MSRI Summer School: Automorphic forms and the Langlands program

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1 Lecture One

We begin with a brief overview of some of the topics to be covered in this summer school.

Let $N \in \mathbb{Z}_{\geq 1}$ be an integer and $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$ a Dirichlet character modulo N, i.e., a group homomorphism. Attached to χ is a Galois representation ρ_χ of the absolute Galois group

$$\rho_\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_1(\mathbb{C}),$$

arising from the universal property of quotients; specifically, we take the unique lift $\bar{\chi} \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which equals the composition of χ with the surjection $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}\big(\mathbb{Q}(\zeta_N)/\mathbb{Q}\big) = (\mathbb{Z}/N\mathbb{Z})^*$, where ζ_N is a primitive Nth root of unity, the isomorphism given by

$$n \in (\mathbb{Z}/N\mathbb{Z})^\times \longmapsto (\zeta_N \mapsto \zeta_N^n).$$

In our notation, $=$ will denote a canonical isomorphism.

In particular, suppose f is a cuspidal modular form for GL_2 which is a simultaneous eigenvector for the Hecke operators T_p; we will write

$$T_p f = \lambda_p f, \lambda_p \in \mathbb{C}.$$

It happens that the subfield E_f of $\mathbb{C}$ generated by $\{\lambda_p : p \text{ prime}\}$ is a number field.

If $\ell \in \mathbb{Z}$ is prime and $\lambda | \ell$ is a prime of E_f, then (following a suggestion of Serre) Deligne was able to prove a two-dimensional analogue of the above; namely, the construction of the map

$$\rho_f : \mathrm{Gal}\big(\overline{\mathbb{Q}}/\mathbb{Q}\big) \to \mathrm{GL}_2\big(\overline{E_{f,\lambda}}\big),$$

where $E_{f,\lambda}$ is the localization of E_f at λ.

The representation ρ_f is "attached to f" in some way. More precisely: if f has level $N \geq 1$, weight $k \geq 1$, and (Dirichlet) character χ, then ρ_f is unramified away from $N\ell$. Furthermore, if p is prime, $p \nmid N\ell$, then $\rho_f(\mathrm{Frob}_p)$ has characteristic polynomial

$$X^2 - \lambda_p X + p^{k-1}\chi(p).$$

The Chebotarev density theorem implies that there exists *at most* one semisimple ρ_f with this property.

For $k \geq 2$, Deligne constructed ρ_f using étale cohomology with non-trivial coefficients; for $k = 1$, Deligne and Serre used another method.

Certain questions arise from Deligne's construction. For instance, if $p | N\ell$, what does ρ_f look like locally at p? We break into two cases:

1. Suppose $p|N, p \neq \ell$. Then the answer comes from the so-called **local Langlands correspondence**. This will be explained (hopefully) later this week.

2. Suppose $p = \ell$; then we will use the p-adic local Langlands correspondence.

We consider now an easier variant on the same theme. Instead of asking for the representation ρ_f itself, we might ask for its reduction modulo ℓ, i.e. the map

$$\overline{\rho_f} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}_\ell}),$$

recalling that $\overline{\mathbb{F}_\ell}$ is the residue field of $\overline{E_{f,\lambda}}$ at λ. It happens that $\overline{\rho_f}$ lies in the ℓ-torsion of an appropriate abelian variety.

We might reasonably ask the question: are ρ_χ and ρ_f special cases of a more general phenomenon?

> **Theorem 1.1.** *(Harris, Lan, Taylor, Thorne, 2013; Scholze, 2017?) Let E be a totally real or CM number field, π a cuspidal, automorphic representation of $GL_n(\mathbb{A}_E)$. Assume that π_∞ is "cohomological" (to be thought of for now as a strong "algebraicity" assumption); then there exists some representation*
>
> $$\rho_\pi : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$$
>
> *attached to π in some canonical way.*

This theorem is analogous to giving the characteristic polynomial of $\rho_f(\mathrm{Frob}_p)$; more details on all these objects will follow in the coming lectures.

The constructions we have seen so far follow a general theme: namely, to associate to a given algebraic or analytic object (i.e. χ, f, π) a representation of a Galois group (i.e. $\rho_\chi, \rho_f, \rho_\pi$). We might ask: is it possible to classify the image? That is, given a representation ρ of some Galois group in GL_n, does there exist some representation ρ_f (say) arising from the algebraic/analytic object, which is isomorphic to ρ?

We will begin by considering the one-dimensional case. Say $K/\mathbb{Q}$ is a finite Galois extension with one-dimensional representation

$$\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_1(\mathbb{C}).$$

When is this ρ isomorphic to some ρ_χ, for χ a Dirichlet character?

By the universal property of quotients, we may assume that ρ is injective; otherwise we replace K by an appropriate subfield. Because the image of ρ lies in an abelian group, it follows that $\mathrm{Gal}(K/\mathbb{Q})$ is itself abelian – we say that K is an **abelian** extension of $\mathbb{Q}$. If

$$\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = \mathrm{Gal}\big(\mathbb{Q}(\zeta_N)/\mathbb{Q}\big) \to \mathbb{C}^\times$$

is the character above and ρ_χ as before, then ρ_χ gives rise to an injection

$$\mathrm{Gal}(L/\mathbb{Q}) \hookrightarrow \mathbb{C}^\times,$$

for some subfield L of $\mathbb{Q}(\zeta_N)$.

Our question now becomes: If K is a number field which is an abelian Galois extension of $\mathbb{Q}$, does there exist a positive integer N such that $K \subseteq \mathbb{Q}(\zeta_N)$?

> **Theorem 1.2.** *(Kronecker-Weber theorem; "explicit case of global class field theory"): The answer to the above question is "yes."*

Thus: for all continuous $\rho : \mathrm{Gal}\big(\overline{\mathbb{Q}}/\mathbb{Q}\big) \to \mathrm{GL}_1(\mathbb{C})$, there exists a character $\chi : \big(\mathbb{Z}/N\mathbb{Z}\big)^\times \to \mathbb{C}^\times$ such that $\rho \cong \rho_\chi$.

Moving to the two-dimensional case: suppose f is a cuspidal modular eigenform as before, then the associated representation $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$ has the following properties:

1. ρ_f is absolutely irreducible.
2. ρ_f is **odd**, i.e., $\det(z \mapsto \bar{z}) = -1$.
3. ρ_f is unramified away from a finite set of primes, and is *potentially semi-stable* (a p-adic Hodge-theoretic condition) at ℓ.

The third property is sometimes abbreviated by saying ρ_f is **geometric**, but we probably won't use this terminology.

In the early 1990s, Fontaine and Mazur asked the converse question: If $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}_l})$ satisfies the conditions 1., 2., and 3., then is $\rho \cong \rho_f$ for some f?

The answer to this question is now basically known by the work of Kisin and Emerton, but is still being consolidated and published; see "The Fontaine-Mazur conjecture for GL_2" and "Local-global compatibility in the p-adic Langlands program for GL_2," both of which written circa 2012.

In the general case, we ask: If $\rho : \mathrm{Gal}(\overline{E}/E) \to \mathrm{GL}_n(\overline{\mathbb{Q}_l})$ satisfies certain assumptions, is it the case that $\rho \cong \rho_\pi$ as in theorem 1.2?

This is proven by Barnet-Lamb, Gee, Geraghty, and Taylor for many cases, but not generally.

Starting with the next talk, we will begin to include far more details and computations.

2 Lecture Two

The aim of this course is to give us a feeling for the statements of the theorems that we just saw.

We'll see precise definitions and examples, and learn to work with these objects. And maybe we'll work with groups other than GL_n.

2.1 Part 1: the local Langlands Correspondence

(The first lecture was global.)

The local Langlands correspondence for GL_n / K is, vaguely speaking, a canonical bijection between

1. Certain typically infinite-dimensional irreducible $\mathbb{C}$-representations of $\mathrm{GL}_n(K)$, and
2. certain n-dimensional complex representations of a group related to $\mathrm{Gal}(\overline{K}/K)$.

If $n = 1$, this is the so-called "local class field theory".

Remark. Local class field theory has sort of a funny history. Most of the results in global class field theory were proved in the early 20th century, before p-adic numbers had really been discovered. Various local results were deduced from the global results, but then people realized that it was more natural to prove the local statements first and then deduce the global statements.

When $n > 1$, this is the local Langlands conjectures for GL_n / K, which are a 2000 theorem of Harris and Taylor (proofs are global).

Now we'll start being rigorous.

2.1.1 Infinite Galois groups

Let's remind ourselves of what happens in the finite case. Let K be any field, and say L/K is a finite extension. We say that L/K is *Galois* if L/K is normal and separable.

Then $\mathrm{Gal}(L/K)$ is the group of field automorphisms of L fixing K pointwise, with $\#\mathrm{Gal}(L/K) = \dim_K(L)$ (i.e. the dimension of L as a K-vector space).

Furthermore, there is an inclusion-reversing correspondence (the *Galois correspondence*) between subgroups $H \subseteq \mathrm{Gal}(L/K)$ and intermediate extensions $M, K \subseteq M \subseteq L$.

Alright, now say that K is a field and L/K is an algebraic extension of possibly infinite degree. We're going to convince ourselves that

$$\mathrm{Gal}(L/K) \simeq \varprojlim \mathrm{Gal}(M/K),$$

where M runs over the finite Galois extensions of K in L.

We say that L/K is Galois if it's normal and separable. Set $\mathrm{Gal}(L/K)$ to be the field automorphisms $\varphi : L \to L$ such that ϕ fixes K pointwise.

The key observation about the (possibly infinite) Galois group is that if φ is in $\mathrm{Gal}(L/K)$, then φ is determined by its values on the elements of L, each of which lies in a *finite* extension of K.

That is, if $\lambda \in L$, then there exists $M, L \supseteq M \supseteq K$ such that M/K is finite and Galois and $\lambda \in M$.

In particular, there is a canonical quotient map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(M/K)$ and $\varphi(\lambda)$ is determined by the image of φ in $\mathrm{Gal}(M/K)$.

In particular, φ is determined by $\varphi|_M$ for all $M : L \supseteq M \supseteq K$, where M is finite and Galois over K.

Therefore, this induces an embedding

$$\mathrm{Gal}(L/K) \hookrightarrow \prod_{\substack{L \supseteq M \supseteq K \\ M/K \text{ finite, Galois}}} \mathrm{Gal}(M/K).$$

The product topology induces a nontrivial subspace topology on $\mathrm{Gal}(L/K)$ (the *Krull topology.*) (This is really just the topology of pointwise convergence.)

Exercise 1. *Check that*$\mathrm{Gal}(L/K)$ *is a closed subspace of this product.*

Of course, no recap of Galois theory is complete without mentioning the fundamental theorem of Galois theory: there is an inclusion-reversing bijective correspondence between the closed subgroups of $\mathrm{Gal}(L/K)$ and the intermediate fields $L \supseteq M \supseteq K$.

Example 1. Here are some examples.

1. Let $K = \mathbb{Q}$ and let $L = \bigcup_{n \geq 1} \mathbb{Q}(\zeta_{p^n})$, where ζ_{p^n} is $e^{2\pi i/p^n}$, p prime. Set $L_n = \mathbb{Q}(\zeta_{p^n})$. We know that $\mathrm{Gal}(L_n/\mathbb{Q})$ is $(\mathbb{Z}/p^n\mathbb{Z})^\times$, and so

$$\mathrm{Gal}(L/\mathbb{Q}) \hookrightarrow \prod_{n \geq 1} (\mathbb{Z}/p^n\mathbb{Z})^\times,$$

sending $\varphi \mapsto (\varphi_n)_{n\geq 1}$. There is an infinite tower

$$\mathbb{Q} \subseteq L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots \subseteq L_n \subseteq \ldots,$$

with $\bigcup L_i = L$ and each $\varphi_n : L_n \to L_n$.

Hence, if we know φ_n, we know what φ_m is for all $m \leq n$. More precisely, if $\varphi \leftrightarrows (\varphi_n)$, and $\varphi_n \in (\mathbb{Z}/p^n\mathbb{Z})^\times$, then $\varphi_m = \varphi_n$ modulo p^m.

In particular, $\mathrm{Gal}(L/\mathbb{Q}) = \varprojlim (\mathbb{Z}/p^n\mathbb{Z})^\times = \mathbb{Z}_p^\times$, equipped with the subspace topology from $\mathbb{Z}_p \overset{\mathrm{df}}{=} \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

2. Let's do an example closer to what we'll be interested in: local fields. Well, before we do local fields, let's do a finite field.

   Let K be finite, and let $L = \overline{K}$. Say that $\#K = q$. Then L is the union of the unique q^nth-order extensions of K.

   Remark. Note that $\mathbb{F}_q \subseteq \mathbb{F}_{q^2}$, but $\mathbb{F}_{q^2} \not\subseteq \mathbb{F}_{q^3}$, simply by dimension + counting elements of a finite-dimensional vector space over a finite field. How it works is that this inclusion only happens if the smaller exponent divides the larger.

   Let's remind ourselves how Galois theory over finite fields works. Write $L_n \overset{\mathrm{df}}{=} \mathbb{F}_{q^n}$. L_n/K is Galois over n and one checks that $\mathrm{Gal}(L_n/K) = \mathbb{Z}/n\mathbb{Z}$ where the generator 1 on the right hand side goes to Frob_q, which is a completely explicit member of the Galois group of L_n/K.

   So, we can send $g \in \mathrm{Gal}(\overline{K}/L)$ to $(g_n) \in \prod \mathbb{Z}/n\mathbb{Z}$.

   (g_n) is in the image of $\mathrm{Gal}(\overline{K}/K)$, if and only if g_n modulo m is g_m for all m dividing n.

   Therefore, $\mathrm{Gal}(\overline{K}/K)$ is $\varprojlim \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, the limit taken over the inverse system of positive integers, ordered by divisibility.

3. Now, local fields. Let's stick to the case where $K/\mathbb{Q}_p$ is finite. Choose an algebraic closure $\overline{K}$ over K. We want to understand $\mathrm{Gal}(\overline{K}/K)$. This group is only defined up to inner automorphism, i.e. equivalence in the 2-category **Grp**.

   (We will fail to do this, but we'll get some scraps.)

   So, $\mathbb{Q}_p$ is the p-adic numbers, and $\mathbb{Q}_p$ contains $\mathbb{Z}_p$ the p-adic integers. There is a valuation $\mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ the *normalized valation*, where $v(x)$ is just the number of times p divides x, i.e. $v(p^n u) = n$ if $u \in \mathbb{Z}_p^\times$.

$K \supseteq \mathcal{O}_K$ the integers of K, where $\mathcal{O}_K$ is a local ring, whose unique maximal ideal is $\mathfrak{p}_K = (\pi_K)$ principal.

Therefore, we have $v : K^\times \to \mathbb{Z}$ with $v(\pi_K) = 1$ and $v(\pi_K^n u) = n$, where $u \in \mathcal{O}_K^\times$.[1]

Say that L/K is an algebraic extension (possibly infinite).

The valuation $v_K : K^\times \to \mathbb{Z}$ extends to $L^\times \to \mathbb{Q}$, and $L \supseteq \mathcal{O}_L \supseteq \mathfrak{p}_L$, where $\mathcal{O}_L = \{0\} \cup \{\lambda \in L \mid v(\lambda) \geq 0\}$.

The residue field $\mathcal{O}_L/\mathfrak{p}_L = k_L =$ algebraic extension of $k_K = \mathcal{O}_K/\mathfrak{p}_K =$ a finite field.

If L/K is Galois, then we get a map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k_K)$ and this is surjective (and not injective in general).

Definition. We say that L/K is unramified if the natural map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k_K)$ is an isomorphism.

Here is the set-up: $K/\mathbb{Q}_p$ is finite, $K \supseteq \mathcal{O}_K \supseteq \mathfrak{p}_K = (\pi_K)$.

Exercise 2. *Show that the following are equivalent:*

(a) $\mathfrak{p}_L = \pi_K \mathcal{O}_L$.

(b) $v_K(L^\times) = \mathbb{Z}$.

(c) L/K *is unramified.*

Exercise 3. *Show that the compositum of two unramified extensions of* K *is unramified.*

Therefore, if L/K is algebraic, then there exists a unique maximal unramified subextension, so an M where $L \supseteq M \supseteq K$, M/K unramified and maximal, and $\mathrm{Gal}(M/K) = \mathrm{Gal}(k_M/k_K)$ is cyclic or pro-cyclic.

[1] Reference: Serre's *Local Fields*.

3 Lecture Three

3.1 Inertia subgroup

Recall from last time: We had $K/\mathbb{Q}_p$ a finite extension. Let L/K be an algebraic, normal extension (hence Galois).

The Galois group $\mathrm{Gal}(L/K)$ surjects onto the Galois group $\mathrm{Gal}(k_L/k_K)$. We can then define the **inertia subgroup** $I_{L/K}$ to be the kernel of the map $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k_K)$. Since k_K is finite, we see that the quotient of $\mathrm{Gal}(L/K)$ by $I_{L/K}$ is procyclic (topologically generated by one element).

To understand $\mathrm{Gal}(L/K)$, we wish to focus on $I_{L/K}$.

(In theory we don't lose much by pretending in our heads that L/K is a finite extension.)

$I_{L/K}$ is a closed subgroup of $\mathrm{Gal}(L/K)$ since it is the preimage of a closed set under a "sufficiently nice" map. By the fundamental theorem of Galois theory, $I_{L/K}$ corresponds to a subextension M/K, which is the union of all unramified subextensions K'/K.

An interesting special case follows. When $L = \overline{K}$, the maximal unramified subextension of $\overline{K}$ (which we denote K^{nr}) satisfies $\mathrm{Gal}(K^{nr}/K) \cong \mathrm{Gal}(\overline{k_K}/k_K) \cong \hat{\mathbb{Z}}$ canonically; this is due to the fundamental theorem of Galois theory again.

Consider for example $K = \mathbb{Q}_p$; then we will find that

$$K^{nr} = \bigcup_{\substack{m \geq 1 \\ p \nmid m}} \mathbb{Q}_p(\zeta_m).$$

Exercise 4. *Is this true for general K?*

Now we consider L/K Galois with *finite* inertia group $I_{L/K}$. (For example, this is true when L/K is finite.)

We know that $I_{L/K}$ is a normal subgroup of $\mathrm{Gal}(L/K)$. We will put a filtration on $I_{L/K}$ as follows: if $\sigma \in I_{L/K}$, we consider σ as a map $L \to L$.

Exercise 5. *Check that $\sigma(\mathcal{O}_L) = \mathcal{O}_L$ and $\sigma(\mathfrak{p}_L) = \mathfrak{p}_L$.*

Because $I_{L/K}$ is finite, we in fact have $\mathfrak{p}_L$ principal. Consider the maximal unramified subextension M; we note that $\mathfrak{p}_M = \pi_K \mathcal{O}_M$ (where $\mathfrak{p}_K = \pi_K \mathcal{O}_K$) because M/K unramified. Then $v_L : L \to \mathbb{Z}$ the discrete valuation satisfies $v_L = (\#I_{L/K}) v_K$ on $K^\times$.

Let π_L be the appropriate uniformizer. If $i \geq 1$, we define $I_{L/K,i} = \{\sigma \in I_{L/K} : \sigma(\pi_L)/\pi_L \in 1 + \mathfrak{p}_L^i\}$.

Set $I_{L/K,0} = I_{L/K}$.

> **Exercise 6.** *Check that each $I_{L/K,i}$ is a subgroup of $I_{L/K}$ and that each is a normal subgroup of* $\mathrm{Gal}(L/K)$.

Furthermore, if σ is not the identity, $\sigma(\pi_L)/\pi_L \neq 1$. Hence for $i \gg 0$ we see that $I_{L/K,i} = \{1\}$.

We note that $I_{L/K}/I_{L/K,1} \hookrightarrow k_L^\times$ via $\sigma \mapsto \sigma(\pi_L)/\pi_L$. In particular, $I_{L/K}/I_{L/K,1}$ is cyclic of order prime to p.

> **Exercise 7.** *If $i \geq 1$, show that*
> $$I_{L/K,i}/I_{L/K,i+1} \hookrightarrow \mathfrak{p}_L^i/\mathfrak{p}_L^{i+1}$$
> *via $\sigma \mapsto \sigma(\pi_L)/\pi_L - 1$, as abelian groups. (Recall that $\mathfrak{p}_L^i/\mathfrak{p}_L^{i+1}$ is isomorphic to the additive group of k_L).*

> **Exercise 8.** *In particular, if $i \geq 1$, show that $I_{L/K,i}/I_{L/K,i+1} \cong (\mathbb{Z}/p\mathbb{Z})^{n_i}$ for some n_i, so that its order is a power of p and each non-identity element has order p.*

The upshot is that $I_{L/K,1}$ is the unique Sylow p-subgroup of $I_{L/K}$, with the quotient $I_{L/K}/I_{L/K,1}$ cyclic of order prime to p. (In particular $I_{L/K}$ is a solvable group.)

We say that L/K is **tamely ramified** if $I_{L/K,1}$ is trivial. (Unramified extensions are tamely ramified.) Otherwise we say L/K is **wildly ramified**.

We're really interested in the case $L = \overline{K}$, in which case $I_{L/K}$ is not finite. The "lower numbering" $I_{L/K,i}$ does not behave nicely with regard to extensions of L.

This means that if we have extensions $L'/L/K$ with both L, L' Galois over K and $I_{L'/K}$ finite, we have a natural surjection $I_{L'/K} \to I_{L/K}$, but $I_{L'/K,i}$ is not identified with $I_{L/K,i}$ in general.

However, we can fix this! We introduce a new relabelling of the filtration.

Say we have L/K Galois and $I_{L/K}$ finite as before. Let $g_i = \#I_{L/K,i}$. Then $g_0 \geq g_1 \geq \dots$. Define $\phi : [0,\infty) \to [0,\infty)$ a piecewise linear and continuous function; it is linear on each interval $(i, i+1)$ with $\phi(0) = 0$ and the slope on $(i, i+1)$ is g_{i+1}/g_0.

Then eventually ϕ becomes linear with slope $1/g_0$. It is clearly a strictly increasing bijection.

Now for any $v \in \mathbb{R}_{\geq 0}$, define $I_{L/K,v} = I_{L/K,\lceil v \rceil}$ where $\lceil \cdot \rceil$ is the ceiling function.

