q - (c, r)$ is a square submatrix of A, hence it has determinant 1, 0 or -1. Hence, $Det(Q)$ is 1,0 or -1.

Hence in all cases, $Det(Q)$ is 1,0 or -1. So $\bar{A}$ is totally unimodular matrix.
Hence, proved. □

Thus by adding more unit rows to the column of a totally unimodular matrix, it remains totally unimodular.

3 Integral Optimal Solution in LP

We will now see how to find the integral optimal solution by performing LP relaxation.

Lemma. *Consider a LP in standard form*

$$max\ c^T x$$

$$subject\ to\ Ax \leq b$$

$$x \geq 0$$

$$b \in Z^m$$

and A is a totally unimodular matrix. Then, If LP has an optimal solution, this implies, it also has an integral optimal solution.

Proof. First, we convert the problem to equation form. The constraints now become.

$$A\bar{x} = b$$

Let us consider that $\bar{x^*}$ is the optimal solution found using the simplex method and B is the optimal basis. Then,

$$\bar{x^*} = (\bar{x^*_B}, \bar{x^*_N})$$

$$\bar{x^*_B} = \bar{A_B}^{-1} b\ and\ \bar{x^*_N} = (0, 0, ...)$$

Since $\bar{A_B}$ is a sub matrix of A and A is totally unimodular, $|\bar{A_B}|$ would be 1 or -1 as it can't be 0 if solution exist. Also, we know

$$\bar{A_B}^{-1} = \frac{adj(\bar{A_B})}{|\bar{A_B}|}$$

Each entry in *adjoint* of a matrix is ± 1 multiplied by determinant of submatrix [3]. Thus, each entry in *adjoint* of $\bar{A_B}$ is also 1, 0 or -1.

This implies all entries in $\bar{A_B}^{-1}$ are integers.

Hence, we can conclude that $\bar{x^*_B}$ has all integers.

Hence, proved. □

4 Bipartite Graphs

A graph is a set of vertices connected by edges. Edges can be directed or undirected.

There are numerous ways to represent graph. Some of them are by representing them as Adjacency matrix and Incidence Matrix.

Definition. *Adjacency Matrix The Adjacency matrix is a square matrix that is used to represent finite graphs. Each entry in the matrix indicate whether there is edge between vertices, represented by row and column index, or not.*

Definition. *Incidence Matrix The Incidence matrix is a matrix in which row indices represent vertex and column indices represent edges.*

Let A be the incidence matrix of Graph G.

$$a_{ij} = \begin{cases} 1, & \textit{if jth edge incident on ith vertex} \\ 0, & \textit{otherwise} \end{cases} \tag{3}$$

It is easy to see that each column will have two 1's as an edge can have two end points.

Definition. *Bipartite Graphs A bipartite graph, also called a bigraph, is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent (having edge in common).*

All acyclic graphs are bipartite. A cyclic graph is bipartite if all its cycles are of even length [4]. It is intuitive to see that the cycle in bipartite graph has to be of even length. As, if an edge is incident in one set of vertices, the other end point has to be in the other set.

Definition. *Maximum Matching A matching in a Bipartite Graph is a set of edges such that no two edges share a common end point i.e each vertex has less that or equal to one incident edge. A Maximum Matching is the matching having maximum number of edges. There can be more than one maximum matchings.*

Now we would see the importance of taking Bipartite graph into consideration and how we can solve its LP relaxation to find optimal solution.

4.1 Lemma

Let $G = (X \cup Y, E)$ be a bipartite graph. Then the incidence matrix of the graph A is totally unimodular.

Proof: By induction, we prove that every $l \times l$ sub-matrix Q of A has determinant 0 or ± 1.

- $l = 1$: The matrix and determinant are given as:

$$Q = [1],\ det(Q) = 1 \tag{4}$$

$$Q = [0],\ det(Q) = 0 \tag{5}$$

- $l > 1$: The columns of $l \times l$ sub-matrix Q correspond to edges and each column of Q has at most two non-zero entries (1).

  1. If there are no 1's, then all the entries in the column are 0. Therefore, $det(Q) = 0$.
  2. If there is a column with only one non-zero entry, we can expand along that column using Laplace expansion. $det(Q)$ will be equal to the determinant of the $(l-1) \times (l-1)$ sub-matrix Q'. By using induction $det(Q') = 0, 1, -1$. Therefore, $det(Q) = 0, 1, -1$.
  3. If all the columns contain two 1's, then by adding the rows corresponding to X, we get a row of 1's [1.....1]. Likewise, by adding the rows corresponding to Y, we get a row of 1's [1.....1]. Since the matrix has two equal rows, this implies that the det(Q) = 0.

5 Maximum matching and minimum vertex cover in Bipartite Graphs

5.1 Maximum matching

A matching in a graph $G = (V, E)$ is a set of $E' \subseteq E$ of edges with the property that each vertex is incident on at-most one edge in E'. Maximum matching occurs if it has the largest number of edges among all the matchings in G.

5.2 Minimum Vertex Cover

A vertex cover of $G = (V, E)$ is a set of $V' \subseteq V$ of vertices with the property that each edge is incident on at least one vertex in V'. A vertex cover is minimum if it has the smallest number of vertices among all the covers in G.

6 König's Theorem

Let $G = (V, E)$ be a bipartite graph. Then the size of maximum matching in G equals the size of minimum vertex cover in G.

Proof: Let us consider the integer programming formulation for the maximum matching.

$$maximize \sum_{j=1}^{m} x_j \tag{6}$$

$$\begin{aligned} subject\ to\ Ax &\leq B \\ x &\geq 0 \\ x &\in Z^m \end{aligned}$$

A is the incidence matrix of G. The row of A corresponding to the vertex v_i induces the constraint:

$$\sum x_j \leq 1 \tag{7}$$

This means that $x_j \in \{0, 1\}$ and that the edges e_j with $\tilde{x}_j = 1$ in an optimal solution $\tilde{x}$ form a maximum matching.

Next we formulate the minimum vertex cover as an integer programming problem.

$$maximize \sum_{i=1}^{n} y_i \tag{8}$$

$$\begin{aligned} subject\ to\ A^T y &\geq B \\ y &\geq 0 \\ y &\in Z^n \end{aligned}$$

where A is the incidence matrix of G. The row of A^T corresponding to edge e_j will exhibit

$$\sum y_i \geq 1 \tag{9}$$

This shows that in any optimal solution $\tilde{y}$, we have $\tilde{y}_j \in \{0, 1\}$ for all i, since any larger value could be decreased to 1. But then the vertices v_i with $\tilde{y} = 1$ in an optimal solution $\tilde{y}$ form a minimum vertex cover.

In both maximum matching and minimum vertex cover, we can *drop the integrality constraints* without affecting the optimum values A and A^T and both are totally unimodular. But the resulting linear programs are dual to each other, thus duality theorem shows that their optimal values are equal.

References

[1] https://en.wikipedia.org/wiki/Integer_programming#Proof_of_NP-hardness

[2] https://en.wikipedia.org/wiki/Laplace_expansion

[3] https://en.wikipedia.org/wiki/Adjugate_matrix

[4] https://proofwiki.org/wiki/Graph_is_Bipartite_iff_No_Odd_Cycles