BOLIB: Bilevel Optimization LIBrary of test problems

SHENGLONG ZHOU, ALAIN B. ZEMKOHO, AND ANDREY TIN

School of Mathematics, University of Southampton, UK
{shenglong.zhou, a.b.zemkoho, a.tin}@soton.ac.uk

ABSTRACT. To help accelerate the development of numerical solvers for bilevel optimization, BOLIB aims at presenting a collection of academic and real-world examples or case studies on the problem. This first version of the library is made of 124 academic examples of nonlinear bilevel optimization problems, tidied up from a wide range of publications. To the best of our knowledge, this is the first time that such a number of examples are provided to render a uniform basis on which algorithms proposed to deal with nonlinear bilevel optimization can be tested and compared. All the collected examples are programmed via Matlab and the library will be made freely available online.

1. INTRODUCTION

The general bilevel optimization problem can take the form

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \leq 0, H(x,y) = 0, \\ & y \in S(x) := \arg\min_{y} \{f(x,y) : \ g(x,y) \leq 0, h(x,y) = 0\}, \end{aligned} \tag{1.1}$$

where the functions $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_G}$ and $H : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_H}$ define the upper-level constraints, while $g : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_g}$ and $h : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_h}$ describe the lower-level constraints. On the other hand, $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ and $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$ denote the upper-and lower-level objective/cost functions, respectively. The set-valued map $S : \mathbb{R}^{n_x} \rightrightarrows \mathbb{R}^{n_y}$ represents the optimal solution/argminimum mapping of the lower-level problem. Further recall that problem (1.1) as a whole is often called upper-level problem.

As the medium and long term goal of this library is to include various classes of bilevel optimization problems, in particular, simple, linear, nonlinear, and real-world examples or case studies, we intentionally consider our model (1.1) to be broad, as it is likely that the overwhelming majority of these problems will be of this form. However, motivated by the numerical developments that have led to this initial version of the library [54], we just focus on nonlinear academic examples here. Recall that problem (1.1) is linear if all the functions involved are linear; otherwise, it is nonlinear.

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Another important restriction made in the codes of this version of the library is that equality constraint are not considered. This is because in the test set presented here, only four examples have equality constraints; see Table 3 or the detailed formulas of the problems in Appendix A. Hence, we use the fact that $H(x,y) = 0$ (similarly to $h(x,y) = 0$) can be expressed as $H(x,y) \leq 0$ and $-H(x,y) \leq 0$. Changes will be introduced in future versions of the library, provided that a significant number of new examples include equality constraints in the upper or lower-level of the problem. Hence, our focus here will be on nonlinear bilevel optimization problems of the form

$$\begin{aligned} \min_{x,y} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \leq 0, \\ & y \in S(x) := \arg\min_{y} \{f(x,y) : \ g(x,y) \leq 0\}. \end{aligned} \tag{1.2}$$

This paper provides a unique platform for the development of numerical methods, as well as theoretical results for nonlinear bilevel optimization problems. The main contributions of the paper can be summarized as follows:

(1) This first version of BOLIB provides codes for 124 examples of nonlinear bilevel optimization problems, ready to be used to test numerical algorithms.
(2) BOLIB provides the true or best known solutions and the corresponding references for all the examples included in the library. Hence, can therefore serve as a benchmark for numerical accuracy for methods designed to solve (1.2).
(3) All the mathematical formulas of the examples present in the library are also given, in order to allow researchers to test theoretical properties or build codes for the examples in software different from MATLAB, if necessary. For each example, the formulas of the functions F, G, f, and g, involved in (1.2), are put together in Appendix A, as well as some useful background details.
(4) This library also provides access to `SNBO`, a MATLAB based solver for nonlinear bilevel optimization problems, developed in the paper [55]. It can be used to test the examples presented here, as well as to solve other nonlinear bilevel programs.

To the best of our knowledge, this is the largest library of test examples for bilevel optimization, especially for the nonlinear class of the problem. It includes problems from Benoit Colson's BIPA [11], Sven Leyffer's MacMPEC [33], as well as from Mitsos and Barton's technical report [36]. Special classes of examples from the latter test sets not in this version of BOLIB will be included in future versions, where corresponding classes of problems are expanded to a reasonable size.

The main goal that we hope to achieve with BOLIB is the acceleration of numerical software development for bilevel optimization, as it is our opinion that the level of expansion of applications of the problem has outpaced the development rate for numerical solvers, especially for the nonlinear class of the problem.

For the remainder of the outline of this paper, note that in the next section, we describe the library and in Section 3, we discuss how to use it.

2. Descriptions of Library

This section mainly describes inputs and outputs of each example, and also lists all 124 examples, together with their true or best known solutions and the corresponding references. Before we proceed, note that each `m-file` contains information about the corresponding example, which include the first and second order derivatives of the input functions. For the upper-level objective function $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}$, these derivatives are defined as follows

$$\begin{aligned}
\nabla_x F(x,y) &= \begin{bmatrix} \nabla_{x_1} F \\ \vdots \\ \nabla_{x_{n_x}} F \end{bmatrix} \in \mathbb{R}^{n_x}, \\
\nabla^2_{xx} F(x,y) &= \begin{bmatrix} \nabla^2_{x_1x_1} F & \cdots & \nabla^2_{x_{n_x}x_1} F \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1x_{n_x}} F & \cdots & \nabla^2_{x_{n_x}x_{n_x}} F \end{bmatrix} \in \mathbb{R}^{n_x \times n_x}, \qquad (2.1) \\
\nabla^2_{xy} F(x,y) &= \begin{bmatrix} \nabla^2_{x_1y_1} F & \cdots & \nabla^2_{x_{n_x}y_1} F \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1y_{n_x}} F & \cdots & \nabla^2_{x_{n_x}y_{n_x}} F \end{bmatrix} \in \mathbb{R}^{n_y \times n_x}.
\end{aligned}$$

Similar expressions are valid for $\nabla_y F(x,y) \in \mathbb{R}^{n_y}, \nabla^2_{yy} F(x,y) \in \mathbb{R}^{n_y \times n_y}$ and the lower-level objective function f. As the constraint functions are vector-valued, we have for instance, in the context of the upper-level constraint function $G : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n_G}$, that

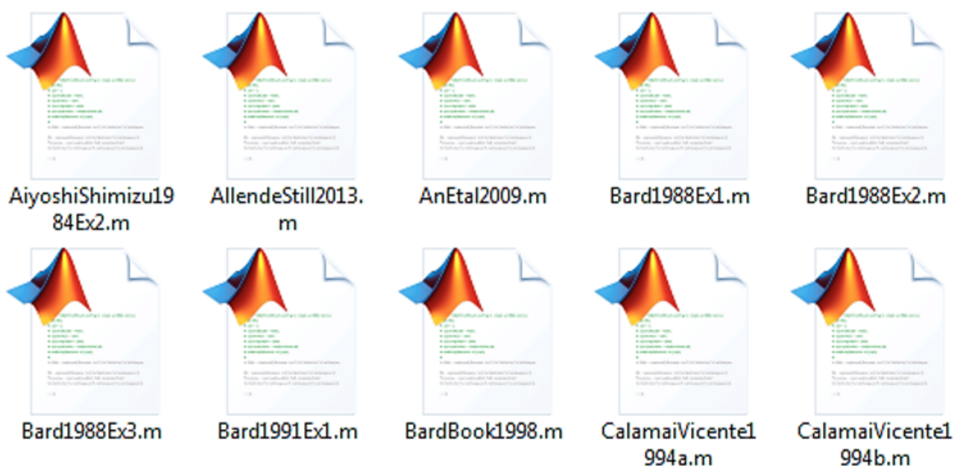
$$\nabla_x G(x,y) = \begin{bmatrix} \nabla_x G_1 \\ \vdots \\ \nabla_x G_{n_G} \end{bmatrix} = \begin{bmatrix} \nabla_{x_1} G_1 & \cdots & \nabla_{x_{n_x}} G_1 \\ \vdots & \ddots & \vdots \\ \nabla_{x_1} G_{n_G} & \cdots & \nabla_{x_{n_x}} G_{n_G} \end{bmatrix} \in \mathbb{R}^{n_G \times n_x}, \qquad (2.2)$$

$$\nabla^2_{xx} G(x,y) = \begin{bmatrix} \nabla^2_{xx} G_1 \\ \vdots \\ \nabla^2_{xx} G_{n_G} \end{bmatrix} = \begin{bmatrix} \nabla^2_{x_1x_1} G_1 & \cdots & \nabla^2_{x_{n_x}x_1} G_1 \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1x_{n_x}} G_1 & \cdots & \nabla^2_{x_{n_x}x_{n_x}} G_1 \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1x_1} G_{n_G} & \cdots & \nabla^2_{x_{n_x}x_1} G_{n_G} \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1x_{n_x}} G_{n_G} & \cdots & \nabla^2_{x_{n_x}x_{n_x}} G_{n_G} \end{bmatrix} \in \mathbb{R}^{(n_G n_x) \times n_x}, \qquad (2.3)$$

$$\nabla^2_{xy} G(x,y) = \begin{bmatrix} \nabla^2_{xy} G_1 \\ \vdots \\ \nabla^2_{xy} G_{n_G} \end{bmatrix} = \begin{bmatrix} \nabla^2_{x_1y_1} G_1 & \cdots & \nabla^2_{x_{n_x}y_1} G_1 \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1y_{n_y}} G_1 & \cdots & \nabla^2_{x_{n_x}y_{n_y}} G_1 \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1y_1} G_{n_G} & \cdots & \nabla^2_{x_{n_x}y_1} G_{n_G} \\ \vdots & \ddots & \vdots \\ \nabla^2_{x_1y_{n_y}} G_{n_G} & \cdots & \nabla^2_{x_{n_x}y_{n_y}} G_{n_G} \end{bmatrix} \in \mathbb{R}^{(n_G n_y) \times n_x}. \qquad (2.4)$$

Similar formulas will be valid for $\nabla_y G(x,y) \in \mathbb{R}^{n_G \times n_y}$, $\nabla^2_{yy} G(x,y) \in \mathbb{R}^{n_G n_y \times n_y}$ and the lower-level constraint g. Here we need to emphasize that when $n_G = 1$, $\nabla_x G(x,y) \in \mathbb{R}^{1\times n_x}$ which is a row vector whilst $\nabla_x F(x,y) \in \mathbb{R}^{n_x}$ or $\nabla_x f(x,y) \in \mathbb{R}^{n_x}$ are column vectors.

2.1. **Inputs and outputs.** Downloading the BOLIB zip folder (http://www.southampton.ac.uk/~abz1e14/bolib.html) and following the path `BOLIB` > `BOLIB-main`, one can find the folder named `BOLIBExample`, which contains 124 MATLAB m-files. Each one specifies a test example, named by a combination of authors' surnames, year of publication, and when necessary, the order of the example in the corresponding reference. For example, as in following figure (showing a partial list of the examples), `AiyoshiShimizu1984Ex2.m` stands for Example 2 in the paper by Aiyoshi and Shimizu published in 1984; see the corresponding reference [1] for more details.



Now we describe the inputs and outputs for a given m-file example. All files have the uniform function handle as

$$w = \texttt{example\_name}(x, y, \texttt{keyf}, \texttt{keyxy}).$$

For the inputs, we have

$$\begin{array}{rcl} x & \in & \mathbb{R}^{n_x}, \quad y \in \mathbb{R}^{n_y}, \\ \texttt{keyf} & \in & \{\texttt{`F'}, \texttt{`G'}, \texttt{`f'}, \texttt{`g'}\}, \\ \texttt{keyxy} & \in & \{[\,], \texttt{`x'}, \texttt{`y'}, \texttt{`xx'}, \texttt{`xy'}, \texttt{`yy'}\}, \end{array}$$

where `F'`, `G'`, `f'`, and `g'` respectively stand for the four functions involved in (1.2). `x'` and `y'` represent the first order derivative with respect to x and y, respectively. Finally, `xx'`, `xy'`, and `yy'` correspond to the second order derivative of the function F, G, f, and g, with respect to xx, xy, and yy, respectively.

For the outputs, $w = \texttt{example\_name}(x, y, \texttt{keyf})$ or $w = \texttt{example\_name}(x, y, \texttt{keyf}, [\,])$ returns the function value of `keyf`, and $w = \texttt{example\_name}(x, y, \texttt{keyf}, \texttt{keyxy})$ returns the first or second order derivative of `keyf` with respect to choice of `keyxy` as describe above. We can summarize the input-inputs scenarios in the following table:

keyf \ keyxy	[ ]	\`x′	\`y′	\`xx′	\`xy′	\`yy′
\`F′	$F(x,y)$	$\nabla_x F(x,y)$	$\nabla_y F(x,y)$	$\nabla^2_{xx} F(x,y)$	$\nabla^2_{xy} F(x,y)$	$\nabla^2_{yy} F(x,y)$
\`G′	$G(x,y)$	$\nabla_x G(x,y)$	$\nabla_y G(x,y)$	$\nabla^2_{xx} G(x,y)$	$\nabla^2_{xy} G(x,y)$	$\nabla^2_{yy} G(x,y)$
\`f′	$f(x,y)$	$\nabla_x f(x,y)$	$\nabla_y f(x,y)$	$\nabla^2_{xx} f(x,y)$	$\nabla^2_{xy} f(x,y)$	$\nabla^2_{yy} f(x,y)$
\`g′	$g(x,y)$	$\nabla_x g(x,y)$	$\nabla_y g(x,y)$	$\nabla^2_{xx} g(x,y)$	$\nabla^2_{xy} g(x,y)$	$\nabla^2_{yy} g(x,y)$

For the dimension of w in each scenario, see (2.1)–(2.4). If $n_G = 0$ (or $n_g = 0$), all outputs related to G (or g) should be empty, namely, $w = [\,]$. Let us look at some specific usage:

- w = example_name(x,y, \`F′) or $w$ = example_name($x,y$, \`F′,[ ]) returns the function value of F, i.e., $w = F(x,y)$; this is similar for G, f, and g;
- w = example_name(x,y, \`F′, \`x′) returns the partial derivative of F with respect to x, i.e., $w = \nabla_x F(x,y)$;
- w = example_name(x,y, \`G′, \`y′) returns the Jacobian matrix of G with respect to y, i.e., $w = \nabla_y G(x,y)$;
- w = example_name(x,y, \`f′, \`xy′) returns the Hessian matrix of f with respect to xy, i.e., $w = \nabla^2_{xy} f(x,y)$;
- w = example_name(x,y, \`g′, \`yy′) returns the second order derivative of g with respect to yy, i.e., $w = \nabla^2_{yy} g(x,y)$.

We use one example to illustrate the definitions above.

