IWASAWA INVARIANTS OF ABELIAN VARIETIES IN EXTENSIONS OF NUMBER FIELDS

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Abstract. We establish a Kida-type formula for the behaviour of the Iwasawa invariants of abelian varieties in finite Galois p-extensions of number fields on the algebraic side generalizing the results of Hachimori–Matsuno for elliptic curves.

1. Introduction

The classical Riemann–Hurwitz formula describes the relationship of the Euler characteristics of two Riemann surfaces when one is a ramified covering of the other. Suppose $\pi : R_1 \to R_2$ is an n-fold covering of compact, connected Riemann surfaces and g_1, g_2 are their respective genera. The classical Riemann–Hurwitz formula is the statement

$$2g_1 - 2 = (2g_2 - 2)n + \sum \left(e(\mathcal{P}_2) - 1\right),$$

where the sum is over all points $\mathcal{P}_2$ on R_2 and $e(\mathcal{P}_2)$ denotes the ramification index of $\mathcal{P}_2$ for the covering π. There is a generalization of this Riemann–Hurwitz formula to finite separable morphisms of smooth projective geometrically connected curves over an arbitrary field provided that the ramification is tame, see [Har83, IV §2]. An analogue of the above formula for CM fields was proven by Y. Kida in [Kid80] and is referred to as Kida's formula. This formula describes the change of the classical Iwasawa λ-invariants in a p-extension in terms of the degree and the ramification index. In [Iwa81], K. Iwasawa proved this formula using the theory of Galois cohomology for extensions of $\mathbb{Q}$ which are not necessarily finite. More precisely, he proved the following theorem.

Theorem 1.1. [Iwa81, Theorem 6] Let $p \geqslant 2$ and F be a number field. Let F_{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of F and $\mathcal{L}/F_{\text{cyc}}$ be a cyclic extension of degree p, unramified at every infinite place of F_{cyc}. Assume that the classical μ-invariant, denoted by $\mu(F_{\text{cyc}})$ is 0. Then

$$\lambda(\mathcal{L}) = p\lambda\left(F_{\text{cyc}}\right) + \sum_{w \nmid p} \Big(e\left(w \mid v\right) - 1\Big) + (p-1)\left(h_2 - h_1\right)$$

where w ranges over all non-p places of $\mathcal{L}$, h_i is the rank of the abelian group $\mathrm{H}^i(\mathcal{L}/F_{\text{cyc}}, E_{\mathcal{L}})$, and $E_{\mathcal{L}}$ is the group of all units of $\mathcal{L}$.

The Selmer group plays a crucial role in the study of rational points of abelian varieties (in particular, elliptic curves). In [Maz72], B. Mazur initiated the study of the growth of the p-primary part of the Selmer group in the cyclotomic $\mathbb{Z}_p$-extension of number fields. In [HM99], Y. Hachimori–K. Matsuno proved an analogue of Kida's formula for Selmer groups of *elliptic curves* in p-extensions

Date: January 2, 2024.
2020 *Mathematics Subject Classification.* 11R23 (primary).
Key words and phrases. Iwasawa invariant, abelian varieties, abelian varieties which are not modular.
DK was partially supported by a PIMS postdoctoral fellowship. We thank Timo Keller for stimulating discussions in the early stages of the project that contributed to the ideas in this paper and Antonio Lei for always generously answering our questions. We are grateful to Meng Fai Lim for his comments on an earlier draft of this paper.

of the cyclotomic $\mathbb{Z}_p$-extension of a number field under the assumption that the associated μ-invariant is 0. In [PW06], R. Pollack–T. Weston proved a similar statement for Selmer groups of a general class of Galois representations including the case of p-ordinary Hilbert modular forms and p-supersingular modular forms. Recently, M. F. Lim has made progress towards proving a Kida-type formula for Galois representations without any hypothesis on the μ-invariant (see [Lim21, Lim22]).

The scope of this article is to prove a Kida-type formula for abelian varieties which are not necessarily modular (see Theorem 3.1). Even though the strategy of our proof is the same as that of [HM99], the calculation of the local Herbrand quotients is significantly different from the elliptic curve case and uses the theory of Néron models. The complications arise because one does not have a product decomposition of (the special fiber of) the Néron model into its good reduction, multiplicative and unipotent part, but only a filtration. Hence, our proof differs significantly from [HM99] and is much more involved. As an application of this algebraic result, we provide an upper bound on the growth of the Mordell–Weil rank of an abelian variety at each finite layer of a uniform p-adic Lie extensions in Corollary 5.4. In Section 4, we adapt the strategy of [Lim22] and prove a general Kida-type result with a slight weakening of the assumption that the μ-invariant is zero.

Outlook: In Iwasawa theory, there is a general philosophy that (an appropriate analogue of) the Iwasawa–Greenberg Main Conjecture should relate the structure of of the arithmetic object (such as the Selmer group) to an associated p-adic L-function. In particular, the invariants of the p-adic L-function associated with a modular abelian variety should be equal to the invariants of the characteristic polynomial attached to the Selmer group of the said abelian variety over the cyclotomic $\mathbb{Z}_p$-extension. More precisely, let A_f be a modular abelian variety defined over $\mathbb{Q}$ and f be the associated modular form with coefficients in $\mathbb{Q}(f)/\mathbb{Q}$. By the work of Amice–Vélu (see [AV75]) and/or Mazur–Tate–Teitelbaum (see [MTT86]) there is a construction of a p-adic L-function with very precise interpolation properties. The Main Conjecture in this setting asserts that the characteristic ideal of the Selmer group over $\mathbb{Q}_{\text{cyc}}$ is equal to the ideal generated by the appropriate L-function. One expects that appropriate generalizations hold upon base-changing A_f to number fields. Studying the algebraic side of the Main Conjecture upon base change is relatively straightforward. An ambitious future project is to to appropriately define the correct p-adic L-function in this setting and prove an analytic version of the Kida formula generalizing the work of [Mat00]. If the shape of the formulas match (which one would certainly expect) on both sides, this would provide concrete evidence towards the Main Conjecture for modular abelian varieties.

In [BKR21], the authors had studied questions pertaining to rank stability of elliptic curves in degree p and degree p-power extensions over $\mathbb{Q}$. The key idea was to use the Kida-type formula in the elliptic curve case proven by [HM99] and measure 'how often' the λ-invariant remains trivial upon base-change. We are confident that the results in this paper can be used to generalize the results in [BKR21] to the abelian variety setting.

Organization: Including the introduction, the article has five sections. Section 2 is preliminary in nature. We discuss the basics of Iwasawa theory of abelian varieties and facts about Néron models of abelian varieties. In Section 3, we prove a Kida-type formula for the algebraic λ-invariant of abelian varieties. In Section 4, we prove a more general version under some weakened hypothesis. Using our main result from Section 3, in Section 5 we prove results towards the asymptotic growth of Mordell–Weil ranks of abelian varieties in non-commutative p-adic Lie extensions.

2. Preliminaries

Let F be any number field and A/F be an abelian variety defined over F. Fix an algebraic closure $\overline{F}$ of F and write G_F for the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. For a given integer m, set $\mathsf{A}[m]$

to be the Galois module of all m-torsion points in $\mathsf{A}(\overline{F})$. If v is a prime in F, we write F_v for the completion of F at v. The main object of interest is the Selmer group.

Definition 2.1. For any integer $m \geqslant 2$, the *m-Selmer group* is defined as follows

$$\mathrm{Sel}_m(\mathsf{A}/F) = \ker\left(\mathrm{H}^1\left(G_F, \mathsf{A}[m]\right) \longrightarrow \prod_v \mathrm{H}^1\left(G_{F_v}, \mathsf{A}\right)[m]\right).$$

This m-Selmer group fits into the following short exact sequence

$$0 \longrightarrow \mathsf{A}(F)/m \longrightarrow \mathrm{Sel}_m(\mathsf{A}/F) \longrightarrow Ш(\mathsf{A}/F)[m] \longrightarrow 0. \tag{2.1}$$

Here, $Ш(\mathsf{A}/F)$ is the *Shafarevich–Tate group* which is conjecturally finite. Throughout this article, we will assume the finiteness of the Shafarevich–Tate group, wherever required.

2.1. **Recollections from Iwasawa theory.** For details, we refer the reader to standard texts in Iwasawa theory (e.g. [Was97, Chapter 13]). Let p be a fixed prime. Consider the *cyclotomic* $\mathbb{Z}_p$-extension of a number field F, denoted by F_{cyc}. Set $\Gamma := \mathrm{Gal}(F_{\mathrm{cyc}}/F) \simeq \mathbb{Z}_p$. The *Iwasawa algebra* $\Lambda = \Lambda(\Gamma)$ is the completed group algebra $\mathbb{Z}_p[\![\Gamma]\!] := \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$. Fix a topological generator γ of Γ; there is the following isomorphism of rings

$$\begin{aligned} \Lambda &\xrightarrow{\sim} \mathbb{Z}_p[\![T]\!] \\ \gamma &\mapsto 1 + T. \end{aligned}$$

Let M be a cofinitely generated cotorsion Λ-module. The *Structure Theorem of* Λ*-modules* asserts that the Pontryagin dual of M, denoted by $M^\vee$, is pseudo-isomorphic to a finite direct sum of cyclic Λ-modules. In other words, there is a map of Λ-modules

$$M^\vee \longrightarrow \left(\bigoplus_{i=1}^{s} \Lambda/(p^{m_i})\right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda/(h_j(T))\right) \tag{2.2}$$

with finite kernel and cokernel. Here, $m_i > 0$ and $h_j(T)$ is a distinguished polynomial (i.e., a monic polynomial with non-leading coefficients divisible by p). The *characteristic ideal* of $M^\vee$ is (up to a unit) generated by the *characteristic element*,

$$f_M^{(p)}(T) := p^{\sum_i m_i} \prod_j h_j(T).$$

The μ-invariant of M is defined as the power of p in $f_M^{(p)}(T)$. More precisely,

$$\mu(M) = \mu_p(M) := \begin{cases} 0 & \text{if } s = 0 \\ \sum_{i=1}^{s} m_i & \text{if } s > 0. \end{cases}$$

When $s > 0$, define $\theta(M) := \max_i\{m_i\}$. The λ-invariant of M is the degree of the characteristic element, i.e.