We can now define the **upper numbering**. For $u \in \mathbb{R}_{\geq 0}$, define $I_{L/K}^u = I_{L/K,\phi^{-1}(u)}$.

Proposition 1. *If $L'/L/K$ are all Galois extensions as usual and $I_{L'/K}$ is finite, then the image of $I^u_{L'/K}$ under the obvious map is exactly $I^u_{L/K}$.*

Proof. See Serre's Local Fields or Cassels-Fröhlich (p.38). □

The point is that the upper numbering extends to an infinite extension.

Theorem 3.1 (Hasse-Arf)**.** *If L/K is abelian, then the jump discontinuities in the upper numbering $I^u_{L/K}$ are at integers.*

Now for any L/K Galois extension, we may define $I^u_{L/K}$ by "gluing together" $I^u_{M/K}$ for M/K algebraic with $I_{M/K}$ finite.

3.1.1 Tamely ramified extensions

Recall that when $I_{L/K}$ is finite, we say L/K is tamely ramified with $I_{L/K,1}$ is trivial - equivalently, $I_{L/K,\varepsilon} = I^{\delta}_{L/K} = \{1\}$ for any $\varepsilon, \delta > 0$. (Using $I^{\delta}_{L/K}$ works to define tame ramification for any Galois L/K.)

The usual story is true: the compositum of two tame extensions is tame. Therefore we can talk about maximal tamely ramified subextension.

We now have the following general picture. Given L/K Galois, we have a maximal unramified subextension K_1 and a maximal tamely ramified subextension K_2, with $K_1 \subset K_2$ and L/K_2 possibly wildly ramified with $\mathrm{Gal}(L/K_2)$ a pro-p group.

Now consider $L = \overline{K}$; we get $\overline{K}/K^t/K^{nr}/K$ a tower of extensions (K^t is the maximal tamely ramified extension). We now may ask: what is K^t?

If L/K is finite and Galois, consider the tower of extensions $L/K_2/K_1/K$ as before. The Galois group $\mathrm{Gal}(K_2/K_1)$ is $I_{L/K}/I_{L/K,1}$, which is cyclic of order prime to p.

By Kummer theory (see article by Birch in Cassels-Fröhlich), K^{nr} contains all the mth roots of unity for m prime to p, so if $\mathrm{Gal}(K_2/K^{nr}) \cong \mathbb{Z}/m\mathbb{Z}$ for m prime to p, then K_2 is $K^{nr}(\sqrt[m]{\alpha})$ for some $\alpha \in K^{nr}$.

Exercise 9. *Check that $K_2 = K^{nr}(\sqrt[m]{\pi_K})$.*

We can now check that

$$K^t = \bigcup_{\substack{m \geq 1 \\ p \nmid m}} K^{nr}(\sqrt[m]{\pi_K}).$$

We find that $\mathrm{Gal}(K^{nr}(\sqrt[m]{\pi_K})/K^{nr})$ is canonically isomorphic to μ_m via $\sigma \mapsto \sigma(\sqrt[m]{\pi_K})/\sqrt[m]{\pi_K}$.

Then $\mathrm{Gal}(K^t/K^{nr}) \cong \varprojlim \mu_m$ where the projective limit is taken over m prime to p. This is noncanonically isomorphic to $\varprojlim \mathbb{Z}/m\mathbb{Z}$ for m prime to p, or $\prod_{\ell \neq p} \mathbb{Z}_\ell$.

So then we have a pro-p group $\mathrm{Gal}(\overline{K}/K^t)$, a noncanonical isomorphism $\mathrm{Gal}(K^t/K^{nr}) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$, and canonically $\mathrm{Gal}(K^{nr}/K) \cong \hat{\mathbb{Z}}$.

Now we want to understand $\mathrm{Gal}(K^t/K)$, which we know as a semidirect product, though we do not know yet how $\hat{\mathbb{Z}}$ acts on $\prod_{\ell \neq p} \mathbb{Z}_\ell$. Let Frob be the canonical generator of $\mathrm{Gal}(K^{nr}/K) \cong \mathrm{Gal}(k_{K^{nr}}/k_K) \cong \hat{\mathbb{Z}}$ (it is the $\#k_K$-power map on the residue fields).

If we are able to lift Frob to $\mathrm{Gal}(K^t/K)$, then it acts by conjugation on the normal subgroup $\mathrm{Gal}(K^t/K^{nr})$ (since $\mathrm{Gal}(K^t/K^{nr})$ is abelian), and we will be able to describe $\mathrm{Gal}(K^t/K)$.

Recall $\mathrm{Gal}(K^t/K^{nr})$ is canonically isomorphic to $\varprojlim \mu_m(\overline{K})$.

> **Exercise 10.** *Check that the map induced by* Frob *is the q-power map $\zeta \mapsto \zeta^q$, where $q = \#k_K$.*

4 Lecture Four

We just saw an attempt to analyze the group $\mathrm{Gal}(\bar{K}/K)$ via an explicit attack on the inertia group. The obstacle is that determining the Sylow-p subgroup of the inertia group is hard. Another approach would be to understand the abelianisation of $\mathrm{Gal}(\bar{K}/K)$.

Let $K/\mathbb{Q}_p$ be finite. We recall the exact sequence

$$1 \to I_{\bar{K}/K} \to \mathrm{Gal}(\bar{K}/K) \to \hat{\mathbb{Z}} \to 1.$$

We have $\hat{\mathbb{Z}} = \langle \mathrm{Frob} \rangle$ topologically, but it has no well-defined lift to $\mathrm{Gal}(\bar{K}/K)$. $\mathrm{Frob} \in \hat{\mathbb{Z}}$ and we put

$$(\mathrm{Frob})^{\mathbb{Z}} = \{\ldots, \mathrm{Frob}^{-1}, 1, \mathrm{Frob}, \ldots\} = \mathbb{Z} \subseteq \hat{\mathbb{Z}}.$$

Definition. The *Weil Group*, W_K is formally defined as

$$W_K = \{g \in \mathrm{Gal}(\bar{K}/K) : \mathrm{Im}(g) \in \hat{\mathbb{Z}} \text{ is in } (\mathrm{Frob})^{\mathbb{Z}} = \mathbb{Z}\}.$$

Pictorially we have the following commutative diagram whose rows are exact:

$$\begin{tikzcd}
1 \arrow[r] & I_{\bar{K}/K} \arrow[r] \arrow[d, equal] & W_K \arrow[r] \arrow[d, hook] & (\mathrm{Frob})^{\mathbb{Z}} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & I_{\bar{K}/K} \arrow[r] & \mathrm{Gal}(\bar{K}/K) \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1
\end{tikzcd}$$

The Weil group is a dense subgroup of the Galois group. We give W_K a topology which is not the subspace topology. We define the topology as follows: $I_{\bar{K}/K}$ is open in W_K with the usual topology so $W_K/I_{\bar{K}/K} = \mathbb{Z}$ with the discrete topology. Consider the following diagram

$$\begin{tikzcd}
W_K \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathrm{Gal}(\bar{K}/K) \arrow[r] & \hat{\mathbb{Z}}
\end{tikzcd}$$

Since $\mathbb{Z}$ has discrete topology, $\mathbb{Z} \to \hat{\mathbb{Z}}$ is continuous. $\hat{\mathbb{Z}}$ has profinite topology and the map $\mathrm{Gal}(\bar{K}/K) \to \hat{\mathbb{Z}}$ is also continuous. W_K is the pull-back of the Galois group in the category of topological groups. W_K surjects onto $\mathrm{Gal}(L/K)$ for finite extensions L/K; this follows from the observations that

$$\hat{Z}/n\hat{\mathbb{Z}} \simeq \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}.$$

Define for G a topological group, the subgroup G^c to be the topological closure of the normal subgroup of G generated by $ghg^{-1}h^{-1}$ for $g, h \in G$. Then $G^{ab} = G/G^c$ is the maximal abelian Hausdorff quotient of G.

Theorem 4.1. *(Main theorem of Local Class Field Theory) Let $K/\mathbb{Q}_p$ be a finite extension. Then there is a canonical isomorphism (called the Artin map)*

$$r_k : K^\times \to W_K^{ab}.$$

(For details refer to Serre's article in Cassels-Fröhlich). We now list out the properties of this Artin map:

1. $r_K(\mathcal{O}_K) = \mathrm{Im}(I_{\bar{K}/K})$.
2. $r_K(1 + \mathfrak{p}_K^i) = \mathrm{Im}(I^i_{\bar{K}/K})$.
3. $r_K(\pi_K) \in \mathrm{Frob}^{-1} \cdot \mathrm{Im}(\bar{K}/K)$.

   Remark. If X, Y are two abelian groups and $\varphi : X \to Y$ is a canonical isomorphism then $\psi : X \to Y$ defined by $\psi(x) = \varphi(x)^{-1}$ is just as canonical. So there are *two* canonical isomorphisms between $K^\times$ and W_K^{ab}. These are r_K and $r_K \circ (x \mapsto x^{-1})$. The way to tell them apart is by the last condition; the one we use sends $\pi_K \in K^\times$ to the inverse of the Frob.

   As per Deligne's definition we call Frob^{-1} the *geometric Frobenius* and Frob the *arithmetic Frobenius.*

4. If L/K is finite then $\mathrm{Gal}(\bar{K}/L) \hookrightarrow \mathrm{Gal}(\bar{K}/K)$. Thus there is a canonical injection of the Weil groups. We have the following commutative diagram (where we note that the bottom arrow need not be injective)

$$\begin{tikzcd} W_L \arrow[r, hook] \arrow[d] & W_K \arrow[d] \\ W_L^{ab} \arrow[r] & W_K^{ab} \end{tikzcd}$$

5. The following diagram commutes:

$$\begin{tikzcd} L^\times \arrow[r, "r_L"] \arrow[d, "N_{L/K}"'] & W_L^{ab} \arrow[d] \\ K^\times \arrow[r, "r_K"] & W_K^{ab} \end{tikzcd}$$

Since we know that $K^\times \hookrightarrow L^\times$ there must be a map $W_K^{ab} \to W_L^{ab}$. There exists the *Verlagerung/Transfer map*: let $H \leq G$ of finite index. There exists

$$V : G^{ab} \to H^{ab}; \qquad g \mapsto \prod_i \gamma_i g \gamma_i^{-1}$$

where $\{\gamma_i\}$ is the set of coset representatives of H in G. Then we have the following commutative diagram

$$\begin{tikzcd} L^\times \arrow[r, "r_L"] & W_L^{ab} \\ K^\times \arrow[r, "r_K"] \arrow[u, hook] & W_K^{ab} \arrow[u, "\text{Transfer}"'] \end{tikzcd}$$

6. Local class field theory tells about cohomology groups. Let L/K be Galois and finite, then W_L is a normal subgroup of W_K (of finite index) and $W_K/W_L = \text{Gal}(L/K)$. In fact, $W_L^c \leq W_L \leq W_K$ and W_L^c is a characteristic subgroup and hence normal in W_K. We define

$$W_{L/K} = W_K/W_L^c$$

and this gives us the exact sequence

$$1 \to L^\times = W_L^{ab} \hookrightarrow W_{L/K} \to \text{Gal}(L/K) \to 1.$$

This gives rise to an element of $H^2(\text{Gal}(L/K), L^\times)$ called $\alpha_{L/K}$ such that $H^2(\text{Gal}(L/K), L^\times) = \langle \alpha_{L/K} \rangle$ is cyclic of order $n = [L : K]$. This element is called the *fundamental class*.

Remark. There are lots of cohomology groups one can compute using the fundamental class, using the cup product (for example).

Upshot: We now understand Galois groups of maximal abelian extensions of K.

> **Exercise 11.** *Show that the compositum of two abelian extensions is abelian, and therefore that there exists a maximal abelian extension* $K^{ab} \subseteq \bar{K}$. *Show that* $\mathrm{Gal}(K^{ab}/K) = \mathrm{Gal}(\bar{K}/K)^{ab}$.

We have the following commutative diagram:

$$\begin{tikzcd} I_{K^{ab}/K} \arrow[r] & \mathrm{Gal}(K^{ab}/K) \arrow[r] & \mathrm{Gal}(K^{nr}/K) = \hat{\mathbb{Z}} \\ I_{K^{ab}/K} \arrow[u, equal] \arrow[r] & W_K^{ab} \simeq K^{\times} \arrow[u, hook] \arrow[r] & \mathbb{Z} \arrow[u, hook] \end{tikzcd}$$

The isomorphism $\mathrm{Gal}(K^{ab}/K) \simeq \mathcal{O}_K^{\times} \times \hat{\mathbb{Z}}$ is non-canonical, as it depends on the choice of Frob.

5 Lecture Five

Today we will work towards the statement of the local Langlands conjecture. Vaguely speaking, the local Langlands conjecture for GL_n over K (where $K/\mathbb{Q}_p$ is a finite extension) is that there is a one-to-one correspondence between certain n-dimensional representations of a particular Galois group, and representations of $\mathrm{GL}_n(K)$.

Let us consider the first of these sets. The Galois group in which we are interested is a "Weil-Deligne group" (related to the Weil group W_K), and so we are motivated to begin investigating representations of W_K.

Recall the definition of W_K. We have the following commutative diagram whose rows are exact:

$$\begin{tikzcd}
1 \arrow[r] & I_{\overline{K}/K} \arrow[r] \arrow[d, "\mathrm{id}"'] & W_K \arrow[r] \arrow[d, hook] & \mathrm{Frob}^{\mathbb{Z}} \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & I_{\overline{K}/K} \arrow[r] & \mathrm{Gal}(\overline{K}/K) \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1
\end{tikzcd}$$

Now, let E be a field, and equip both E and $\mathrm{GL}_n(E)$ (for some fixed $n \geq 1$) with the discrete topology. We consider what it means for a homomorphism of groups $\rho : W_K \to \mathrm{GL}_n(E)$ to be continuous: we have that ρ will be continuous if and only if $\ker \rho$ is open in W_K. Because $I_{\overline{K}/K}$ is compact in W_K and ρ is continuous, we know that $\rho(I_{\overline{K}/K})$ is a compact subset of $\mathrm{GL}_n(E)$, which is therefore finite.

Thus we can use the theory of lower numbering on $\rho(I_{\overline{K}/K})$: because $I_{\overline{K}/K} = \mathrm{Gal}(\overline{K}/K^{nr})$, we deduce the existence of a subextension $K^{nr} \subseteq L = L(\rho) \subseteq \overline{K}$ such that $\mathrm{Gal}(L/K^{nr}) = \rho(I_{\overline{K}/K})$. Thus

$$\rho(I_{\overline{K}/K}) = I_{L/K} \supseteq I_{L/K,1} \supseteq \cdots$$

Now, define $\mathfrak{f}(\rho)$ to be the **conductor** of ρ, i.e.

$$\mathfrak{f}(\rho) = \sum_{i=0}^{\infty} \frac{1}{[I_{L/K} : I_{L/K,i}]} \dim(V/V^{I_{L/K,i}}),$$

where

$$\rho : W_K \to \mathrm{GL}_n(F) = \mathrm{Aut}_E(V), V = E^n,$$

and V^H is the set of points of V fixed by H for $H \leq \mathrm{Aut}(V)$.

Note that our sum is finite, as for $i \gg 0$ we know $I_{L/K,i} = 1$ and so the dimension of the quotients vanishes for large enough i.

Exercise 12. *Show that* $\mathfrak{f}(\rho) = 0$ *if and only if* ρ *is unramified, if and only if* $\rho(I_{\overline{K}/K}) = 1$. *Show that* $\mathfrak{f}(\rho)$ *is an integer.*

Note also that

$$\mathfrak{f}(\rho) = \dim(V/V^{I_{L/K}}) + \sum_{i \geq 1} \frac{1}{[I_{L/K} : I_{L/K,i}]} \dim(V/V^{I_{L/K,i}})$$

For example, recall the isomorphism $r_K : K^\times \to W_K^{ab}$ from before; this, together with the surjection $W_K \to W_K^{ab}$ gives us another surjection $W_K \to K^\times$. In addition to the normalized valuation $v_K : K^\times \to \mathbb{Z}$, we also have the notion of a **norm** on K, given by

$$\|\lambda\| = \varepsilon^{v_K(\lambda)},$$

where $0 < \varepsilon < 1$. Two norms $\|\cdot\|_1, \|\cdot\|_2$ are called **equivalent** if $\|x\|_1 < 1$ if and only if $\|x\|_2 < 1$ for every $x \in L$; with our norm, any different choice $\tilde{\varepsilon}$ will yield an equivalent norm.

Now, if K is any field which is complete with respect to a non-trivial, non-archimedean norm, we can build the theory of *rigid geometry* over K. For instance, let $K = \mathbb{C}((t))$ be the field of formal Laurent series in t over $\mathbb{C}$, and let

$$v\left(\sum_{n=-m}^{\infty} a_n t^n\right) = -m,$$

if $a_{-m} \neq 0$. We put a norm on K by fixing some $\varepsilon, 0 < \varepsilon < 1$ and putting $\|f\| = \varepsilon^{v(f)}$. Here, there is no real "canonical" choice of ε. In our situation, however, where $K/\mathbb{Q}_p$ is finite, a canonical choice of ε is given by $\varepsilon = q^{-1}$, where $q = |k_K|$.

Indeed, because k_K is finite, we know that K is locally compact, and indeed a compact open neighbourhood of the identity is given by $\mathcal{O}_K \cong \mathbb{Z}_p^d$ (for some $d \geq 1$). As such, there exists an additive Haar measure μ on K. Because $\mathcal{O}_K$ is compact, we have that $\mu(\mathcal{O}_K)$ is finite, so we will scale μ so that $\mu(\mathcal{O}_K) = 1$. Then what is $\mu(\mathfrak{p}_K)$?

Recall that a **Haar measure** on a locally compact abelian group G is any translation-invariant measure, and furthermore that all Haar measures on G are proportional. Clearly

$$\mu\big(X \coprod Y\big) = \mu(X) + \mu(Y)$$

if $X, Y \subseteq G$ are compact; therefore, since

$$\mathcal{O}_K = \coprod_{\lambda \in k_K} \tilde{\lambda} + \mathfrak{p}_K$$

(where λ is any lift of λ to $\mathcal{O}_K$), we have $\mu(\mathcal{O}_K) = q\mu(\mathfrak{p}_K)$; thus $\mu(\mathfrak{p}_K) = q^{-1}$.

Now, if $a \in K^\times$, then define $\|a\|$ to be the factor by which multiplication by a scales the Haar measure; for instance, if $K = \mathbb{R}$, let $X = [0, 1], \mu(X) = 1$ and choose $\lambda \in \mathbb{R}^\times$. Then

$$\mu(\lambda X) = \mu\big([0, \lambda]\big) = |\lambda|\mu(X),$$

thus $\|\lambda\| = |\lambda|$ (i.e. the usual absolute value).

> **Exercise 13.** *Compute the norm $\|z\|$ for $z \in \mathbb{C}$, using this method. (Hint: for compact X with nonzero measure, we must have $\|\lambda\| = \mu(\lambda X)/\mu(X)$).*

Back to the case of K a finite extension of $\mathbb{Q}_p$: we have our normalized valuation v so that $v(\pi_K) = 1$, we want $\|\pi_K\|$ to equal simply our choice of ε; does there exist a canonical choice? Indeed, we have $\mu(\mathcal{O}_K) = 1$, thus

$$\|\pi_K\| = \mu(\pi_K\mathcal{O}_K) = \mu(\mathfrak{p}_K) = q^{-1},$$

so we have a canonical choice in $\varepsilon = q^{-1}$, as claimed. That is, there is a *natural norm* on K

$$\|\cdot\| : K^\times \to \mathbb{R}_{>0}, \|\lambda\| = q^{-v_K(\lambda)},$$

and $\|0\| = 0$. In fact, if v_K is normalized, then $\|\cdot\|$ takes only rational values, and a group homomorphism $K^\times \to \mathbb{Q}_{>0}$ is obtained. Thus we have the following commutative diagram:

$$\begin{tikzcd}
W_K \arrow[r] \arrow[rrd, bend right, "\|\cdot\|"'] & W_K^{ab} \arrow[r, "r_K"] & K^\times \arrow[d, "\|\cdot\|"] \\
 & & \mathbb{Q}_{>0}
\end{tikzcd}$$

This defines a map $\|\cdot\| : W_K \to \mathbb{Q}_{>0}$.

For example: if $E = \mathbb{Q}$, then $\|\cdot\| : W_K \to \mathrm{GL}_1(E)$ is a representation; more generally, all integer powers of $\|\cdot\|$ will yield a representation of W_K.

For another example, suppose $\mathfrak{f}(\|\cdot\|^m) = 0$; what is $\|\widetilde{\mathrm{Frob}}\|$? (As before, $\widetilde{\mathrm{Frob}} \in W_K$ is a lift of $\mathrm{Frob} \in \mathbb{Z}$). Then for the above diagram to commute, we must have that $r_K(\pi_K) = \mathrm{Frob}^{-1}(I_{K^{ab}/K})$ maps to q^{-1}, and hence $\|\widetilde{\mathrm{Frob}}\| = q$.

Now, we introduce the notion of a **Weil-Deligne representation**. This is a pair (ρ_0, N), where $\rho_0 : W_K \to \mathrm{Aut}_E(V) \cong \mathrm{GL}_n(E)$ is a continuous representation as before (E is a field of characteristic 0 with the discrete topology, and $V = E^n$), and N is a nilpotent endomorphism of V such that, for all $\sigma \in W_K$, one has

$$\rho_0(\sigma) N \rho_0(\sigma)^{-1} = \|\sigma\| N. \tag{$*$}$$

For a trivial example: let ρ_0 be any continuous representation, and $N = 0$. Thus every representation gives rise to a Weil-Deligne representation.

