RELATING THE CLASSICAL $\mu = 0$ CONJECTURE WITH COATES-SUJATHA CONJECTURE A

DEBANJANA KUNDU

Abstract. The main goal of this article is to provide more evidence on the relationship between the Classical Iwasawa $\mu = 0$ Conjecture and the $\mu = 0$ Conjecture for fine Selmer groups (Conjecture A). We give sufficient conditions to prove the Classical $\mu = 0$ Conjecture that improves upon previously known results. Furthermore, we prove isogeny invariance of Conjecture A in some previously unknown cases. We also provide a class of examples for which Conjecture A holds independent of the Classical $\mu = 0$ Conjecture.

1. Introduction

Classical Iwasawa theory is concerned with the growth of class groups in towers of number fields. In [7], Iwasawa showed that in a $\mathbb{Z}_p$-extension of a number field F, the growth of the p-part of the class group is regular. In particular,

Theorem. *There exist constant non-negative integers λ and μ and a constant integer ν such that for large enough n,*

$$|A_n| = p^{\mu p^n + \lambda n + \nu}$$

where A_n is the class group of F_n, the n-th layer in the tower.

Further, the following conjecture was made.

Classical Conjecture. *For the cyclotomic $\mathbb{Z}_p$-extension $F_{\rm cyc}/F$, $\mu = 0$.*

The conjecture is known to hold for Abelian extensions $F/\mathbb{Q}$ (see [4], [15]).

In [11], Mazur introduced the Iwasawa theory of Selmer groups of Abelian varieties and described the growth of the size of the p^∞-Selmer group in $\mathbb{Z}_p$-extensions F_∞/F. For an Abelian variety A/F, the dual Selmer group over F_∞, denoted by $X(A/F_\infty)$, is a finitely generated $\Lambda(\Gamma)$-module; here $\Gamma = \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p$ and $\Lambda(\Gamma)$ is the associated Iwasawa algebra. However, $X(A/F_\infty)$ is *not always* $\Lambda(\Gamma)$-torsion, depending on the reduction type at p.

When $X(A/F_\infty)$ is $\Lambda(\Gamma)$-torsion, it affords a structure theorem like in the classical case. But an analogue of the classical conjecture is known to be false. For the cyclotomic extension of $\mathbb{Q}$, there are examples of elliptic curves where the associated μ-invariant is positive at a prime of good ordinary reduction.

In [3], Coates and Sujatha studied a subgroup of the Selmer group, called the *fine Selmer group*. They made the following conjecture.

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Conjecture A. *Let p be an odd prime and E be an elliptic curve over a number field F. When $F_\infty = F_{\mathrm{cyc}}$, the Pontryagin dual of the fine Selmer group, denoted $Y(E/F_{\mathrm{cyc}})$, is a finitely generated $\mathbb{Z}_p$-module i.e. $Y(E/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion and the associated μ-invariant, $\mu(Y(E/F_{\mathrm{cyc}})) = 0$.*

A priori, a relation between Galois modules coming from class groups and those coming from elliptic curves is not obvious. However, there is growing evidence that in a cyclotomic tower, fine Selmer groups and class groups have similar growth patterns [3], [10]. We provide more evidence for this by proving the following theorems.

Theorem 1.1. *Let F be a number field such that the Classical Conjecture holds. For E an elliptic curve over F with $E(F)[p] \neq 0$, Conjecture A holds for $Y(E/F_{\mathrm{cyc}})$.*

Remark

(i) [3, Corollary 3.6] is a particular case of out theorem when $F \supset \mu_p$. Indeed, this is a consequence of the following two facts. First, if $F/\mathbb{Q}$ is an Abelian extension, so is $F(\mu_p)/\mathbb{Q}$ and the Classical Conjecture holds. Second, for finite extensions of number fields L/F, Conjecture A for $Y(E/L_{\mathrm{cyc}})$ implies Conjecture A for $Y(E/F_{\mathrm{cyc}})$.

(ii) Tools from Galois cohomology are used to provide new evidence for the Classical Conjecture in Section 4. An application of Theorem 1.1 provides new evidence for Conjecture A.

