Optimization Methods		Date:	*January 16, 2018*
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Lecture 5: Linear Programming - Convex Hull and Introduction to Basic Feasible Solutions

1 Recap

Last lecture, we learnt about *convex sets* and *convex functions* and looked at proofs and some of the examples around them. Also, talked about *hyperplanes* which divides the space into two halves called as *halfspaces*. *Polyhedron* is the intersection of halfspaces. Looked briefly on separation hyperplane theorem. When *polyhedron* is closed we call it as bounded *polyhedron* or *polytope*.We also defined what is vertex of a polyhedron.

2 Convex Hull

The *Convex Hull* of a given set X, where $X \subset R^n$, can be defined as intersection of all convex sets that contain X. Thus it is the smallest convex set containing X. It implies that any convex set containing X also contains its *convex hull*. *Convex Hull* $C(X)$, can be written in mathematical form as:

$$C(X) = \cap_\alpha C_\alpha \quad where \quad \forall \quad \alpha \quad and \quad C_\alpha \supseteq X \tag{1}$$

The convex hull can also be described using convex combinations. This can be captured mathematically as below:

$$\tilde{C}(X) = \left\{ \sum_{i=1}^{m} t_i x_i : m \geq 1, x_1, ..., x_m \in X, \quad t_1, ..., t_m \geq 0, \quad \sum_{i=1}^{m} t_i = 1 \right\} \tag{2}$$

$\tilde{C}(X)$ is a convex combination of $x_1, x_2 ... x_m$. A convex combination is a particular kind of a linear combination, in which the coefficients are non- negative and sum to 1.
The *convex hull* of X is shown in the Figure 1.

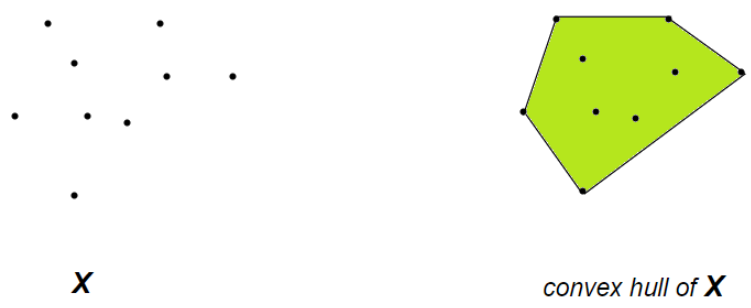

Figure 1: Convex Hull

If Eq. (1) and Eq. (2) are equal, we need to show that $\tilde{C}(X) \subseteq C(X) \quad and \quad C(X) \subseteq \tilde{C}(X)$

Lemma 1. *The* convex hull *of a set* $X \subseteq R_n$, $\quad C(X) = \cap_\alpha C_\alpha$ *equals the set*

$$\tilde{C}(X) = \left\{ \sum_{i=1}^{m} t_i x_i : m \geq 1, x_1, ..., x_m \in X, \quad t_1, ..., t_m \geq 0, \quad \sum_{i=1}^{m} t_i = 1 \right\}$$

$\tilde{C}(X)$ *is the collection of all convex combinations of points in* X.

Proof. Proving by induction on m that each convex combination has to lie in the *convex hull* $C(X)$. For $m = 1 \quad and \quad m = 2 \quad (finite)$, these are points in X and hence are part of C(X). In other words, it follows directly from the convexity of $C(X)$.
Let $m \geq 3$ and let $x = t_1x_1 + ... + t_mx_m$ be a convex combination of points of X
If $t_m = 1$, then we have $x = x_m \in C(X)$
For $t_m < 1$, let $t^{'} = {}^{t_i}/_{1-t_m} \quad i = 1, 2..., m-1$
Then $x^{'} = t^{'}_1x_1 + ... + t^{'}_{m-1}x_{m-1}$ is a convex combination of points $x_1, ..., x_{m-1}$ ($t^{'}_i$ sum to 1).
By the inductive hypothesis, $x^{'} \in C(X)$
So $x = (1 - t_m)x^{'} + t_mx_m$ is a convex combination of two points of the (convex) set $C(X)$ and it also lies in $C(X)$.
Thus $\tilde{C}(X) \subseteq C(X)$.