For a nontrivial example: let $E = \mathbb{Q}$ and $V = \langle e_1, e_0 \rangle_{\mathbb{Q}}$. Let

$$\rho_0(\sigma) = \begin{pmatrix} \|\sigma\| & 0 \\ 0 & 1 \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

so that

$$\rho_0(\sigma) e_1 = \|\sigma\| e_1, \rho_0(\sigma) e_0 = e_0,$$

and

$$N e_0 = e_1, N e_1 = 0.$$

We check that property $(*)$ holds:

$$\begin{pmatrix} \|\sigma\| & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \|\sigma\|^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \|\sigma\| \\ 0 & 0 \end{pmatrix} = \|\sigma\| N.$$

Exercise 14. *Check that property $(*)$ holds when $V = \mathbb{Q}^n$ with basis $e_0, \ldots, e_{n-1}$ with representation $\rho_0(\sigma) e_i = \|\sigma\|^i e_i$ and nilpotent operator $N e_i = i + 1, i = 0, \ldots, n-2, N e_{n-1} = 0$.*

We say that a Weil-Deligne representation (ρ_0, N) is **F-semisimple** if $\rho_0(\widetilde{\mathrm{Frob}})$ is a semisimple matrix (i.e. diagonalizable over $\overline{E}$); this notion is independent of choice of lift $\widetilde{\mathrm{Frob}}$.

Remark. In general, we call a representation of a group G is **semisimple** if it is a direct sum of simple representations. With a little work, one can show that in the case $G = W_K$, and ρ_0 is continuous, (ρ_0, N) is F-semisimple if and only if $\rho_0(G)$ is semisimple.

With this machinery, we can now define one half of the local Langlands correspondence for GL_n: it will be the set of isomorphism classes of all n-dimensional, F-semisimple Weil-Deligne representations of W_K. We will define the other half in the next lecture.

6 Lecture Six

6.1 Representations of $\mathrm{GL}_n(K)$

In stark contrast to previous lectures, almost *all* representations here will be infinite-dimensional.

Here's the set-up. Let E be a field, discretely topologized. Let V be an E-vector space (possibly infinite-dimensional). Let $K/\mathbb{Q}_p$ finite.

Let $\pi : GL_n(K) \to \mathrm{Aut}_E(V)$ be a group homomorphism.

There are too many π's. We want a sensible notion of continuity.

Definition. Say that π is *smooth* if for all $v \in V$, the stabilizer $\mathrm{Stab}_\pi(v) \overset{\mathrm{df}}{=} \{g \in \mathrm{GL}_n(K) \mid \pi(g)(v) = v\}$ is open. (We have put the p-adic topology on $\mathrm{GL}_n(K)$.)

Remark. This is equivalent to the map $\mathrm{GL}_n(V) \times V \to V$ being continuous. On one hand, if the stabilizer is open, then the preimage of any v along the action splits into its cosets; on the other hand, if the preimage of v along the action is open, then you can intersect it with the open set $\mathrm{GL}_n(V) \times \{v\}$ to obtain $\mathrm{Stab}_\pi(v) \times \{v\}$; since the product topology is the box topology for a finite product, $\mathrm{Stab}_\pi(v)$ is open.

Definition. Say that a smooth π is **smooth-admissible** or just plain **admissible**, if for all $U \subseteq \mathrm{GL}_n(K)$ an open subgroup, the fixed points V^U of U is finite-dimensional.

If we say "π-admissible", we mean that π is smooth-admissible.

For example, if the dimension of V is 1, then the trivial representation $\pi(g) = 1$ for all g is smooth and admissible. If the dimension of V is infinite, then the trivial representation is smooth, but not admissible.

This is tricky to show, but true: π irreducible and smooth implies π admissible.[2]

Let's describe the topology on $\mathrm{GL}_n(K)$.

Definition. A basis of open neighborhoods of the identity in $\mathrm{GL}_n(K)$ is given by the matrices

$$\left\{M \in \mathrm{GL}_n(\mathcal{O}_K) \,\middle|\, M \equiv I_n \bmod \mathfrak{p}_k^m\right\}_{m \in \mathbb{Z}_{\geq 0}}.$$

Remark. Recall that π is called **irreducible** if there exist exactly two $\mathrm{GL}_n(K)$-invariant subspaces, namely 0 and V.

Here's a vague statement of the local Langlands conjectures for GL_n.

[2] Question: why? Who proved this?

- Local class field theory tells us that $K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$.

  But we want to understand *all* of W_K. Langland's insight was that you could reinterpret this local class field theory isomorphism: if you see the abelianization of a group, this tells you something about its 1-dimensional representation theory. Therefore, irreducible 1-dimensional representations of $K^\times$ should be the same as irreducible 1-dimensional representations of W_K, and the trick is that we're going to study the rest of W_K by studying the rest (i.e. the higher-dimensional representation theory of $K^\times$.)

- **Local Landlands conjecture for GL_n.** There exists a canonical (functorial, definable, bi-interpretable, etc...) bijection

$$\left\{\begin{array}{l}\text{irreducible, smooth-}\\ \text{admissible representa-}\\ \text{tions of } \mathrm{GL}_n(K)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}n\text{-dimensional, } F\text{-semi-}\\ \text{simple Weil-Deligne rep-}\\ \text{resentations of } W_K.\end{array}\right\}$$

Remark. Of course, we have to be careful about the criteria by which we define "canonical". It's easy to check that either side of this correspondence has the sae cardinality, for example, but any randomly chosen bijection won't do.

Next time, we're going to convince ourselves is that when $n = 1$, this is precisely local class field theory, and then we're going to see lots of examples in the case of $n = 2$ of elements on both sides.

7 Lecture Seven

7.1 Some comments

Last time we finished by stating the local Langlands correspondence for GL_n. Today we're going to think about $E = \mathbb{C}$-representations (on both sides of the correspondence) for concreteness.

We can interpret the word 'canonical' in the statement of the LLC as meaning that the bijection satisfies a long list of nice properties – e.g. there is a notion of duality on both sides, we may define L-functions and ε-factors, etc.

A historical note: the list of 'nice properties' for the bijection in the LLC for GL_n became sufficiently long that it became a theorem that there was at most one bijection satisfying the nice properties. It turns out that there is at least one such bijection satisfying these properties. In the function field case, the existence of this bijection is a theorem of Laumon, Rapoport, and Stuhler; the p-adic field case is a theorem of Harris and Taylor. The proofs in both cases are global.

Two obvious observations:

1. This is a brilliant generalization of local class field theory. Given a p-adic field K we have to choose an algebraic closure $\overline{K}$, and it is the representations of $\mathrm{Gal}(\overline{K}/K)$ that turn out to be important.

2. This is completely pointless as it relates two uninteresting sets, since we have seen neither Weil-Deligne representations nor smooth admissible irreducible representations of $\mathrm{GL}_n(K)$ showing up elsewhere in mathematics. We're in some sense bijecting two rather pathological collections of objects.

Today we will begin to check that the local Langlands correspondence is useful.

Remark. For any connected reductive group G over K (meaning, roughly, over $\overline{K}$ our group G is isomorphic to some nice group such as $\mathrm{GL}_n, O(n), Sp(n)$, etc.), we can formulate a local Langlands correspondence for G between smooth irreducible admissible representations of $G(\mathbb{C})$ and certain Weil-Deligne representations of some Weil group associated to the L-group of $G(\mathbb{C})$. However, this is not a bijection but a surjection with finite fibers, which again satisfies some long list of natural properties (though in this case the list does not uniquely characterize the map). The finite fibers of the map are called 'L-packets'.

7.2 Local Langlands for GL_1

One side of local Langlands for $n = 1$ consists of 1-dimensional Weil-Deligne representations $(\rho_0, N) : W_K \to \mathrm{GL}_1(\mathbb{C})$ (note that we must have $N = 0$ and $\ker \rho_0$ closed - hence ρ_0 factors through $W_K^{ab} \cong K^\times$). Hence this side of the correspondence is just 1-dimensional continuous complex representations of W_K^{ab}.

The other side of the correspondence consists of smooth irreducible admissible representations of $K^\times$. It is not difficult to check that an admissible irreducible representation of $K^\times$ must have finite dimension - in fact, it must be one-dimensional. Furthermore continuity turns out to be equivalent to smoothness. Hence this side of the correspondence is just continuous group homomorphisms $K^\times \to \mathbb{C}^\times$.

Then r_K of local class field theory gives us the desired canonical bijection.

More explicitly, we have a bijection of sets

$$\left\{\begin{array}{l}\text{irreducible, admissible rep-}\\ \text{resentations of } GL_1(K)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{one-dimensional Weil-}\\ \text{Deligne representations}\\ \text{of } W_K\end{array}\right\}$$

given by $\pi \mapsto (\pi^G, 0)$ and $\chi^A \leftarrow\!\shortmid (\chi, 0)$, where π^G is the composition

$$\pi^G : W_K \longrightarrow W_K^{ab} \xrightarrow{r_K^{-1}} K^\times \xrightarrow{\pi} \mathbb{C}^\times = GL_1(\mathbb{C})$$

and χ^A is the composition

$$\chi^A : K^\times \xrightarrow{r_K} W_K^{ab} \xrightarrow{\chi} \mathbb{C}^\times = GL_1(\mathbb{C})$$

If $\pi : K^\times \to \mathbb{C}^\times$ is a smooth admissible irreducible one-dimensional representation, we may define the **conductor** $\mathfrak{f}(\pi)$ to be 0 if $\pi|_{\mathcal{O}_K^\times}$ is trivial and otherwise $\mathfrak{f}(\pi) = r$, for r the smallest positive integer so $\pi|_{1+\mathcal{P}_K^r\mathcal{O}_K}$ is trivial.

One can check that the conductors of ρ_0 and the corresponding π are the same.

7.3 Weil-Deligne representations arising from ℓ-adic representations

As ever, $K/\mathbb{Q}_p$ is a finite extension. Say $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ is a continuous representation of the absolute Galois group of K ($\mathrm{GL}_n(\mathbb{Q}_\ell)$ carries the ℓ-adic topology for $\ell \neq p$ a prime).

The wild part of the inertia subgroup will end up being finite under this representation, but the same cannot be said of the tame part.

These ℓ-adic representations show up in 'nature': for example, the ℓ-adic Tate module of an elliptic curve E/K, or more generally, ℓ-adic étale cohomology $H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell)$ of an algebraic variety X/K. Furthermore ℓ-adic deformations of examples give new examples.

Remark. If E/K is an elliptic curve with split multiplicative reduction, then $E(\overline{K}) \cong \overline{K}^\times/(q^{\mathbb{Z}})$ for some $q \in K, |q| < 1$ canonically. This allows one to explicitly compute the ℓ-adic Tate module of E as $T_\ell E/I_{\overline{K}/K} = \left(\begin{smallmatrix} cyclo & * \\ 0 & Id \end{smallmatrix}\right)$ where $*$ can be nontrivial and infinite.

Recall that $\ell \neq p$, so if ρ is an ℓ-adic representation as above, $\rho(I_{\overline{K}/K})$ can be infinite, but $\rho(I^\varepsilon_{\overline{K}/K})$ is finite for $\varepsilon > 0$ since $I^\varepsilon_{\overline{K}/K}$ is a pro-p-group. Also recall that

$$\mathrm{Gal}(K^t/K^{nr}) \cong \prod_{\substack{r \text{ prime} \\ r \neq p}} \mathbb{Z}_r,$$

so the only part we need to worry about is the $\mathbb{Z}_\ell$-part.

Fix $t : \mathrm{Gal}(K^t/K^{nr}) \to \mathbb{Z}_\ell$ a surjection. Also fix $\phi \in \mathrm{Gal}(\overline{K}/K)$ lifting $\mathrm{Frob} \in \mathrm{Gal}(K^{nr}/K)$.

Proposition 2 (Grothendieck). *If $\rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(E)$ for $E = \mathbb{Q}_\ell$ is a continuous ℓ-adic representation, then there exists a unique Weil-Deligne representation $(\rho_0, N) : W_K \to \mathrm{GL}_n(E)$ such that $\rho(\phi^m\sigma)$ for any $\sigma \in I_{\overline{K}/K}, m \in \mathbb{Z}$ is equal to $\rho_0(\phi^m\sigma) \cdot \exp(N.t(\sigma))$. In fact the isomorphism class of (ρ_0, N) is independent of the choice of ϕ, t.*

Note that taking ρ_0 to be the restriction of ρ would not necessarily work due to the fact ρ is continuous with respect to the ℓ-adic topology on E while ρ_0 must be continuous with respect to the discrete topology on E. We may define the exponential of a nilpotent matrix via the usual power series.

Remark. In the Tate curve example, it turns out that N is nonzero; indeed, $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

Remark. Not all Weil-Deligne representations arise in this way, since those (ρ_0, N) that do must have that all eigenvalues of $\rho_0(\phi)$ be ℓ-adic units. However, this is in some sense the only obstruction - if the eigenvalues of $\rho_0(\phi)$ are ℓ-adic units for some Weil-Deligne representation (ρ_0, N), then there exists ρ so the Weil-Deligne representation arises from ρ.

7.4 Smooth admissible representations of $\mathrm{GL}_2(K)$

Say we have $\chi_1, \chi_2 : K^\times \to \mathbb{C}^\times$ continuous admissible characters. Define $I(\chi_1, \chi_2)$ to be the vector space of functions $\phi : \mathrm{GL}_2(K) \to \mathbb{C}$ such that (i) ϕ is locally constant for the p-adic topology, and (ii) for all $a, d \in K^\times$, one has

$$\phi\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) g\right) = \chi_1(a)\chi_2(d)\|a/d\|^{1/2}\phi(g).$$

This can also be constructed as an induced representation from the upper triangular subgroup B to $\mathrm{GL}_2(K)$.

Define $\pi : \mathrm{GL}_2(K) \to \mathrm{Aut}_{\mathbb{C}}(I(\chi_1, \chi_2))$ by $(\pi(g)\phi)(h) = \phi(hg)$ for $g, h \in \mathrm{GL}_2(K), \phi \in I(\chi_1, \chi_2)$.

8 Lecture Eight

Lemma 1. *Let $B(K)$ be the upper triangular matrices. Then*

$$\mathrm{GL}_2(K) = B(K) \cdot \mathrm{GL}_2(\mathcal{O}_K) = \{bg : \ b \in B(K), \ g \in \mathrm{GL}_2(\mathcal{O}_K)\}.$$

Proof. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K)$. We want $\beta \in B(K)$ and $\gamma \in \mathrm{GL}_2(\mathcal{O}_K)$ such that $M = \beta\gamma$.

Without loss of generality, we may assume $M \in \mathrm{SL}_2(K)$ as we can left multiply by $\begin{pmatrix} \det(M)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. We may further assume $c, d \in \mathcal{O}_K$ and at least one of them is a unit. This is because we can choose $\alpha \in K^\times$ such that $\alpha c, \alpha d \in \mathcal{O}_K$ and at least one is a unit by left multiplication by $\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}$. If c is not a unit, we can multiply on the right by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_K)$. We can switch c, d so without loss of generality we assume that c is a unit in $\mathcal{O}_K$. We see

$$\begin{pmatrix} 1 & -a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1/c \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_K).$$

□

Remark. Let $\varphi : \mathrm{GL}_2(K) \to \mathbb{C}$ be locally constant; then φ is continuous with respect to the discrete topology on $\mathbb{C}$ and hence its image must be finite. $\varphi(B(K))$ is controlled by the definition of $I(\chi_1, \chi_2)$.

Recall: $I(\chi_1, \chi_2)$ is a vector space, and is smooth and admissible. We have the norm $\| \cdot \| : K^\times \to \mathbb{Q}_{\geq 0}$, defined by $\pi_K \mapsto q_K^{-1}$.

Question: Is $I(\chi_1, \chi_2)$ irreducible?
No, this is easy to see if we study $I^{naive}(\chi_1, \chi_2) = \{\varphi(Mg) = \chi_1(a)\chi_2(d)\varphi(g)\}$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. and noticing that $I^{naive}(\chi_1, \chi_2) = I(\chi_1 \| \cdot \|^{-1/2}, \chi_2 \| \cdot \|^{1/2})$.

If $\chi_1 = \chi_2$ are both the trivial representation, using Frobenius reciprocity we can see that $I^{naive}(\chi_1, \chi_2)$ contains the constant functions, and therefore contains a one-dimensional G-invariant subspace. More generally, we can show that if $\chi_1 = \chi_2$ then I^{naive} contains a one-dimensional G-invariant subspace. There is another case when

$I(\chi_1, \chi_2)$ is not irreducible, this is when $\chi_1/\chi_2 = \|\cdot\|^{\pm 2}$. In this case $I^{naive}(\chi_1, \chi_2)$ has a one-dimensional quotient.

This is what happened: There's a duality, i.e., there is a natural pairing

$$I(\chi_1, \chi_2) \times I(\chi^{-1}, \chi_2^{-1}) \to \mathbb{C}$$

involving an integral on G and B. At some point we change left Haar measure on B to a right Haar measure. These don't coincide and differ by the "fudge factor" $\|a/d\|^{1/2}$. Thus, the dual is $I(\chi_1^{-1}, \chi_2^{-1})$.

Remark. (Theorem 1.21 Bernstein-Zelevinsky) Turns out $I(\chi_1, \chi_2)$ is irreducible if $\chi_1/\chi_2 \neq \|\cdot\|^{\pm 1}$. If $\chi_1/\chi_2 = \|\cdot\|^{-1}$ then we have the exact sequence

$$0 \to \rho \to I(\chi_1, \chi_2) \to S(\chi_1, \chi_2) \to 0$$

where $\rho = (\rho, V)$ is the one-dimensional representation of $\mathrm{GL}_2(K)$ given by $g \mapsto (\chi_1 \times \|\cdot\|_K^{1/2})(\det g)$. It is a fact that $S(\chi_1, \chi_2)$ is irreducible. It follows from duality that if $\chi_1/\chi_2 = \|\cdot\|^1$ then there is an exact sequence

$$0 \to S(\chi_2, \chi_1) \to I(\chi_1, \chi_2) \to \rho \to 0$$

There is a completely different construction (cf. Jacquet-Langlands; for understanding the articles in this book, Bump's book is helpful). It was an observation of Weil that for any field k one can write a precise presentation of $\mathrm{SL}_2(k)$:

$$\left\langle A(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, B(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;\middle|\; \text{explicit obvious relations} \right\rangle$$

Upshot: We can construct representations of $\mathrm{SL}_2(k)$ by giving explicit actions of the generators on some $\mathbb{C}$-vector space, and then checking relations. Weil observed that we can use the vector space of L^2 functions on k. These are another huge source of representations of $\mathrm{SL}_2(k)$, and hence $\mathrm{GL}_2(k)$.

(Theorem 4.6 in J-L, page 72) Let L/K be a quadratic extension. If $\chi : L^\times \to \mathbb{C}^\times$ is admissible, with $\chi \neq \chi \circ \sigma$ $(1 \neq \sigma \in \mathrm{Gal}(L/K))$, then one can construct an irreducible, infinite-dimensional representation $\mathrm{BC}_L^K(\chi)$ of $\mathrm{GL}_2(K)$ on the space of square-integrable functions on L.

Remark. 1. $I(\chi_1, \chi_2), S(\chi, \chi \times \|\cdot\|), \mathrm{BC}_L^K(\psi)$ are all infinite-dimensional, smooth-admissible representations. The last two are always irreducible, and the first is irreducible for the cases mentioned in the above discussion.

2. The only isomorphisms between the representations $I(\chi_1, \chi_2)$ have the form $I(\chi_1, \chi_2) = I(\chi_2, \chi_1)$, when $\chi_1/\chi_2 \neq \| \cdot \|^{\pm 1}$.

3. When $\mathrm{char}(\kappa_K) = p > 2$, then these are all the infinite-dimensional irreducible smooth-admissible representations of $\mathrm{GL}_2(K)$. This is *not* true for $p = 2$.

4. The only finite-dimensional representations π of $\mathrm{GL}_2(K)$ are one-dimensional, and hence factor through the abelianisation. Thus, it's of the form $\chi \circ \det$ where $\chi : K^\times \to \mathbb{C}^\times$.

9 Lecture Nine

Recall the examples we have seen so far of smooth, irreducible, admissible representations of $\mathrm{GL}_2(K)$, namely:

- $I(\chi_1, \chi_2)$, when $\chi_1/\chi_2 \neq \|\cdot\|^{\pm 1}$;
- $S(\chi, \chi \times \|\cdot\|)$;
- $\chi \circ \det$; and
- $BC_L^K(\psi)$.

It is a fact that, if $p = \mathrm{char}(k_K) > 2$, these are the only smooth irreducible admissible representations of $\mathrm{GL}_2(K)$.

If G is a connected, reductive group over K and π is a smooth admissible representation of $G(K)$, then there is a notion of "genericness" of π: if $G = \mathrm{GL}_2$, then π is called **generic** if $\dim(\pi)$ is infinite.

Hence, suppose π is an irreducible, admissible representation of $\mathrm{GL}_2(K)$, and assume that π is infinite-dimensional.

Theorem 9.1. *(Casselman) For $n \geq 0$, define*

$$U_1(\mathfrak{p}_K^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_K) : c \equiv 0 \bmod \mathfrak{p}_K^n, d \equiv 1 \bmod \mathfrak{p}_K^n \right\}.$$

Then every $U_1(\mathfrak{p}_K^n)$ is compact-open, and the quantity $d(\pi, n) := \dim\left(\pi^{U_1(\pi_K^n)}\right)$ is finite. Furthermore,

$$d(\pi, n) = \max\left(0, 1 + n - \mathfrak{f}(\pi)\right)$$

for all $n \geq 0$.