A converse of Theorem 1.1 is true: given a number field F, the Classical Conjecture holds for F_{cyc}/F, if there exists *one* elliptic curve E/F, with $E(F)[p] \neq 0$ for which Conjecture A holds. It is known that the Classical Conjecture holds if there exists one elliptic curve E/F, with $E(F)[p] \neq 0$ for which the dual *Selmer group* is $\Lambda(\Gamma)$-torsion and the corresponding μ-invariant is 0 (see [2], [9]). Our result weakens the hypothesis significantly and is proved using a different machinery.

Theorem 1.2. *Let E be an elliptic curve defined over the number field F. Let p be any odd prime. Further assume that $E(F)[p] \neq 0$. If Conjecture A holds for $Y(E/F_{\mathrm{cyc}})$, then the Classical Conjecture holds for F_{cyc}/F.*

Theorems 1.1 and 1.2 prove isogeny invariance of Conjecture A in some previously unknown cases (see [17]).

Corollary 1.3. *Let F be a number field contianing μ_p or be a totally real field. Let E and E' be isogenous elliptic curves such that both E and E' have non-trivial p-torsion points over F. Then, Conjecture A holds for $Y(E/F_{\mathrm{cyc}})$ if and only if Conjecture A holds for $Y(E'/F_{\mathrm{cyc}})$.*

Remark. All statements hold for Abelian varieties of dimension d. The only property of cyclotomic $\mathbb{Z}_p$-extensions required in the proofs is that primes in a certain finite set decompose finitely. The theorems are stated for elliptic curves over the cyclotomic $\mathbb{Z}_p$-extension as the original conjectures are in this setting.

2. Preliminaries

Throughout this paper, F will denote a number field and p an odd prime.

Let A/F be a d-dimensional Abelian variety and S be a finite set of primes of F containing the Archimedean primes, primes above p, and primes where A has bad reduction. Fix an algebraic closure $\overline{F}/F$ and set F_S to be the maximal subfield of

$\overline{F}$ containing F which is unramified outside S. Denote the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ by G_F and the Galois group $\mathrm{Gal}(F_S/F)$ by $G_S(F)$.

Definitions of p^∞-Selmer group and p^∞-fine Selmer group are as in [19].

Definition 2.1. *The p-**fine Selmer group**, with respect to the finite set S, is*

$$R_S(A[p]/F) := \ker\left(H^1(G_S(F), A[p]) \to \bigoplus_{v\in S} H^1(F_v, A[p])\right). \tag{1}$$

*The p^∞-**fine Selmer group** is*

$$R(A/F) := \ker\left(H^1(G_S(F), A[p^\infty]) \to \bigoplus_{v\in S} H^1(F_v, A[p^\infty])\right). \tag{2}$$

The definition of $R(A/F)$ is independent of S. For a $\mathbb{Z}_p$-extension F_∞/F,

$$R(A/F_\infty) = \varinjlim_{L} R(A/L)$$

where the inductive limit is over all finite extensions L/F contained in F_∞.

Recall, the Pontryagin dual of a discrete p-primary (resp. compact pro-p) Abelian group is a compact (resp. discrete) module over the assoiated Iwasawa algebra. For G a profinite group and M a G-module, M^G is the subgroup of elements fixed by G and M_G is the largest quotient of M with trivial G action.

Definition 2.2. *For an Abelian group N, its p-**rank** is the $\mathbb{Z}/p\mathbb{Z}$-dimension of $N[p]$, denoted by $r_p(N)$. If G is a pro-p group, write $h_i(G) = r_p\left(H^i(G, \mathbb{Z}/p)\right)$.*

Lemma 2.3. [10, Lemma 3.1] *Let G be a pro-p group and M be a discrete G-module which is co-finitely generated over $\mathbb{Z}_p$. If $h_1(G)$ is finite, $r_p\left(H^1(G, M)\right)$ is finite. Furthermore, the following inequalities hold*

$$\begin{aligned} h_1(G)r_p(M^G) - r_p\left((M/M^G)^G\right) &\le r_p\left(H^1(G, M)\right) \\ &\le h_1(G)(\mathrm{corank}_{\mathbb{Z}_p}(M) + \log_p\left(\left|M/M_{div}\right|\right) \end{aligned}$$

Lemma 2.4. [10, Lemma 3.2] *Consider an exact sequence of co-finitely generated Abelian groups,*

$$W \to X \to Y \to Z.$$

Then

$$\left|r_p(X) - r_p(Y)\right| \le 2r_p(W) + r_p(Z).$$

Definition 2.5. *The p-**Hilbert S-class field** of F, denoted $H_S(F)$, is the maximal Abelian unramified p-extension of F in which all primes in S split completely. By class field theory, the Galois group $\mathrm{Gal}(H_S(F)/F) = \mathrm{Cl}_S(F)$, is the S-class group.*

