Summary of Findings: Exponential Convergence of Steiner Symmetrizations

Adithya Chakravarthy

May to August 2019

Supervisor: Professor Almut Burchard

Contents

Rate of Side Increase for Polygons

Proposition 0.0.1. *Let $A \subset \mathbb{R}^2$ be an n-sided polygon (irregular or regular) and let $u \in S^1$. Then $S_u A$ is a $(2n-2)$-sided polygon, as long as A is not already symmetrical about the unit vector u.*

So if our original set is a triangle, then the sides increase in the pattern 3, 4, 6, 10, 18, 34, and so on. There are two observations we can make:

1. Any polygon, once symmetrized, will have an even number of sides because $2n-2$ is even for all integers $n > 2$.

2. The number of sides increase *exponentially* fast. In the above example, the general formula is $f(n) = 2^n + 2$.

We also found that the sequence of regular n-sided polygons (where n increases at the above exponential rate) converges exponentially fast to the ball: both with respect to symmetric difference and moment of inertia. To show the proof for moment of inertia, we look at the following table of values.

Notation: Let P_n denote a regular n-sided polygon with area 1 centered at the origin. Let $m(A)$ denote the moment of inertia of a set $A \in \mathbb{R}^2$. If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of sets all with volume 1, let

$$d_n = \frac{m(A_{n+1}) - m(A^*)}{m(A_n - m(A^*)}$$

Table 1: Regular Polygon Moment of Inertia Values

n	$m(P_n)$	d_n
3	0.19245	0.22561
4	0.16667	0.16243
6	0.16038	0.118481
10	0.1593	0.092143
18	0.15916	0.07774
34	0.15915	0.070299
66	0.15915	0.067561

Notice that $d_n < 1$ for all n in this table. Furthermore, the sequence of d_n's is strictly decreasing so we be fairly sure that $d_n \leq r < 1$ for all $n \in \mathbb{N}$, where we let $r = 0.22561$.

A natural strategy, therefore, would be to keep the polygon regular at each symmetrization (i.e: to make the regular 4-gon into a regular 6-gon, and that into a regular 10-gon, and so on). If we could do that, we could ensure exponential convergence. But from the data we collected for regular polygons, we found that for $n = 4, 6, 10, 18$ there seemed to be no symmetrals $u \in S^1$ for which $S_u P_n$ was a regular polygon!

For each symmetral u, we measured three quantities of $S_u P_n$: the standard deviation of the angles, the standard deviation of the side lengths, and the moment of inertia. These are clearly all continuous functions of the symmetral angle. They all achieved minima at certain angles, but those angles were *different* for the three criteria! So there was no one angle where the resulting shape was exactly regular.

Conjecture: *Let $A \subset \mathbb{R}^2$ be a regular n-sided polygon. There is no vector $u \in S^1$ (where A is not already symmetric about u) such that $S_u A$ is a regular polygon.*

The best case scenario of making our polygon regular at each stage is out of the picture. So we thought of various strategies to make our polygon "as regular as possible" at each stage. We outline our findings below.

Results from Simulations

Throughout this section, we obtained data about our strategies using a Python simulator of Steiner Symmetrization. The full code is available here. (We used the SymPy Library to write the code and the SageMath "Polyhedron" Module to view the polygon.) Using the code, we computed the moment of inertia values of the various sets at each stage.

Greedy Strategy

The first strategy we tried was the "greedy" strategy: where at each step we choose the angle that reduced the moment of inertia the most.

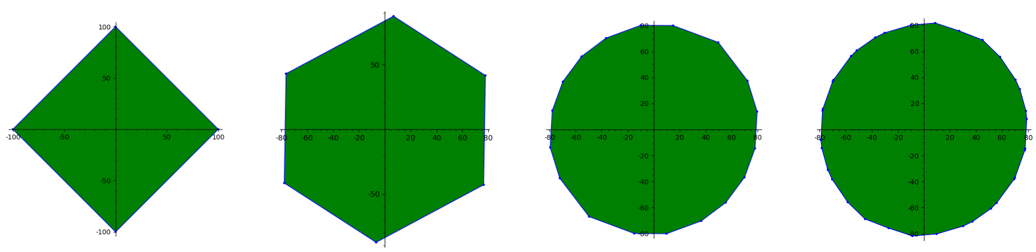
(*Aside*: there exists a so-called "best angle" to symmetrize along since the moment of inertia of our symmetrized set $S_u A$ is a continuous function of the angle of the unit vector u. Thus, if we consider angles on the closed interval $[0, 2\pi]$, then by the Extreme Value Theorem there must exist a minimum moment of inertia of $S_u A$.)

We began with a square $A \subset \mathbb{R}^2$, where $m(A) = 3333.333$ and $m(A^*) = 3183.098$. Here is the table of values for the moment of inertia values of the resulting symmetrized sets, along with their differences from the ball. (pictures below!)

Table 2: "Greedy Strategy" Moment of Inertia Values

Symmetrization (k)	$m(A_k)$	$m(A_k) - m(A^*)$	d_k
0	3333.3333	150.23	0.107
1	3199.1798	16.08	0.625
2	3193.1445	10.05	0.5572
3	3188.7016	5.60	0.8732
4	3187.988	4.89	???

This suggests that this strategy does *not* converge exponentially. The d_k's do not seem like they will satisfy $d_k \leq r < 1$ for all k, for some r. As $k \to \infty$, the values of $m(A_k) - m(A*)$ will likely decrease arbitrarily slowly since $m(A_k) - m(A*) > 0$ for all $k \in \mathbb{N}$. This suggests that $d_k \to 1$.



The problem with this strategy is that it is, in some sense, "too effective" at the beginning. At the start, it quickly gets the set into a ring. Once we are a few symmetrizations in, however, there is less progress to be made so the strategy inevitably slows down. This strategy is unlikely to converge exponentially.