CS 771A: Intro to Machine Learning, IIT Kanpur				**Quiz I** (28 Jan 2025)
Name	MELBO			**20 marks** Page **1** of **2**
Roll No	250007	Dept.	AWSM	

Instructions:
1. This question paper contains 1 page (2 sides of paper). Please verify.
2. Write your name, roll number, department above in **block letters neatly with ink**.
3. Write your final answers neatly **with a blue/black pen**. Pencil marks may get smudged.
4. Don't overwrite/scratch answers especially in MCQ – such cases may get straight 0 marks.
5. Do not rush to fill in answers. You have enough time to solve this quiz.

Q1. (**True-False**) Write **T** or **F** for True/False in the **box on the right** and a **brief justification** in the space below. **Note**: $L \in \mathbb{R}^{2\times 2}$ is not necessarily positive semidefinite. (**3 x (1+2) = 9 marks**)

1	For any $\mathbf{w} \in \mathbb{R}^2, b \in \mathbb{R}, L \in \mathbb{R}^{2\times 2}$, the set $\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{w}^\top(L\mathbf{x}) + b = 0\}$ is either a line or the entire $\mathbb{R}^2$ or else empty. If **T**, give a brief proof. If **F**, give a counterexample.	T

The set is simply $\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{v}^\top\mathbf{x} + b = 0\}$ for $\mathbf{v} = L^\top\mathbf{w}$. If $\mathbf{v} \neq \mathbf{0}$ then the set is a line (that passes through the origin if $b = 0$). If $\mathbf{v} = \mathbf{0}$ and $b = 0$ too then the set is the entire $\mathbb{R}^2$. If $\mathbf{v} = \mathbf{0}$ but $b \neq 0$ then the set is empty. Please note that proof-by-example or proof-by-picture are not admissible.

2	For any $L \in \mathbb{R}^{2\times 2}$ and any convex set $\mathcal{C} \subset \mathbb{R}^2$, if we define $\mathcal{D} \stackrel{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^2 : L\mathbf{x} \in \mathcal{C}\}$, then $\mathcal{D}$ is always convex or empty. If **T**, give a brief proof. If **F**, give a counterexample.	T

Suppose $\mathbf{x}, \mathbf{y} \in \mathcal{D}$ i.e. $L\mathbf{x} \in \mathcal{C}$ and $L\mathbf{y} \in \mathcal{C}$. Since $\mathcal{C}$ is convex, this means that $\frac{1}{2}(L\mathbf{x} + L\mathbf{y}) \in \mathcal{C}$. However, that means that $L\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right) \in \mathcal{C}$ i.e. $\frac{\mathbf{x}+\mathbf{y}}{2} \in \mathcal{D}$. By midpoint convexity, we have shown that the set $\mathcal{D}$ is convex. Please note that proof-by-example or proof-by-picture are not admissible.

3	If circles are sets of points of the form $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 = r\}$ for some $r \geq 0$, then for any $L \in \mathbb{R}^{2\times 2}$, the set $\{\mathbf{x} \in \mathbb{R}^2 : \|L\mathbf{x}\|_2 = 1\}$ is either a circle or else empty. If **T**, give a brief proof. If **F**, give a counterexample (where it is non-empty but not a circle).	F

Consider $L = \mathbf{1}\mathbf{1}^\top$ the all-ones matrix. The set is $\left\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{1}^\top\mathbf{x}| = \frac{1}{\sqrt{2}}\right\}$ as $\|\mathbf{1}\mathbf{1}^\top\mathbf{x}\|_2 = \sqrt{2} \cdot |\mathbf{1}^\top\mathbf{x}|$ The set of points where $|\mathbf{1}^\top\mathbf{x}| = \frac{1}{\sqrt{2}}$ is a pair of lines since they correspond to the two lines, namely $\mathbf{1}^\top\mathbf{x} = \frac{1}{\sqrt{2}}$ and $\mathbf{1}^\top\mathbf{x} = -\frac{1}{\sqrt{2}}$. Thus, this set is clearly not a circle. Other examples exist e.g., if L is a diagonal matrix with unequal entries then the set becomes an oblong ellipse and not a circle. Note that $L = \mathbf{0}\mathbf{0}^\top$, i.e. the all-zero matrix is not a valid counterexample since in that case the set would be empty, and the question demands a counterexample with a non-empty set.