Example 2.1. Shimizu et al. (1997), see [47], considered the bilevel program (1.2) with

$$\begin{array}{rcl} F(x,y) & := & (x-5)^2 + (2y+1)^2, \\ f(x,y) & := & (y-1)^2 - 1.5xy, \\ g(x,y) & := & \begin{bmatrix} -3x+y+3 \\ x-0.5y-4 \\ x+y-7 \end{bmatrix}. \end{array}$$

Clearly, $n_x = 1, n_y = 1, n_G = 0, n_g = 3$. The m-file is named by ShimizuEtal1997a (i.e., exmaple_name = ShimizuEtal1997a), which was coded through MATLAB as follows

```
function w=ShimizuEtal1997a(x,y,keyf,keyxy)
if nargin<4 || isempty(keyxy)
    switch keyf
    case `F'; w = (x-5)^2+(2*y+1)^2;
    case `G'; w = [];
    case `f'; w = (y-1)^2-1.5*x*y;
    case `g'; w = [-3*x+y+3; x-0.5*y-4; x+y-7];
    end
else
    switch keyf
    case `F'
        switch keyxy
        case `x' ; w = 2*(x-5);
```

```
        case `y' ; w = 4*(2*y+1);
        case `xx'; w = 2;
        case `xy'; w = 0;
        case `yy'; w = 8;
        end
    case `G'
       switch keyxy
        case `x' ; w = [];
        case `y' ; w = [];
        case `xx'; w = [];
        case `xy'; w = [];
        case `yy'; w = [];
        end
    case `f'
        switch keyxy
        case `x' ; w = -1.5*y;
        case `y' ; w = 2*(y-1)-1.5*x;
        case `xx'; w = 0;
        case `xy'; w = -1.5;
        case `yy'; w = 2;
        end
    case `g'
        switch keyxy
        case `x' ; w = [-3;  1;  1];
        case `y' ; w = [ 1;-0.5; 1];
        case `xx'; w = [ 0;  0;  0];
        case `xy'; w = [ 0;  0;  0];
        case `yy'; w = [ 0;  0;  0];
        end
     end
end
end
```

If we are given some inputs (as in left column of the table below), then `ShimizuEtal1997a` will return us corresponding results as in the right column of the table:

Inputs	Outputs
`x = 4`	`x = 4`
`y = 0`	`y = 0`
`F = ShimizuEtal1997a(x,y,'F')`	`F = 2`
`Fx = ShimizuEtal1997a(x,y,'F','x')`	`Fx = -2`
`Gy = ShimizuEtal1997a(x,y,'G','y')`	`Gy = []`
`fxy = ShimizuEtal1997a(x,y,'f','xy')`	`fxy = -1.5`
`gyy = ShimizuEtal1997a(x,y,'g','yy')`	`gyy = [0;0;0]`

2.2. **List of test examples.** For the examples presented here, at least one of the functions involved in the problem is nonlinear. The details related to each example presented in the BOLIB library are in a column of Table 3 below. The first column of the table provides the list of problems, as they appear in the `BOLIBExample` folder. The second column gives the reference in the literature where the example might have first appeared. The third column combines the labels corresponding to the nature of the functions involved in (1.2). Precisely, "N" and "L" will be used to indicate whether the functions F, G, f, and g are nonlinear (N) or linear (L), while "O" is used to symbolize that there is either no function G or g present in problem (1.2). Then follows the column with n_x and n_y for the upper and lower-level variables dimensions, as well as n_G (resp. n_g) to denote the the number of components of the upper (resp. lower)-level constraint function. On the other hand, F^* and f^* denote the best known optimal upper and lower-level objective function values, respectively, according to the literature that is listed in the last column **RefII**. Notice that example `CalamaiVicente1994c` has unknown lower-level objective function value, and `Zlobec2001b` has no optimal solutions. Moreover, 3 examples involve parameters. They are `CalamaiVicente1994a` with $\rho \geq 1$ (its F^* and f^* listed in the table are under $\rho = 1$, other cases can be found in Appendix A), `HenrionSurowiec2011` with $c \in \mathbb{R}$ and `IshizukaAiyoshi1992a` with $M > 1$.

TABLE 3. List of nonlinear bilevel programs with their labels.

Example name	**RefI**	F-G-f-g	n_x	n_y	n_G	n_g	F^*	f^*	**RefII**
AiyoshiShimizu1984Ex2	[1]	L-L-N-L	2	2	5	6	5	0	[1]
AllendeStill2013	[2]	N-L-N-N	2	2	5	2	-1.5	-0.5	[2]
AnEtal2009	[3]	N-L-N-L	2	2	6	4	2251.6	165.2	[3]
Bard1988Ex1	[4]	N-L-N-L	1	1	1	4	17	1	[4]
Bard1988Ex2	[4]	N-L-N-L	4	4	9	12	-6600	54	[54]
Bard1988Ex3	[4]	N-N-N-N	2	2	3	4	-12.68	-1.02	[8]
Bard1991Ex1	[5]	L-L-N-L	1	2	2	3	2	12	[5]
BardBook1998	[6]	N-L-L-L	2	2	4	7	0	5	[54]
CalamaiVicente1994a($\rho = 1$)	[7]	N-O-N-L	1	1	0	3	0	0	[7]
CalamaiVicente1994b	[7]	N-O-N-L	4	2	0	6	0.17	-0.45	[54]
CalamaiVicente1994c	[7]	N-O-N-L	4	2	0	6	0.3125	×	[7]
CalveteGale1999P1	[9]	L-L-L-N	2	3	2	6	-29.2	0.31	[9, 22]
ClarkWesterberg1990a	[10]	N-L-N-L	1	1	2	3	5	4	[44]
Colson2002BIPA1	[11]	N-L-N-L	1	1	3	3	250	0	[54]
Colson2002BIPA2	[11]	N-L-N-L	1	1	1	4	17	2	[8]
Colson2002BIPA3	[11]	N-L-N-L	1	1	2	2	2	24	[8]
Colson2002BIPA4	[11]	N-L-N-L	1	1	2	2	88.79	-0.77	[8]
Colson2002BIPA5	[11]	N-L-N-N	1	2	1	6	2.75	0.57	[8]
Dempe1992a	[12]	L-N-N-N	2	2	1	2	-0.5	0.63	[54]
Dempe1992b	[12]	N-O-N-N	1	1	0	1	31.25	4	[8]
DempeDutta2012Ex24	[13]	N-O-N-N	1	1	0	1	0	0	[13]
DempeDutta2012Ex31	[13]	L-N-N-N	2	2	4	2	-1	4	[13]
DempeEtal2012	[14]	L-L-N-L	1	1	2	2	-1	-1	[14]
DempeFranke2011Ex41	[15]	N-L-N-L	2	2	4	4	5	-2	[15]
DempeFranke2011Ex42	[15]	N-L-N-L	2	2	4	3	2.13	-3.5	[15]
DempeFranke2014Ex38	[16]	L-L-N-L	2	2	4	4	-1	-4	[16]

DempeLohse2011Ex31a	[17]	N-O-N-L	2	2	0	4	-6	1	[17]
DempeLohse2011Ex31b	[17]	N-O-N-L	3	3	0	5	-6	0	[17]
DeSilva1978	[18]	N-O-N-L	2	2	0	4	-1	0	[8]
EdmundsBard1991	[19]	L-L-N-L	2	2	4	7	0	200	[49]
FalkLiu1995	[20]	N-O-N-L	2	2	0	4	-2.27	0	[8, 54]
FloudasZlobec1998	[21]	N-L-L-N	1	2	2	6	1	-1	[22, 36]
GumusFloudas2001Ex1	[22]	N-L-N-L	1	1	3	3	2250	197.75	[36]
GumusFloudas2001Ex3	[22]	L-L-N-L	2	3	4	9	-29.2	0.31	[36]
GumusFloudas2001Ex4	[22]	N-L-N-L	1	1	5	2	9	0	[36]
GumusFloudas2001Ex5	[22]	L-L-N-N	1	2	2	6	0.19	-7.23	[36]
HatzEtal2013	[23]	L-O-N-L	1	2	0	2	0	0	[23]
HendersonQuandt1958	[24]	N-L-N-L	1	1	2	1	-3266.7	-711.11	[24]
HenrionSurowiec2011	[25]	N-O-N-O	1	1	0	0	$-c^2/4$	$-c^2/8$	[26]
IshizukaAiyoshi1992a	[27]	N-L-L-L	1	2	1	5	0	-M	[27]
KleniatiAdjiman2014Ex3	[28]	L-L-N-L	1	1	2	2	-1	0	[28]
KleniatiAdjiman2014Ex4	[28]	N-N-N-N	5	5	13	11	-10	-3.1	[28]
LampariielloSagratella2017Ex23	[29]	L-L-N-L	1	2	2	2	-1	1	[29]
LampariielloSagratella2017Ex31	[30]	N-L-L-L	1	1	1	1	1	0	[30]
LampariielloSagratella2017Ex32	[30]	N-O-N-O	1	1	0	0	0.5	0	[30]
LampariielloSagratella2017Ex33	[30]	N-L-L-L	1	2	1	3	0.5	0	[30]
LampariielloSagratella2017Ex35	[30]	N-L-L-L	1	1	2	3	0.8	-0.4	[30]
LucchettiEtal1987	[31]	N-L-N-L	1	1	2	2	0	0	[31]
LuDebSinha2016a	[32]	N-L-N-O	1	1	4	0	1.94	1.96	[32]
LuDebSinha2016b	[32]	N-L-N-O	1	1	4	0	0	1.66	[32]
LuDebSinha2016c	[32]	N-L-N-O	1	1	4	0	1.12	0.06	[32]
LuDebSinha2016d	[32]	L-N-L-N	2	2	11	3	-192	-192	[54]
LuDebSinha2016e	[32]	N-L-L-N	1	2	6	3	1.1	-18.57	[54]
LuDebSinha2016f	[32]	L-N-N-O	2	1	9	0	0	0.13	[54]
MacalHurter1997	[34]	N-O-N-O	1	1	0	0	81.33	-0.33	[34]
Mirrlees1999	[35]	N-O-N-O	1	1	0	0	0.01	-1.04	[35]
MitsosBarton2006Ex38	[36]	N-L-N-L	1	1	4	2	0	0	[36]
MitsosBarton2006Ex39	[36]	L-L-N-L	1	1	3	2	-1	-1	[36]
MitsosBarton2006Ex310	[36]	L-L-N-L	1	1	2	2	0.5	-0.1	[36]
MitsosBarton2006Ex311	[36]	L-L-N-L	1	1	2	2	-0.8	0	[36]
MitsosBarton2006Ex312	[36]	N-L-N-L	1	1	2	2	0	0	[36]
MitsosBarton2006Ex313	[36]	L-L-N-L	1	1	2	2	-1	0	[36]
MitsosBarton2006Ex314	[36]	N-L-N-L	1	1	2	2	0.25	-0.08	[36]
MitsosBarton2006Ex315	[36]	L-L-N-L	1	1	2	2	0	-0.83	[36]
MitsosBarton2006Ex316	[36]	L-L-N-L	1	1	2	2	-2	0	[36]
MitsosBarton2006Ex317	[36]	N-L-N-L	1	1	2	2	0.19	-0.02	[36]
MitsosBarton2006Ex318	[36]	N-L-N-L	1	1	2	2	-0.25	0	[36]
MitsosBarton2006Ex319	[36]	N-L-N-L	1	1	2	2	-0.26	0	[36]
MitsosBarton2006Ex320	[36]	N-L-N-L	1	1	2	2	0.31	-0.08	[36]
MitsosBarton2006Ex321	[36]	N-L-N-L	1	1	2	2	0.21	-0.07	[36]
MitsosBarton2006Ex322	[36]	N-L-N-N	1	1	2	3	0.21	-0.07	[36]
MitsosBarton2006Ex323	[36]	N-N-L-N	1	1	3	3	0.18	-1	[36]
MitsosBarton2006Ex324	[36]	N-L-N-L	1	1	2	2	-1.75	0	[36]

MitsosBarton2006Ex325	[36]	N-N-N-N	2	3	6	9	-1	-2	[36]
MitsosBarton2006Ex326	[36]	N-N-N-L	2	3	7	6	-2.35	-2	[36]
MitsosBarton2006Ex327	[36]	N-N-N-N	5	5	13	13	2	-1.1	[36]
MitsosBarton2006Ex328	[36]	N-N-N-N	5	5	13	13	-10	-3.1	[36]
MorganPatrone2006a	[37]	L-L-N-L	1	1	2	2	-1	0	[37]
MorganPatrone2006b	[37]	L-O-N-L	1	1	0	4	-1.25	0	[37]
MorganPatrone2006c	[37]	L-O-N-L	1	1	0	4	-1	-0.25	[37]
MuuQuy2003Ex1	[38]	N-L-N-L	1	2	2	3	-2.08	-0.59	[38]
MuuQuy2003Ex2	[38]	N-L-N-L	2	3	3	4	0.64	1.67	[38]
NieEtal2017Ex34	[39]	L-L-N-N	1	2	2	2	2	0	[39]
NieEtal2017Ex52	[39]	N-N-N-N	2	3	5	2	-1.71	-2.23	[39]
NieEtal2017Ex54	[39]	N-N-N-N	4	4	3	2	-0.44	-1.19	[39]
NieEtal2017Ex57	[39]	N-N-N-N	2	3	5	2	-2	-1	[39]
NieEtal2017Ex58	[39]	N-N-N-N	4	4	3	2	-3.49	-0.86	[39]
NieEtal2017Ex61	[39]	N-N-N-N	2	2	5	1	-1.02	-1.08	[39]
Outrata1990Ex1a	[40]	N-O-N-L	2	2	0	4	-8.92	-6.05	[40]
Outrata1990Ex1b	[40]	N-O-N-L	2	2	0	4	-7.56	-0.58	[40]
Outrata1990Ex1c	[40]	N-O-N-L	2	2	0	4	-12	-112.71	[40]
Outrata1990Ex1d	[40]	N-O-N-L	2	2	0	4	-3.6	-2	[40]
Outrata1990Ex1e	[40]	N-O-N-L	2	2	0	4	-3.15	-16.29	[40]
Outrata1990Ex2a	[40]	N-L-N-L	1	2	1	4	0.5	-14.53	[40]
Outrata1990Ex2b	[40]	N-L-N-L	1	2	1	4	0.5	-4.5	[40]
Outrata1990Ex2c	[40]	N-L-N-L	1	2	1	4	1.86	-10.93	[40]
Outrata1990Ex2d	[40]	N-L-N-N	1	2	1	4	0.92	-19.47	[40]
Outrata1990Ex2e	[40]	N-L-N-N	1	2	1	4	0.90	-14.94	[40]
Outrata1993Ex31	[41]	N-L-N-N	1	2	1	4	1.56	-11.67	[41]
Outrata1993Ex32	[41]	N-L-N-N	1	2	1	4	3.21	-20.53	[41]
Outrata1994Ex31	[42]	N-L-N-N	1	2	2	4	3.21	-20.53	[42]
OutrataCervinka2009	[43]	L-L-N-L	2	2	1	3	0	0	[43]
PaulaviciusEtal2017a	[44]	N-L-N-L	1	1	4	2	0.25	0	[44]
PaulaviciusEtal2017b	[44]	L-L-N-L	1	1	4	2	-2	-1.5	[44]
SahinCiric1998Ex2	[45]	N-L-N-L	1	1	2	3	5	4	[45]
ShimizuAiyoshi1981Ex1	[46]	N-L-N-L	1	1	3	3	100	0	[46]
ShimizuAiyoshi1981Ex2	[46]	N-L-N-L	2	2	3	4	225	100	[46]
ShimizuEtal1997a	[47]	N-O-N-L	1	1	0	3	25	-14	[54]
ShimizuEtal1997b	[47]	N-L-N-L	1	1	2	2	2250	197.75	[47]
SinhaMaloDeb2014TP3	[48]	N-N-N-N	2	2	3	4	-18.68	-1.02	[48]
SinhaMaloDeb2014TP6	[48]	N-L-N-L	1	2	1	6	-1.21	7.62	[48]
SinhaMaloDeb2014TP7	[48]	N-N-N-L	2	2	4	4	-1.96	1.96	[48]
SinhaMaloDeb2014TP8	[48]	N-L-N-L	2	2	5	6	0	100	[48]
SinhaMaloDeb2014TP9	[48]	N-O-N-L	10	10	0	20	0	1	[48]
SinhaMaloDeb2014TP10	[48]	N-O-N-L	10	10	0	20	0	1	[48]
TuyEtal2007	[49]	N-L-L-L	1	1	2	3	22.5	-1.52	[49]
Vogel2002	[50]	N-L-N-L	1	1	2	1	0	2	[50]
WanWangLv2011	[51]	N-O-L-L	2	3	0	8	10.63	-0.5	[51]
YeZhu2010Ex42	[52]	N-L-N-L	1	1	2	1	1	-2	[52]
YeZhu2010Ex43	[52]	N-L-N-L	1	1	2	1	1.25	-2	[52]

Yezza1996Ex31	[53]	N-L-N-L	1	1	2	2	1.5	-2.5	[53]
Yezza1996Ex41	[53]	N-O-N-L	1	2	0	2	0.5	2.5	[53]
Zlobec2001a	[56]	N-O-L-L	1	2	0	3	-1	-1	[56]
Zlobec2001b	[56]	L-L-L-N	1	1	2	4	×	×	[56]

It is worth mentioning that some examples contain equalities constraints $H(x,y) = 0$ or $h(x,y) = 0$. As we mentioned in the beginning of this manuscript, equalities constraints can be transformed to inequalities ones. However, for the sake of clarity, we list those examples whose constraints contain equalities below.