Proof. Omitted (see Casselman's Antwerp conference proceedings). □

Exercise 15. *Check that the theorem holds for $I(\chi_1, \chi_2)$, with $\chi_1/\chi_2 \neq \|\cdot\|^{\pm 1}$.*

Exercise 16. *Check that*

$$\mathfrak{f}\big(I(\chi_1, \chi_2)\big) = \mathfrak{f}(\chi_1) + \mathfrak{f}(\chi_2)$$

and that

$$\mathfrak{f}\big(S(\chi, \chi \times \|\cdot\|\big) = \begin{cases} 1 & \text{if } f(\chi) = 0, \\ 2\mathfrak{f}(\chi) & \text{if} f(\chi) > 0. \end{cases}$$

Exercise 17. *Prove Schur's lemma: if π is an irreducible admissible representation of* $\mathrm{GL}_n(K)$*, then there exists an admissible character* $\chi_\pi : K^\times \to \mathbb{C}^\times$ *such that, for all* $\lambda \in Z\big(\mathrm{GL}_n(K)\big) \cong K^\times$*, the action of* λ *on* π *is that of the scalar* $\chi_\pi(\lambda)$*; the character* χ_π *is called the* ***central character*** *of* π*.*

Exercise 18. *Show that*

$$\chi_{I(\chi_1,\chi_2)} = \chi_1\chi_2 = \chi_{S(\chi_1,\chi_2)} \text{ and } \chi_{\phi\circ\det} = \phi^2$$

for any $\phi : K^\times \to \mathbb{C}^\times$*.*

Now, returning again to the local Langlands correspondence, we introduce some notation: assuming that the correspondence is true for GL_1, write χ_i for the character $K^\times \to \mathbb{C}^\times$ associated to the representation $\rho_i : W_K \to \mathbb{C}^\times$ for $i \geq 1$, and conversely. Then the correspondence for GL_2 associates:

1. $I(\chi_1, \chi_2)$ to the Weil-Deligne representation $(\rho_0 = \rho_1 \oplus \rho_2, N = 0)$,
2. $S(\chi_1, \chi_1 \times \|\cdot\|)$ to
$$\left(\rho_0 = \begin{pmatrix} \|\cdot\|\rho_1 & 0 \\ 0 & \rho_1 \end{pmatrix}, N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right),$$
3. $\chi_1 \circ \det$ to
$$\left(\rho_0 = \begin{pmatrix} \rho_1 \times \|\cdot\|^{1/2} & 0 \\ 0 & \rho_1 \times \|\cdot\|^{-1/2} \end{pmatrix}, N = 0\right), \text{ and}$$
4. $BC_L^K(\psi)$ to $\big(\rho_0 = \mathrm{Ind}_{W_L}^{W_K}\sigma, N = 0\big)$, where Ind_H^G is the induced representation, $\psi : L^\times \to \mathbb{C}^\times$ and $1 \neq \sigma \in \mathrm{Gal}(L/K)$.

In the case $p = \mathrm{char}(k_K) = 2$, a problem arises from the fact that there is a Galois extension $K/\mathbb{Q}_2$ with Galois group isomorphic to S_4, which has no abelian subgroup of index 2.

For GL_2, the local Langlands correspondence was proven by explicitly writing down these associations; for general GL_n, representation-theoretic techniques are needed to reduce the problem to matching irreducible pairs (ρ_0, N) with so-called *supercuspidal* representations π, using a global argument.

We now introduce some terminology regarding Weil-Deligne representations (ρ_0, N). Define the **conductor** of (ρ_0, N) to be

$$\mathfrak{f}(\rho_0, N) = \mathfrak{f}(\rho_0) + \dim\big(V^{I_{\overline{K}/K}}/(\ker N)^{I_{\overline{K}/K}}\big).$$

Note that $\mathfrak{f}(\rho_0, 0) = \mathfrak{f}(\rho_0)$ and that $\det(\rho_0) : W_K \to \mathbb{C}^\times$.

Exercise 19. *Check that if π corresponds to the Weil-Deligne representation (for the* GL_2 *Langlands correspondence), then* $\mathfrak{f}(\pi) = \mathfrak{f}(\rho_0, N)$ *and that* χ_π *corresponds to* $\det(\rho_0)$ *via local class field theory.*

Exercise 20. *Check that, if $p > 2$, then the only F-semisimple, two-dimensional Weil-Deligne representations of W_K are the ones we have listed above. (Hint: look up result 2.2.5.2 in Tate's "Number theoretic background").*

We will close our discussion of the local Langlands correspondence with an explicit consideration of the "unramified" case (i.e., when $\mathfrak{f}(\pi) = 0$) in dimension two. If (ρ_0, N) is the corresponding Weil-Deligne representation, then $\mathfrak{f}(\rho_0, N) = 0$ also, and there are precisely two options: either

1. $\pi = I(\chi_1, \chi_2)$ for some $\chi_i : K^\times \to \mathbb{C}^\times$ factoring through $K^\times/\mathcal{O}_K^\times$ with $\chi_1/\chi_2 \neq \|\cdot\|^{\pm 1}$; or

2. $\pi = \chi \circ \det$ for some $\chi : K^\times \to \mathbb{C}^\times$ factoring through $K^\times/\mathcal{O}_K^\times$.

If $\dim \pi = \infty$, so that $\pi = I(\chi_1, \chi_2)$ as in case 1, then the fact that $\mathfrak{f}(\pi) = 0$ implies that $\pi^{\mathrm{GL}_2(\mathcal{O}_K)}$ is one-dimensional.

More generally, if G is a connected, reductive, *unramified* group over K (such as GL_n), and π is a smooth admissible irreducible representation of $G(K)$, then we say that π is **unramified** if there exists a *hyperspecial* maximal compact subgroup H of $G(K)$ such that $\pi^H \neq 0$ (such as $H = \mathrm{GL}_n(\mathcal{O}_K)$).

If we want to perform calculations with π, a good place to start is with the invariant subspace $\pi^{\mathrm{GL}_2(\mathcal{O}_K)}$, which we note is *not* $\mathrm{GL}_2(K)$-invariant. We will do this by introducing the *Hecke operators*.

Let $G = \mathrm{GL}_2(K)$; or, more generally, any locally compact, totally disconnected topological group (maybe satisfying some additional properties). Then if π is a representation of G and $U, V \subseteq G$ are compact open subgroups (for instance, $U = U_1(\mathfrak{p}_K^n)$ or $\mathrm{GL}_2(\mathcal{O}_K)$) and $g \in G$, there exists a **Hecke operator**

$$[UgV] : \pi^V \to \pi^U,$$

which is $\mathbb{C}$-linear, and is defined as follows: write the (compact) subset UgV as a disjoint union of its cosets

$$UgV = \coprod_{i=1}^{r} g_i V,$$

and put

$$[UgV]x := \sum_{i=1}^{r} g_i x.$$

This may be thought of as a sort of averaging process.

Exercise 21. *Check that $[UgV]x$ does indeed lie in π^U, and is independent of choice of coset representatives g_i.*

Returning to the $\mathrm{GL}_2(K)$ case, and assuming $\mathfrak{f}(\pi) = 0$, let us take $U = V = \mathrm{GL}_2(\mathcal{O}_K)$, and define the operators S, T to be

$$S := \left[U\left(\begin{smallmatrix} \pi_K & 0 \\ 0 & \pi_K \end{smallmatrix}\right)V\right] : \mathrm{GL}_1(\mathcal{O}_K) \to \mathrm{GL}_1(\mathcal{O}_K).$$

and

$$T := \left[U\left(\begin{smallmatrix} \pi_K & 0 \\ 0 & 1 \end{smallmatrix}\right)V\right] : \mathrm{GL}_2(\mathcal{O}_K) \to \mathrm{GL}_2(\mathcal{O}_K).$$

Then $\dim \pi^{\mathrm{GL}_1(\mathcal{O}_K)} = 1$, so by Schur's lemma, T acts via a scalar $t \in \mathbb{C}$ and $s \in \mathbb{C}$.

Exercise 22. *If $\pi = I(\chi_1, \chi_2), \chi_1/\chi_2 \neq \|\cdot\|^{\pm 1}$, and $\mathfrak{f}(\pi) = 0$, then show that*

$$t = \sqrt{q_K} \times (\alpha + \beta) \text{ and } s = \chi_\pi(\pi_K) = \alpha\beta,$$

where $\alpha = \chi_1(\pi_K), \beta = \chi_2(\pi_K)$, and $q_K = |k_K|$. Also check the case when π is one-dimensional and unramified.

Exercise 23. *Show that if π is an irreducible admissible representation of $\mathrm{GL}_2(K)$ and $\mathfrak{f}(\pi) = 0$ (where π is $I(\chi_1, \chi_2)$ or is one-dimensional, then π corresponds to the Weil-Deligne representation $(\rho_0, 0)$, where*

$$\rho_0 : W_K \to W_K/I_{\overline{K}/K} \cong \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C}),$$

and $\rho_0(\mathrm{Frob})$ has characteristic polynomial $X^2 - \frac{t}{\sqrt{q_K}}X + s$ and $\mathrm{Frob} \in W_K/I_{\overline{K}/K}$.

Exercise 24. *If $G = \mathrm{GL}_n$ and*

$$T_i = \mathrm{GL}_n(\mathcal{O}_K)\begin{pmatrix}\pi_K &&&&&& \\ & \pi_K &&&&& \\ && \ddots &&&& \\ &&& \pi_K &&& \\ &&&& 1 && \\ &&&&& \ddots & \\ &&&&&& 1\end{pmatrix}\mathrm{GL}_n(\mathcal{O}_K)$$

(with i copies of π_K on the diagonal), corresponding to the scalars t_i, then compute the characteristic polynomial of $\rho_0(\mathrm{Frob})$.

If $G = G(K)$ with G/K unramified, and if π is an unramified representation of G, then Langlands reinterpreted the Satake isomorphism to associate to π a semisimple conjugacy class in ${}^L G(\mathbb{C})$; in particular, the conjugacy class is that of $\rho_0(\mathrm{Frob})$, where π corresponds to (ρ_0, N).

10 Lecture Ten

Part 2: The global Langlands correspondence.

In this part, K will be a number field, i.e. a finite extension of $\mathbb{Q}$.

We'll start by talking about the structure $\mathrm{Gal}(L/K)$ of a finite Galois extension L/K, and in particular its relationship to local Galois groups. Taking limits, we get structure on $\mathrm{Gal}(\overline{K}/K)$. In the local situation, the Weil group wasn't quite the right answer because we needed Weil-Deligne representations; in the global case, we need the "global Langlands group", but no one knows what that is...[3]

However, we still have ℓ-adic representations of $\mathrm{Gal}(\overline{K}/K)$, and these are maybe a working definition of the ρ-side of the correspondence. And then on the π-side is where we see automorphic representations.

The global Langlands philosophy is that every automorphic representation of $\mathrm{GL}_2(K)$ should yield a 2-dimensional representation of the global Langlands group (whatever this is...)

"Uncheckable conjecture": all automorphic representations for π should correspond to n-dimensional representations of a global Langlands group.

Here's a checkable conjecture instead: all algebraic automorphic representations π of $\mathrm{GL}_n(K)$ should correspond to "motivic" n-dimensional representation of $\mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ ("unramified outside a finite set, de Rham...") and these should correspond to "motives":

$$\begin{tikzcd}
\left\{\begin{array}{l}\text{algebraic automorphic rep-}\\ \text{resentations } \pi \text{ of } \mathrm{GL}_n(K)\end{array}\right\} \arrow[rr, leftrightarrow] & & \left\{\begin{array}{l}\text{“motivic” representations}\\ \rho : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})\end{array}\right\} \\
& \{\text{motives}\} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Furthermore, if we abelianize the global Langlands group, we should get $K^\times \backslash \mathbb{A}_K^\times$.

In $n = 1$, the "uncheckable conjecture" will be global class field theory.[4]

[3] Quote: 'You can begin to see why this is called a "philosophy", because some parts are very well-defined, and for other parts you have to be... a little bit of a dreamer. I think I read in a paper of Arthur that in 100 or so years time after we've proved all the various Langlands correspondences, maybe the very last theorem will be that the global Langlands group exists. Maybe we'll have to construct all the representations and then say, "ha ha! by the Tannakian formalism..." '

[4] Quote: 'I don't know if this is worth mentioning, but I've had this dream—the automorphic representations of $\mathrm{GL}_n(K)$ should be a subset of a new p-adic space of p-adic automorphic representations of $\mathrm{GL}_n(K)$.'

10.1 Galois groups

Return to a finite extension $K/\mathbb{Q}$.

K contains $\mathcal{O}_K$, the algebraic integers in K; for instance if $K = \mathbb{Q}$, then $\mathcal{O}_K = \mathbb{Z}$.

Choose $0 \neq \mathfrak{p}$, a prime (and hence maximal). This will be an ideal of $\mathcal{O}_K$, with $\mathcal{O}_K/\mathfrak{p} =: k_\mathfrak{p}$ a finite residue field.

We can complete K at p. Here's one way of doing it: we can define

$$\mathcal{O}_{K_\mathfrak{p}} \stackrel{\text{df}}{=} \varprojlim \mathcal{O}_K/\mathfrak{p}^n$$

and we put

$$K_\mathfrak{p} \stackrel{\text{df}}{=} \operatorname{Frac}\left(\mathcal{O}_{K_\mathfrak{p}}\right).$$

Alternatively, with $\mathfrak{p}$ fixed, then if $\lambda \in K^\times$, then $\lambda\mathcal{O}_K$ is a fractional ideal of K and therefore factors as the product of $\mathfrak{p}^{v_\mathfrak{p}(\lambda)}$ with other prime ideals to various powers, where $v_\mathfrak{p} : K^\times \twoheadrightarrow \mathbb{Z}$, and we can define $\|\lambda\|_\mathfrak{p}$ a norm on K with $\|0\|_\mathfrak{p} = 0$ ad $\|\lambda\|_\mathfrak{p} \stackrel{\text{df}}{=} (q_\mathfrak{p})^{-v_\mathfrak{p}(\lambda)}$.

The norm on K induces a metric, and now we can complete K with respect to this metric and get $K_\mathfrak{p}$ a local field. $K_\mathfrak{p}$ is a finite extension of $\mathbb{Q}_p$, where $\mathfrak{p} \cap \mathbb{Z} = p$.

Now, let L/K be a finite Galois extension of number fields. We get a finite Galois group $\operatorname{Gal}(L/K)$.

Let's say we've got $\mathfrak{p} \subseteq \mathcal{O}_K$ as above, and we've got this number field L with $\mathfrak{p}\mathcal{O}_L \subseteq L$:

$$\begin{tikzcd} \mathcal{O}_K \arrow[r, hook] & \mathcal{O}_L \\ \mathfrak{p} \arrow[u, hook] \arrow[r, mapsto] & \mathfrak{p}\mathcal{O}_L \arrow[u, hook] \end{tikzcd}$$

We know from undergraduate abstract algebra that $\mathfrak{p}\mathcal{O}_L$ will be an ideal, but may not be a prime ideal. However, it factors into prime ideals of L, say

$$\mathfrak{p}\mathcal{O}_L = \mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}\cdots\mathfrak{P}_g^{e_g},$$

with each $\mathfrak{P}$ a prime ideal of $\mathcal{O}_L$.

$\operatorname{Gal}(L/K)$ acts on L, and on $\mathcal{O}_L$ (by exercise 5), and acts trivially on K and hence on $\mathfrak{p}$.

Therefore, $\operatorname{Gal}(L/K)$ maps $\mathfrak{p}\mathcal{O}_L$ to $\mathfrak{p}\mathcal{O}_L$.

Exercise 25. *Show that, if* $\sigma \in \mathrm{Gal}(L/K)$ *then* $\sigma(\mathfrak{P}_i)$ *is a prime ideal of* $\mathcal{O}_L$*, so* $\sigma(\mathfrak{P}_i)$ *lies in the set* $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_g\}$.[5]

Exercise 26. *Show that the action of* $\mathrm{Gal}(L/K)$ *on the set of primes of* $\mathcal{O}_L$ *dividing* $\mathfrak{p}$ *is transitive.*

As a consequence we see that $e_1 = \cdots = e_g$. This also means in some sense that the $\mathfrak{P}_i$ are all isomorphic, since they are Galois-conjugate. Therefore,

$$L_{\mathfrak{P}_1} \simeq \cdots \simeq L_{\mathfrak{P}_g},$$

where $L_{\mathfrak{P}} = \mathcal{O}_L/\mathfrak{P}$.

Here's the set-up. We have L/K finite Galois, $\mathfrak{p}$ as before, so that $\mathfrak{p}\mathcal{O}_K$ factors into $\prod_{i \leq g} \mathfrak{P}_i^{e_i}$. Set $\mathfrak{P} = \mathfrak{P}_1$, fixed choice of a prime of L.[6]

Define $\mathcal{D}_{\mathfrak{P}} \stackrel{\text{df}}{=} \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma(\mathfrak{P}) \subseteq \mathfrak{P}\}$. Then $\mathrm{Gal}(L/K)/\mathcal{D}_{\mathfrak{P}} = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_g\}$.

Our task is to understand this $\mathcal{D}_{\mathfrak{P}}$. If $\sigma \in \mathcal{D}_{\mathfrak{P}}$, then $\sigma : L \to L$, $\mathcal{O}_L \to \mathcal{O}_L$, and $\mathfrak{P} \to \mathfrak{P}$. Then again by transport de structure, we get $\sigma : L_{\mathfrak{P}} \to L_{\mathfrak{P}}$ fixing $K_{\mathfrak{p}}$.

It turns out that $L_{\mathfrak{P}}/K_{\mathfrak{p}}$ is Galois and that $\mathcal{D}_{\mathfrak{P}} \simeq \mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}})$, so $\mathrm{Gal}(L_{\mathfrak{P}}/L_{\mathfrak{p}})$ lives inside $\mathrm{Gal}(L/K)$.

(This is in Serre's *Local fields*).

Recap: we start with $\mathrm{Gal}(L/K)$ and we choose some $\mathfrak{p} \subseteq \mathcal{O}_K$. Then we have to choose a $\mathfrak{P} \mid \mathfrak{p}\mathcal{O}_L$, which induces an identification $\mathcal{D}_{\mathfrak{P}} \subseteq \mathrm{Gal}(L/K)$, and we can identify $\mathcal{D}_{\mathfrak{P}}$ with $\mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}})$.

Inside $\mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}})$ is an inertia subgroup. A global fact: if $\mathfrak{p}$ does not divide the discriminant of L/K, then this inertia subgroup is trivial.

If $L_{\mathfrak{P}}$ is an unramified extension of $K_{\mathfrak{p}}$, then there exists $\mathsf{Frob}_{\mathfrak{P}} \in \mathcal{D}_{\mathfrak{P}} \subseteq \mathrm{Gal}(L/K)$.

$\mathsf{Frob}_{\mathfrak{P}}$ is slightly annoying: because it depends not only on $\mathfrak{p}$ but also on the choice of $\mathfrak{P} \mid \mathfrak{p}\mathcal{O}_L$. Say $\mathfrak{P}'$ is another choice.

[5]Here's a trivial observation: if X and Y are objects in mathematics, topological spaces or groups or whatever, and you do some kind of calculation in X which involves taking elements in X and using axioms or whatever, then because this calculation is definable in X, an appropriate notion of isomorphism will show the analogous calculation in Y to be valid also. This is called by Deligne, in French, "transport de structure," and it's a completely trivial observation. But sometimes mathematics is about giving the right definitions or thinking about things the right way. It turns out that thinking clearly about this trivial observation in the particular case where X happens to be equal to Y but the isomorphism is *not* the identity map can sometimes really help, e.g. $X = Y = L$ and σ the isomorphism is in $\mathrm{Gal}(L/K)$.

[6]'Somehow you remember where you were when you learn things. I learned about modules at a bar. I was reading Atiyah-Macdonald.'

By transitivity of the action of $\mathrm{Gal}(L/K)$ on $\{\mathfrak{P}_1, \dots, \mathfrak{P}_g\}$, there exists a $\sigma \in \mathrm{Gal}(L/K)$ such that $\sigma(\mathfrak{P}) = \mathfrak{P}'$, and then by transport de structure, $D_{\mathfrak{P}'} = \sigma D_{\mathfrak{P}} \sigma^{-1}$ and $\mathsf{Frob}_{\mathfrak{P}'} = \sigma \, \mathsf{Frob}_{\mathfrak{P}} \, \sigma^{-1}$.

Upshot from these considerations: you can define $\mathsf{Frob}_{\mathfrak{p}}$ to be the conjugacy class of $\mathsf{Frob}_{\mathfrak{P}} = \{\mathsf{Frob}_{\mathfrak{P}'} \text{ such that } \mathfrak{P}' \mid \mathfrak{p}\mathcal{O}_L\}$, and this works for all $\mathfrak{p}$ not dividing the discriminant of L over K.

11 Lecture Eleven

11.1 Decomposition and inertia subgroups

Recall that for L/K a finite Galois extension and $\mathfrak{p}$ a nonzero prime ideal of $\mathcal{O}_K$, the ideal $\mathfrak{p}\mathcal{O}_L$ Is probably not prime. We may factor $\mathfrak{p}\mathcal{O}_L$ as $\prod_{i=1}^r \mathfrak{P}_i^{e_i}$, a product of prime ideals of $\mathcal{O}_L$.

As we have seen, $\mathrm{Gal}(L/K)$ acts on the set of these $\mathfrak{P}_i$ - in fact, it acts transitively on this set.

Fix $\mathfrak{P} = \mathfrak{P}_i$ for some I and set $D_{\mathfrak{P}/\mathfrak{p}} = \{\sigma \in \mathrm{Gal}(L/K) : \sigma(\mathfrak{P}) = \mathfrak{P}\}$. This is the **decomposition group**.

> **Exercise 27.** *Check that, if $\sigma \in D_{\mathfrak{P}/\mathfrak{p}}$, then σ acts on the completion $L_{\mathfrak{P}}$ of L at $\mathfrak{P}$ and fixes $K_{\mathfrak{p}}$, and so $\sigma \in \mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}}) = D_{\mathfrak{P}/\mathfrak{p}} \subseteq \mathrm{Gal}(L/K)$.*

We can then define again an inertia subroup $I_{\mathfrak{P}/\mathfrak{p}} = I_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} \subseteq \mathrm{Gal}(L_{\mathfrak{P}}/K_{\mathfrak{p}}) = D_{\mathfrak{P}/\mathfrak{p}}$.

Now we consider a miracle. There is an ideal $\Delta = \mathrm{disc}(L/K) \subseteq \mathcal{O}_K$, the **discriminant** of L/K, such that $\mathfrak{p} \nmid \Delta$ if and only if $I_{\mathfrak{P}/\mathfrak{p}} = \{1\}$ for any $\mathfrak{P}$ lying over $\mathfrak{p}$. (The inertia subgroups are independent of the choice of $\mathfrak{P}$ up to isomorphism, by transport de structure.)

Thus for all $\mathfrak{p}$ excepting some finite set $S = \{\mathfrak{p} : \mathfrak{p} \mid \Delta_{L/K}\}$ we get for any $\mathfrak{P}$ lying over $\mathfrak{p}$ a cyclic group $D_{\mathfrak{P}/\mathfrak{p}} = \langle \mathrm{Frob}_{\mathfrak{P}} \rangle = D_{\mathfrak{P}/\mathfrak{p}}/I_{\mathfrak{P}/\mathfrak{p}} = \mathrm{Gal}(k_{\mathfrak{P}}/k_{\mathfrak{p}})$.

