1. Introduction

My research lies at the intersection of number theory, algebra, arithmetic algebraic geometry and complex analysis; more specifically I study the arithmetic of elliptic curves. These are precisely the genus 1 algebraic curves with rational points, and hence land in a sweet spot for mathematical enquiry. The arithmetic of simpler, genus 0, algebraic curves has been studied for centuries. On the other hand, higher genus curves are less accessible. A major application was seen in A. Wiles' proof of Fermat's Last Theorem.

I am interested in Iwasawa theory: an area of number theory that emerged from the foundational work of K. Iwasawa in the 1950s. The main goal is to study growth of arithmetic objects, like Galois modules over infinite towers of number fields. The central theme is that it is often hard to study objects like class groups or Selmer groups in isolation; but in 'nice' families, they are more amenable since their properties stabilize.

A key observation is that a part of this growth exhibits regularity which can be described in terms of values of meromorphic functions known as L-functions, such as the Riemann zeta function. Through such descriptions, Iwasawa theory unveiled intricate links between algebraic, geometric, and analytic objects of arithmetic nature. The existence of such links is a common theme in many areas within arithmetic geometry. So Iwasawa theory has found itself to be a subject of continued great interest. The literature on Iwasawa theory is often technical, but the underlying ideas possess an undeniable beauty!

Iwasawa theory is heavily used in work on the Birch and Swinnerton-Dyer (BSD) conjecture, a Clay Math Millennium Problem. The Iwasawa invariants associated to elliptic curves epitomize their arithmetic and Iwasawa theoretic properties. Moreover, there is a deep relationship between the behaviour of Iwasawa invariants and the p-adic BSD formula. The first positive result to be proved in this direction, the *Coates–Wiles theorem* that analytic rank 0 implies algebraic rank 0 for elliptic curves over $\mathbb{Q}$ with complex multiplication (CM), was shown using Iwasawa theory. More generally, almost all the results on BSD that we now have use the machinery of Iwasawa theory.

My research uses tools from Galois cohomology, module theory, algebraic and analytic number theory, and arithmetic statistics to answer questions on the structure and growth of Selmer groups, fine Selmer groups, and class groups in infinite (and finite) field extensions. My research can be divided into three broad themes.

In §3, I will discuss results on growth questions of *fine* Selmer groups. The notion of *fine Selmer group* was introduced by J. Coates and R. Sujatha in [7]. This is a subgroup of the classical Selmer group obtained by imposing stronger vanishing conditions at primes above p (see §3.1 below). I have made progress towards proving two of the main conjectures (formulated by Coates–Sujatha) which drive the research in this area. In particular, my results indicate that the fine Selmer group "interpolates" between the class group and the Selmer group.

In my work [36], we studied the average behaviour of the Iwasawa invariants for the Selmer groups of elliptic curves, setting out new directions in arithmetic statistics and Iwasawa theory. The broad goal of this program is to understand the variation of the Iwasawa invariants 'on average' as the triple (E, F, p) varies such that $\mathsf{E}_{/F}$ has good reduction at p. In §4, I discuss the progress I have made towards answering questions at the intersection of these two subjects.

I am also interested in studying questions pertaining to rank growth of elliptic curves upon base-change. I primarily use tools from Iwasawa theory to attack such problems. My results towards such questions are discussed in §5.

Finally, in §6 I mention ongoing/ future projects which I plan to pursue in the coming years.

2. Definitions and Notations

Let p be a fixed prime. Consider the *cyclotomic* $\mathbb{Z}_p$-extension of $\mathbb{Q}$, denoted by $\mathbb{Q}_{\text{cyc}}$. Set $\Gamma := \text{Gal}\left(\mathbb{Q}_{\text{cyc}}/\mathbb{Q}\right) \simeq \mathbb{Z}_p$. The *Iwasawa algebra* Λ is the completed group algebra $\mathbb{Z}_p[\![\Gamma]\!] := \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^{p^n}]$ and can be identified with $\mathbb{Z}_p[\![T]\!]$.

Let M be a finitely generated torsion Λ-module. The *Structure Theorem of* Λ*-modules* asserts that M is pseudo-isomorphic to a finite direct sum of cyclic Λ-modules. In other words, there is a map of Λ-modules

$$M \longrightarrow \left(\bigoplus_{i=1}^{s} \Lambda/(p^{m_i})\right) \oplus \left(\bigoplus_{j=1}^{t} \Lambda/(h_j(T))\right)$$

with finite kernel and cokernel. Here, $m_i > 0$ and $h_j(T)$ is a a monic polynomial with non-leading coefficients divisible by p. There are two invariants of study; namely the μ and the λ-invariants of M which are defined as follows:

$$\mu(M) = \mu_p(M) := \begin{cases} 0 & \text{if } s = 0 \\ \sum_{i=1}^{s} m_i & \text{if } s > 0 \end{cases} \qquad \text{and} \qquad \lambda(M) = \lambda_p(M) := \sum_{j=1}^{t} \deg h_j(T).$$

3. Research Focus I: Fine Selmer Groups

A deep result of K. Kato shows that the fine Selmer group of an elliptic curve over $\mathbb{Q}_{\mathrm{cyc}}$ is always Λ-cotorsion, regardless of the reduction type at p, a fact that is *not* true for classical Selmer groups. The fine Selmer group is a fundamental object in the study of Iwasawa theory and plays a crucial role in the reformulation of the Iwasawa Main Conjecture for elliptic curves without p-adic L-functions (see [19, Conjecture 12.10] and [61, Conjecture 7]).

3.1. **Definition of fine Selmer groups.** Fix a number field F, an elliptic curve $\mathsf{E}_{/F}$, and an odd prime p. Let S be a finite set of primes of F containing p, the primes dividing the conductor of E, and the archimedean primes. If L/F is a finite extension and w is a place of L, write L_w for its completion at w; when L/F is infinite, it is the union of completions of all finite sub-extensions of L.

Definition. Let L/F be a finite algebraic extension. The p-primary fine Selmer group is a subgroup of the classical Selmer group with additional vanishing conditions at $v \mid p$. More precisely,

$$0 \to \mathrm{Sel}_0\left(\mathsf{E}/L\right) \to \mathrm{Sel}\left(\mathsf{E}/L\right) \to \bigoplus_{w|p} \mathsf{E}(L_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p.$$

When F_∞/F is an infinite extension,

$$\mathrm{Sel}_0\left(\mathsf{E}/F_\infty\right) = \varinjlim_{L} \mathrm{Sel}_0\left(\mathsf{E}/L\right),$$

where the limit is taken with respect to the restriction maps and L runs over all finite extensions of F in F_∞.