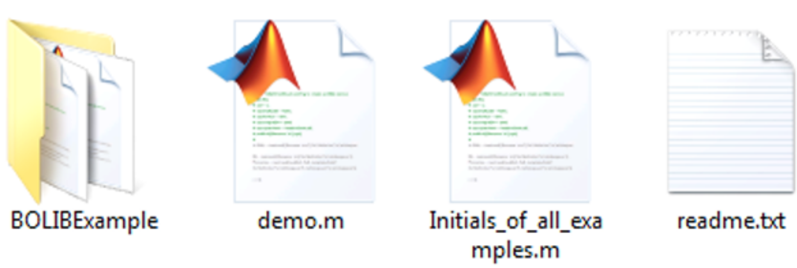
TABLE 4. List of nonlinear bilevel programs with equalities constraints.

Example name	**Ref.**	$F/G/H/f/g/h$	n_x	n_y	n_G	n_H	n_g	n_h
DempeDutta2012Ex31	[13]	L-L-N-N-N-0	2	2	2	1	2	0
DempeFranke2011Ex41	[15]	N-L-L-N-L-0	2	2	2	1	4	0
DempeFranke2011Ex42	[15]	N-L-L-N-L-0	2	2	2	1	3	0
Zlobec2001b	[56]	L-L-0-L-L-N	1	1	2	0	2	1

3. How to use the library

When a solver is introduced to solve some examples from the BOLIB library, how to operate it? This section demonstrates how to do this. First, recall that the library can be accessed at http://www.southampton.ac.uk/~abz1e14/bolib.html

3.1. **Files in library.** To get more insights about the files, first open the folder `BOLIB-main` (sub-folder of `BOLIB`), and there you will find another folder, two m-files, and a txt-file, as can be seen in the following picture:



Recall that the `BOLIBExample` folder contains all the 124 nonlinear examples; `readme.txt` describes what files are contained in the `BOLIB-main` folder; `Initials_of_all_examples.m` provides the dimensions involved in problem (1.2), namely n_x, n_y, n_G, and n_g, and two starting points for each example. This file is useful when using BOLIB and has the format

```
[xy, dim]=Initials_of_all_examples(exname,s)
```

with inputs and outputs respectively defined by

- `exname` is the name of example, such as `exname = 'ShimizuEtal1997a'`. If `s`$= 1$, it returns the first starting point, and returns the second one otherwise;

- `xy`=$[x^\top\ y^\top]^\top$ and `dim`=$[n_x\ n_y\ n_G\ n_g]$.

As for `demon.m`, it demonstrates how to call an example in the `BOLIBExample` folder. This file contains the code

```
1 clc; close all;
2
3 % add the path
4 addpath(genpath(pwd));
5
6 % call example to be tested
7 exname = 'ShimizuEtal1997a';
8 [xy, dim] = Initials_of_all_examples(exname,1);
9 fun = str2func(exname);
```

Let us briefly explain some lines in this code. The purpose of line 4 is to add `BOLIBExample` folder to the path. Line 7 defines the name of tested example, i.e., `ShimizuEtal1997a`, in this case (`ShimizuEtal1997a` can be replaced by a different example). Line 8 returns the dimensions `dim` and the first starting point `xy` (since `s`= 1) of the example. The most important sentence is line 9, in which `str2func(exname)` calls the function in `BOLIBExample` folder with the function name being the example name `ShimizuEtal1997a`. The whole line constructs a new function handle, `fun`, such that

`fun(x,y,keyf,keyxy)` = `ShimizuEtal1997a(x,y,keyf,keyxy)`.

Here, `fun` can well be changed, but having a uniform function name is a more convenient format to run experiments.

3.2. **Two demonstrations.** We use two examples to demonstrate how to operate BOLIB. The first one is to see the outputs of examples from `BOLIBExample` folder. To proceed, open `demon.m` file and type codes after line 9 as shown below:

```
1 clc; close all;
2
3 % add the path
4 addpath(genpath(pwd));
5
6 % call example to be tested
7 exname = 'ShimizuEtal1997a';
8 [xy, dim] = Initials_of_all_examples(exname,1);
9 fun = str2func(exname);
10
11 x = xy(1:dim(1))
12 y = xy(dim(1)+1:dim(1)+dim(2))
13 F = fun(x,y,'F')
14 Fx = fun(x,y,'F','x')
15 Gy = fun(x,y,'G','y')
16 fxy = fun(x,y,'f','xy')
17 gyy = fun(x,y,'g','yy')
```

Then running those codes, the results are

```
x=4 y=3 F=50 Fx=-2 Gy=[] fxy=-1.5 gyy=[0 0 0]'
```

The second one is to call a solver (we use `SNBO` [55] here) to solve examples from `BOLIBExample` folder. The solver `SNBO` can be found by opening folders as “`BOLIB` ” --> “`Solvers`”. The procedure can be summarized as follows. First put the solver `SNBO` in the `BOLIB-mian` folder as shown in the figure below:

Then proceed with `demon.m` file within MATLAB, and type codes after line 9 as

```
1  clc; close all;
2
3  % add the path
4  addpath(genpath(pwd));
5
6  % call example to be tested
7  exname = 'ShimizuEtal1997a';
8  [xy, dim] = Initials_of_all_examples(exname,1);
9  fun = str2func(exname);
10
11 pars.xy = xy;
12 pars.lam = 1;
13 Out = SNBO(dim,fun,pars)
```

Then running these codes, the results are

```
    x:  5.0000
    y:  2.0000
    F:  25.0000
    f:  -14.0000
 iter:  5
 time:  0.0439
error:  3.2706e-11
alpha:  1.0000
```

Another approach for the second demonstration is to change line 7; we apply this while testing a different example, namely, `AiyoshiShimizu1984Ex2`. Similarly, we type the following code lines:

```
1  clc; close all;
2
3  % add the path
4  addpath(genpath(pwd));
5
6  % call example to be tested
7  exname = 'AiyoshiShimizu1984Ex2';
8  [xy, dim] = Initials_of_all_examples(exname,1);
9  fun = str2func(exname);
10
11 pars.xy = xy;
12 pars.lam = 1;
13 Out = SNBO(dim,fun,pars)
```

Then run them and get the results as

```
    x:  [2x1 double]
    y:  [2x1 double]
    F:  25.0000
    f:  5.5625
 iter:  7
 time:  0.1137
error:  3.0644e-06
alpha:  1.0000
```

As it can be seen above from the process of running examples `ShimizuEtal1997a` and `AiyoshiShimizu1984Ex2`, only line 7 is altered. This is because, as we explained before, line 9 unifies two different names as the same one (i.e., `fun`), which greatly enhances the convenience of the usage.

4. Conclusion

This manuscript guides readers on the understanding of BOLIB, a benchmark library for bilevel optimization test examples. This version of the library provides a collection of 124 nonlinear bilevel optimization test examples, describes some main files (including the MATLAB m-files defining the examples and the m-file providing initials of all examples), and offers instructions on how to access and use the BOLIB library. Using the SNBO solver developed in [55], this manuscript also provides a clear path on how to use BOLIB to test a numerical algorithm programmed with MATLAB. In the next version of the library, nonlinear bilevel programs will be expanded and a collection of academic linear and simple bilevel programs, as well as real-world examples and case studies on bilevel optimization, will be included. This will allow researchers and practitioners to test their algorithms on problems of larger sizes.

References

[1] E. Aiyoshi and K. Shimizu, A solution method for the static constrained Stackelberg problem via penalty method, IEEE Transactions on Automatic Control, 29, 1111-1114, 1984.
[2] G.B. Allende and G. Still, Solving bilevel programs with the KKT-approach. Mathematical Programming, 138(1-2), 309-332, 2013.
[3] L.T.H. An, P.D. Tao, N.N. Canh and N.V. Thoai, DC programming techniques for solving a class of nonlinear bilevel programs, Journal of Global Optimization, 44(3), 313-337, 2009.
[4] J.F. Bard, Convex two-level optimization, Mathematical Programming, 40, 15-27, 1988.
[5] J.F. Bard, Some properties of the bilevel programming problem, Journal of Optimization Theory and Applications, 68(2), 371-378, 1991.
[6] J.F. Bard, Practical bilevel optimization: Algorithms and applications, Dordrecht: Kluwer Academic Publishers, 1998.
[7] P.H. Calamai and L.N. Vicente, Generating quadratic bilevel programming test problems, ACM Transactions on Mathematical Software, 20(1), 103-119, 1994.
[8] B. Colson, P. Marcotte and G. Savard, A trust-region method for nonlinear bilevel programming: algorithm and computational experience, Computational Optimization and Applications, 30(3), 211-227, 2005.
[9] H.I. Calvete and C. Galé, The bilevel linear/linear fractional programming problem, European Journal of Operational Research, 114(1), 188-197, 1999.
[10] P.A. Clark and A.W. Westerberg, A note on the optimality conditions for the bilevel programming problem, Naval Research Logistics, 35, 414-418, 1988.
[11] B. Colson, BIPA (Bilevel Programming with Approximation Methods): Software guide and test problems, Technical report, 2002.
[12] S. Dempe, A necessary and a sufficient optimality condition for bilevel programming problems, Optimization, 25, 341-354, 1992.
[13] S. Dempe and J. Dutta, Is bilevel programming a special case of a mathematical program with complementarity constraints? Mathematical programming, 131(1-2), 37-48, 2012.
[14] S. Dempe, B. Mordukhovich, and A.B. Zemkoho, Sensitivity analysis for two-level value functions with applications to bilevel programming, SIAM Journal on Optimization, 22(4), 1309-1343, 2012.
[15] S. Dempe and S. Franke, An algorithm for solving a class of bilevel programming problems, Preprint 2011-04, TU Bergakademie, 2011.
[16] S. Dempe and S. Franke, Solution algorithm for an optimistic linear stackelberg problem, Computers & Operations Research, 41, 277-281, 2014.
[17] S. Dempe and S. Lohse, Dependence of bilevel programming on irrelevant data, Dekan der Fakultät für Mathematik und Informatik, 2011.
[18] A.H. De Silva, Sensitivity formulas for nonlinear factorable programming and their application to the solution of an implicitly defined optimization model of US crude oil production, Ph.D. thesis, George Washington University, 1978.
[19] T.A. Edmunds and J.F. Bard, Algorithms for nonlinear bilevel mathematical programs, IEEE transactions on Systems, Man, and Cybernetics, 21(1), 83-89, 1991.
[20] J.E. Falk and J. Liu, On bilevel programming, Part I: general nonlinear cases, Mathematical Programming, 70(1), 47-72, 1995.
[21] C.A. Floudas and S. Zlobec, Optimality and duality in parametric convex lexicographic programming, In Multilevel Optimization: Algorithms and Applications, pp. 359-379, Springer, 1998.
[22] Z.H. Gumus and C.A. Floudas, Global optimization of nonlinear bilevel programming problems, Journal of Global Optimization, 20(1), 1-31, 2001.
[23] K. Hatz, S. Leyffer, J.P. Schlöder and H.G. Bock, Regularizing bilevel nonlinear programs by lifting, Argonne National Laboratory, USA, 2013.
[24] J.M. Henderson and R.E. Quandt, Microeconomic theory: a mathematical approach, McGraw-Hill: New York, 1958.
[25] R. Henrion and T. Surowiec, On calmness conditions in convex bilevel programming, Applicable Analysis, 90(6), 951-970, 2011.

[26] F. Facchinei, H. Jiang and L. Qi, A smoothing method for mathematical programs with equilibrium constraints, Mathematical programming, 85(1), 107-134, 1999.

[27] Y. Ishizuka and E. Aiyoshi, Double penalty method for bilevel optimization problems, Annals of Operations Research, 34(1), 73-88, 1992.

[28] P.-M. Kleniati and C.S. Adjiman, Branch-and-sandwich: a deterministic global optimization algorithm for optimistic bilevel programming problems, Part II: Convergence analysis and numerical results, Journal of Global Optimization, 60(3), 459-481, 2014.

[29] L. Lampariello and S. Sagratella, Numerically tractable optimistic bilevel problems, Optimization-Online, 2017.

[30] L. Lampariello and S. Sagratella, A bridge between bilevel programs and nash games, Journal of Optimization Theory and Applications, 174(2), 613-635, 2017.

[31] R. Lucchetti, F. Mignanego, and G. Pieri, Existence theorems of equilibrium points in Stackelberg games with contraints, Optimization, 18(6), 857-866, 1987.

[32] Z.C. Lu, K. Deb, and A. Sinha, Robust and reliable solutions in bilevel optimization problems under uncertainties, COIN Report 2016026, Retrived on 19 November 2017 from http://www.egr.msu.edu/kdeb/papers/c2016026.pdf.

[33] S. Leyffer, MacMPEC: AMPL collection of MPECs, Argonne National Laboratory, 2000. Available at https://wiki.mcs.anl.gov/leyffer/index.php/MacMPEC.

[34] C.M. Macal and A.P.Hurter, Dependence of bilevel mathematical programs on irrelevant constraints, Computers & operations research, 24(12), 1129-1140, 1997.

[35] J.A. Mirrlees, The theory of moral hazard and unobservable behaviour: Part I, The Review of Economic Studies, 66(1), 3-21, 1999.

[36] A. Mitsos and P.I. Barton, A Test Set for Bilevel Programs, Technical report, MIT, 2006. Available at https://www.researchgate.net/publication/228455291_A_test_ set_for_bilevel_programs.

[37] J. Morgan and F. Patrone, Stackelberg problems: Subgame perfect equilibria via tikhonov regularization, Advances in dynamic games, 209-221, 2006.

[38] L.D. Muu and N.V. Quy, A global optimization method for solving convex quadratic bilevel programming problems, Journal of Global Optimization, 26(2), 199-219, 2003.

[39] J.W. Nie, L. Wang and J.J. Ye, Bilevel polynomial programs and semidefinite relaxation methods, SIAM Journal on Optimization, 27(3), 1728-1757, 2017.

[40] J.V. Outrata, On the numerical solution of a class of Stackelberg problems, Zeitschrift für Operations Research, 34(4), 255-277, 1990.

[41] J.V. Outrata, Necessary optimality conditions for Stackelberg problems, Journal of Optimization Theory and Applications, 76(2), 305-320, 1993.

[42] J.V. Outrata, On optimization problems with variational inequality constraints, SIAM Journal on optimization, 4(2), 340-357, 1994.

[43] J.V. Outrata and M. Cervinka, On the implicit programming approach in a class of mathematical programs with equilibrium constraints, Control and Cybernetics, 38(4B), 1557-1574, 2009.

[44] R. Paulavicius, P.M. Kleniati and C.S. Adjiman, BASBL: Branch-and-sandwich bilevel solver, Part II: Implementation and computational study with the BASBLib test sets, Technical report, 2017.

[45] K.H. Sahin and A.R. Ciric. A dual temperature simulated annealing approach for solving bilevel programming problems. Computers and Chemical Engineering, 23(1), 11-25, 1998.

[46] K. Shimizu and E. Aiyoshi, A new computational method for Stackelberg and min-max problems by use of a penalty method, IEEE Transactions on Automatic Control, 26(2), 460-466, 1981.

[47] K. Shimizu, Y. Ishizuka and J.F. Bard, Nondifferentiable and two-level mathematical programming, Dordrecht: Kluwer Academic Publishers, 1997.

[48] A. Sinha, P. Malo and K. Deb, An improved bilevel evolutionary algorithm based on quadratic approximations, In Evolutionary Computation (CEC), 2014 IEEE Congress, pp. 1870-1877, 2014.