$$\lambda(M) = \lambda_p(M) := \sum_{j=1}^{t} \deg h_j(T).$$

Let A/F be an abelian variety with good *ordinary* reduction at all primes above p. We shall assume throughout this article that the prime p is odd. Let S be a finite set of primes in F containing the primes above p, the primes of bad reduction of A, and the archimedean primes. Let F_S be the maximal algebraic extension of F which is unramified at the primes outside S. Set $\mathsf{A}[p^\infty]$

to be the Galois module of all p-power torsion points in $\mathsf{A}(\overline{F})$. For a prime $v \in S$ and any finite extension L/F contained in the unique cyclotomic $\mathbb{Z}_p$-extension of F (denoted by F_{cyc}), write

$$J_v(\mathsf{A}/L) = \bigoplus_{w|v} \mathrm{H}^1\left(G_{L_w}, \mathsf{A}\right)[p^\infty]$$

where the direct sum is over all primes w of L lying above v. Then, the *p-primary Selmer group over F* is defined as follows

$$\mathrm{Sel}_{p^\infty}(\mathsf{A}/F) := \ker\left\{\mathrm{H}^1\left(\mathrm{Gal}\left(F_S/F\right), \mathsf{A}[p^\infty]\right) \longrightarrow \bigoplus_{v\in S} J_v(\mathsf{A}/F)\right\}.$$

It is easy to see that $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F) = \varinjlim_n \mathrm{Sel}_{p^n}(\mathsf{A}/F)$, see for example [CS00, §1.7]. By taking direct limits of (2.1), the p-primary Selmer group fits into a short exact sequence

$$0 \longrightarrow \mathsf{A}(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathrm{Sel}_{p^\infty}(\mathsf{A}/F) \longrightarrow Ш(\mathsf{A}/F)[p^\infty] \longrightarrow 0. \tag{2.3}$$

Next, define

$$J_v(\mathsf{A}/F_{\mathrm{cyc}}) = \varinjlim_L J_v(\mathsf{A}/L)$$

where L ranges over finite extensions contained in F_{cyc} and the inductive limit is taken with respect to the restriction maps. The *p-primary Selmer group over* F_{cyc} is defined as follows

$$\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}}) := \ker\left\{\mathrm{H}^1\left(\mathrm{Gal}\left(F_S/F_{\mathrm{cyc}}\right), \mathsf{A}[p^\infty]\right) \longrightarrow \bigoplus_{v\in S} J_v(\mathsf{A}/F_{\mathrm{cyc}})\right\}.$$

When $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}})$ is a cofinitely generated cotorsion Λ-module, in view of the Structure Theorem of Λ-modules, we can define the algebraic μ and λ-invariants, which we denote as $\mu(\mathsf{A}/F_{\mathrm{cyc}})$ and $\lambda(\mathsf{A}/F_{\mathrm{cyc}})$, respectively.

Remark 2.2. The Λ-cotorsionness hypothesis is a conjecture of B. Mazur [Maz72] and is known to be true for *modular* abelian varieties defined over $\mathbb{Q}$ by the work of K. Kato [Kat04, Theorem 14.4].

2.2. **Facts about Néron models of abelian varieties.** We will briefly review some key facts from the theory of Néron models of abelian varieties following [Art86, BLR90]. The results in this section are stated in the context and notation which we use in the rest of the paper; however, note that they hold more generally.

To fix notation, let k be an ℓ-adic local field (where $\ell \neq p$) with ring of integers $\mathcal{O}_k$, and let k_n be the nth layer of the cyclotomic $\mathbb{Z}_p$-extension k_{cyc} of k.

Definition 2.3. Let A be an abelian variety over k. The Néron model $\mathscr{A}$ of A is a smooth commutative group scheme over k such that for any smooth morphism $\mathcal{S} \to \mathcal{O}_k$, the natural map of abelian groups $\mathrm{Hom}_{\mathcal{O}_k}(\mathcal{S}, \mathscr{A}) \to \mathrm{Hom}_k(\mathcal{S} \times_{\mathcal{O}_k} k, \mathsf{A})$ is a bijection. This condition is known as the *Néron mapping property.*

It is a theorem of Néron [Nér64] (see also [Art86, Theorem 1.2]) that the Néron model of A/k exists. Since $\mathcal{O}_{k_n}/\mathcal{O}_k$ is étale, in particular smooth, it follows from the Néron mapping property that $\mathsf{A}(k_n) = \mathscr{A}(\mathcal{O}_{k_n})$.

Let κ denote the residue field of k of characteristic ℓ ($\neq p$). We will also later make use of the following result concerning the kernel of the reduction from the $\mathcal{O}_k$-points of $\mathscr{A}$ (equal to the k-points of A) to the κ-points of $\mathscr{A}$.

Theorem 2.4. Let k be as above with maximal ideal $\mathfrak{m}$. Let A be a d-dimensional abelian variety with associated formal group $\widehat{\mathsf{A}}$. The kernel of the reduction $\mathsf{A}(k) = \mathscr{A}(\mathcal{O}_k) \to \mathscr{A}(\kappa)$ is $\widehat{\mathsf{A}}(\mathfrak{m}) \simeq \mathfrak{m}^d$. Moreover, this latter group is a pro-ℓ group.

Proof. For the first assertion see [HS00, Theorem C.2.6]. The second statement is proven in [Ser06, Corollary 2, p. 118]. □

We find it useful to consider the Néron models of abelian varieties when we are interested in different types of reduction. For elliptic curves, we have the notion of good reduction, multiplicative reduction, and additive reduction which can be generalized to abelian varieties using the theory of Néron models. First, we recall the following theorem of C. Chevalley [Che60]:

Theorem 2.5 (Chevalley)**.** For a perfect field k, every smooth connected k-group G is an extension of an abelian variety A by a smooth connected affine k-group N:

$$1 \longrightarrow N \longrightarrow G \longrightarrow \mathsf{A} \longrightarrow 1.$$

Let A be an abelian variety over k with Néron model $\mathscr{A}$. Denote by $\mathscr{A}_0$ the special fiber of $\mathscr{A}$ and by $\mathscr{A}_0^\circ$ its identity component. Applying Chevalley's theorem to $\mathscr{A}_0^\circ$, we have

$$1 \longrightarrow H \longrightarrow \mathscr{A}_0^\circ \longrightarrow B \longrightarrow 1,$$

where B is an abelian variety and H is a smooth connected affine k-group. The group H contains a maximal torus T such that $U := H/T$ is a unipotent group variety (where $H = T \oplus U$ is split because H is commutative), $B := \mathscr{A}_0^\circ/H$ is an abelian variety, and $\pi_0^{\text{ét}}(\mathscr{A}/\kappa) = \mathscr{A}_0/\mathscr{A}_0^\circ$ is a finite étale group scheme.

Definition 2.6. With notation as above, we say

- A has *good reduction* if $\mathscr{A}_0^\circ \simeq B$ (or $\mathscr{A}_0 \simeq B$).
- A has *semi-stable reduction* if $H = T$, or in other words $\mathscr{A}_0^\circ$ has no unipotent part.
- A has *anisotropic reduction* (or *unipotent reduction* or *additive reduction*) if $\mathscr{A}_0^\circ \simeq U$ (and T is trivial).

3. Algebraic Kida-Type Formulae for Abelian Varieties

In this section, we prove a generalization of [HM99, Theorem 3.1] to abelian varieties. Throughout this discussion, we fix a number field F and assume that A/F is a d-dimensional abelian variety with good ordinary reduction at a prime p. We further suppose that the p-primary Selmer group $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-cotorsion, see Remark 2.2. Since p is fixed throughout this discussion, we drop it from the subscript.

The main theorem in this section is the following which relates the λ-invariant of the Selmer group of A/F to that of the base-change curve A/L where L/F is a finite p-power Galois extension.

Theorem 3.1 (algebraic Kida-type formula for abelian varieties)**.** Let L/F be a finite Galois extension of degree a power of p. Let A/F be a d-dimensional abelian variety with good ordinary reduction at p and such that **(Add)** is satisfied. Suppose that $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-cotorsion and that the associated μ-invariant, denoted by $\mu(\mathsf{A}/F_{\mathrm{cyc}})$, is zero. Then, $\mathrm{Sel}_{p^\infty}(\mathsf{A}/L_{\mathrm{cyc}})$ is a cotorsion $\Lambda(\Gamma)$-module with $\mu(\mathsf{A}/L_{\mathrm{cyc}}) = 0$. Furthermore, the respective λ-invariants, denoted by $\lambda(\mathsf{A}/F_{\mathrm{cyc}})$ and $\lambda(\mathsf{A}/L_{\mathrm{cyc}})$ satisfy the following formula:

$$\lambda(\mathsf{A}/L_{\mathrm{cyc}}) = [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + \sum_{w \in P_1} c_s(w)\left(e(w) - 1\right) + \sum_{w \in P_2} 2c'\left(e(w) - 1\right).$$

Here, $e(w)$ is the ramification degree in $L_{\mathrm{cyc}}/F_{\mathrm{cyc}}$, and the sets P_1, P_2 are the sets of primes in L_{cyc} defined as

$$P_1 = \{w \nmid p : \ \mathsf{A} \text{ has non-trivial split toric multiplicative reduction part at } w\},$$

$$P_2 = \{w \nmid p : \ \mathsf{A} \text{ has non-trivial good reduction part at } w \text{ and } \mathsf{A}(L_{\mathrm{cyc},w})[p] \neq 0\}.$$

The constant $c_s(w)$ is the dimension of the split toric part of the special fiber $\mathscr{A}_0$ of the Néron model $\mathscr{A}$ of A at w and $0 \leqslant c' \leqslant c_g(w)$ where $c_g(w)$ is the dimension of the abelian part of the special fiber $\mathscr{A}_0$ of the Néron model $\mathscr{A}$ of A at w.

We often call the maximal abelian variety subquotient of the Néron model or its fibers the "good reduction part", the (split) toric reduction subquotient, the "(split) toric part", etc.

Strategy of proof: We begin by observing that to prove this theorem, it suffices to consider the special case $G = \mathrm{Gal}(L/F) \simeq \mathbb{Z}/p\mathbb{Z}$. For details of this reduction step, see [HM99, Lemma 3.2].