3.2. $\mu = 0$ **Conjecture for fine Selmer groups.** Over the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$, even when the Selmer group is Λ-cotorsion (i.e., at an odd prime p of good *ordinary* reduction), Mazur constructed examples of elliptic curves where the μ-invariant of the Selmer group is *positive*, see [48]. Thus, providing evidence that class groups and Selmer groups often differ in growth behaviour in infinite extensions.

Motivated by the *classical Iwasawa* $\mu = 0$ *conjecture* for the *cyclotomic* $\mathbb{Z}_p$-extension and the observation that the growth behaviour of the fine Selmer group parallels that of the class group [7, Lemma 3.8], Coates–Sujatha formulated the following conjecture. Henceforth, this will be referred to as *Conjecture A*.

Conjecture A. Let p be an odd prime and $\mathsf{E}_{/F}$ be an elliptic curve. Over F_{cyc}/F, the fine Selmer group $\mathrm{Sel}_0(\mathsf{E}/F_{\mathrm{cyc}})$ is Λ-cotorsion and the associated μ-invariant, denoted by $\mu_{\mathrm{fine}}(\mathsf{E}/F_{\mathrm{cyc}})$, is 0.

We are far from proving this conjecture in full generality. In my doctoral thesis, I provided a class of examples where Conjecture A is true (see [28, 34]). More precisely,

Theorem 3.1 (K.). *Let $\mathsf{E}_{/F}$ be the base-change of a rational elliptic curve E. Suppose that $\mathrm{Sel}(\mathsf{E}/F)$ is finite. If E has CM by an order of an imaginary quadratic field K, assume further that the Galois closure of F, denoted by F^c, contains K. Then there exists a set of prime numbers of density at least $\frac{1}{[F^c:\mathbb{Q}]}$ such that Conjecture A holds for $\mathsf{E}_{/F}$.*

I have now extended this to higher rank elliptic curves and primes of good supersingular reduction (see §4).

3.3. **Close relationship with class groups.** In [46], it was shown that the class groups of number fields and fine Selmer groups are closely related in $\mathbb{Z}_p$-extensions where primes are finitely decomposed.

3.3.1. **Arbitrarily large μ-invariant.** In my doctoral thesis, a question of interest was the following: do the growth properties of fine Selmer groups mimic those of class groups in p-adic Lie extensions where primes *are not finitely decomposed*? In [26], I proved that there exist $\mathbb{Z}_p$-extensions of number fields where the μ_{fine}-invariant can be made *arbitrarily large*. This was proven by comparing p-ranks of fine Selmer groups and class groups in towers of number fields, and observing that the μ-invariant is closely related to the p-rank. More precisely,

Theorem 3.2 (K.). *Let $F = \mathbb{Q}(\zeta_p)$ be the cyclotomic field of p-th roots of unity for $p > 2$. Let $\mathsf{E}_{/F}$ be an elliptic curve such that $\mathsf{E}(F)[p] \neq 0$. Given an integer $N \geq 1$, there exists a cyclic Galois extension L/F of degree p and a* non-cyclotomic *$\mathbb{Z}_p$-extension L_∞/L such that $\mu_{\mathrm{fine}}(\mathsf{E}/L_\infty) \geq N$.*

In [27], I developed a strategy to show that the (generalized) μ-invariant of fine Selmer groups can be arbitrarily large in extensions where the (generalized) μ-invariant associated to the class group is arbitrarily large. I provided *explicit examples* of commutative and non-commutative p-adic Lie extensions with arbitrarily large (generalized) μ-invariant of fine Selmer groups. A striking feature in *all* these examples is that there are infinitely many primes which are infinitely decomposed in these extensions. This raises the following question which is worth understanding:

Question: For a $\mathbb{Z}_p$-extension where primes are finitely decomposed, is the classical Iwasawa μ-invariant always 0? More generally, if F_∞/F is a (uniform) pro-p p-adic Lie extension where primes are finitely decomposed, is the (generalized) μ-invariant trivial?

3.3.2. **Anti-cyclotomic $\mathbb{Z}_p$-extension.** Our result in [35] highlights the relationship between the Iwasawa classical μ-invariant and that of fine Selmer groups in the *anti-cyclotomic* $\mathbb{Z}_p$-extension of an imaginary quadratic field, K.

Theorem 3.3 (K.–Ray). *Let E be an elliptic curve defined over an imaginary quadratic field K such that $\mathsf{E}(K)[p] \neq 0$ and the Heegner hypothesis is satisfied. Then, $\mu(K_{\mathrm{ac}}/K) = 0$ if and only if $\mathrm{Sel}_0(\mathsf{E}/K_{\mathrm{ac}})$ is a cotorsion Λ-module with $\mu_{\mathrm{fine}}(\mathsf{E}/K_{\mathrm{ac}}) = 0$.*

3.3.3. **Pseudonullity Conjectures for fine Selmer groups.** A long-standing conjecture of Greenberg (see [14, Conjecture 3.5]) predicts that the unramified Iwasawa module of the compositum of all $\mathbb{Z}_p$-extensions of a number field F is pseudonull; this morally means that the module is smaller than expected. This is referred to as the *Generalized Greenberg's Conjecture*. Motivated by this, Coates–Sujatha formulated an analogue for fine Selmer groups.

Conjecture B. Let $\mathsf{E}_{/F}$ be an elliptic curve and let $\mathcal{L}/F$ be a pro-p p-adic Lie extension of dimension at least 2 which contains F_{cyc}. Suppose that Conjecture A holds for $\mathsf{E}_{/F}$. Then the dual fine Selmer group is a pseudonull module over the Iwasawa algebra $\Lambda\left(\mathrm{Gal}\left(\mathcal{L}/F\right)\right)$.

Progress towards these conjectures has been slow. In [34], joint with F. Nuccio and R. Sujatha, we provide evidence for Conjecture B for elliptic curves with complex multiplication and clarify the relationship between the two conjectures.

3.3.4. $p \neq q$ **Iwasawa theory.** For distinct primes p and q, the p-part of the class number stabilizes in the *cyclotomic* $\mathbb{Z}_q$-extension of an abelian number field, see [59, 60]. These results were extended by J. Lamplugh in [42] to other $\mathbb{Z}_q$-extensions where primes are finitely decomposed.

