Camlin Page
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9/8/2020

MODULE I

Theoretical Distributions

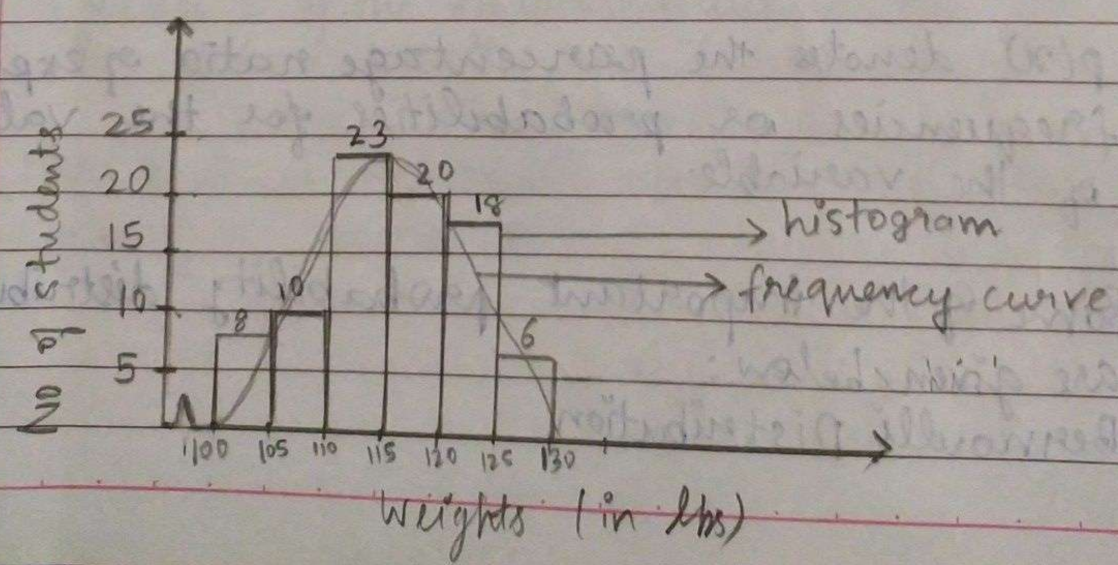
The frequency distributions can be broadly classified into two types namely

(a) Observed frequency distribution
(b) Theoretical or expected frequency distributions or probability distributions.

The observed frequency distributions are based on actual observations and experimentations. For instance, consider a study of weights of students of a class represented in the form of a Table as well as a Histogram and Frequency curve.

Weights of 85 students of a class:

Weights (in lbs)	100-105	105-110	110-115	115-120	120-125	125-130	Total
No of Students	8	10	23	20	18	6	85



After studying a large number of such frequency distributions which are based on actual observations, it is possible to mathematically deduce a model distribution.

This model distribution is called Probability distribution or mathematical or expected frequency distribution, since these are expected on the basis of experience or theoretical considerations.

Suppose a random variable assumes value 1, 2, 3, 4, 5 and the percentage ratio of expected frequencies corresponding to various values of the variables are 0.1, 0.2, 0.3, 0.2, 0.2. Then the theoretical distribution or probability distribution is

x	$p(x)$
1	0.1
2	0.2
3	0.3
4	0.2
5	0.2

$p(x)$ denotes the percentage ratio of expected frequencies or probabilities for the values of the variable.

Some of the important probability distributions are given below:

1. Bernoulli Distribution
2. Binomial distribution (B.D)
3. Poisson distribution (P.D)
4. Uniform distribution (U.D)
5. Normal distribution (N.D)

Among these, Bernoulli, Binomial and Poisson are discrete probability distributions where as Uniform and Normal distributions are continuous probability distributions.

Bernoulli Distribution

Bernoulli trial

A trial is said to be Bernoulli in nature if it satisfies the following conditions:

1. The trial must result either in a success or in a failure
2. The probability of success should remain constant for any trial.

Examples:

1. An unbiased coin is tossed once: Let X take value 1 if the throw results in Head or 0 if the throw results in Tail respectively.

Here X is a Bernoulli variate with parameter $p = \frac{1}{2}$. Here $x = 0, 1$

Definition

A discrete random variable X is said to follow a Bernoulli distribution, if its probability mass function is given by

$P(X=x) = p^x q^{1-x}$ where $x = 0, 1$, with parameter $= p$

Here p = probability of success, $p > 0$
q = probability of failure, $q = p-1$
x = number of success

Mean and Variance of Bernoulli Distribution

$\text{mean} = E(x) = \sum_{}^{1} x f(x) = \sum^{1} x p^x q^{1-x}$.
$\text{Mean} = {}^{0}0 + 1 \cdot p^1 \cdot q^{0} = 0 + p = \underline{p}$
$\text{Variance} = E(x^2) - (E(x))^2$
$= (\sum x^2 \cdot p^x \cdot q^{1-x}) - p^2 = pq$
$= (0^2 \times p^0 \cdot q^1 + 1^2 \cdot p^1 \cdot q^0) - p^2 = p - p^2 = p(1-p)$

Relationship between mean and variance.

$pq < p$ (since $q < 1$)

ie. variance < mean

Properties of Bernoulli distribution

1. p is the parameter of the distribution
2. The mean of the distribution is p.
3. The variance is pq where $q = 1-p$
4. Variance is less than mean.

5. If $x_1, x_2, \ldots x_n$ are n independently and identically distributed Bernoulli variates with parameter p, then $(x_1 + x_2 + \cdots + x_n)$ follows Bernoulli distribution with n and p as parameters.

Ex-1: x follows a Bernoullian distribution with parameter $= 0.8$. Find mean and variance of x. Also find mean and variance of $2x+3$

ans:- Given $p = 0.8$
$q = 1 - p$
$= 1 - 0.8$
$q = \underline{0.2}$

$\text{Mean} = p$
$= 0.8$

$\text{Variance} = pq$
$= 0.8 \times 0.2$
$= \underline{0.16}$

Mean of $2x+3$
$= E(2x+3)$
$= E(2x) + E(3)$
$= 2E(x) + 3$
$= 2 \times 0.8 + 3$
$= 1.6 + 3$
$= \underline{4.6}$

$V(2X+3) = 2^2 V(X)$
$= 4 \times 0.16$
$= \underline{0.64}$

$\therefore$ Mean of $(2x+3) = 4.6$

Variance of $(2x+3) = 0.64$

Ex. 2 If x is a Bernoulli variate taking values 1 or 0 with probabilities 0.6 and 0.4 respectively, then find mean and variance

ans:- Given $p = 0.6$

$\therefore$ Mean $= p = 0.6$

Variance of the distribution is pq

$= (0.6)(0.4)$

$= \underline{0.24}$

Ex. 3 Write down the probability function with its range of a Bernoulli's distribution with parameter 0.35.

ans Given $p = 0.35$

$q = 1-p$

$= 1 - 0.35$

$q = 0.65$

The probability function of the Bernoulli distribution is given by

$f(x) = p^x q^{1-x}$

$= (0.35)^x (0.65)^{1-x}$ with range $x = 0, 1$

Ex 4. If $p = 0.4$ find mean

given $p = 0.4$

$\therefore q = 1 - p = 1 - 0.4 = \underline{0.6}$

$\because p = 0.4$; mean $= \underline{0.4}$

Ex 5. If $p = 0.6$, write down the probability function

$p = 0.6$; $q = 1 - p = 1 - 0.6 = 0.4$

$f(x) = p^x \cdot q^{1-x}$

$= (0.6)^x \cdot (0.4)^{1-x}$

NOTE:- (relation b/w Mean and Variance)

Mean   Variance

$p > pq$

eg:- $p = 0.6$; $q = 0.4$

$\underset{p}{0.6} > \underset{pq}{0.24}$

Binomial Distribution

defn of binomial distribution:-

A random variable X is said to follow Binomial distribution with parameters n and p if its probability function is

$$f(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

where $x = 0, 1, 2, 3 \ldots n$

$x \to$ no. of successes

in a single trial $p \to$ probability for success

$q \to$ probability for failure

p lies between 0 and 1 and $q + p = 1$ ie

$q = 1 - p$

Success and failure:

Consider an event associated to random experiment. When the random experiment is repeated a number of times, the event may or may not occur in each of those experiments.

The occurrence of the event is named success and the non-occurrence is named failure

A random experiment has only two possible outcomes: "success" and "failure"

eg:- In tossing a coin, there are two events, 'Head' and 'tail'. One of them is a success and the other a failure.

eg:- In throwing a die, we can say getting a six and not getting a six are the two events. One of them is success and other is failure.

Binomial distribution is discrete.

Binomial distribution is often very useful in decision making situations in business. It can be used ~~whether~~ to judge whether a coin or a die is unbiased or not by comparing the frequencies.

Situations where Binomial distribution can be applied.

when:-

(1) Random experiment has two outcomes, i.e. 'success' and 'failure'.

(2) Probability ~~for~~ success in a single trial remains constant from trial to trial of the experiment

(3) The experiment is repeated finite number of times.

