Control Theorems for Fine Selmer Groups

Debanjana Kundu

Quebec-Maine Number Theory Conference 2020

September 28, 2020

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The key player in today's talk will be a subgroup of the Selmer group called the **fine Selmer group**. This subgroup interpolates the growth of class group and the Selmer group.

Definition: Selmer Groups of Elliptic Curves

Let F be a number field. Consider an elliptic curve E/F and p be any prime. Define the classical Selmer group of E relative to p^n by

$$0 \to \mathrm{Sel}_{p^n}(E/F) \to H^1(F, E[p^n]) \to \prod_v H^1(F_v, E)$$

where v runs through all the non-archimedean places of K. Then

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where v runs through all the non-archimedean places of K. Then

$$\mathrm{Sel}(E/F) = \mathrm{Sel}_{p^\infty}(E/F) = \varinjlim_n \mathrm{Sel}_{p^n}(E/F),$$

$$\mathrm{Sel}(E/\mathcal{L}) = \mathrm{Sel}_{p^\infty}(E/\mathcal{L}) := \varinjlim_L \mathrm{Sel}_{p^\infty}(E/L)$$

where L runs over all finite extensions of F contained in a pro-p p-adic Lie extension, $\mathcal{L}$.

Definition: Fine Selmer Group

We define

$$R_{p^n}(E/F) := \ker\left(\mathrm{Sel}_{p^n}(E/F) \to \bigoplus_{v|p} H^1(F_v, E[p^n])\right).$$

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Control Problem

Let F be a number field and $\mathcal{L}/F$ be a pro-p p-adic Lie extension with Galois group $\mathrm{Gal}(\mathcal{L}/F) \simeq G$. Let E be an elliptic curve defined over F. The study of the natural restriction map

$$s_{\mathcal{L}/F} : \mathrm{Sel}\left(E/F\right) \to \mathrm{Sel}\left(E/\mathcal{L}\right)^G$$

is called the **control problem**.

Mazur's Control Theorem

Theorem (Mazur (1972))

Let $\mathcal{L}/F$ be a $\mathbb{Z}_p$-extension and let E be an elliptic curve defined over F with good ordinary reduction at primes above p. Then both $\ker(s_{\mathcal{L}/L})$ and $\mathrm{coker}(s_{\mathcal{L}/L})$ are finite and bounded as L/F varies over all finite extensions inside $\mathcal{L}$.

Application: Growth of the Shafarevich-Tate Group

Theorem

Assume that E has good, ordinary reduction at all primes of F lying over p. Assume that $\mathrm{Sel}\left(E/\mathcal{L}\right)$ is Λ-cotorsion and that $\text{Ш}(F_n)[p^\infty]$ is finite for all $n \geq 0$. Then $\left|\text{Ш}(E/F_n)[p^\infty]\right| = p^{e_n}$ and there exist constants λ, μ, and ν such that

$$e_n = \lambda n + \mu p^n + \nu \textit{ for all } n \gg 0.$$

Greenberg's Control Theorem(s)

Theorem (Greenberg (2003))

Assume E has potentially ordinary reduction at all primes of F lying over p. Assume that $\mathcal{L}/F$ is a p-adic Lie extension satisfying the property that $\mathfrak{d}'_{\mathfrak{p}} = \mathfrak{i}'_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ above p. Further suppose that $\mathfrak{g}$ is reductive or $E(\mathcal{L})[p^\infty]$ is finite. Then both $\ker(s_{\mathcal{L}/L})$ and $\operatorname{coker}(s_{\mathcal{L}/L})$ are finite as L varies over all finite extensions of F inside $\mathcal{L}$.

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Some examples of p-adic Lie extensions $\mathcal{L}/F$, where the property $\mathfrak{d}'_{\mathfrak{p}} = \mathfrak{i}'_{\mathfrak{p}}$ holds for all primes $\mathfrak{p} \mid p$, include:

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- when the inertia subgroup has finite index in G for all $\mathfrak{p} \mid p$.
- when G admits a faithful, finite-dimensional p-adic representation of Hodge-Tate type at $\mathfrak{p} \mid p$.

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is called the **control problem**.

Known Results

Theorem (Rubin (2000?))