In [30], we use a theorem of H. Hida on the non-vanishing modulo p of algebraic L-functions to extend the previous results to a class of *anti-cyclotomic* $\mathbb{Z}_p$-extensions. Even in this $p \neq q$ setting, the growth of the p-part of the class group is related to the p-part of fine Selmer group of a fixed elliptic curve E over a $\mathbb{Z}_q$-tower. More precisely,

Theorem 3.4 (K.–Lei). *Let p and q be distinct odd primes. Let F be any number field and let $\mathsf{E}_{/F}$ be an elliptic curve such that $\mathsf{E}(F)[p] \neq 0$. Let F_∞/F be* any *$\mathbb{Z}_q$-extension such that the primes above q and the primes of bad reduction of E are* finitely decomposed. *If there exists $N \gg 0$ such that p-part of the class number stabilizes for all layers $n \geq N$, then $\mathrm{Sel}_0(\mathsf{E}/F_n) = \mathrm{Sel}_0(\mathsf{E}/F_N)$.*

A related problem for exploration by early graduate students: I expect that the strategy explained in §3.3.1 can be combined with the above results. More precisely, it should be possible to show that in $\mathbb{Z}_q$-extensions where primes of bad reduction of E split completely, the fine Selmer group has unbounded (but predictable) growth.

3.4. **Close relationship with classical Selmer groups.** Since the fine Selmer group is a subgroup of the classical Selmer group, they exhibit similar growth behaviour. One instance where this has been observed is in the formula of the cyclotomic λ_{fine}-invariants via a Kida-type formula, see [29].

3.4.1. **Control Theorems.** The *Control Theorem* has a ubiquitous role in Iwasawa theory. For example, it was used to provide the first evidence towards Mazur's co-torsion conjecture.

Via the *Iwasawa Main Conjecture*, the Selmer group $\mathrm{Sel}(\mathsf{E}/F_{\mathrm{cyc}})$ can be related to a p-adic L-function. Therefore, Mazur's Control Theorem provides a channel to extract information on $\mathrm{Sel}(\mathsf{E}/F)$ from the main conjecture thereby providing an invaluable approach towards studying the Birch and Swinnerton-Dyer Conjecture (see [19, 54, 58]). The Control Theorem connects the Selmer groups at the finite layers with that over the infinite tower, allowing one to deduce properties of this arithmetic object over the infinite tower from those at the finite layers, and vice versa.

In [33], which is joint work with M. F. Lim, we prove Control Theorems for fine Selmer groups. More precisely,

(a) we establish estimates on the $\mathbb{Z}_p$-coranks of the kernel and cokernel of the restriction maps

$$r_{F_\infty/F'} : \mathrm{Sel}_0(\mathsf{E}/F') \longrightarrow \mathrm{Sel}_0(\mathsf{E}/F_\infty)^{\mathrm{Gal}\left(F_\infty/F'\right)}$$

for a p-adic Lie extension F_∞/F with intermediate sub-fields F'/F.

(b) we obtain sharper results by specializing to three cases of p-adic Lie extensions: $\mathbb{Z}_p^d$-extensions, multi-false-Tate extensions, and the trivializing extension obtained by adjoining to F all the p-power division points of the elliptic curve, E. In each of these cases, we show (under appropriate assumptions) that the kernel and cokernel of the restriction map are *finite*, and also establish growth estimates for their orders.

Influence: Our results go through almost verbatim for abelian varieties, see [23]. These results have been used to prove new asymptotic formulas for the growth of ideal class groups and fine Selmer groups in multi-$\mathbb{Z}_p$-extensions, see [23]. Ideas from our paper have been extended to modular forms in [24] to study generalized Iwasawa invariants.

3.4.2. **Fine Selmer groups and duality.** In [12], Greenberg established a criteria for when two finitely generated Λ-modules are pseudo-isomorphic. This has been used to show that the Selmer groups of ordinary (resp. non-ordinary) representations satisfy a functional equation in [12] (resp. [20, 1, 44]). A key ingredient is that the local Selmer conditions at p are *exact annihilator of each other.* This is *not true* for fine Selmer groups, since the local conditions at p are trivial. In [17], which is joint work with J. Hatley, A. Lei, J. Ray, we investigated the link between fine Selmer groups of weight k modular forms and their dual. More precisely, let f be a weight $k(\geq 2)$ modular form and $\overline{f}$ be the conjugate modular form. Then,

(a) for an integer i, we proved control theorems for the fine Selmer groups of $f(i)$ and $\overline{f}(k-i)$.
(b) we showed, via global duality and global Euler characteristic formulae, that the criteria established by Greenberg can be reinterpreted in terms of growth conditions on the localization maps.
(c) under hypotheses which could be verified computationally, using the control theorems we showed that the growth conditions on certain localization maps suffice to study the relation between the fine Selmer groups of $f(i)$ and $\overline{f}(k-i)$ over $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$. In particular, if certain naturally arising growth conditions of localization maps are satisfied, the μ-invariants of $\mathrm{Sel}_0(f(i)/\mathbb{Q}_{\mathrm{cyc}})$ and $\mathrm{Sel}_0(\overline{f}(k-i)/\mathbb{Q}_{\mathrm{cyc}})$ are equal.

4. Research Focus II: Arithmetic Statistics and Iwasawa Theory

I will now explain the progress I have made in studying the following interrelated problems.

Question 4.1. (i) For a fixed elliptic curve $\mathsf{E}_{/F}$, how do the Iwasawa invariants vary as p varies over all odd primes p at which E has good reduction?
(ii) For a fixed prime p and fixed number field F, how do the Iwasawa invariants vary as E varies over all elliptic curves (with good reduction at p)?
(iii) Fix an elliptic curve $\mathsf{E}_{/\mathbb{Q}}$ with good reduction at p. How do the Iwasawa invariants of $\mathsf{E}_{/F}$ vary when F varies over a family of number fields?

4.1. **Iwasawa invariants in $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$.** In [36], we started exploring questions at the intersection of arithmetic statistics and Iwasawa theory using the *Euler characteristic.* The Euler characteristic is defined as an alternating product of Galois cohomology groups. By the work of P. Schneider (see [56, 57]) and B. Perrin-Riou (see [51]), this invariant is known to be given by the *p-adic BSD formula* for primes of good ordinary reduction. Thus, it captures information about the size of the Tate–Shafarevich group, the Tamagawa number, the anomalous primes, and the (global) torsion points of the elliptic curve; thereby providing information about the Iwasawa invariants.