(4) Trials are independent.

Properties of Binomial distribution

- It is a discrete probability distribution
- Mean of the Binomial Distribution increases as 'n' increases, with 'p' remaining constant

Mean = np where n is the number of times repeated.

$SD = \sqrt{npq}$

eg:- Four coins are tossed; the success is getting 2 heads

Success = getting 2 heads

$n = 4$

$x = 2$

$p = 1/2$

$q = 1 - p = 1 - 1/2 = 1/2$

Binomial probability Law

Let p be the probability of success in a single trial and q be the probability of failure in a single trial of the experiment.

Let there be success 'x' times out of 'n' trials and failure in the remaining 'n - x' times.

then

$$f(x) = {}^nC_x \, p^x q^{n-x}$$

where n - number of times the experiment is repeated

x - the success

p - the probability of success in a single trial

q - the probability of failure in a single trial.

NOTE:

1. $^nC_x = \frac{n!}{(n-x)!}$
2. $n! = n \times (n-1) \times (n-2) \times \dots \times 2 \times 1$
3. $^nC_0 = 1$
4. $^nC_n = 1$
5. $^nC_x = {}^nC_{n-x}$

eg:- $^5C_0 = 1$ eg:- $^5C_2 = \frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3} =$

eg:- $^{100}C_0 = 1$

eg:- $^7C_5 = {}^7C_2$

eg:- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 180$

eg: ${}^{7}C_{5} = {}^{7}C_{2} = \frac{7 \times 6}{1 \times 2}$

${}^{15}C_{13} = {}^{15}C_{2} = \frac{15 \times 14}{1 \times 2}$

Ex I. 5 coins are tossed simultaneously. What is the probability of getting 3 heads?

$n = 5$
$x = 3$
$p = 1/2$

$q = 1 - p = 1 - 1/2$
$q = 1/2$

$f(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$f(x=3) = {}^{5}C_{3} (1/2)^{3} (1/2)^{2}$

$= 5C_2 \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^2$

$= \frac{5 \times 4}{1 \times 2} \times \frac{1}{8} \times \frac{1}{4}$

$= 10 \times \frac{1}{8 \times 4}$

$f(x) = \frac{5}{16}$

Ex 2 Four coins are tossed simultaneously. What is the probability of getting 2 heads.

$n = 4$ (no. of coins tossed)
$x = 2$
$p = 1/2$
$q = 1 - 1/2$
$q = 1/2$

$f(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$f(x=2) = {}^{4}C_{2} \left(\frac{1}{2}\right)^{2} \cdot \left(\frac{1}{2}\right)^{2}$

$= 4C_2 \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$

$= \frac{4 \times 3}{1 \times 2} \times \frac{1}{4} \times \frac{1}{4}$

$f(x) = \frac{3}{8}$

Ex 3 Probability that a batsman scores a century in a cricket match is 1/3. Find the probability that out of 5 matches, he may score century in (i) exactly 2 matches (ii) no matches

(i) Success - getting a century

$p = 1/3$

$q = 1 - p = 1 - 1/3 = 2/3$

$n = 5$ $\qquad x = 2$

$P(X=2) = {}^nC_x\, p^x\, q^{n-x}$

$= {}^5C_2 \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^3$

$= \frac{5\times4}{1\times2}\ \frac{1}{9} \times \frac{4}{9} \times \frac{2}{3}$

$= \frac{20\times4}{81\times3} = \frac{80}{243}$

(ii) Success - getting a century

$p = 1/3 \qquad q = 2/3$

$n=5 \qquad x = 0$

$P(X=0) = {}^nC_x\, p^x\, q^{n-x}$

$= {}^5C_0\, p^0\, q^{5-0}$

$= 1 \times \left(\frac{1}{3}\right)^0 \times \left(\frac{2}{3}\right)^5$

$= \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$

$= \frac{32}{243}$

EX4. Eight biased coins were tossed simultaneously. Find the probability of getting
(a) exactly 4 heads (b) no heads at all
(c) 6 or more heads (d) utmost 2 heads
~~(e) utmost 2 heads~~ (e) number of heads from 3 to 5

$p = \frac{1}{2}$

$q = 1 - p = 1 - 1/2 = 1/2$

$n = 8$

(a)

$P(x=4) = {}^nC_x\, p^x\, q^{n-x}$

$= {}^8C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^4$

$= \frac{8\times7\times6\times5}{1\times2\times3\times4} \times \frac{1}{2}\times\frac{1}{2}\times\frac{1}{4} \times \frac{1}{2}\times\frac{1}{2}\times\frac{1}{4}$

$= \frac{10\times7}{16\times16} = \frac{70}{256}$

(b) $P(x=0) = {}^8C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^8$

$= 1\times1\times \frac{1\times1\times1}{2\times2\times2\times2\times2\times2\times2\times2}$

$= \frac{1}{256}$

(c) $P(x=6 \text{ or } x=7 \text{ or } x=8)$

$P(x=6) + P(x=7) + P(x=8)$

$P(X=6) = {}^8C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^2$

$= \cancel{8\times 7 \times}$

${}^8C_2 \left(\frac{1}{2}\right)^6 \cdot \left(\frac{1}{2}\right)^2$

$= {}^8C_2 \left(\frac{1}{2}\right)^8$

$P(X=7) = {}^8C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^1$

$= {}^8C_7 \left(\frac{1}{2}\right)^8$

$P(X=8) = {}^8C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^0$

$= {}^8C_8 \left(\frac{1}{2}\right)^8$

P(6 or more heads)

$= {}^8C_2 \times \left(\frac{1}{2}\right)^8 + {}^8C_1 \left(\frac{1}{2}\right)^8 + 1 \times \left(\frac{1}{2}\right)^8$

$= \left(\frac{1}{2}\right)^8 \left({}^8C_2 + {}^8C_1 + 1\right)$

$= \frac{1}{256}\left(\frac{\overset{4}{\cancel{8}}\times 7}{1\times \cancel{2}} + \frac{8}{1} + 1\right)$

$= \frac{1}{256}(28+9)$

$= \frac{37}{256}$

(d) $P(x=0$ or $x=1$ or $x=2)$

$P(X=0) + P(X=1) + P(X=2)$

$P(X=0) = {}^8C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^8$

$P(X=1) = {}^8C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^7$

$P(X=2) = {}^8C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^6$

P(utmost 2 heads) =

$= {}^8C_0 \left(\frac{1}{2}\right)^8 + {}^8C_1 \left(\frac{1}{2}\right)^8 + {}^8C_2 \left(\frac{1}{2}\right)^8$

$= \left(\frac{1}{2}\right)^8 \left({}^8C_0 + {}^8C_1 + {}^8C_2\right)$

$= \left(\frac{1}{2}\right)^8 \left(1 + 8 + \frac{\overset{4}{\cancel{8}}\times 7}{1\times \cancel{2}}\right)$

$= \frac{1}{256}(9+28)$

$= \frac{37}{256}$

(e) $P(x=3$ or $x=4$ or $x=5)$

$= P(x=3) + P(x=4) + P(x=5)$

$P(x=3) = {}^8C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^5 = {}^8C_3 \left(\frac{1}{2}\right)^8$

$P(x=4) = {}^8C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^4 = {}^8C_4 \left(\frac{1}{2}\right)^8$

$P(x=5) = {}^8C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^3 = {}^8C_5 \left(\frac{1}{2}\right)^8$

P(no heads from 2 to 5)

$= 8C_3\left(\frac{1}{2}\right)^8 + 8C_4\left(\frac{1}{2}\right)^8 + 8C_5\left(\frac{1}{2}\right)^8$

$= \left(\frac{1}{2}\right)^8 \left[8C_3 + 8C_4 + 8C_3\right)$

$= \frac{1}{256}\left(\frac{8\times7\times\cancel{6}}{1\times\cancel{2}\times3} + \frac{\cancel{8}\times7\times\cancel{6}\times5}{1\times\cancel{2}\times3\times\cancel{4}} + \frac{8\times7\times\cancel{6}}{1\times\cancel{2}\times\cancel{3}}\right)$

$= \frac{1}{256}\left(56 + 70 + 56\right)$

$= \frac{182}{256}$

Ex5: Consider families with 4 children each. What ~~prob~~ percentage of families would you expect to have
(a) 2 boys and 2 girls (b) atleast one boy
(c) no girls (d) atmost two girls

$n = 4$ % = probability ×100

success = boy

$p = 1/2$ $q = 1 - p = 1 - 1/2 = 1/2$

(a) $n = 2$

$P(x=2) = 4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$

$= 4C_2\left(\frac{1}{2}\right)^4$

$= \frac{\cancel{4}\times3}{1\times2} \times \frac{1}{\cancel{16}\,4}$

$P(x=2) = \frac{3}{8}$

% 2 boys and 2 girls $= \frac{3}{8} \times 100 = \frac{75}{2}$

$= 37.5\%$

(b) $x = 1, 2, 3, 4$

$P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$P(x=1) = 4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3$

