Anticyclotomic μ-invariants of residually reducible Galois Representations

Debanjana Kundu and Anwesh Ray

Department of Mathematics
University of British Columbia
Vancouver BC, V6T 1Z2, Canada.

Abstract

Let E be an elliptic curve over an imaginary quadratic field K, and p be an odd prime such that the residual representation $E[p]$ is reducible. In this article, we study the μ-invariant of the fine Selmer group of E over the anticyclotomic $\mathbb{Z}_p$-extension of K. We do not impose the Heegner hypothesis on E, thus allowing certain primes of bad reduction to decompose infinitely in the anticyclotomic $\mathbb{Z}_p$-extension. It is shown that the fine μ-invariant vanishes if certain explicit conditions are satisfied. Further, a partial converse is proven.

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1. Introduction

Iwasawa theory is the study of objects of arithmetic interest over infinite towers of number fields. K. Iwasawa conjectured that for the *cyclotomic* $\mathbb{Z}_p$-extension of a number field F, the Galois group of its maximal abelian unramified pro-p extension is a finitely generated $\mathbb{Z}_p$-module (see [21]). This is known as Iwasawa's $\mu = 0$ conjecture for the number field F. In [13], B. Ferrero and L. Washington proved this conjecture for all finite abelian extensions of $\mathbb{Q}$. The Iwasawa theory of abelian varieties (in particular, elliptic curves) was initiated by B. Mazur in [30]. The main object of study is the p-primary Selmer group of an elliptic curve E defined over a number field F, with good *ordinary* reduction at all primes of F above p. The Selmer group over a $\mathbb{Z}_p$-extension of F is a cofinitely generated module over the Iwasawa algebra, Λ. In [15], R. Greenberg analyzed the algebraic structure of these Selmer groups. For elliptic curves E over $\mathbb{Q}$, it is conjectured that if the residual representation on $E[p]$ is irreducible, then μ-invariant of the p-primary Selmer group over the cyclotomic $\mathbb{Z}_p$-extension, denoted $\mu(E/\mathbb{Q}^{\text{cyc}})$, vanishes (see [16, Conjecture 1.11]).

The *fine* Selmer group is a subgroup of the classical Selmer group obtained by imposing trivial conditions at primes above p. It had first been studied by K. Rubin [37] and B. Perrin-Riou [33, 34], under various guises. The analysis of the fine Selmer groups is an essential part of K. Kato's seminal work on the Iwasawa Main Conjecture for elliptic curves and modular forms (see [23]). In recent years, the study of fine Selmer groups has gained considerable momentum (see for instance [7, 41, 40, 22, 1]). J. Coates and R. Sujatha conjectured that the fine Selmer group of E over F^{cyc} is Λ-cotorsion with associated μ-invariant, $\mu^{\text{fine}}(E/F^{\text{cyc}})$, equal to zero [7, Conjecture A]. The formulation of this conjecture makes no hypothesis on the reduction type at primes above p or the residual representation on $E[p]$. However, such a statement need not be true for the (classical) Selmer group even for elliptic curves over $\mathbb{Q}$ with good ordinary reduction at p. In fact, Mazur provided examples of elliptic curves $E_{/\mathbb{Q}}$ for which the residual representation $E[p]$ is reducible and $\mu(E/\mathbb{Q}^{\text{cyc}})$ is non-zero. There is a systematic approach towards finding examples of elliptic curves with positive $\mu(E/F^{\text{cyc}})$ (see [8, 9]).

On the other hand, the conjecture by Coates and Sujatha predicts a close relationship in the growth of ideal class groups and fine Selmer groups in cyclotomic $\mathbb{Z}_p$-extensions. Some evidence towards this is provided in [7, Theorem 3.4]. In particular, the result of Ferrero and Washington implies that for an elliptic curve over an abelian number field, $\mu^{\text{fine}}(E/F^{\text{cyc}})$ is zero when $E[p]$ is reducible (see [7, Corollary 3.6]). Subsequently, the relation in the growth of ideal class groups and fine Selmer groups has been studied in more general settings (see [27, 24, 25, 26]). In this paper, we study the relationship in the growth of fine Selmer groups and class groups in *anticyclotomic* $\mathbb{Z}_p$-extensions of imaginary quadratic fields.

Several authors have studied classical Selmer groups of elliptic curves (more generally, abelian varieties or modular forms) in anticyclotomic $\mathbb{Z}_p$-extensions of imaginary quadratic fields (see for example [39, 3, 35]). Over the anticyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field, it is known that the classical Selmer group need not be Λ-cotorsion (see [2]). However, in line with the conjecture of Coates and Sujatha, it was conjectured by A. Matar that for an elliptic curve over the imaginary quadratic field K, its fine Selmer group over the anticyclotomic $\mathbb{Z}_p$-extension is Λ-cotorsion with $\mu^{\text{fine}}(E/K^{\text{ac}}) = 0$ (see [28, Conjecture B]). We remark that for an odd prime p, the cotorsion property of the fine Selmer group is equivalent to the elliptic curve analogue of the weak Leopoldt conjecture [34, Theorem 1.3.2] (see also [29, Theorem 2.2]) and is always conjectured to be true (see [34, §1.4 (Leop(V))]). Even though there is limited (computational) evidence towards the conjecture of Matar, previously known results in the class group setting provide reasons to expect such a conjecture to be true. In [38], J. Sands proved that if p is an odd prime that does not divide the class number of the imaginary quadratic field K, and the classical (cyclotomic) λ-invariant of K is at most 2, then for *every* $\mathbb{Z}_p$-extension of K the classical μ-invariant is 0. Therefore, the close relationship between the class group and the fine Selmer group provides a strong moral reason to expect that $\mu^{\text{fine}}(E/K^{\text{ac}}) = 0$ for elliptic curves over K.

Most results in literature focus on the case when the residual representation is irreducible. In contrast, this paper primarily studies the case when the residual representation is *reducible*. The first main result (see Theorem 3.2) shows that if certain conditions are satisfied then the vanishing of the μ-invariant of the p-primary fine Selmer group may be detected by an analogous fine Selmer group associated to the residual representation, $E[p]$. Further, if $E[p]$ is reducible then there are characters

$$\varphi_1, \varphi_2 : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_1(\mathbb{F}_p)$$

which fit into a short exact sequence

$$0 \to \mathbb{F}_p(\varphi_1) \to E[p] \to \mathbb{F}_p(\varphi_2) \to 0.$$

Note that $\varphi_2 = \overline{\chi}\varphi_1^{-1}$, where $\overline{\chi}$ denotes the mod-p cyclotomic character. This short exact sequence makes it possible to analyze the structure of the fine Selmer group associated to $E[p]$. Next, we prove a partial converse to the above theorem (see Theorem 3.4). A key difference between the cyclotomic and anticyclotomic $\mathbb{Z}_p$-extensions is the following: in the cyclotomic extension, all primes are finitely decomposed, whereas in the anticyclotomic extension, there are infinitely many primes which split completely. For the aforementioned result of Coates and Sujatha in the cyclotomic extension case, this fact is crucially used (in the proofs of [7, Lemmas 3.2, 3.7, Theorem 3.4]). The elliptic curve E is said to satisfy the Heegner hypothesis if all primes $v \nmid p$ of K at which it has bad reduction are split completely in K. By Fact 2.3, such a prime number is finitely decomposed in the anticyclotomic $\mathbb{Z}_p$-extension, K^{ac}. Our results improve previously known results by a slight weakening of the usual Heegner hypothesis.

Two elliptic curves, E_1 and E_2, are said to be p-congruent if the p-torsion subgroups $E_1[p]$ and $E_2[p]$ are isomorphic as Galois modules. In [18], Greenberg and V. Vatsal studied the relation between cyclotomic invariants of elliptic curves over $\mathbb{Q}$ which are p-congruent. In [17, Proposition 4.1.6], Greenberg reformulated the conjecture of Coates and Sujatha in terms of the vanishing of a second cohomology group with coefficients in $E[p]$. From this, it is immediate that if elliptic curves E_1, E_2 are p-congruent then $\mu^{\mathrm{fine}}(E_1/\mathbb{Q}^{\mathrm{cyc}}) = 0$ if and only if $\mu^{\mathrm{fine}}(E_2/\mathbb{Q}^{\mathrm{cyc}}) = 0$. In the same spirit, we prove a result for fine μ-invariants over K^{ac} (see Theorem 5.1), without imposing any hypothesis on the reducibility of the residual representations.

The authors expect that the methods in this paper should generalize to residually reducible Galois representations arising from abelian varieties. The results shall however be more technical and the arguments more cumbersome in the higher dimensional setting. The authors choose a less general framework in which the inherent simplicity of the underlying ideas come across easily.

The paper is organized into six sections. Preliminary notions are discussed in §2. In §3 and §4, we state and prove the main results: we establish a criterion for the vanishing of the μ-invariant of the fine Selmer group in the anticyclotomic $\mathbb{Z}_p$-extension. We also prove a partial converse to the above theorem. In §5, we compare the anticyclotomic μ-invariant for two elliptic curves which are p-congruent. Finally, in §6 we list examples illustrating the results in this article.