[49] H. Tuy, A. Migdalas and N.T. Hoai-Phuong, A novel approach to bilevel nonlinear programming, Journal of Global Optimization, 38(4), 527-554, 2007.

[50] S. Vogel, Zwei-Ebenen-Optimierungsaufgaben mit nichtkonvexer Zielfunktion in der unteren Ebene, PhD thesis, Department of Mathematics and Computer Science, TU Bergakademie Freiberg, Freiberg, Germany, 2012.

[51] Z.P. Wan, G.M. Wang and Y.B. Lv, A dual-relax penalty function approach for solving nonlinear bilevel programming with linear lower level problem, Acta Mathematica Scientia, 31(2), 652-660, 2011.
[52] J.J. Ye and D.L. Zhu, New necessary optimality conditions for bilevel programs by combining the mpec and value function approaches, SIAM Journal on Optimization, 20(4), 1885-1905, 2010.
[53] A. Yezza, First-order necessary optimality conditions for general bilevel programming problems, Journal of Optimization Theory and Applications, 89(1), 189-219, 1996.
[54] A.B. Zemkoho, S.L. Zhou and A. Fischer, Detailed numerical experiments for the paper "Semismooth Newton-type method for bilevel optimization: Theory and extensive numerical experiments", Technical Report, School of Mathematics, University of Southampton, Southampton, UK (under preparation)
[55] A.B. Zemkoho, S.L. Zhou and A. Fischer, Semismooth Newton-type method for bilevel optimization: Theory and extensive numerical experiments, In preparation, 2018.
[56] S. Zlobec, Bilevel programming: optimality conditions and duality, pp. 180-185, Springer US, Boston, MA, 2001.

Appendix A. Formulas of the problems

Here we provide formulas of the functions F, G, f, and g involved in problem (1.2) for all the 124 examples presented in this paper, together with true or best known values of the solutions, and some useful background information in some cases.

Problem name: `AiyoshiShimizu1984Ex2`

Source: [1]

Description: AiyoshiShimizu1984Ex2 is defined as follows

$$\begin{aligned}
F(x,y) &:= 2x_1+2x_2-3y_1-3y_2-60\\
G(x,y) &:= \begin{pmatrix} x_1+x_2+y_1-2y_2-40\\ x-50_2\\ -x \end{pmatrix}\\
f(x,y) &:= (y_1-x_1+20)^2+(y_2-x_2+20)^2\\
g(x,y) &:= \begin{pmatrix} 2y-x+10_2\\ -y-10_2\\ y-20_2 \end{pmatrix}
\end{aligned}$$

Comment: The global optimal solution of the problem is $(25,30,5,10)$ according to [1]. A local optimal one is $(0,0,-10,-10)$ by [27].

Problem name: `AllendeStill2013`

Source: [2]

Description: AllendeStill2013 is defined as follows

$$\begin{aligned}
F(x,y) &:= -x_1^2-2x_1+x_2^2-2x_2+y_1^2+y_2^2\\
G(x,y) &:= \begin{pmatrix} -x\\ -y\\ x_1-2 \end{pmatrix}\\
f(x,y) &:= y_1^2-2x_1y_1+y_2^2-2x_2y_2\\
g(x,y) &:= \begin{pmatrix} (y_1-1)^2-0.25\\ (y_2-1)^2-0.25 \end{pmatrix}
\end{aligned}$$

Comment: The best known solution from [2] is $(0.5,0.5,0.5,0.5)$.

Problem name: AnEtal2009

Source: [3]

Description: AnEtal2009 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & \frac{1}{2}(x^\top, y^\top)H(x^\top, y^\top)^\top + c_1^\top x + c_2^\top y \\
G(x,y) & := & \begin{pmatrix} -x \\ -y \\ Ax + By + d \end{pmatrix} \\
f(x,y) & := & y^\top P x + \frac{1}{2} y^\top Q y + q^\top y \\
g(x,y) & := & Dx + Ey + b
\end{array}
$$

with H, c_1, c_2, A, B, d, P, Q, q, D, E, and b, respectively defined as follows

$$
H := \begin{bmatrix} -3.8 & 4.4 & 1.2 & -2.2 \\ 4.4 & -2.2 & 0.6 & 1.8 \\ 1.2 & 0.6 & 0.0 & 0.4 \\ -2.2 & 1.8 & 0.4 & 0.0 \end{bmatrix}, \quad c_1 := \begin{bmatrix} 935.74474 \\ 87.53654 \end{bmatrix}, \quad c_2 := \begin{bmatrix} 121.96196 \\ 299.24825 \end{bmatrix}
$$

$$
A := \begin{bmatrix} 0.00000 & 3.88889 \\ -2.00000 & 8.77778 \end{bmatrix}, \quad B := \begin{bmatrix} 4.88889 & 7.44444 \\ -5.11111 & 0.88889 \end{bmatrix}, \quad d := \begin{bmatrix} -61.57778 \\ -0.80000 \end{bmatrix}
$$

$$
P := \begin{bmatrix} -17.85000 & 6.57500 \\ 30.32500 & 30.32500 \end{bmatrix}, \quad Q := \begin{bmatrix} 21.10204 & 11.81633 \\ -5.11111 & -14.44898 \end{bmatrix}, \quad q := \begin{bmatrix} -18.21053 \\ 13.05263 \end{bmatrix}
$$

$$
D := \begin{bmatrix} 5.00000 & 7.44444 \\ -8.33333 & 3.00000 \\ -8.66667 & -8.55556 \\ 6.44444 & -5.11111 \end{bmatrix}, \quad E := \begin{bmatrix} 3.88889 & 1.77778 \\ 6.88889 & 6.11111 \\ -5.33333 & -7.00000 \\ 1.44444 & 4.44444 \end{bmatrix}, \quad b := \begin{bmatrix} -39.62222 \\ -60.00000 \\ 72.37778 \\ -17.28889 \end{bmatrix}
$$

Comment: $(0.200001, 1.999997, 3.999998, 4.600005)$ is the best known approximation of the solution of the problem; cf. [3].

Problem name: Bard1988Ex1

Source: [4]

Description: Bard1988Ex1 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & (x-5)^2 + (2y+1)^2 \\
G(x,y) & := & -x \\
f(x,y) & := & (y-1)^2 - 1.5xy \\
g(x,y) & := & \begin{pmatrix} -3x + y + 3 \\ x - 0.5y - 4 \\ x + y - 7 \\ -y \end{pmatrix}
\end{array}
$$

Comment: $(1, 0)$ is the global optimum and $(5, 2)$ is a local optimal point.

Problem name: Bard1988Ex2

Source: [4]

Description: Bard1988Ex2 is defined as follows

$$\begin{aligned}
F(x,y) &:= (200-y_1-y_3)(y_1+y_3)+(160-y_2-y_4)(y_2+y_4)\\
G(x,y) &:= \begin{pmatrix} x_1+x_2+x_3+x_4-40\\ x_1-10\\ x_2-5\\ x_3-15\\ x_4-20\\ -x \end{pmatrix}\\
f(x,y) &:= (y_1-4)^2+(y_2-13)^2+(y_3-35)^2+(y_4-2)^2\\
g(x,y) &:= \begin{pmatrix} 0.4y_1+0.7y_2-x_1\\ 0.6y_1+0.3y_2-x_2\\ 0.4y_3+0.7y_4-x_3\\ 0.6y_3+0.3y_4-x_4\\ y_1-20\\ y_2-20\\ y_3-40\\ y_4-40\\ -y \end{pmatrix}
\end{aligned}$$

Comment: This version of the problem is taken from [11]. The original one in [4] has two lower-level problem. The upper-and lower-level optimal value are respectively obtained as −6600.00 and 57.48 in the former paper. However, based on [54], the upper-and lower-level optimal value are respectively obtained as −6600.00 and 54.

Problem name: Bard1988Ex3

Source: [4]

Description: Bard1988Ex3 is defined as follows

$$\begin{aligned}
F(x,y) &:= -x_1^2-3x_2-4y_1+y_2^2\\
G(x,y) &:= \begin{pmatrix} x_1^2+2x_2-4\\ -x \end{pmatrix}\\
f(x,y) &:= 2x_1^2+y_1^2-5y_2\\
g(x,y) &:= \begin{pmatrix} -x_1^2+2x_1-x_2^2+2y_1-y_2-3\\ -x_2-3y_1+4y_2+4\\ -y \end{pmatrix}
\end{aligned}$$

Comment: The upper-and lower-level optimal value are respectively obtained as −12.68 and −1.02 in the paper [8].

Problem name: Bard1991Ex1

Source: [5]

Description: Bard1991Ex1 is defined as follows

$$\begin{aligned} F(x,y) &:= x+y_2 \\ G(x,y) &:= \begin{pmatrix} -x+2 \\ x-4 \end{pmatrix} \\ f(x,y) &:= 2y_1+xy_2 \\ g(x,y) &:= \begin{pmatrix} x-y_1-y_2+4 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(2,6,0)$; cf. [5].

Problem name: `BardBook1998`

Source: [6]

Description: BardBook1998 is defined as follows

$$\begin{aligned} F(x,y) &:= (y_1-x_1+20)^2+(y_2-x_2+20)^2 \\ G(x,y) &:= \begin{pmatrix} x-50_2 \\ -x \end{pmatrix} \\ f(x,y) &:= 2x_1+2x_2-3y_1-3y_2-60 \\ g(x,y) &:= \begin{pmatrix} x_1+x_2+y_1-2y_2-40 \\ 2y-x+10_2 \\ y-20_2 \\ -y-10_2 \end{pmatrix} \end{aligned}$$

Comment: The best known solution for the problem is $(25,30,5,10)$; cf. [54].

Problem name: `CalamaiVicente1994a`

Source: [7]

Description: CalamaiVicente1994a is defined as follows

$$\begin{aligned} F(x,y) &:= \tfrac{1}{2}(x-1)^2+\tfrac{1}{2}y^2 \\ f(x,y) &:= \tfrac{1}{2}y-xy \\ g(x,y) &:= \begin{pmatrix} x-y-1 \\ -x-y+1 \\ x+y-\rho \end{pmatrix} \end{aligned}$$

Comment: It is assumed in [7] that the parameter $\rho \geq 1$. We consider the following scenarios studied in the latter reference:

(i) For $\rho = 1$, the point $(1,0)$ is global optimum of the problem.
(ii) For $1 < \rho < 2$, the point $\frac{1}{2}(1+\rho, -1+\rho)$ is a global optimal solution, while $\frac{1}{2}(1,1)$ is a local optimal solution of the problem.
(iii) For $\rho = 2$, the points $\frac{1}{2}(1,1)$ and $\frac{1}{2}(3,1)$ are global optimal solution.
(iv) For $\rho > 2$, the point $\frac{1}{2}(1,1)$ is global optimum of the problem.

Problem name: `CalamaiVicente1994b`

Source: [7]

Description: CalamaiVicente1994b is defined as follows

$$
\begin{array}{lll}
F(x,y) & := & \frac{1}{2}\sum_{i=1}^{4}(x_i-1)^2+\sum_{i=1}^{2}y_i^2 \\
f(x,y) & := & \sum_{i=1}^{2}\left(\frac{1}{2}y_i^2-x_iy_i\right) \\
g(x,y) & := & \begin{pmatrix} x-y-1_2 \\ -x-y+1_2 \\ x_1+y_1-1.5 \\ x_1+y_2-3 \end{pmatrix}
\end{array}
$$

Comment: The point $(1.25, 0.67, 1.00, 1.00, 0.25, 0.33)$ is the best known optimal solution of the problem according to [54].

Problem name: `CalamaiVicente1994c`

Source: [7]

Description: CalamaiVicente1994c is defined as follows

$$
\begin{array}{lll}
F(x,y) & := & \frac{1}{2}x^\top Ax+\frac{1}{2}y^\top By+a^\top x+2 \\
f(x,y) & := & \frac{1}{2}y^\top By+x^\top Cy \\
g(x,y) & := & Dx+Ey+d
\end{array}
$$

with A, B, a, B and C respectively as follows

$$
A:=\begin{bmatrix} 197.2 & 32.4 & -129.6 & -43.2 \\ 32.4 & 110.8 & -43.2 & -14.4 \\ -129.6 & -43.2 & 302.8 & -32.4 \\ -43.2 & -14.4 & -32.4 & 289.2 \end{bmatrix},\ B:=\begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix},\ a:=\begin{bmatrix} -8.56 \\ -9.52 \\ -9.92 \\ -16.64 \end{bmatrix}
$$

$$
C:=\begin{bmatrix} -132.4 & -10.8 \\ -10.8 & -103.6 \\ 43.2 & 14.4 \\ 14.4 & 4.8 \end{bmatrix},\ D:=\begin{bmatrix} 13.24 & 1.08 & -4.32 & -1.44 \\ 1.08 & 10.36 & -1.44 & -0.48 \\ 13.24 & 1.08 & -4.32 & -1.44x_4 \\ 1.08 & 10.36 & -1.44 & -0.48 \\ -13.24 & -1.08 & +4.32 & +1.44 \\ -1.08 & -10.36 & +1.44 & +0.48 \end{bmatrix}
$$

$$
E:=\begin{bmatrix} -10 & 0 \\ 0 & -10 \\ 10 & 0 \\ 0 & 10 \\ -10 & 0 \\ 0 & -10 \end{bmatrix},\ d:=\begin{bmatrix} -1 \\ -1 \\ -1.5 \\ -3 \\ 1 \\ 1 \end{bmatrix}
$$

Comment: According to [7], the problem has a unique global optimal solution with the corresponding upper-level objective function value being 0.3125.

Problem name: `CalveteGale1999P1`

Source: [9]

Description: CalveteGale1999P1 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & -8x_1 - 4x_2 + y_1 - 40y_2 - 4y_3 \\
G(x,y) & := & -x \\
f(x,y) & := & \dfrac{1 + x_1 + x_2 + 2y_1 - y_2 + y_3}{6 + 2x_1 + y_1 + y_2 - 3y_3} \\
g(x,y) & := & \left[\begin{array}{c} -y \\ -y_1 + y_2 + y_3 - 1 \\ 2x_1 - y_1 + 2y_2 - \frac{1}{2}y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - \frac{1}{2}y_3 - 1 \end{array}\right]
\end{array}
$$

Comment: The best known value of the upper-level objective function is -29.2 and can be achieved at $(0.0, 0.9, 0.0, 0.6, 0.4)$, for example; cf. [22].

Problem name: `ClarkWesterberg1990a`

Source: [10]

Description: ClarkWesterberg1990a is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & (x-3)^2 + (y-2)^2 \\
G(x,y) & := & \begin{pmatrix} x-8 \\ -x \end{pmatrix} \\
f(x,y) & := & (y-5)^2 \\
g(x,y) & := & \begin{pmatrix} -2x + y - 1 \\ x - 2y - 2 \\ x + 2y - 14 \end{pmatrix}
\end{array}
$$

Comment: The best known optimal solution of the problem is $(1.0, 3.0)$; cf. [44].

Problem name: `Colson2002BIPA1`

Source: [11]

Description: Colson2002BIPA1 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & (10-x)^3 + (10-y)^3 \\
G(x,y) & := & \begin{pmatrix} x-5 \\ -x+y \\ -x \end{pmatrix} \\
f(x,y) & := & (x + 2y - 15)^4 \\
g(x,y) & := & \begin{pmatrix} x + y - 20 \\ y - 20 \\ -y \end{pmatrix}
\end{array}
$$

Comment: The best known solution is $(5, 5)$ obtained in [54].

Problem name: `Colson2002BIPA2`

Source: [11]

Description: Colson2002BIPA2 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^2+(2y+1)^2 \\ G(x,y) &:= -x \\ f(x,y) &:= (y-1)^2-1.5xy+x^3 \\ g(x,y) &:= \begin{pmatrix} -3x+y+3 \\ x-0.5y-4 \\ x+y-7 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The best known solution is $(1,0)$; cf. [8, 54].

