Optimization Methods	Date:	*23-01-2018*
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Lecture 7: Simplex Algorithm - Finding an initial feasible basis, Pivot rules and Simplex algorithm summary

1 Recap

1.1 Problem 1:

Maximize

$$x_1 + x_2$$

Subject to

$$x_1 + x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

Looking at the problem, it can be said that all the points on the line $x_1 + x_2$ are optimal. Solving through the simplex method:

1. Add slack variables to convert standard form to equational form.

$$x_1 + x_2 + x_3 = 1$$

2. Simplex tableau would be:

$$x_3 = 1 \text{ - } x_1 \text{ - } x_2$$

$$z = 0 + x_1 + x_2$$

Basis elements are $\{3\}$

3. Let x_1 be the entering variable

$$x_1 = 1 \text{ - } x_2 \text{ - } x_3$$

$$z = 1 - x_3$$

Basis elements are $\{1\}$

4. All the coefficients of the variables in z are negative, so the simplex algorithm can be stopped and the maximum value of $z = 1$. Here, we have the coefficients of one of the non-basis vectors as zero. Hence, the whole line segment is an optimal solution.

1.2 Problem 2:

Maximize

$$x_1 + x_2$$

Subject to

$$x_1 + x_2 + x_3 \leq 1$$
$$x_1, x_2, x_3 \geq 0$$

1. Add slack variables to convert standard form to equational form.

$$x_1 + x_2 + x_3 + x_4 = 1$$

2. Simplex tableau would be:

$$x_4 = 1 \text{ - } x_1 \text{ - } x_2 \text{ - } x_3$$

$$z = 0 + x_1 + x_2$$

Basis elements are $\{4\}$

3. Let x_1 be the entering variable

$$x_1 = 1 \text{ - } x_2 \text{ - } x_3 \text{ - } x_4$$

$$z = 1 - x_3 - x_4$$

Basis elements are $\{1\}$

4. All the coefficients of the variables in z are negative, so the simplex algorithm can be stopped and the maximum value of z = 1. Similar to the above case, one of the non-basis variables have are zero. Hence, the whole line is a feasible solution.

1.3 Problem 3:

Maximize

$$x_1 + x_2 + x_3$$

Subject to

$$x_1 + x_2 + x_3 \leq 1$$
$$x_1, x_2, x_3 \geq 0$$

1. Add slack variables to convert standard form to equational form.

$$x_1 + x_2 + x_3 + x_4 = 1$$

2. Simplex tableau would be:

$$x_4 = 1 \text{ - } x_1 \text{ - } x_2 \text{ - } x_3$$

$$z = 0 + x_1 + x_2 + x_3$$

Basis elements are {4}

3. Let x_1 be the entering variable

$$x_1 = 1 \text{ - } x_2 \text{ - } x_3 \text{ - } x_4$$

$$z = 1 - x_4$$

Basis elements are {1}

4. In this case, two of the non-basis variables have a zero coefficient. Hence, the optimal solution is a localised 2-d plane.

1.4 Problem 4:

Maximize

$$x_1 + x_2$$

Subject to

$$x_1 + x_2 = 1$$
$$x_1, x_2 \geq 0$$

This can be rewritten as

Maximize

$$x_1 + x_2 - x_3 - x_4$$

Subject to

$$x_1 + x_2 - x_3 - x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

1. Simplex tableau would be:

$$x_1 = 1 \text{ - } x_2 + x_3 + x_4$$

$$z = 1$$

Basis elements are {1}

2. In this case, there are 3 degrees of freedom i.e 3 non-basic variables have zero coefficient. The optimal solutions is a localized 3-dimensional region.

If the matrix $A = m \times n$, there can be at most $n - m$ degrees of freedom.