Since we suppose that $\mu(\mathsf{A}/F_{\mathrm{cyc}}) = 0$, the same argument as [HM99, Corollary 3.4] implies that $\mathrm{Sel}_{p^\infty}(\mathsf{A}/L_{\mathrm{cyc}})$ is a cotorsion $\Lambda(\Gamma)$-module with $\mu(\mathsf{A}/L_{\mathrm{cyc}}) = 0$.

The rest of the argument for this proof proceeds exactly as in [HM99, Section 4], which we omit. In particular, we have the following analogue of (4.3) in *op. cit.*

$$\lambda(\mathsf{A}/L_{\mathrm{cyc}}) = [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\lambda(\mathsf{A}/F_{\mathrm{cyc}}) - \sum_{w \in T'_{\mathrm{cyc}}} b_w \,(p-1)\,.$$

Here, we use the notation introduced in *op. cit.* and write T'_{cyc} to denote the (finite) subset of primes in L_{cyc} which lie above the primes of bad reduction of A *and do not split* in $L_{\mathrm{cyc}}/F_{\mathrm{cyc}}$ (equivalently, in L/F). The integer b_w is the maximum power of p dividing the Herbrand quotient of the p-primary torsion points of $\mathsf{A}(L_{\mathrm{cyc},w})$, i.e.,

$$b_w := \mathrm{ord}_p\left(h_G\left(\mathsf{A}(L_{\mathrm{cyc},w})[p^\infty]\right)\right).$$

The only place where our proof differs from the original proof, is in the calculation these integers b_w. In the remainder of this section, we will carry out the computation for b_w.

We compute these b_w in four steps.

Step 1: compute the p^∞-torsion of the connected component of the identity of the special fiber of the Néron model $\mathscr{A}$ of A over the residue field κ_{cyc}, see Section 3.1.
Step 2: compute the Herbrand quotients over the residue field, see Section 3.2.
Step 3: show that this is the same as the Herbrand quotient over $F_{\mathrm{cyc},v}$, see Section 3.3.
Step 4: compare the Herbrand quotient over $F_{\mathrm{cyc},v}$ to the Herbrand quotient over $L_{\mathrm{cyc},w}$ and compute b_w, see Section 3.4.

3.1. **p^∞-torsion of abelian varieties.** Let ℓ be a prime number distinct from p. Let k be an ℓ-adic local field, and let k_n be the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension k_{cyc} of k. This is the unique unramified $\mathbb{Z}_p$-extension of k. Indeed, any $\mathbb{Z}_p$-extension of k must be at most tamely ramified because the wild inertia group is an ℓ-group and $p \neq \ell$. By local class field theory, any $\mathbb{Z}_p$-extension of an ℓ-adic field is unramified. As before, we denote the ring of integers of k_n by $\mathcal{O}_{k_n}$ and its finite residue field by κ_n.

Recall that for an abelian variety A/k, there exists a smooth group scheme $\mathscr{A}/\mathcal{O}_k$ called the Néron model of A/k, such that $\mathsf{A}(k_n) = \mathscr{A}(\mathcal{O}_{k_n})$. Further, we have that the special fiber $\mathscr{A}_0$ has a filtration by smooth commutative group schemes,

$$0 \subseteq T \subseteq H \subseteq \mathscr{A}_0^\circ \subseteq \mathscr{A}_0.$$

Here, $\mathscr{A}_0^\circ$ is the connected component of the identity of $\mathscr{A}_0$, T is a torus, $U := H/T$ is a unipotent group variety (where $H = T \oplus U$ is split because H is commutative), $B := \mathscr{A}_0^\circ/H$ is an abelian variety, and $\pi_0^{\text{ét}}(\mathscr{A}/\kappa) = \mathscr{A}_0/\mathscr{A}_0^\circ$ is a finite étale group scheme. We can decompose the torus T into a split (resp. non split) part, which we denote by $\mathbb{G}_m^{c_s(k)}$ (resp. T_{an}).

Since we would like to compute the Herbrand quotient of $\mathsf{A}[p^\infty]$, it will suffice to compute the Herbrand quotients of B, $\mathbb{G}_m^{c_s(k)}$, and $H = T_{\mathrm{an}} \oplus U$. In the following proposition, we compute the p^∞-torsion of B, H, and $\mathbb{G}_m^{c_s(k)}$ over κ_{cyc}.

Proposition 3.2. Let ℓ and p be distinct primes, and $k/\mathbb{Q}_\ell$ be a finite extension such that μ_p is contained in k, with finite residue field κ of characteristic $\ell \neq p$. Let $k_{\mathrm{cyc}} = k(\mu_{p^\infty})$ be the cyclotomic $\mathbb{Z}_p$-extension of k. Let A be a d-dimensional abelian variety over k.

(1) Writing B to denote the good reduction part,
$$B[p^\infty](\kappa_{\mathrm{cyc}}) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{2c'} \text{ where } 0 \leqslant c' \leqslant c_g(k).$$
(2) Let $T^{\mathrm{split}} \simeq \mathbb{G}_m^{c_s(k)}$ be the split toric part of $\mathscr{A}_0^\circ$. Then
$$T^{\mathrm{split}}[p^\infty](\kappa_{\mathrm{cyc}}) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{c_s(k)}.$$
(3) Let $H = T_{\mathrm{an}} \oplus U$ be the anisotropic (i.e., non-split toric and unipotent) part of $\mathscr{A}_0^\circ$. Then $H[p^\infty](\kappa_{\mathrm{cyc}})$ is finite. If $p > 2d+1$ or if $H = T_{\mathrm{an}}$, then
$$H[p^\infty](\kappa_{\mathrm{cyc}}) = 0.$$

Proof. First, we make the following observation: since k contains μ_p, the order of the residue field of k_n is congruent to $1 \pmod p$.

(1) Consider $B[p](\kappa)$ as a finite étale subgroup scheme of $B[p]$. Define the finite étale group scheme C to be the quotient $B[p]/B[p](\kappa)$. Consider the Galois submodule M which consists of all elements of $B[p^\infty](\overline{\kappa})$ such that some non-zero multiple lies in $B[p](\kappa)$. It is $\mathbb{Z}_p$-cofree of rank equal to $c := \dim_{\mathbb{F}_p} B[p](\kappa)$.

Note that
$$\ker\left(\mathrm{GSp}_c(\mathbb{Z}_p) \twoheadrightarrow \mathrm{GSp}_c(\mathbb{F}_p)\right) \hookrightarrow \mathrm{Mat}_{c\times c}(\mathbb{Z}_p)$$
is a pro-p group. Hence, $\kappa(M)/\kappa$ is a pro-p extension. Further, since κ has absolute Galois group $\widehat{\mathbb{Z}}$, we know that
$$\mathrm{Gal}(\kappa(M)/\kappa) = \mathbb{Z}_p = \mathrm{Gal}(\kappa_{\mathrm{cyc}}/\kappa).$$
Hence, $M(\kappa_{\mathrm{cyc}}) = M(\overline{\kappa}_{\mathrm{cyc}})$.

Observe that $C(\kappa) = 0$. Applying the topological Nakayama's Lemma (see [NSW13, Corollary 5.2.18 (ii)]) we know that $C(\kappa_{\mathrm{cyc}}) = 0$. Hence $B[p^\infty](\kappa_{\mathrm{cyc}}) = M(\kappa_{\mathrm{cyc}}) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^c$.

Now, to prove that c is even, consider the perfect, alternating and Galois-equivariant Weil pairing
$$B[p](\overline{\kappa}) \times B[p](\overline{\kappa}) \longrightarrow \mu_p(\kappa).$$
Since $\mu_p \subseteq k^\times$ and $p \neq \ell$, $\mu_p(\kappa)$ is a *trivial* Galois module. Hence the dual of the trivial Galois module $B[p](\kappa)$ is trivial again, and hence equal to it because it is the maximal trivial Galois submodule. Since the dual $B[p](\kappa)^\vee = \mathrm{Hom}(B[p](\kappa), \mathbb{Z}/p)$ contains for every non-zero element of $B[p](\kappa)$ a linear form not vanishing on that element, the Weil pairing restricts to a *perfect* alternating pairing on $B[p](\kappa)$.

(2) Recall that we denote the split part of the torus, written as T^{split}, by $\mathbb{G}_m^{c_s(k)}$; in other words, $\mathbb{G}_m^{c_s(k)} \simeq (\mu_{p^\infty})^{c_s(k)}$ as p-divisible groups. Thus, we have $\mathbb{G}_m^{c_s(k)}(\kappa_{\mathrm{cyc}}) = (\mu_{p^\infty})^{c_s(k)}(\kappa_{\mathrm{cyc}})$. Because κ_{cyc} is the residue field of k_{cyc}, which is by definition $k(\mu_{p^\infty})$, it contains all of the p^n roots of unity. It follows that
$$(\mu_{p^\infty})(\kappa_{\mathrm{cyc}}) \simeq \varinjlim \mathbb{Z}/p^n\mathbb{Z} \simeq \mathbb{Q}_p/\mathbb{Z}_p.$$
Thus, $(\mu_{p^\infty})^{c_s(k)}(\kappa_{\mathrm{cyc}}) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{c_s(k)}$.

(3) Let α, $c_s(k)$, and $c_a(k)$ be the dimensions of the unipotent, toric and abelian parts of the special fiber of the Néron model over k. For the *anisotropic part*, it is proven in [CX08, Main Theorem (b)] that the prime-to-ℓ torsion part of $\#H(k)$ is finite and bounded above by an absolute constant $C = C(\alpha, c_s(k), c_a(k), \ell)$. Therefore, if $p \gg 0$, then $H(k)[p^\infty] = 0$ and by the topological Nakayama's Lemma we know that $H(k_{\mathrm{cyc}})[p^\infty] = 0$, as well.

In the *non-split toric case*, $\alpha = c_a(k) = 0$ and the Néron special fiber does not contain a copy of $\mathbb{G}_m$. The non-split toric part T_{an} corresponds to a twist of a totally split torus $\mathbb{G}_m^n$ by an element f in

$$\mathrm{H}^1\left(\mathrm{Gal}(\overline{\kappa}/\kappa), \mathrm{GL}_n(\mathbb{Z})\right) = \mathrm{Hom}_{\mathrm{cont}}\left(\mathrm{Gal}(\overline{\kappa}/\kappa), \mathrm{GL}_n(\mathbb{Z})\right).$$

Since $\mathrm{Gal}(\overline{\kappa}/\kappa)$ is pro-cyclic, this is the same as twisting by a homomorphism

$$f\colon\ \mathrm{Gal}(\kappa'/\kappa) \longrightarrow \mathrm{GL}_n(\mathbb{Z})$$

with κ'/κ a finite extension. Or, equivalently by a torsion element g in $\mathrm{GL}_n(\mathbb{Z})$. Since T_{an} has no split sub-torus, g fixes only the zero element of $\mathbb{Z}^n$. As a Galois module, $T_{\mathrm{an}}(\overline{\kappa}) \simeq (\overline{\kappa}^\times)^n$ but it is twisted by an action of g.