Problem name: `Colson2002BIPA3`

Source: [11]

Description: Colson2002BIPA3 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-5)^4+(2y+1)^4 \\ G(x,y) &:= \begin{pmatrix} x+y-4 \\ -x \end{pmatrix} \\ f(x,y) &:= \exp(-x+y)+x^2+2xy+y^2+2x+6y \\ g(x,y) &:= \begin{pmatrix} -x+y-2 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The best known solution is $(4,0)$; cf. [8, 54].

Problem name: `Colson2002BIPA4`

Source: [11]

Description: Colson2002BIPA4 as defined as follows

$$\begin{aligned} F(x,y) &:= x^2+(y-10)^2 \\ G(x,y) &:= \begin{pmatrix} x+2y-6 \\ -x \end{pmatrix} \\ f(x,y) &:= x^3+2y^3+x-2y-x^2 \\ g(x,y) &:= \begin{pmatrix} -x+2y-3 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The best known solution is $(0,0.6039)$; cf. [8, 54].

Problem name: `Colson2002BIPA5`

Source: [11]

Description: Colson2002BIPA5 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-y_2)^4+(y_1-1)^2+(y_1-y_2)^2 \\ G(x,y) & := & -x \\ f(x,y) & := & 2x+\exp y_1+y_1^2+4y_1+2y_2^2-6y_2 \\ g(x,y) & := & \begin{pmatrix} 6x+y_1^2+\exp y_2-15 \\ 5x+y_1^4-y_2-25 \\ y_1-4 \\ y_2-2 \\ -y \end{pmatrix} \end{array}$$

Comment: The best known solution is $(1.94, 0, 1.21)$; cf. [8, 54].

Problem name: Dempe1992a

Source: [12]

Description: Dempe1992a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & y_2 \\ G(x,y) & := & x_1^2+(x_2+1)^2-1 \\ f(x,y) & := & \frac{1}{2}(y_1-1)^2+\frac{1}{2}y_2^2 \\ g(x,y) & := & \begin{pmatrix} y_1+y_2x_1+x_2 \\ y_1 \end{pmatrix} \end{array}$$

Comment: One possible solution reported in [54] is $(0, 0, 0, -0.5)$.

Problem name: Dempe1992b

Source: [12]

Description: Dempe1992b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-3.5)^2+(y+4)^2 \\ f(x,y) & := & (y-3)^2 \\ g(x,y) & := & y^2-x \end{array}$$

Comment: The upper-and lower-level optimal values are respectively obtained as 31.25 and 4.00 in the paper [8].

Problem name: DempeDutta2012Ex24

Source: [13]

Description: DempeDutta2012Ex24 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-1)^2+y^2 \\ f(x,y) & := & x^2y \\ g(x,y) & := & y^2 \end{array}$$

Comment: The global optimal solution of the problem is $(1, 0)$; cf. [13].

Problem name: DempeDutta2012Ex31

Source: [13]

Description: DempeDutta2012Ex31 is defined as follows

$$\begin{aligned}
F(x,y) &:= -y_2\\
G(x,y) &:= \begin{pmatrix} -x \\ y_1y_2 \\ -y_1y_2 \end{pmatrix}\\
f(x,y) &:= y_1^2+(y_2+1)^2\\
g(x,y) &:= \begin{pmatrix} (y_1-x_1)^2+(y_2-x_1-1)^2-1 \\ (y_1+x_2)^2+(y_2-x_2-1)^2-1 \end{pmatrix}
\end{aligned}$$

Comment: The point $(0.71, 0.71, 0, 1)$ is the best known solution of the problem provided in [13, 39].

Problem name: DempeEtal2012

Source: [14]

Description: DempeEtal2012 is defined as follows

$$\begin{aligned}
F(x,y) &:= x\\
G(x,y) &:= \begin{pmatrix} -1-x \\ x-1 \end{pmatrix}\\
f(x,y) &:= xy\\
g(x,y) &:= \begin{pmatrix} -y \\ y-1 \end{pmatrix}
\end{aligned}$$

Comment: The best known optimal solution is $(-1, 1)$; cf. [14].

Problem name: DempeFranke2011Ex41

Source: [15]

Description: DempeFranke2011Ex41 is defined as follows

$$\begin{aligned}
F(x,y) &:= x_1+y_1^2+y_2^2\\
G(x,y) &:= \begin{pmatrix} -1-x_1 \\ -1+x_1 \\ -1-x_2 \\ 1+x_2 \end{pmatrix}\\
f(x,y) &:= x^\top y\\
g(x,y) &:= \begin{pmatrix} -2y_1+y_2 \\ y_1-2 \\ -y_2 \\ y_2-2 \end{pmatrix}
\end{aligned}$$

Comment: The best known optimal solution is $(0, -1, 1, 2)$; cf. [15].

Problem name: DempeFranke2011Ex42

Source: [15]

Description: DempeFranke2011Ex42 is defined as follows

$$
\begin{aligned}
F(x,y) &:= x_1 + (y_1 - 1)^2 + y_2^2 \\
G(x,y) &:= \begin{pmatrix} -1 - x_1 \\ -1 + x_1 \\ -1 - x_2 \\ 1 + x_2 \end{pmatrix} \\
f(x,y) &:= x^\top y \\
g(x,y) &:= \begin{pmatrix} -y_1 + y_2 - 1 \\ y_1 + y_2 - 3.5 \\ y_2 - 2 \end{pmatrix}
\end{aligned}
$$

Comment: The best known optimal solution is $(1, -1, 0, 1)$; cf. [15].

Problem name: `DempeFranke2014Ex38`

Source: [16]

Description: DempeFranke2014Ex38 is defined as follows

$$
\begin{aligned}
F(x,y) &:= 2x_1 + x_2 + 2y_1 - y_2 \\
G(x,y) &:= \begin{pmatrix} -1 - x_1 \\ -1 + x_1 \\ -1 - x_2 \\ x_2 + 0.75 \end{pmatrix} \\
f(x,y) &:= x^\top y \\
g(x,y) &:= \begin{pmatrix} -2y_1 + y_2 \\ y_1 - 2 \\ -y_2 \\ y_2 - 2 \end{pmatrix}
\end{aligned}
$$

Comment: The best known optimal solution is $(-1, -1, 2, 2)$; cf. [16].

Problem name: `DempeLohse2011Ex31a`

Source: [17]

Description: DempeLohse2011Ex31a is defined as follows

$$
\begin{aligned}
F(x,y) &:= (x_1 - 0.5)^2 + (x_2 - 0.5)^2 - 3y_1 - 3y_2 \\
f(x,y) &:= x_1 y_1 + x_2 y_2 \\
g(x,y) &:= \begin{pmatrix} y_1 + y_2 - 2 \\ -y_1 + y_2 \\ -y \end{pmatrix}
\end{aligned}
$$

Comment: The point $(0.5, 0.5, 1, 1)$ is the unique global optimal solution of the problem according to [17].

Problem name: `DempeLohse2011Ex31b`

Source: [17]

Description: DempeLohse2011Ex31b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x_1-0.5)^2+(x_2-0.5)^2+x_3^2-3y_1-3y_2-6x_3 \\ f(x,y) & := & x_1y_1+x_2y_2+x_3y_3 \\ g(x,y) & := & \begin{pmatrix} y_1+y_2+y_3-2 \\ -y_1+y_2 \\ -y \end{pmatrix} \end{array}$$

Comment: The point $(0.5, 0.5, 0, 1, 1, 0)$ is a local optimal solution of the problem according to [17]. While based on [54], the solution is $(0.5, 0.5, 0, 0, 0, 2)$.

Problem name: `DeSilva1978`

Source: [18]

Description: DeSilva1978 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x_1^2-2x_1+x_2^2-2x_2+y_1^2+y_2^2 \\ f(x,y) & := & (y_1-x_1)^2+(y_2-x_2)^2 \\ g(x,y) & := & \begin{pmatrix} -y+(0.5)_2 \\ y-(1.5)_2 \end{pmatrix} \end{array}$$

Comment: The best known upper-and lower-level values obtained in [8] are -1.00 and 0.00, respectively.

Problem name: `EdmundsBard1991`

Source: [19]

Description: EdmundsBard1991 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & 2x_1+2x_2-3y_1-3y_2-60 \\ G(x,y) & := & \begin{pmatrix} -x \\ x-50_2 \end{pmatrix} \\ f(x,y) & := & (y_1-x_1+20)^2+(y_2-x_2+20)^2 \\ g(x,y) & := & \begin{pmatrix} x_1+x_2+y_1-2y_2-40 \\ 2y-x+10_2 \\ -10_2-y \\ y-20_2 \end{pmatrix} \end{array}$$

Comment: The best know solution of the problem is $(0, 0, -10, -10)$; [49].

Problem name: `FalkLiu1995`

Source: [20]

Description: FalkLiu1995 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x_1^2-3x_1+x_2^2-3x_2+y_1^2+y_2^2 \\ f(x,y) & := & (y_1-x_1)^2+(y_2-x_2)^2 \\ g(x,y) & := & \begin{pmatrix} -y+(0.5)_2 \\ y-(1.5)_2 \end{pmatrix} \end{array}$$

Comment: The point $(0.7537, 0.7537, 0.7463, 0.7463)$ is the best known solution for the problem according to [8, 54].

Problem name: `FloudasZlobec1998`

Source: [21]

Description: FloudasZlobec1998 is defined as follows

$$
\begin{aligned}
F(x,y) &:= x^3 y_1 + y_2 \\
G(x,y) &:= \begin{pmatrix} x-1 \\ -x \end{pmatrix} \\
f(x,y) &:= -y_2 \\
g(x,y) &:= \begin{pmatrix} -y_1 - 1 \\ y_1 - 1 \\ -y_2 \\ y_2 - 100 \\ xy_1 - 10 \\ y_1^2 + xy_2 - 1 \end{pmatrix}
\end{aligned}
$$

Comment: Notice that explicit bounds on the variable y were added. This is same as [36]. The global optimal solution is $(1, 0, 1)$ according to [22, 36].

Problem name: `GumusFloudas2001Ex1`

Source: [22]

Description: GumusFloudas2001Ex1 is defined as follows

$$
\begin{aligned}
F(x,y) &:= 16x^2 + 9y^2 \\
G(x,y) &:= \begin{pmatrix} -x \\ x - 12.5 \\ -4x + y \end{pmatrix} \\
f(x,y) &:= (x + y - 20)^4 \\
g(x,y) &:= \begin{pmatrix} -y \\ y - 50 \\ 4x + y - 50 \end{pmatrix}
\end{aligned}
$$

Comment: The optimal solution of the problem is $(11.25, 5)$; cf. [36].

Problem name: `GumusFloudas2001Ex3`

Source: [22]

Description: GumusFloudas2001Ex3 is defined as follows

$$\begin{aligned}
F(x,y) &:= -8x_1 - 4x_2 + y_1 - 40y_2 - 4y_3 \\
G(x,y) &:= \begin{bmatrix} -x \\ x - 2_2 \end{bmatrix} \\
f(x,y) &:= \frac{1 + x_1 + x_2 + 2y_1 - y_2 + y_3}{6 + 2x_1 + y_1 + y_2 - 3y_3} \\
g(x,y) &:= \begin{bmatrix} -y \\ y - 2_3 \\ -y_1 + y_2 + y_3 - 1 \\ 2x_1 - y_1 + 2y_2 - \frac{1}{2}y_3 - 1 \\ 2x_2 + 2y_1 - y_2 - \frac{1}{2}y_3 - 1 \end{bmatrix}
\end{aligned}$$

Comment: The optimal solution of the problem is $(0, 0.9, 0, 0.6, 0.4)$; cf. [36].

Problem name: `GumusFloudas2001Ex4`

Source: [22]

Description: GumusFloudas2001Ex4 is defined as follows

$$\begin{aligned}
F(x,y) &:= (x-3)^2 + (y-2)^2 \\
G(x,y) &:= \begin{pmatrix} -x \\ x - 8 \\ -2x + y - 1 \\ x - 2y + 2 \\ x + 2y - 14 \end{pmatrix} \\
f(x,y) &:= (y-5)^2 \\
g(x,y) &:= \begin{pmatrix} -y \\ y - 10 \end{pmatrix}
\end{aligned}$$

Comment: The optimal solution of the problem is $(3, 5)$; cf. [36].

Problem name: `GumusFloudas2001Ex5`

Source: [22]

Description: GumusFloudas2001Ex5 is defined as follows

$$\begin{aligned}
F(x,y) &:= x \\
G(x,y) &:= \begin{pmatrix} -x + 0.1 \\ x - 10 \end{pmatrix} \\
f(x,y) &:= -y_1 + 0.5864 y_1^{0.67} \\
g(x,y) &:= \begin{pmatrix} -y + (0.1)_2 \\ y - 10_2 \\ \frac{0.0332333}{y_2} + 0.1y_1 - 1 \\ 4\frac{x}{y_2} + 2\frac{x^{-0.71}}{y_2} + 0.0332333x^{-1.3} - 1 \end{pmatrix}
\end{aligned}$$

Comment: The optimal solution is $(0.193616, 9.9667667, 10)$; cf. [36].

Problem name: `HatzEtal2013`

Source: [23]

Description: HatzEtal2013 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -x+2y_1+y_2 \\ f(x,y) & := & (x-y_1)^2+y_2^2 \\ g(x,y) & := & -y \end{array}$$

Comment: The optimal solution of the problem is $(0,0,0)$; cf. [23].

Problem name: `HendersonQuandt1958`

Source: [24]

Description: HendersonQuandt1958 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}x^2+\frac{1}{2}xy-95x \\ G(x,y) & := & \begin{pmatrix} x-200 \\ -x \end{pmatrix} \\ f(x,y) & := & y^2+(\frac{1}{2}x-100)y \\ g(x,y) & := & -y \end{array}$$

Comment: The best solution from [26] is $(93.33333, 26.667)$.

Problem name: `HenrionSurowiec2011`

Source: [25]

Description: HenrionSurowiec2011 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+cy \\ f(x,y) & := & 0.5y^2-xy \end{array}$$

Comment: Here, c is a real-valued parameter. The optimal solution of the problem is $-0.5c(1,1)$; cf. [25].

Problem name: `IshizukaAiyoshi1992a`

Source: [27]

Description: IshizukaAiyoshi1992a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & xy_2^2 \\ G(x,y) & := & -x-M \\ f(x,y) & := & y_1 \\ g(x,y) & := & \begin{pmatrix} -x \\ -x-y_1 \\ y_1-x \\ -M-y_1-y_2 \\ y_1+y_2-M \end{pmatrix} \end{array}$$

Comment: Here, M is assumed to be an arbitrarily large number such that $M > 1$. From [27], $(x^*, -M, 0)$ is the optimal solution where $x^* \in [0, M]$.

Problem name: `KleniatiAdjiman2014Ex3`

Source: [28]

Description: KleniatiAdjiman2014Ex3 is defined as follows

$$
\begin{aligned}
F(x,y) &:= x-y \\
G(x,y) &:= \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\
f(x,y) &:= 0.5xy^2 - xy^3 \\
g(x,y) &:= \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix}
\end{aligned}
$$

Comment: $(0,1)$ is the optimal solution of the problem according to [28, 44].

Problem name: `KleniatiAdjiman2014Ex4`

Source: [28]

Description: KleniatiAdjiman2014Ex4 is defined as follows

$$
\begin{aligned}
F(x,y) &:= \sum_{j=1}^{5} -\left(x_j^2 + y_j^2\right) \\
G(x,y) &:= \begin{pmatrix} y_1y_2 - x_1 \\ x_2y_1^2 \\ x_1 - \exp x_2 + y_3 \\ -x - 1_5 \\ x - 1_5 \end{pmatrix} \\
f(x,y) &:= y_1^3 + y_2^2x_1 + y_2^2x_2 + 0.1y_3 + (y_4^2 + y_5^2)x_3x_4x_5 \\
g(x,y) &:= \begin{pmatrix} x_1 - y_3^2 - 0.2 \\ -y - 1_5 \\ y - 1_5 \end{pmatrix}
\end{aligned}
$$

Comment: $(1.0, -(1.0)_9)$ is a solution of the problem according to [28, 44].