2 Finding an initial feasible basis

So far, all the examples were of the form-

$$\text{maximize } c^T x$$
$$\text{subject to } Ax \leq b \text{ and } x \geq 0$$

In these cases, the indices of the slack variables introduced to transform it to equational form served as the feasible basis. But, a lot of times it is not possible to determine the basis looking at the problem. This problem can be solved by first applying the simplex method over an auxiliary problem. For example, consider the following linear program in equational form:

maximize

$$x_1 + 2x_2$$

subject to

$$\begin{gathered} x_1 + 3x_2 + x_3 = 4 \\ 2x_2 + x_3 = 2 \\ x_1, x_2, x_3 \geq 0 \end{gathered}$$

It is not easy to tell the initial basis. Now, let us introduce auxiliary variables x_4 and x_5, and call them as "corrections of infeasibility": $x_4 = 4 - x_1 - 3x_2 - x_3$, $x_5 = 2 - 2x_2 - x_3$. Now, if we could come up with a feasible solution in terms of x_1, x_2, x_3 being positive and the auxiliary variables zero, we have found an initial feasible basis. That is we must solve the linear program given by:

maximize

$$-x_4 - x_5$$

subject to

$$\begin{gathered} x_1 + 3x_2 + x_3 + x_4 = 4 \\ 2x_2 + x_3 + x_5 = 2 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{gathered}$$

Now, we know that x_4 and x_5 form a feasible basis to begin the simplex method. If a feasible solution exists for our original linear program, our objective function $-x_4 - x_5$ will turn out to be zero. Solving the linear program, we can obtain (2,0,2,0,0) as optimal solution for the linear program(this can be verfied).

Thus, we can start our original program with (x1,x2,x3) = (2,0,2) as the basic feasible solution. The initial simplex tableu would be:

$$\begin{array}{c} x_1 = 2 - x_2 \\ x_3 = 2 - 2x_2 \\ \hline z = 2 - x_2 \end{array}$$

and continue with the algorithm.

Thus, a general procedure to finding an initial feasible basis for a linear program of the form for a linear program of the form

$$\begin{gathered} \text{maximize } c^T x \\ \text{subject to } Ax \leq b \text{ and } x \geq 0 \end{gathered}$$

is as follows:

1. Rearrange the matrix A to obtain $b \geq 0$. This can be obtained by multiplying the equations with $b_i < 0$ by -1.

2. Introduce m new variables and solve the auxiliary linear program given by

   maximize

   $$-(x_{\mathrm{n+1}} + x_{\mathrm{n+2}} + ... + x_{\mathrm{n+m}})$$

   subject to

$$\bar{E}\bar{y} = b$$
$$\bar{x} \geq 0,$$

where,

$\bar{y} = (x_1, ..., x_{\text{n+m}})$ is the vector of all varibles including the new ones,

$\bar{E} = (A|I_m)$, i.e, the matrix obtained by appending A with an identiy $m * m$ matrix to the right.

3. The auxiliary program can be solved by choosing $x_{\text{n+1}}$ through $x_{\text{n+m}}$ as the initial feasible basis.

4. The original linear program is feasible if and only if every optimal solution in the linear program satifies $x_{\text{n+1}} = x_{\text{n+2}} = ... = x_{\text{n+m}} = 0$.

5. If the linear program has a feasible solution, feaible basis for the original linear program can be obtained from one of the optimal solutions of the auxiliary program.

3 Pivot Rules

A lot of times, there are multiple variables that can that could be chosen as entering variable in simplex algorithm. Pivot rules help us to choose the entering variables. Some common pivot rules are:

1. *Largest Positive Coeffient:* According to this rule, the variable corresponding to the maximum positive coefficient in the row of the objective function z is chosen. The intuition in this rule is that the variable is chosen so that there is a higher increase in objective z for unit increase in the entering variable.

2. *Largest Increase in the objective value:* According to this rule, the entering variable should be chosen such that there is a maximum increase in the value of the objective function Z. Though this rule is computationally more expensive than the *Largest Positive Coeffient* rule, it locally maximizes the process.