$P(x=2) = 4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$

$P(x=3) = 4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1$

$P(x=4) = 4C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0$

$P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$=$

$= \left(\frac{1}{2}\right)^4\left[4C_1 + 4C_2 + {}^4C_3 + {}^4C_4\right]$

$= \frac{1}{16}\left[4 + 6 + 4 + 1\right)$

$= \frac{1}{16} \times 15 = \frac{15}{16}$

% of atleast one boy $= \frac{15}{16} \times 100$

$= 93.75\%$

(c) $x = 4$

$P(x=4) = {}^4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0$

$= 1 \times \frac{1}{16}$

$= \frac{1}{16}$

% of no girls $= \cancel{0.0625\%}\ \frac{1}{16} \times 100$

$= 6.25\%$

Ex

(d) $P(x=4) + P(x=3) + P(x=2)$

$= {}^4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0 + {}^4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1$

$+ {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$

girls 0, 1, 2

boy 4, 3, 2

$= \left(\frac{1}{2}\right)^4 \left[{}^4C_4 + {}^4C_3 + {}^4C_2 \right]$

$= \frac{1}{16} \left(1 + 4 + \frac{4 \times 3}{1 \times 2} \right)$

$= \frac{5+6}{16} = \frac{11}{16}$

% of atmost 2 girls $= \frac{11}{16} \times 100$

$= 68.75\%$

Ex5. Four dice are thrown 162 times. The occurrence of 2 or 3 is considered as success. In how many throws do you expect (a) exactly 2 success
(b) atleast 1 success

(total no)
$N = 162$

$P \times N$

$n = 4 \qquad p = \frac{1}{3}$

$x = 2 \qquad q = 1 - \frac{1}{3} = 2/3$

$P(x=2) = {}^4C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2$

$= \frac{4 \times 3}{1 \times 2} \times \frac{1}{9} \times \frac{4}{9}$

$= \frac{4 \times 2}{27} = \frac{8}{27}$

No. of throws $= \frac{8}{27} \times 162$

$= 48$

(b) $P(x=1)$ or $P(x=2)$ or $P(x=3)$ or $P(x=4)$

$= 1 - P(x) = 0)$

$$= 1 - \left({}^{4}C_0 \left(\frac{1}{3}\right)^0 \cdot \left(\frac{2}{3}\right)^4 \right)$$

$$= 1 - \left(1 \times 1 \times \frac{2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3} \right)$$

$$= 1 - \frac{16}{81} = \frac{81 - 16}{81} = \frac{65}{81}$$

no of throws $= \frac{65}{81} \times 162$

$$= \underline{\underline{130}}$$

Problems related to mean & variance

E.

Ex1. For a binomial distribution, mean = 4, variance = 12/9. write down all the terms of the distribution

$m = np$

$SD = \sqrt{npq}$

$V = npq$

Mean = 4

variance $= \frac{12}{9}$

$np = 4$ —— ①

$npq = \frac{12}{9}$ —— ②

$\frac{②}{①} \Rightarrow \frac{npq}{np} = \frac{12}{9} \times \frac{1}{4}$

$$q = \frac{1}{3}$$

$p = 1 - q = 1 - 1/3 = \frac{2}{3}$

$np = 4$; $p = \frac{2}{3}$

$\frac{np}{p} = 4 \times \frac{3}{2}$

$$\underline{\underline{n = 6}}$$

$P(x=0) = {}^{6}C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^6$

$= 1 \times 1 \times \frac{1}{729}$

$P(x=0) = \underline{\underline{\frac{1}{729}}}$

$P(x=1) = {}^{6}C_1 \left(\frac{2}{3}\right)^1 \left(\frac{1}{3}\right)^5$

$= 6 \times \frac{2}{3} \times \left(\frac{1}{3}\right)^5$

$= 6 \times \frac{2}{729}$

$P(x=1) = \underline{\underline{\frac{12}{729}}}$

$P(x=2) = {}^{6}C_2 \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right)^4$

$= \frac{\cancel{6}^{3} \times 5}{1 \times 2} \times \frac{4}{729}$

$P(x=2) = \underline{\underline{\frac{60}{729}}}$

$P(x=3) = {}^6C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^3$

$= \frac{6 \times 5 \times 4}{1 \times 2 \times 3} \times \frac{8}{729}$

$P(x=3) = \frac{160}{729}$

$P(x=4) = {}^6C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2$

$= \frac{6 \times 5}{1 \times 2} \; \frac{16}{729}$

$P(x=4) = \frac{240}{729}$

Ex

$P(x=5) = {}^6C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^1$

$= 6 \times \frac{16 \times 2}{729}$

$P(x=5) = \frac{192}{729}$

$P(x=6) = {}^6C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^0$

$= 1 \times \frac{64}{729}$

$P(x=6) = \frac{64}{729}$

Ex2. In a binomial distribution consisting of 5 independent trials first and second terms are 0.4096 and 0.2048. Find p

$n = 5 \qquad p = ?$

$P(x=0) = {}^nC_0 \, p^0 q^{n-0}$

$= {}^5C_0 \, p^0 \cdot q^5$

$= 1 \times 1 \times q^5$

$P(x=0) = q^5$

$P(x=1) = {}^nC_1 \, p^1 q^{n-1}$

$= {}^5C_1 \, p^1 q^5$

$P(x=1) = 5 \times p \times q^4$

$q^5 = 0.4096$ — ①

$5pq^4 = 0.2048$ — ②

② ÷ ①

$\frac{5pq^4}{q^5} = \frac{0.2046}{0.4096}$

$\frac{5p}{q} = \frac{0.2048}{0.4096}$

$\frac{5p}{q} = \frac{1}{2}$

$10p = q \qquad (\because q = 1-p)$

$10p = (1-p)$

$11p = 1 \qquad p = 1/11$

Ex3. Bring out the fallacy: The mean of a binomial distribution is 5 and SD = 3

mean, $np = 5$ — ① $SD = \sqrt{npq}$

$\sqrt{npq} = 3$

② ÷ ① $npq = 9$ — ②

$\frac{npq}{np} = \frac{9}{5}$

$q = 1.8$ (this is wrong because q and p must be less than 1 ie $0 \leq p \leq 1$)

E. This is not possible.

Ex 4 Find the binomial distribution with mean 3 and variance 2 ?

mean, $np = 3$
Variance, $npq = 2$

$\frac{npq}{np} = \frac{2}{3}$

$q = \frac{2}{3}$

$p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$

$np = 3$

$n \times \frac{1}{3} = n$; $n = 9$ value of $n = 9$

Binomial destrubution $P(x) = {}^nC_x\, p^x \cdot q^{n-x}$

$x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.$

$P(x=0) = {}^9C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^9$

$= 1 \times 1 \times \frac{512}{19683}$

$P(x=0) = \frac{512}{19683}$

$P(x=1) = {}^9C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^8$

$= 9 \times \left(\frac{1}{3}\right)^9 \times 256$

$P(x=1) = \frac{2304}{19683}$

$P(x=2) = {}^9C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^7$

$= \frac{9 \times \cancel{8}^{4}}{1 \times \cancel{2}} \times \frac{128}{19683}$

$P(x=2) = \frac{4608}{19683}$

$P(x=3) = {}^9C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^6$

$= \frac{\cancel{9}^{3} \times \cancel{8}^{4} \times 7}{1 \times \cancel{2} \times \cancel{3}} \times \frac{64}{19683}$

$P(x=3) = \frac{5376}{19683}$

$P(x=4) = {}^9C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^5$

$= \frac{9 \times \cancel{7} \times \cancel{6} \times 5}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \times \frac{\cancel{32}\ 8}{19683}$

$= \frac{2520}{19683}$

$P(x=5) = {}^9C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^4$

$= \frac{9 \times 7 \times \cancel{6} \times 5}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \times \frac{\cancel{16}\ 4}{19683}$

$P(x=5) = \frac{1260}{19683}$

$P(x=6) = {}^9C_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^3$

$= \frac{9 \times 7 \times \cancel{6}}{1 \times \cancel{2} \times \cancel{3}} \times \frac{8}{19683}$

$P(x=6) = \frac{504}{19683}$

$P(x=7) = {}^9C_7 \left(\frac{1}{3}\right)^7 \left(\frac{2}{3}\right)^2$

$= \frac{9 \times 7}{1 \times \cancel{2}} \times \frac{\cancel{4}\ 2}{19683}$

$= \frac{126}{19683}$

$P(x=8) = {}^9C_8 \left(\frac{1}{3}\right)^8 \left(\frac{2}{3}\right)^1$

$P(x=8) = 9 \times \frac{2}{19683} = \frac{18}{19683}$

$=$

$P(x=9) = {}^9C_9 (1/3)^9 (2/3)^0$

$P(x=9) = 1 \times \frac{1}{19683} = \frac{1}{19683}$

Ex 5. For a binomial distribution with $n=6$, the 3rd term is 9 times the 5th term. Find p