Lemma 2.6. [10, Lemma 5.2, 5.3] *Let F_{cyc}/F be the cyclotomic $\mathbb{Z}_p$-extension and F_n be the subfield of F_{cyc} such that $[F_n : F] = p^n$. Let A be an Abelian variety over F and S be as defined before. Then*

$$\left|r_p\left(\mathrm{Cl}(F_n)\right) - r_p\left(\mathrm{Cl}_S(F_n)\right)\right| = O(1), \tag{3}$$

$$\left|r_p\left(R_S(A[p]/F_n)\right) - r_p\left(R(A/F_n)\right)\right| = O(1). \tag{4}$$

3. Proof of the Main Results

We first prove Corollary 1.3. It follows from the main theorems in Section 1.

Proof of Corollary 1.3. Let F be a number field that contains μ_p or $F/\mathbb{Q}$ be a totally real field. Let E be an elliptic curve isogenous to E' over F with the additional property that both $E(F)[p], E'(F)[p]$ are non-trivial. WLOG if Conjecture A holds for $Y(E/F_{\mathrm{cyc}})$ then by Theorem 1.2 the Classical Conjecture holds for F_{cyc}/F. Now by Theorem 1.1 Conjecture A holds for $Y(E'/F_{\mathrm{cyc}})$. This proves the corollary. $\square$

3.1. **Proof of Theorem 1.2.** Theorem 1.2 follows from the following lemma when A is an elliptic curve.

Lemma 3.1. *Let F_{cyc}/F be the cyclotomic $\mathbb{Z}_p$-extension and F_n be the subfield of F_{cyc} such that $[F_n : F] = p^n$. Let A be a d-dimensional Abelian variety over F and S be as defined before. Assume $A(F)[p] \neq 0$. Then for some positive constant k_1 that depends on $A(F)[p]$,*

$$k_1 r_p\left(\mathrm{Cl}_S(F_n)\right) \leq r_p\left(R(A/F_n)\right) + O(1) \tag{5}$$

Proof. For the ease of notation, set $H_n = H_S(F_n)$ and $H_{n,w} = H_S(F_n)_w$. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \to & R(A/F_n) & \to & H^1(G_S(F_n), A[p^\infty]) & \to & \bigoplus_{v_n} H^1(F_{n,v_n}, A[p^\infty]) \\
 & & \downarrow r_n & & \downarrow f_n & & \downarrow \gamma_n \\
0 & \to & R(A/H_n) & \to & H^1(G_S(H_n), A[p^\infty]) & \to & \bigoplus_{v_n}\bigoplus_{w|v_n} H^1(H_{n,w}, A[p^\infty])
\end{array}$$

Here v_n runs over all primes in $S(F_n)$, the finite set of primes in F_n that lie above the primes in S. Observe

$$\ker \gamma_n = \bigoplus_{v_n} \ker \gamma_{n,v_n}.$$

Each $\ker \gamma_{n,v_n} = H^1(G_{n,v_n}, A(H_{n,v_n})[p^\infty])$ where G_{n,v_n} is the decomposition group of $G_n := \mathrm{Gal}(H_n/F_n)$. By definition of p-Hilbert S-class field, $G_{n,v_n} = 1$. So, $\ker \gamma_n = \operatorname{coker} \gamma_n = 0$.

By inflation-restriction, $\ker(f_n) = H^1(G_n, A(H_n)[p^\infty])$ and by diagram chasing, one obtains $\ker(f_n) \hookrightarrow R(A/F_n)$. Thus,

$$r_p\left(H^1(G_n, A(H_n)[p^\infty])\right) \leq r_p\left(R(A/F_n)\right).$$

Combining this with Lemma 2.3 gives the following inequality

$$h_1(G_n) r_p\left(A(F_n)[p^\infty]\right) - 2d \leq r_p\left(R(A/F_n)\right). \tag{6}$$