3. *Steepest Edge:* According to this rule, the entering variable should be chosen in such a way that the current basic feasible solution moves to a direction closest to the direction of vector c. That is, we should maximize

$$\frac{c^T(x_{\text{new}} - x_{\text{old}})}{||x_{\text{new}} - x_{\text{old}}||}$$

   where x_{old} is the basic feasible solution for the current simplex tableau and x_{new} is the feasible solution that would be obtained after choosing an entering variable.

4. *Random Selection:* According to this rule, a variable is randomly selected from all the possible entering variables. The problem with this rule is that it does not prevent cycling of variables.

5. *Bland's rule:* According to this rule, there is an initial total ordering of variables. When there are several possibilities for the entering variable, variable that comes first in the order is chosen. This rule is significant since it prevents cycling.

4 Simplex Algorithm

The following summarizes the procedure to be followed in simplex algorithm-

1. Convert the input linear program to equational form

   maximize $\mathbf{c^T x}$

   subject to $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$

   with n variables and m equations, where A has rank m

2. If no feasible basis is available, arrange for $\mathbf{b} \geq \mathbf{0}$, and solve the following auxiliary linear program by the simplex method:

   maximize -$(y_{n+1} + y_{n+2} + ... + y_{n+m})$

   subject to $\overline{E}\overline{y} = \mathbf{b}$

   $\overline{y} \geq \mathbf{0}$

   where $y_{n+1}, ..., y_{n+m}$ are new variables If the optimal value of the objective function comes out negative, the original linear program is infeasible; **stop**. Otherwise, the first n components of the optimal solution form a basic feasible solution of the original linear program.

3. For a feasible basis $B \subseteq$ {1,2,3,...,n} compute the simplex tableau *T(B)*, of the form

   $$\mathbf{x_B} = \mathbf{p} + \mathbf{Q x_N}$$

   $$\overline{\qquad\qquad\qquad}$$

   $$z = z_0 + \mathbf{r^T x_N}$$

4. If $\mathbf{r} \leq \mathbf{0}$ in the current simplex tableau, return an optimal solution ($\mathbf{p}$ specifies the basic components, while the nonbasic components are 0); **stop.**

5. Otherwise, select an *entering variable* x_v whose coefficient in the vector $\mathbf{r}$ is positive. If there are several possibilities, use some pivot rule.

6. If the column of the entering variable x_v in the simplex tableau is non negative, the linear program is unbounded; **stop.**

7. Otherwise select a *entering variable* x_u. Consider all rows of the simplex tableau where the coefficient of x_u is negative. and in each such row divide the corresponding component of the vector $\mathbf{p}$ by that coefficient and invert sign. The leaving variable is the one for which this ratio in its corresponding row in the tableu is minimal. If there are several possibilities, decide by a pivot rule, or arbitrarily if the pivot rule doesn't specify how to break ties in this case.

8. Replace the current feasible basis B by the new feasible basis $(B \backslash \{u\}) \cup$ {v}. Update the simplex tableau so that it corresponds to this new basis. Go to Step 4.

4.1 Problem:

Simplex algorithm is demonstrated with *Largest Positive Coefficient* as pivot rule here:

Maximize using largest coefficient pivot rule-

$$9x_1 + 3x_2 + x_3$$

Subject to

$x_1 \leq 1$

$6x_1 + x_2 \leq 9$

$18x_1 + 6x_2 + x_3 \leq 81$

$x_1,\ x_2,\ x_3 \geq 0$

4.2 Solution:

1. Add slack variables to convert standard form to equational form.

   $x_1 + x_4 = 1$

   $6x_1 + x_2 + x_5 = 9$

   $18x_1 + 6x_2 + x_3 + x_6 = 81$

2. Let the initial basic feasible solution be

   $x_4 = 1 - x_1$

   $x_5 = 9 - 6x_1 - x_2$

   $x_6 = 81 - 18x_1 - 6x_2 - x_3$

   $z = 0 + 9x_1 + 3x_2 + x_3$

   Basis is {4,5,6}

3. x_1 has the largest coefficient and let x_1 be the *entering variable*

   $x_1 = 1 - x_4$

   $x_5 = 3 + 6x_4 - x_2$

   $x_6 = 63 + 18x_4 - 6x_2 - x_3$

   $z = 9 - 9x_4 + 3x_2 + x_3$

   Basis is {1,5,6}

4. x_2 has the largest coefficient and let x_2 be the *entering variable*

   $x_1 = 1 - x_4$

   $x_2 = 3 + 6x_4 - x_5$

   $x_6 = 45 - 18x_4 + 6x_5 - x_3$

   $z = 18 + 9x_4 - 3x_5 + x_3$

   Basis is {1,2,6}

5. x_4 has the largest coefficient and let x_4 be the *entering variable*

   $x_4 = 1 - x_1$

   $x_2 = 9 - 6x_1 - x_5$

   $x_6 = 27 + 18x_1 + 6x_5 - x_3$

   $z = 27 - 9x_1 - 3x_5 + x_3$

   Basis is {4,2,6}

6. x_3 has the largest coefficient and let x_3 be the *entering variable*

   $x_4 = 1 - x_1$

   $x_2 = 9 - 6x_1 - x_5$

   $x_3 = 27 + 18x_1 + 6x_5 - x_6$

   $z = 54 + 9x_1 + 3x_5 - x_6$

   Basis is {4,2,3}

7. x_1 has the largest coefficient and let x_1 be the *entering variable*

$$x_1 = 1 - x_4$$
$$x_2 = 3 + 6x_4 - x_5$$
$$x_3 = 45 - 18x_4 + 6x_5 - x_6$$

$$z = 63 - 9x_4 + 3x_5 - x_6$$

Basis is {1,2,3}

8. x_5 has the largest coefficient and let x_5 be the *entering variable*

$$x_1 = 1 - x_4$$
$$x_5 = 3 + 6x_4 - x_2$$
$$x_3 = 63 + 18x_4 - 6x_2 - x_6$$

$$z = 72 + 9x_4 - 3x_2 - x_6$$

Basis is {1,3,5}

9. x_4 has the largest coefficient and let x_4 be the *entering variable*

$$x_4 = 1 - x_1$$
$$x_5 = 9 - 6x_1 - x_2$$
$$x_3 = 81 - 18x_1 - 6x_2 - x_6$$

$$z = 81 - 6x_1 - 6x_2 - x_6$$

10. All the coefficients of the varibales in z are negative, so the simplex algorithm can be stopped and the maximum value of $z = 81$ and the basis is {3,4,5}.

4.3 Effiiency of the Simplex Method:

The simplex method performs very satisfactorily even for large linear programs. Computational experiments indicate that for linear programs in equational form with m equations it typically reaches an optimal solution in something between $2m$ and $3m$ pivot steps.

It was thus a great surprise when Klee and Minty constructed a linear program with n nonnegative variables and n inequalities for which the simplex method with Dantzig's original pivot rule needs exponentially many pivot steps, namely 2^n - 1!

The set of feasible solutions is an ingeniously deformed n - dimensional cube, called the *Klee-Mint cube*, constructed in such a way that above problem solved using simplex method passes through all of its vertices. The deformed cube is inscribed in an ordinary cube in order to better convey the shape.

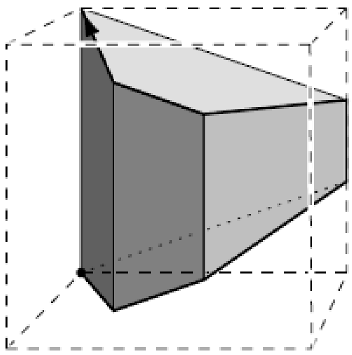

Figure 1: Klee Minty Cube with $n = 3$

References

[Matousek and Gartner(2006)] Jiri Matousek and Bernd Gartner. *Understanding and Using Linear Programming*. Springer, 2006.