2. Preliminaries

Throughout, let p be an odd prime and K be an imaginary quadratic field. Let S_p be the set of primes of K above p. Fix an elliptic curve E over K. Set $\mathrm{G}_K := \mathrm{Gal}(\overline{K}/K)$ to be the absolute Galois group of K. Denote by $E[p^n]$ (resp. $E[p^\infty]$) the p^n-torsion (resp. p-primary torsion) subgroup of $E(\overline{K})$. The Tate module $\mathrm{T}_p(E)$ is the inverse limit with respect to multiplication by p maps,

$$\mathrm{T}_p(E) := \varprojlim_n E[p^n].$$

The $\mathbb{Z}_p$-module $\mathrm{T}_p(E)$ is free of rank 2 and the group of $\mathbb{Z}_p$-linear automorphisms of $\mathrm{T}_p(E)$ is identified with $\mathrm{GL}_2(\mathbb{Z}_p)$. The action of G_K on $\mathrm{T}_p(E)$ induces a continuous Galois representation, $\rho_E : \mathrm{G}_K \to \mathrm{GL}_2(\mathbb{Z}_p)$. Write $\bar{\rho}_E$ to denote the mod-p reduction of ρ_E, as depicted

$$\begin{tikzcd} & \mathrm{GL}_2(\mathbb{Z}_p) \arrow[d] \\ \mathrm{G}_K \arrow[ur, "\rho_E"] \arrow[r, "\bar{\rho}_E"'] & \mathrm{GL}_2(\mathbb{F}_p). \end{tikzcd}$$

The residual representation $\bar{\rho}_E$ is induced by the action of G_K on $E[p]$. Let $\mathcal{N}$ be the conductor of E and S be the set of primes that divide $\mathcal{N}p$. It is known that ρ_E is unramified at all primes $v \notin S$. Let K_S denote the maximal algebraic extension of K in which all primes $v \notin S$ are unramified and set $\mathrm{G}_{K,S} := \mathrm{Gal}(K_S/K)$. In this paper, we primarily focus on the case when the Galois representation ρ_E is residually reducible. However, some of the results will apply to the residually irreducible case as well. We shall be careful to make precise when the following hypothesis is required.

Hypothesis 2.1. Assume that the residual representation $\bar{\rho}_E$ is reducible, i.e. $E[p]$ contains a proper non-zero G_K-stable submodule.

This is indeed the case when $E(K)[p] \neq 0$. Though the condition is in fact far more general. In this setting, there are characters

$$\varphi_1,\ \varphi_2 : \mathrm{G}_{K,S} \to \mathrm{GL}_1(\mathbb{F}_p)$$

and a 1-cocycle

$$\beta : \mathrm{G}_{K,S} \to \mathbb{F}_p(\varphi_1\varphi_2^{-1})$$

such that

$$\bar{\rho}_E \simeq \begin{pmatrix} \varphi_1 & \varphi_2\beta \\ & \varphi_2 \end{pmatrix}.$$

The residual representation $\bar{\rho}_E$ is said to be *indecomposable* if the cohomology class $[\beta] \in H^1(\mathrm{G}_{K,S}, \mathbb{F}_p(\varphi_1\varphi_2^{-1}))$ is non-zero, and *split* otherwise. If $\bar{\rho}_E$ is split, then $\bar{\rho}_E \simeq \begin{pmatrix} \varphi_1 & \\ & \varphi_2 \end{pmatrix}$, and the characters φ_1 and φ_2 may be interchanged. If $\bar{\rho}_E$ is indecomposable, then φ_1 is the unique character such that $E[p]$ contains a Galois submodule isomorphic to $\mathbb{F}_p(\varphi_1)$.

Remark 2.2. By a result of Fontaine (see [11, Theorem 2.6]), if $E_{/\mathbb{Q}}$ is an elliptic curve with good supersingular reduction at p, then $\bar{\rho}_{E\restriction \mathrm{G}_{\mathbb{Q}_p}}$ is irreducible. Therefore, if $E_{/\mathbb{Q}}$ is an elliptic curve such that $E[p]$ is reducible as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module, then E has either good ordinary reduction or bad reduction at p.

Suppose $\mathcal{K}$ is a $\mathbb{Z}_p$-extension of K which is Galois over $\mathbb{Q}$. Then, there are exactly two cases to consider: either $\mathcal{K}$ is the cyclotomic $\mathbb{Z}_p$-extension or the anticyclotomic $\mathbb{Z}_p$-extension. Note that $\Gamma_{\mathcal{K}} := \mathrm{Gal}(\mathcal{K}/K)$ is an index two normal subgroup of $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$. The group $\mathrm{Gal}(K/\mathbb{Q})$ acts on $\Gamma_{\mathcal{K}}$, as is explained. Let $\tau \in \mathrm{Gal}(K/\mathbb{Q})$

and $x \in \Gamma_{\mathcal{K}}$; choose a lift $\widetilde{\tau} \in \mathrm{Gal}(\mathcal{K}/\mathbb{Q})$ of τ and set $\tau \cdot x := \widetilde{\tau} x \widetilde{\tau}^{-1}$. Since $\Gamma_{\mathcal{K}}$ is abelian, $\tau \cdot x$ does not depend on the choice of the lift $\widetilde{\tau}$. If the action of $\mathrm{Gal}(K/\mathbb{Q})$ on $\Gamma_{\mathcal{K}}$ is via the trivial (resp. non-trivial) character, then $\mathcal{K}$ is the cyclotomic (resp. anticyclotomic) extension of K. Thus, the cyclotomic extension is pro-cyclic and the anticyclotomic extension is pro-dihedral. We only consider the anticyclotomic extension in this article, and denote it by K^{ac}. Set Γ to denote $\mathrm{Gal}(K^{\mathrm{ac}}/K)$. For $n \geq 0$, the *n-th layer* is the unique number field K_n such that $K \subseteq K_n \subset K^{\mathrm{ac}}$ and $[K_n : K] = p^n$. Note that K_n is Galois over $\mathbb{Q}$ and its Galois group $\mathrm{Gal}(K_n/\mathbb{Q})$ is (isomorphic to) the dihedral group of order $2p^n$.

For any set of primes S', write $S'(K^{\mathrm{ac}})$ (resp. $S'(K_n)$) to denote the primes of K^{ac} (resp. K_n) which lie above some prime $v \in S'$. For instance, $v(K^{\mathrm{ac}})$ (resp. $v(K_n)$) will denote the primes $\eta | v$ of K^{ac} (resp. of K_n). The set of primes above a given prime of K in the cyclotomic $\mathbb{Z}_p$-extension K^{cyc} is finite. This is not the case for K^{ac}; the following characterization is well known.

Fact 2.3. Let v be a prime of K and l the prime number such that $v | l$. The set of primes $v(K^{\mathrm{ac}})$ is finite if and only if $l = p$ or l splits in K.

Indeed, there is a large enough value of n such that all primes of K_n above p are totally ramified in K^{ac}, see the last three lines of [4, p. 2131]. A prime $l \neq p$ which does not split in K must split completely in K^{ac}, see the first paragraph of p. 2132 of *loc. cit.* Prime numbers l which split in K must be finitely decomposed in K^{ac}, see for example, Corollary 1 of *loc. cit.*

Recall that $\mathcal{N}$ is the conductor of E. Denote by $\overline{\mathcal{N}}$ the Artin conductor of $\bar{\rho}_E$. Note that $\overline{\mathcal{N}}$ divides $\mathcal{N}$, denote by $(\mathcal{N}/\overline{\mathcal{N}})$ the quotient. We make an assumption on a certain subset of primes in S which is described below.

Definition 2.4. Denote by $\Sigma \subseteq S$ the set of primes $v \nmid p$ at which *all* of the following conditions are satisfied.

1. $v | (\mathcal{N}/\overline{\mathcal{N}})$.
2. If $p \geq 5$ and $\mu_p \subset K_v$, then E has split multiplicative reduction at v.

3. If $p = 3$ and $\mu_3 \subset K_v$, then E has split multiplicative reduction or additive reduction at v.

Hypothesis 2.5. Let E be an elliptic curve over an imaginary quadratic field K. Let $v \in \Sigma$ and l be the prime number such that $v|l$. Then l is split in K.

Definition 2.6. Suppose that $\bar{\rho}_E$ is reducible and indecomposable. Let $\Sigma(\varphi_2)$ be the set of primes $v \in S \setminus S_p$ such that $\varphi_{2\restriction \mathrm{G}_{\mathbb{Q}_v}} = 1$.

Hypothesis 2.7. Suppose that $\bar{\rho}_E$ is reducible and indecomposable. Let $v \in \Sigma(\varphi_2)$ and l be the prime number in such that $v|l$. Then l is split in K.

Remark 2.8. The above hypotheses are all weaker than the Heegner hypothesis. Some of our results will require the above hypotheses; they will be assumed only when explicitly stated. For instance, Theorem 3.2 will require Hypothesis 2.5 (and Hypothesis 2.7 when $\bar{\rho}_E$ is indecomposable). On the other hand, Theorem 3.4 does not require these hypotheses.