Claim: The $\mathrm{Gal}(\overline{\kappa}/\kappa)$-invariants of $T_{\mathrm{an}}(\overline{\kappa})$ have no p-torsion. In other words, every element of $(\overline{\kappa}^\times)^n$ of order exactly p is not fixed by g.

Justification: Since $\mu_p \subseteq \kappa$, therefore

$$(\overline{\kappa}^\times)^n[p] \simeq (\mathbb{Z}/p)^n$$

where $(\overline{\kappa}^\times)^n[p]$ has the standard $\mathrm{Gal}(\overline{\kappa}/\kappa)$-action. But, we have noted above that in our case $(\overline{\kappa}^\times)^n[p]$ is further twisted by a character which does not fix any element except 0. Therefore, the twisted $(\overline{\kappa}^\times)^n$ has no p-torsion. In other words, $T_{\mathrm{an}}[p](\kappa) = 0$. This implies that $T_{\mathrm{an}}[p^\infty](\kappa) = 0$.

In the *purely unipotent case*, we have that $c_s(k) = c_a(k) = 0$. By [CX08, Main Theorem (c)] it is known that $U(k)[p^\infty] = 0$ for primes $p > 2d + 1$. An application of the topological Nakayama's Lemma implies that $U(k_{\mathrm{cyc}})[p^\infty] = 0$.

When $p \leqslant 2d+1$, it is still possible to show that $U(k_{\mathrm{cyc}})[p^\infty]$ is finite. To prove the claim, it is enough to show that when $c_s(k) = c_a(k) = 0$ and $n \geqslant 0$, the p^∞-torsion part of $\mathsf{A}(k_n)$ is independent of n. For each n, let κ_n denote the residue field of k_n. By Theorem 2.4, we know that there exists an exact sequence

$$0 \longrightarrow \widehat{\mathsf{A}}(\mathfrak{m}) \longrightarrow \mathsf{A}(k_n) \longrightarrow \mathscr{A}(\kappa_n),$$

such that $\widehat{\mathsf{A}}(\mathfrak{m})[p^\infty]$ is trivial. Further, a result of H. W. Lenstra and F. Oort [LJO85, Theorem 1.15] implies that if $c_s(k) = c_a(k) = 0$ then $\pi_0^{\text{ét}}(\mathscr{A}/\kappa_n)[p^\infty]$ has order bounded by 2^{2d}. In particular, this quantity is independent of n. Therefore, for our purposes it is now enough to show that $\mathscr{A}^0_{\kappa_n}$ has order prime to p, which is true in this setting (see for example, [Mil08, p. 141 proof of Theorem 3.5]). □

Our result in Proposition 3.2 (1) is consistent with [HM99, Proposition 5.1 (i)] for elliptic curves.

3.2. **Herbrand quotients over the residue field.** Throughout this section, we suppose that $G \simeq \mathbb{Z}/p\mathbb{Z}$. We make an observation that $\mathsf{A}[p^\infty](\kappa_{\mathrm{cyc}})$ is a direct sum of copies of $\mathbb{Q}_p/\mathbb{Z}_p$ with trivial G-action. In this section, we compute the Herbrand quotient $h_G\left(\mathsf{A}[p^\infty](\kappa_{\mathrm{cyc}})\right)$.

Definition 3.3. Let M be a divisible $\mathbb{Z}_p[G]$-module of cofinite type. The *Herbrand quotient* is defined as follows

$$h_G(M) = \frac{\#\mathrm{H}^2(G, M)}{\#\mathrm{H}^1(G, M)}.$$

Lemma 3.4. With notation as above, $\operatorname{ord}_p\left(h_G(\mathbb{Q}_p/\mathbb{Z}_p)\right) = -1$.

Proof. First, we simplify the denominator. Since the G-action is trivial,

$$\mathrm{H}^1(G, \mathbb{Q}_p/\mathbb{Z}_p) = \operatorname{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}_p/\mathbb{Z}_p) = \tfrac{1}{p}\mathbb{Z}/\mathbb{Z}.$$

In particular,

$$\#\mathrm{H}^1(G, \mathbb{Q}_p/\mathbb{Z}_p) = p.$$

Next, we simplify the numerator. Since G is a cyclic group, we know that

$$\mathrm{H}^2(G, \mathbb{Q}_p/\mathbb{Z}_p) = \widehat{\mathrm{H}}^0(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}_p/\mathbb{Z}_p).$$

Here, we have used the notation $\widehat{\mathrm{H}}^0(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}_p/\mathbb{Z}_p)$ for the 0-th Tate cohomology group, also called the norm residue group (see [NSW13, p. 21]). Now, by definition of the 0-th Tate cohomology group

$$\widehat{\mathrm{H}}^0(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}_p/\mathbb{Z}_p) = (\mathbb{Q}_p/\mathbb{Z}_p)^{\mathbb{Z}/p\mathbb{Z}}/N_G(\mathbb{Q}_p/\mathbb{Z}_p) = (\mathbb{Q}_p/\mathbb{Z}_p)/p(\mathbb{Q}_p/\mathbb{Z}_p),$$

where N_G is the image of the norm map. Finally, we know that $\mathbb{Q}_p/\mathbb{Z}_p$ is a divisible group; hence, $\mathbb{Q}_p/\mathbb{Z}_p = p\mathbb{Q}_p/\mathbb{Z}_p$. Therefore,

$$\#\mathrm{H}^2(G, \mathbb{Q}_p/\mathbb{Z}_p) = 1.$$

The result is now immediate. $\square$

Recall that α, $c_s(k)$, and $c_a(k)$ are the dimensions of the unipotent, toric and abelian parts of the special fiber of the Néron model, and $c_g(k)$ is $\dim_{\mathbb{F}_p} B[p](\kappa)$ with B the good reduction part.

Theorem 3.5. With notation introduced above,

$$\operatorname{ord}_p\Big(h_G\left(\mathscr{A}[p^\infty](\kappa_{\mathrm{cyc}})\right)\Big) = -2c' - c_s(k).$$

Proof. As before, let B denote the good reduction part, $\mathbb{G}_m^{c_s(k)}$ be the totally split part, T_{an} be the anisotropic toric part, and U be the unipotent part. From Proposition 3.2, one has the p^∞-torsion of each component. By applying Lemma 3.4, we obtain their respective Herbrand quotients. More precisely:

- Since $B[p^\infty](\kappa_{\mathrm{cyc}}) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{2c'}$ where $0 \leqslant c' \leqslant c_g(k)$, we can conclude that the Herbrand quotient $h_G(B[p^\infty](\kappa_{\mathrm{cyc}})) = p^{-2c'}$.
- As $\mathbb{G}_m^{c_s(k)}[p^\infty](\kappa_{\mathrm{cyc}}) \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{c_s(k)}$, the Herbrand quotient $h_G(\mathbb{G}_m^{c_s(k)}[p^\infty](\kappa_{\mathrm{cyc}})) = p^{-c_s(k)}$.
- Since $H[p^\infty](\kappa_{\mathrm{cyc}})$ is finite, it follows that $h_G(H[p^\infty](\kappa_{\mathrm{cyc}})) = 1$.
- Since $T_{\mathrm{an}}[p^\infty](\kappa_{\mathrm{cyc}})$ is finite, it follows that $h_G(T_{\mathrm{an}}[p^\infty](\kappa_{\mathrm{cyc}})) = 1$.
- Since $U[p^\infty] = 0$, once again $h_G(U[p^\infty](\kappa_{\mathrm{cyc}})) = 1$.

The result is now immediate since the Herbrand quotient is additive in exact sequences. $\square$

3.3. **Herbrand quotients over $F_{\mathrm{cyc},v}$.** In this section, we go from the Herbrand quotient over the residue field κ_{cyc} to the Herbrand quotient over $F_{\mathrm{cyc},v}$. We begin with the following result.

Proposition 3.6. With notation as above,

$$h_G\Big(\mathsf{A}[p^\infty]\left(F_{\mathrm{cyc},v}\right)\Big) = h_G\Big(\mathscr{A}[p^\infty]\left(\kappa_{\mathrm{cyc}}\right)\Big).$$

Proof. Recall that $G = \mathbb{Z}/p\mathbb{Z}$ for our purpose. To prove the assertion it suffices to show that the specialization morphism

$$\mathrm{H}^i\left(G, \mathsf{A}[p^\infty](F_{\mathrm{cyc},v})\right) \longrightarrow \mathrm{H}^i\left(G, \mathscr{A}[p^\infty](\kappa_{\mathrm{cyc}})\right) \quad \text{for } i = 1, 2$$

is an isomorphism. For ease of notation, we write $M_{F,\mathrm{cyc}} = \mathsf{A}[p^\infty](F_{\mathrm{cyc},v})$ and $M_{\kappa,\mathrm{cyc}} = \mathscr{A}[p^\infty](\kappa_{\mathrm{cyc}})$. Further writing $M_{F,n} = \mathsf{A}[p^n](F_{\mathrm{cyc},v})$ and $M_{\kappa,n} = \mathscr{A}[p^n](\kappa_{\mathrm{cyc}})$ we note that

$$M_{F,\mathrm{cyc}} = \varinjlim_n M_{F,n} \text{ and } M_{\kappa,\mathrm{cyc}} = \varinjlim_n M_{\kappa,n}.$$

Since G is a finite group, we know that

$$\mathrm{H}^n(G, M_{F,\mathrm{cyc}}) = \varinjlim_n \mathrm{H}^n(G, M_{F,n}) \text{ and } \mathrm{H}^n(G, M_{\kappa,\mathrm{cyc}}) = \varinjlim_n \mathrm{H}^n(G, M_{\kappa,n})$$

for all $n \geqslant 1$. Thus to show $h_G(\mathsf{A}[p^\infty](F_{\mathrm{cyc},v})) = h_G(\mathscr{A}[p^\infty](\kappa_{\mathrm{cyc}}))$ it suffices to show that the specialization morphism $\mathsf{A}[p^n](F_{\mathrm{cyc},v}) \to \mathscr{A}[p^n](\kappa_{\mathrm{cyc}})$ is an isomorphism for all $n \geqslant 1$. By [Con, Lemma 1.3], the quasi-finite étale group scheme $E := \mathscr{A}[p^n]/\mathcal{O}_k$ splits into $E = E_f \coprod E_\eta$ with $E_\eta(\overline{\kappa}) = \varnothing$, and E_f is finite étale. In order to show the statement of the corollary, it suffices to show that $E(F_{\mathrm{cyc},v}) = E(\kappa_{\mathrm{cyc}})$. But note that [Con, Lemma 1.3] asserts that $E_f(\overline{\kappa}) = E_f(\overline{F}_v)^{I_v}$, so $E_f(\kappa_{\mathrm{cyc}}) = E_f(F_{\mathrm{cyc},v})$. Thus, it remains to show that $E_\eta(\kappa_{\mathrm{cyc}}) = E_\eta(F_{\mathrm{cyc},v})$. Since E_η has empty special fiber by definition, we have $E_\eta(\overline{\kappa}) = \varnothing$ and so $E_\eta(\kappa_{\mathrm{cyc}}) = \varnothing$. We are now required to show that $E_\eta(F_{\mathrm{cyc},v}) = \varnothing$.