By definition of S-class group, $\mathrm{Gal}(H_n/F_n) = \mathrm{Cl}_S(F_n)$. So

$$\begin{aligned}
h_1(G_n) &= h_1\left(\mathrm{Gal}(H_n/F_n)\right) \\
&= r_p\left(\mathrm{Cl}_S(F_n)/p\right) \\
&= r_p\left(\mathrm{Cl}_S(F_n)\right)
\end{aligned}$$

where the last equality follows from the finiteness of the S-class group. Also,

$$\begin{aligned}
r_p\left(A(F_n)[p^\infty]\right) &\geq r_p\left(A(F)[p^\infty]\right) \\
&= r_p\left(A(F)[p]\right).
\end{aligned}$$

By Inequality 6 and the above discussion it follows,

$$r_p\left(A(F)[p]\right) r_p\left(\mathrm{Cl}_S(F_n)\right) \leq r_p\left(R(A/F_n)\right) + O(1). \tag{7}$$

This proves the lemma as the hypothesis forces $r_p\left(A(F)[p]\right) \neq 0$. □

We now provide a proof of Theorem 1.2.

Proof. By an application of [18, Lemma 13.20], Conjecture A holds for $Y(E/F_{\mathrm{cyc}})$ if and only if $r_p\left(R(E/F_n)\right) = O(1)$. In other words, Conjecture A holds if and only if the p-rank remains bounded in the cyclotomic tower.

By hypothesis, Conjecture A holds for $Y(E/F_{\mathrm{cyc}})$ so $r_p\left(R(E/F_n)\right) = O(1)$. Also by hypothesis, $E(F)[p] \neq 0$. Inequality 7 implies $r_p\left(\mathrm{Cl}_S(F_n)\right)$ is bounded independent of n. By Equation 3, so is $r_p\left(\mathrm{Cl}(F_n)\right)$.

This is enough to prove the Classical Conjecture. Indeed, the Classical Conjecture holds for F_{cyc}/F if and only if $r_p\left(\mathrm{Cl}(F_n)/p\right)$ is bounded independent of n [18, Proposition 13.23]. Since class groups are finite, $r_p\left(\mathrm{Cl}(F_n)\right) = r_p\left(\mathrm{Cl}(F_n)/p\right)$. Thus, the Classical Conjecture is equivalent to $r_p\left(\mathrm{Cl}(F_n)\right)$ being independent of n. This finishes the proof. □

3.2. **Proof of Theorem 1.1.** Recall the following well-known facts [8]. If L/F is a p-power Galois extension, $\mu(F_{\mathrm{cyc}}/F) = 0$ implies $\mu(L_{\mathrm{cyc}}/L) = 0$. On the other hand, for any extension L/F, $\mu(L_{\mathrm{cyc}}) = 0$ implies $\mu(F_{\mathrm{cyc}}) = 0$.

Case (i) Suppose $F \supset \mu_p$.

Since $E(F)[p] \neq 0$, it follows from the Weil pairing that $F(E[p])/F$ is either trivial or cyclic of order p. This is precisely the situation of [10, Theorem 5.5]. There is nothing left to prove.

Case(ii) Suppose $F \not\supset \mu_p$.

Theorem 1.1 follows from the following inequality where k_2 is a positive constant,

$$r_p\left(R(E/F_n)\right) \leq k_2 r_p\left(\mathrm{Cl}(F_n)\right) + O(1). \tag{8}$$

Indeed, the Classical Conjecture is equivalent to $r_p\left(\mathrm{Cl}(F_n)\right)$ being bounded independent of n [18, Proposition 13.23]. If the Classical Conjecture holds, Inequality 8 for an elliptic curve E implies $r_p\left(R(E/F_n)\right)$ is bounded independent of n. Therefor,e Conjecture A holds.

Observe that by Equations 3 and 4, Theorem 1.1 follows from the following variant of Inequality 8,

$$r_p\left(R_S(E[p]/F_n)\right) \leq k_2 r_p\left(\mathrm{Cl}_S(F_n)\right) + O(1). \tag{9}$$