We now introduce the fine Selmer group. At each prime $v \in S$, set

$$\mathcal{H}_v(K^{\mathrm{ac}}, E[p^\infty]) := \prod_{\eta \in v(K^{\mathrm{ac}})} H^1(K^{\mathrm{ac}}_\eta, E[p^\infty]).$$

Definition 2.9. The *fine Selmer group* associated to $E[p^\infty]$ is defined as follows

$$\mathcal{R}_{p^\infty}(E/K^{\mathrm{ac}}) := \ker\left\{ H^1(K_S/K^{\mathrm{ac}}, E[p^\infty]) \longrightarrow \bigoplus_{v \in S} \mathcal{H}_v(K^{\mathrm{ac}}, E[p^\infty]) \right\}.$$

Recall that $\Gamma := \mathrm{Gal}(K^{\mathrm{ac}}/K) \simeq \mathbb{Z}_p$. The Iwasawa algebra Λ is the completed group algebra $\mathbb{Z}_p[\![\Gamma]\!] := \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$. After fixing a topological generator γ of Γ, there is an isomorphism of rings $\Lambda \cong \mathbb{Z}_p[\![X]\!]$, by sending $\gamma - 1$ to the formal variable X. The fine Selmer group is a cofinitely generated Λ-module.

The Pontryagin dual $\mathcal{R}_{p^\infty}(E/K^{\mathrm{ac}})^\vee$ is, up to pseudo-isomorphism, a finite direct sum of cyclic Λ-modules:

$$\mathcal{R}_{p^\infty}(E/K^{\mathrm{ac}})^\vee \sim \Lambda^r \oplus \left(\bigoplus_{i=1}^{s} \Lambda/(p^{\mu_i}) \right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda/(f_j(T)) \right).$$

Here, $\mu_i > 0$, $f_j(T)$ is a distinguished polynomial (i.e. a monic polynomial with non-leading coefficients divisible by p), and r is the Λ-corank of the fine Selmer group.

The μ-invariant of the fine Selmer group is defined as follows,

$$\mu^{\text{fine}}(E/K^{\text{ac}}) := \begin{cases} 0 & \text{if } s = 0 \\ \sum_{i=1}^{s} \mu_i & \text{if } s > 0. \end{cases}$$

The number of summands s is a well-defined invariant and we refer to it as the *μ-multiplicity*. We now introduce the fine Selmer group associated to the residual representation on $E[p]$. At each prime v of K, set

$$\mathcal{H}_v(K^{\text{ac}}, E[p]) := \prod_{\eta \in v(K^{\text{ac}})} H^1(K^{\text{ac}}_\eta, E[p]).$$

Definition 2.10. Let T be a finite set of primes containing S. The fine Selmer group associated to $E[p]$ and the set of primes T is defined as follows

$$\mathcal{R}^T(E[p]/K^{\text{ac}}) := \ker \left\{ H^1(K_T/K^{\text{ac}}, E[p]) \longrightarrow \bigoplus_{v \in T} \mathcal{H}_v(K^{\text{ac}}, E[p]) \right\}.$$

Set $\mathcal{R}(E[p]/K^{\text{ac}}) = \mathcal{R}^S(E[p]/K^{\text{ac}})$; this is the mod-$p$ fine Selmer group.

Set Ω to denote the mod-p Iwasawa algebra, $\Omega := \Lambda/(p) \simeq \mathbb{F}_p[\![X]\!]$. Note that both $\mathcal{R}(E[p]/K^{\text{ac}})$ and $\mathcal{R}_{p^\infty}(E/K^{\text{ac}})[p]$ are Ω-modules. Since $\mathcal{R}_{p^\infty}(E/K^{\text{ac}})$ is cofinitely generated over Λ, it follows that $\mathcal{R}_{p^\infty}(E/K^{\text{ac}})[p]$ is cofinitely generated over Ω. The following is an easy consequence of the structure theory of Λ-modules.

Lemma 2.11. The Ω-corank of $\mathcal{R}_{p^\infty}(E/K^{\text{ac}})[p]$ is equal to $r + s$, where r and s are defined above. In particular,

$$\mathcal{R}_{p^\infty}(E/K^{\text{ac}}) \text{ is } \Lambda\text{-cotorsion with } \mu^{\text{fine}}(E/K^{\text{ac}}) = 0 \Leftrightarrow \mathcal{R}_{p^\infty}(E/K^{\text{ac}})[p] \text{ is finite.}$$

Proof. Note that $\left(\mathcal{R}_{p^\infty}(E/K^{\text{ac}})[p]\right)^\vee$ is isomorphic to $\mathcal{R}_{p^\infty}(E/K^{\text{ac}})^\vee/p$. Let Φ be a pseudo-isomorphism of Λ-modules, i.e. a homomorphism,

$$\Phi : \mathcal{R}_{p^\infty}(E/K^{\text{ac}})^\vee \to \Lambda^r \oplus \left(\bigoplus_{i=1}^{s} \Lambda/(p^{\mu_i}) \right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda/(f_j(T)) \right)$$

with finite kernel and cokernel. The mod-p reduction $\overline{\Phi}$ is the following map

$$\overline{\Phi} : \left(\mathcal{R}_{p^\infty}(E/K^{\text{ac}})[p]\right)^\vee \to \Omega^{r+s} \oplus \left(\bigoplus_{j=1}^{t} \Omega/(\overline{f}_j(T)) \right),$$

where $\overline{f}_j(T)$ is the mod-p reduction of $f_j(T)$. Since $f_j(T)$ is a distinguished polynomial, $\overline{f}_j(T) = T^{\deg f_j}$ and $\Omega/(\overline{f}_j(T))$ is finite. The result follows. $\square$

3. A Criterion for the Vanishing of the fine μ-invariant

In this section, we establish a criterion for the vanishing of the fine μ-invariant in the anticyclotomic $\mathbb{Z}_p$-extension. Before stating the result, let us introduce some notation. We shall not assume that $\bar{\rho}_E$ is reducible in this section, unless when explicitly stated. However, when it is reducible, recall that the characters φ_1 and φ_2 are such that $E[p]$ fits into a short exact sequence of Galois-modules,

$$0 \to \mathbb{F}_p(\varphi_1) \to E[p] \to \mathbb{F}_p(\varphi_2) \to 0.$$

Let $K(\varphi_i)$ be an extension of K fixed by $\ker \varphi_i$. For an algebraic extension $\mathcal{F}$ of K, let $\mathcal{F}(\varphi_i)$ be the composite $\mathcal{F} \cdot K(\varphi_i)$. Let $\mathcal{L}_n^{(i)}$ be the maximal abelian unramified p-extension of $K_n(\varphi_i)$ in which the primes in $S(K_n(\varphi_i))$ split. This extension is called the *p-Hilbert S-class field extension*. Class field theory prescribes a natural isomorphism

$$\mathrm{Gal}(\mathcal{L}_n^{(i)}/K_n(\varphi_i)) \simeq \mathrm{Cl}_S(K_n(\varphi_i))[p^\infty],$$

where $\mathrm{Cl}_S(K_n(\varphi_i))[p^\infty]$ is the p-primary part of the S-class group of $K_n(\varphi_i)$. Set $\mathcal{L}^{(i)} := \bigcup_n \mathcal{L}_n^{(i)}$ and $\mathcal{X}^{(i)} = \mathrm{Gal}(\mathcal{L}^{(i)}/K^{\mathrm{ac}}(\varphi_i))$. The field diagram is drawn below

$$\begin{tikzcd}[arrows=dash]
 & \mathcal{L}^{(i)} \arrow[dl, "\mathcal{X}^{(i)}"'] \arrow[dr] & \\
K^{\mathrm{ac}}(\varphi_i) \arrow[dr] & & \mathcal{L}_n^{(i)} \arrow[dl] \\
 & K_n(\varphi_i) \arrow[d] & \\
 & K(\varphi_i). &
\end{tikzcd}$$

Let M be any $\mathbb{Z}_p$-module on which $\Delta := \mathrm{Gal}(K(\varphi_i)/K)$ acts by $\mathbb{Z}_p$-linear automorphisms. Let $\psi : \Delta \to \mathbb{F}_p^\times$ be a character, and consider its Teichmüller lift, $\widetilde{\psi} : \Delta \to \mathbb{Z}_p^\times$. Set

$$\mathrm{M}_\psi := \{x \in \mathrm{M} | g \cdot x = \widetilde{\psi}(g) x\}.$$

Standard arguments show that $\mathcal{X}^{(i)}$ is a finitely generated torsion Λ-module. Note that $\mathcal{X}^{(i)}$ decomposes into a direct sum of Λ-submodules

$$\mathcal{X}^{(i)} = \bigoplus_{\psi} \mathcal{X}^{(i)}_{\psi}, \tag{3.1}$$

where ψ ranges over the characters $\Delta \to \mathbb{F}_p^\times$. Recall that the ψ-eigenspace $\mathcal{X}^{(i)}_{\psi}$ is defined as follows

$$\mathcal{X}^{(i)}_{\psi} := \{x \in \mathcal{X}^{(i)} \mid g \cdot x = \widetilde{\psi}(g)x\}.$$

There is a unique field extension $\mathcal{E}^{(i)}_n$ of $K_n(\varphi_i)$ which is Galois over K and satisfying the additional property that

$$\mathrm{Gal}(\mathcal{E}^{(i)}_n/K_n(\varphi_i)) \simeq \left(\mathrm{Cl}_S(K_n(\varphi_i))[p^\infty]\right)_{\varphi_i}.$$

