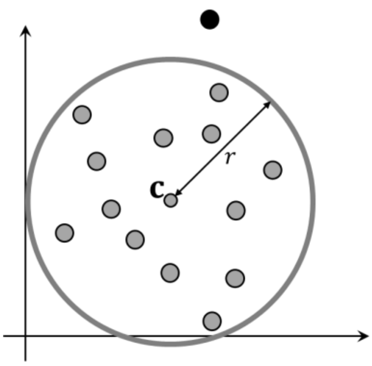
Instructions:

1. This question paper contains 2 pages (4 sides of paper). Please verify.
2. Write your name, roll number, department in **block letters** with **ink** on **each page**.
3. Write your final answers neatly **with a blue/black pen**. Pencil marks may get smudged.
4. Don't overwrite/scratch answers especially in MCQ – ambiguous cases will get 0 marks.

Q1 (Anomalous Thresholds) Melba is designing an anomaly detection algorithm, using a neural network to embed all data points as 2D vectors and finding the mean vector $\mathbf{c}$. Melba now wants to find a radius threshold r such that given a test data point with 2D embedding $\mathbf{x}$, Melba can label it as $\hat{y} = +1$ (anomalous) if $\|\mathbf{x} - \mathbf{c}\|_2 > r$ else label it as $\hat{y} = -1$ (normal). Melba has the following test data containing normal and anomalous points. To simplify, the Euclidean distance of all test points from mean $\mathbf{c}$ and their ground truth label y ($y = +1$ for anomalous, $y = -1$ for normal) is given.



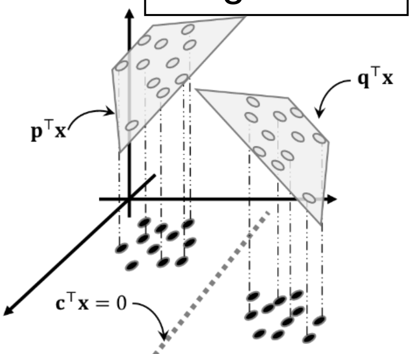
$\|\mathbf{x} - \mathbf{c}\|_2$	1.0	2.5	3.0	4.5	5.0	6.5	7.0	8.5	9.0	10.5
true label y	-1	-1	-1	$+1$	-1	-1	-1	$+1$	-1	$+1$

For each part, answer in the box and also justify briefly in space provided. **((1+1) x 3 = 6 marks)**

Is there a radius threshold that achieves perfect test accuracy i.e. i.e. $\mathbb{P}[\hat{y} = y] = 1$? Write **T** or **F**. Justify your answer briefly.	

Give (any) one radius threshold for which Melba's classifier achieves perfect true-positive rate (TPR) on test data i.e. $\mathbb{P}[\hat{y} = 1 \mid y = 1] = 1$. Justify your answer briefly.	

Among radius thresholds that achieve perfect TPR, find (any) one that minimizes the false-positive rate (FPR) i.e. $\mathbb{P}[\hat{y} = 1 \mid y = -1]$ is smallest. Justify your answer briefly.	



Q2 (Probabilistic DT) Melbo wants to solve a mixed regression problem using two regression models $\mathbf{p}, \mathbf{q} \in \mathbb{R}^d$. A classifier $\mathbf{c} \in \mathbb{R}^d$ is also needed to decide which model to use at test time (see figure). For a data point $\mathbf{x} \in \mathbb{R}^d$, if $\mathbf{c}^\top\mathbf{x} \geq 0$, Melbo will predict $\mathbf{p}^\top\mathbf{x}$. If $\mathbf{c}^\top\mathbf{x} < 0$, predict $\mathbf{q}^\top\mathbf{x}$. Assume that bias is hidden inside the models. Note that this is nothing but a regression tree with one root and two leaves. **(4 x 4 = 16 marks)**