Remark. Say L/K is finite, then what is $\langle \mathrm{Frob}_{\mathfrak{P}} \rangle$? To see that, pick $\mathfrak{P}/\mathfrak{p}$, then there exists a unique element $\sigma \in \mathrm{Gal}(L/K)$, s.t. $\sigma(x) \equiv x^{\# k_{\mathfrak{p}}}$ for all $x \in \mathcal{O}_L$. Then we denote $\mathrm{Frob}_{\mathfrak{P}}$ to be the element σ and choose $\langle \mathrm{Frob}_{\mathfrak{P}} \rangle$ to be the conjugacy class of that element. One could show that the class is independent of the choice of the prime ideal $\mathfrak{P}$ lying above $\mathfrak{p}$.

However we now encounter an issue: $\mathrm{Frob}_{\mathfrak{P}}$ depends on choice of $\mathfrak{P}$ lying over $\mathfrak{p}$, so $\mathfrak{p}$ determines a conjugacy class of automorphisms $\mathrm{Frob}_{\mathfrak{P}}$ for $\mathfrak{P}|\mathfrak{p}$. By abuse of notation, we will write $\mathrm{Frob}_{\mathfrak{p}}$ for that conjugacy class.

Here is another important fact. Every conjugacy class C of $\mathrm{Gal}(L/K)$ is equal to $\mathrm{Frob}_{\mathfrak{p}}$ for infinitely many $\mathfrak{p}$. This is a consequence of the **Chebotarev density theorem.**

In fact, the density of primes $\mathfrak{p}$ such that $\mathrm{Frob}_{\mathfrak{p}} = C$ is $\#C/\#\mathrm{Gal}(L/K)$.

11.2 Infinite extensions

For K a number field with fixed algebraic closure $\overline{K}$, $\mathrm{Gal}(\overline{K}/K)$ is ramified at every prime of K.

Let S be some finite set of maximal ideals of $\mathcal{O}_K$. We want to be able to define $\mathrm{Frob}_{\mathfrak{p}}$ for $\mathfrak{p} \notin S$.

It turns out that if $\overline{K}/L_1/K, \overline{K}/L_2/K$ are towers of extensions so that L_i/K finite and unramified outside S, then the same is true of the compositum.

Thus we can define

$$K^S = \bigcup_{\substack{L/K \text{ finite Galois} \\ \text{unramified outside } S}} L.$$

For example, if S is empty, $\mathbb{Q}^S = \mathbb{Q}$. However for general K a number field, even if S is empty the extension K^S/K may be infinite.

Theorem 11.1. *Let S be a finite set of primes of K, let $d \in \mathbb{Z}_{>0}$, then there exists only finite many L/K satisfying*

(1). $[L : K] \leq d$;

(2). L/K is unramified outside S.

Theorem 11.2. *(**Chebotarev**) Let L/K be a finite Galois extension. Let $C \subset \mathrm{Gal}(L/K)$ be a conjugacy class. Then there are infinitely many places v of K, unramified in L, s.t. $Frob_v = C$. In fact,*

$$\frac{|C|}{|\mathrm{Gal}(L/K)|} = \textit{Drichlet density of number of such primes}$$

Now consider $K = \mathbb{Q}, S = \{p\}$. It is immediately apparent that K^S contains $\mathbb{Q}(\zeta_{p^n})$ for any $n \geq 1$.

For $K = \mathbb{Q}, N \in \mathbb{Z}_{\geq 1}$, define $S = \{p : p \mid N\}$; then similarly K^S contains $\mathbb{Q}(\zeta_N)$. If $p \notin S$ is a prime, there should be a canonical conjugacy class Frob_p in $(\mathbb{Z}/N\mathbb{Z})^\times$, which is abelian. It is immediately apparent that we *should* have $\mathrm{Frob}_p = p \bmod N$. This can be seen by considering the action on ζ_N and the action on the residue field.

Let's stick to the case now $K = \mathbb{Q}, S = \{p\}$. It is immediately apparent that K^S contains $\bigcup_{n=1}^{\infty} \mathbb{Q}(\zeta_{p^n}) =: \mathbb{Q}(\zeta_{p^\infty})$. Hence $\mathrm{Gal}(K^S/K)$ surjects onto $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) = \mathbb{Z}_p^\times$.

If r is prime and $r \neq p$, we want to find Frob_r in $\mathbb{Z}_p^\times$: similar considerations as before make it clear that in each $(\mathbb{Z}/p^n\mathbb{Z})^\times = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$ we have $\mathrm{Frob}_r = r \bmod p^n$, so $\mathrm{Frob}_r \in \mathbb{Z}_p^\times$ is easily defined.

In the more general case, we may write $\mathrm{Gal}(K^S/K)$ as a projective limit of $\varprojlim \mathrm{Gal}(L/K)$ for L/K finite Galois and unramified outside S – in which case for all $\mathfrak{p} \notin S$ we get conjugacy classes $\mathrm{Frob}_{\mathfrak{p},L/K} \subseteq \mathrm{Gal}(L/K)$ for each L which 'glue together' in a nice way to give a conjugacy class $\mathrm{Frob}_{\mathfrak{p},K^S/K} \subseteq \mathrm{Gal}(K^S/K)$.

The Chebotarev density theorem does not apply to infinite extensions. However, if L/K is an infinite Galois extension unramified outside S, the map $\mathfrak{p} \mapsto \mathrm{Frob}_{\mathfrak{p}}$ for $\{\mathfrak{p} \notin S\}$ has dense image; that is, $\bigcup_{\mathfrak{p}\notin S} \mathrm{Frob}_{\mathfrak{p}} \subseteq \mathrm{Gal}(L/K)$ is dense.

As a corollary, if we have some map $F : \mathrm{Gal}(L/K) \to X$ continuous and constant on conjugacy classes, we may be able to recover F from the data $F(\mathrm{Frob}_{\mathfrak{p}})$ for $\mathfrak{p} \notin S$.

11.3 Representations of Galois groups

Theorem 11.3 (Brauer-Nesbitt). *Let G be a group and E a field. Recall that a representation $\rho : G \to \mathrm{GL}_n(E)$ is called semisimple if ρ is a direct sum of irreducible representations. If we have two semisimple representations $\rho_1, \rho_2 : G \to \mathrm{GL}_n(E)$ and for any $g \in G$, $\{\rho_i(g)\}_{i=1,2}$ have the same characteristic polynomials, then in fact $\rho_1 \cong \rho_2$.*

Proof. Algebra. □

Remark. The above is *not* true that if we replace $\mathrm{GL}_n(E)$ with an arbitrary group and 'having the same characteristic polynomial' with being conjugate to one another.

Remark. If $\mathrm{char}(E) = 0$ it suffices that $\mathrm{tr}(\rho_1) = \mathrm{tr}(\rho_2)$ (though both must still be semisimple). If $n!$ is invertible, this is true since we can compute the characteristic polynomial of $\rho(g)$ from $\mathrm{tr}(\rho(g^i))$ for $i = 1, \dots, n$.

Exercise 28. *For $G = \mathbb{Z}/3\mathbb{Z}, E = \overline{\mathbb{F}}_2$, find non-isomorphic ρ_1, ρ_2 which are semisimple and reducible such that* $\mathrm{tr}(\rho_1) = \mathrm{tr}(\rho_2)$*.*

The upshot of all this is that if we have an ℓ-adic representation $\rho : \mathrm{Gal}(K^S/K) \to \mathrm{GL}_n(E)$ (where E some finite extension of $\mathbb{Q}_\ell$), continuous with regard to the ℓ-adic topology and semisimple (or just irreducible), the characteristic polynomial of $\rho(\mathrm{Frob}_{\mathfrak{p}})$ for all $\mathfrak{p} \notin S$ is a well-defined polynomial $F_{\mathfrak{p}} \in E[x]$. IF we know all $F_{\mathfrak{p}}$ for $\mathfrak{p} \notin S$, then ρ is uniquely determined by this data.

Now, as an example, let's do the cyclotomic character.

Let $K = \mathbb{Q}, S = \{p\}$. Let $L = \mathbb{Q}(\zeta_{p^\infty})$. Then $\mathrm{Gal}(L/K) = \mathbb{Z}_p^\times \subseteq \mathrm{GL}_1(\mathbb{Q}_p)$.

By the fundamental theorem of Galois theory, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ surjects onto $\mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \subseteq \mathrm{GL}_1(\mathbb{Q}_p)$. Hence we get a one-dimensional p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. ρ

is called the **p-adic cyclotomic character**, ω_p. It is determined by the fact that $\rho(\mathrm{Frob}_r) = r \in \mathbb{Z}_p^\times$ for all primes r not equal to p.

A confusing thing: if p, ℓ are distinct primes, with $S = \{p, \ell\}$, we have two representations $\omega_p : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}^S/\mathbb{Q}) \to \mathbb{Z}_p^\times$ and $\omega_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times$. Note that Frob_r for $r \notin S$ give a dense subset of $\mathrm{Gal}(\mathbb{Q}^S/\mathbb{Q})$.

Then ω_p, ω_ℓ are two Galois representations that are equal on some dense subset of $\mathrm{Gal}(\mathbb{Q}^S/\mathbb{Q})$. Does Brauer-Nesbitt imply $\omega_p = \omega_\ell$? No, because these representations are over different fields.

In fact we will find that $\ker \omega_p, \ker \omega_\ell$ are very different.

12 Lecture Twelve

Another weird example: Let $K/\mathbb{Q}$ be a number field and E be an elliptic curve over K, and let S_0 be the set of finite places (maximal ideals of $\mathcal{O}_K$) where E has bad reduction. Let ℓ be a prime number; the Galois group $\mathrm{Gal}(\bar{K}/K)$ acts on $E[\ell^n](\bar{K})$. The *Tate module* is defined as

$$\varprojlim_n E[\ell^n] = T_\ell[E] \simeq \mathbb{Z}_\ell \times \mathbb{Z}_\ell.$$

We can realize the Galois action by looking at the map

$$\rho_{E,L} : \mathrm{Gal}(\bar{K}/K) \to \mathrm{GL}_2(\mathbb{Z}_\ell) = \mathrm{Aut}(\mathbb{Z}_\ell \times \mathbb{Z}_\ell).$$

The above map is well defined up to conjugation. $\rho_{E,L}$ factors through $\mathrm{Gal}(K^{S_0 \cup \{\mathfrak{p}|\ell\}}/K)$. The characteristic polynomial of $\rho_{E,\ell}(\mathrm{Frob}_\mathfrak{p})$ is $X^2 - a_\mathfrak{p} X + N(\mathfrak{p}) \in \mathbb{Q}[X] \hookrightarrow \mathbb{Q}_\ell[X]$ where $a_\mathfrak{p} = 1 + N(\mathfrak{p}) - \#\bar{E}(\kappa_\mathfrak{p})$ and $N(\mathfrak{p})$ is the norm. The trace of $\rho_{E,L}(\mathrm{Frob}_\mathfrak{p})$ is $a_\mathfrak{p}$, which is independent of ℓ. What we need to observe is that $\rho_{E,\ell}, \rho_{E,\mathfrak{p}}$ are not isomorphic when $\mathfrak{p} \neq \ell$. This is because $\rho_{E,\ell}$ is infinitely ramified at ℓ, but $\rho_{E,\mathfrak{p}}$(wild inertia at ℓ) is finite.

12.1 ℓ-adic Representations

Let K be a number field and $E/\mathbb{Q}_\ell$ be a finite extension. Let S be a finite set of maximal ideals of $\mathcal{O}_K$.

Definition. If $\rho : \mathrm{Gal}(K^S/K) \to \mathrm{GL}_n(E)$ is continuous with respect to the ℓ-adic topology on the right-hand side, and the profinite topology on the left-hand side, we call ρ an *ℓ-adic representation* of $\mathrm{Gal}(\bar{K}/K)$. We also say that ρ is *unramified* outside S.

Definition. We say that ρ is **rational** over E_0 (a subfield of E) if, for all $\mathfrak{p} \notin S$, the characteristic polynomial $\rho(\mathrm{Frob}_\mathfrak{p})$ lies in $E_0[X]$.

Example 2. 1. Consider the cyclotomic character

$$\begin{aligned} \omega_\ell : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) &\to \mathrm{GL}_1(\mathbb{Q}_\ell) \\ \mathrm{Frob}_r &\mapsto r, \qquad r \neq \ell. \end{aligned}$$

Then ω_ℓ is rational over $\mathbb{Q}$.

2. Tate modules of elliptic curves are rational over $\mathbb{Q}$.

3. (Deligne) ℓ-adic étale cohomology of smooth proper algebraic varieties (over K), $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ are rational over $\mathbb{Q}$.

Definition. We say that ρ is **pure of weight** w if ρ is rational over a number field E and, for all embeddings $i : \bar{E} \hookrightarrow \mathbb{C}$ and all eigenvalues α of $\rho(\mathrm{Frob}_\mathfrak{p})$ ($\mathfrak{p} \notin S$), we have

$$|i(\alpha)| = (\#k_\mathfrak{p})^{-w/2}.$$

Example 3. 1. (Deligne) $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$ is pure of weight i when X is a smooth proper algebraic variety.

2. Cyclotomic characters are pure of weight -2.

3. Tate modules are pure of weight -1; in this case, the roots of $X^2 - a_\mathfrak{p} X + N(\mathfrak{p})$ are complex conjugates, and $|a_\mathfrak{p}| \leq 2\sqrt{N(\mathfrak{p})}$ (Hasse inequality).

Now ℓ will vary. Let K be a number field, E_0 be a subfield of E, a finite extension of $\mathbb{Q}$ and S_0 be a set of finite places. We are given a huge amount of data, as for all $\mathfrak{p} \notin S_0$ we have a polynomial $F_\mathfrak{p} \in E_0[X]$. Also, for all maximal ideals $\lambda \subseteq \mathcal{O}_{E_0}$ we have an ℓ-adic representation

$$\rho_\lambda : \mathrm{Gal}(K^{S_0 \cup \{\mathfrak{p}|\ell\}}/K) \to \mathrm{GL}_n(\overline{(E_0)_\lambda})$$

where the completions $(E_0)_\lambda$ for $\lambda \mid \ell$ are finite extensions of $\mathbb{Q}_\ell$.

Definition. We say ρ_λ is a *compatible system of λ-adic representations* if for all $\lambda \mid \ell$ and $\mathfrak{p} \notin S_0$, $\mathfrak{p} \nmid \ell$, the characteristic polynomial $\rho_\lambda(\mathrm{Frob}_\mathfrak{p}) = F_\mathfrak{p}(X)$ independent of λ.

Example 4. 1. For cyclotomic characters, $F_p(X) = X - p$ and so is a compatible system.

2. For Tate modules $T_\ell E$, for all ℓ and $E_0 = \mathbb{Q}$, one has

$$F_\mathfrak{p}(X) = X^2 - a_\mathfrak{p} X + N(\mathfrak{p}),$$

and hence this is a compatible system.

3. $H^i_{\text{ét}}$ are known to be compatible systems.

The main aim of global class field theory is to understand $\mathrm{Gal}(\bar{K}/K)^{ab}$.

12.2 Adèles

Let K be a number field of degree d. There are d embeddings $\sigma : K \hookrightarrow \mathbb{C}$. Let the number of real embeddings be r_1 and the number of complex embeddings be $2r_2$, then $d = r_1 + 2r_2$.

Definition. An *infinite place* of K is either a real place $v = \sigma : K \to \mathbb{R}$ or $v = \{\sigma, \sigma \circ c\} : K \to \mathbb{C} \setminus \mathbb{R}$, where c is complex conjugation. We then define

$$K_\infty = \prod_v K_v,$$

the product taken over all infinite places.

Remark.

1. $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R}$.
2. $K_\infty = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ as rings.
3. $K_\infty^\times = (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$ are not connected in general.

13 Lecture Thirteen

Last time we introduced the notion of an infinite place of a number field K. For instance, if $K = \mathbb{Q}(\sqrt[3]{2})$, then we have

$$K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R} = \big(\mathbb{Q}[X]/(X^3-2)\big) \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}[X]/(X^3-2).$$

The roots of our polynomial are $\alpha = \sqrt[3]{2} \in \mathbb{R}$, $w = \zeta\alpha$, and $\bar{w} = \zeta^2\alpha$, where ζ_3 is a primitive third root of unity; thus

$$K_\infty = \mathbb{R}[X]/(X-\alpha)(X^2+\alpha X+\alpha^2).$$

The Chinese remainder theorem implies that this is isomporphic to

$$\big(\mathbb{R}[X]/(X-\alpha)\big) \times \big(\mathbb{R}[X]/(X^2+\alpha X+\alpha^2)\big) \cong \mathbb{R} \times \mathbb{C};$$

the isomorphism with the real field is canonical, whereas the isomorphism with $\mathbb{C}$ is only unique up to complex conjugation.

Definition. Let K be a number field and $\mathfrak{p} \subseteq \mathcal{O}_K$ a maximal ideal (i.e. *finite place*), and $K_\mathfrak{p}$ the completion of K at $\mathfrak{p}$. The **finite adèle ring** of K (or the **finite adèles** of K) is the restricted product of all finite places $K_\mathfrak{p}$ with respect to the family $\mathcal{O}_{K_\mathfrak{p}}$; that is,

$$\mathbb{A}_{K,f} := \prod_{\mathfrak{p}}{}' K_\mathfrak{p} \subseteq \prod_{\mathfrak{p}} K_\mathfrak{p},$$

the prime on the product indicating that, if $x = (x_\mathfrak{p})_\mathfrak{p} \in \mathbb{A}_{K,f}$ with every $x_\mathfrak{p} \in K_\mathfrak{p}$, then $x_\mathfrak{p} \in \mathcal{O}_{K_\mathfrak{p}}$ for all but finitely many places $\mathfrak{p}$. Set theoretically:

$$\mathbb{A}_{K,f} = \Big\{(x_\mathfrak{p})_\mathfrak{p} \in \prod_{\mathfrak{p}} K_\mathfrak{p} : x_\mathfrak{p} \in \mathcal{O}_{K_\mathfrak{p}} \text{ for all but finitely many } \mathfrak{p}\Big\}.$$

> **Exercise 29.** *Show that $\mathbb{A}_{K,f}$ is a ring under pointwise addition and multiplication, and that there is a "diagonal embedding" $K \hookrightarrow \mathbb{A}_{K,f}$.*

The restricted product construction endows $\mathbb{A}_{K,f}$ with a topology: an open neighbourhood of 0 in $\mathbb{A}_{K,f}$ is a product $\prod_\mathfrak{p} U_\mathfrak{p}$, with every $U_\mathfrak{p} \subseteq K_\mathfrak{p}$ an open neighbourhood of 0 and all but finitely many $U_\mathfrak{p} = \mathcal{O}_{K_\mathfrak{p}}$. In particular, $\prod_\mathfrak{p} \mathcal{O}_{K_\mathfrak{p}}$ is a compact open neighbourhood of 0.

We now define the **adèle ring** (or **adèles**) of K to be

$$\mathbb{A}_K := K_\infty \times \mathbb{A}_{K,f} = \prod_v{}' K_v,$$

the restricted product taken over all (finite and infinite) places v, with the product topology. The diagonal inclusion obviously extends to an embedding $K \hookrightarrow \mathbb{A}_K$.

We can create an analogous construction for function fields: for instance, take $K = \mathbb{C}(t)$ and put $\mathcal{O}_K = \mathbb{C}[t]$. The maximal ideals of $\mathcal{O}_K$ take the form $t - \lambda, \lambda \in \mathbb{C}$, with completion $K_\lambda = \mathbb{C}((t-\lambda))$, i.e., the ring of formal Laurent series. There is then an inclusion

$$K \hookrightarrow {\prod_{v \in \mathbb{C}}}' \mathbb{C}((t-v)),$$

the restricted product taken with respect to the power series rings $\mathbb{C}[[t-v]]$.

Exercise 30. *Just as $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R} = K \otimes_{\mathbb{Q}} \mathbb{Q}_\infty$, show that*

$$\mathbb{A}_{K,f} = K \otimes_{\mathbb{Q}} A_{\mathbb{Q},f} \text{ and } \mathbb{A}_K = K \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}.$$

Theorem 13.1. *One has*

$$\mathbb{A}_{\mathbb{Q},f} = \mathbb{Q} + \prod_p \mathbb{Z}_p;$$

that is, that if $x = (x_p)_p \in \mathbb{A}_{\mathbb{Q},f}$, then there exist $\lambda \in \mathbb{Q}\,(\subseteq \mathbb{A}_{\mathbb{Q},f})$ and $\mu \in \prod_p \mathbb{Z}_p$ such that $x = \lambda + \mu$.

Proof. Let S be the (finite) set of primes such that $x_p \in \mathbb{Z}_p$ for all $p \notin S$; we will induct on the order of S. If $S = \emptyset$, then $x \in \prod_p \mathbb{Z}_p$ and we are done.

Otherwise, choose $p \in S$ and write

$$x_p = \sum_{m=-n}^{\infty} a_m p^m \in \mathbb{Q}_p,$$

where $a_{-n} \neq 0$; put $\lambda = a_{-n} p^{-n} + \cdots + a_{-1} p^{-1}$, and write $\lambda = \frac{N}{p^n}$ where N is an integer. Then clearly $\lambda \in \mathbb{Z}_\ell$ for all $\ell \neq p$, and so $y = (y_p)_p = x - \lambda$ has $y_p \in \mathbb{Z}_p$ for all $p \notin S \setminus \{p\}$, which has strictly smaller cardinality; so by induction, we are done. $\square$

Exercise 31. *Show that $\mathbb{A}_{K,f} = K + \prod_{\mathfrak{p}} \mathcal{O}_{K_{\mathfrak{p}}}$ for a general number field K.*

We turn our attention for the moment to the group of units of the topological ring $\mathbb{A}_K$, i.e., $\mathbb{A}_K^\times$. This is comparatively small with regard to the ambient ring:

for instance, if $x_p = p$ for all finite places and $x_\infty = 1$ for the infinite place, then $x \in \mathbb{A}_\mathbb{Q}$, but $x^{-1} \notin \mathbb{A}_\mathbb{Q}$. One can check that

$$\mathbb{A}_{K,f}^\times = \prod_{\mathfrak{p}}{}' K_\mathfrak{p}^\times,$$

the restricted product taken with respect to the subgroups $\mathcal{O}_{K_\mathfrak{p}}^\times$, and hence that

$$\mathbb{A}_K^\times = \prod_v{}' K_v^\times.$$

We call $\mathbb{A}_K^\times$ the **idèle group** (or simply the **idèles**) of K. Again, the restricted product gives us a topology on the idèles, which is *not* the subspace topology of $\mathbb{A}_K^\times$ in $\mathbb{A}_K$.