For ease of notation, set $Y_i = \mathcal{X}^{(i)}_{\varphi_i}$. Let $\mathcal{M}^{(i)}$ denote the maximal abelian unramified pro-p field extension of $K^{\mathrm{ac}}(\varphi_i)$. Let $\widetilde{Y}_i$ be the φ_i-component of $\mathrm{Gal}(\mathcal{M}^{(i)}/K^{\mathrm{ac}}(\varphi_i))$. Define $\mathcal{E}^{(i)} := \bigcup_n \mathcal{E}^{(i)}_n$. Then,

$$\begin{aligned}
\mathrm{Gal}(\mathcal{E}^{(i)}/K^{\mathrm{ac}}(\varphi_i)) &= \varprojlim \mathrm{Gal}(\mathcal{E}^{(i)}_n/K_n(\varphi_i)) \\
&\simeq \varprojlim \left(\mathrm{Cl}_S(K_n(\varphi_i))[p^\infty]\right)_{\varphi_i} \\
&\simeq \mathcal{X}^{(i)}_{\varphi_i} = Y_i.
\end{aligned}$$

Note that Y_i is a quotient of $\widetilde{Y}_i$. Let $\widetilde{Y}_i^{\perp} \subseteq \mathrm{Gal}(\mathcal{M}^{(i)}/K^{\mathrm{ac}}(\varphi_i))$ be the complement of $\widetilde{Y}_i$, taken to be the direct sum over eigenspaces $\mathrm{Gal}(\mathcal{M}^{(i)}/K^{\mathrm{ac}}(\varphi_i))_{\psi}$ as ψ ranges over the characters not equal to φ_i. We have that

$$\widetilde{Y}_i \oplus \widetilde{Y}_i^{\perp} = \mathrm{Gal}(\mathcal{M}^{(i)}/K^{\mathrm{ac}}(\varphi_i)).$$

Denote by $\widetilde{\mathcal{E}}^{(i)}$ denote the fixed field of $\widetilde{Y_i}^{\perp}$ and note that

$$\widetilde{Y}_i = \mathrm{Gal}(\widetilde{\mathcal{E}}^{(i)}/K^{\mathrm{ac}}(\varphi_i)).$$

Note that Y_i satisfies the additional property that $\mathcal{E}^{(i)}$ is the maximal subfield of $\widetilde{\mathcal{E}}^{(i)}$ at which all primes of $K^{\mathrm{ac}}(\varphi_i)$ above S split completely. The fields are depicted below

$$\begin{tikzcd}
\mathcal{M}^{(i)} \arrow[ddd, dash] \arrow[dr, dash] & \\
& \widetilde{\mathcal{E}}^{(i)} \arrow[d, dash] \arrow[ddl, dash, "\widetilde{Y}_i"'] \\
& \mathcal{E}^{(i)} \arrow[dl, dash, "Y_i"] \\
K^{\mathrm{ac}}(\varphi_i) \arrow[d, dash] & \\
K(\varphi_i). &
\end{tikzcd}$$

Finally, we denote the μ-invariants of the Galois modules Y_i and $\widetilde{Y}_i$ (with respect to the $\mathbb{Z}_p$-extension $K^{\mathrm{ac}}(\varphi_i)/K(\varphi_i)$) by $\mu(Y_i)$ and $\mu(\widetilde{Y}_i)$, respectively.

The next result follows from the work of H. Hida [20, Theorem I] (see also [14, Theorem 1.1]) and Rubin [36, Theorem 4.1].

Theorem 3.1. Suppose that $p = v\overline{v}$ in K. Let $\mathrm{G}_v = \mathrm{Gal}(\overline{K_v}/K_v)$ and assume that $\varphi_{1\restriction \mathrm{G}_v} \neq 1,\ \overline{\chi}_{\restriction \mathrm{G}_v}$ (or equivalently, $\varphi_{2\restriction \mathrm{G}_v} \neq 1,\ \overline{\chi}_{\restriction \mathrm{G}_v}$). Then,

$$\mu(\widetilde{Y}_1) = \mu(\widetilde{Y}_2) = \mu(Y_1) = \mu(Y_2) = 0.$$

Proof. Since $\varphi_1\varphi_2 = \overline{\chi}$, the condition on φ_1 is equivalent to that on φ_2. Let φ_i be either of the characters, φ_1 or φ_2. Let $\mathcal{Q}^{(i)}$ be the maximal abelian pro-p extension of $K^{\mathrm{ac}}(\varphi_i)$ which is unramified away from $v(K^{\mathrm{ac}})$. The anticyclotomic analogue of Rubin's results identifies $\mathcal{Q}^{(i)}$ with an anticyclotomic p-adic L-function (see for example [5, text surrounding (2.10)]). Moreover, Hida (also T. Finis) proves that the μ-invariant of such an anticyclotomic p-adic L-function vanishes. Indeed, in [20], Hida computes μ-invariant associated to the anticyclotomic $\mathbb{Z}_p$-extension of a CM field if conductor of the branch character is a product of primes split over the maximal real subfield. Since we are working with imaginary quadratic fields, the maximal real subfield is $\mathbb{Q}$. Hence the condition on branch characters is automatically satisfied. He notes that the μ-invariant vanishes except for certain rare cases (denoted by (V) in *loc. cit*) This condition is equivalent to three conditions (M1)-(M3) (see p. 45 of *loc. cit.*). Notice that in our case, (V) can not be satisfied. Since $F = \mathbb{Q}$, (M1) can never hold, i.e., there does not exist any (imaginary quadratic) number field which is unramified at every finite place.

Combining their results, we deduce that $\mu(\mathcal{Q}^{(i)}) = 0$ for $i = 1, 2$. Since $\widetilde{Y}_i$ is a quotient of $\mathcal{Q}^{(i)}$, it follows that $\mu(\widetilde{Y}_i) = 0$ (hence, $\mu(Y_i) = 0$) for $i = 1, 2$. $\square$

The most explicit examples of reducible Galois representations arise from elliptic curves with p-torsion points over the base field. However, in this setting, the conditions of Theorem 3.1 are not satisfied since $\{\varphi_1, \varphi_2\} = \{1, \overline{\chi}\}$. We now state the main theorems of this article.

Theorem 3.2. Suppose the following conditions hold.

1. $\bar{\rho}_E$ is reducible.
2. Hypothesis 2.5 is satisfied.
3. If $\bar{\rho}_E$ is indecomposable, then Hypothesis 2.7 is satisfied.
4. For $i = 1, 2$, the μ-invariant $\mu(Y_i) = 0$ (for instance, if the conditions of Theorem 3.1 are satisfied).

Then, the fine Selmer group $\mathcal{R}_{p^\infty}(E/K^{\mathrm{ac}})$ is Λ-cotorsion with $\mu^{\mathrm{fine}}(E/K^{\mathrm{ac}}) = 0$.

The proof of this theorem is provided in Section 4. This result is illustrated by Example 6.1. For elliptic curves over $\mathbb{Q}$ where hypotheses of Theorem 3.1 are satisfied, a similar result is proven in [6].

Remark 3.3. When all primes $v \in S \setminus S_p$ are split in K, the vanishing of the μ-invariant for Y_i is equivalent to the vanishing of the μ-invariant of $\widetilde{Y}_i$ (see the proof of Proposition 4.3). However, if a prime $v \in S \setminus S_p$ does not split in K, Fact 2.3 asserts that there are infinitely many primes above v. Recall that $\mathcal{E}^{(i)}$ is the subfield of $\widetilde{\mathcal{E}}^{(i)}$ in which the primes above v are split. Note that since Y_i is a quotient of $\widetilde{Y}_i$, the condition $\mu(Y_i) = 0$ is a weaker condition than requiring $\mu(\widetilde{Y}_i) = 0$.

Our next result is a (partial) converse to Theorem 3.2. We prove it in Section 4.

Theorem 3.4. Let E be an elliptic curve defined over K. Assume that

1. $\bar{\rho}_E$ is reducible.
2. $\mathcal{R}_{p^\infty}(E/K^{\mathrm{ac}})$ is a cotorsion Λ-module with $\mu^{\mathrm{fine}}(E/K^{\mathrm{ac}}) = 0$.

If $\bar{\rho}_E$ is indecomposable, then $\mu(Y_1) = 0$. If $\bar{\rho}_E$ is split, then $\mu(Y_1) = \mu(Y_2) = 0$.

The next corollary takes note of a special case of interest.

Corollary 3.5. Let E be an elliptic curve defined over K and S be the set of primes dividing $\mathcal{N}p$, where $\mathcal{N}$ is the conductor of E. Assume that

1. $E(K)[p] \neq 0$.
2. $\mathcal{R}_{p^\infty}(E/K^{\mathrm{ac}})$ is a cotorsion Λ-module with $\mu^{\mathrm{fine}}(E/K^{\mathrm{ac}}) = 0$.
3. the Heegner hypothesis is satisfied: for $v \in S \setminus S_p$, let l be the prime number such that $v|l$, then l splits in K.