Let F be a number field and E be an elliptic curve defined over F. Let $\mathcal{L}/F$ be a $\mathbb{Z}_p^d$-extension where $d \geq 1$, and suppose all primes of bad reduction of E and all primes above p are finitely decomposed. Then both $\ker(r_{\mathcal{L}/L})$ and $\operatorname{coker}(r_{\mathcal{L}/L})$ are finite as L varies over all finite extensions of F inside $\mathcal{L}$.

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Remarks.

1. The Control Theorem for fine Selmer groups is independent of the reduction type at p.
2. When $d = 1$, the Control Theorem is proved for *all* $\mathbb{Z}_p$-extensions by Wuthrich (2004). Moreover, the order of $\ker(r_{\mathcal{L}/L})$ and $\operatorname{coker}(r_{\mathcal{L}/L})$ are *bounded* independent of L.

Our Results

- Prove a very general Control Theorem for fine Selmer groups (without any hypothesis).

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- We can give growth estimates for the order of the kernel and cokernel when specializing to
  - multi $\mathbb{Z}_p$-extensions
  - multi false Tate extensions
  - trivializing extensions.
- Asymptotic growth formula in finite layers.*

*Thank you Antonio Lei!

Our Results: Multi $\mathbb{Z}_p$-Extension Case

Theorem

Let E be an elliptic curve defined over F, and $\mathcal{L} = F_\infty$ be a $\mathbb{Z}_p^d$-extension of F, with $d \geq 2$. Then the kernel and cokernel of the restriction map

$$r_n : R\left(E/F_n\right) \longrightarrow R\left(E/F_\infty\right)^{G_n}$$

are finite. Furthermore,

$$\mathrm{ord}_p|\ker r_n| = O(n) \quad \textit{and} \quad \mathrm{ord}_p|\mathrm{coker}\, r_n| = O(p^{(d-1)n})^{*}.$$

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*can do better if additional properties are known!

Application: Asymptotic Growth

For any finitely generated (not necessarily torsion) $\mathbb{Z}_p[\![G]\!]$-module, M, denote by $e(M)$ the p-exponent of the *torsion subgroup* of M.

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Corollary

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$$e\left(R\left(E/F_n\right)\right) = \mu_G\left(\left(R\left(E/F_\infty\right)^\vee\right)\right)p^{dn} + O(np^{(d-1)n}).$$

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Unfortunately, this does not automatically translate to an asymptotic growth formula for the fine Shafarevich-Tate group.

Our Results: Trivializing Extension (CM) Case

Theorem

Let E be an elliptic curve with complex multiplication defined over the number field, F. Suppose that $F_\infty = F\left(E[p^\infty]\right)$ and $G = \mathrm{Gal}(F_\infty/F)$ is uniform. Then the kernel and cokernel of the restriction maps

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*If p is a prime of potential ordinary reduction, then $\mathrm{ord}_p|\ker r_n| = O(1)$.

Our Results: Trivializing Extension (non-CM) Case

Theorem

Let E be an elliptic curve without complex multiplication defined over F. Suppose that $F_\infty = F\left(E[p^\infty]\right)$ and $G = \mathrm{Gal}(F_\infty/F)$ is uniform. Then the kernel and cokernel of the restriction maps

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are finite. Further, the $\mathrm{ord}_p|\ker r_n| = O(n)^*$ *and* $\mathrm{ord}_p|\mathrm{coker}\, r_n| = O(np^{2n})$.

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Thank you!

Idea of the Proof

Consider the following fundamental diagram

$$
\begin{array}{ccccccc}
0 & \to & R(E/F_n) & \to & H^1\left(G_S\left(F_n\right),\ E[p^\infty]\right) & \to & \bigoplus_{v_n \in S(F_n)} H^1\left(F_{n,v_n},\ E[p^\infty]\right) \\
 & & \downarrow r_n & & \downarrow h_n & & \downarrow g_n \\
0 & \to & R\left(E/F_\infty\right)^{G_n} & \to & H^1\left(G_S\left(F_\infty\right),\ E[p^\infty]\right)^{G_n} & \to & \bigoplus_{w \in S(F_\infty)} H^1\left(F_{\infty,w},\ E[p^\infty]\right)^{G_n}
\end{array}
$$

with exact rows.