Melbo has training data $(\mathbf{x}^i, y^i), i \in [N]$ where $\mathbf{x}^i \in \mathbb{R}^d, y^i \in \mathbb{R}$ but does not know which data point belongs to which model so Melba advises using latent variables. For each data point $i \in [N]$, Melbo uses a binary latent variable $z^i \in \{-1, +1\}$ and the following likelihoods: $\mathbb{P}\left[z^i \mid \mathbf{x}^i, \mathbf{p}, \mathbf{q}, \mathbf{c}\right] = \sigma(z^i \cdot \mathbf{c}^\top\mathbf{x}^i)$ and $\mathbb{P}\left[y^i \mid \mathbf{x}^i, z^i = +1, \mathbf{p}, \mathbf{q}, \mathbf{c}\right] = \frac{1}{\sqrt{\pi}}\exp\left(-\left(y^i - \mathbf{p}^\top\mathbf{x}^i\right)^2\right)$ and $\mathbb{P}\left[y^i \mid \mathbf{x}^i, z^i = -1, \mathbf{p}, \mathbf{q}, \mathbf{c}\right] = \frac{1}{\sqrt{\pi}}\exp\left(-\left(y^i - \mathbf{q}^\top\mathbf{x}^i\right)^2\right)$ where $\sigma(t) \stackrel{\text{def}}{=} \frac{1}{(1+\exp(-t))}$ is the sigmoid.

Give brief derivation of an expression for the exact likelihood $\mathbb{P}\left[y^i \mid \mathbf{x}^i, \mathbf{p}, \mathbf{q}, \mathbf{c}\right]$ by eliminating z^i.

As exact MLE is hard, Melbo instead tries to solve $\underset{\mathbf{p},\mathbf{q}}{\arg\max}\ \underset{\{z^i\}}{\arg\max}\ \underset{\mathbf{c}}{\arg\max}\ \left\{\mathcal{L}\left(\mathbf{p}, \mathbf{q}, \mathbf{c}, \{z^i\}\right)\right\}$ with $\mathcal{L}(\mathbf{p}, \mathbf{q}, \mathbf{c}, z^i) = \sum_{i \in [N]} \ln\left(\mathbb{P}\left[y^i \mid \mathbf{x}^i, z^i, \mathbf{p}, \mathbf{q}, \mathbf{c}\right]\right) + \sum_{i \in [N]} \ln\left(\mathbb{P}\left[z^i \mid \mathbf{x}^i, \mathbf{p}, \mathbf{q}, \mathbf{c}\right]\right)$ using alternating optimization. **You are free to use simple operations like least squares, logistic regression directly**.

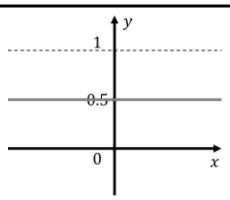
Step 1: Freeze $\mathbf{c}, \{z^i\}$ and give brief derivation on how to find $\arg\max_{\mathbf{p},\mathbf{q}} \mathcal{L}\left(\mathbf{p}, \mathbf{q}, \mathbf{c}, \{z^i\}\right)$.

CS 771A: Intro to Machine Learning, IIT Kanpur			**Endsem Exam** (27 Apr 2025)
Name			**40 marks**
Roll No		Dept.	Page **3** of **4**

Step 2: Freeze $\mathbf{p}, \mathbf{q}, \mathbf{c}$ and give brief derivation on how to find $\arg\max_{\{z^i\}} \mathcal{L}(\mathbf{p}, \mathbf{q}, \mathbf{c}, \{z^i\})$.

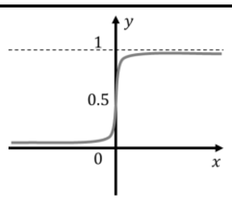
Step 3: Freeze $\mathbf{p}, \mathbf{q}, \{z^i\}$ and give brief derivation on how to find $\arg\max_{\mathbf{c}} \mathcal{L}(\mathbf{p}, \mathbf{q}, \mathbf{c}, \{z^i\})$.

Q3 (Sharp Sigmoids) The sigmoid function becomes more expressive by introducing a bandwidth parameter B as $\sigma(x; B) \stackrel{\text{def}}{=} 1/(1 + \exp(-B \cdot x))$. For each of the following five curves, select the value of B that best generates that curve. **Shade only one circle in each part**. (**5 x 1 = 5 marks**)



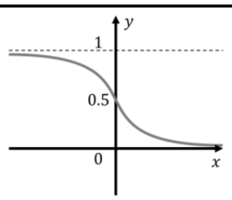
(a)

- ○ $B \to -\infty$
- ○ $B = -1$
- ○ $B = 0$
- ○ $B = +1$
- ○ $B \to +\infty$



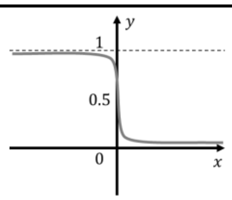
(b)