We proceed by contradiction. Suppose $E_\eta(F_{\mathrm{cyc},v}) \neq \varnothing$, then there exists a section x such that $x \in E_\eta(F_{\mathrm{cyc},v})$. This $x \in E_\eta(F_{\mathrm{cyc},v}) \subset E(F_{\mathrm{cyc},v}) \subset \mathsf{A}(F_{\mathrm{cyc},v})$ gives rise to a map from $\mathrm{Spec}\,(F_{\mathrm{cyc},v}) \to E_\eta \subset \mathsf{A}$. By the Néron mapping property the section x extends uniquely to a map $\overline{x}$ from $\mathrm{Spec}\,(\mathcal{O}_{F_{\mathrm{cyc},v}}) \to E_\eta \subset \mathscr{A}$ and thus $E_\eta(\overline{\kappa}) \neq \varnothing$, which is a contradiction. So, we have that $E_\eta(F_{\mathrm{cyc},v}) = \varnothing$.

Now we have shown that $E(F_{\mathrm{cyc},v}) = E(\kappa_{\mathrm{cyc}})$, which means that $\mathsf{A}[p^n](F_{\mathrm{cyc},v}) = \mathscr{A}[p^n](\kappa_{\mathrm{cyc}})$ for all $n \geqslant 1$. The statement of the corollary follows. □

3.4. **Herbrand quotients over** $L_{\mathrm{cyc},w}$. The main result of this section is Theorem 3.7 which compares the Herbrand quotients of $\mathsf{A}[p^\infty](F_{\mathrm{cyc},v})$ and $\mathsf{A}[p^\infty](L_{\mathrm{cyc},w})$. Recall that we are only interested in understanding the Herbrand quotient of $\mathsf{A}[p^\infty](L_{\mathrm{cyc},w})$ when $w \mid v$ is (tamely) ramified in $L_{\mathrm{cyc}}/F_{\mathrm{cyc}}$. In other words, there is a unique prime $w \mid v$ throughout our discussion as L/F (and $L_{\mathrm{cyc}}/F_{\mathrm{cyc}}$) is a $\mathbb{Z}/p\mathbb{Z}$-extension. Furthermore, recall that L_w/F_v (resp. $L_{\mathrm{cyc},w}/F_{\mathrm{cyc},v}$) is non-trivial of degree p precisely when F_v (resp. $F_{\mathrm{cyc},v}$) contains μ_p (resp. μ_{p^∞}).

Next, we state and prove the main result of this section. We make the following hypothesis.

(Add) For an Abelian variety A/F of dimension d, any prime v above ℓ in F, and any finite Galois extension L_w/F_v, we have the following:

(1) the dimension of the non-toric reduction part (i.e. the dimension of T_{an} and U) over F_v remains the same over L_w, and
(2) for the good reduction part B of the Néron model of A over F_v, we have that $\dim_{\mathbb{F}_p} B[p](\kappa_v) = \dim_{\mathbb{F}_p} B[p](\kappa_w)$ where κ_v (resp. κ_w) is the residue field of F_v (resp. L_w).

Theorem 3.7. Keep the notation introduced above and assume that **(Add)** holds. Then,

$$\mathrm{ord}_p\left(h_G\left(\mathsf{A}[p^\infty]\left(F_{\mathrm{cyc},v}\right)\right)\right) = \mathrm{ord}_p\left(h_G\left(\mathsf{A}[p^\infty]\left(L_{\mathrm{cyc},w}\right)\right)\right)$$

Proof. Recall the setting: we have a degree-p Galois extension of number fields, L/F. Let v be a prime above ℓ ($\neq p$) in F and let π_{F_v} be the uniformizer of the local field F_v. Throughout this proof we focus on the case that $L_{\mathrm{cyc},w} \neq F_{\mathrm{cyc},v}$, else there is nothing to prove. Let $w \mid v$ be a prime in L. The extension L_w/F_v is tamely ramified of degree p. Since L/F is a Galois extension, so is L_w/F_v.

Since L_w/F_v is a finite Galois extension, we can apply **(Add)**. Over F_v, we adapt the following notation: we have that $\mathscr{A}_0^\circ$ decomposes into a good reduction part B of dimension $c_g(v)$, a split toric part $\mathbb{G}_m^{c_s(v)}$, and an anisotropic part T_{an} and U. Now looking over the residue field $\kappa_{\text{cyc},v}$ of $F_{\text{cyc},v}$, Theorem 3.5 implies that

$$\operatorname{ord}_p\left(h_G\left(\mathsf{A}[p^\infty](\kappa_{\text{cyc},v})\right)\right) = -2c_v - c_s(v)$$

where $c_v = \dim_{\mathbb{F}_p} B[p](\kappa_v)$ and κ_v is the residue field of F_v. Similarly, over L_w, we have that $\mathscr{A}_0^\circ$ decomposes into a good reduction part B' of dimension $c_g(w)$, a split toric part $\mathbb{G}_m^{c_s(w)}$, and an anisotropic part T'_{an} and U'.

(Add)-(1) tells us that $\dim T'_{\text{an}} + \dim U' = \dim T_{\text{an}} + \dim U$. We also know that $\mathscr{A}_0^\circ$ has the same dimension over L_w as over F_v. Thus

$$\dim \mathscr{A}_0^\circ = c_g(v) + c_s(v) + \dim T_{\text{an}} + \dim U = c_g(w) + c_s(w) + \dim T'_{\text{an}} + \dim U'.$$

Given that $\dim T'_{\text{an}} + \dim U' = \dim T_{\text{an}} + \dim U$, we can rewrite this as

$$c_g(v) + c_s(v) = c_g(w) + c_s(w).$$

Now the dimension of B must be the same over L_w and F_v (that is, $c_g(v) = c_g(w)$). From the equation above we conclude that $c_s(v) = c_s(w)$ as well.

Using Theorem 3.5 over the residue field of $L_{\text{cyc},w}$, denoted by $\kappa_{\text{cyc},w}$, we have that

$$\operatorname{ord}_p\left(h_G\left(\mathsf{A}[p^\infty](\kappa_{\text{cyc},w})\right)\right) = -2c_w - c_s(w)$$

where $c_w = \dim_{\mathbb{F}_p} B[p](\kappa_w)$ and κ_w is the residue field of L_w. Moreover, **(Add)**-(2) implies that $c_w = c_v$ for all $w \mid v$, and hence we have that

$$\operatorname{ord}_p\left(h_G\left(\mathsf{A}[p^\infty](\kappa_{\text{cyc},w})\right)\right) = \operatorname{ord}_p\left(h_G\left(\mathsf{A}[p^\infty](\kappa_{\text{cyc},v})\right)\right).$$

Now, by Proposition 3.6 applied to L_w, we have

$$h_G\left(\mathsf{A}[p^\infty]\left(L_{\text{cyc},w}\right)\right) = h_G\left(\mathscr{A}[p^\infty]\left(\kappa_{\text{cyc},w}\right)\right).$$

Thus, we have

$$h_G\left(\mathsf{A}[p^\infty]\left(L_{\text{cyc},w}\right)\right) = h_G\left(\mathsf{A}[p^\infty]\left(F_{\text{cyc},v}\right)\right).$$

□

This completes the proof of Theorem 3.1.

3.5. **Special case: modular abelian varieties.** When $\mathsf{A}/\mathbb{Q}$ is a d-dimensional *modular* abelian variety (i.e., of GL_2-type), the result is more precise. This is because for modular abelian varieties, it is not possible to have mixed reduction type. More precisely, if A is associated to a newform $f \in S_2(\Gamma_0(N))$ then the reduction type is

- purely good i.e., $p \nmid N$ *or*
- purely semistable i.e., $p \mid N$ and $\left|a_p(f)\right| = 1$ *or*
- purely unipotent i.e., $p \mid N$ and $a_p(f) = 0$.

Over any finite Galois extension $L/\mathbb{Q}$, for a modular Abelian variety A of dimension d, the dimension of the non-toric reduction part over $\mathbb{Q}$ remains the same over L.

Theorem 3.8 (algebraic Kida-type formula for modular abelian varieties)**.** Let L/F be a finite Galois extension of degree a power of p. Let A/F be a d-dimensional abelian variety which is the base extension of a modular abelian variety over $\mathbb{Q}$. Assume it has good ordinary reduction at primes above p and that **(Add)** holds.