Problem name: `LamparielloSagratella2017Ex23`

Source: [29]

Description: LamparielloSagratella2017Ex23 is defined as follows

$$
\begin{aligned}
F(x,y) &:= x \\
G(x,y) &:= \begin{bmatrix} -x-1 \\ x-1 \end{bmatrix} \\
f(x,y) &:= (x-y_1)^2 + (y_2+1)^2 \\
g(x,y) &:= \begin{bmatrix} y_1^3 - y_2 \\ -y_2 \end{bmatrix}
\end{aligned}
$$

Comment: The best known optimal solution is $(-1,-1,0)$; cf. [29].

Problem name: `LamparielloSagratella2017Ex31`

Source: [30]

Description: LamparielloSagratella2017Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ G(x,y) & := & -x+1 \\ f(x,y) & := & y \\ g(x,y) & := & -x-y+1 \end{array}$$

Comment: The best known optimal solution is $(1,0)$; cf. [30].

Problem name: `LamparielloSagratella2017Ex32`

Source: [30]

Description: LamparielloSagratella2017Ex32 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ f(x,y) & := & (x+y-1)^2 \end{array}$$

Comment: The best known optimal solution is $(0.5,0.5)$; cf. [30].

Problem name: `LamparielloSagratella2017Ex33`

Source: [30]

Description: LamparielloSagratella2017Ex33 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+(y_1+y_2)^2 \\ G(x,y) & := & -x+0.5 \\ f(x,y) & := & y_1 \\ g(x,y) & := & \begin{pmatrix} -x-y_1-y_2+1 \\ -y \end{pmatrix} \end{array}$$

Comment: The best known optimal solution is $(0.5,0,0.5)$; cf. [30].

Problem name: `LamparielloSagratella2017Ex35`

Source: [30]

Description: LamparielloSagratella2017Ex35 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ G(x,y) & := & \begin{pmatrix} -1-x \\ x-1 \end{pmatrix} \\ f(x,y) & := & -y \\ g(x,y) & := & \begin{pmatrix} 2x+y-2 \\ -y \\ y-1 \end{pmatrix} \end{array}$$

Comment: The best known optimal solution is $(\frac{4}{5},\frac{2}{5})$; cf. [30].

Problem name: `LucchettiEtal1987`

Source: [31]

Description: LucchettiEtal1987 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & 0.5(1-x)+xy \\ G(x,y) & := & \begin{pmatrix} -x \\ x-1 \end{pmatrix} \\ f(x,y) & := & (x-1)y \\ g(x,y) & := & \begin{pmatrix} -y \\ y-1 \end{pmatrix} \end{array}$$

Comment: The optimal solution of the problem is $(1,0)$; cf. [31].

Problem name: `LuDebSinha2016a`

Source: [32]

Description: LuDebSinha2016a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & 2-\exp\left[-\left(\frac{0.2y-x+0.6}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\frac{0.15y-0.4+x}{0.3}\right)^{2}\right] \\ G(x,y) & := & \begin{pmatrix} -x \\ x-1 \\ -y \\ y-2 \end{pmatrix} \\ f(x,y) & := & 2-\exp\left[-\left(\frac{1.5y-x}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\frac{2y-3+x}{0.5}\right)^{2}\right] \end{array}$$

Comment: $(1.4,0.2)$ is a possible solution for the problem; [32]. Clearly, the $x^* = 1.4$ violates the constraints. One possible solution derived by [54] is $(0,1.5)$.

Problem name: `LuDebSinha2016b`

Source: [32]

Description: LuDebSinha2016b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-0.5)^2+(y-1)^2 \\ G(x,y) & := & \begin{pmatrix} -x \\ x-1 \\ -y \\ y-2 \end{pmatrix} \\ f(x,y) & := & 2-\exp\left[-\left(\frac{1.5y-x}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\frac{2y-3+x}{0.5}\right)^{2}\right] \end{array}$$

Comment: $(0.5,1)$ is a possible solution for the problem; [32].

Problem name: `LuDebSinha2016c`

Source: [32]

Description: LuDebSinha2016c is defined as follows

$$\begin{aligned} F(x,y) &:= 2-\exp\left[-\left(\tfrac{0.2y-x+0.6}{0.055}\right)^{0.4}\right]-0.8\exp\left[-\left(\tfrac{0.15y-0.4+x}{0.3}\right)^{2}\right]\\ G(x,y) &:= \begin{pmatrix} -x\\ x-1\\ -y\\ y-2 \end{pmatrix}\\ f(x,y) &:= (x-0.5)^2+(y-1)^2 \end{aligned}$$

Comment: $(0.26, 1)$ is a possible solution for the problem; [32].

Problem name: `LuDebSinha2016d`
Source: [32]
Description: LuDebSinha2016d is defined as follows

$$\begin{aligned} F(x,y) &:= -x_2\\ g(x,y) &:= \begin{pmatrix} -\left(\frac{y_1}{14}+\frac{16}{7}\right)(x_1-2)^2+x_2\\ -x_2+\left(\frac{y_1}{14}+\frac{16}{7}\right)(x_1-5)\\ -\left[x_1+4-\left(\frac{y_1}{14}+\frac{16}{7}\right)\right]\left[x_1+8-\left(\frac{y_1}{14}+\frac{16}{7}\right)\right]+x_2\\ -4-x_1\\ x_1-10\\ -100-x_2\\ x_2-200\\ -4-y_1\\ y_1-10\\ -100-y_2\\ y_2-200 \end{pmatrix}\\ f(x,y) &:= -y_2\\ g(x,y) &:= \begin{pmatrix} -\left(\frac{x_1}{14}+\frac{16}{7}\right)(y_1-2)^2+y_2\\ -y_2+12.5\left(\frac{x_1}{14}+\frac{16}{7}\right)(y_1-5)\\ -5\left[y_1+4-\left(\frac{x_1}{14}+\frac{16}{7}\right)\right]\left[y_1+8-\left(\frac{x_1}{14}+\frac{16}{7}\right)\right]+y_2 \end{pmatrix} \end{aligned}$$

Comment: The best known solution of this problem is $(10, 192, 10, 192)$ according to [54].

Problem name: `LuDebSinha2016e`
Source: [32]

Description: LuDebSinha2016e is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & \left(\frac{y_2-50}{30}\right)^2 + \left(\frac{x-2.5}{0.2}\right)^2 \\
G(x,y) & := & \begin{pmatrix} 2-x \\ x-3 \\ -4-y_1 \\ y_1-10 \\ -100-y_2 \\ y_2-200 \end{pmatrix} \\
f(x,y) & := & -y_2 \\
g(x,y) & := & \begin{pmatrix} -x(y_1-2)^2+y_2 \\ -y_2+12.5x(y_1-5) \\ -5(y_1+4-x)(y_1+8-x)+y_2 \end{pmatrix}
\end{array}
$$

Comment: The best known solution of this problem is $(2.49, -0.74, 18.6)$ according to [54].

Problem name: `LuDebSinha2016f`

Source: [32]

Description: LuDebSinha2016f is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & -x_2 \\
G(x,y) & := & \begin{pmatrix} 2-y \\ y-4 \\ -80-x_1 \\ x_1-200 \\ -100-x_2 \\ x_2-200 \\ -y\left(\frac{x_1}{20}-2\right)+x_2 \\ -x_2+12.5y\left(\frac{x_1}{20}-5\right) \\ -5\left(\frac{x_1}{20}+4-y\right)\left(\frac{x_1}{20}+8-y\right)+x_2 \end{pmatrix} \\
f(x,y) & := & \left(\frac{x_1-50}{28}\right)^2 + \left(\frac{y-2.5}{0.2}\right)^2
\end{array}
$$

Comment: The best known solution of this problem is $(40, 0, 2.5)$ according to [54].

Problem name: `MacalHurter1997`

Source: [34]

Description: MacalHurter1997 is defined as follows

$$
\begin{array}{rl}
F(x,y) & := (x-1)^2 + (y-1)^2 \\
f(x,y) & := 0.5y^2 + 500y - 50xy
\end{array}
$$

Comment: The optimal solution is $(10.0163, 0.8197)$; cf. [34].

Problem name: `Mirrlees1999`

Source: [35]
Description: Mirrlees1999 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & (x-2)^2+(y-1)^2 \\
f(x,y) & := & -x\exp[-(y+1)^2]-\exp[-(y-1)^2]
\end{array}
$$

Comment: This problem is known in the literature as Mirrlees problem. It is usually used to illustrate how the KKT reformulation of the bilevel optimization problem is not appropriate for problems with nonconvex lower-level problems. The best known optimal solution for the problem is $(1, 0.95753)$; [35].

Problem name: `MitsosBarton2006Ex38`
Source: [36]
Description: MitsosBarton2006Ex38 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & y^2 \\
G(x,y) & := & \begin{pmatrix} -x-1 \\ x-1 \\ -y-0.1 \\ y-0.1 \end{pmatrix} \\
f(x,y) & := & (x+\exp x)y \\
g(x,y) & := & \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix}
\end{array}
$$

Comment: The optimal solution of the problem is $(-0.567, 0)$; cf. [36].

Problem name: `MitsosBarton2006Ex39`
Source: [36]
Description: MitsosBarton2006Ex39 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & x \\
G(x,y) & := & \begin{pmatrix} -x+y \\ -x+10 \\ x-10 \end{pmatrix} \\
f(x,y) & := & y^3 \\
g(x,y) & := & \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix}
\end{array}
$$

Comment: The optimal solution of the problem is $(-1, -1)$; cf. [36].

Problem name: `MitsosBarton2006Ex310`
Source: [36]

Description: MitsosBarton2006Ex310 is defined as follows

$$\begin{aligned} F(x,y) &:= y \\ G(x,y) &:= \begin{pmatrix} -x+0.1 \\ x-1 \end{pmatrix} \\ f(x,y) &:= x(16y^4+2y^3-8y^2-1.5y+0.5) \\ g(x,y) &:= \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix} \end{aligned}$$

Comment: The set of all optimal solution is $[0.1, 1]\times\{0.5\}$; cf. [36].

Problem name: `MitsosBarton2006Ex311`

Source: [36]

Description: MitsosBarton2006Ex311 is defined as follows

$$\begin{aligned} F(x,y) &:= y \\ G(x,y) &:= \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\ f(x,y) &:= x(16y^4+2y^3-8y^2-1.5y+0.5) \\ g(x,y) &:= \begin{pmatrix} -y-0.8 \\ y-1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0,-0.8)$; cf. [36].

Problem name: `MitsosBarton2006Ex312`

Source: [36]

Description: MitsosBarton2006Ex312 is defined as follows

$$\begin{aligned} F(x,y) &:= -x+xy+10y^2 \\ G(x,y) &:= \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\ f(x,y) &:= -xy^2+0.5y^4 \\ g(x,y) &:= \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0,0)$; cf. [36].

Problem name: `MitsosBarton2006Ex313`

Source: [36]

Description: MitsosBarton2006Ex313 is defined as follows

$$\begin{aligned} F(x,y) &:= x-y \\ G(x,y) &:= \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\ f(x,y) &:= 0.5xy^2-x^3y \\ g(x,y) &:= \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0, 1)$; cf. [36].

Problem name: `MitsosBarton2006Ex314`

Source: [36]

Description: MitsosBarton2006Ex314 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-0.25)^2+y^2 \\ G(x,y) &:= \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\ f(x,y) &:= \tfrac{1}{3}y^3-xy \\ g(x,y) &:= \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0.25, 0.5)$; cf. [36].

Problem name: `MitsosBarton2006Ex315`

Source: [36]

Description: MitsosBarton2006Ex315 is defined as follows

$$\begin{aligned} F(x,y) &:= x+y \\ G(x,y) &:= \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\ f(x,y) &:= \tfrac{1}{2}xy^2-\tfrac{1}{3}y^3 \\ g(x,y) &:= \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(-1, 1)$; cf. [36].

Problem name: `MitsosBarton2006Ex316`

Source: [36]

Description: MitsosBarton2006Ex316 is defined as follows

$$\begin{aligned} F(x,y) &:= 2x+y \\ G(x,y) &:= \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\ f(x,y) &:= -\tfrac{1}{2}xy^2-\tfrac{1}{4}y^4 \\ g(x,y) &:= \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix} \end{aligned}$$

Comment: The points $(-1, 0)$ and $(-0.5, -1)$ are the two optimal solutions of the problem; cf. [36].

Problem name: `MitsosBarton2006Ex317`

Source: [36]

Description: MitsosBarton2006Ex317 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x+\frac{1}{2})^2+\frac{1}{2}y^2 \\ G(x,y) & := & \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\ f(x,y) & := & \frac{1}{2}xy^2+\frac{1}{4}y^4 \\ g(x,y) & := & \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix} \end{array}$$

Comment: The points $(-0.25, 0.5)$ and $(-0.25, -0.5)$ are the two optimal solutions of the problem; cf. [36].

Problem name: `MitsosBarton2006Ex318`

Source: [36]

Description: MitsosBarton2006Ex318 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -x^2+y^2 \\ G(x,y) & := & \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\ f(x,y) & := & xy^2-\frac{1}{2}y^4 \\ g(x,y) & := & \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix} \end{array}$$

Comment: The optimal solution of the problem is $(0.5, 0)$; cf. [36].

Problem name: `MitsosBarton06Ex319`

Source: [36]

Description: MitsosBarton2006Ex319 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & xy-y+\frac{1}{2}y^2 \\ G(x,y) & := & \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\ f(x,y) & := & -xy^2+\frac{1}{2}y^4 \\ g(x,y) & := & \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix} \end{array}$$

Comment: The optimal solution of the problem is $(0.189, 0.4343)$; cf. [36].

Problem name: `MitsosBarton2006Ex320`

Source: [36]

Description: MitsosBarton2006Ex320 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & (x-\frac{1}{4})^2+y^2 \\
G(x,y) & := & \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\
f(x,y) & := & \frac{1}{3}y^3-x^2y \\
g(x,y) & := & \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix}
\end{array}
$$

Comment: The optimal solution of the problem is $(0.5, 0.5)$; cf. [36].

Problem name: `MitsosBarton2006Ex321`

Source: [36]

Description: MitsosBarton2006Ex321 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & (x+0.6)^2+y^2 \\
G(x,y) & := & \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\
f(x,y) & := & y^4+\frac{4}{30}(-x+1)y^3+(-0.02x^2+0.16x-0.4)y^2 \\
 & & +(0.004x^3-0.036x^2+0.08x)y \\
g(x,y) & := & \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix}
\end{array}
$$

Comment: The optimal solution of the problem is $(-0.5545, 0.4554)$; cf. [36].

Problem name: `MitsosBarton2006Ex322`

Source: [36]

Description: MitsosBarton2006Ex322 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & (x+0.6)^2+y^2 \\
G(x,y) & := & \begin{pmatrix} -x-1 \\ x-1 \end{pmatrix} \\
f(x,y) & := & y^4+\frac{2}{15}(-x+1)y^3+(-0.02x^2+0.16x-0.4)y^2 \\
 & & +(0.004x^3-0.036x^2+0.08x)y \\
g(x,y) & := & \begin{pmatrix} -y-1 \\ y-1 \\ 0.01(1+x)^2-y^2 \end{pmatrix}
\end{array}
$$

Comment: The optimal solution of the problem is $(-0.5545, 0.4554)$; cf. [36].

Problem name: `MitsosBarton2006Ex323`

Source: [36]

Description: MitsosBarton2006Ex323 is defined as follows

$$\begin{aligned} F(x,y) &:= x^2 \\ G(x,y) &:= \begin{pmatrix} -x-1 \\ x-1 \\ 1+x-9x^2-y \end{pmatrix} \\ f(x,y) &:= y \\ g(x,y) &:= \begin{pmatrix} -y-1 \\ y-1 \\ y^2(x-0.5) \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(-0.4191, -1)$; cf. [36].