Q2. (Subcalculus) Melba came across a function $f\colon \mathbb{R} \to \mathbb{R}$ described on the right and wants to analyse its properties. For parts a,b,c,d, **fill only one circle**. For parts d,e, **answer in the space provided**. No proofs/derivations needed in any part. **Note**: the subdifferential at a point is a set in general (singleton set if the func. is differentiable at that point). (**1 x 6 = 6 marks**)

$$f(x) = \begin{cases} -x & x \le 0 \\ -\ln(1+x) & 0 < x \le 1 \\ -\ln(2) & 1 < x \end{cases}$$

		True	False
a.	Is f a continuous function over all of $\mathbb{R}$?	●	○
b.	Is f a convex function over all of $\mathbb{R}$?	●	○
c.	Is f differentiable at $x = 0$?	●	○
d.	Is f differentiable at $x = 1$?	○	●
e.	What is the subdifferential of f at $x = 0$?	$\{-1\}$ The subdifferential is a singleton set	
f.	What is the subdifferential of f at $x = 1$?	$[-0.5,0]$ The subdifferential is an interval	

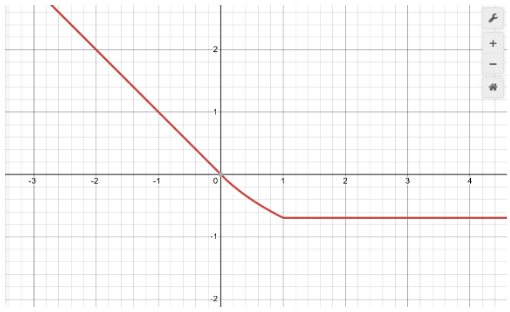


Figure courtesy desmos.com

Q4. (Too many prototypes) Melbu has a learning-with-prototypes (LwP) model for a binary problem with two labels $+$ and $-$ and 2D features. Every point on the circle $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 = 1\}$ is a $-$ prototype and every point on the circle $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 = 2\}$ is a $+$ prototype. Write down the equation for the decision boundary of this classifier and give justification below. (**2 + 3 = 5 marks**)

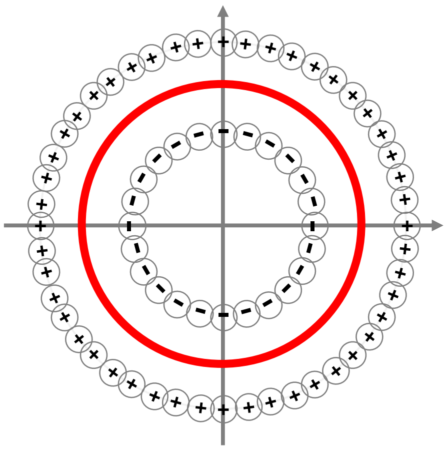
Write equation of decision boundary here

$$\|\mathbf{x}\|_2 = 1.5$$

Give justification here

Polar coordinates make life simpler here. The squared Euclidean distance between two points (r, θ) and (s, ϕ) is

$$r^2 + s^2 - 2rs \cdot \cos(\theta - \phi)$$



Using the above, we deduce that for a point (r, θ), the squared distance to the closest $+$ prototype is $(r-2)^2$ and the squared distance to the closest – prototype is $(r-1)^2$.
The decision boundary is situated where these two are equal i.e. $(r-2)^2 = (r-1)^2$ i.e. $r - 2 = \pm(r-1)$ since this is where the classifier gets confused. The first case $r - 2 = r - 1$ yields an absurdity but the second case $r - 2 = 1 - r$ yields $r = 1.5$ which is the answer. The decision boundary is shown in the diagram (although it is not required to draw the decision boundary to get full marks).