To prove, $C(X) \subseteq \tilde{C}(X)$, we need to show that $\tilde{C}(X)$ is convex.
Let $x, y \in \tilde{C}(X)$ be two convex combinations of $x_1, x_2, ...x_n$

$x = \lambda_1x_1 + \lambda_2x_2 + ...\lambda_nx_n \quad where \quad \lambda_1, \lambda_2...\lambda_n \geq 0 \quad and \quad \sum_{i=1}^{n} \lambda_i = 1$

$y = t_1x_1 + t_2x_2 + ...t_mx_m \quad where \quad t_1, t_2...t_m \geq 0 \quad and \quad \sum_{j=1}^{m} t_i = 1$

If $0 \geq a, b \leq 1$ and $a + b = 1$, then
$ax + by$ is a convex combination of $x_i's \quad and \quad y_j's$
As n and m are finite and the new combination point is in $\tilde{C}(X)$ then, $\tilde{C}(X)$ is a convex set containing X.
Thus, $C(X) \subseteq \tilde{C}(X)$

Since $\tilde{C}(X) \subseteq C(X)$ and $C(X) \subseteq \tilde{C}(X)$ Then, $C(X) = \tilde{C}(X)$

□

3 Basic Feasible Solutions

Consider the following LP in standard form :

$$a_{11}x_1 + a_{12}x_2 \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 \leq b_2$$

$$x_1, x_2 \geq 0$$

The equational form of the above standard form is :

$$a_{11}x_1 + a_{12}x_2 + x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + x_4 = b_2$$

$$x_1, x_2, x_3, x_4 \geq 0$$

In the Figure 2, the points co-ordinates written in black color are points of intersection of lines in standard form, whereas the co-ordinates in red color are points of intersection of lines in equational form, where the added dimensions are due to the slack variables introduced in equational form.

Let A matrix be of dimension m x n. This implies that we have m constraints excluding the trivial constraints [$x_i \geq 0$]. To obtain a point in $\mathbb{R}^n$, we need the equality of n distinct hyperplanes. Since

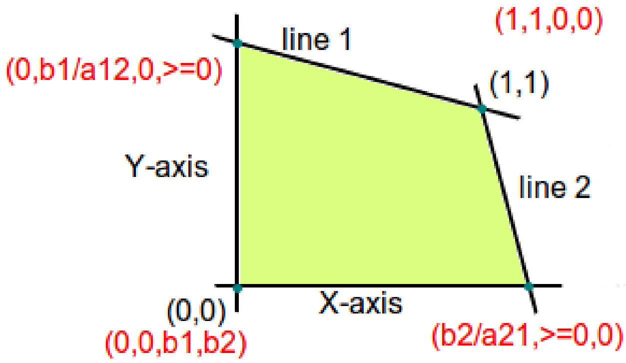

Figure 2: Basic Feasible Solutions

matrix A will give us only m constraints, to get the remaining $n-m$ constraints , we will use the trivial constraints [$x_i \geq 0$].
If we are in equational form ; $m \geq n$ can happen only if we have some redundant constraints (e.g $x+y \geq 2$ and $x+y \geq 1$), or if some constraint is a linear combination of other two constraints. If $m = n$, (we will assume A is full rank matrix), there will be exactly one feasible point. In general we are interested in looking for the corners/vertices of the feaisble region and as the above description, it should have $n-m$ variable to be zero. This is called as basic feasible solution and more formally defined as,

Definition 3.1. Basic Feasible Solution A *Basic Feasible Solution* of linear program
maximize $c^T X$ subject to $Ax = b$ and $x \geq 0$

is a feasible solution of $x \in \mathbb{R}^n$,for which there exists an m-element set $B \subseteq \{1, 2, ..., n\}$ such that

- The (square)matrix A_B is non-singular, i.e. the columns indexed by B are linearly independent
- $x_j = 0 \ \forall j \notin B$.

The variables corresponding to the set B are called basic-variables x_B. Remaining variables are called non-basic variables x_N. Set B is called basis.

Consider the LP in equational form :

$$x_1 + x_2 + x_3 = 1$$

- For this equation along with trivial constraints , corners in 3-D are $(1,0,0)$; $(0,1,0)$ *and* $(0,0,1)$. Here without any additional constraints , we have $m = 1$, so we have a point as out solution and according to definition of Basic Feasible solution, we have $B = \{1\}$, $x_B = \{x_1\}$ *and* $x_N = \{x_2, x_3\}$ for first point , $B = \{2\}$, $x_B = \{x_2\}$ *and* $x_N = \{x_1, x_3\}$ for second point and $B = \{3\}$, $x_B = \{x_3\}$ *and* $x_N = \{x_1, x_2\}$ for the third point.
- If we add 1 more constraint $x_3 \geq \frac{1}{2}$ then since now we have two constraints our $m = 2$, and our feasible solution is a line segment. The Corner points are $(\frac{1}{2}, 0, \frac{1}{2})$ *and* $(0, \frac{1}{2}, \frac{1}{2})$. Accordingly we have $B = \{1,3\}$, $x_B = \{x_1, x_3\}$ *and* $x_N = \{x_2\}$ for first point , and $B = \{2,3\}$, $x_B = \{x_2, x_3\}$ *and* $x_N = \{x_1\}$ for second point.
- If we add 1 more constraint $x_1 = x_2$ to the above LP , we have $m = 3$, and our feasible region is a single point : $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, we have $B = \{1,2,3\}$, $x_B = \{x_1, x_2, x_3\}$ *and* $x_N = \{\}$.