Problem name: `MitsosBarton2006Ex324`

Source: [36]

Description: MitsosBarton2006Ex324 is defined as follows

$$\begin{aligned} F(x,y) &:= x^2-y \\ G(x,y) &:= \begin{pmatrix} -x \\ x-1 \end{pmatrix} \\ f(x,y) &:= [(y-1-0.1x)^2-0.5-0.5x]^2 \\ g(x,y) &:= \begin{pmatrix} -y \\ y-3 \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(0.2106, 1.799)$; cf. [36].

Problem name: `MitsosBarton2006Ex325`

Source: [36]

Description: MitsosBarton2006Ex325 is defined as follows

$$\begin{aligned} F(x,y) &:= x_1y_1+x_2y_1^2-x_1x_2y_3 \\ G(x,y) &:= \begin{pmatrix} -x-1_2 \\ x-1_2 \\ 0.1y_1y_2-x_1^2 \\ x_2y_1^2 \end{pmatrix} \\ f(x,y) &:= x_1y_1^2+x_2y_2y_3 \\ g(x,y) &:= \begin{pmatrix} -y-1_3 \\ y-1_3 \\ y_1^2-y_2y_3 \\ y_2^2y_3-y_1x_1 \\ -y_3^2+0.1 \end{pmatrix} \end{aligned}$$

Comment: The best known optimal value for the upper-level objective function is -1 and a corresponding optimal point is $(-1, -1, -1, 1, 1)$; cf. [36].

Problem name: `MitsosBarton2006Ex326`

Source: [36]

Description: MitsosBarton2006Ex326 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x_1y_1 + x_2y_2^2 + x_1x_2y_3^3 \\ G(x,y) & := & \begin{pmatrix} 0.1 - x_1^2 \\ 1.5 - y_1^2 - y_2^2 - y_3^2 \\ 2.5 + y_1^2 + y_2^2 + y_3^2 \\ -x - 1_2 \\ x - 1_2 \end{pmatrix} \\ f(x,y) & := & x_1y_1^2 + x_2y_2^2 + (x_1 - x_2)y_3^2 \\ g(x,y) & := & \begin{pmatrix} -y - 1_3 \\ y - 1_3 \end{pmatrix} \end{array}$$

Comment: The optimal solution of the problem is $(-1, -1, 1, 1, -0.707)$; cf. [36].

Problem name: `MitsosBarton2006Ex327`

Source: [36]

Description: MitsosBarton2006Ex327 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \sum_{j=1}^{5}(x_j^2 + y_j^2) \\ G(x,y) & := & \begin{pmatrix} -x - 1_5 \\ x - 1_5 \\ y_1y_2 - x_1 \\ x_2y_1^2 \\ x_1 - \exp x_2 + y_3 \end{pmatrix} \\ f(x,y) & := & y_1^3 + y_2^2x_1 + y_2^2x_2 + 0.1y_3 + (y_4^2 + y_5^2)x_3x_4x_5 \\ g(x,y) & := & \begin{pmatrix} -y - 1_5 \\ y - 1_5 \\ y_1y_2 - 0.3 \\ x_1 - y_3^2 - 0.2 \\ -\exp y_3 + y_4y_5 - 0.1 \end{pmatrix} \end{array}$$

Comment: The best known optimal value for the upper-level objective function is 2 and a corresponding optimal point is[1] $(0_5, -1, 0, -1, 0, 0)$; cf. [36].

Problem name: `MitsosBarton2006Ex328`

Source: [36]

[1] 0_5 is used to indicate the origin of $\mathbb{R}^5$

Description: MitsosBarton2006Ex328 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & -\sum_{j=1}^{5}(x_j^2+y_j^2) \\
G(x,y) & := & \begin{pmatrix} -x-1_5 \\ x-1_5 \\ y_1y_2-x_1 \\ x_2y_1^2 \\ x_1-\exp x_2+y_3 \end{pmatrix} \\
f(x,y) & := & y_1^3+y_2^2x_1+y_2^2x_2+0.1y_3+(y_4^2+y_5^2)x_3x_4x_5 \\
g(x,y) & := & \begin{pmatrix} -y-1_5 \\ y-1_5 \\ y_1y_2-0.3 \\ x_1-y_3^2-0.2 \\ -\exp y_3+y_4y_5-0.1 \end{pmatrix}
\end{array}
$$

Comment: The best known optimal values for the upper and lower-level objective functions are -10 and -3.1 respectively, and a corresponding optimal point is $(1,(-1)_5,1,-1,-1,1)$; cf. [36]. Another possible solution gotten in [54] is $((-1)_5,1,-1,-1,-1,1)$ with the same the upper and lower-level objective function values.

Problem name: `MorganPatrone2006a`

Source: [37]

Description: MorganPatrone2006a is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & -(x+y) \\
G(x,y) & := & \begin{pmatrix} -0.5-x \\ x-0.5 \end{pmatrix} \\
f(x,y) & := & xy \\
g(x,y) & := & \begin{pmatrix} -1-y \\ y-1 \end{pmatrix}
\end{array}
$$

Comment: The best known optimal solution is $(0,1)$; cf. [37].

Problem name: `MorganPatrone2006b`

Source: [37]

Description: MorganPatrone2006b is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & -(x+y) \\
f(x,y) & := & \begin{cases} (x+0.25)y & \text{if } x\in[-0.5,-0.25] \\ 0 & \text{if } x\in[-0.25,0.25] \\ (x-0.25)y & \text{if } x\in[0.25,0.5] \end{cases} \\
g(x,y) & := & \begin{pmatrix} -0.5-x \\ x-0.5 \\ -1-y \\ y-1 \end{pmatrix}
\end{array}
$$

Comment: The best known optimal solution is $(0.25, 1)$; cf. [37].

Problem name: `MorganPatrone2006c`

Source: [37]

Description: MorganPatrone2006c is defined as follows

$$
\begin{aligned}
F(x,y) &:= -(x+y) \\
f(x,y) &:= \begin{cases} \left(x+-\frac{7}{4}\right)y & \text{if } x \in \left[-2, -\frac{7}{4}\right] \\ 0 & \text{if } x \in \left[-\frac{7}{4}, \frac{7}{4}\right] \\ \left(x--\frac{7}{4}\right)y & \text{if } x \in \left[\frac{7}{4}, 2\right] \end{cases} \\
g(x,y) &:= \begin{pmatrix} -2-x \\ x-2 \\ -1-y \\ y-1 \end{pmatrix}
\end{aligned}
$$

Comment: The best known optimal solution is $(2, -1)$; cf. [37].

Problem name: `MuuQuy2003Ex1`

Source: [38]

Description: MuuQuy2003Ex1 is defined as follows

$$
\begin{aligned}
F(x,y) &:= y_1^2 + y_2^2 + x^2 - 4x \\
G(x,y) &:= \begin{pmatrix} -x \\ x-2 \end{pmatrix} \\
f(x,y) &:= y_1^2 + \tfrac{1}{2}y_2^2 + y_1y_2 + (1-3x)y_1 + (1+x)y_2 \\
g(x,y) &:= \begin{pmatrix} 2y_1 + y_2 - 2x - 1 \\ -y \end{pmatrix}
\end{aligned}
$$

Comment: The best known solution is $(0.8438, 0.7657, 0)$ according to [38].

Problem name: `MuuQuy2003Ex2`

Source: [38]

Description: MuuQuy2003Ex2 is defined as follows

$$
\begin{aligned}
F(x,y) &:= y_1^2 + y_3^2 - y_1y_3 - 4y_2 - 7x_1 + 4x_2 \\
G(x,y) &:= \begin{pmatrix} -x \\ x_1 + x_2 - 1 \end{pmatrix} \\
f(x,y) &:= y_1^2 + \tfrac{1}{2}y_2^2 + \tfrac{1}{2}y_3^2 + y_1y_2 + (1-3x_1)y_1 + (1+x_2)y_2 \\
g(x,y) &:= \begin{pmatrix} 2y_1 + y_2 - y_3 + x_1 - 2x_2 + 2 \\ -y \end{pmatrix}
\end{aligned}
$$

Comment: The best known solution is $(0.609, 0.391, 0.000, 0.000, 1.828)$; cf. [38].

Problem name: `NieEtal2017Ex34`

Source: [39]

Description: NieEtal2017Ex34 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x+y_1+y_2 \\ G(x,y) & := & \begin{pmatrix} -x+2 \\ -3+x \end{pmatrix} \\ f(x,y) & := & x(y_1+y_2) \\ g(x,y) & := & \begin{pmatrix} -y_1^2+y_2^2+(y_1^2+y_2^2)^2 \\ -y_1 \end{pmatrix} \end{array}$$

Comment: The global optimal solution of the problem is $(2,0,0)$; cf. [39].

Problem name: `NieEtal2017Ex52`

Source: [39]

Description: NieEtal2017Ex52 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x_1y_1+x_2y_2+x_1x_2y_1y_2y_3 \\ G(x,y) & := & \begin{pmatrix} -1_2-x \\ x-1_2 \\ y_1y_2-x_1^2 \end{pmatrix} \\ f(x,y) & := & x_1y_1^2+x_2^2y_2y_3-y_1y_3^2 \\ g(x,y) & := & \begin{pmatrix} 1-y_1^2-y_2^2-y_3^2 \\ y_1^2+y_2^2+y_3^2-2 \end{pmatrix} \end{array}$$

Comment: The point $(-1,-1,1.1097,0.3143,-0.8184)$ is the best known solution of the problem provided in [39].

Problem name: `NieEtal2017Ex54`

Source: [39]

Description: NieEtal2017Ex54 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x_1^2y_1+x_2y_2+x_3y_3^2+x_4y_4^2 \\ G(x,y) & := & \begin{pmatrix} \|x\|^2-1 \\ y_1y_2-x_1 \\ y_3y_4-x_3^2 \end{pmatrix} \\ f(x,y) & := & y_1^2-y_2(x_1+x_2)-(y_3+y_4)(x_3+x_4) \\ g(x,y) & := & \begin{pmatrix} \|y\|^2-1 \\ y_2^2+y_3^2+y_4^2-y_1 \end{pmatrix} \end{array}$$

Comment: $(0,-0,-0.7071,-0.7071,0.6180,0,-0.5559,-0.5559)$ is the best known solution of the problem obtained in [39].

Problem name: `NieEtal2017Ex57`

Source: [39]

Description: NieEtal2017Ex57 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}x_1^2y_1 + x_2y_2^2 - (x_1 + x_2^2)y_3 \\ G(x,y) & := & \begin{pmatrix} -1_2 - x \\ x - 1_2 \\ -x_1 - x_2 + x_1^2 + y_1^2 + y_2^2 \end{pmatrix} \\ f(x,y) & := & x_2(y_1y_2y_3 + y_2^2 - y_3^3) \\ g(x,y) & := & \begin{pmatrix} -x_1 + y_1^2 + y_2^2 + y_3^2 \\ -1 + 2y_2y_3 \end{pmatrix} \end{array}$$

Comment: The point $(1,1,0,0,1)$ is the best known solution of the problem provided in [39].

Problem name: `NieEtal2017Ex58`

Source: [39]

Description: NieEtal2017Ex58 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x_1 + x_2 + x_3 + x_4)(y_1 + y_2 + y_3 + y_4) \\ G(x,y) & := & \begin{pmatrix} \|x\|^2 - 1 \\ y_3^2 - x_4 \\ y_2y_4 - x_1 \end{pmatrix} \\ f(x,y) & := & x_1y_1 + x_2y_2 + 0.1y_3 + 0.5y_4 - y_3y_4 \\ g(x,y) & := & \begin{pmatrix} y_1^2 + 2y_2^2 + 3y_3^2 + 4y_4^2 - x_1^2 - x_3^2 - x_2 - x_4 \\ -y_2y_3 + y_1y_4 \end{pmatrix} \end{array}$$

Comment: $(0.5135, 0.5050, 0.4882, 0.4929, -0.8346, -0.4104, -0.2106, -0.2887)$ is the best known solution of the problem obtained in [39].

Problem name: `NieEtal2017Ex61`

Source: [39]

Description: NieEtal2017Ex61 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & y_1^3(x_1^2 - 3x_1x_2) - y_1^2y_2 + y_2x_2^3 \\ G(x,y) & := & \begin{pmatrix} -x - 1_2 \\ x - 1_2 \\ -y_2 - y_1(1 - x_1^2) \end{pmatrix} \\ f(x,y) & := & y_1y_2^2 - y_2^3 - y_1^2(x_2 - x_1^2) \\ g(x,y) & := & y_1^2 + y_2^2 - 1 \end{array}$$

Comment: The point $(0.5708, -1, -0.1639, 0.9865)$ is the best known solution of the problem provided in [39].

Problem name: `Outrata1990Ex1a`

Source: [40]

Description: Outrata1990Ex1a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y_1^2+y_2^2)-3y_1-4y_2+r(x_1^2+x_2^2) \\ f(x,y) & := & \frac{1}{2}\langle y,Hy\rangle-\langle b(x),y\rangle \\ g(x,y) & := & \begin{pmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y \end{pmatrix} \end{array}$$

with $r:=0.1$, $H:=\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $b(x):=x$.

Comment: Outrata1990Ex1b, Outrata1990Ex1c, Outrata1990Ex1d, and Outrata1990Ex1e are obtained by respectively replacing r, H, and $b(x)$ in the lower-level objective function of Outrata1990a by

$$\begin{array}{l} r:=1,\ H:=\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix},\ \text{and}\ \ b(x):=x, \\ r:=0,\ H:=\begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix},\ \text{and}\ \ b(x):=x, \\ r:=0.1,\ H:=\begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix},\ \text{and}\ \ b(x):=x, \\ r:=0.1,\ H:=\begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix},\ \text{and}\ \ b(x):=\begin{bmatrix} -1 & 2 \\ 3 & -3 \end{bmatrix}x. \end{array}$$

According to [40], the best known optimal solution for problems Outrata1990Ex1a, Outrata1990Ex1b, Outrata1990Ex1c, Outrata1990Ex1d, and Outrata1990Ex1e are respectively

$$\begin{array}{l}(0.97,3.14,2.6,1.8),\ (0.28,0.48,2.34,1.03),\\ (20.26,42.81,3,3),\ (2,0.06,2,0),\ \text{and}\ (2.42,-3.65,0,1.58).\end{array}$$

Note that for Outrata1990Ex1b and Outrata1990Ex1c, the solutions above change with a different starting point for the algorithm used in [40].

Problem name: `Outrata1990Ex2a`

Source: [40]

Description: Outrata1990Ex2a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}[(y_1-3)^2+(y_2-4)^2] \\ G(x,y) & := & -x \\ f(x,y) & := & \frac{1}{2}\langle y,H(x)y\rangle-(3+1.333x)y_1-xy_2 \\ g(x,y) & := & \begin{pmatrix} -0.333y_1+y_2-2 \\ y_1-0.333y_2-2 \\ -y \end{pmatrix} \end{array}$$

with the matrix $H(x)$ defined by $H(x):=I$.