4.1.1. As a first step, Theorem 3.1 was extended to *higher rank* elliptic curves and to *supersingular primes.*

When E has *supersingular reduction* at p, the p-primary Selmer group, $\mathrm{Sel}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}})$, is *not* Λ-cotorsion. Instead, one considers the *plus and minus* Selmer groups, denoted by $\mathrm{Sel}^{\pm}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}})$, which were introduced in [25] and are known to be Λ-cotorsion. The associated Iwasawa invariants $\mu^{\pm}$ and $\lambda^{\pm}$ are defined analogously. Under standard hypotheses on the Shafarevich–Tate group, the λ-invariant associated to a (Λ-cotorsion) Selmer group is *always* at least as large as the Mordell–Weil rank of E. Using $\pm$-Euler characteristic formula we proved the following theorem.

Theorem 4.2 (K.–Ray)**.** *Let $\mathsf{E}_{/\mathbb{Q}}$ be an elliptic curve such that* $\mathrm{Sel}(\mathsf{E}/\mathbb{Q})$ *is finite. Then, for* all but finitely many *primes of good supersingular reduction* $\mathrm{Sel}^{\pm}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}})$ *is trivial.*

Since $\mathrm{Sel}_0(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}})$ is a subgroup of $\mathrm{Sel}^{\pm}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}})$ when p is a prime of supersingular reduction, the next result is immediate on combining Theorems 3.1 and 4.2.

Corollary 4.3. *For an elliptic curve $\mathsf{E}_{/\mathbb{Q}}$ with finite* $\mathrm{Sel}(\mathsf{E}/\mathbb{Q})$, *Conjecture A holds for density 1 primes.*

In the *higher rank* setting, the difficulty in answering the question is because of the presence of the (normalized) p-adic regulator term in the Euler characteristic formula. In any case, the following theorem can be proven.

Theorem 4.4 (K.–Ray)**.** *Let $\mathsf{E}_{/\mathbb{Q}}$ be an elliptic curve such that its Mordell–Weil rank, $r_{\mathsf{E}} \geq 1$. Then there exists an explicitly determined set of good ordinary primes such that $\mu(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = 0$ and $\lambda(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = r_{\mathsf{E}}$.*

In the direction of Question 4.1(ii), the following result from [36] allows distinguishing between when the λ-invariant is *exactly equal* to the Mordell–Weil rank and when it is *strictly greater* than the rank.

Theorem 4.5 (K.–Ray)**.** *Let $p \geq 5$ be a fixed prime number. Let $\mathcal{E}(X)$ be the set of isomorphism classes of all elliptic curves over $\mathbb{Q}$ with height $\leq X$. Let $\mathcal{J}(X)$ be the subset of $\mathcal{E}(X)$ containing rank 0 elliptic curves E with good reduction at p, and $\mathcal{Z}(X)$ be a subset for which either of the following hold:*

(*i*) *if E has good ordinary reduction at p, then* $\mathrm{Sel}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = 0$ or
(*ii*) *if E has good supersingular reduction at p, then* $\mathrm{Sel}^{\pm}(\mathsf{E}/\mathbb{Q}^{\mathrm{cyc}}) = 0$.

Then, for an explicit positive *constant $\epsilon(p)$ which approaches 0 as $p \to \infty$,*

$$\limsup_{X\to\infty} \frac{\#\mathcal{Z}(X)}{\#\mathcal{E}(X)} \geq \limsup_{X\to\infty} \frac{\#\mathcal{J}(X)}{\#\mathcal{E}(X)} - \epsilon(p).$$

In view of standard conjectures, one expects that

$$\limsup_{X\to\infty} \frac{\#\mathcal{J}(X)}{\#\mathcal{E}(X)} = \frac{1}{2}\left(1 - \frac{1}{p}\right).$$

Theorem 4.5 indicates that for a *positive proportion* of elliptic curves $\mathrm{Sel}(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = 0$; the proportion approaches $1/2$ as $p \to \infty$. In [37], we refined the results to prove the following:

Theorem 4.6 (K.–Ray)**.** *Let $n > 0$ be an integer and p be an odd prime number. Assume that the Shafarevich–Tate group is finite for all elliptic curves $\mathsf{E}_{/\mathbb{Q}}$. The set of elliptic curves $\mathsf{E}_{/\mathbb{Q}}$ with good* ordinary *reduction at p* and *the additional property that $\mu + \lambda \geq n$, has* positive density *which can be explicitly determined.*

4.2. **Iwasawa invariants in anti-cyclotomic $\mathbb{Z}_p$-extensions.** In [18], which is joint work with A. Ray and J. Hatley, we study the problems raised in Question 4.1 for rank 0 elliptic curves with good *ordinary* reduction at p over the *anti-cyclotomic* $\mathbb{Z}_p$-extensions of an *imaginary quadratic field* in both the *definite* and the *indefinite* setting.

4.2.1. **Definite Case: Heegner hypothesis is *not* satisfied.** This setting was studied by R. Pollack and T. Weston in [53], and in their joint work with C.-H. Kim [21]. Here, the number of bad inert primes is odd, preventing the existence of Heegner points. Consequently, there should be few rational points. Their work confirms this and shows that under various hypotheses, the anti-cyclotomic Selmer group is cotorsion. Hence, the story is somewhat similar to the cyclotomic one. In particular, we prove an Euler characteristic formula for $\mathrm{Sel}(\mathsf{E}/K_{\mathrm{ac}})$.

- In the direction of Question 4.1(i), we prove that for *non-CM elliptic curves*, the exact order of growth for the number of primes at which $\mu = 0$ is closely related to the Lang–Trotter Conjecture.
- In response to Question 4.1(ii) we show that for rank 0 elliptic curves the answer is primarily dependent on the variation of Shafarevich–Tate groups, which can be studied via the heuristics of C. Delaunay.
- Question 4.1(iii) however is more subtle. This question is largely dependent on the divisibility by p of the order of the Shafarevich–Tate group upon base-change to $\mathbb{Q}(\sqrt{-d})$ (as $d > 0$ varies). Even though it appears difficult to provide (unconditional) theoretical results, there is computational data which suggests that "often" large values of p *do not* divide the order of the Shafarevich–Tate group.

Refinements. Question 4.1(iii) was revisited in [39] with the aim to study the average behaviour of Iwasawa invariants at primes of *good reduction* without using the Euler characteristic formula. Here, we measure how mild the assumptions in [21] and [53] are from a statistical point of view. More precisely, we prove that the proportion of such imaginary quadratic fields is halved for each prime of bad reduction that is *split* that would violate the key hypothesis of [21] *were it inert*. The main result is the following.

