CS 771A: Intro to Machine Learning, IIT Kanpur				**Quiz II** (01 Apr 2025)
Name				**20 marks**
Roll No		Dept.		Page **1** of **2**

Instructions:
1. This question paper contains 1 page (2 sides of paper). Please verify.
2. Write your name, roll number, department above in **block letters neatly with ink**.
3. Write your final answers neatly **with a blue/black pen**. Pencil marks may get smudged.
4. Don't overwrite/scratch answers especially in MCQ – such cases may get straight 0 marks.
5. Do not rush to fill in answers. You have enough time to solve this quiz.

Q1. (Tracing stuff) Let $\mathbf{u}_1, \ldots, \mathbf{u}_{100} \in \mathbb{R}^{100}$ be the left singular vectors of a full-rank matrix $B \in \mathbb{R}^{100 \times 100}$. Let $\mathbf{p}, \mathbf{q}$ be random vectors with support $\{\mathbf{u}_1, \ldots, \mathbf{u}_{100}\}$. Find the following quantities (no derivations). *Hint 1*: linearity of expectation. *Hint 2*: trace is linear but rank is not.(**1 x 5 = 5 marks**)

2.1	$\mathbb{E}[\mathrm{trace}(\mathbf{p}\mathbf{q}^\top)]$ if $\mathbf{p}, \mathbf{q}$ are chosen uniformly but without replacement i.e. $\mathbf{p}$ is chosen uniformly randomly from $\{\mathbf{u}_1, \ldots, \mathbf{u}_{100}\}$ and then $\mathbf{q}$ is chosen uniformly randomly from $\{\mathbf{u}_1, \ldots, \mathbf{u}_{100}\} \backslash \{\mathbf{p}\}$.	
2.2	$\mathrm{trace}(\mathbb{E}[\mathbf{p}\mathbf{q}^\top])$ if $\mathbf{p}, \mathbf{q}$ are chosen uniformly but without replacement i.e. $\mathbf{p}$ is chosen uniformly randomly from $\{\mathbf{u}_1, \ldots, \mathbf{u}_{100}\}$ and then $\mathbf{q}$ is chosen uniformly randomly from $\{\mathbf{u}_1, \ldots, \mathbf{u}_{100}\} \backslash \{\mathbf{p}\}$.	
2.3	$\mathrm{trace}(\mathbb{E}[\mathbf{p}\mathbf{q}^\top])$ if $\mathbf{p}, \mathbf{q}$ are chosen uniformly but with replacement i.e. $\mathbf{p}$ is chosen uniformly randomly from $\{\mathbf{u}_1, \ldots, \mathbf{u}_{100}\}$ and then $\mathbf{q}$ is chosen uniformly randomly from $\{\mathbf{u}_1, \ldots, \mathbf{u}_{100}\}$ independently of $\mathbf{p}$.	
2.4	$\mathrm{rank}(\mathbb{E}[\mathbf{p}\mathbf{p}^\top])$ if $\mathbf{p}$ is chosen uniformly randomly from $\{\mathbf{u}_1, \ldots, \mathbf{u}_{100}\}$	
2.5	$\mathbb{E}[\mathrm{rank}(\mathbf{p}\mathbf{p}^\top)]$ if $\mathbf{p}$ is chosen uniformly randomly from $\{\mathbf{u}_1, \ldots, \mathbf{u}_{100}\}$	

Q2. (Kernel Smash) $K_1, K_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Mercer kernels i.e., for any $x, y \in \mathbb{R}$, we have $K_i(x, y) = \langle \phi_i(x), \phi_i(y) \rangle$ with the feature maps given below. Design a map $\phi_3 : \mathbb{R} \to \mathbb{R}^9$ for kernel K_3 s.t. $K_3(x, y) = \left(K_1(x, y) + K_2(x, y)\right)^2$ for all $x, y \in \mathbb{R}$. No derivation needed. **Note that $\boldsymbol{\phi_3}$ must not use more than 9 dimensions. If your solution does not require 9 dimensions then fill the rest of the dimensions with zero.** **(9 marks)**

$$\phi_1(x) = \left(\frac{1}{x}, x\right), \phi_2(x) = \left(\frac{1}{x^2}, x^2\right)$$

$$\phi_3(x) = \left(\boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{}, \boxed{}\right)$$

Q3. (**True-False**) Write **T** or **F** for True/False in the **box on the right** and a **brief justification** in the space below (brief proof if **T**, counterexample if **F**). A square matrix is termed *diagonal* if all of its off-diagonal entries are zero (its diagonal entries can be zero/-ve/+ve). (**3 x (1+1) = 6 marks**)

1	A diagonal matrix $A \in \mathbb{R}^{3\times3}$ with all diagonal entries being non-zero must always have rank exactly equal to 3.	
2	It is possible for a diagonal matrix $B \in \mathbb{R}^{3\times3}$ to be positive semi-definite if it has two strictly positive diagonal entries and one strictly negative diagonal entry.	
3	For any Mercer kernel $K: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and any vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^2$, the matrix $G = [g_{ij}] \in \mathbb{R}^{n\times n}$ defined as $g_{ij} \overset{\text{def}}{=} K(\mathbf{x}_i, \mathbf{x}_j)$ is always positive semi-definite.	