Comment: Outrata1990Ex2b, Outrata1990Ex2c, Outrata1990Ex2d and Outrata1990Ex2e are respectively obtained by performing some changes on the terms $H(x)$ and

$g(x,y)$ in the lower-level objective function of Outrata1990Ex2a:

$$H(x) := \begin{bmatrix} 1+x & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{pmatrix} -0.333y_1 + y_2 - 2 \\ y_1 - 0.333y_2 - 2 \\ -y \end{pmatrix},$$
$$H(x) := \begin{bmatrix} 1+x & 0 \\ 0 & 1+0.1x \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{pmatrix} -0.333y_1 + y_2 - 2 \\ y_1 - 0.333y_2 - 2 \\ -y \end{pmatrix},$$
$$H(x) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{pmatrix} (-0.333+0.1x)y_1 + y_2 - x \\ y_1 + (-0.333-0.1x)y_2 - 2 \\ -y \end{pmatrix},$$
$$H(x) := \begin{bmatrix} 1+x & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad g(x,y) := \begin{pmatrix} (-0.333+0.1x)y_1 + y_2 - x \\ y_1 + (-0.333-0.1x)y_2 - 2 \\ -y \end{pmatrix},$$

According to [40], the best known optimal solution for problems Outrata1990Ex2a, Outrata1990Ex2b, Outrata1990Ex2c, Outrata1990Ex2d, Outrata1990Ex2e are respectively

$$\begin{array}{l} (2.07,3,3), \;\; (0,3,3), \;\; (3.456,1.707,2.569), \\ (2.498,3.632,2.8) \;\; \text{and} \;\; (3.999,1.665,3.887),. \end{array}$$

Problem name: Outrata1993Ex31

Source: [41]

Description: Outrata1993Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y_1-3)^2 + \frac{1}{2}(y_2-4)^2 \\ G(x,y) & := & -x \\ f(x,y) & := & \frac{1}{2}(1+0.2x)y_1^2 + \frac{1}{2}(1+0.1x)y_2^2 - (3+1.33x)y_1 - xy_2 \\ g(x,y) & := & \begin{pmatrix} (-0.333+0.1x)y_1 + y_2 + 0.1x - 2 \\ y_1 + (-0.333-0.1x)y_2 + 0.1x - 2 \\ -y \end{pmatrix} \end{array}$$

Comment: Outrata1993Ex32 is obtained by replacing the lower-level constraint by

$$g(x,y) \;\; := \;\; \begin{pmatrix} -0.333y_1 + y_2 + 0.1x - 1 \\ y_1^2 + y_2^2 - 0.1x - 9 \\ -y \end{pmatrix}.$$

For Outrata1993Ex1 and Outrata1993Ex2, the best known solutions from [41] are $(1.90910, 2.97836, 2.23182)$ and $(4.06095, 2.68227, 1.48710)$, respectively.

Problem name: Outrata1994Ex31

Source: [42]

Description: Outrata1994Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y_1-3)^2+\frac{1}{2}(y_2-4)^2 \\ G(x,y) & := & \begin{pmatrix} -x \\ x-10 \end{pmatrix} \\ f(x,y) & := & \frac{1}{2}(1+0.2x)y_1^2+\frac{1}{2}(1+0.1x)y_2^2-(3+1.333x)y_1-xy_2 \\ g(x,y) & := & \begin{pmatrix} -0.333y_1+y_2+0.1x-1 \\ y_1^2+y_2^2-0.1x-9 \\ -y \end{pmatrix} \end{array}$$

Comment: According to [42], the best known optimal solution is (4.0604, 2.6822, 1.4871).

Problem name: `OutrataCervinka2009`

Source: [43]

Description: OutrataCervinka2009 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -2x_1-0.5x_2-y_2 \\ G(x,y) & := & x_1 \\ f(x,y) & := & y_1-y_2+x^\top y+\frac{1}{2}y^\top y \\ g(x,y) & := & \begin{pmatrix} y_2 \\ y_2-y_1 \\ y_2+y_1 \end{pmatrix} \end{array}$$

Comment: The point 0_4 is a solution of the problem according to [43].

Problem name: `PaulaviciusEtal2017a`

Source: [44]

Description: PaulaviciusEtal2017a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ G(x,y) & := & \begin{pmatrix} -x-1 \\ x-1 \\ -y-1 \\ y-1 \end{pmatrix} \\ f(x,y) & := & xy^2-\frac{1}{2}y^4 \\ g(x,y) & := & \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix} \end{array}$$

Comment: This problem a slight modification of MitsosBarton2006Ex318, just with the upper-level objective function there replaced by x^2+y^2. Doing so, the point $(0.5,0)$ remains optimal for the new problem [44].

Problem name: `PaulaviciusEtal2017b`

Source: [44]

Description: PaulaviciusEtal2017b is defined as follows

$$\begin{aligned} F(x,y) &:= x+y \\ G(x,y) &:= \begin{pmatrix} -x-1 \\ x-1 \\ -y-1 \\ y-1 \end{pmatrix} \\ f(x,y) &:= 0.5xy^2 - x^3y \\ g(x,y) &:= \begin{pmatrix} -y-1 \\ y-1 \end{pmatrix} \end{aligned}$$

Comment: This problem a slight modification of MitsosBarton2006Ex313, just with the *minus* in upper-level objective function replaced by a *plus*. Doing so, the optimal solution the new problem above is $(-1,-1)$ according to [44].

Problem name: `SahinCiric1998Ex2`

Source: [45]

Description: SahinCiric1998Ex2 is defined as follows

$$\begin{aligned} F(x,y) &:= (x-3)^2 + (y-2)^2 \\ G(x,y) &:= \begin{pmatrix} -x \\ x-8 \end{pmatrix} \\ f(x,y) &:= (y-5)^2 \\ g(x,y) &:= \begin{pmatrix} -2x+y-1 \\ x-2y+2 \\ x+2y-14 \end{pmatrix} \end{aligned}$$

Comment: The best known optimal value for the upper-level objective function is 5 and a corresponding optimal point is $(1,3)$; cf. [45].

Problem name: `ShimizuAiyoshi1981Ex1`

Source: [46]

Description: ShimizuAiyoshi1981Ex1 is defined as follows

$$\begin{aligned} F(x,y) &:= x^2 + (y-10)^2 \\ G(x,y) &:= \begin{pmatrix} x-15 \\ -x+y \\ -x \end{pmatrix} \\ f(x,y) &:= (x+2y-30)^2 \\ g(x,y) &:= \begin{pmatrix} x+y-20 \\ y-20 \\ -y \end{pmatrix} \end{aligned}$$

Comment: The optimal solution of the problem is $(10,10)$ according to [46].

Problem name: `ShimizuAiyoshi1981Ex2`

Source: [46]

Description: ShimizuAiyoshi1981Ex2 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & (x_1-30)^2+(x_2-20)^2-20y_1+20y_2 \\
G(x,y) & := & \begin{pmatrix} -x_1-2x_2+30 \\ x_1+x_2-25 \\ x_2-15 \end{pmatrix} \\
f(x,y) & := & (x_1-y_1)^2+(x_2-y_2)^2 \\
g(x,y) & := & \begin{pmatrix} y-10_2 \\ -y \end{pmatrix}
\end{array}
$$

Comment: The optimal solution of the problem is $(20, 5, 10, 5)$ according to [46].

Problem name: ShimizuEtal1997a
Source: [47]
Description: ShimizuEtal1997a is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & (x-5)^2+(2y+1)^2 \\
f(x,y) & := & (y-1)^2-1.5xy \\
g(x,y) & := & \begin{bmatrix} -3x+y+3 \\ x-0.5y-4 \\ x+y-7 \end{bmatrix}
\end{array}
$$

Comment: $(5, 2)$ is the best known solution according to [54].

Problem name: ShimizuEtal1997b
Source: [47]
Description: ShimizuEtal1997b is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & 16x^2+9y^2 \\
G(x,y) & := & \begin{pmatrix} -4x+y \\ -x \end{pmatrix} \\
f(x,y) & := & (x+y-20)^4 \\
g(x,y) & := & \begin{pmatrix} 4x+y-50 \\ -y \end{pmatrix}
\end{array}
$$

Comment: $(11.25, 5)$ is the global optimal solution of the problem and $(7.2, 12.8)$ is a local optimal solution [47].

Problem name: SinhaMaloDeb2014TP3
Source: [48]

Description: SinhaMaloDeb2014TP3 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -x_1^2 - 3x_2^2 - 4y_1 + y_2^2 \\ G(x,y) & := & \begin{pmatrix} x_1^2 + 2x_2 - 4 \\ -x \end{pmatrix} \\ f(x,y) & := & 2x_1^2 + y_1^2 - 5y_2 \\ g(x,y) & := & \begin{pmatrix} -x_1^2 + 2x_1 - x_2^2 + 2y_1 - y_2 - 3 \\ -x_2 - 3y_1 + 4y_2 + 4 \\ -y \end{pmatrix} \end{array}$$

Comment: The best known values of the upper-level and lower-level objective values are -18.6787 and -1.0156, respectively; cf. [48].

Problem name: `SinhaMaloDeb2014TP6`

Source: [48]

Description: SinhaMaloDeb2014TP6 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-1)^2 + 2y_1 - 2x \\ G(x,y) & := & -x \\ f(x,y) & := & (2y_1 - 4)^2 + (2y_2 - 1)^2 + xy_1 \\ g(x,y) & := & \begin{pmatrix} 4x + 5y_1 + 4y_2 - 12 \\ 4y_2 - 4x - 5y_1 + 4 \\ 4x - 4y_1 + 5y_2 - 4 \\ 4y_1 - 4x + 5y_2 - 4 \\ -y \end{pmatrix} \end{array}$$

Comment: The best known values of the upper-level and lower-level objective values are -1.2091 and 7.6145, respectively; cf. [48].

Problem name: `SinhaMaloDeb2014TP7`

Source: [48]

Description: SinhaMaloDeb2014TP7 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -\frac{(x_1+y_1)(x_2+y_2)}{1+x_1y_1+x_2y_2} \\ G(x,y) & := & \begin{pmatrix} x_1^2 + x_2^2 - 100 \\ x_1 - x_2 \\ -x \end{pmatrix} \\ f(x,y) & := & \frac{(x_1+y_1)(x_2+y_2)}{1+x_1y_1+x_2y_2} \\ g(x,y) & := & \begin{pmatrix} y - x \\ -y \end{pmatrix} \end{array}$$

Comment: The best known values of the upper-level and lower-level objective values are -1.96 and 1.96, respectively; cf. [48].

Problem name: `SinhaMaloDeb2014TP8`

Source: [48]

Description: SinhaMaloDeb2014TP8 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & |2x_1+2x_2-3y_1-3y_2-60| \\
g(x,y) & := & \begin{bmatrix} x_1+x_2+y_1-2y_2-40 \\ x-50_2 \\ -x \end{bmatrix} \\
f(x,y) & := & (y_1-x_1+20)^2+(y_2-x_2+20)^2 \\
g(x,y) & := & \begin{bmatrix} 2y-x+10_2 \\ y-20_2 \\ -y-10_2 \end{bmatrix}
\end{array}
$$

Comment: The best known values of the upper-level and lower-level objective values are 0 and 100.0, respectively; cf. [48].

Problem name: `SinhaMaloDeb2014TP9`

Source: [48]

Description: SinhaMaloDeb2014TP9 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & \sum_{i=1}^{10}\left[(x_i-1)^2+y_i^2\right] \\
f(x,y) & := & \exp\left\{\left[1+\frac{1}{400}\sum_{i=1}^{10}y_i^2-\prod_{i=1}^{10}\cos\left(\frac{y_i}{\sqrt{i}}\right)\right]\sum_{i=1}^{10}x_i^2\right\} \\
g(x,y) & := & \begin{pmatrix} y-\pi_{10} \\ -y-\pi_{10} \end{pmatrix}
\end{array}
$$

Comment: The best known values of the upper-level and lower-level objective values are 0.0 and 1.0, respectively; cf. [48].

Problem name: `SinhaMaloDeb2014TP10`

Source: [48]

Description: SinhaMaloDeb2014TP10 is defined as follows

$$
\begin{array}{rcl}
F(x,y) & := & \sum_{i=1}^{10}\left[(x_i-1)^2+y_i^2\right] \\
f(x,y) & := & \exp\left[1+\frac{1}{4000}\sum_{i=1}^{10}x_i^2y_i^2-\prod_{i=1}^{10}\cos\left(\frac{x_iy_i}{\sqrt{i}}\right)\right] \\
g(x,y) & := & \begin{pmatrix} y-\pi_{10} \\ -y-\pi_{10} \end{pmatrix}
\end{array}
$$

Comment: The best known values of the upper-level and lower-level objective values are 0.0 and 1.0, respectively; cf. [48].

Problem name: `TuyEtal2007`

Source: [49]

Description: TuyEtal2007 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x^2+y^2 \\ G(x,y) & := & \begin{pmatrix} -x \\ -y \end{pmatrix} \\ f(x,y) & := & -y \\ g(x,y) & := & \begin{pmatrix} 3x+y-15 \\ x+y-7 \\ x+3y-15 \end{pmatrix} \end{array}$$

Comment: The best know solution of the problem is $(4.492188, 1.523438)$; cf. [49].

Problem name: Vogel2002

Source: [50]

Description: Vogel2002 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (y+1)^2 \\ G(x,y) & := & \begin{pmatrix} -3-x \\ x-2 \end{pmatrix} \\ f(x,y) & := & y^3-3y \\ g(x,y) & := & x-y \end{array}$$

Comment: The point $(-2,-2)$ is the optimal solution of the problem; cf. [50].

Problem name: WanWangLv2011

Source: [51]

Description: WanWangLv2011 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (1+x_1-x_2+2y_2)(8-x_1-2y_1+y_2+5y_3) \\ f(x,y) & := & 2y_1-y_2+y_3 \\ g(x,y) & := & \begin{pmatrix} -y_1+y_2+y_3-1 \\ 2x_1-y_1+2y_2-0.5y_3-1 \\ 2x_2+2y_1-y_2-0.5y_3-1 \\ -x \\ -y \end{pmatrix} \end{array}$$

Comment: The optimal solution is $(0, 0.75, 0, 0.5, 0)$ according to [51].

Problem name: YeZhu2010Ex42

Source: [52]

Description: YeZhu2010Ex42 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & (x-1)^2+y^2 \\ G(x,y) & := & \begin{pmatrix} -x-3 \\ x-2 \end{pmatrix} \\ f(x,y) & := & y^3-3y \\ g(x,y) & := & x-y \end{array}$$

Comment: YeZhu2010Ex43 is obtained by replacing the upper-level objective function by

$$(x-1)^2 + (y-2)^2$$

Note that YeZhu2010Ex42 is a slightly modified version of Vogel2002, with the term y^2 added to the upper-level objective function. The point $(1,1)$ is the optimal solution for both YeZhu2010Ex42 and YeZhu2010Ex43; cf. [52].

Problem name: `Yezza1996Ex31`

Source: [53]

Description: Yezza1996Ex31 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -(4x-3)y+2x+1 \\ G(x,y) & := & \begin{pmatrix} -x \\ x-1 \end{pmatrix} \\ f(x,y) & := & -(1-4x)y-2x-2 \\ g(x,y) & := & \begin{pmatrix} -y \\ y-1 \end{pmatrix} \end{array}$$

Comment: The optimal solution of the problem is $(0.25, 0)$; cf. [53].

Problem name: `Yezza1996Ex41`

Source: [53]

Description: Yezza1996Ex41 is defined as follows

$$\begin{array}{rcl} F(x,y) & := & \frac{1}{2}(y-2)^2 + \frac{1}{2}(x-y-2)^2 \\ f(x,y) & := & \frac{1}{2}y^2 + x - y \\ g(x,y) & := & \begin{pmatrix} -y \\ y-x \end{pmatrix} \end{array}$$

Comment: The optimal solution of the problem is $(3,1)$; cf. [53].

Problem name: `Zlobec2001a`

Source: [56]

Description: Zlobec2001a is defined as follows

$$\begin{array}{rcl} F(x,y) & := & -y_1/x \\ f(x,y) & := & -y_1 - y_2 \\ g(x,y) & := & \begin{pmatrix} x+y_1 \\ y_2-1 \\ -y \end{pmatrix} \end{array}$$

Comment: This example is used in [56] to illustrate that the objective function of the problem can be discontinuous. As stated in [56] , an optimal solution is $(1,1,0)$.

Problem name: `Zlobec2001b`

Source: [56]
Description: Zlobec2001b is defined as follows

$$\begin{array}{rcl} F(x,y) & := & x+y \\ G(x,y) & := & \begin{pmatrix} x-1 \\ -x \end{pmatrix} \\ f(x,y) & := & -y \\ g(x,y) & := & \begin{pmatrix} y-1 \\ -y \\ xy \\ -xy \end{pmatrix} \end{array}$$

Comment: This example is used in [56] to illustrate that the feasible set of a bilevel optimization problem is not necessarily closed. As stated in [56], this problem does not have an optimal solution.