Theorem 4.7 (K.–Sprung)**.** *Fix a pair $(\mathsf{E}_{/\mathbb{Q}}, p)$ so that*

(*i*) *$\mathsf{E}_{/\mathbb{Q}}$ is an elliptic curve with square-free conductor $N_{\mathsf{E}} = \prod_{i=1}^{r} q_i$, and*
(*ii*) *$p > 3$ is a prime at which E has good reduction, $\overline{\rho}_{\mathsf{E},p}$ is surjective, and $k < r$.*

Then the proportion of imaginary quadratic fields such that $\gcd\left(\left|\operatorname{disc}\mathbb{Q}(\sqrt{-d})\right|, pN_{\mathsf{E}}\right) = 1$*, the prime* p *splits in* $\mathbb{Q}(\sqrt{-d})$*, and* $\mathrm{Sel}^{\pm}(\mathsf{E}/\mathbb{Q}(\sqrt{-d})_{\mathrm{ac}})$ *with associated* μ*-invariant equal to zero is at least is* Λ*-cotorsion is* at least

$$\frac{pN_{\mathsf{E}}}{2^{k+\delta}(p+1)\prod_{q_i \mid N_{\mathsf{E}}}(q_i+1)} \cdot \left(1 - c_p^*\right),$$

with $\delta = 1$ *(or* 2*) when* p *is ordinary (or supersingular). Here,* c_p^* *is related to the Cohen–Lenstra heuristics.*

4.2.2. **Indefinite Case: Heegner hypothesis holds.** When the Heegner hypothesis holds, the p-primary Selmer group $\mathrm{Sel}(\mathsf{E}/K_{\mathrm{ac}})$ is *not* Λ-cotorsion. The theory in this setting is vastly different. Many arguments used in proving the earlier results fail, and their analogues are often false. Importantly, in this setting, there is no known formula for the Euler characteristic of $\mathrm{Sel}(\mathsf{E}/K_{\mathrm{ac}})$. This issue is circumvented by relating this Selmer group to an *auxiliary* Selmer group which is Λ-cotorsion and then using recent progress towards the anti-cyclotomic Iwasawa Main Conjectures to obtain an Euler characteristic formula for the auxiliary Selmer group.

Answering Questions 4.1(i)-(iii) in this setting is deeply intertwined with the theory of the BDP p-adic L-function. In [18], we provide partial answers and supplement the results with computational data.

4.3. **Iwasawa invariants in non-commutative p-adic Lie extensions.** In [31], which is work with A. Lei and A. Ray, we extended the study of average Iwasawa invariants to the non-commutative setting. The key challenge is that in a non-commutative p-adic Lie extension F_∞/F, it is possible for primes other than p to ramify. Given a triple $(\mathsf{E}, p, F_\infty)$, we studied the variation of the structure of the Selmer group $\mathrm{Sel}(\mathsf{E}/F_\infty)$ in three different contexts.

(a) Fix the pair (E, p) and let F_∞ vary over a family of admissible extensions.
(b) Fix the pair (p, F_∞) and let E vary over a subset of elliptic curves $\mathsf{E}_{/\mathbb{Q}}$ of rank 0.
(c) Fix an elliptic curve E and associate to each prime p, an extension F_∞ in a natural way. Then, vary p over the primes at which E has good *ordinary* reduction.

Crucially using the Euler characteristic formula, we studied these three questions in distinct settings.

(1) First, consider the $\mathbb{Z}_p^2$-extension of imaginary quadratic fields. This is a 2-dimensional *abelian* extension and a *metabelian* extension over $\mathbb{Q}$. We prove that this case parallels the cyclotomic theory and in fact, the Euler characteristics for the $\mathbb{Z}_p^2$-extension and the cyclotomic $\mathbb{Z}_p$-extension coincide.
(2) The next step is to work with the simplest non-commutative 2-dimensional p-adic Lie extension, which is the false Tate curve extension. Given primes p and ℓ, write

$$F_\infty^{(\ell)} := \mathbb{Q}(\mu_{p^\infty}, \ell^{\frac{1}{p^n}} : n = 1, 2, \cdots)$$

and set $G = \mathrm{Gal}\left(F_\infty/F\right) \simeq \mathbb{Z}_p \rtimes \mathbb{Z}_p$. The three questions of interest in this case can be explained as follows:
(a) Fix a non-CM elliptic curve E of conductor N_{E} and a prime p of good *ordinary* reduction of E. Consider the family of false Tate curve extensions obtained by varying $\ell \nmid N_{\mathsf{E}}p$. We prove that if $\mathrm{rank}_{\mathbb{Z}}\,\mathsf{E}(\mathbb{Q}(\mu_p)) = 0$, the density of primes ℓ for which the Selmer group is trivial over $F_\infty^{(\ell)}$ is at least $\frac{5}{8}$.
(b) Fix the primes p and ℓ, and let E vary over all elliptic curves defined over $\mathbb{Q}$ ordered by height. To answer the second question we calculate an upper bound for the proportion of elliptic curves for which the Selmer group is *not* trivial. This upper bound has the following shape

$$1 - \prod_{i \geq 1}\left(1 - \frac{1}{p^{2i-1}}\right) + (\zeta(p) - 1) + \frac{(\ell-1)}{2\ell^2} + \zeta(10)C\left(\frac{4\sqrt{\ell}}{p} + 1\right)\sqrt{\ell}\log\ell\,(\log\log\ell)^2,$$

where $C > 0$ is effective constant.
(c) Finally, fix a rank 0 elliptic curve $\mathsf{E}_{/\mathbb{Q}}$, a good prime ℓ, and let p vary over the primes at which E has good *ordinary* reduction. We show that for *at least half* of the primes p, the G-Euler characteristic coincides with the Γ-Euler characteristic. On the other hand, when E has good *supersingular* reduction at ℓ, this happens for *exactly two-third* of the primes p. For such primes p, the Selmer group over the false Tate curve extension is trivial if and only if that over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}(\mu_p)$ is trivial.

5. Applications of Iwasawa Theory and Diophantine Stability

We say that an elliptic curve E is *Diophantine stable* if it does not gain any new points upon base-change. Questions related to Diophantine stability have been explored in great detail by B. Mazur–K. Rubin in [49, 50].

5.1. **Diophantine stability and Iwasawa theory.** In [3], we studied questions pertaining to Diophantine stability using tools from Iwasawa theory. To answer questions on rank jump of elliptic curves upon base change, the natural thing to do was to study the growth of a more tractable arithmetic object, i.e., the p-primary Selmer groups upon base change. In particular, the following two questions were investigated using a Kida-type formula for λ-invariants.