Remark. We could generalize idèle group in the following way: for fixed number n,

$$GL_n(\mathbb{A}_K) := \{(d, g) \in \mathbb{A}_K \times M_{n\times n(\mathbb{A}_K)} | d \det(g) = 1 \in \mathbb{A}_K\}$$

In particular, if $n = 1$, we have $GL_1(\mathbb{A}_K) = \mathbb{A}_K^\times$.

Both the adèles and the idèles are natural objects in global class field theory.

Theorem 13.2. *There exists a continuous, surjective group homomorphism*

$$K^\times \backslash \mathbb{A}_K^\times \overset{r_K}{\twoheadrightarrow} \mathrm{Gal}(K^{ab}/K),$$

called the ***global Artin map****, with nontrivial kernel.*

We know *a priori* that the global Artin map cannot be an isomorphism, because its domain contains $K_\infty^\times$, while its codomain is profinite. In particular, *the image of* $\left(K_\infty^\times\right)^\circ$ *(i.e. the connected component of the identity) lies in a totally disconnected group*, and so must be a singleton.

It turns out that, if C_K is the image of $\left(K_\infty^\times\right)^\circ$ in $K^\times \backslash \mathbb{A}_K^\times$, then $\ker r_K$ is precisely the closure of C_K. We now list without proof two other properties of r_K:

1. For every finite place $\mathfrak{p}$ of K, there is a commutative diagram

$$\begin{tikzcd}
K^\times \backslash \mathbb{A}_K^\times \arrow[r, "r_K"] & \mathrm{Gal}(K^{ab}/K) \\
K_\mathfrak{p}^\times \arrow[u, hook] \arrow[r, "r_{K_\mathfrak{p}}"] & \mathrm{Gal}(\overline{K_\mathfrak{p}}/K_\mathfrak{p})^{ab} \arrow[u]
\end{tikzcd}$$

where the inclusion $K_\mathfrak{p} \hookrightarrow K^\times \backslash \mathbb{A}_K^\times$ sends x to $(x)_\mathfrak{p} \times \prod_{v \neq \mathfrak{p}} (1)_v$.

2. If L/K is a finite extension, then there is a commutative diagram

$$\begin{tikzcd}
L^\times \backslash \mathbb{A}_L^\times \arrow[r, "r_L"] \arrow[d, "N"'] & \mathrm{Gal}(L^{ab}/L) \arrow[d] \\
K^\times \backslash \mathbb{A}_K^\times \arrow[r, "r_K"] & \mathrm{Gal}(K^{ab}/K)
\end{tikzcd}$$

where N is the norm map.

14 Lecture Fourteen

Remark. We know the kernel of the global Artin homomorphism onto $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, so we "know" the group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. In general, however, we don't know K^{ab}.

In the case where K is $\mathbb{Q}$ or imaginary quadratic, we do know K^{ab} (see Serre's other Cassels-Fröhlich article.)

Let's analyze $\mathbb{Q}^\times \backslash \mathbb{A}^\times$.

Lemma 2.

$$\mathbb{A}_{\mathbb{Q}}^\times = \mathbb{Q}^\times . \left(\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0} \right),$$

i.e., if $(x_v) \in \mathbb{A}_{\mathbb{Q}}^\times$, then there exists $\lambda \in \mathbb{Q}^\times$ such that $x_p/\lambda \in \mathbb{Z}_p^\times$ for all p, and $x_\infty/\lambda > 0$.

Proof. We have to look at the bad set from $(x_v) \in \mathbb{A}_{\mathbb{Q}}^\times$, $S = \{p \mid x_p \notin \mathbb{Z}_p^\times\}$. We induct on the cardinality of S. If $S = \emptyset$, then $\lambda = \pm 1$ depending on the sign of x_∞. For general $S \neq \emptyset$, then choose some prime $p \in S$, so that $x_p \in \mathbb{Q}_p^\times$ and $x_p \notin \mathbb{Z}_p^\times$.

Rewrite $x_p = p^n u$ where $u \in \mathbb{Z}_p^\times$, and set $\lambda = p^n$. Then $\lambda \in \mathbb{Z}_r^\times$, for all $r \neq p$.

Dividing by λ, we reduce to the case $S \backslash \{p\}$, and by induction we are done. □

Remark. In the case when K is a number field, this is hard to push through. Suppose we have some $x_{\mathfrak{p}}$, where $\mathfrak{p}$ is some finite place. What happens if $v_{\mathfrak{p}}(x_{\mathfrak{p}}) = n$? Then we want to set $\lambda \in K^\times$, with $\mathcal{O}_K \lambda = \mathfrak{p}^n$.

But what if $\mathfrak{p}^n$ is not principal? Then the lemma is no longer true.

Exercise 32. *Show that the double quotient*

$$K^\times \backslash \mathbb{A}_K^\times / \left(\prod_{\mathfrak{p}} \mathcal{O}_{K_{\mathfrak{p}}}^\times \times K_\infty^\times \right)$$

is precisely the ideal class group of K.

Remark. In fact, we've shown that

$$\mathbb{A}_{\mathbb{Q}}^\times = (\mathbb{Q}^\times) \times \left(\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0} \right),$$

because

$$\mathbb{Q}^\times \cap \left(\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0} \right) = \{1\}$$

In fact, we've gotten something for free!

Corollary 1. $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times$ *is canonically isomorphic to* $\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$.

Corollary 2. $\ker r_\mathbb{Q} = \mathbb{R}_{>0}$

Corollary 3. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *is canonically isomorphic to*

$$\prod_p \mathbb{Z}_p^\times = \widehat{\mathbb{Z}}^\times = \varprojlim_{n\geq 1} (\mathbb{Z}/n\mathbb{Z})^\times = \hat{\mathbb{Z}}^\times,$$

which of course suggests that $\mathbb{Q}^{\mathrm{ab}}$ *is just the maximal cyclotomic extension of* $\mathbb{Q}$ *(which is true.)*

Definition. Let K be a number field. A Groessencharacter (also sometimes called a Hecke character, and we will write it as GC) is a continuous group homomorphism $K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$.[7]

Example 5. Let $K = \mathbb{Q}$. $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times = \widehat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$. If we want to describe the GCs for $\mathbb{Q}$, we just need to figure out what all the continuous group homomorphisms to $\mathbb{C}^\times$ from either component of $\widehat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$.

Exercise 33. *Show that the continuous group homomorphisms* $\mathbb{R}_{>0} \to \mathbb{C}^\times$ *are all of the form* $x \mapsto x^s \overset{\text{df}}{=} \exp(s \log x)$.

What about $\widehat{\mathbb{Z}}^\times$? Note that $\mathbb{C}^\times$ has no "small subgroups".

Let $\alpha : \widehat{\mathbb{Z}}^\times \to \mathbb{C}^\times$ be a continuous group homomorphism.

Write $\widehat{\mathbb{Z}}^\times = \varprojlim (\mathbb{Z}/n\mathbb{Z})^\times$, so for U open, we have $\alpha^{-1}(U)$ open, hence $\alpha^{-1}(U) \supseteq \ker\left(\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/n\mathbb{Z})^\times\right) \overset{\text{df}}{=} K_N$.

Now, $a(K_n) \subseteq U$, and is a subgroup. Therefore, $\alpha(K_n) = 1$ and α factors as

$$\widehat{\mathbb{Z}}^\times \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times.$$

The upshot is: a GC for $\mathbb{C}$ is a map $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ and for each GC, there exists a pair (χ, s) where χ is a Dirichlet character and $s \in \mathbb{C}$, and a GC on $\widehat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$ factors as $\widehat{\mathbb{Z}}^\times \times \mathbb{R}_{>0} \to (\mathbb{Z}/n\mathbb{Z})^\times \times \mathbb{R}_{>0}$ by $(n, x) \mapsto \chi(n) \cdot x^s$.

[7] Why do we care about these things? Here's a secret: GCs are automorphic representations for $\mathrm{GL}_1(K)$. (This was Langland's insight—now we have to figure out how to change the 1 to a 2.)

Corollary 4. (Due to Tate.) *The set of GCs for $\mathbb{Q}$ has the structure of a Riemann surface.*

Fix a Dirichlet character χ and embed $\mathbb{C} \hookrightarrow \{GCs\}$ by $s \mapsto (\chi, s)$.

In Tate's thesis, he takes a GC ψ and defines a number $L(\psi) \in \mathbb{C} \cup \{\infty\}$, which induces a complex-valued function L on the Riemann surface of GCs, which admits a meromorphic extension to all of the $\mathbb{C}$-eigencurve.

Tate checks that the restriction of L to the copy of $\mathbb{C}$ attached to χ is $L(\chi, s)$.

Here's a generalization of this result to K. Recall for $K_{\mathfrak{p}}/\mathbb{Q}_p$ finite, there's a canonical norm where the norm for the uniformizer π is $\frac{1}{q}$ (where q is the size of the residue field $\#k_{\mathfrak{p}}$), in terms of scaling the additive Haar measure.

The same trick extends to $\mathbb{A}_K$; namely, there exists a canonical Haar measure on $\mathbb{A}_K$ for which multiplication by $x \in \mathbb{A}_K^\times$ scales this Haar measure by $\|x\|$, which is some positive real $\|x\| \in \mathbb{R}_{>0}$.

This therefore defines a norm $\|\cdot\| : \mathbb{A}_K^\times \to \mathbb{R}_{>0}$. Since $K^\times$ diagonally embeds into $\mathbb{A}_K^\times$, this canonical norm must be trivial on $K^\times$.

The upshot is that $\|\cdot\| : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{R}_{>0}$ restricts to the p-adic norm $\|\cdot\|_{\mathfrak{p}}$ on all $K_{\mathfrak{p}}$, to the usual norm on copies of $\mathbb{R}^\times$, and to the square of the usual norm on copies of $\mathbb{C}^\times$.

So we now have an analogue of what we had for the rational numbers. Hence, the set of all GCs for K also becomes a Riemann surface.

The idea behind the global Langlands philosophy is that there exists a global Langlands group $L_{\mathbb{Q}}$ (in general L_K for general K), such that $L_K^{\mathrm{ab}} = K^\times \backslash \mathbb{A}_K^\times$.

And now we have shown the 1-dimensional global Langlands correspondence: there exists a canonical bijection between automorphic representations of $\mathrm{GL}_1(K)$ with 1-dimensional representations of L_K.

Remark. Here's something to think about: if $s \in \mathbb{Z}$, life would be better: $K = \mathbb{Q}$ and a GC $\psi = (\chi, s)$, where χ is a Dirichlet character, so that $E_0 = \mathbb{Q}(\zeta_m) \subseteq \mathbb{C}$, so that χ induces $\chi_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_1((E_0)_\lambda)$, which is a compatible system of λ-adic Galois representations.

So when $s = 1$, α is trivial, and $\psi = \|\cdot\|^{-1}$, $\|\pi_{\mathfrak{p}}\| = \frac{1}{q_{\mathfrak{p}}}$.

15 Lecture Fifteen

15.1 Recalling definitions

15.1.1 Compatible systems of λ-adic representations

Recall the following definition. For K, E_0 number fields, S a finite set of maximal ideals of $\mathcal{O}_K$ (a finite set of finite places of $\mathcal{O}_K$), $\mathfrak{p}$ any finite place of K with $\mathfrak{p} \notin S$ and $F_{\mathcal{P}}(X)$ a polynomial in $E_0[X]$ which is monic of degree n, we define a **compatible system of λ-adic Galois representations** as a collection of representations

$$\{\rho_\lambda : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\overline{(E_0)_\lambda})\},$$

the index running over all finite places λ of E_0, such that: for all ρ_λ and for all $\mathfrak{p} \nmid \ell$ with $\lambda \mid \ell$ and $\mathfrak{p} \notin S$, the representation ρ_λ is unramified at $\mathfrak{p}$ and $\rho_\lambda(\mathrm{Frob}_{\mathfrak{p}})$ has characteristic polynomial $F_{\mathfrak{p}}(X)$, independent of λ.

We give some examples:

(i) A Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ in fact has image in $E_0 = \mathbb{Q}(\zeta_d)$, where $d = \#(\mathbb{Z}/N\mathbb{Z})^\times = \phi(N)$ (i.e. the Euler totient function). So if we let $K = \mathbb{Q}, S = \{p \mid N\}$, we may define $\rho_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \to \mathrm{GL}_1(\overline{(E_0)_\lambda})$, which gives a "stupid" example of a compatible system of Galois representations.

It is immediately apparent that $\rho_\lambda(\mathrm{Frob}_p)$ for $p \notin S$ does not depend on λ since it will in fact lie in $\mathrm{GL}_1(E_0) \subseteq \mathrm{GL}_1(\overline{(E_0)_\lambda})$ and its characteristic polynomial is by definition $X - \chi(p)$.

(ii) Consider, for any integer $n \in \mathbb{Z}$, the nth power of the cyclotomic character $\omega_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\zeta_{\ell^\infty})/\mathbb{Q}) \cong \mathbb{Z}_\ell^\times \hookrightarrow \mathrm{GL}_1(\mathbb{Q}_\ell)$. Then for $K = \mathbb{Q} = E_0, S = \emptyset, F_p(X) = X - p$, we see that $\omega_\ell(\mathrm{Frob}_p) = p$ independent of ℓ for $p \neq \ell$.

Then the nth power of the cyclotomic character also gives a compatible system as desired.

(iii) The Tate module $T_\ell E$ of an elliptic curve E also gives an example.

15.1.2 Grossencharacters

We also will recall the definition of a Grossencharacter. For $K = \mathbb{Q}$ we have $\mathbb{A}_K^\times = \mathbb{Q}^\times \times \hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$; we note that any continuous group homomorphism $K^\times \backslash A_K^\times \to \mathbb{C}^\times$ must factor through $\hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0} \to (\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{R}_{>0}$ for some N. For example:

(i) Recall that any Grossencharacter for $K = \mathbb{Q}$ is defined by a complex number s and a Dirichlet character $(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ such that on the real component of $\mathbb{Q}^\times\backslash\mathbb{A}_\mathbb{Q}^\times$ we have $x \mapsto x^s$, and on the profinite part the map is given by the Dirichlet character.

(ii) Consider the case $K = \mathbb{Q}(i)$.

The adèles $\mathbb{A}_K$ are equal to $\mathbb{A}_{K,f} \times K_\infty$. In this case, we have an isomorphism $K_\infty \cong \mathbb{C}$. We may define $\mathbb{A}_{K,f}^\times = \prod'_\mathfrak{p} K_\mathfrak{p}^\times$, the restricted product indicating that if $(x_\mathfrak{p})_\mathfrak{p} \in \mathbb{A}_{K,f}^\times$, then $v_\mathfrak{p}(x_\mathfrak{p}) = 0$ for all but finitely many $\mathfrak{p}$.

Note the following construction: given a finite idèle $x = (x_\mathfrak{p})_\mathfrak{p}$, define $n_\mathfrak{p} := v_\mathfrak{p}(x_\mathfrak{p}) \in \mathbb{Z}$ for each $\mathfrak{p}$. Clearly $n_\mathfrak{p} = 0$ for all but finitely many $\mathfrak{p}$.

Define a fractional ideal $I(x) = \prod_\mathfrak{p} \mathfrak{p}^{n_\mathfrak{p}} \subseteq K$. This is in fact a finite product, which is an ideal if all $n_\mathfrak{p} \geq 0$.

Thus we get a map from $\mathbb{A}_{K,f}^\times$ to the set of fractional ideals of K, under which the image of $K^\times$ is the set of principal ideals of K – so, the quotient maps to the class group, in some sense.

We return to our case $K = \mathbb{Q}(i)$. We want to write down a Grossencharacter.

We see that $\mathbb{A}_K^\times = \mathbb{A}_{K,f}^\times \times K_\infty^\times$. We claim that $\mathbb{A}_K^\times = K^\times \cdot (\prod_\mathfrak{p} \mathcal{O}_{K_\mathfrak{p}}^\times \times K_\infty^\times)$. This is a consequence of the fact that K has class number 1.

Proof. Write $x \in \mathbb{A}_K^\times$ as a product $x_f \times x_\infty$ where x_f is a finite idele and $x_\infty \in \mathbb{C}^\times$.

Then since K has class number 1, $I(x_f) = (\lambda)$ for some $\lambda \in K^\times$. Then one can check easily that $v_\mathfrak{p}(x_\mathfrak{p}/\lambda) = 0$ for all $\mathfrak{p}$; hence $x_f/\lambda \in \prod_\mathfrak{p} \mathcal{O}_{K_\mathfrak{p}}^\times$, as desired. □

This proof works for any K with class number 1.

Then for any K a number field of class number 1 we see that to give a Grossencharacter $\psi : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ it suffices to define a continuous ψ on $K_\infty^\times \times \prod_\mathfrak{p} \mathcal{O}_{K_\mathfrak{p}}^\times$, such that ψ is trivial on $K^\times \cap (\prod_\mathfrak{p} \mathcal{O}_{K_\mathfrak{p}}^\times \times K_\infty^\times) = \mathcal{O}_K^\times$.

Thus for $K = \mathbb{Q}(i)$, take any ideal $\mathcal{N} \subseteq \mathbb{Z}[i]$. We then take any group homomorphism $\chi : (\mathbb{Z}[i]/\mathcal{N})^\times \to \mathbb{C}^\times$.

Then χ gives us a map $\prod_\mathfrak{p} \mathcal{O}_{K_\mathfrak{p}}^\times = (\hat{\mathcal{O}}_K)^\times \to (\mathcal{O}_K/\mathcal{N})^\times \to \mathbb{C}^\times$.

Abstractly as a group, $\mathbb{R}_{>0} \times S^1 \cong \mathbb{C}^\times$. We can define a group homomorphism on the real part by $x \mapsto x^s$ for some $s \in \mathbb{C}$ and on the S^1-part by $z \mapsto z^n$ for some $n \in \mathbb{Z}$. This exhausts the group homomorphisms $\mathbb{C}^\times \to \mathbb{C}^\times$.

The upshot so far is that given such χ, s, n, we get some $\psi_0 : \prod_{\mathfrak{p}} \mathcal{O}_{K_\mathfrak{p}}^\times \times K_\infty^\times \to \mathbb{C}^\times$.

However, ψ_0 may not extend to a Grossencharacter. When will it? Exactly when χ and $\mathbb{C}^\times \to \mathbb{C}^\times$ given by s, n are both trivial on $\mathcal{O}_K^\times = \{\pm 1, \pm i\}$.

Exercise 34. *With ψ_0 defined as above, check that ${\psi_0}^4$ always extends to a Grossencharacter.*

(iii) Now we will consider $K = \mathbb{Q}(\sqrt{2})$. We note that $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$ and $\mathcal{O}_K^\times = \pm 1 \times \langle 1 + \sqrt{2} \rangle$.

Here $A_K^\times = K^\times \cdot (\prod_{\mathfrak{p}} \mathcal{O}_{K_\mathfrak{p}}^\times \times K_\infty^\times)$ since we again have class number 1.

A character of $\prod_{\mathfrak{p}} \mathcal{O}_{K_\mathfrak{p}}^\times$ will factor through some map $\alpha : (\mathbb{Z}[\sqrt{2}]/\mathcal{N})^{|} times \to \mathbb{C}^\times$ (same argument with profinite completion).

In this case $K_\infty^\times \cong \mathbb{R}^\times \times \mathbb{R}^\times$. A character will look like

$$\chi_\infty(x_1, x_2) = |x_1|^{s_1} |x_2|^{s_2} \operatorname{sgn}(x_1)^{e_1} \operatorname{sgn}(x_2)^{e_2},$$

for two complex numbers s_i and a choice of sign $e_i \in \{0, 1\}$ for each $i = 1, 2$.

We can then define $\psi_0 : \prod_{\mathfrak{p}} \mathcal{O}_{K_\mathfrak{p}}^\times \times K_\infty^\times \to \mathbb{C}^\times$.

How do we get ψ_0 to extend to a Grossencharacter? This happens when $\psi_0(1 + \sqrt{2})$ is trivial - then $|1 + \sqrt{2}|^{s_1} |1 - \sqrt{2}|^{s_2}$ is a root of unity. If $1 + \sqrt{2} = x$, then $|1 - \sqrt{2}| = 1/x$; hence it is clear that we must have $x^{s_1 - s_2}$ a root of unity, so $s_1 - s_2$ must be in $\frac{\pi i}{\log(1+\sqrt{2})}\mathbb{Q}$, which is a discrete set.

The upshot is that if we have $\chi : (\mathbb{Z}[\sqrt{2}]/\mathcal{N})^\times \to \mathbb{C}^\times$ and two complex numbers $s_1, s_1 + q$ for q in some discrete set, and some signs, then the corresponding ψ_0 to some finite power does extend to a Grossencharacter.

16 Lecture Sixteen

16.1 Global Langlands Conjecture for GL_1

We've seen examples of compatible systems of one-dimensional Galois representations and GCs.

Definition. A GC $\psi : K^\times \backslash \mathbb{A}_K{}^\times \to \mathbb{C}^\times$ is said to be *algebraic* if its restriction to the connected component of the identity $(K_\infty{}^\times)^o \simeq (\mathbb{R}_{>0})^r \times \mathbb{C}^s$ is defined by

$$\psi(x_1, \dots, x_r, z_1, \dots, z_s) = x_1^{n_1} \cdots x_r^{n_r} z_1^{n_{r+1}} \bar{z}_1^{n_{r+2}} \cdots z_s^{n_{r+2s-1}} \bar{z}_s^{n_{r+2s}},$$

with each $n_i \in \mathbb{Z}$.

Example 6.

1. $\|\cdot\| : K^\times \backslash \mathbb{A}_K{}^\times \to \mathbb{C}^\times$ is algebraic with all $n_i = 1$.
2. For $K = \mathbb{Q}$ the GC $x \mapsto x^s$ is algebraic if and only if $s \in \mathbb{Z}$.