- ○ $B \to -\infty$
- ○ $B = -1$
- ○ $B = 0$
- ○ $B = +1$
- ○ $B \to +\infty$



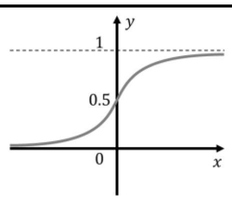
(c)

- ○ $B \to -\infty$
- ○ $B = -1$
- ○ $B = 0$
- ○ $B = +1$
- ○ $B \to +\infty$



(d)

- ○ $B \to -\infty$
- ○ $B = -1$
- ○ $B = 0$
- ○ $B = +1$
- ○ $B \to +\infty$



(e)

- ○ $B \to -\infty$
- ○ $B = -1$
- ○ $B = 0$
- ○ $B = +1$
- ○ $B \to +\infty$

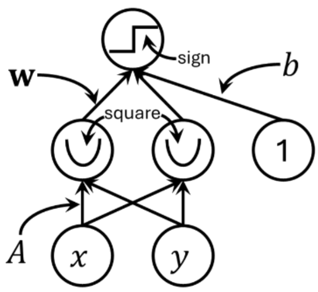
Q4 (Total Confusion) Melbu used a linear model $\text{sign}(\mathbf{w}^\top\mathbf{x} + b)$ to solve a binary classification problem with model vector $\mathbf{w} \in \mathbb{R}^d$ and bias $b \in \mathbb{R}$. The classifier was evaluated on 1000 test data points and the following confusion matrix was obtained. Note that y denotes the true label of a test point and $\hat{y}$ denotes the label predicted by the classifier. The entries in the confusion matrix show how many points of a particular class were classified in a particular manner by the classifier. There are only two classes namely $-1, +1$. Calculate the following quantities for the classifier based on its test performance (no derivations needed) **(4 x 1 + 2 = 6 marks)**

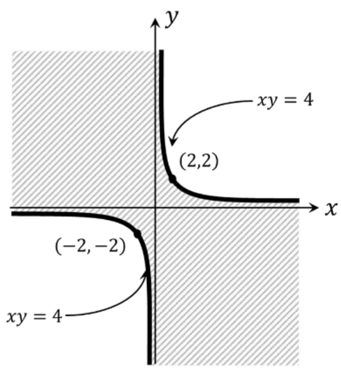
	$\hat{y} = 1$	$\hat{y} = -1$
$y = 1$	100	700
$y = -1$	100	100

Accuracy $\mathbb{P}[\hat{y} = y]$		Recall $\mathbb{P}[\hat{y} = 1 \mid y = 1]$	
Precision $\mathbb{P}[y = 1 \mid \hat{y} = 1]$		False omission rate $\mathbb{P}[\hat{y} \neq y \mid \hat{y} = -1]$	

Melbu's classifier is not very good. Suggest a simple change to the model parameters so that the new classifier's accuracy goes up to at least 75% (maybe more). You are not allowed to (re)train on original or additional data or change the training algorithm. All you can do is make modifications directly to the model parameters $\mathbf{w}, b$ learnt by Melbu. Briefly justify your answer.

Q5 (Hyperbolic Networks) We wish to use a neural network with architecture shown on the left, to solve a binary classification problem shown on the right. The bold decision boundaries in the figure depict the hyperbola $xy = 4$. The NN has parameters a 2×2 matrix $A \in \mathbb{R}^{2\times2}$, a 2D vector $\mathbf{w} \in \mathbb{R}^2$ and a bias $b \in \mathbb{R}$. The output of the NN is $\text{sign}(\mathbf{w}^\top\phi(\mathbf{x}) + b)$ where $\phi(\mathbf{x}) = (A\mathbf{x})^2$ with square activation being applied coordinate-wise i.e. for a vector $\mathbf{v} = (v_1, v_2)$, the square activation is applied as $(\mathbf{v})^2 \stackrel{\text{def}}{=} (v_1^2, v_2^2)$. Find values of the parameters $A, \mathbf{w}, b$ so that the NN gives output $+1$ in the shaded region and -1 in the white region. Be careful to write value of A and not $A^\top$ (note that we have $\phi(\mathbf{x}) = (A\mathbf{x})^2$ not $(A^\top\mathbf{x})^2$). **Write only in the space provided**. **(7 marks)**





$$A = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \qquad \mathbf{w} = [\,\square \quad \square\,] \qquad b = \square$$