Question 5.1. (i) Given an elliptic curve $\mathsf{E}_{/\mathbb{Q}}$ with trivial p-primary Selmer group, for what proportion of degree-p cyclic extensions does the p-primary Selmer group remain trivial upon base-change.
(ii) Given $p \neq 2, 3$, for how many elliptic curves over $\mathbb{Q}$ does there exist *at least one* $\mathbb{Z}/p\mathbb{Z}$-extension where the p-primary Selmer group remains trivial upon base-change.

The answer to the first question is the following and its proof involves two steps. Using Iwasawa theory, we first controlled the rank growth in $\mathbb{Z}/p\mathbb{Z}$-extensions and then used a result on torsion growth, see [10].

Theorem 5.2 (Beneish–K.–Ray)**.** *Let $p \geq 7$ and $\mathsf{E}_{/\mathbb{Q}}$ be an elliptic curve with $\mu(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = \lambda(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) = 0$. There are infinitely many $\mathbb{Z}/p\mathbb{Z}$-extensions of $\mathbb{Q}$ where the Mordell–Weil group does not grow.*

The following theorem answers Question 5.1(ii).

Theorem 5.3 (Beneish–K.–Ray)**.** *For a* positive proportion *of rank 0 rational elliptic curves, there exists* at least one *$\mathbb{Z}/p\mathbb{Z}$-extension over $\mathbb{Q}$ disjoint from $\mathbb{Q}_{\mathrm{cyc}}$, such that the p-primary Selmer group upon base-change is trivial.*

5.2. **Hilbert's 10th Problem.** In [32], we studied the analogue of Hilbert's 10th Problem for rings of integers of number fields, which is the following question:

Is $\mathbb{Z}$ a Diophantine subset of the ring of integers of a number field L?

The strategy to prove such results is inspired by the work of N. Garcia-Fritz–H. Pasten, see [9]. However, there are some important differences which will be explained after mentioning the key ingredients that go into the proof.

Improving the results of Garcia-Fritz–Pasten. Here are the main differences between our work and previous results:

(I) Using a result from our previous work [31, Section 8 and Appendix], we refine the results of [9] and improve the densities of both $\mathcal{P}$ and $\mathcal{Q}$.

Theorem 5.4 (K.–Lei–Sprung)**.** *There are Chebotarev sets of primes $\mathcal{P}$ and $\mathcal{Q}$, of density $\frac{9}{16}$ and $\frac{7}{48}$, such that for all $p \in \mathcal{P}$ and $q \in \mathcal{Q}$, the analogue of Hilbert's 10th Problem is unsolvable for the ring of integers of $\mathbb{Q}(\sqrt[3]{p}, \sqrt{-q})$.*

(II) We provide a direct proof of the main theorem of [9]. We prove the vanishing of certain 3-Selmer groups (rather than the finiteness of 3^∞-Selmer group). This allows significant weakening of the hypotheses and the possibility to find *many auxiliary elliptic curves* (not necessarily of positive minimal discriminant).

The direct approach permits working with the congruent number curve with good supersingular reduction at 3.

Theorem 5.5 (K.–Lei–Sprung)**.** *There is a Chebotarev set of primes $\mathcal{P}$ with density $\frac{11}{16}$ such that Hilbert's 10th Problem is unsolvable for the ring of integers of $\mathbb{Q}(\sqrt[3]{p}, \sqrt{q})$ whenever q is a congruent number.*

Further Questions:

(i) Find a method that generates infinitely many such auxiliary curves.
(ii) Is it then possible to ensure that the resulting $\mathcal{Q}$ is non-empty? Or even have positive density?

6. Ongoing and Future Projects

In this section, I outline some ongoing and future problems which I will be interested in working on. The first *three* projects are ongoing work. For the *two* projects outlined after that, I have concrete ideas of how to proceed. I will be taking up these projects once the ongoing articles are completed. The last *five* questions will be interesting to work on (in my opinion) and are natural questions that have arisen from my previous work. I have not spent much time thinking about these questions and do not have a plan of attack for some of them.

6.1. **Modelling λ-Invariants for anti-cyclotomic $\mathbb{Z}_p$-extensions.** In [40], we study the following question: how often is the λ-invariant (of the anti-cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field) equal to m?

If p does not divide the class number of K and p is inert in K, then there is no class group growth up the $\mathbb{Z}_p$-extension. In particular, $\mu = \lambda = \nu = 0$. Therefore, the case worth investigating is when p divides the class number of K.

A key feature distinguishing the cyclotomic case from the anti-cyclotomic one is that the Hilbert class field of K may have non-trivial intersection with K_{ac}/K. Based on computer experiments we propose the following two models.

Conjecture (Intersection Heuristics)**.** *Fix a finite abelian p-group, G. Let K vary over imaginary quadratic fields such that p is non-split in K and the p-part of its Hilbert class group is isomorphic to G. Let $L(K)$ denote the p-Hilbert class field of K. The probability that $L(K)\cap K_{\mathrm{ac}} = K_n$ is given by*

$$\frac{\#(\textit{surjective homomorphisms } G \longrightarrow \mathbb{Z}/p^n\mathbb{Z})}{\#G}$$

For the anti-cyclotomic Iwasawa invariants we make the following prediction.

Conjecture (Invariants Heuristics)**.** *Among the imaginary quadratic fields K in which p does not split, the proportion for which the μ and the λ-invariant is 0 is at least*

$$\left(1-\frac{1}{p}\right)+\frac{1}{p-1}\prod_{j\geq 1}\left(1-p^{-j}\right).$$

For $p=3$, this predicts that $\lambda=0$ at least 94% of the time, and for $p=5$ at least 99% of the time. It is hard to imagine that there is such a small number of actual exceptions. This raises the following question.

Question: Is $\lambda(K_{\mathrm{ac}}/K)=\mu(K_{\mathrm{ac}}/K)=0$ always true for an imaginary quadratic field K in which p does not split?

In addition, we prove the following result which asserts that the p-Hilbert class field of the n-th layer of the anti-cyclotomic extension, denoted by A_n, is non-cyclic if A_0 is non-trivial and disjoint from the anti-cyclotomic tower.

Theorem 6.1 (K.–Washington)**.** *Let K be an imaginary quadratic field and let p be an odd prime that does not split in K. If the p-Hilbert class field of K is non-trivial and disjoint from K_{ac}, then A_n is non-cyclic for all $n\geq 1$.*

Next steps: We now hope to extend our results and heuristics to the case $p=2$.