Suppose that $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-cotorsion and that $\mu(\mathsf{A}/F_{\mathrm{cyc}}) = 0$. Then, $\mathrm{Sel}_{p^\infty}(\mathsf{A}/L_{\mathrm{cyc}})$ is a cotorsion $\Lambda(\Gamma)$-module with $\mu(\mathsf{A}/L_{\mathrm{cyc}}) = 0$. Furthermore, the respective λ-invariants, denoted by $\lambda(\mathsf{A}/F_{\mathrm{cyc}})$ and $\lambda(\mathsf{A}/L_{\mathrm{cyc}})$ satisfy the following formula:

$$\lambda(\mathsf{A}/L_{\mathrm{cyc}}) = [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + d \sum_{w\in P_1} \left(e(w) - 1\right) + \sum_{w\in P_2} 2c' \left(e(w) - 1\right).$$

Here, $e(w)$ is the ramification degree in $L_{\mathrm{cyc}}/F_{\mathrm{cyc}}$, and the sets P_1, P_2 are the sets of primes in L_{cyc} defined as

$$P_1 = \{w \nmid p : \ \mathsf{A} \text{ has split toric reduction at } w\},$$
$$P_2 = \{w \nmid p : \ \mathsf{A} \text{ has good reduction at } w \text{ and } \mathsf{A}(L_{\mathrm{cyc},w})[p] \neq 0\}.$$

Remark 3.9. In [Lim21], M. F. Lim proves a general result about Kida type formulas without assuming any condition on the μ-invariant; instead he needs to assume that the $\mathfrak{M}_H(G)$-property holds. The results in that paper are proven for very general Galois modules. On the other hand, the calculations in our paper are very explicit and precise. In particular, our main result concretely calculates the terms appearing in [Lim21, Theorem 4.3] in the case of abelian varieties.

4. Algebraic Kida-type Formula (Without Assuming $\mu = 0$)

In this section, we prove a mild generalization of Theorem 3.1. We continue to assume that $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-cotorsion (equivalently, the Galois cohomology group $H^2(G_S(F_{\mathrm{cyc}}), \mathsf{A}[p^\infty])$ is trivial and the defining exact sequence of the usual Selmer group is surjective) but we remove the additional hypothesis that $\mu(\mathsf{A}/F_{\mathrm{cyc}}) = 0$. Instead, we impose a condition on the m_i in (2.2). We adapt the strategy of the proof of [Lim22, Proposition 5.2].

4.1. Setup.

For this general setting, we need to work with modules which will be *quasi-projective* over $\mathbb{Z}_p[G]$ where G is a finite p-group.

Definition 4.1. Let G be a fixed finite p-group. A $\mathbb{Z}_p[G]$-module X is called *projective* if

- it is a free $\mathbb{Z}_p$-module of finite rank *and*
- if $H^i(H, \mathrm{Hom}(X, \mathbb{Q}_p/\mathbb{Z}_p)) = 0$ for $i = 1, 2$, for every subgroup H of G.

A finitely generated $\mathbb{Z}_p[G]$-module Y is called *strictly quasi-projective* if there exists a projective $\mathbb{Z}_p[G]$-module X and a $\mathbb{Z}_p[G]$-homomorphism $Y \to X$ with finite kernel and cokernel.

A finitely generated $\mathbb{Z}_p[G]$-module Z is called *quasi-projective* if there exists a short exact sequence

$$0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow Z \longrightarrow 0$$

of finitely generated $\mathbb{Z}_p[G]$-modules with Y_1, Y_2 being strictly quasi-projective.

We remind the readers (see [Lim22, Lemma 2.6]) that when M is a finitely generated torsion $\Lambda(\Gamma)$ with $\theta(M) \leqslant 1$, it follows from the structure theorem that $M/M[p]$ is finitely generated as a $\mathbb{Z}_p$-module. Moreover,

$$\lambda(M) = \mathrm{rank}_{\mathbb{Z}_p}(M/M[p]). \tag{4.1}$$

If we further know that G is a finite p-group and $M/M[p]$ is a quasi-projective $\mathbb{Z}_p[G]$-module, then

$$\lambda(M) = |G|\,\lambda(M_G).$$

Here, M_G denotes the largest quotient of M on which G acts trivially. For a proof, we refer the reader to [Lim22, Corollary 2.9].

Suppose that A is an abelian variety defined over the number field $\mathcal{F}$ and $F/\mathcal{F}$ is a finite extension. Suppose for the sake of simplicity that L/F is a Galois extension of degree a power of p which is disjoint from F_{cyc}/F. In particular, $\mathrm{Gal}(L_{\mathrm{cyc}}/F) \simeq \mathrm{Gal}(L_{\mathrm{cyc}}/L) \times \mathrm{Gal}(L/F)$. Henceforth, we set $G = \mathrm{Gal}(L/F)$[1]. Choose a set S of primes in F which contains precisely all the primes above p, the ramified primes of $F/\mathcal{F}$ and L/F, the primes of bad reduction of A, and the archimedean primes. We also set $S_1 = S \setminus S_p$, where S_p is the usual notation to mean the set of primes above p.

Lemma 4.2. The natural map

$$\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}}) \longrightarrow \mathrm{Sel}_{p^\infty}(\mathsf{A}/L_{\mathrm{cyc}})^G$$

has finite kernel and cokernel.

Proof. The proof in [HM99, Lemma 3.3] goes through even for abelian varieties. □

Remark 4.3.

(i) In view of the above lemma, it is straightforward to note that $\mathrm{Sel}_{p^\infty}(\mathsf{A}/L_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-cotorsion whenever $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-cotorsion.

(ii) The calculations in [HM99, Section 4, pp. 8–9] go through when we replace the elliptic curve E in *op. cit.* by an abelian variety A. In particular, under the assumption that $\mathrm{Sel}_{p^\infty}(A/L_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-cotorsion we can conclude that $H^i(G, \mathrm{Sel}_{p^\infty}(A/L_{\mathrm{cyc}}))$ is finite for all $i \geqslant 1$.

4.2. **Non-Primitive Selmer Groups.** For proving the main result in this section, we follow the approach initiated by R. Greenberg in [Gre11] and work with non-primitive Selmer groups. These modified Selmer groups were introduced in [GV00] and have played a crucial role in the advancement of Iwasawa theory in the last two decades.

Definition 4.4. For a subset S_0 of S_1, define the *non-primitive* Selmer group with respect to S_0 as

$$\mathrm{Sel}^{S_0}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}}) := \ker\left\{\mathrm{H}^1\left(\mathrm{Gal}\left(F_S/F_{\mathrm{cyc}}\right), \mathsf{A}[p^\infty]\right) \longrightarrow \bigoplus_{v\in S(F_{\mathrm{cyc}})\setminus S_0(F_{\mathrm{cyc}})} J_v(\mathsf{A}/F_{\mathrm{cyc}})\right\}.$$

We now record facts relating the usual Selmer group and the non-primitive Selmer group.

Lemma 4.5. For any number field F and a cotorsion $\Lambda(\Gamma)$-module $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}})$, we have

(1) $0 \to \mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}}) \to \mathrm{Sel}^{S_0}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}}) \to \bigoplus_{v\in S_0(F_{\mathrm{cyc}})} H^1(F_{\mathrm{cyc},v}, \mathsf{A}[p^\infty]) \to 0$.

(2) $0 \to \mathrm{Sel}^{S_0}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}}) \to \bigoplus_{v\in S_0(F_{\mathrm{cyc}})} H^1(F_{\mathrm{cyc},v}, \mathsf{A}[p^\infty]) \to \bigoplus_{v\in S(F_{\mathrm{cyc}})\setminus S_0(F_{\mathrm{cyc}})} J_v(\mathsf{A}/F_{\mathrm{cyc}}) \to 0$.

Proof. These results are well-known over $\mathbb{Q}$ and can be extended to the number field setting under the cotorsion hypothesis.

(1) See [GV00, Corollary 2.3].

(2) See [GV00, Proof of Proposition 2.5, p. 39].

□

[1]As explained in the strategy of proof in Section 3, we may assume without loss of generality that $G \simeq \mathbb{Z}/p\mathbb{Z}$.

We now prove a quasi-projectivity result for the non-primitive Selmer group. This is an auxiliary result which will allow us to prove our main theorem in the next section.

Theorem 4.6. Choose S_0 to contain the set of primes $v \in S_1$ such that the inertia degree of v in L/F is divisible by p. Suppose that $\mathrm{Sel}_{p^\infty}(\mathsf{A}/L_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-cotorsion and that $\theta(\mathrm{Sel}^\vee_{p^\infty}(\mathsf{A}/L_{\mathrm{cyc}})) \leqslant 1$. Then $\mathrm{Sel}^{S_0,\vee}_{p^\infty}(\mathsf{A}/L_{\mathrm{cyc}})/\mathrm{Sel}^{S_0,\vee}_{p^\infty}(\mathsf{A}/L_{\mathrm{cyc}})[p]$ is a quasi-projective $\mathbb{Z}_p[G]$-module where $G = \mathrm{Gal}(L/F)$.

Proof. We can imitate the proof of [Lim22, Theorem 4.7]. The same strategy of proof goes through for abelian varieties without any change. We note that at every prime above p, the abelian variety A has good *ordinary* reduction (recall that we are assuming our abelian varieties A have good ordinary reduction). The detailed calculations for the P-Herbrand quotients (as defined in *op. cit.*) for the ordinary case can be found in [Gre11, Section 3.2] which can once again be adapted without any change for abelian varieties. □

4.3. **Towards a Generalized Kida-type Formula.** Keeping the same notation as introduced before, if the μ-invariant of $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}})$ is not equal to zero, then the relation between the Selmer groups upon base-change is well known; see for example, [CS05, Equation 4] or [Lim21, Theorem 4.1]. In particular,

$$\mu(\mathsf{A}/L_{\mathrm{cyc}}) = [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\mu(\mathsf{A}/F_{\mathrm{cyc}}).$$

In what follows we will derive a formula for the λ-invariant of the Selmer group $\mathrm{Sel}_{p^\infty}(\mathsf{A}/L_{\mathrm{cyc}})$ which we denote by $\lambda(\mathsf{A}/L_{\mathrm{cyc}})$. Our proof follows along the lines of [Lim22, Theorem 5.2]. First, in view of (4.1), we obtain a straightforward relation between the λ-invariants of the imprimitive Selmer groups over L_{cyc} and F_{cyc}, i.e.,

$$\lambda(\mathrm{Sel}^{S_0}_{p^\infty}(\mathsf{A}/L_{\mathrm{cyc}})) = [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\lambda\left(\mathrm{Sel}^{S_0}_{p^\infty}\left(\mathsf{A}/L_{\mathrm{cyc}}\right)^G\right) = [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\lambda(\mathrm{Sel}^{S_0}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}})).$$

Using both short exact sequences in Lemma 4.5 we can conclude that

$$\begin{aligned}\lambda(\mathsf{A}/L_{\mathrm{cyc}}) = [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}]\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + [L_{\mathrm{cyc}} : F_{\mathrm{cyc}}] &\sum_{\substack{v\nmid p\\ v\in F_{\mathrm{cyc}}}} \mathrm{corank}_{\mathbb{Z}_p}(H^1(F_{\mathrm{cyc}}, \mathsf{A}[p^\infty])) \\ &- \sum_{\substack{w\nmid p\\ w\in L_{\mathrm{cyc}}}} \mathrm{corank}_{\mathbb{Z}_p}(H^1(L_{\mathrm{cyc}}, \mathsf{A}[p^\infty])).\end{aligned}$$

Before proceeding any further, we introduce a slightly modified notation to denote sets of primes we have discussed previously. The modified notation is self-explanatory and is useful for the following discussion as we work with sets of primes in F_{cyc} and L_{cyc}. We write $P_i(L)$ in place of P_i introduced in Theorem 3.1 and write $P_i(F)$ to denote the primes below those in $P_i(L)$ which lie in F_{cyc}.