Then the classical (anticyclotomic) Iwasawa μ-invariant, $\mu(K^{\mathrm{ac}}/K) = 0$.

Proof. Note that $E(K)[p] \neq 0$ when the residual representation is of the form $\bar{\rho}_E \simeq \begin{pmatrix} 1 & * \\ & \overline{\chi} \end{pmatrix}$. The result follows from Theorem 3.4 and Proposition 4.3. □

Example 6.2 illustrates Corollary 3.5. Before proving the above results, a brief sketch of the method is provided. The Kummer sequence

$$0 \to E[p] \to E[p^\infty] \xrightarrow{\times p} E[p^\infty] \to 0 \tag{3.2}$$

induces a comparison map

$$\Psi : \mathcal{R}(E[p]/K^{\mathrm{ac}}) \to \mathcal{R}_{p^\infty}(E/K^{\mathrm{ac}})[p]. \tag{3.3}$$

The main theorem is deduced from the following statements.

1. The kernel of Ψ is finite. If Hypothesis 2.5 is satisfied, then $\operatorname{cok}\Psi$ is also finite. This step does not require any assumption on $\bar{\rho}_E$.
2. Suppose that $\bar{\rho}_E$ is reducible. Further, if the conditions of Theorem 3.2 are satisfied, then the mod-p fine Selmer group $\mathcal{R}(E[p]/K^{\mathrm{ac}})$ is finite.

We now analyze the map Ψ and deduce a criterion for the vanishing of the fine μ-invariant. To show that $\operatorname{cok}\Psi$ is finite when Hypothesis 2.5 is satisfied, we prove the following result.

Lemma 3.6. Let $v \in S \setminus \Sigma$ be a prime such that $v \nmid p$ and η be a prime of K^{ac} above v. Then, the group $E(K^{\mathrm{ac}}_\eta)[p^\infty]$ is p-divisible.

Proof. Since $v \notin \Sigma$, (at least) one of the following conditions is satisfied:

1. $v \nmid (\mathcal{N}/\overline{\mathcal{N}})$,
2. $p \geq 5$, $\mu_p \subset K_v$, and E has non-split multiplicative reduction or additive reduction at v,
3. $p = 3$, $\mu_3 \subset K_v$, and E has non-split multiplicative reduction at v.

When E has good reduction at v, the condition (1) is satisfied. The claim for (1) is follows from the proof of [12, Lemma 4.1.2]. On the other hand, the claim for cases (2) and (3) follows from [19, Proposition 5.1 (iii)]. $\square$

For $v \in S$, let h_v denote the natural map

$$h_v : \mathcal{H}_v(K^{\mathrm{ac}}, E[p]) \to \mathcal{H}_v(K^{\mathrm{ac}}, E[p^\infty])[p]$$

induced from (3.2). For each prime $\eta | v$ of K^{ac}, set h_η to denote the natural map

$$h_\eta : H^1(K_\eta^{\mathrm{ac}}, E[p]) \to H^1(K_\eta^{\mathrm{ac}}, E[p^\infty])[p],$$

also induced from (3.2).

Corollary 3.7. If Hypothesis 2.5 is satisfied, then $\ker h_v$ is finite for $v \in S$.

Proof. The kernel of h_v is the product of $\ker h_\eta$, as η ranges over $v(K^{\mathrm{ac}})$. By Kummer theory, $\ker h_\eta$ is isomorphic to $H^0(K_\eta^{\mathrm{ac}}, E[p^\infty])/p$; hence, it is finite.

First, consider the case when $v \in \Sigma \cup S_p$. Let l be the prime number for which $v|l$. By Hypothesis 2.5, if $v \in \Sigma$, then l is split in K. Therefore, by Fact 2.3, the set of primes $v(K^{\mathrm{ac}})$ is finite for $v \in \Sigma \cup S_p$. Since there are only finitely many primes $\eta \in v(K^{\mathrm{ac}})$, $\ker h_v$ is finite in this case.

Next, consider the case when $l \in S \setminus (\Sigma \cup S_p)$. Lemma 3.6 asserts that the group $E(K_\eta^{\mathrm{ac}})[p^\infty]$ is p-divisible for all $\eta \in v(K^{\mathrm{ac}})$. In this case,

$$\ker h_\eta = H^0(K_\eta^{\mathrm{ac}}, E[p^\infty])/p = 0.$$

Therefore, h_v is injective in this case. This completes the proof of the corollary. $\square$

Proposition 3.8. The kernel of Ψ is finite. If Hypothesis 2.5 is satisfied, then $\operatorname{cok} \Psi$ is also finite.

Proof. The Kummer sequence (3.2) induces the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{R}(E[p]/K^{\mathrm{ac}}) \arrow[r] \arrow[d, "\Psi"] & H^1(K_S/K^{\mathrm{ac}}, E[p]) \arrow[r] \arrow[d, "g"] & \mathrm{im}(\overline{\Phi}_E) \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & \mathcal{R}_{p^\infty}(E/K^{\mathrm{ac}})[p] \arrow[r] & H^1(K_S/K^{\mathrm{ac}}, E[p^\infty])[p] \arrow[r] & \bigoplus_{v\in S} \mathcal{H}_v(K^{\mathrm{ac}}, E[p^\infty])[p]. \end{tikzcd}$$

Here, $\overline{\Phi}_E$ is the natural map

$$\overline{\Phi}_E : H^1(K_S/K^{\mathrm{ac}}, E[p]) \to \bigoplus_{v\in S} \mathcal{H}_v(K^{\mathrm{ac}}, E[p]).$$

Clearly, the map g is surjective. The snake lemma yields an exact sequence,

$$0 \to \ker \Psi \to \ker g \to \ker h \to \operatorname{cok} \Psi \to 0. \tag{3.4}$$

Since $\ker g \simeq H^0(K^{\mathrm{ac}}, E[p^\infty])/p$, it is finite (with cardinality at most p^2). From (3.4), we deduce that $\ker \Psi$ is finite. It follows from Corollary 3.7 that $\ker h$ is finite. Therefore, $\operatorname{cok} \Psi$ is finite as well. □

Remark 3.9. The proof of the above proposition shows that $\ker h$ must be finite if $\mu^{\mathrm{fine}}(E/K^{\mathrm{ac}}) = 0$ (even if Hypothesis 2.5 is not satisfied). This point is of considerable interest to the authors, since there is no apparent reason to suggest why this should be true in general.

Proposition 3.10. Let $E_{/K}$ be an elliptic curve. The following assertions hold.

1. If $\mathcal{R}_{p^\infty}(E/K^{\mathrm{ac}})$ is Λ-cotorsion with $\mu^{\mathrm{fine}}(E/K^{\mathrm{ac}}) = 0$, then $\mathcal{R}(E[p]/K^{\mathrm{ac}})$ is finite.

2. Assume that Hypothesis 2.5 is satisfied. Then, the fine Selmer group is Λ-cotorsion with $\mu^{\mathrm{fine}}(E/K^{\mathrm{ac}}) = 0$ if and only if $\mathcal{R}(E[p]/K^{\mathrm{ac}})$ is finite.

Proof. The assertion is a consequence of Lemma 2.11 and Proposition 3.8. □

Let $F = K(E[p])$ be the field extension of K left fixed by the kernel of $\bar{\rho}_E$. Set F_∞ to denote the compositum $K^{\mathrm{ac}} \cdot F$. In the remainder of this section, let $\mathcal{X}_E^S$ be the S-Hilbert class field extension of F_∞. In other words, it is the maximal unramified abelian pro-p extension of F_∞ in which the primes of $S(F_\infty)$ are split. The μ-invariant of $\mathcal{X}_E^S$ is taken with respect to the $\mathbb{Z}_p$-extension F_∞/F. Recall that in this section, there is no hypothesis on (the reducibility) of the residual representation $\bar{\rho}_E$.

Corollary 3.11. Let $E_{/K}$ be an elliptic curve such that

1. Hypothesis 2.5 is satisfied.
2. $\mu(\mathcal{X}_E^S) = 0$.

Then, the fine μ-invariant, $\mu^{\text{fine}}(E/K^{\text{ac}}) = 0$.

Proof. By inflation-restriction, we have

$$0 \to H^1\left(\text{Gal}(F_\infty/K^{\text{ac}}), E[p]\right) \to H^1(K^{\text{ac}}, E[p]) \to \text{Hom}\left(\text{Gal}(\overline{F}_\infty/F_\infty), E[p]\right)^{\text{Gal}(F/K)}.$$

It follows from Definition 2.10 that if $f \in \mathcal{R}(E[p]/K^{\text{ac}})$, then it is trivial at each prime $\eta \in S(K^{\text{ac}})$. Hence, the image of f in $\text{Hom}\left(\text{Gal}(\overline{F}_\infty/F_\infty), E[p]\right)^{\text{Gal}(F/K)}$ must factor through $\mathcal{X}_E^S$. Since $E[p]$ is p-torsion, it must factor through $\mathcal{X}_E^S/p$. Since $\text{Gal}(F_\infty/K^{\text{ac}})$ is finite, it follows that the above exact sequence induces a map with finite kernel

$$\mathcal{R}(E[p]/K^{\text{ac}}) \to \text{Hom}\left(\mathcal{X}_E^S/p, E[p]\right)^{\text{Gal}(F/K)}. \tag{3.5}$$

Since $\mu(\mathcal{X}_E^S) = 0$, the group $\mathcal{X}_E^S/p$ is finite. By (3.5), the mod-p fine Selmer group $\mathcal{R}(E[p]/K^{\text{ac}})$ is finite. The result follows from Proposition 3.10. It is in applying Proposition 3.10, that we require Hypothesis 2.5. $\square$

4. Finiteness of the residual fine Selmer group

Throughout this section $\bar{\rho}_E$ is assumed to be reducible. Therefore, $E[p]$ fits into the short exact sequence,

$$0 \to \mathbb{F}_p(\varphi_1) \to E[p] \to \mathbb{F}_p(\varphi_2) \to 0.$$

This allows the analysis of the algebraic structure of the mod-p fine Selmer group $\mathcal{R}(E[p]/K^{\text{ac}})$ in terms of the mod-p fine Selmer groups $\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{\text{ac}})$ for $i = 1, 2$ (definitions given below). This completes the proof of Theorems 3.2 and 3.4.