$n = 6$

3rd term = 9 x 5th term

$x = 0, 1, 2, 3, 4, 5, 6$

$P(x=2) = 9 \times P(x=4)$

${}^6C_2 p^2 q^4 = 9 \times {}^6C_4 p^4 q^2$

$\cancel{\frac{6 \times 5}{1 \times 2}} \times p^2 \times q^4 = 9 \times \cancel{\frac{6 \times 5}{1 \times 2}} \times p^4 \times q^2$

$\cancel{p^2} \times q^2 \times \cancel{q^2} = 9 \times \cancel{p^2} \times \cancel{q^2} \times p^2$

$q^2 = 9 \times p^2$

$q = 3p$ $\therefore$

$3p = 1 - p$

$4p = 1$

$p = 1/4$

Ques 3% of a given lot of manufactured parts are defective. what is the probability that in a sample of 4 items none will be defective?

success = getting non defective

$n = 4$

$q = 0.03 \quad p = 0.97 \quad x = 4$

$P(x=4) = {}^nC_x\, p^x q^{n-x}$

$= {}^4C_4 \cdot (.97)^4 (.03)^0$

$= 1 \times .88529281 \times 1$

$= .88529281$

(1) The probability that a batsman scores a century in a cricket match is 1/3. What is the probability that in 4 matches he will score centuries in atleast 3 innings?

success → scoring a century

$n = 4, \; p = 1/3, \; q = 1 - 1/3 = 2/3$

$x = 3, 4$

$P(x = 3 \text{ or } x = 4) = P(x=3) + P(x=4)$

$= {}^nC_x\, p^x q^{n-x}$

$= 4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{2}{3}\right)^1 + 4C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0$

$= 4 \times 2 \times \left(\frac{1}{3}\right)^4 + 1 \cdot \left(\frac{1}{3}\right)^4$

$= \left(\frac{1}{3}\right)^4 (8+1)$

$= \frac{9}{81} = \frac{1}{9}$

(2) A box containing 100 transistors, 20 of which are defective, 10 are selected for inspection. Indicate what is the probability that (i) all 10 are defective (ii) 10 are good (iii) at least one is defective and (iv) at the most three are defective?

success → getting defective

$p = \frac{20}{100} = \frac{1}{5} \qquad q = 1 - \frac{1}{5} = \frac{4}{5}$

$n = 10$

(a) $P(x=10) = {}^{10}C_{10} \left(\frac{1}{5}\right)^{10} \left(\frac{4}{5}\right)^0$

$= 1 \times \frac{1}{9765625} = \frac{1}{9765625}$

(b) $P(x=0) = {}^{10}C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{10}$

$= 1 \times \frac{1048576}{9765625}$

$= \frac{1048576}{9765625}$

(c) P (atleast 1 defective) ie P (atleast one success)

$= 1 - P(x=0)$

$= 1 - {}^{10}C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{15}\right)^{10}$

$= 1 - 1 \times 1 \times \frac{1048576}{9765625}$

$= \frac{9765625 - 1048576}{9765625}$

$= \frac{8717049}{9765625}$

(d) $P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= {}^{10}C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{10} + {}^{10}C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^9 + {}^{10}C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^8$

$+ {}^{10}C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^7$

$= 1 \times \frac{4\times4\times4\times4\times4\times4\times4\times4\times4\times4}{5\times5\times5\times5\times5\times5\times5\times5\times5\times5} + \overset{2}{\cancel{10}} \times \frac{1}{\cancel{5}} \times \frac{4\times4\times4\times4\times4\times4\times4\times4\times4}{5\times5\times5\times5\times5\times5\times5\times5\times5} + \frac{\overset{5}{\cancel{10}}\times9}{1\times\cancel{2}} \times$

$\frac{4\times4\times4\times4\times4\times4\times4\times4}{5\times5\times5\times5\times5\times5\times5\times5\times5\times5} + \frac{\overset{5}{\cancel{10}}\times\overset{3}{\cancel{9}}\times8}{1\times\cancel{2}\times\cancel{3}} \times \frac{4\times4\times4\times4\times4\times4\times4}{5\times5\times5\times5\times5\times5\times5\times5\times5\times5}$

$= \left(\frac{1}{5}\right)^{10} \left[\begin{array}{l} 1\times1048576 + 2\times262144 + \\ 45\times65536 + 120\times16384 \end{array} \right]$

$= \frac{1}{9765625} \left[\begin{array}{l} 1048576 + 524288 + 2949120 \\ + 1966080 \end{array} \right]$

$= \frac{6488064}{9765625}$

③ The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. What is the probability that out of six workmen, 4 or more will contract the disease?

success → suffering from disease

$n = 6 \quad p = \frac{10}{100} = 0.1 \quad q = 1 - 0.1 = 0.9$

$x = 4, 5, 6$

$P(x=4) + P(x=5) + P(x=6)$

$= {}^6C_4 (\cdot1)^4 (0.9)^2 + {}^6C_5 (\cdot1)^5 (0.9)^1 + {}^6C_6 (\cdot1)^6 (\cdot9)^0$

$= 6C_2 \times 0.0001 \times 0.81 + 6C_1 \times 0.00001 \times 0.9$

$+ 1 \times 0.000001 \times 1$

$$= \frac{\cancel{6}^{3} \times 5}{1 \times 2} \times 0.000081 + 6 \times 0.000009 + 0.000001$$

$$= 0.01215 + 0.000054 + 0.000001$$

$$= 0.012205$$

(4) The administrator of a large airport is interested in the number of aircraft departure delays that are attributable to inadequate control facilities. A random sample of 8 aircrafts take offs is to be throughly investigated. If true proportion of such delays in all departures is 0.40, what is the probability that 3 of the sample departure are delayed because of control inadequacies?

$n = 8$ $p \rightarrow$ delayed departure

$p = 0.40$ $q = 1 - p = 0.40$

$= 0.60$

$x = 3$

$$P(x=3) = {}^{8}C_3 (0.4)^3 (0.6)^5$$

$$= \frac{\cancel{8}^{4} \times 7 \times \cancel{6}^{2}}{1 \times \cancel{2} \times \cancel{3}} \times 0.064 \times 0.07776$$

$$= 56 \times 0.064 \times 0.07776$$

$$= 0.27869$$

(5) A coin is tossed 6 times. What is the probability of obtaining five more heads.

$n = 6$ $p \rightarrow$ obtaining head

$p = \frac{1}{2}$ $q = \frac{1}{2}$

$x = 5, 6$

$$P(x=5) + P(x=6)$$

$$= {}^{6}C_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^1 + {}^{6}C_6 \left(\tfrac{1}{2}\right)^6 \left(\tfrac{1}{2}\right)^0$$

$$= 6 \cdot \left(\tfrac{1}{2}\right)^6 + 1 \times \left(\tfrac{1}{2}\right)^6$$

$$= \left(\tfrac{1}{2}\right)^6 (6+1)$$

$$= \frac{7}{64}$$

(6) 16 unbiased coins are tossed once. What are the probabilities of obtaining (i) exactly 5 heads (ii) 14 or more heads

$n = 16$ $p = 1/2$ $q = 1/2$ $p \rightarrow$ obtaining head

(i) $P(x=5)$ ie $x = 5$

$$= {}^{16}C_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^{11}$$

$$= \frac{\cancel{16}^{4} \times \cancel{15}^{3} \times 14 \times 13 \times \cancel{12}^{2}}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5}} \times \left(\tfrac{1}{2}\right)^{16}$$

$$= \frac{4368}{65536}$$

(i) $x = 14, 15, 16$

$P(x=14) + P(x=15) + P(x=16)$

$= {}^{16}C_{14}\left(\frac{1}{2}\right)^{14}\left(\frac{1}{2}\right)^{2} + {}^{16}C_{15}\left(\frac{1}{2}\right)^{15}\left(\frac{1}{2}\right)^{1} +$

${}^{16}C_{16}\left(\frac{1}{2}\right)^{16}\left(\frac{1}{2}\right)^{0}$

$= \frac{\cancel{16}^{8} \times 15}{1 \times \cancel{2}} \times \left(\frac{1}{2}\right)^{16} + 16 \times \left(\frac{1}{2}\right)^{16} + 1 \times \left(\frac{1}{2}\right)^{16}$

$= \left(\frac{1}{2}\right)^{16}\left(120 + 16 + 1\right)$

$= \frac{137}{65536}$

② The incidence of occupational disease in an industry is such that the workmen have a 20% chance of suffering from it. 5 workers are chosen at random. Find the probability that (i) none of the 5 chosen workers would be suffering from the disease (ii) exactly 2 of them would be suffering (iii) atleast 3 of them would be suffering

$n = 5$ $\quad$ $p \rightarrow$ suffering

$p = \frac{20}{100} = \frac{1}{5}$ $\quad$ $q = 1 - \frac{1}{5} = \frac{4}{5}$

(i) $x = 0$

$P(x=0) = {}^{5}C_{0}\left(\frac{1}{5}\right)^{0}\left(\frac{4}{5}\right)^{5}$

$= 1 \times 1 \times \left(\frac{4}{5}\right)^{5}$

$= \frac{1024}{3125}$

(ii) $x = 2$

$P(x=0) = {}^{5}C_{2}\left(\frac{1}{5}\right)^{2}\left(\frac{4}{5}\right)^{3}$

$= \frac{5 \times \cancel{4}^{2}}{1 \times \cancel{2}} \times \left(\frac{1}{5}\right)^{5} \times (4)^{3}$