Define $R_S(E(F_n)[p]/F_n)$ by replacing $E[p]$ with $E(F_n)[p]$ in Equation 1. $G_S(F_n)$ acts trivially on $E(F_n)[p]$; hence it is possible to relate $R_S(E(F_n)[p]/F_n)$ with $\mathrm{Cl}_S(F_n)$ and similarly their p-ranks. Since the Galois action is trivial,

$$H^1(G_S(F_n), E(F_n)[p]) = \mathrm{Hom}(G_S(F_n), E(F_n)[p])$$

and there are similar identifications for the local cohomology groups. It follows

$$R_S(E(F_n)[p]/F_n) = \mathrm{Hom}(\mathrm{Cl}_S(F_n), E(F_n)[p]) \simeq \mathrm{Cl}_S(F_n)[p]^{r_p\left(E(F_n)[p]\right)}$$

where the isomorphism is as Abelian groups. This gives the following inequality

$$\begin{aligned} r_p\left(R_S(E(F_n)[p]/F_n)\right) &= r_p\left(E(F_n)[p]\right) r_p\left(\mathrm{Cl}_S(F_n)\right) \\ &\leq 2 r_p\left(\mathrm{Cl}_S(F_n)\right). \end{aligned}$$

Note that Inequality 9 follows from the above inequality provided the p-ranks of $R_S(E[p]/F_n)$ and $R_S(E(F_n)[p]/F_n)$ have the same order of growth in F_{cyc}. This is the content of the following lemma. This completes the proof of Theorem 1.1.

Lemma 3.2. *Let F be a totally real number field. Let E be an elliptic curve over F with $E(F)[p] \neq 0$. Let F_{cyc}/F be the cyclotomic $\mathbb{Z}_p$-extension and suppose the Classical $\mu = 0$ Conjecture holds for F_{cyc}. Let S be as defined before. Then*

$$\left| r_p\left(R_S(E(F_n)[p]/F_n)\right) - r_p\left(R_S(E[p]/F_n)\right)\right| = O(1). \tag{10}$$

Proof. Notice that if $E(F)[p] = E[p]$, then the Lemma is trivial. The focus is on the case $E(F)[p] \neq 0, E[p]$. Set $B_n = E(F_n)[p]$. Consider the commutative diagram

$$\begin{array}{ccccccc}
0 & \rightarrow & R_S(B_n/F_n) & \rightarrow & H^1(G_S(F_n),\ B_n) & \rightarrow & \bigoplus_{v_n} H^1(F_{n,v_n},\ B_n) \\
 & & \Big\downarrow s_n & & \Big\downarrow f_n & & \Big\downarrow g_n \\
0 & \rightarrow & R_S(E[p]/F_n) & \rightarrow & H^1(G_S(F_n),\ E[p]) & \rightarrow & \bigoplus_{v_n} H^1(F_{n,v_n},\ E[p])
\end{array}$$

where v_n runs over all the primes in the finite set $S(F_n)$.

By hypothesis, E has an F_n-rational p-torsion point. This gives the short exact sequence

$$0 \rightarrow B_n \rightarrow E[p] \rightarrow \mu_p \rightarrow 0. \tag{11}$$

This is because, if E has an F_n-rational p-torsion point, this point gives an injection $\mathbb{Z}/p\mathbb{Z} \hookrightarrow E[p]$. Therefore,

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow E[p] \rightarrow M \rightarrow 0.$$

By Cartier duality and the Weil pairing, the above short exact sequence turns into

$$0 \rightarrow M^\vee \rightarrow E[p] \rightarrow \mu_p \rightarrow 0,$$

where μ_p is viewed as a quotient of $E[p]$. Since the Weil pairing is alternating, the orthogonal complement of $\mathbb{Z}/p\mathbb{Z}$ is $\mathbb{Z}/p\mathbb{Z}$, thus $M^\vee = \mathbb{Z}/p\mathbb{Z}$ as a subgroup of $E[p]$.

Taking the $G_S(F_n)$-cohomology of Equation 11, $\ker(f_n) \subseteq H^0(G_S(F_n),\ \mu_p)$. Since μ_p is finite, therefore $r_p\left(\ker(f_n)\right) = O(1)$ and hence $r_p\left(\ker(s_n)\right) = O(1)$. A similar argument for the local cohomology gives $r_p\left(\ker(g_n)\right) = O(1)$.