Definition 4.1. Let S be the set of primes dividing $\mathcal{N}p$, where $\mathcal{N}$ is the conductor of E. The fine Selmer group $\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{\text{ac}})$ is defined as follows

$$\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{\text{ac}}) := \ker\left\{H^1(K_S/K^{\text{ac}}, \mathbb{F}_p(\varphi_i)) \to \prod_{\eta \in S(K^{\text{ac}})} H^1(K_\eta^{\text{ac}}, \mathbb{F}_p(\varphi_i))\right\}.$$

The dependence on S is suppressed in the notation.

Proposition 4.2. Suppose that $\bar{\rho}_E$ is reducible and let φ_i be one of the characters on the diagonal of $\bar{\rho}_E$. Then, $\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{\text{ac}})$ is finite if and only if $\mu(Y_i) = 0$.

Proof. Set $\Delta := \mathrm{Gal}(K(\varphi_i)/K)$. Since the order of Δ is coprime to p,

$$\mathrm{Gal}(K^{\mathrm{ac}}(\varphi_i)/K) \simeq \Delta \times \mathrm{Gal}(K^{\mathrm{ac}}/K).$$

The group $\mathrm{Gal}(K^{\mathrm{ac}}(\varphi_i)/K)$ acts on $\mathcal{X}^{(i)}$. Let $g \in \mathrm{Gal}(K^{\mathrm{ac}}(\varphi_i)/K)$ and $x \in \mathcal{X}^{(i)}$. Choose a lift $\widetilde{g} \in \mathrm{Gal}(\mathcal{L}^{(i)}/K)$, and set $g \cdot x := \widetilde{g}x\widetilde{g}^{-1}$. Since $\mathcal{X}^{(i)}$ is abelian, $g \cdot x$ is independent of the choice of lift, $\widetilde{g}$. This induces the action of Δ on $\mathcal{X}^{(i)}$. Recall from (3.1) that $\mathcal{X}^{(i)} = \bigoplus_\psi \mathcal{X}^{(i)}_\psi$ and thus,

$$\mathrm{Hom}(\mathcal{X}^{(i)}, \mathbb{F}_p(\varphi_i))^\Delta = \bigoplus_\psi \mathrm{Hom}(\mathcal{X}^{(i)}_\psi, \mathbb{F}_p(\varphi_i))^\Delta.$$

Note that $Y_i := \mathcal{X}^{(i)}_{\varphi_i}$, and the action of Δ on Y_i is via the Teichmüller lift of φ_i. Thus, $gx = \widetilde{\varphi}_i(g)x$ for $g \in \Delta$ and $x \in Y_i$. It is easy to see that if $\psi \neq \varphi_i$, then, $\mathrm{Hom}(\mathcal{X}^{(i)}_\psi, \mathbb{F}_p(\varphi_i))^\Delta = 0$. On the other hand, $\mathrm{Hom}(Y_i, \mathbb{F}_p(\varphi_i))^\Delta$ may simply be identified with $\mathrm{Hom}(Y_i, \mathbb{F}_p)$. Indeed, if $f : Y_i \to \mathbb{F}_p$ is any group homomorphism, it is automatically $\mathbb{Z}_p$-linear and thus

$$f(gx) = f(\widetilde{\varphi}_i(g)x) = \widetilde{\varphi}_i(g)f(x) = \varphi_i(g)f(x).$$

The last equality follows since $\varphi = \widetilde{\varphi}_i \mod p$. Summarizing the above discussion, we have that

$$\mathrm{Hom}(\mathcal{X}^{(i)}_\psi, \mathbb{F}_p(\varphi_i))^\Delta = \begin{cases} \mathrm{Hom}(Y_i, \mathbb{F}_p) & \text{if } \psi = \varphi_i, \\ 0 & \text{otherwise.} \end{cases}$$

There are thus natural isomorphisms

$$\mathrm{Hom}(\mathcal{X}^{(i)}, \mathbb{F}_p(\varphi_i))^\Delta \simeq \bigoplus_\psi \mathrm{Hom}(\mathcal{X}^{(i)}_\psi, \mathbb{F}_p(\varphi_i))^\Delta \simeq \mathrm{Hom}(Y_i, \mathbb{F}_p).$$

Since the order of $\mathrm{Gal}(K^{\mathrm{ac}}(\varphi_i)/K^{\mathrm{ac}})$ is prime to p, it follows (from restriction-corestriction) that the cohomology group $H^j(\mathrm{Gal}(K^{\mathrm{ac}}(\varphi_i)/K^{\mathrm{ac}}), \mathbb{F}_p(\varphi_i)) = 0$ for all $j > 0$. It follows from the inflation-restriction sequence that

$$H^1(K_S/K^{\mathrm{ac}}, \mathbb{F}_p(\varphi_i)) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Gal}(K_S/K^{\mathrm{ac}}(\varphi_i)), \mathbb{F}_p(\varphi_i))^\Delta,$$

where Δ is identified with $\mathrm{Gal}(K^{\mathrm{ac}}(\varphi_i)/K^{\mathrm{ac}})$. This induces an isomorphism

$$\iota : \mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{\mathrm{ac}}) \xrightarrow{\sim} \mathrm{Hom}(Y_i/pY_i, \mathbb{F}_p).$$

It is an easy consequence of the structure theory of Λ-modules (see the argument in the proof of Lemma 2.11) that the μ-invariant of Y_i is zero if and only if Y_i/pY_i is finite. The result follows. $\square$

Proposition 4.3. Suppose that $\bar{\rho}_E$ is reducible and φ_i is one of the characters on the diagonal. Suppose that the following assumptions hold.

1. $\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{\mathrm{ac}})$ is finite.
2. The Heegner hypothesis is satisfied: let $v \in S \setminus S_p$ and l be the prime number such that $v|l$, then l splits in K.

Then, we have that $\mu(\widetilde{Y}_i) = 0$.

We remind the reader that this result was used in the proof of Corollary 3.5 where it was further assumed that $\varphi_i = 1$. In this special case, the assertion is that the (classical) Iwasawa invariant $\mu(K^{\mathrm{ac}}/K)$ vanishes.

Proof. Recall the definitions of Galois extensions $\mathcal{M}^{(i)}$ and $\mathcal{X}^{(i)}$ of $K^{\mathrm{ac}}(\varphi_i)$ from (the beginning of) §3. Consider the natural action of $\Delta := \mathrm{Gal}(K(\varphi_i)/K)$ on $\mathcal{M}^{(i)}$ (resp. $\mathcal{X}^{(i)}$) and recall that $\widetilde{Y}_i$ (resp. Y_i) denotes φ_i-component of $\mathcal{M}^{(i)}$ (resp. $\mathcal{X}^{(i)}$) with respect to this action. At each prime η of K^{ac}, set $H^1_{\mathrm{nr}}(K^{\mathrm{ac}}_\eta, \mathbb{F}_p(\varphi_i))$ to denote the subspace of unramified cohomology classes in $H^1(K^{\mathrm{ac}}_\eta, \mathbb{F}_p(\varphi_i))$. Set $\mathcal{R}'(\mathbb{F}_p(\varphi_i)/K^{\mathrm{ac}})$ to consist of cohomology classes $f \in H^1(K_S/K^{\mathrm{ac}}, \mathbb{F}_p(\varphi_i))$ which are unramified at every prime $\eta \in S(K^{\mathrm{ac}})$. By an application of inflation-restriction (see the argument in the proof of Proposition 4.2),

$$\begin{aligned}
\mathcal{R}'(\mathbb{F}_p(\varphi_i)/K^{\mathrm{ac}}) &\simeq \mathrm{Hom}(\mathcal{M}^{(i)}/p, \mathbb{F}_p(\varphi_i))^{\Delta} \simeq \mathrm{Hom}(\widetilde{Y}_i/p, \mathbb{F}_p),\\
\mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{\mathrm{ac}}) &\simeq \mathrm{Hom}(\mathcal{X}^{(i)}/p, \mathbb{F}_p(\varphi_i))^{\Delta} \simeq \mathrm{Hom}(Y_i/p, \mathbb{F}_p).
\end{aligned}$$