Philosophy: Let χ be a GC for K (an automorphic representation for GL_1/K); it corresponds to one-dimensional representations of the (conjectural) global Langlands group L_K. Since we can't even define the global Langlands group, this correspondence is not so much a conjecture as it is a philosophy.

> **Theorem 16.1.** *(Weil, Basic Number Theory?) Let χ be an algebraic GC, then there exists a compatible system of λ-adic Galois representations attached to χ, and conversely.*

Idea: Let $\psi : K^\times \backslash \mathbb{A}_K{}^\times \to \mathbb{C}^\times$, then $\psi|_{(K_\infty{}^\times)^o} : x \mapsto x^n$. We want a representation

$$\mathrm{Gal}(\bar{K}/K)^{ab} = K^\times \backslash \mathbb{A}_K{}^\times / \overline{(K_\infty{}^\times)^\circ} \to \mathrm{GL}_1(\overline{\mathbb{Q}_\ell}).$$

Proof. Given an algebraic GC, χ, define

$$\chi|_{(K_\infty{}^\times)^o}(x_\infty) = \prod_{v \text{ real}} x_v^{n_v} \times \prod_{v \leftrightarrow \{\sigma, c\circ\sigma\}} (\sigma x_v)^{n_{v,1}} (\overline{\sigma x_v})^{n_{v,2}}$$

with $n_v \in \mathbb{Z}$, and the second product taken over the complex embeddings $\sigma, c \circ \sigma : K \to \mathbb{C}$ (as usual, c is complex conjugation).

We further define

$$\begin{aligned} \chi_0 : \mathbb{A}_K{}^\times &\to \mathbb{C}^\times \\ x &\mapsto \chi(x) \Big/ \Bigg(\prod_{v \text{ real}} x_v^{n_v} \times \prod_{v \leftrightarrow \{\sigma, c\circ\sigma\}} (\sigma x_v)^{n_{v,1}} (\overline{\sigma x_v})^{n_{v,2}} \Bigg). \end{aligned}$$

We note that χ_0 is trivial on the connected component but is non-trivial on $K^\times$ (but earlier it was trivial). For $\lambda \in K^\times$, $\chi_0(\lambda) = \prod_{\sigma : K \to \mathbb{C}} \sigma(\lambda)^{n_\sigma}$ where $n_\sigma \in \mathbb{Z}$, thus non-trivial.

One checks that $\mathrm{Im}(\chi_0) \subseteq E_0$ (a number field). Say λ is a finite place of E_0, so that the image of

$$\begin{aligned} \chi_0|_{K^\times} : \mathbb{Q}^\times &\to {\mathbb{Q}_\ell}^\times \\ \mu &\mapsto \prod_\sigma \sigma(\mu)^{n_\sigma} \end{aligned}$$

lies in $E_0 \subset E_{0,\lambda}$ with $n_\sigma \in \mathbb{Z}$. It extends to a continuous group homomorphism

$$\chi_\ell : (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times = \prod_{\mathfrak{p} \mid \ell} {K_\mathfrak{p}}^\times \to (E_{0,\lambda})^\times .$$

We further define, for $\lambda \mid \ell$,

$$\begin{aligned} \psi_\lambda : {\mathbb{A}_K}^\times &\to (E_{0,\lambda})^\times \\ x &\mapsto \frac{\chi_0(x)}{\chi_\ell(x_\ell)} \end{aligned}$$

which is very complicated at $K_\mathfrak{p}$ for $\mathfrak{p} \mid \ell$, but is $\psi_\lambda = \chi$ at all places $\mathfrak{p} \nmid \ell$.

Since ψ_λ is trivial on the connected component and on $K^\times$, it extends to $\mathrm{Gal}(\bar{K}/K)^{ab} = K^\times \backslash {\mathbb{A}_K}^\times / \overline{({K_\infty}^\times)^\circ}$.

$F_\mathfrak{p}(X) = X - \chi_0(\pi_{K_\mathfrak{p}})$ for a uniformizer $\pi_{K_\mathfrak{p}}$ and hence it is a compatible system.

Conversely, if ψ_λ is a compatible system, to show that it comes from an algebraic GC we use Waldschmidt's theorem from transcendence theory. □

Remark. The big picture:

$$\begin{aligned}
\left\{\begin{array}{l}\text{automorphic representa-}\\ \text{tions of } \mathrm{GL}_1/K\end{array}\right\} &\longleftrightarrow \left\{\begin{array}{l}\text{one-dimensional rep-}\\ \text{resentations of } L_K\end{array}\right\} \\
\left\{\begin{array}{l}\text{algebraic automorphic rep-}\\ \text{resentations of } \mathrm{GL}_1/K\end{array}\right\} &\longleftrightarrow \left\{\begin{array}{l}\text{compatible systems of}\\ \text{one-dimensional repre-}\\ \text{sentations of } \mathrm{Gal}(\bar{K}/K)\end{array}\right\} \\
\left\{\begin{array}{l}p\text{-adic automorphic rep-}\\ \text{resentations of } \mathrm{GL}_1/K\end{array}\right\} &\longleftrightarrow \left\{\begin{array}{l}\text{continuous } p\text{-adic repre-}\\ \text{sentations } \mathrm{Gal}(\bar{K}/K) \to\\ \mathrm{GL}_1(\overline{\mathbb{Q}_p})\end{array}\right\}
\end{aligned}$$

The second correspondence is the theorem we just proved above. There is an n-dimensional analogue of the above statements, where 1 is replaced by n in each case. However, for general $n \geq 1$, most of the terms in this picture aren't even well-defined and, hence it's just a philosophy.

17 Lecture Seventeen

Last time, we discussed a series of correspondences related to the global Langlands program; one was the correspondence between the set of algebraic automorphic representations for G over K (where G is any connected, reductive algebraic group), and the set of compatible systems of semisimple ℓ-adic Galois representations $\mathrm{Gal}(\overline{K}/K) \to {}^L G(\mathbb{Q}_p)$ (i.e. the *Langlands dual group*).

In the case $G = \mathrm{GL}_n$, we will have ${}^L G = \mathrm{GL}_n$, although for general G there are some subtleties. One is that there is more than one notion of *algebraicity*, two of which we might call C-algebraic and L-algebraic, respectively; we will discuss these more below. Another issue is that the correspondence may fail to be a bijection for general G, and may instead be a finite-to-one correspondence; this arises from the fact that the L-packets, introduced above, are collections of automorphic representations which may still give the same ℓ-adic representation ρ. A further issue arises from the fact that different global Langlands parameters may be isomorphic everywhere locally.

These obstacles are not insurmountable; however, they complicate the understanding of our "correspondence," which must then be investigated more carefully.

Philosophically, we want the ℓ-adic representation ρ_π associated to a given automorphic representation π of G to be defined only up to some *Tate-Shafarevich group*, which is trivial when $G = \mathrm{GL}_n$, but not generally. We will not pursue these considerations deeply, and will restrict our attention for the moment to the group $G = \mathrm{GL}_n$.

The correspondence in this case was worked out in a series of papers by Langlands-Corvallis, and further by Clozel. Explicit correspondences were worked out in the case when π is an automorphic representation of GL_n over K, where K is totally real (or has complex multiplication), and satisfies both a certain strong self-duality condition, and a strong algebraicity (or cohomological) condition; Clozel was able to write down precisely a compatible system ρ_π.

Clozel's proof goes roughly as follows: given π, we find an appropriate *Shimura variety* X, then relate the cohomology of X to automorphic forms, using the *Eichler-Shimura congruence relation*. More recently (c. 2013), a paper by Harris-Lan-Taylor-Thorne were able to remove the self-duality condition by (roughly speaking) considering the representation $\pi \oplus \pi^\vee$ rather than π itself. Taking limits of cohomology of Shimura varieties yields the desired compatible system ρ.

These ideas are rooted in Weil's construction (which we saw yesterday) of an appropriate one-dimensional ℓ-adic representation ρ_χ attached to a given Größencharakter χ. Eichler, Shimura, Deligne, and Serre were able to show that, if f

is a weight k modular eigenform, then there exists a compatible system of two-dimensional Galois representations $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Now, before we introduce the notion of an *automorphic representation*, we will investigate what properties these representations will *not* satisfy. Recall that the local Langlands conjectures concern themselves with the set of all smooth-admissible, irreducible representations of $\mathrm{GL}_n(K)$ for a finite extension $K/\mathbb{Q}_p$, and the set of certain n-dimensional Weil-Deligne representations.

By contrast, the global Langlands conjectures will be concerned with automorphic representations of $\mathrm{GL}_n(\mathbb{A}_K)$, where K is a number field; by definition, an automorphic representation will be irreducible. It is not hard to show that

$$\mathrm{GL}_n(\mathbb{A}_K) = \prod_v{}' \mathrm{GL}_n(K_v),$$

the restricted product taken with respect to the subgroups $\mathrm{GL}_n(\mathcal{O}_{K_v})$. By *Flath's theorem*, we deduce that certain irreducible, well-behaved representations of $\mathrm{GL}_n(\mathbb{A}_K)$ correspond to tensor products of representations of the factors $\mathrm{GL}_n(K_v)$; we will write

$$\pi = \bigotimes_v{}' \pi_v,$$

each π_v an irreducible, admissible representation of $\mathrm{GL}_n(K_v)$.

The idea is now as follows: if π is such a representation of $\mathrm{GL}_n(\mathbb{A}_K)$, corresponding to the compatible system ρ under the global Langlands correspondence, then for every finite place, the representation π_v will correspond (by the *local* Langlands correspondence) to a Weil-Deligne representation $\rho_{0,v} : \mathrm{Gal}(\overline{K_v}/K_v) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$, obtained by restriction.

Consequently: the definition of an automorphic representation of GL_n over K *cannot* simply be an arbitrary smooth-admissible irreducible representation of $\mathrm{GL}_n(\mathbb{A}_K)$; here the analogy with the local Langlands correspondence begins to fall apart.

To illustrate this point, we consider an example in dimension one. Suppose on the contrary that an *automorphic representation* of GL_1/K is simply a representation of $\mathbb{A}_K^\times$, and let $K = \mathbb{Q}$. By Flath's theorem it suffices to give representations of $\mathbb{Q}_v^\times$ for every place v.

For the first twenty-five primes p (say), let us define our representation of $\mathbb{Q}_p^\times$ via $\mathbb{Z}_p^\times \mapsto 1$ and $p \mapsto 7$; for the remaining primes we will take the trivial representation, and our representation of $\mathbb{Q}_\infty^\times = \mathbb{R}^\times$ will also be trivial. Finally, we put

$$\pi = \bigotimes_v{}' \pi_v,$$

as above, and we have a completely valid representation π of $\mathrm{GL}_1(\mathbb{A}_\mathbb{Q})$. If π is indeed automorphic, then it corresponds (under global Langlands) to some compatible system

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_1(\mathbb{Q}_\ell),$$

which we want to respect the local Langlands correspondence. In particular, this will mean that $\rho_\ell(\mathrm{Frob}_p) = 1$ for all $p > 97$, and so by the Chebotarev density theorem we must have that ρ_ℓ is the trivial one-dimensional representation. But this implies that

$$\rho_\ell(\mathrm{Frob}_2) = 1$$

and *not* seven, as we have demanded. Thus we see that we will need to take a more restrictive definition of an *automorphic representation*, in stark contrast to the local case.

The main failure of our pathological example π is that it will fail to be trivial on $\mathbb{Q}^\times$, as is the case for Größencharakteren; indeed, we have

$$\pi(2) = \pi(2_\infty, 2_2, 2_3, 2_5, \ldots) = 1 \cdot 7 \cdot 1 \cdot 1 \cdots = 7 \neq 1.$$

Now: if G is a *finite* group, suppose we want to find all irreducible representations of G. Clearly it will suffice to investigate the group ring

$$\mathbb{C}[G] \cong \bigoplus_{\pi \text{ irreducible}} \pi^{\dim \pi}.$$

In fact, if $H \leq G$ is a subgroup, it is enough to look at $\mathbb{C}[H \backslash G]$, which we identify with the set of functions $H \backslash G \to \mathbb{C}$ on the coset space. We may view $\mathbb{C}[H \backslash G]$ as a subspace of $\mathbb{C}[G]$, and there will be some set S (generally not the entire set of irreducible representations) and integers $m(\pi)$ such that

$$\mathbb{C}[H \backslash G] \cong \bigoplus_{\pi \in S} \pi^{m(\pi)} \subseteq \mathbb{C}[G].$$

Generalizing this idea: taking instead $G = \mathrm{GL}_n(\mathbb{A}_K)$, we consider the functions $\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K) \to \mathbb{C}$, satisfying certain (not too restrictive) conditions – call these the *nice* functions ϕ, and let $\mathcal{A}_0 = \mathcal{A}_0\big(\mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}_K)\big)$ denote the vector space of all such nice ϕ. There is an action of $\mathrm{GL}_n(\mathbb{A}_K)$ on $\mathcal{A}_0$ via

$$(g.\phi)(\gamma) = \phi(\gamma g).$$

Exercise 35. *Check that this is indeed a group action, and is well-defined in the sense that* $(g.\phi)(\alpha\gamma) = (g.\phi)(\gamma)$ *for* $\alpha \in \mathrm{GL}_n(K)$.

In the case $n = 1$, a Größencharakter will be a nice function in $\mathcal{A}_0$, and indeed, a finite sum of arbitrary Größencharakteren will also be nice.

In general, the space $\mathcal{A}_0$ will not be irreducible, but may turn out to be a direct sum of irreducible representations π of $\mathrm{GL}_n(\mathbb{A}_K)$; thus we are motivated to investigate *these* representations π, and we consider these possible candidates for the definition of an *automorphic representation*. We will make this more precise in the next lecture.

18 Lecture Eighteen

Let K be a number field and let S be a finite set of finite places.

Consider $\mathrm{Gal}(K^S/K)$. We have conjugacy classes $\mathrm{Frob}_\mathfrak{p}$ for each $\mathfrak{p} \notin S$. If $\mathrm{Gal}(K^S/K)$ happened to be a free group, freely generated by these $\mathrm{Frob}_\mathfrak{p}$'s, then we could just send Frobenius anywhere we like, i.e. we can define $\rho : \mathrm{Gal}(K^S/K) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ by choosing random matrices $\rho(\mathrm{Frob}_\mathfrak{p}) = M_\mathfrak{p} \in \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ for all $\mathfrak{p} \notin S$, and then we're finished.

And then the global Langlands conjectures would say: "take any old irreducible representation π of $\mathrm{GL}_n(\mathbb{A}_K)$ and we'll get ρ_π."

How this would work is that we'd have a decomposition $\pi = \bigotimes \pi_v$, sending $\pi_\mathfrak{p}$ unramified to $M_\mathfrak{p}$.

But this is not the truth, because $\mathrm{Gal}(K^S/K)$ is not *freely* generated by the $\mathrm{Frob}_\mathfrak{p}$, and these $\mathrm{Frob}_\mathfrak{p}$ are all related in some vastly complex way which no human understands.

For example, we could just let K be the rational numbers and let S be the primes $\{2, 3, 5\}$. And in this case we could ask: could we choose a member of the Frob_7-conjugacy class and a member of the Frob_{11}-conjugacy class which multiply to the Frob_{23}-conjugacy class? Things like that.

But! The Chebotarev density theorem gives us something. It says that the $\mathrm{Frob}_\mathfrak{p}$ are *dense*. In fact it tells us that if we remove a single $\mathrm{Frob}_\mathfrak{p}$, then what's left is still dense, so it tells us that this single one that we removed can be written as a limit of the others.

So it's complicated, and figuring out these relations is really hard.

So on the automorphic representation side, we don't get this nice tensor product decomposition, because this would correspond to having this freely-generated story on the $\mathrm{Gal}(K^S/K)$-side.

Here's another approach. Based on our successes for GL_1, we will restrict to representations π of $\mathrm{GL}_n(\mathbb{A}_K)$ which show up in $\mathcal{A}_0(\mathrm{GL}_n(K))$ which will be "nice functions" $\mathrm{GL}_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K) \to \mathbb{C}$.

Question 1. What's a nice function?

Well, when $n = 1$, Groessencharacters were nice. Just compose by the inclusion $\mathbb{C}^\times \hookrightarrow \mathbb{C}$.

Let χ be a Groessencharacter, $\chi\colon \mathrm{GL}_1(\mathbb{A}_K) \to \mathbb{C}$.

Define $(g * \chi)(\gamma) = \chi(\gamma g) = \chi(g)\chi(\gamma)$, and therefore

$$g * \chi = \chi(g) \cdot \chi.$$

Hence $\mathbb{C}\chi$ is a one-dimensional vector space on which $\mathrm{GL}_1(\mathbb{A}_K)$ acts via χ.

Also, Groessencharacters are locally constant at finite places and smooth at x. They're of the form $x \mapsto x^s$. In fact, they're more than smooth. They're also group homomorphisms. But that doesn't help us here, because nice functions out of $\mathrm{GL}_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)$ are functions on a coset space, which *a priori* might not be a group (because $\mathrm{GL}_n(K)$ is generally not a normal subgroup).

Let's differentiate $f(x) = x^s = \exp(s \log x)$, so $f'(x) = \exp(s \log x) \cdot \frac{s}{x} = x^s \cdot \frac{s}{x}$.

Therefore, $xf'(x) = sx^s = sf(x)$.

So it satisfies this natural differential equation.

We need machinery which spits out differential equations.

Definition. "Nice" means:

- locally constant at finite places
- left $\mathrm{GL}_n(K)$-invariant
- smooth at infinite places
- satisfies a certain differential equation
- boundedness

18.1 Interlude on differential equations

A Lie group is a group object in the category of smooth manifolds. Given a Lie group G, we can form its Lie algebra $\mathfrak{g}$, which is the tangent space of the identity; $\mathfrak{g}$ can be thought of as comprising "differential operators on G".

There is a natural *exponential map* $\mathfrak{g} \to G$.

If we have an $X \in \mathfrak{g}$, then we view this as inducing a differential operator on the space of

$$\{C^\infty\text{-functions } G \to \mathbb{C}\}$$

via

$$(X * f)(g) \overset{\mathrm{df}}{=} \frac{\mathrm{d}}{\mathrm{d}t} \left(f(g \cdot \exp(tX)) \right) \Big|_{t=0}.$$

Let's see an example of this. Let $G = \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$.

Then $\mathfrak{g} = \mathbb{R}$, so let $X = 1$. Then

$$(X * f)(g) = \frac{\mathrm{d}}{\mathrm{d}t} \left(f(g \cdot e^t) \right) \Big|_{t=0},$$

and differentiating, we obtain

$$f'(g)g.$$

The point here is that there is a canonical differential operator coming from the Lie algebra, and this canonical differential operator is apparently not the usual derivative. To recap, we have a *differential equation*

$$(X * f)(x) = xf'(x), x \in \mathbb{R}^\times.$$

So, returning to our differential equation from before

$$xf'(x) = sx^s,$$

we then want to write $X * f = s \cdot f$.

What does this look like for $\mathrm{GL}_2(\mathbb{R})$? In this case $\mathfrak{g} = M_2(\mathbb{R})$ has basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Each of these gives a differential operator on the vector space of C^∞-functions $\mathrm{GL}_2(\mathbb{R}) \to \mathbb{C}$; call the space of such functions V.

Now, we have operators $E, F, H, Z : V \to V$, and these operators do not commute (e.g. $EF - FE = H$. So it's a bad idea to ask for simultaneous eigenfunctions.

So here's the plan: we're going to find a bunch of differential operators that commute!

Let's start with a Lie algebra $\mathfrak{g}$ (keep in mind $\mathfrak{gl}_2(\mathbb{R})$), which is some finite-dimensional vector space $\mathfrak{g}$ over $\mathbb{R}$, equipped with a Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Definition. The **universal enveloping algebra** of a Lie algebra $\mathfrak{g}$ is the associative algebra $U(\mathfrak{g})$ such that the functor $\mathfrak{g} \mapsto U(\mathfrak{g})$ is left-adjoint to the forgetful functor from the category of unital associative algebras to the category of Lie algebras (i.e. the functor which takes the underlying vector space and attaches the commutator as bracket, $[X, Y] := XY - YX$).

Let $\mathfrak{g}$ be a Lie algebra with $\mathbb{R}$-basis $\{X_1, \ldots, X_d\}$. This contains all the first order differential operators. The associative algebra $\mathbb{R}\langle X_1, \ldots, X_d \rangle$ of non-commutative polynomials in d variables contains (non-commuting) higher order differential operators. Taking the quotient by the bi-ideal generated by relations $(X_i X_j - X_j X_i) - [X_i, X_j]$, we obtain the universal enveloping algebra $U(\mathfrak{g})$.

What's the centre of this algebra? It's a whole bunch of commuting differential operators, for which we might hope that our nice functions are simultaneous eigenforms.

Harish-Chandra figured out what $Z(U(\mathfrak{g}_{\mathbb{C}}))$ is, where $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

$\mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{h}$, and $U(\mathfrak{h}_{\mathbb{C}})$ is the commutative polynomial ring $\mathbb{C}[X_1, \ldots, X_d]$.

Theorem 18.1. (Harish-Chandra homomorphism) *Let* $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$. *Then, there exists a canonical injection*

$$Z\left(U(\mathfrak{g}_{\mathbb{C}})\right) \hookrightarrow U(\mathfrak{h}_{\mathbb{C}}),$$

and the image is $U(\mathfrak{g}_{\mathbb{C}})^W$, *where* W *is the Weyl group.*

Specializing to $n = 2$, we get get $\mathfrak{g}_{\mathbb{C}} = \mathbb{C}E \oplus \mathbb{C}F \oplus \mathbb{C}Z \oplus \mathbb{C}H$, with $U(\mathfrak{g}_{\mathbb{C}}) \ni EF + FFH$, and

$$Z(U(\mathfrak{g}_{\mathbb{C}})) \simeq U(\mathbb{C} \oplus \mathbb{C})^W = \mathbb{C}[X, Y]^{\mathbb{Z}/2\mathbb{Z}}.$$

So now we have a problem: what are the polynomials in 2 variables which are the same when we interchange X and Y?