6.2. **Congruent modular forms and jump in λ-invariant.** In [15], R. Greenberg and V. Vatsal proved the following remarkable theorem: if $\mathsf{E}_1,\mathsf{E}_2$ are two elliptic curves defined over $\mathbb{Q}$ with good ordinary reduction at p, $\mathsf{E}_1[p]\simeq\mathsf{E}_2[p]$ as Galois modules, and these modules are irreducible, then

$$\lambda(\mathsf{E}_1/\mathbb{Q}_{\mathrm{cyc}})+\sum_{\ell\in S_0}\sigma_{\mathsf{E}_1}(\ell)=\lambda(\mathsf{E}_2/\mathbb{Q}_{\mathrm{cyc}})+\sum_{\ell\in S_0}\sigma_{\mathsf{E}_2}(\ell).$$

Here, S_0 is a finite set of primes containing ∞, the primes of bad reduction of E_1 and E_2, and not containing p. The non-negative integers $\sigma_{\mathsf{E}_i}(\ell)$ are determined by the Euler factor of $L(\mathsf{E}_i/\mathbb{Q},s)$ at ℓ. For the case of elliptic curve, it is now conceptually understood that two congruent elliptic curves have different λ-invariants, precisely when they lie on two branches of the associated Hida family with different ramification behaviour.

These results were then extended to modular forms in [8]. A natural question of interest is to quantify and understand from a statistical point of view, *how often* do the λ-invariants of congruent modular forms differ? More precisely, consider the following set-up: Let p be an odd prime number. Let $\mathsf{E}_{/\mathbb{Q}}$ be an elliptic curve and f be the associated weight 2 modular form of level N (coprime to p). Combining results of H. Carayol and F. Diamond–R. Taylor, one can write down all the possible levels M of weight 2 modular forms g such that f,g are congruent modulo p.

Question: Varying over all possible pairs (g,M) arising from the works of Carayol and Diamond–Taylor, for what proportion of pairs is $\lambda(f)\neq\lambda(g)$?

In [16], we take the first steps towards understanding this. Here is a sample result in a simplified setting:

Theorem 6.2 (K.–Hatley)**.** *Let $\mathsf{E}_{/\mathbb{Q}}$ be an elliptic curve with good ordinary reduction at $p\geq 3$ such that $\lambda(\mathsf{E})=0$ and any prime ℓ of bad reduction satisfies one of the following conditions:*

(*i*) *ℓ is a prime of additive reduction.*
(*ii*) *if ℓ is a prime of split multiplicative reduction then $\ell\not\equiv 1 \pmod p$.*
(*iii*) *if ℓ is a prime of non-split multiplicative reduction then $\ell\not\equiv -1 \pmod p$.*

Then, there exists a set of prime numbers $\mathcal{R}_1$ of density $\frac{(p-3)}{(p-1)^2}$ and another set of prime numbers $\mathcal{R}_2$ of density $\frac{p^2-p-1}{(p-1)(p^2-1)}$ such that for every integer M of the form

$$M=N\prod_{\ell_1\in\mathcal{R}_1}\ell_1^{\alpha(\ell_1)}\prod_{\ell_2\in\mathcal{R}_2}\ell_2^{\alpha(\ell_2)}\ \textit{where } 0\leq\alpha(\ell_1)\leq 1 \textit{ and } 0\leq\alpha(\ell_2)\leq 2, \tag{1}$$

there exists a modular form f of level M with $\overline{\rho}_{\mathsf{E}}\simeq\overline{\rho}_f$ and $\lambda(\mathsf{E})=\lambda(f)$.

Next steps: It seems possible that several of the hypotheses included to simplify calculations can be relaxed. In a follow-up project, we will weaken the hypotheses and consider applications.

6.3. **An Euler system for the symmetric square of a Hida family.** In my efforts to learn about Euler systems and Hida families, I have started a project with J. Ray and S. Vigni. The main goal is to prove the existence of an Euler system for the symmetric square of a Hida family and prove one-divisibility of the Iwasawa Main Conjecture.

As a first step, we have already proven the existence of a non-trivial Euler system using ideas and methods introduced in [22, 47]. We are currently working on constructing a Kolyvagin system from the Euler system. This requires carefully verifying the set of hypotheses laid out by Mazur–Rubin which guarantees this translation. A crucial input will be proving a *big image* result in the symmetric square setting. The final step is to conclude one-divisibility of the Iwasawa Main Conjecture, which should follow using standard techniques once the Kolyvagin system has been constructed. Our work depends on a recent work on the Iwasawa Main Conjecture for Hida families [5].

6.4. **Stabilization of rank in cyclotomic $\mathbb{Z}_p$-extensions.**

Question (Greenberg)**.** *Let p be a fixed prime number and let n_0 denote the minimum integer such that for all $n \geq n_0$ in $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$, the following equality holds*

$$\operatorname{rank}_{\mathbb{Z}} \mathsf{E}(\mathbb{Q}_n) = \operatorname{rank}_{\mathbb{Z}} \mathsf{E}(\mathbb{Q}_{n_0}).$$

Can n_0 be chosen independently of E and/or p?

In [6], G. Chinta proved that n_0 can be chosen independently of p. The question of whether n_0 can be chosen independently of E is still open. It would be interesting to answer Greenberg's question in full generality. A modest first step is to give a description of n_0 in terms of p and E. Here is a brief plan to study this question:

(1) Let $\mathbb{Q}_{(n)}$ denote the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Recall [13, Exercise 1.9] that for $p > 2$

$$\operatorname{rank}_{\mathbb{Z}} \mathsf{E}(\mathbb{Q}_{(n)}) \equiv \operatorname{rank}_{\mathbb{Z}} \mathsf{E}(\mathbb{Q}_{(n-1)}) \pmod{\phi(p^n)}.$$

In other words, there exist integers $d_i \geq 0$ such that

$$\operatorname{rank}_{\mathbb{Z}} \mathsf{E}(\mathbb{Q}_{(n)}) = \operatorname{rank}_{\mathbb{Z}} \mathsf{E}(\mathbb{Q}) + \sum_{i\geq 1}^{n} (d_i p^{i-1})(p-1).$$

(2) In $\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}$, it is easy to show that

$$\lambda(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) \geq \operatorname{rank}_{\mathbb{Z}} \mathsf{E}(\mathbb{Q}) + \sum_{i\geq 1}^{n_0} (d_i p^{i-1})(p-1).$$

(3) It is then possible to show that n_0 is the maximum of all N such that $\frac{\lambda(\mathsf{E}/\mathbb{Q}_{\mathrm{cyc}}) - \operatorname{rank}_{\mathbb{Z}} \mathsf{E}(\mathbb{Q})}{(p-1)p^N} \geq 1$.

(4) The aim would then be to use Theorem 4.6 to make a precise statement.