The groups fit into an exact sequence,

$$0 \to \mathcal{R}(\mathbb{F}_p(\varphi_i)/K^{\mathrm{ac}}) \to \mathrm{Hom}(\widetilde{Y}_i/p, \mathbb{F}_p(\varphi_i)) \to \prod_{\eta \in S(K^{\mathrm{ac}})} H^1_{\mathrm{nr}}(K^{\mathrm{ac}}_\eta, \mathbb{F}_p(\varphi_i)).$$

By Fact 2.3, the set $S(K^{\mathrm{ac}})$ is finite. Hence, the group on the right is finite. We deduce that $\widetilde{Y}_i/p$ is finite, therefore the μ-invariant of $\widetilde{Y}_i$ is zero. ☐

Proof of Theorem 3.2. By Proposition 3.10, it suffices to show that $\mathcal{R}(E[p]/K^{\mathrm{ac}})$ is finite. The short exact sequence

$$0 \to \mathbb{F}_p(\varphi_1) \to E[p] \to \mathbb{F}_p(\varphi_2) \to 0,$$

induces the long exact sequence

$$\cdots \to H^0(K^{\mathrm{ac}}_\eta, \mathbb{F}_p(\varphi_2)) \xrightarrow{\delta^0_\eta} H^1(K^{\mathrm{ac}}_\eta, \mathbb{F}_p(\varphi_1)) \to H^1(K^{\mathrm{ac}}_\eta, E[p]) \to \cdots .$$

Set $S_{\mathrm{ac}} := S(K^{\mathrm{ac}})$ and

$$U := \prod_{\eta \in S_{\mathrm{ac}}} \operatorname{image} \delta_\eta^0 \subset \prod_{\eta \in S_{\mathrm{ac}}} H^1(K_\eta^{\mathrm{ac}}, \mathbb{F}_p(\varphi_1)).$$

If $\bar{\rho}_E$ is split, then δ_η^0 is the 0 map; hence, $U = 0$.
Next, consider the case when $\bar{\rho}_E$ is indecomposable. When $v \in S \setminus (\Sigma(\varphi_2) \cup S_p)$, note that $H^0(K_v, \mathbb{F}_p(\varphi_2)) = 0$. Let $\eta \in v(K^{\mathrm{ac}})$. Since K_η^{ac}/K_v is a p-extension, for all $v \in S \setminus (\Sigma(\varphi_2) \cup S_p)$, the product $\prod_{\eta \in v(K^{\mathrm{ac}})} H^0(K_\eta^{\mathrm{ac}}, \mathbb{F}_p(\varphi_2)) = 0$. On the other hand, for all primes $v \in \Sigma(\varphi_2) \cup S_p$, it follows from Hypothesis 2.7 and Fact 2.3 that there are only finitely many primes η in $v(K^{\mathrm{ac}})$. Therefore, the assumptions imply that U is finite.

Consider the diagram

$$\begin{array}{ccccccc}
& & H^1(K_S/K^{\mathrm{ac}}, \mathbb{F}_p(\varphi_1)) & \longrightarrow & H^1(K_S/K^{\mathrm{ac}}, E[p]) & \longrightarrow & H^1(K_S/K^{\mathrm{ac}}, \mathbb{F}_p(\varphi_2)) \\
& & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
0 & \longrightarrow & \Big(\prod_{\eta \in S_{\mathrm{ac}}} H^1(K_\eta^{\mathrm{ac}}, \mathbb{F}_p(\varphi_1))\Big)/U & \longrightarrow & \prod_{\eta \in S_{\mathrm{ac}}} H^1(K_\eta^{\mathrm{ac}}, E[p]) & \longrightarrow & \prod_{\eta \in S_{\mathrm{ac}}} H^1(K_\eta^{\mathrm{ac}}, \mathbb{F}_p(\varphi_2))
\end{array}$$

and the associated short exact sequence

$$\ker \alpha \to \mathcal{R}(E[p]/K^{\mathrm{ac}}) \to \mathcal{R}(\mathbb{F}_p(\varphi_2)/K^{\mathrm{ac}}). \tag{4.1}$$

The group $\ker \alpha$ fits into an exact sequence

$$0 \to \mathcal{R}(\mathbb{F}_p(\varphi_1)/K^{\mathrm{ac}}) \to \ker \alpha \to U. \tag{4.2}$$

Proposition 4.2 asserts that $\mathcal{R}(\mathbb{F}_p(\varphi_i), K^{\mathrm{ac}})$ are finite for $i = 1, 2$. This requires the assumption that $\mu(Y_i) = 0$ for $i = 1, 2$. It follows from (4.1) and (4.2) that $\mathcal{R}(E[p]/K^{\mathrm{ac}})$ is also finite. By Lemma 2.11, we have that $\mathcal{R}_{p^\infty}(E/K^{\mathrm{ac}})$ is Λ-cotorsion with $\mu^{\mathrm{fine}}(E/K^{\mathrm{ac}}) = 0$. This completes the proof. □

Proof of Theorem 3.4. Refer to the argument in the proof of Theorem 3.2. Lemma 2.11 asserts that if $\mathcal{R}_{p^\infty}(E/K^{\mathrm{ac}})$ is Λ-torsion with $\mu^{\mathrm{fine}}(E/K^{\mathrm{ac}}) = 0$, then $\mathcal{R}(E[p]/K^{\mathrm{ac}})$ is finite. Refer to (4.1). Since $H^0(K_S/K^{\mathrm{ac}}, \mathbb{F}_p(\varphi_1))$ is finite, finiteness of $\mathcal{R}(E[p]/K^{\mathrm{ac}})$ implies that $\ker \alpha$ is finite. By (4.2), it is seen that $\mathcal{R}(\mathbb{F}_p(\varphi_1)/K^{\mathrm{ac}})$ is finite. Therefore, by Proposition 4.2, it follows that $\mu(Y_1) = 0$. This completes the proof in the case when $\bar{\rho}_E$ is indecomposable. When $\bar{\rho}_E$ is split, the proof is completed by interchanging the roles of φ_1 and φ_2. □

5. Congruent Galois representations

In this section, we consider two elliptic curves E_1 and E_2 defined over K which are p-congruent, i.e. their residual representations are isomorphic,

$$\bar{\rho}_{E_1} \simeq \bar{\rho}_{E_2}.$$

We make no assumption on the reducibility of the residual representations. However, it is assumed that both E_1 and E_2 satisfy the Heegner hypothesis.

Heegner hypothesis. Let $E_{/K}$ be an elliptic curve with conductor $\mathcal{N}$, and S be the set of primes dividing $\mathcal{N}p$. Let $v \in S$ and l be the prime number such that $v|l$. Then, l splits in K if $l \neq p$.

We prove the following main result in this section.

Theorem 5.1. Let E_1 and E_2 be elliptic curves over K. Suppose that the following assumptions hold.

1. Both E_1 and E_2 satisfy the Heegner hypothesis.
2. The residual Galois representations, $\bar{\rho}_{E_1}$ and $\bar{\rho}_{E_2}$, are isomorphic.

Then,

$$\begin{aligned} &\mathcal{R}_{p^\infty}(E_1/K^{\mathrm{ac}}) \text{ is } \Lambda\text{-cotorsion with } \mu^{\mathrm{fine}}(E_1/K^{\mathrm{ac}}) = 0 \\ \Leftrightarrow &\mathcal{R}_{p^\infty}(E_2/K^{\mathrm{ac}}) \text{ is } \Lambda\text{-cotorsion with } \mu^{\mathrm{fine}}(E_2/K^{\mathrm{ac}}) = 0. \end{aligned}$$

The key result in proving this theorem is Proposition 3.10, which states that

$$\mathcal{R}_{p^\infty}(E_i/K^{\mathrm{ac}}) \text{ is } \Lambda\text{-cotorsion with } \mu^{\mathrm{fine}}(E_i/K^{\mathrm{ac}}) = 0 \Leftrightarrow \mathcal{R}(E_i[p]/K^{\mathrm{ac}}) \text{ is finite.}$$

In view of Lemma 2.11, it suffices to prove the following result.

Proposition 5.2. Let E_1 and E_2 be elliptic curves satisfying the conditions of Theorem 5.1. Then $\mathcal{R}(E_1[p]/K^{\mathrm{ac}})$ is finite if and only if $\mathcal{R}(E_2[p]/K^{\mathrm{ac}})$ is finite.