$= \frac{5 \times 64}{3125} = \frac{320}{3125}$

(iii) $x = 3, 4, 5$

$P(x=3) + P(x=4) + P(x=5)$

$= {}^{5}C_{3}\left(\frac{1}{5}\right)^{3}\left(\frac{4}{5}\right)^{2} + {}^{5}C_{4}\left(\frac{1}{5}\right)^{4}\left(\frac{4}{5}\right)^{1} + {}^{5}C_{5}\left(\frac{1}{5}\right)^{5}$

$= \frac{5 \times \cancel{4}^{2}}{1 \times \cancel{2}}(4^{2}) \times \left(\frac{1}{5}\right)^{5} + 5 \times 4 \times \left(\frac{1}{5}\right)^{5} + 1 \times \left(\frac{1}{5}\right)^{5}$

$= \left(\frac{1}{5}\right)^{5}\left(160 + 20 + 1\right)$

$= \frac{181}{3125}$

(8) The odds in favour of X's winning a game against Y are 4:3. Find the probability of Y's winning
(i) exactly 3 games (ii) atleast 5 games (iii) atmost 2 games, out of 7 games played.

(hint $p = 3/7,\ q = 4/7$)

$n = 7 \quad p = 3/7 \quad q = 4/7$

(i) $x = 3$

$P(x=3) = {}^7C_3 \left(\frac{3}{7}\right)^3 \left(\frac{4}{7}\right)^4$

$= \frac{7\times6\times5}{1\times2\times3} \times \frac{3\times3\times3\times4\times4\times4\times4}{823543}$

$= \frac{35\times27\times256}{823543}$

$= \frac{241920}{823543}$

(ii) $x = 5, 6, 7$

$P(x=5) + P(x=6) + P(x=7)$

$= {}^7C_5 \left(\frac{3}{7}\right)^5 \left(\frac{4}{7}\right)^2 + {}^7C_6 \left(\frac{3}{7}\right)^6 \left(\frac{4}{7}\right)^1 + {}^7C_7 \left(\frac{3}{7}\right)^7 \left(\frac{4}{7}\right)^0$

$= \frac{7\times6}{1\times2} \times \frac{3^5\times4^2}{7^7} + \frac{7\times3^6\times4}{7^7} + \frac{1\times3^7}{7^7}$

$= \left(\frac{1}{7}\right)^7 \left((21\times243\times16) + (28\times729) + 2187\right)$

$= \frac{1}{823543} \times (81648 + 20412 + 2187)$

$= \frac{104247}{823543}$

(iii) $x = 0, 1, 2$

$P(x=0) + P(x=1) + P(x=2)$

$= {}^7C_0 \left(\frac{3}{7}\right)^0 \left(\frac{4}{7}\right)^7 + {}^7C_1 \left(\frac{3}{7}\right)^1 \left(\frac{4}{7}\right)^6 + {}^7C_2 \left(\frac{3}{7}\right)^2 \left(\frac{4}{7}\right)^5$

$= \left(\frac{1}{7}\right)^7 \left(1\times(4)^7 + 7\times3\times(4)^6 + \frac{7\times6}{1\times2}\times3^2\times4^5\right)$

$= \left(\frac{1}{7}\right)^7 \left(16384 + 21\times4096 + 21\times9\times1024\right)$

$= \left(\frac{1}{7}\right)^7 \left(16384 + 86016 + 193536\right)$

$= \frac{295936}{823543}$

(9) The overall percentage of failure in a certain examination is 40, what is the probability that out of a group of 6 candidates atleast 4 passed examination?

$n = 6$ $\qquad$ p → candidates pass exam

$p = \frac{60}{100} = 0.6$

$q = 0.4$

$n = 4, 5, 6$

$P(x=4) + P(x=5) + P(x=6)$

$= {}^6C_4 (0.6)^4 (0.4)^2 + {}^6C_5 (0.6)^5 (0.4)^1 + {}^6C_6 (0.6)^6$

$= \frac{\cancel{6}^{3} \times 5}{\cancel{2}} \times 0.1296 \times 0.16 + 6 \times 0.07776 \times 0.4 + 1 \times 0.0466$

$= 0.31104 + 0.18662 + 0.0466$

$= 0.54426$

(10) From the production process which turns 5% defective on an overage, a simple of size 10 is drawn. Find the probability that the sample contains
(i) no defective (ii) atmost one defective
(iii) atleast one defective

$n = 10$ $\qquad p \to$ defective

$p = 5/100 = 0.05$ $\qquad q = 1 - 0.05$
$q = 0.95$

(i) $n = 0$

$P(x=0) = {}^{10}C_0 (0.05)^0 (0.95)^{10}$

$= 1 \times 0.5987$
$= 0.5987$

(ii) $n = 1$

$P(x=1) = {}^{10}C_1 (0.05)^1 (0.95)^9$
$= 10 \times 0.05 \times 0.630249$
$= 0.315$

(iii) atleast one defective
P(atleast 1 defective)
$= 1 - P$(no defective
$= 1 - P(0)$
$= 1 - 0.5987$
$= 0.4013$

(11) In how many cases should we expect to get 6 heads and 4 tails if 10 coins are simultaneously tossed 1000 times.

$n = 10$ $\qquad p \to$ getting head

$p = 1/2$ $\qquad q = 1/2$

$n = 6$

$P(x=6) = {}^{10}C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^4$

$= \frac{\cancel{10}^{5} \times \cancel{9}^{3} \times \cancel{8}^{2} \times 7}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \times \left(\frac{1}{2}\right)^{10}$

$= \frac{210}{1024}$

1000 times tossed then,

$= \frac{210}{1024} \times 1000$

$= \frac{210000}{1024}$

$= 205 \cdot \cancel{078125}$

(12) 5 dice are thrown 150 times. The occurence of an odd face is considered a success. In how many throws do you expect (i) less than 4 success (ii) atleast 3 success (iii) exactly one success

$n = 5$ $\quad$ $p \rightarrow$ success of odd face occurs

$p = 1/2$

$q = 1 - 1/2 = 1/2$

(i) $x = 0, 1, 2, 3$

$P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 + 5C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4 + 5C_2 \left(\frac{1}{2}\right)^5 + 5C_3 \left(\frac{1}{2}\right)^5$

$= 1 \times \left(\frac{1}{2}\right)^5 + 5 \times \left(\frac{1}{2}\right)^5 + \frac{5 \times 4^2}{1 \times 2} \left(\frac{1}{2}\right)^5 + \frac{5 \times 4^2}{1 \times 2}$

$= \left(\frac{1}{2}\right)^5 \left(1 + 5 + 10 + 10\right)$

$= \frac{26}{32}$

how many throws

$= 150 \times \frac{26}{32}$

$= \frac{3900}{32}$

$= 122$ throws

(ii) $x = 3, 4, 5$

$P(x=3) + P(x=4) + P(x=5)$

$= {}^5C_3 \left(\frac{1}{2}\right)^5 + 5C_4 \left(\frac{1}{2}\right)^5 + 5C_5 \left(\frac{1}{2}\right)^5$

$= \left(\frac{1}{2}\right)^5 \left(\frac{5 \times 4^2}{1 \times 2} + 5 + 1\right)$

$= \left(\frac{1}{2}\right)^5 (10 + 6)$

$= \frac{16}{32}$

how many throws $= \frac{16}{32} \times 150$

$= \frac{2400}{32}$

$= 75$ throws

(iii) $x = 1$

$P(x=1) = {}^5C_1 \left(\frac{1}{2}\right)^5$

$= \frac{5 \times 1}{32}$

$= \frac{5}{32}$

how many throws $= \frac{5}{32} \times 150$

$= \frac{750}{32}$

$= 23$ throws

(5) Take 100 sets of 10 tosses of an unbiased coin. In how many cases do you expect to get (i) 6 H and 4 T (ii) atleast 9 H

$n = 10$ $\quad$ $p \to$ getting head

$N = 100$ $\quad$ $p = 1/2$ $\quad$ $q = 1/2$

i) $x = 6$

$P(x=6) = {}^{10}C_6 \cdot \left(\frac{1}{2}\right)^{10}$

$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2} \times 0.03125 \times 0.03125$

$= 0.205$

$\Rightarrow 0.205 \times 100$

205 cases

(ii) $x = 9, 10$

$P(x=10) + P(x=9)$

$= {}^{10}C_9 (1/2)^{10} + {}^{10}C_{10} (1/2)^{10}$

$= 10 + 1 \; (1/2)^{10}$

$= 11 \times 0.03125 \times 0.03125$

$= 0.0107$

no of cases $\Rightarrow$ 0.0107×100

$= 10.2$

11 cases

Fitting a Binomial Distribution

1. Determine the values of p and q and n and substitute them in the function ${}^nC_x \, p^x q^{n-x}$, we get the probability function of the Binomial Distribution

2. Put $x = 0, 1, 2 \cdots$ in the function ${}^nC_x \, p^x q^{n-x}$, we get $n+1$ terms

3. Multiply each such term by N (total frequency) to obtain the expected frequency

Ex 1. 8 coins were tossed together, 256 times. Find the expected frequencies of Heads. Find mean and SD.