By Lemma 2.3 applied to the map s_n,

$$\begin{aligned}
\left| r_p\left(R_S(B_n/F_n)\right) - r_p\left(R_S(E[p]/F_n)\right)\right| &\leq 2r_p\left(\ker(s_n)\right) + r_p\left(\operatorname{coker}(s_n)\right) \\
&= r_p(\operatorname{coker}(s_n)) + O(1).
\end{aligned}$$

If $r_p\left(\operatorname{coker}(s_n)\right) = O(1)$, the proof is complete. Observe, $\operatorname{coker}(f_n) \subseteq H^1\left(G_S\left(F_n\right),\ \mu_p\right)$ and $\operatorname{coker}(g_n) \subseteq \bigoplus_{v_n} H^1(F_{n,v_n},\ \mu_p)$. Further, note that

$$r_p\left(\ker\left(H^1\left(G_S\left(F_n\right),\ \mu_p\right) \rightarrow \bigoplus_{v_n} H^1\left(F_{n,v_n},\ \mu_p\right)\right)\right) = O(1) \Rightarrow r_p\left(\operatorname{coker}(s_n)\right) = O(1).$$

For ease of notation, refer to the kernel as a *fine Selmer group*, $R_S(\mu_p/F_n)$.

By hypothesis, the Classical Conjecture holds for F_{cyc}/F. By [17, Proposition 4.10(1)] it is equivalent to $H^2\left(G_S\left(F_{\text{cyc}}\right),\ \mu_p\right) = 0$. This latter statement is often referred to as *Conjecture A for* μ_p. Using equivalent reformulations (see [3], [17]),

$$\begin{aligned} H^2\left(G_S\left(F_{\text{cyc}}\right),\ \mu_p\right) = 0 &\Leftrightarrow \varprojlim_n \left(H^2\left(G_S\left(F_n\right),\ \mu_p\right)\right) \text{ is finite} \\ &\Leftrightarrow R_S\left(\mu_p/F_{\text{cyc}}\right) \text{ is finite} \\ &\Leftrightarrow \varinjlim_n R_S\left(\mu_p/F_n\right) \text{ is finite} \\ &\Leftrightarrow R_S\left(\mu_p/F_n\right) \text{ is finite and bounded} \end{aligned}$$

In particular, $r_p\left(\text{coker}\left(s_n\right)\right) = O(1)$ which is what we needed to prove. □

4. Illustrating the Results with Examples

In this section, we show that the Classical Conjecture holds for p-rational number fields. This allows us to provide evidence for Conjecture A.

Remark. For *totally real* p-rational fields, the classical Iwasawa module is trivial [13]. In particular, classical $\mu = \lambda = 0$ for such fields. It appears that a proof of the Classical Conjecture for *all* p-rational fields has not been written down in literature. We suspect it is because examples of non-Abelian p-rational fields have come to light only recently (see [1]).

Let F be a number field. Let S be a finite set of primes of F *containing* the primes above p and the Archimedean primes. The weak Leopoldt conjecture in the classical setting is the assertion

$$H^2\left(\text{Gal}\left(F_S/F_{\text{cyc}}\right),\ \mathbb{Q}_p/\mathbb{Z}_p\right) = 0. \tag{12}$$

It holds for the cyclotomic extension of a number field [12, Theorem 10.3.25]. If Equation 12 holds for a finite set S as mentioned above, it holds for the set $S = \Sigma = S_p \cup S_\infty$ where S_p is the set of primes of F above p and S_∞ are the Archimedean primes [12, Theorem 11.3.2]. Therefore, the weak Leopoldt Conjecture is independent of the choice of S. From here on, fix $S = \Sigma$.

Let F_{S_p} denote the maximal p-ramifed extension of F. Consider the Galois group $\text{Gal}(F_{S_p}/F)$ and let $\mathcal{G}_{S_p}(F) = \text{Gal}\left(F_{S_p}(p)/F\right)$ be its maximal pro-p quotient.

Definition 4.1. [13] *Let F be a number field, p be an odd prime. F is called p-**rational** if and only if $\mathcal{G}_{S_p}(F)$ is pro-p-free.*

Given a number field F, it is conjectured to be p-rational for all primes outside a set of Dirichlet density 0 [6].

The following theorem is well-known.