Proposition 1. *A basic feasible solution is uniquely determined by the set B i.e, for every m-element set B with A_B non-singular, there exists at most one feasible solution $x \in \mathbb{R}^n$ with $x_j = 0 \quad \forall \quad j \notin B, \quad B \subseteq \{1, 2...n\}$*

Proof:
Expressing in $Ax = b$ form, we have:
$\begin{bmatrix} A_B & A_N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$

$\implies A_B x_B + A_N x_N = b$
Since x_N's are 0, then, $A_B x_B = b$
As A_B is non-singular, AB^{-1} exists. Thus, $x_B = A_B^{-1} b$
(when $x_i < 0$, it is not feasible and hence we say at most and not exactly). Given a basis, a solution may not exist as some of the variables in X_B may take negative values which does not satisfy the constraint $x \geq 0$. However, if solution exists, it is unique. Given a basis, a solution may not exist as some of the variables in X_B may take negative values which does not satisfy the constraint $x \geq 0$. However, if solution exists, it is unique.

Theorem 1. *For a linear program in equational form, if an optimal solution exists, then there is a basic feasible solution that is optimal*

To given an intuition behind the above theorem, let us consider a simple linear program

$$max \quad x_1 + x_2$$

$$s.t \quad x_1 + x_2 \leq 1$$

$$x_1 \geq 0 \quad , \quad x_2 \geq 0$$

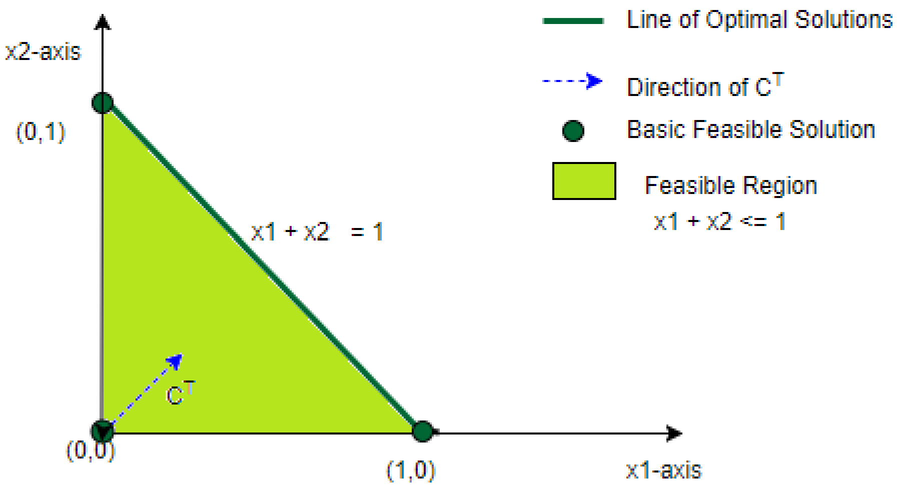

Figure 3: Basic feasible solution that is optimal

From the above Fig. (3), it is clear that basic feasible solutions are (0,0), (1,0) and (0,1). Out of these basic feasible solutions, we have (1,0) and (0,1) on the optimal line. Thus, we can infer that if an optimal solution exists, then there at least one basic solution that is optimal.

4 Homework/Additional Problems

4.1 Problem 1

Find the basic solution: given matrix A as below, $B = \{2,4\}$ and $b = [14 \quad 7]^T$

$A = \begin{bmatrix} 1 & 5 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 6 \end{bmatrix}$

$b = \begin{bmatrix} 14 \\ 7 \end{bmatrix}$

Set of columns in basic, $B = \{2,4\}$

Basic non-singular matrix with columns set {2,4}, $A_B = \begin{bmatrix} 5 & 4 \\ 1 & 5 \end{bmatrix}$

$x = \begin{bmatrix} x_2 \\ x_5 \end{bmatrix}$

Solving for x_2 and x_4:

$\begin{bmatrix} 5 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 14 \\ 7 \end{bmatrix}$

Solving above gives: $x_2 = 2 \quad and \quad x_4 = 1$

The basic solution with $B = \{2,4\}$ is $x = (0,2,0,1,0)^T$

4.2 Problem 2

Analyse the number of extreme points present in an L.P.

The feasible region for a linear programming problem, given by,

$$\begin{aligned} \max \quad & c'x \\ \text{s.t.} \quad & Ax \leq b \\ & \;\; x \geq 0 \end{aligned}$$

where, A is an m x n matrix, $c \in R^n$, and $b \in R^m$, $m < n$, is represented by a polyhedron. A polyhedron has a finite number of extreme points or vertices which are bounded by,

$$\binom{m}{n} = \frac{m!}{n!\,(m-n)!}$$

which represents m equations and choosing n variables if $m > n$. As m and n increase, the value of $\binom{m}{n}$ increases. hence, for a general **L.P.** problem, the number of vertices can be very large.

References

[1] Jiri Matousek and Bernd Gartner. *Understanding and Using Linear Programming.* Springer, 2007.