$p = P(\text{getting head in a toss}) = 1/2$

$q = 1 - p = 1/2$ $\qquad N = 256$

$n = 8$

The binomial distribution is $P(x) = {}^8C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{8-x}$

putting $x = 0, 1, 2, 3 \ldots 8$ we get all the terms of Binomial distribution

No of heads (x)	$P(x)$	Expected frequency $P(x) \times 256$
0	${}^8C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^8 = 1/256$	1
1	${}^8C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^7 = 8/256$	8
2	${}^8C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^6 = 28/256$	28
3	${}^8C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^5 = 56/256$	56
4	${}^8C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^4 = 70/256$	70
5	${}^8C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3 = 56/256$	56
6	${}^8C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^2 = 28/256$	28
7	${}^8C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^1 = 8/256$	8
8	${}^8C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^0 = 1/256$	1

Mean $= np = 8 \times \frac{1}{2} = \underline{\underline{4}}$

$SD = \sqrt{npq} = \sqrt{8 \times (1/2) \times (1/2)} = \sqrt{2} = \underline{\underline{1.414}}$

Ex 2. The following data shows the number of seeds germinating out of 5 lb damp filter for 80 sets of seeds. Fit a binomial distribution of this data and find the expected frequencies

x	0	1	2	3	4	5
f	6	20	28	12	8	6

→ Since p is not given, find mean of given data

x	f	fx
0	6	0
1	20	20
2	28	56
3	12	36
4	8	32
5	6	30
	80	174

Mean $= \frac{\Sigma fx}{\Sigma f} = \frac{174}{80} = \underline{\underline{2.175}}$

ie $np = 2.175$, we know $n = 5$

$\therefore p = \frac{2.175}{5}$

$p = 0.435$ $\quad ; \quad q = 1 - p = 0.565$

$\therefore$ Binomial distribution is ${}^5C_x (0.435)^x (0.565)^{5-x}$ for $x = 0, 1, 2, 3, 4, 5$

Expected frequencies are obtained from $N \times {}^nC_x\, p^x q^{n-x}$ where

$N = 80$ $n = 5$ $p = 0.435$ $q = 0.565$

and $x = 0, 1, 2 \ldots 5$.

X	P(X)	Expected frequency $f = P(x) \times 80$
0	${}^5C_0 (0.435)^0 (0.565)^5 = 0.0625$	5
1	${}^5C_1 (0.435)^1 (0.565)^4 = 0.225$	18
2	${}^5C_2 (0.435)^2 (0.565)^3 = 0.3375$	27
3	${}^5C_3 (0.435)^3 (0.565)^2 = 0.2625$	21
4	${}^5C_4 (0.435)^4 (0.565)^1 = 0.075$	6
5	${}^5C_5 (0.435)^5 (0.565)^0 = 0.0875$	7

EX 3 The screws produced by certain machine were checked by examining samples. The following table shows the distribution of 128 sample according to the number of defective items they contained.

No of defective :	0	1	2	3	4	5	6	7
No of samples :	7	6	19	35	30	23	7	1

Fit a binomial distribution to find the mean and variance of the distribution.

x	f	fx
0	7	0
1	6	6
2	19	38
3	35	105
4	30	120
5	23	115
6	7	42
7	1	7
	128	433

$n = 7$

$N = 128$

$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{433}{128} = 3.383$

mean, $np = 3.383$

$n \times p = 3.383$

$p = \frac{3.383}{7}$

$p = 0.483$

Variance, $npq = 0.483 \times 0.517 = 0.249711$

$q = 1 - p = 1 - 0.483 = 0.517$

$\therefore$ Binomial distribution is ${}^7C_x (0.483)^x (0.517)^{7-x}$ where $x = 0, 1, 2, 3, 4, 5, 6, 7$

X	P(X)	Expected frequency $P(x) \times 128$
0	${}^7C_0 (0.483)^0 (0.517)^7 = .00987$	1.26336
1	${}^7C_1 (0.483)^1 (0.517)^6 = .06454$	8.26112
2	${}^7C_2 (0.483)^2 (0.517)^5 = .18081$	23.14368
3	${}^7C_3 (0.483)^3 (0.517)^4 = .28175$	36.064
4	${}^7C_4 (0.483)^4 (0.517)^3 = .2632$	33.6896
5	${}^7C_5 (0.483)^5 (0.517)^2 = .14742$	18.86976
6	${}^7C_6 (0.483)^6 (0.517)^1 = .04592$	5.87776
7	${}^7C_7 (0.483)^7 (0.517)^0 = .00613$	0.78464

Ex4 5 coins are tossed 3200 times. Find the expected frequencies of success (getting a head). Tabulate the results

$p(\text{getting a head}) = p = 1/2$

$q = 1-p = 1/2$

$N = 3200$

$n = 5$

$P(x) = {}^5C_x (1/2)^x (1/2)^{5-x}$

x	$P(x)$	Expected frequency $P(x) \times 3200$
0	${}^5C_0 (1/2)^0 (1/2)^5 = 1/32$	100
1	${}^5C_1 (1/2)^1 (1/2)^4 = 5/32$	500
2	${}^5C_2 (1/2)^2 (1/2)^3 = 10/32$	1000
3	${}^5C_3 (1/2)^3 (1/2)^2 = 10/32$	1000
4	${}^5C_4 (1/2)^4 (1/2)^1 = 5/32$	500
5	${}^5C_5 (1/2)^5 (1/2)^0 = 1/32$	100

Ex5 For a binomial distribution, mean is 6 and standard deviation is $\sqrt{2}$. Find the parameters. Write down all the terms of distribution

$\text{mean} = np = 6$ ——— (1)

$SD = \sqrt{npq} = \sqrt{2}$

$npq = 2$ ——— (2)

(2) ÷ (1)

$\frac{npq}{np} = \frac{2}{6} = \frac{1}{3}$

$q = 1/3 \qquad p = 1 - q = 1 - 1/3 = \underline{2/3}$

$np = 6$

$n = \frac{6}{p} = 6 \times \frac{3}{2}$

$\underline{\underline{n = 9}}$

$P(x) = {}^nC_x\, p^x \cdot q^{n-x}$

$P(x=0) = {}^9C_0 (2/3)^0 (1/3)^9 = \frac{1}{19683}$

$P(x=1) = {}^9C_1 (2/3)^1 (1/3)^8 = \frac{2}{19683}$

$P(x=2) = {}^9C_2 (2/3)^2 (1/3)^7 = \frac{4}{19683}$

$P(x=3) = {}^9C_3 (2/3)^3 (1/3)^6 = \frac{8}{19683}$

$P(x=4) = {}^9C_4 (2/3)^4 (1/3)^5 = \frac{16}{19683}$

$P(x=5) = {}^9C_5 (2/3)^5 (1/3)^4 = \frac{32}{19683}$

$P(x=6) = {}^9C_6 (2/3)^6 (1/3)^3 = \frac{64}{19683}$

$P(x=7) = {}^9C_7 (2/3)^7 (1/3)^2 = \frac{128}{19683}$

$P(x=8) = {}^9C_8 (2/3)^8 (1/3)^1 = \frac{256}{19683}$

$P(x=9) = {}^9C_9 (2/3)^9 (1/3)^0 = \frac{512}{19683}$

Ex6 Write down the fallacy if any in " The mean of a binomial distribution is 10 and SD is 4."

mean, $np = 10$ ——— (1)

SD $= \sqrt{npq} = 4$

$= npq = 16$ ——— (2)

$\frac{(2)}{(1)} = \frac{npq}{np} = \frac{16}{10}$

$q = 1.6$

(this is wrong ie q cannot be greater than 1 and in this case $q > 1$)

Ex7 Write down the form of binomial distribution with parameters $n = 4$, $p = 1/3$

$n = 4$

$p = 1/3 \quad q = 1 - 1/3 = 2/3$

$x = 0, 1, 2, 3, 4$

$P(x) = 4C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}$

$P(x=0) = {}^4C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

$P(x=1) = 4C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^3 = \frac{32}{81}$

$P(x=2) = 4C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = \frac{24}{81}$

$P(x=3) = {}^4C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^1 = \frac{8}{81}$

$P(x=4) = 4C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0 = \frac{1}{81}$

Poisson Distribution

conditions:-

(1) The number of trials is very large. (ie $n \to \infty$)

(2) The probability of success for each trial is very small ie ($p \to 0$)

(3) np is finite (say 'm' ie np = 'm')

If $p \to 0$ and $n \to \infty$

$P(x) = \frac{e^{-m} \cdot m^x}{x!}$

Q1 If 3% of electric bulbs manufactured by a company are defective, find the probability that in a sample of 100 bulbs, exactly 5 bulbs are defective?

defective - 3 %. success - getting a defective

$p = \frac{3}{100} = 0.03 \quad (p \to 0)$

$n = 100 \quad (n \to \text{large})$

$x = 5$

$P(x) = \dfrac{e^{-m} m^x}{x!}$

$\left\{ \begin{aligned} m &= np \\ &= 100 \times 0.03 \\ m &= 3 \end{aligned} \right\}$

$P(x) = \dfrac{e^{-m} m^x}{x!}$

$P(x=5) = \dfrac{e^{-3} \cdot 3^5}{5!} = \dfrac{0.0498 \times 243}{120}$

$= \dfrac{12.1014}{120} = \underline{\underline{0.100845}}$

Q2 It is known from past experience that in a certain plant there are on the average 4 industrial accidents per month. Find the probability that in a given year, there will be less than 3 accidents?