Theorem 4.2. [12, Theorem 11.3.7] *The Classical Conjecture holds for F_{cyc} if and only if $\mathcal{G}_\Sigma(F_{\text{cyc}}) = \text{Gal}(F_\Sigma(p)/F_{\text{cyc}})$ is a free pro-p group.*

Definition 4.3. [14, Page 23] *A pro-p group G is free if and only if its p-cohomological dimension $\text{cd}_p(G) \leq 1$.*

By a standard fact in Galois cohomology of pro-p groups [14, Chapter I, Section 4, Proposition 21], an equivalent formulation of Theorem 4.2 is the following: the Classical Conjecture holds for F_{cyc} if and only if

$$H^2(\mathcal{G}_\Sigma(F_{\text{cyc}}), \mathbb{Z}/p\mathbb{Z}) = 0. \tag{13}$$

Corollary 4.4. *Let p be an odd prime and F be a p-rational number field. The Classical Conjecture holds for F_{cyc}/F.*

Proof. Since $p \neq 2$, we replace S_p by Σ in the definition of p-rational fields. For p-rational number fields, $\mathcal{G}_\Sigma(F) = \mathrm{Gal}(F_\Sigma(p)/F)$ has p-cohomological dimension at most 1. Equivalently,

$$H^2(\mathcal{G}_\Sigma(F), \mathbb{Z}/p\mathbb{Z}) = 0.$$

Since $\mathcal{G}_\Sigma(F_{\text{cyc}})$ is a closed normal subgroup of $\mathcal{G}_\Sigma(F)$; by [14, Proposition 14]

$$\mathrm{cd}_p\left(\mathcal{G}_\Sigma(F_{\text{cyc}})\right) \leq \mathrm{cd}_p\left(\mathcal{G}_\Sigma(F)\right) \leq 1.$$

Thus,

$$H^2(\mathcal{G}_\Sigma(F_{\text{cyc}}), \mathbb{Z}/p\mathbb{Z}) = 0.$$

By Equation 13, the result follows. $\square$

This provides new evidence for Conjecture A.

Corollary 4.5. *Let F be a p-rational number field such that either*

(i) $F \supseteq \mu_p$ or

(ii) F is a totally real number field.

Suppose E is an elliptic curve over F with $E(F)[p] \neq 0$. Then Conjecture A holds for $Y(E/F_{\text{cyc}})$.

Proof. This follows from Theorem 1.1 along with Corollary 4.4. $\square$

In special cases, a proof of Conjecture A can be provided *independent* of the Classical Conjecture. For this, we need the following equivalent formulation of Conjecture A, established independently by Greenberg [5] and Sujatha [16].

Proposition 4.6. *Assume the weak Leopoldt conjecture for elliptic curves holds, i.e. $H^2(G_S(F_{\text{cyc}}), E[p^\infty]) = 0$. Then Conjecture A for $Y(E/F_{\text{cyc}})$ is equivalent to the assertion*

$$H^2(G_S(F_{\text{cyc}}), E[p]) = 0.$$

Proposition 4.7. *Let p be an odd prime, F be a p-rational field, and E be an elliptic curve with good reduction everywhere over F (or bad reduction at primes above p) such that $E[p] \subset E(F)$. Then Conjecture A holds for $Y(E/F_{\text{cyc}})$.*

Proof. Choose $S = \Sigma = S_p \cup S_\infty$. By p-rationality of F and the isomorphism of the inflation map [12, Corollary 10.4.8], we have

$$H^2(\mathcal{G}_\Sigma(F), E[p]) = H^2(G_\Sigma(F), E[p]) = 0. \tag{14}$$

By an application of the Hochschild-Serre Spectral sequence, we obtain the following exact sequence [12, Page 119]

$$H^2(\mathcal{G}_\Sigma(F), E[p]) \to H^0(\Gamma, H^2(\mathcal{G}_\Sigma(F_{\text{cyc}}), E[p])) \to 0,$$

where $\Gamma = \mathrm{Gal}(F_{\text{cyc}}/F)$. The first term is 0, thus $H^0\left(\Gamma, H^2(\mathcal{G}_\Sigma(F_{\text{cyc}}), E[p])\right)$ is trivial. Since $H^2(\mathcal{G}_\Sigma(F_{\text{cyc}}), E[p])$ is a discrete module, it must be 0. Once again by the isomorphism of the inflation map,

$$0 = H^2(\mathcal{G}_\Sigma(F_{\text{cyc}}), E[p]) = H^2(G_\Sigma(F_{\text{cyc}}), E[p]).$$

By Proposition 4.6, Conjecture A holds for $Y(E/F_{cyc})$. $\square$

Remark. It appears to be hard to weaken the hypothesis $E[p] \subseteq F$, in proving Conjecture A independent of the Classical Conjecture.

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Department of Mathematics, University of Toronto, Bahen Centre, 40 St. George St., Room 6290, Toronto, Ontario, Canada, M5S 2E4

E-mail address: `dkundu@math.utoronto.ca`