6.5. **Filtration of fine Selmer groups.** Fine Selmer groups interpolate between the classical Selmer groups and the class groups. In ongoing joint work with V. K. Murty, we are exploring whether there is a possible filtration of (higher) fine Selmer groups which can further refine these interpolation properties.

More precisely: Let K be a number field. The *p-primary fine Selmer group*, can be viewed in the following way

$$\operatorname{Sel}_0(\mathsf{E}/K) = \operatorname{Sel}^{(G_v)}(\mathsf{E}/K) \simeq \ker\left(\operatorname{Sel}(\mathsf{E}/K) \longrightarrow \bigoplus_{v|p} H^1\left(G_v, \mathsf{E}[p^\infty]\right)\right).$$

We know that there is a filtration of filtration of Galois groups at $v \mid p$,

$$G_v \supseteq I_v = I_{v,0} =: I_0 \supseteq I_{v,1} =: I_1 \supseteq \ldots \supseteq .$$

Here, I_v is the *inertia subgroup* and I_1 is the *tame inertia.* Now, we define a filtration of the p-primary Selmer group which respects the above filtration. Define the *j-th filtered (p-primary) Selmer group* as follows

$$\operatorname{Sel}^{(I_j)}(\mathsf{E}/K) := \ker\left(\operatorname{Sel}(\mathsf{E}/K) \longrightarrow \bigoplus_{v|p} \frac{H^1\left(G_v, \mathsf{E}[p^\infty]\right)}{H^1\left(G_v/I_j, \mathsf{E}(\overline{K_v}^{I_j})[p^\infty]\right)}\right).$$

Therefore, we have the filtration

$$\operatorname{Sel}_0(\mathsf{E}/K) \subseteq \operatorname{Sel}^{(I_0)}(\mathsf{E}/K) \subseteq \operatorname{Sel}^{(I_1)}(\mathsf{E}/K) \subseteq \ldots \subseteq \operatorname{Sel}(\mathsf{E}/K).$$

In order to understand whether this filtration is in fact 'meaningful' we have started thinking about the cohomology groups of units (of local fields) in wildly ramified extensions.

6.6. **Density of primes of supersingular reduction for non-CM elliptic curves.** When $\mathsf{E}_{/F}$ is a non-CM elliptic curve, one can show using the original ideas J. P. Serre that there are density 0 primes of supersingular reduction *in* F. However, I believe that the answer to the following question is most likely not known.

Question: What proportion of primes in $\mathbb{Q}$ lie below the supersingular primes of a non-CM elliptic curve $\mathsf{E}_{/F}$?

Heuristics suggest that the primes in $\mathbb{Q}$ should also be of density 0. But this is not obvious (at least to me) because non-split primes in F have density 0, but primes lying under the non-split primes have positive density.

6.7. **Class groups of Kummer extensions.** Tools from Galois cohomology have been used to study the p-ranks of class groups in extensions of the form $\mathbb{Q}(N^{1/p})$ where $N \equiv 1 \pmod{p}$ is a prime, see [55, 43]. I want to study how the p-rank of $\mathbb{Q}(\zeta_p, N^{1/p})$ varies with varying N. It is easy to get an upper and a lower bound for this p-rank. This upper bound is independent of N, see [11, Proposition 2.6]. A question to ask is the following

Question: Are all the values in between actually be attained as N varies? If yes, how often is each value attained?

6.8. **Arithmetic statistics of fine Selmer groups.** An obstruction to studying the arithmetic statistics of fine Selmer groups using the machinery discussed in Section 4 is that the Euler characteristic formula is not as neat. This is because the defining sequence of the fine Selmer group over the cyclotomic $\mathbb{Z}_p$-extension is *not* short exact.

It is reasonable to expect that the one should be able to extend the methods and understand what happens on analytic side of the Iwasawa theory "on average". A concrete problem is the following: in [45], it was shown that for an elliptic curve $\mathsf{E}_{/\mathbb{Q}}$ such that p is a prime of supersingular reduction, the μ-invariant of the fine Selmer group is 0 if and only if p does not divide the gcd of the $\pm$ p-adic L-functions. One expects (see [41, Problem 3.2]) that the gcd of the pair of p-adic L-functions is exactly

$$X^{\operatorname{rank} \mathsf{E}(\mathbb{Q})} \prod_{e_n \geq 1, n \geq 1} \Phi_n^{e_n - 1}.$$

This problem is closely related to one raised by Greenberg [41, Problem 0.7]. I want to understand if the methods and results attained on the Euler characteristic of $\pm$-Selmer groups can be used to understand either of these analytic questions. This might allow us to say something more about Conjecture A in the case of supersingular primes.

6.9. **Refinement of Iwasawa invariants via p-adic L-function.** Let p be a fixed odd prime.

Question: As E varyies over all elliptic curves over $\mathbb{Q}$ and $m > 0$ is fixed, how often is $\lambda(\mathsf{E}/\mathbb{Q}_{\text{cyc}}) = m$?

Answering this question will involve refining the methods of [36, 37]. Even though a complete answer might be out of reach, it would be worthwhile to carry out computations and establish reasonable heuristics.

In [4], F. Bianchi has calculated the sub-leading term of the p-adic L-function of a given elliptic curve $\mathsf{E}_{/\mathbb{Q}}$ in terms of the leading term. I would like to understand if this is likely to have some bearing on the Iwasawa invariants.

6.10. **λ-invariants of fine Selmer groups in anti-cyclotomic extensions.** As pointed out in Section 3.4, the fine Selmer group often behaves like the classical Selmer group. In [29], I showed that in special cases, the cyclotomic λ_{fine}-invariants satisfy a Kida-type formula. On the other hand, in [2, 38] we studied a Kida-type formula for the classical Selmer group in more general situations (extending the work of [52]).

Question: Does the λ_{fine}-invariant satisfy a Kida-type formula in more general $\mathbb{Z}_p$-extensions?

This involves overcoming obvious hurdles since the defining sequence over a $\mathbb{Z}_p$-extension is not short exact.

In [38], we also showed that the anti-cyclotomic λ-invariant of Selmer groups can become arbitrarily large. This raises another obvious question which will also involve some new ideas for solving:

Question: Can the cyclotomic λ_{fine}-invariant be arbitrarily large as E varies over all elliptic curves over $\mathbb{Q}$?

The standard approach is to find a family of elliptic curves which are isomorphic mod p to a given elliptic curve and then use results of [15] to compare their λ-invariants. Since fine Selmer groups can contain proper finite index Λ-submodules, the results of [15] do not generalize in an obvious way for fine Selmer groups.

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