Mean $= np = m = 4$

$x = 0, 1, 2$

$P(x=0) + P(x=1) + P(x=2)$

$= \dfrac{e^{-4} \cdot 4^0}{0!} + \dfrac{e^{-4} \cdot 4^1}{1!} + \dfrac{e^{-4} \cdot 4^2}{2!}$

$= e^{-4} \left\{ 1 + 4 + \dfrac{16}{2} \right\}$

$= e^{-4} \{ 5 + 8 \}$

$= e^{-4} \times 13$

$= 0.0183 \times 13$

$= \underline{0.2379}$

Q3 A manufactures of blades knows that 5% of his product is defective. If he sells blades in boxes of 100 and guarentees that no more than 10 blades will be defective what is the probability that a box will fail to meet the guarenteed quality?

guarentee → more than 10 will not be defective
fail → more than 10 will be defective

defective = 5%. $\quad x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

$p = 5\% = \dfrac{5}{100} = 0.05 \qquad n = 100$

$m = np = 100 \times 0.05 = \underline{\underline{5}}$

$P(x > 10) = 1 - P(x \leq 10)$

$= 1 - \left\{ \dfrac{e^{-5} \times 5^x}{x!} \right\}$

$= 1 - \left\{ e^{-5} \left(\dfrac{5^0}{0!} + \dfrac{5^1}{1!} + \dfrac{5^2}{2!} + \dfrac{5^3}{3!} + \dfrac{5^4}{4!} + \dfrac{5^5}{5!} + \dfrac{5^6}{6!} + \dfrac{5^7}{7!} + \dfrac{5^8}{8!} + \dfrac{5^9}{9!} + 5^{10}/10! \right\}$

$$= 1 - \left\{ 6.74 \left(1 + \frac{5}{1} + \frac{25}{2} + \frac{125}{6} + \frac{625}{24} + \frac{3125}{120} + \frac{15625}{720} + \frac{78125}{5040} + \frac{390625}{40320} + \frac{1953125}{362880} + \frac{9765625}{3628800} \right) \right\}$$

$$= 1 - \{6.74 (1+5+12.5+20.83+26.04+26.04+21.70+15.50+9.69+5.38+2.69)$$

Q4 If a random variable x follows poisson distribution such that $P(x=1) = P(x=2)$. Find $P(x=0)$?

$$\frac{e^{-m} \cdot m^1}{1!} = \frac{e^{-m} m^2}{2!} \quad \left(\because \text{given } P(x=1) = P(x=2) \right)$$

$$\frac{1}{1} = \frac{m}{2!}$$

$$\frac{1}{1} = \frac{m}{2}$$

$m = 2$

$$P(x=0) = \frac{e^{-2} \cdot m^0}{0!} = \frac{e^{-2} \cdot 2^0}{0!}$$

$P(x=0) = 0.1353$

Q5 Out of 500 items selected for inspection 0.2% are found to be defective. Find how many lots will contain exactly no defective, if there are 1000 lots?

prob (lots contain no defective) = ?

success - defective

$$p = 0.2\% = \frac{0.2}{100} = 0.002$$

$n = 500$

$m = np = 500 \times 0.002 = 1$

$x = 0$

$$P(x=0) = \frac{e^{-1} \times 1^0}{0!} = e^{-1} = 0.3679$$

Number of lots containing no defective $= P(x=0) \times 1000$

$= 0.3679 \times 1000$

$= 367.9$

ie 368

Q6 Between the hours of 2 pm and 4 pm the average number of phone calls per minute coming into the switchboard of a company is 2.5. Find the probability that during one particular minute, there will be (1) no phone calls at all (2) exactly 2 calls (3) atleast 5 calls success - calls recieved

Here
$m = 2.5$

(1) $x = 0$
$P(\text{no phone calls}) = (x = 0)$

$P(x=0) = \frac{e^{-2.5} \cdot (2.5)^0}{0!}$ $\left(\because P(x) = \frac{e^{-m} \cdot m^x}{x!} \right)$

$= e^{-2.5} \cdot (1)$
$= e^{-2.5}$

$P(x=0) = 0.0821$

(2) $x = 2$

$P(x=2) = \frac{e^{-2.5} \cdot (2.5)^2}{2!}$

$= \frac{e^{-2.5} \cdot (2.5) \cdot (2.5)}{2}$

$= \frac{0.0821 \times 6.25}{2}$ 3.125

$P(x=2) = 0.2566$

(3) $x = 5$ or more calls

$P(x \geq 5) = 1 - P(x \leq 4)$

$P(x \geq 5) = 1 - \left(P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) \right)$

$= 1 - \left[e^{-2.5} \left(\frac{2.5^0}{0!} + \frac{(2.5)^1}{1!} + \frac{(2.5)^2}{2!} + \frac{(2.5)^3}{3!} + \frac{(2.5)^4}{4!} \right) \right]$

$= 1 - \left[0.0821 \left[1 + 2.5 + \frac{6.25}{2} + \frac{15.625}{6} + \frac{39.0625}{24} \right] \right]$

$= 1 - \left[0.0821 \left(1 + 2.5 + 3.125 + 2.604 + 1.6276 \right) \right]$

$= 1 - (0.0821 \times 10.8566)$
$= 1 - 0.8913$
$= 0.1087$

Q7 If $n = 10$ $p = 0.1$ find $P(x=2)$ by both Binomial and Poisson distributions. Show that P.D is an approximation of B.D.

$n = 10, \ p = 0.1, \ q = 0.9$

~~Poiss~~

Poisson: $P(x) = \frac{e^{-m} m^x}{x!}$

Binomial distribution: $P(x) = {}^nC_x\, p^x\, q^{n-x}$

$m = np = 10 \times 0.1$

$m = 1$

$P(x=2) = {}^{10}C_2\,(0.1)^2 (0.9)^8$

$= \frac{10 \times 9}{2} \times 0.01 \times (0.9)^8$

$= 0.19$

$P(x=2) = \frac{e^{-1}\, 1^2}{2!}$

$= 0.18 \cong 0.19$

Poisson Distribution

In Binomial distribution 'n' is finite. So it is possible to count the number of times it does not occur. Suppose an event is rare. For example, number of accidents in a road. Then it may be possible to count the number of times an event occurs but not possible to count the number of times it does not occur.

This is because total number of trials of the experiment is not precisely known. Poisson distribution is a distribution suitable to such situations.

Poisson distribution as a limiting case of Binomial distribution

Poisson distribution may be obtained as a limiting case of Binomial distribution under the following conditions:

(1) The number of trials is very large (ie $n \to \infty$)

(2) The probability of success for each trial is very small (ie $p \to 0$)

(3) 'np' is finite (say 'm')

∴ Poisson distribution can be stated in the following manner

Let the probability for success in a single trial of a random experiment is very small (ie $p \to 0$) and the experiment be repeated larger number of times (ie $n \to \infty$)

Then the probability for x successes out of these 'n' trials is given by :P

$$P(x) = \frac{e^{-m}\, m^x}{x!}$$

This law is called Poisson Law

Meaning & definition

A discrete random variable 'x' is said to follow Poisson distribution if its prob. density function is $P(x) = \frac{e^{-m}\, m^x}{x!}$, for x assuming values 0, 1, 2 . . . .

Poisson distribution may be expected in cases where the chances of happening of any individual event is small.

So the distribution is used to describe the behavior of rare events. ∴ it is called the 'law' of improbable events.

Putting $x = 0, 1, 2, 3 \ldots$ in the probability function

$P(x) = \frac{e^{-m} m^x}{x!}$, we obtain the prob. for $0, 1, 2, 3 \ldots$ times the event to happen.

If we know 'm' all the terms of the Poisson distribution can be obtained.

'm' is the parameter of the Poisson distribution. It is the mean of the poisson distribution.

Uses (or Importance) of Poisson Distribution

Poisson distribution can be used to explain the behavior of the discrete random variables where the probability of occurrence of the event is very small and the total number of possible cases is sufficiently large.

As such, Poisson distribution has found applications in Queuing Theory (waiting line problems) and in the fields of insurance, Physics, Biology, ~~Econo~~ Economics, Industry etc.