Proof. Let $\mathcal{N}_i$ be the conductor of E_i and S_i be the set of primes dividing $\mathcal{N}_i p$. Denote by T the union $S_1 \cup S_2$. Since $E_1[p]$ and $E_2[p]$ are isomorphic as Galois modules, the mod-p fine Selmer groups are isomorphic, i.e.

$$\mathcal{R}^T(E_1[p]/K^{\mathrm{ac}}) \simeq \mathcal{R}^T(E_2[p]/K^{\mathrm{ac}})$$

(recall $\mathcal{R}^T(E_i[p]/K^{\mathrm{ac}})$ from Definition 2.10). Let $v \in T$ and $\eta \in v(K^{\mathrm{ac}})$. Denote by G_η the absolute Galois group of K^{ac}_η and by $\mathrm{I}_\eta \subset \mathrm{G}_\eta$ its inertia subgroup. Set $H^1_{\mathrm{nr}}(K^{\mathrm{ac}}_\eta, E_i[p])$ to denote the kernel of the restriction map,

$$\mathrm{res} : H^1(K^{\mathrm{ac}}_\eta, E_i[p]) \longrightarrow H^1(\mathrm{I}_\eta, E_i[p])^{\mathrm{G}_\eta/\mathrm{I}_\eta}.$$

It follows from inflation-restriction that $H^1_{\mathrm{nr}}(K^{\mathrm{ac}}_\eta, E_i[p])$ can be identified with the cohomology group $H^1(\mathrm{G}_\eta\,/\,\mathrm{I}_\eta, E_i[p]^{\mathrm{I}_\eta})$. Since $\mathrm{G}_\eta\,/\,\mathrm{I}_\eta \simeq \hat{\mathbb{Z}}$, it follows that (see [31, Proposition 1.7.7])

$$H^1(\mathrm{G}_\eta\,/\,\mathrm{I}_\eta, E_i[p]^{\mathrm{I}_\eta}) \simeq H_0(\mathrm{G}_\eta\,/\,\mathrm{I}_\eta, E_i[p]^{\mathrm{I}_\eta})$$

and hence is finite. By the Heegner hypothesis and Fact 2.3, the set $v(K^{\mathrm{ac}})$ is finite for $v \in T$. Let $\mathcal{H}_v^{\mathrm{nr}}(K^{\mathrm{ac}}, E[p]) \subseteq \mathcal{H}_v(K^{\mathrm{ac}}, E[p])$ denote the sum,

$$\mathcal{H}_v^{\mathrm{nr}}(K^{\mathrm{ac}}, E_i[p]) := \bigoplus_{\eta \in v(K^{\mathrm{ac}})} H^1_{\mathrm{nr}}(K^{\mathrm{ac}}_\eta, E_i[p]).$$

It follows from the above discussion that $\mathcal{H}_v^{\mathrm{nr}}(K^{\mathrm{ac}}, E_i[p])$ is finite for $v \in T$.

There is a natural exact sequence,

$$0 \to \mathcal{R}^T(E_i[p]/K^{\mathrm{ac}}) \to \mathcal{R}(E_i[p]/K^{\mathrm{ac}}) \to \bigoplus_{v \in T \setminus S_i} \mathcal{H}_v^{\mathrm{nr}}(K^{\mathrm{ac}}, E_i[p]).$$

Since the last term in the above sequence is finite, it follows that $\mathcal{R}(E_i[p]/K^{\mathrm{ac}})$ is finite if and only if $\mathcal{R}^T(E_i[p]/K^{\mathrm{ac}})$ is. The result follows from this. $\square$

Proof of Theorem 5.1. Note that the Heegner hypothesis implies Hypothesis 2.5. Therefore, Proposition 3.10 applies in this setting and asserts that

$$\mathcal{R}_{p^\infty}(E_i/K^{\mathrm{ac}}) \text{ is } \Lambda\text{-cotorsion with } \mu^{\mathrm{fine}}(E_i/K^{\mathrm{ac}}) = 0 \Leftrightarrow \mathcal{R}(E_i[p]/K^{\mathrm{ac}}) \text{ is finite.}$$

On the other hand, Proposition 5.2 states that

$$\mathcal{R}(E_1[p]/K^{\mathrm{ac}}) \text{ is finite } \Leftrightarrow \mathcal{R}(E_2[p]/K^{\mathrm{ac}}) \text{ is finite.}$$

The result follows. $\square$

6. Examples

In this section, we list some examples to illustrate the results in the article.

Example 6.1. We begin with an example which illustrates Theorem 3.2. In this example, $p = 3$ and E is the elliptic curve 81.1-CMa1 over $K = \mathbb{Q}(\sqrt{-3})$. This is the elliptic curve with Weierstass equation $y^2+y = x^3$ base changed to K. It has complex multiplication and its endomorphism ring is $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$. The group $E(K)[3]$ is the full group $E[3]$ (and isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$). In particular, residual representation $\bar{\rho}_E$ is the trivial representation $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. Therefore, $\bar{\rho}_E$ is split, $\varphi_1 = \varphi_2 = 1$, and Y_1 and Y_2 are both the 3-Hilbert class field extension of K^{ac}. The prime $p = 3$ is a prime of additive reduction and 3 is ramified in K. The conductor of E is $\mathcal{N} = 9\mathcal{O}_K$ and in particular is not divisible by any primes $v \nmid 3$. It follows that $\Sigma = \emptyset$ and Hypothesis 2.5 is satisfied. This unique prime above (3) is totally ramified in the anticyclotomic $\mathbb{Z}_3$-extension of K. Since the class number of K is 1, we know that $\mu(K^{\text{ac}}/K) = 0$ [32, Proposition 4.2]. It follows from Theorem 3.2 that $\mathcal{R}_{3^\infty}(E/K^{\text{ac}})$ is a cotorsion Λ-module with $\mu^{\text{fine}}(E/K^{\text{ac}}) = 0$ (at $p = 3$). In other words, $\mathcal{R}_{3^\infty}(E/K^{\text{ac}})$ is a cofinitely generated $\mathbb{Z}_3$-module.

Example 6.2. This example demonstrates Corollary 3.5. Let $K = \mathbb{Q}(\sqrt{-10})$. We fix $p = 5$ and let E be the elliptic curve $11a1$ over $\mathbb{Q}$ defined by the Weierstrass equation $y^2 + y = x^3 - x^2 - 10x - 20$. This is a rank 0 elliptic curve with the 5-torsion group $E(\mathbb{Q})[5] = \mathbb{Z}/5\mathbb{Z}$. In particular, $E(K)[5] \neq 0$. Over K, the elliptic curve has rank 1 and we assume that $\mathcal{R}_{5^\infty}(E/K^{\text{ac}})$ is a cotorsion Λ-module. In view of [34, Theorem 1.3.2 and §1.4 (Leop(V))], we believe this assumption to be reasonable. Here, the set S consists of the primes above 11 in K. Since 11 splits in K, it follows that condition (3) of Corollary 3.5 is satisfied. Corollary 3.5 asserts that if the μ-invariant of the fine Selmer group $\mathcal{R}_{5^\infty}(E/K^{\text{ac}})$ is zero, then the Iwasawa μ-invariant of the 5-Hilbert class field extension of K^{ac}/K is zero as well.

Even though we do not know if the μ-invariant of $\mathcal{R}_{5^\infty}(E/K^{\text{ac}})$ is zero, it is known that the classical (anticyclotomic) μ-invariant of the 5-Hilbert class field extension of K^{ac}/K is trivial. Indeed, the class number of $K = \mathbb{Q}(\sqrt{-10})$ is 2. By [10, Table 1], we know that $\lambda_5(K^{\text{cyc}}/K) < 2$. Therefore, by a result of Sands mentioned in the introduction (see also [38, Theorem]), the claim is true.

Example 6.3. This example demonstrates an application of Theorem 5.1. Consider the elliptic curves $E_1 = 201c1$ and $E_2 = 469a1$. Both elliptic curves have rank 1 with good ordinary reduction at the prime $p = 5$. Further, there is an isomorphism of $\mathrm{G}_{\mathbb{Q}}$-modules, $E_1[5] \simeq E_2[5]$. Note that E_1 has bad reduction at the primes 3, 67 and E_2 has bad reduction at the primes 7, 67. Set M to be the product of primes of bad reduction $3 \cdot 7 \cdot 67$. The primes of bad reduction of E_1, E_2 split in infinitely many imaginary quadratic number field K of the form $\mathbb{Q}(\sqrt{-Mk+1})$ where $k \in \mathbb{Z}_{\geq 1}$, i.e. the Heegner hypothesis is satisfied. Theorem 5.1 applies; hence

$$\begin{aligned} &\mathcal{R}_{5^\infty}(E_1/K^{\text{ac}}) \text{ is } \Lambda\text{-cotorsion with } \mu^{\text{fine}}(E_1/K^{\text{ac}}) = 0 \\ \Leftrightarrow &\mathcal{R}_{5^\infty}(E_2/K^{\text{ac}}) \text{ is } \Lambda\text{-cotorsion with } \mu^{\text{fine}}(E_2/K^{\text{ac}}) = 0. \end{aligned}$$

Example 6.4. Through this final example we demonstrate that Theorem 5.1 can be used to relate μ-invariants of isogenous elliptic curves over K^{ac}. Consider the isogenous elliptic curves $E_1 = 17a1$ and $E_2 = 17a2$. Both elliptic curves have rank 0 and good supersingular reduction at the prime $p = 11$. Further, there is an isomorphism of $\mathrm{G}_{\mathbb{Q}}$-modules, $E_1[11] \simeq E_2[11]$. Consider the imaginary quadratic number field, $K = \mathbb{Q}(\sqrt{-8})$. Note that the prime 17 splits in K, i.e. both E_1 and E_2 satisfy the Heegner hypothesis. It is known that $\mu^{\text{fine}}(E_1/K^{\text{ac}}) = 0$ (see [28, Table in §4]). It follows from Theorem 5.1 that $\mu^{\text{fine}}(E_2/K^{\text{ac}}) = 0$, as well.

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