Exercise 36. *Show that*

$$Z(U(\mathfrak{g}_{\mathbb{C}})) \simeq U(\mathbb{C} \oplus \mathbb{C})^W = \mathbb{C}[X, Y]^{\mathbb{Z}/2\mathbb{Z}} = \mathbb{C}[S, T],$$

where $S = XY$ *and* $T = X + Y$.

Tomorrow[8] we'll see the definition of an automorphic representation.

19 Lecture 19

For G a connected reductive group over K (in most cases $G = \mathrm{GL}_n$), we want to define $\mathcal{A}(G/K) = \{\phi : G(K)\backslash G(\mathbb{A}_K) | \phi\ nice\}$ for some value of "nice".

These "nice" functions will be **automorphic forms**.

As in the GL_1 case, where $\mathrm{GL}_1(\mathbb{A}_K) \supseteq K^\times \prod_{\mathcal{P}} \mathcal{O}_{K_{\mathcal{P}}}^\times K_\infty^\times$, which is a subgroup of finite index (quotient by which is the class group).

ϕ has to be trivial on $K^\times$, and on $\prod_{\mathcal{P}} \mathcal{O}_{K_{\mathcal{P}}}^\times$ is in practice finite, with $K_\infty^\times$ a real manifold.

The same story is true in the GL_n case. The proof for $\mathrm{GL}_2/\mathbb{Q}$ is coming later.

$$\mathrm{GL}_n(\mathbb{A}_K) \supseteq \mathrm{GL}_n(K) \prod_{\mathcal{P}} \mathrm{GL}_n(\mathcal{O}_{K_{\mathcal{P}}}) \mathrm{GL}_n(K_\infty)$$

[8] "When are we leaving again?" (Note: tomorrow's the last day.)

as a subgroup with finite index; again, ϕ has to be trivial on $\mathrm{GL}_n(K)$, and we get a finite amount of information on $\mathrm{GL}_n(\mathcal{O}_{K_\mathcal{P}})$, with $\mathrm{GL}_n(K_\infty)$ some real manifold again.

etric As a reminder, in the case $\mathrm{GL}_1/\mathbb{Q}$, at ∞ any ϕ is a function on $\mathbb{R}^\times/\mathbb{Z}^\times$, and we want Grossencharacters to exist in $\mathcal{A}(\mathrm{GL}_1/\mathbb{Q})$.

The conclusion we get from this is that $\mathbb{R}_{>0} \to \mathbb{C}^\times$ given by $x \mapsto x^s$ must be nice.

There is some canonical connection with differential operators in this situation. For $G(\mathbb{R})$ the Lie group, we get a Lie algebra $\mathfrak{g}$; for the GL_1-case, we can choose a bvasis vector D of $\mathfrak{g}$ so that $(Df)(x) = xf'(x)$, and $D(x \mapsto x^s) = x \mapsto sx^s$, so $x \mapsto x^s$ is an eigenfunction of D - annihilated by $D - s$.

However, we also want functions like $f(x) = x^s + 7x^t$ to be 'nice' so that $\mathcal{A}$ is a vector space. This function is not annihilated by $D - s$, but it is annihilated by $(D - s)(D - t)$.

Abstractly, the algebra $\mathbb{C}[D]$ acts on C_∞ functions $\mathbb{R}_{>0} \to \mathbb{C}$, with sums of Grossencharacters annihilated by a nonzero idea in $\mathbb{C}[D]$.

If $I = \{D' \in \mathbb{C}[D] : D'(f) = 0\}$ for f a sum of Grossencharacters, this is evidently an ideal. In the cases we have seen so far, I had finite codimension; we see that if we define a map $\mathbb{C}[D] \to \mathbb{C}^2$ by $h(D) \mapsto (h(s), h(t))$, the kernel is $I = ((D - s)(D - t))$, so I has codimension 2.

For general G, we have Lie algebra $\mathfrak{g}$ of $G(K_\infty)$ with basis $E_1, E_2, \ldots, E_d$ of $\mathfrak{g}$ - however, these basis elements do not commute.

Instead we consider the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ as defined before $(\mathbb{C}\langle E_1, \ldots, E_d\rangle/(E_iE_j - E_jE_i - [E_i, E_j])_{i,j}$, the quotient of the noncommutative polynomial algebra in the basis elements by the bi-ideal given by the relations $E_iE_j - E_jE_i = [E_i, E_j]$ for the Lie bracket $[,]$ of $\mathfrak{g}$). Harish-Chandra showed that for $G = \mathrm{GL}_n/\mathbb{Q}$ the center $Z(U(\mathfrak{g}_\mathbb{C})) = \mathbb{C}[T_1, T_2, \ldots, T_n]^W$ where W is the symmetric group S_n acting on the T_i in the usual way.

The upshot is that $Z(U(\mathfrak{g}_\mathbb{C}))$ is a canonical source of higher-order differential operators. Alternatively we can think of these as bi-G-invariant differential operators.

We have a torus T in $\mathrm{GL}_n/\mathbb{Q}$ given by $\begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix} \subseteq \mathrm{GL}_n$. Then we get a Lie subgroup $T(\mathbb{R}) \subseteq G(\mathbb{R})$, with Lie subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$.

$\mathfrak{t}$ has basis $t_1, \ldots, t_n$ and $U(\mathfrak{t}) = \mathbb{C}[t_1, \ldots, t_n]$, with $W = N_G(T)/T$ the Weyl group in this case isomorphic to S_n with the usual action on the t_i.

Then $Z(U(\mathfrak{g})) = \mathbb{C}[t_1, \ldots, t_n]^{S_n} = \mathbb{C}[\sigma_1, \ldots, \sigma_n]$ for σ_i the ith symmetric polynomial.

For example, in the case $\mathrm{GL}_2/\mathbb{Q}$, this machine tells us that $Z(U(\mathfrak{g})) = \mathbb{C}[\Delta, Z]$

for two differential operators Δ, Z.

In this case note that $\sigma_1 = t_1 + t_2, \sigma_2 = t_1 t_2$.

If $\mathfrak{g} = Lie(\mathrm{GL}_2(\mathbb{R}))$, it is $\mathrm{Mat}_2(\mathbb{R})$ with the commutator Lie bracket. Recall its basis $\{\mathcal{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathcal{F} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathcal{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathcal{Z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$. The bracket is given by $[\mathcal{E}, \mathcal{F}] = \mathcal{H}, [\mathcal{E}, \mathcal{H}] = -2\mathcal{E}, \mathcal{F}, \mathcal{H} = 2\mathcal{F}, [*, \mathcal{Z}] = 0$.

Then $U(\mathfrak{g}_{\mathbb{C}}) = \mathbb{C}\langle E, F, H, Z\rangle/(EF - FE - H, EH - HE + 2E, FH - HF - 2F,\ etc.)$.

Exercise: If $\Delta := H^2 + 2EF + 2FE$, Δ commutes with E, F, H, Z.

Then $\Delta, Z \in Z(U(\mathfrak{g}_{\mathbb{C}}))$. It will turn out that they in fact generate the center - $Z(U(\mathfrak{g}_{\mathbb{C}}) = \mathbb{C}[\Delta, Z]$.

Δ is some second order differential operator on $\{f : \mathrm{GL}_2^+(\mathbb{R}) \to \mathbb{C}\}$. As a reminder, $\mathrm{GL}_2^+(\mathbb{R})$ (matrices with positive determinant) acts on the upper half plane $\mathfrak{h}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau+b}{c\tau+d}$; this action is transitive. Then there is a surjection $\mathrm{GL}_2^+(\mathbb{R}) \to \mathfrak{h}$ via $\gamma \mapsto \gamma i$.

The stabilizer of i is $R^\times SO_2(\mathbb{R})$. Then the upper half plane is $\mathrm{GL}_2^+(\mathbb{R})/(\mathbb{R}^\times SO_2(\mathbb{R}))$

So now for $f : \mathfrak{h} \to \mathbb{C}$ and F the associated function on $\mathrm{GL}_2^+(\mathbb{R})$, the desired invariance properties for F carry over to ΔF and we get descent to a function Δf on $\mathfrak{h}$.

What is this operator? Up to a constant, $\Delta f = -y^2(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2})$. The Cauchy-Riemann equations imply that holomorphic and antiholomorphic functions are annihilated by this.

For $\mathrm{GL}_1/\mathbb{Q}$ our definition of nice was that there was some nonzero ideal in $\mathbb{C}[D]$ annihilating f.

Then once again we want f such that some finite codimension ideal in $\mathbb{C}[\Delta, Z] = Z(U(\mathfrak{g}_{\mathbb{C}}))$ annihilates f.

It appears we are interested in functions $f : \mathfrak{h} \to \mathbb{C}$ such that Δ, Z act by scalars λ, μ respectively on f.

Recall a Theorem of Deligne et al. saying that f a modular eigenform corresponds to $\rho_f : \mathrm{Gal}(\overline{Q}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ with $\det \rho_f = -1$.

Now let $f(x)$ be some random irreducible cubic polynomial over $\mathbb{Q}$ with three real roots α, β, γ. Let $K = \mathbb{Q}[\alpha, \beta, \gamma]$. Chances are $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3$.

There is some standard irreducible two-dimensional representation $S_3 \to \mathrm{GL}_2(\mathbb{Q})$. Fix $\rho_0 : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ factoring through this map and the surjection to $\mathrm{Gal}(K/\mathbb{Q})$. Then we find that the determinant of $\rho_0(x \mapsto \overline{x})$ is -1.

For any ℓ prime, this gives $\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_\ell)$. This gives a "bogus" compatible system of ℓ-adic representations.

Attached to this system should be an automorphic representation π by the Langlands philosophy; Maas solved this problem by writing down a specific function on the upper half plane to the complex numbers that is not holomorphic but is invariant under some $\Gamma_1(N)$ for N the conductor of ρ_0, and $\Delta f = \lambda f$ for some $\lambda \neq 0$.

Definition. Take G a connected reductive group over a number field K, a fixed maximal compact subgroup H_∞ (sometimes denoted K_∞) of $G(K_\infty)$. (For example, in our usual examples $G = \mathrm{GL}_n/\mathbb{Q}, G(K_\infty) = \mathrm{GL}_n(\mathbb{R}), H_\infty = O_n(\mathbb{R})$.) We call a function $\phi : G(K)\backslash G(\mathbb{A}_K) \to \mathbb{C}$ is called an **automorphic form** if

1. ϕ is smooth - i.e., if we write $G(\mathbb{A}_K) = G(\mathbb{A}_{K,f}) \times G(K_\infty)$ and write elements as (x, y), then for x fixed, ϕ is C_∞ with regard to y, and for fixed y, ϕ is locally constant with regard to x.

2. ϕ is well-behaved on compact subgroups in a way somehow related to admissibility - i.e.

   (a) There exists $U_f \subseteq G(\mathbb{A}_{K,f})$ compact open such that $\phi(gu) = \phi(g)$ for any $u \in U_f$;

   (b) the $\mathbb{C}$-vector spaced spanned by functions of the form $g \mapsto \phi(gh_\infty)$ for each $h_\infty \in H_\infty$ is finite-dimensional (e.g. ϕ trivial on H_∞).

3. There exists some ideal $I \subseteq Z(U(\mathfrak{g}_\mathbb{C}))$ for $\mathfrak{g}$ the Lie algebra of $G(K_\infty)$ such that I has finite codimension and for any differential operator $D \in I$ we have $D(y \mapsto \phi(x, y)) = 0$ for all $x \in G(\mathbb{A}_{K,f})$.

4. Some "boring" growth condition: $|\phi(x, y)|$ is at most some constant times $||y||^N$ for some N and some sensible norm on $G(K_\infty)$.

20 Lecture 20

More succinctly,

Definition. An *automorphic form* $\varphi : G(K)\backslash G(\mathbb{A}_K) \to \mathbb{C}$ is a smooth, slowly increasing, H_∞-finite, z-finite function.

We have the (hugely ∞-dimensional) $\mathbb{C}$-vector space

$$\mathcal{A}(G) = \{\varphi : \text{ automorphic form for } G\}$$

which is acted on the left by $G(\mathbb{A}_{K,f})$ via $(g * \varphi)(\gamma) = \varphi(\gamma g)$

Unfortunately $G(K_\infty)$ doesn't act on $\mathcal{A}(G)$ because for an arbitrary $g \in G(K_\infty)$, in general $gH_\infty g^{-1} \neq H_\infty$. However H_∞ and $\mathfrak{g}_\mathbb{C}$ act on $\mathcal{A}(G)$ where $\mathfrak{g}$ is the algebra of $G(K_\infty)$.

Remark. There's a second way to define this space using Hilbert spaces,

$$\mathcal{A}(G) = L^2(G(K)\backslash G(\mathbb{A}_K), \psi)$$

where ψ is some character. If we use this definition, $G(K_\infty)$ acts on $\mathcal{A}(G)$. (There is a way to go from one to the other but the relation is not very clear and will not be discussed here). Since we are not going to use the Hilbert space definition we will have to treat $\mathcal{A}(G)$ is a $\mathfrak{g}_\mathbb{C}, H_\infty$-module. Therefore, $\mathcal{A}$ has an action of $G(\mathbb{A}_{K,f}) \times (\mathfrak{g}_\mathbb{C}, H_\infty)$ but we note that $(\mathfrak{g}_\mathbb{C}, H_\infty)$ is not a group (but almost one).

Definition. An *automorphic representation* π for G/K is an irreducible quotient of $\mathcal{A}(G)$.

Remark. The notion of irreducible subquotient is not very clear here: it is *not* the standard definition which says that if G acts on vector space V and subspaces $V_1 \supseteq V_2$ are both G-stable then V_1/V_2 is a subquotient of V.

The definition we use here has to deal with Fourier theory. Consider the action of $\mathbb{R}$ on $L^2(\mathbb{R})$ via $(r * f)(x) = f(x + r)$ for a complex-valued function f on $\mathbb{R}$. Now we fix $y \in \mathbb{R}$, and define a character $\varphi_y : \mathbb{R} \to \mathbb{C}^\times$ by $x \mapsto e^{ixy}$, we have an action given by $(r * \varphi_y) = e^{iry} \cdot \varphi_y$. We observe that φ_y is an eigenvector.

The theory of Fourier transforms says that f is built from functions like φ_y but $|\varphi_y(x)| = 1$ for all x so $\varphi_y \notin L^2$. This φ is somehow supposed to give an irreducible subquotient but we can't seem to understand how here.

We will now try to "fix" the definition in two ways so that we get something which we understand!

Definition. An automorphic representation is called *cuspidal* if

$$\int_{N(K)\backslash N(\mathbb{A}_K)} \varphi(xn)dn = 0$$

where $N(K)$ is a unipotent radical and $MN = P$, the maximal parabolic subgroup.

Example 7. Let $G = \mathrm{GL}_2/\mathbb{Q}$, the maximal proper parabolic P are conjugates of the Borel, B. The normalizer, $N_G(B) = B$ and therefore $G/B \simeq \mathbb{P}^1/\mathbb{Q}$.

Remark. The condition of the integral =0 should remind us (somehow) of cusp forms where some coefficients become 0.

Now, we saw earlier that the map (in fact a character) $x \mapsto x^s \in \ker(D-s)$. $\langle D-s\rangle^2$ is also in $\mathbb{C}[D]$ and for $x \mapsto f(x) = x^3 \log x$, $(D-s)^2 f = 0$. Thus, both the above maps are in the kernel but that means we have "extra stuff" as $\mathbb{R}^\times$ acts non semi-simply. To fix this issue we have to make sure that the center acts by scalars.

Definition. We fix $\psi : Z(K)\backslash Z(\mathbb{A}_K) \to \mathbb{C}^\times$. Then we have

$\mathcal{A}_0(G,\psi) = \{\varphi \imath \mathcal{A}(G) : \varphi$ is cuspidal and $\varphi(gz) = \psi(z)\varphi(g)\ \forall z \in Z(\mathbb{A}_K),\ g \in G(\mathbb{A}_K)\}$.

This splits as an infinite direct sum.

A *cuspidal automorphic representation* π of $G(\mathbb{A}_K)$ is isomorphic to an irreducible subrepresentation of $\mathcal{A}_0(G,\psi)$

Theorem 20.1. *(Langlands) If π is an automorphic representation of G, π not cuspidal, then*

$$\pi \simeq Ind_P^G(\pi_0)$$

where π_0 is cuspidal on some smaller group.

Example 8. Let $G = \mathrm{GL}_1 \times \mathrm{GL}_1$. Since GL_1 has no proper parabolic subgroup, the automorphic representation is cuspidal. In this case the (cuspidal) automorphic representation for G is a pair χ_1, χ_2 of GCs.

We want the global analogue of (subquotient) of $I(\chi_1,\chi_2)$ to be non-cuspidal automorphic representations of GL_2.

Remark. Langlands shows that every automorphic representation of GL_2 is cuspidal or built from cuspidal ones. This is an instance of functoriality.

(*Langlands' Reciprocity*) The GLC for GL_n says

{cuspidal automorphic representaions for GL_n/K} $\leftrightarrow$ {irreducible $n-$dim representaions of L_K}

but since we don't even have a definition for the L-groups, it really doesn't make any sense. Now, any cuspidal representation $\pi = \otimes_\mathfrak{p}\pi_\mathfrak{p} \otimes \pi_\infty$. So, we fix $\mathfrak{p}$ and then restrict to $\mathfrak{p}$. By LLC we have $\pi_\mathfrak{p}$ corresponds to $\rho_\mathfrak{p} : WD(K_\mathfrak{p}) to \mathrm{GL}_n(\mathbb{C})$.

By definition, a *semi-simple representation* is the direct sum of irreducible representations. Thus, by Langlands' philosophy, the semi-simple representations of the L-group "should" correspond to arbitrary automorphic representations of GL_n/K. This is the main motivation for the highly no-trivial theorem stated above (proof involves harmonic analysis).

In general, reciprocity is a philosophy and functoriality is a concrete consequence.

Example 9. Let π be an automorphic representation of GL_2/K. Langlands' philosophy allows us to match-up π with a $\rho : L_K \to \mathrm{GL}_2(\mathbb{C})$. Now $\mathrm{Symm}^2(\rho) : L_K \to \mathrm{GL}_3(\mathbb{C})$ is a 3-dimensional representation and therefore by the philosophy the corresponding $\mathrm{Symm}^2(\pi)$ is an automorphic representation of GL_3/K. If π is cuspidal then $\mathrm{Symm}^2(\pi)$ is also almost always cuspidal. The fact that such a $\mathrm{Symm}^2(\pi)$ does exist is a hard theorem in functional analysis.

20.1 Examples of Automorphic Forms

Let $f : \mathfrak{H} \to \mathbb{C}$ be a holomorphic function and k an integer. Let $\gamma \in \mathrm{GL}_2^+(\mathbb{R})$. We define

$$\begin{aligned} f_k\gamma : \mathfrak{H} &\to \mathbb{C} \\ \tau &\mapsto (\det \gamma)^{k-1}(c\tau + d)^{-k} f(\gamma\tau) \end{aligned}$$

We say f is a cuspidal modular form of level N and weight k, if $f_k\gamma = f$ for all $\gamma \in \Gamma_1(N)$ + some boundedness conditions.

Let f be a cusp form and s be a complex number. We need to define $\varphi : \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$. We have already proved

1. $\mathrm{GL}_2(\mathbb{Q}_p) = B(\mathbb{Q}_p) \cdot \mathrm{GL}_2(\mathbb{Z}_p)$
2. $\mathrm{GL}_2(\mathbb{Z}_p) = \{1\} \cdot \mathrm{GL}_2(\mathbb{Z}_p)$

therefore, $\mathrm{GL}_2(\mathbb{A}_f) = B(\mathbb{A}_f)\mathrm{GL}_2(\hat{\mathbb{Z}})$ where $\mathbb{A}_{\mathbb{Q},f} = \mathbb{Q} + \hat{\mathbb{Z}}$ and $\mathbb{A}_{\mathbb{Q},f}^\times = \mathbb{Q}^\times + \hat{\mathbb{Z}}^\times$.

Therefore we can treat $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) = B(\mathbb{Q}) \cdot \mathrm{GL}_2(\hat{\mathbb{Z}})$.

The last trick: Define $U_f := U_1(N) = \{m \in \mathrm{GL}_2(\hat{\mathbb{Z}}) : \ m \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod N\}$. Then

$$\mathrm{GL}_2(\hat{\mathbb{Z}}) = \coprod_i \tilde{\gamma}_i U_f$$

where $\tilde{\gamma}_i$ are lifts of coset representatives in $\mathrm{GL}_2(\mathbb{Q})$. Therefore, $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f}) = \mathrm{GL}_2(\mathbb{Q})U_1(N)$ which gives

$$\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})U_1(N)\mathrm{GL}_2^+(\mathbb{R}).$$

Given f, s as before we define

$$\begin{aligned} \varphi : \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) &\to \mathbb{C} \\ \varphi(\gamma u h) &= (f_k h)(i) \cdot (\det h)^s. \end{aligned}$$

φ depends only on h. This is okay because φ must vanish on $\mathrm{GL}_2(\mathbb{Q})$ and must be locally constant on $U_1(N)$.

Claim: $\varphi \in \mathcal{A}(G)$.
Justification: It is obvious that φ is $\mathrm{GL}_2(\mathbb{Q})$-invariant and that it is $U_1(N)$-finite. We need to check that the definition is well defined. Indeed, if $\gamma_1 u_1 h_1 = \gamma_2 u_2 h_2$, we see that

$$\gamma_2^{-1}\gamma_1 = u_2 h_2 h_1^{-1} u_1^{-1} \in U_1(N)\mathrm{GL}_2^+(\mathbb{R}) \cap \mathrm{GL}_2(\mathbb{Q}) = \Gamma_1(N)$$

and this is precisely where h is well-behaved.

Furthermore,

$$f_k \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} (i) = (\sin\theta i + \cos\theta)^{-k} f(i).$$

By Cauchy-Riemann equations

$$\Delta\varphi = (k^2 - 2k)\varphi$$
$$Z\varphi = (2s + k - 2)\varphi$$

$\mathrm{GL}_2(\mathbb{A}_K)$ representations spanned by φ is the automorphic representation attached to f.

Let f be an eigenform, this gives us a φ to which we can attach an irreducible cuspidal automorphic representation. Deligne's theorem is a special case of the above example.

(For further details refer to *Buzzard's notes*)