By [Gre89, Proposition 2] and our discussion in Section 3 we obtain the following two equalities

$$\mathrm{corank}_{\mathbb{Z}_p}(H^1(F_{\mathrm{cyc},v}, \mathsf{A}[p^\infty])) = \begin{cases} c_s(v) & \text{if } v \in P_1(F) \\ 2c'_v & \text{if } v \in P_2(F) \\ 0 & \text{otherwise.}\end{cases}$$

$$\mathrm{corank}_{\mathbb{Z}_p}(H^1(L_{\mathrm{cyc},w}, \mathsf{A}[p^\infty])) = \begin{cases} c_s(w) & \text{if } w \in P_1(L) \\ 2c'_w & \text{if } w \in P_2(L) \\ 0 & \text{otherwise.}\end{cases}$$

Therefore,

$$\lambda(\mathsf{A}/L_{\text{cyc}}) = [L_{\text{cyc}} : F_{\text{cyc}}]\lambda(\mathsf{A}/F_{\text{cyc}}) + [L_{\text{cyc}} : F_{\text{cyc}}] \sum_{v \in P_1(F)} c_s(v) + \\ [L_{\text{cyc}} : F_{\text{cyc}}] \sum_{v \in P_2(F)} 2c'_v - \sum_{w \in P_1(L)} c_s(w) - \sum_{w \in P_2(L)} 2c'_w$$

where

$$[L_{\text{cyc}} : F_{\text{cyc}}] \sum_{v \in P_1(F)} c_s(v) = \sum_{v \in P_1(F)} \sum_{w|v} e(w) c_s(v) = \sum_{w \in P_1(L)} e(w) c_s(v).$$

By **(Add)**, we have that $c_s(v) = c_s(w)$ for all $w \mid v$. Similarly for c' ($c'_v = c'_w$ for $w \mid v$, so we drop the subscript) and $v \in P_2(F)$, we have

$$[L_{\text{cyc}} : F_{\text{cyc}}] \sum_{v \in P_2(F)} 2c' = \sum_{v \in P_2(F)} \sum_{w|v} e(w) 2c' = \sum_{w \in P_2(L)} e(w) 2c'.$$

$$\lambda(\mathsf{A}/L_{\text{cyc}}) = [L_{\text{cyc}} : F_{\text{cyc}}]\lambda(\mathsf{A}/F_{\text{cyc}}) + \sum_{w \in P_1(L)} c_s(w)(e(w) - 1) + \sum_{w \in P_2(L)} 2c'(e(w) - 1).$$

We have therefore proven the following theorem:

Theorem 4.7 (generalized algebraic Kida-type formula for abelian varieties)**.** Let L/F be a finite Galois extension of degree a power of p. Let A/F be a d-dimensional abelian variety with good ordinary reduction at p and such that **(Add)** is satisfied. Suppose that $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\text{cyc}})$ is $\Lambda(\Gamma)$-torsion and that $\theta(\mathsf{A}/L_{\text{cyc}}) \leqslant 1$. The respective λ-invariants, denoted by $\lambda(\mathsf{A}/F_{\text{cyc}})$ and $\lambda(\mathsf{A}/L_{\text{cyc}})$ satisfy the following formula:

$$\lambda(\mathsf{A}/L_{\text{cyc}}) = [L_{\text{cyc}} : F_{\text{cyc}}]\lambda(\mathsf{A}/F_{\text{cyc}}) + \sum_{w \in P_1} c_s(w) \left(e(w) - 1\right) + \sum_{w \in P_2} 2c' \left(e(w) - 1\right).$$

Here, $e(w)$ is the ramification degree in $L_{\text{cyc}}/F_{\text{cyc}}$, and the sets P_1, P_2 are the sets of primes in L_{cyc} defined as

$$P_1 = \{w \nmid p : \ \mathsf{A} \text{ has non-trivial split toric multiplicative reduction part at } w\},$$
$$P_2 = \{w \nmid p : \ \mathsf{A} \text{ has non-trivial good reduction part at } w \text{ and } \mathsf{A}(L_{\text{cyc},w})[p] \neq 0\}.$$

The constant $c_s(w)$ is the dimension of the split toric part of the special fiber $\mathscr{A}_0$ of the Néron model $\mathscr{A}$ of A at w and $0 \leqslant c' \leqslant c_g(w)$ where $c_g(w)$ is the dimension of the abelian part of the special fiber $\mathscr{A}_0$ of the Néron model $\mathscr{A}$ of A at w.

5. Applications of the algebraic Kida-type formula

We begin by recording the following standard lemma relating the λ-invariant of the Selmer group of an elliptic curve to its rank is well-known but we include it for the sake of completeness.

Lemma 5.1. Let A/F be an abelian variety with good ordinary reduction at all primes above p and assume that $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\text{cyc}})$ is Λ-cotorsion. Then, $\lambda(\mathsf{A}/F_{\text{cyc}}) \geqslant \mathrm{rank}_{\mathbb{Z}}(\mathsf{A}(F))$.

Proof. Let r_p denote the $\mathbb{Z}_p$-corank of $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\text{cyc}})^\Gamma$, where we recall that $\Gamma = \mathrm{Gal}(F_{\text{cyc}}/F)$. Since $\mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\text{cyc}})$ is cotorsion, r_p is finite. Consider the following short exact sequence

$$0 \longrightarrow \mathsf{A}(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathrm{Sel}_{p^\infty}(\mathsf{A}/F) \longrightarrow Ш(\mathsf{A}/F)[p^\infty] \longrightarrow 0.$$

We deduce that

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(\mathsf{A}/F) \geqslant \mathrm{rank}_{\mathbb{Z}}(\mathsf{A}(F)), \tag{5.1}$$

with equality precisely when $Ш(\mathsf{A}/F)[p^\infty]$ is finite. It follows from the structure theory of Λ-modules that $\lambda(\mathsf{A}/F_{\mathrm{cyc}}) \geqslant r_p$. Hence, it suffices to show that $r_p \geqslant \mathrm{rank}_{\mathbb{Z}}(\mathsf{A}(F))$. This is indeed the case, since Mazur's Control Theorem (see for example [Gre01, Theorem 4.1]) asserts that there is a natural map

$$\mathrm{Sel}_{p^\infty}(\mathsf{A}/F) \longrightarrow \mathrm{Sel}_{p^\infty}(\mathsf{A}/F_{\mathrm{cyc}})^\Gamma$$

with finite kernel and cokernel. From (5.1), we see that $r_p \geqslant \mathrm{rank}_{\mathbb{Z}}(\mathsf{A}(F))$ and the result follows. □

In [CH01, Proposition 6.9], J. Coates and S. Howson used the result of Hachimori–Matsuno to study the asymptotic growth of λ-invariants at each finite layer of a certain (non-commutative) p-adic Lie extension arising from geometry. Unfortunately, using the Kida-type formula provides no indication as to whether the growth in the size of the dual Selmer group in a p-adic Lie extension is due to a large Mordell–Weil group or a large Tate–Shafarevich group. On the other hand, using the theory of Verma modules, M. Harris proved a lower bound for the Mordell–Weil rank of an abelian variety (with good ordinary reduction at p in the) at each finite layer of the trivializing extension of A, see [Har79b]. Some other results where authors have studied the asymptotic growth of Mordell–Weil ranks in pro-p p-adic Lie extensions include [Har79a, HL20, LS20]; however, most of these results are not effective. In contrast, our result succeeds in proving an explicit upper bound for the case of modular abelian varieties.

Growth of Mordell–Weil ranks in pro-p p-adic Lie extensions. Using Theorem 3.1, we can extend the results of Coates–Howson to abelian varieties and all uniform[2] pro-p p-adic Lie extensions. In the case of elliptic curves, these results have been generalized to pro-p p-adic Lie extension in [Ray22]. Under the assumption of the $\mathfrak{M}_H(G)$-conjecture, there are results in this direction due to P. C. Hung–M. F. Lim, see [HL20].

Definition 5.2. A profinite group is said to be *uniform* if it is topologically finitely generated on r generators, and there exists a filtration of subgroups

$$G = G_0 \supset G_1 \supset \dots$$

such that each G_i is normal in G_{i+1}, and $G_i/G_{i+1} \simeq (\mathbb{Z}/p\mathbb{Z})^r$. In particular, a uniform p-adic analytic group is always pro-p.

A field extension F_∞ of F is called a *uniform pro-p p-adic Lie extension of dimension r* if $\mathrm{Gal}(F_\infty/F)$ is uniform, as defined above.

In this section, we consider the following setup. Let F be a given number field and F_∞ be any uniform pro-p p-adic Lie extension of dimension D, containing the cyclotomic $\mathbb{Z}_p$-extension of F. We further suppose that only finitely many primes ramify in F_∞/F[3]. Set $G = \mathrm{Gal}(F_\infty/F)$ and write $G_n = G^{p^n}$, i.e., the group generated by the p^n-th powers of elements in G. Further, we denote by $F_{(n)}$ the fixed field of G_n. This gives a filtration

$$F \subset F_{(1)} \subset \dots \subset F_{(n)} \subset \dots.$$

Observe that by construction, $\mathrm{Gal}(F_{(n)}/F_{(n-1)}) \simeq (\mathbb{Z}/p\mathbb{Z})^D$. We can then consider the cyclotomic $\mathbb{Z}_p$-extensions of each $F_{(n)}$ to get the filtration

$$F_{\mathrm{cyc}} \subset F_{(1),\mathrm{cyc}} \subset \dots \subset F_{(n),\mathrm{cyc}} \subset \dots.$$

Since $\mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$ has dimension $D-1$ and it follows that $\mathrm{Gal}(F_{(n),\mathrm{cyc}}/F_{\mathrm{cyc}}) \simeq (\mathbb{Z}/p\mathbb{Z})^{D-1}$. Consider an abelian variety A/F of dimension d and fix a prime p of good ordinary reduction of A. For

[2]For background details on uniform groups we refer the reader to [DdSMS99].

[3]Such p-adic Lie extensions are abundant and arise naturally in number theory, for example $\mathbb{Z}_p^r$-extensions, false Tate curve extensions etc.

each finite extension $F_{(n)}/F$, we define the sets $P_1(F_{(n)})$ and $P_2(F_{(n)})$ as in Theorem 3.1. Note that the primes in $P_i(F_{(n)})$ lie above the primes in $P_i(F)$ and each of these sets is a collection of finitely many primes. Finally, to simplify notation, for both $i = 1, 2$ set

$$\mathfrak{p}_i(F) = \#P_i(F).$$

Proposition 5.3. With notation as above,

$$\lambda(\mathsf{A}/F_{(n),\mathrm{cyc}}) \leqslant p^{n(D-1)}\left(\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + d\mathfrak{p}_1(F)\right) - p^{n(D-2)}\left(d\mathfrak{p}_1(F) + 2d\mathfrak{p}_2(F)\right).$$

Proof. In particular, we get

$$\sum_{w\in P_1(F_{(n)})} c_s(w)\left(e(w)-1\right) = \sum_{v\in P_1(F)}\left(\sum_{w|v} c_s(w)\left(e(w)-1\right)\right) \leqslant d\sum_{v\in P_1(F)}\left(\sum_{w|v}\left(e(w)-1\right)\right),$$
$$\sum_{w\in P_2(F_{(n)})} c_g(w)\left(e(w)-1\right) = \sum_{v\in P_2(F)}\left(\sum_{w|v} c_g(w)\left(e(w)-1\right)\right) \leqslant 2d\sum_{v\in P_2(F)}\left(\sum_{w|v}\left(e(w)-1\right)\right).$$

Here we have used the fact that both $c_s(w)$ and $\frac{1}{2}c_g(w)$ are always less than or equal to the dimension d of the abelian variety. However, since $F_{(n)}/F$ is a Galois extension, the ramification index $e(w)$ must be the same for all $w \mid v$. Therefore,

$$\begin{aligned}\sum_{w|v}\left(e(w)-1\right) &= \sum_{w|v} e(w) - \sum_{w|v} 1\\ &= [F_{(n),\mathrm{cyc}} : F_{\mathrm{cyc}}] - \sum_{w|v} 1\\ &= p^{n(D-1)} - \sum_{w|v} 1.\end{aligned}$$

Now, observe that for each $v \in P_i(F)$, we know that

$$\sum_{w|v} 1 = \#\{w \mid v\} = \frac{[F_{(n),\mathrm{cyc}} : F_{\mathrm{cyc}}]}{e(w)}.$$

Since $w \nmid p$, and our extension is pro-p, it follows that the prime w must be tamely ramified in the extension. But, recall that tame inertia is cyclic; hence $e(w) \leqslant p^n$. Thus, we conclude that

$$\sum_{w|v}\left(e(w)-1\right) \leqslant p^{n(D-1)} - p^{n(D-2)}.$$

It now follows immediately from Theorem 3.1 that

$$\lambda(\mathsf{A}/F_{(n),cyc}) \leqslant p^{n(D-1)}\left(\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + d\mathfrak{p}_1(F)\right) - p^{n(D-2)}\left(d\mathfrak{p}_1(F) + 2d\mathfrak{p}_2(F)\right).$$

The proof of the proposition is now complete. ∎

An application of Lemma 5.1 yields an upper bound on the Mordell–Weil rank of the abelian variety at the n-th layer of a uniform pro-p p-adic Lie extension. More precisely, we have the following result.

Corollary 5.4. With notation introduced above,

$$\mathrm{rank}_{\mathbb{Z}}(\mathsf{A}/F_{(n)}) \leqslant p^{n(D-1)}\left(\lambda(\mathsf{A}/F_{\mathrm{cyc}}) + d\mathfrak{p}_1(F)\right) - p^{n(D-2)}\left(d\mathfrak{p}_1(F) + 2d\mathfrak{p}_2(F)\right).$$

References

[Art86] M. Artin. Néron models. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 213–230. Springer, New York, 1986.

[AV75] Yvette Amice and Jacques Vélu. Distributions p-adiques associées aux séries de Hecke. *Astérisque*, 24(25):119–131, 1975.

[BKR21] Lea Beneish, Debanjana Kundu, and Anwesh Ray. Rank jumps and growth of Shafarevich–Tate groups for elliptic curves in-extensions. *J. Aust. Math. Soc.*, pages 1–38, 2021.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1990.

[CH01] John Coates and Susan Howson. Euler characteristics and elliptic curves II. *J. Math. Soc. Japan*, 53(1):175–235, 2001.

[Che60] Claude Chevalley. Une démonstration d'un théoreme sur les groupes algébriques. *J. Math. Pures Appl.(9)*, 39:307–317, 1960.

[Con] Brian Conrad. Classification of quasi-finite etale separated schemes. http://math.stanford.edu/~conrad/vigregroup/vigre03/zmt.pdf.

[CS00] John Coates and Ramdorai Sujatha. *Galois cohomology of elliptic curves*. Narosa, 2000.

[CS05] John Coates and Ramdorai Sujatha. Fine Selmer groups of elliptic curves over p-adic Lie extensions. *Math. Ann.*, 331(4):809–839, 2005.

[CX08] Pete L Clark and Xavier Xarles. Local bounds for torsion points on abelian varieties. *Canadian J. Math.*, 60(3):532–555, 2008.

[DdSMS99] John Dixon, Marcus du Sautoy, Avinoam Mann, and Dan Segal. *Analytic pro-p groups*, volume 61 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1999.

[Gre89] Ralph Greenberg. Iwasawa theory for p-adic representations. In *Algebraic Number Theory—in Honor of K. Iwasawa*, pages 97–137. Mathematical Society of Japan, 1989.

[Gre01] Ralph Greenberg. Introduction to Iwasawa theory for elliptic curves. In *IAS/Park City Mathematics Series*, volume 9, pages 407–464, 2001.

[Gre11] Ralph Greenberg. Iwasawa theory, projective modules, and modular representations, 2011.

[GV00] Ralph Greenberg and Vinayak Vatsal. On the Iwasawa invariants of elliptic curves. *Invent. math.*, 142(1):17–63, 2000.

[Har79a] Michael Harris. p-adic representations arising from descent on abelian varieties. *Compositio Math*, 39(2):177–245, 1979.

[Har79b] Michael Harris. Systematic growth of Mordell–Weil groups of abelian varieties in towers of number fields. *Invent. math.*, 51(2):123–141, 1979.

[Har83] Robin Hartshorne. *Algebraic geometry. Corr. 3rd printing.* Graduate Texts in Mathematics, **52**. New York-Heidelberg-Berlin: Springer-Verlag. xvi + 496 p., 1983.

[HL20] Pin-Chi Hung and Meng Fai Lim. On the growth of Mordell–Weil ranks in p-adic Lie extensions. *Asian J. Math*, 24(4):549–570, 2020.

[HM99] Yoshitaka Hachimori and Kazuo Matsuno. An analogue of Kida's formula for the Selmer groups of elliptic curves. *J. Alg. Geom.*, 8(3):581–601, 1999.

[HS00] Marc Hindry and Joseph H Silverman. *Diophantine geometry: an introduction*, volume 201. Springer-Verlag New York, Inc., 2000.

[Iwa81] Kenkichi Iwasawa. Riemann-Hurwitz formula and p-adic Galois representations for number fields. *Tohoku Mathematical Journal, Second Series*, 33(2):263–288, 1981.

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, 295:117–290, 2004.

[Kid80] Yûji Kida. ℓ-extensions of CM-fields and cyclotomic invariants. *J. Number Theory*, 12:519–528, 1980.

[Lim21] Meng Fai Lim. Some remarks on Kida's formula when $\mu \neq 0$. *The Ramanujan J.*, pages 1–18, 2021.

[Lim22] Meng Fai Lim. On the cohomology of Kobayashi's plus/minus norm groups and applications. *Math. Proc. Camb.*, 173(1):1–24, 2022.

[LJO85] HW Lenstra Jr and F Oort. Abelian varieties having purely additive reduction. *Journal of Pure and Applied Algebra*, 36:281–298, 1985.

[LS20] Antonio Lei and Florian Sprung. Ranks of elliptic curves over $\mathbb{Z}_p^2$-extensions. *Israel J. Math*, 236(1):183–206, 2020.

[Mat00] Kazuo Matsuno. An analogue of Kida's formula for the p-adic L-functions of modular elliptic curves. *J. Number Theory*, 84(1):80–92, 2000.

[Maz72] Barry Mazur. Rational points of abelian varieties with values in towers of number fields. *Invent. Math.*, 18(3-4):183–266, 1972.

[Mil08] James S. Milne. Abelian varieties (v2.00), 2008. Available at www.jmilne.org/math/.

[MTT86] Barry Mazur, John Tate, and Jeremy Teitelbaum. On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. *Invent. math*, 84(1):1–48, 1986.

[Nér64] André Néron. Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. *Inst. Hautes Études Sci. Publ. Math.*, (21):128, 1964.

[NSW13] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323 of *Graduate Texts in Mathematics*. Springer, 2013.

[PW06] Robert Pollack and Tom Weston. Kida's formula and congruences. *Doc. Math., Special*, 2006:615–630, 2006.

[Ray22] Anwesh Ray. Asymptotic growth of Mordell–Weil ranks of elliptic curves in noncommutative towers. *Canadian Math. Bull.*, 2022.

[Ser06] Jean-Pierre Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition.

[Was97] Lawrence C Washington. *Introduction to cyclotomic fields*, volume 83 of *Graduate Texts in Mathematics*. Springer, 1997.

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