Practical situations where Poisson distribution can be used

(1) to count the number of telephone calls arriving at a telephone switch board in unit time (say, per minute)

(2) to count the number of customers arriving at the super market (say per hour)

(3) to count the number of defects per unit of a manufactured product, (in Statistical Quality control)

(4) to count the number of radio-active disintegrations of a radio-active element per unit of time (in physics)

(5) to count the number of bacterias per unit (in Biology)

(6) to count the number of defective materials say, pins, blades etc (in a packing of manufactured goods by a concern)

(7) to count the number of casualities (persons

dying) due to a rare disease such as heart attack in a year

(g) to count the number of accidents taking place in a day on a busy road.

Characteristics (or properties) of Poisson Distribution

1. Poisson distribution is a discrete probability distribution

2. If 'x' follows a Poisson distribution, then 'x' takes values 0, 1, 2 ... to infinity

3. Poisson Distribution has a single parameter, m. When 'm' is known all the terms can be found out

4. Mean and variance of Poisson distribution are equal to m.

5. Poisson distribution is a positively skewed distribution

NOTE !

$x! = x(x-1)(x-2) \cdots 2 \times 1$

eg.- $3! = 3 \times 2 \times 1 = 6$

Mean $= \mu_1 = m$

Variance $= \mu_2 = m$

Standard deviation $= \sqrt{m}$

$$\sum_{x=0}^{\infty} \frac{m^x}{x!} = e^m$$

$$\sum_{x=0}^{\infty} \frac{m^{x-1}}{(x-1)!} = e^m$$

$$\sum_{x=0}^{\infty} \frac{m^{x-2}}{(x-2)!} = e^m$$

$\mu_1' = \sum x f(x)$ Mean = Variance in poisson distribution

$\mu_2' = \sum x^2 f(x)$

$\mu_2 = \mu_2' - (\mu_1')^2$

Mean of a poisson distribution

Mean $= \mu_1 = \sum x f(x)$ $\left(\because x! = x \times (x-1) \times (x-2) \cdots 2 \times 1\right)$

$$= \sum_{x=0}^{\infty} \frac{\cancel{x} \cdot e^{-m} \cdot m^x}{\cancel{x}\,(x-1)!} \quad \left(\because m^1 \times m^{x-1} = m^{1+x-1} = m^x\right)$$

$$= \sum_{x=0}^{\infty} \frac{e^{-m} \cdot m^1 \times m^{x-1}}{(x-1)!}$$

$$= e^{-m} \cdot m^1 \sum_{x=0}^{\infty} \frac{m^{x-1}}{(x-1)!} \quad \left(\because \sum_{x=0}^{\infty} \frac{m^{x-1}}{(x-1)!} = e^m \right)$$

$$= e^{-m} \cdot m^1 \cdot e^m \quad \left(\because e^{-m} \cdot e^m = e^{-m+m} = e^0 = 1 \right)$$

$$= m$$

Variance of a poisson distribution

$$\mu_2' = \sum_{x=0}^{\infty} x^2 \cdot f(x) \quad \left(\because x^2 = x(x-1)+x = x^2 - x + x \right)$$

$$= \sum_{x=0}^{\infty} \left[x(x-1) + x \right] f(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) f(x) + \underbrace{\sum_{x=0}^{\infty} x f(x)}_{m}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-m} \cdot m^x}{x!} + m$$

$$= \sum_{x=0}^{\infty} e^{-m} \cdot \frac{m^2 \cdot m^{x-2}}{(x-2)!} + m \quad \left(\because m^x = m^{x-2} \cdot m^2 = m^{x-2+2} \right)$$

$$= e^{-m} \cdot m^2 \left[\sum_{x=0}^{\infty} \frac{m^{x-2}}{(x-2)!} \right] + m \quad \left(\text{since } \sum = e^m \right)$$

$$= e^{-m} \cdot m^2 \times e^m + m$$

$$\mu_2' = m^2 + m$$

Variance, $\mu_2 = m^2 + m - m^2$

$$\mu_2 = m$$

Q1 If mean of a Poisson distribution is 1·5 Find Variance and S.D?

Ans: Mean = 1·5, $m = 1·5$

Variance = m = 1·5

Standard deviation = $\sqrt{\text{variance}}$
$= \sqrt{m}$
$= \sqrt{1·5}$

Q2 Comment on the following: For a poisson distribution :-

Mean = 8, Variance = 7

Ans Mean = Variance = m

This is not possible as $8 \neq 7$

NOTE

Coefficient of skewness = $\sqrt{1/m}$

Measure of Kurtosis = $3 + 1/m$

Mode

If m is integer, mode is m, m-1

If m is not integer, the integral part is mode

eg 1.7, the integral part 1 is the mode

Q3 The mean of a poisson distribution is 1.5. Find mode and standard deviation

mean = m = 1.5

mode = 1 (∵ the integral part of 1.5 is 1)

$SD = \sqrt{m} = \sqrt{1.5} = 1.225$

NOTE Binomial distribution is the approximation of Poisson distribution.

Fitting Poisson Distribution

If we want to fit a poisson distribution to a given frequency distribution, we have to compute the mean of the given distribution and take it as m.

Once m is known, the poisson distribution is obtained by putting that value of m in $\frac{e^{-m} m^x}{x!}$. The expected (or theoretical) frequencies can be obtained by putting $x = 0, 1, 2, 3 \dots$ in $N \times \frac{e^{-m} m^x}{x!}$

Q4 Fit a poisson distribution to the following data and calculate the theoretical frequencies

x :	0	1	2	3	4
f :	123	59	14	3	1

Ans

x	f	fx
0	123	0
1	59	59
2	14	28
3	3	9
4	1	4
	200	100

$\therefore \bar{x} = \frac{\Sigma fx}{N} = \frac{100}{200} = 0.5$

$\Sigma f = N$

Since Mean = 0.5, m = 0.5

∴ The poisson distribution $P(x) = \frac{e^{-0.5} (1.5)^x}{x!}$

Theoretical frequencies are

x	P(x)	Theoretical frequency $= N \times P(x)$
0	$e^{-0.5} \frac{(0.5)^0}{0!} = 0.6065$	$= 200 \times 0.6065 = \underline{121}$
1	$e^{-0.5} \frac{(0.5)^1}{0!} = 0.30325$	$= 0.303\cancel{25} \times 200 = \underline{61}$

2	$\frac{e^{-0.5}\cdot(0.5)^2}{2!} = 0.0766$	$= 200 \times .0766$ $= 15$
3	$\frac{e^{-0.5}\cdot(0.5)^3}{3!} = 0.0127$	$= 200 \times 0.0127$ $= 3$
4	$\frac{e^{-0.5}(0.5)^4}{4!} = 0.0016$	$= 200 \times 0.0016$ $= 0$

Q5 A systematic sample of 100 pages was taken from the concise Oxford dictionary and the observed frequency distribution of foreign words per page was found to be as follows. Calculate the expected frequencies using Poisson distribution. Also compute variance of fitted distribution

No of foreign words per page (x)

0	1	2	3	4	5	6	
48	27	12	7	4	1	1	: Frequency (f)

Ans

x	f	fx
0	48	0
1	27	27
2	12	24
3	7	21
4	4	16
5	1	5
6	1	6
	100	99

Mean, $\bar{x} = \frac{\Sigma fx}{\Sigma f}$

$= \frac{99}{100}$

$= 0.99$

$\therefore m = 0.99$

Poisson distribution $P(x) = \frac{e^{-0.99}(0.99)^x}{x!}$

Calculation of theoretical frequencies

x	$P(x)$	Theoretical frequency $N \times P(x)$
0	$\frac{e^{-0.99}(0.99)^0}{0!} = 0.3716$	$100 \times .3716$ $= 37.2$
1	$\frac{e^{-0.99}(0.99)^1}{1!} = 0.3679$	$100 \times .3679$ $= 36.8$
2	$\frac{e^{-0.99}(0.99)^2}{2!} = 0.1821$	100×0.1821 $= 18.2$
3	$\frac{e^{-0.99}(0.99)^3}{3!} = 0.0601$	100×0.0601 $= 6$
4	$\frac{e^{-0.99}(0.99)^4}{4!} = 0.0149$	100×0.0149 $= 1.5$
5	$\frac{e^{-0.99}(0.99)^5}{5!} = 0.0029$	100×0.0029 $= 0.3$
6	$\frac{e^{-0.99}(0.99)^6}{6!} = 0.0005$	100×0.0005 $= 0.1$

Hence the theoretical (expected) frequencies of the poisson distribution are

x	0	1	2	3	4	5	6
Expected frequency rounded	37	37	18	6	2	0	0

For poisson distribution, mean and variance are equal to m

$\therefore$ Variance $= m = 0.99$

Q1 In a town 10 accidents take place in a span of 50 days. Assuming that the number of accidents follow Poisson pattern, find the probability that there are 2 accidents in a day

Ans $m = \frac{10}{50} = 0.2$

$x = 2$

$P(x) = \frac{e^{-m} \cdot m^x}{x!}$

$= \frac{e^{-0.2} \cdot (0.2)^2}{2!}$ 0.02

$= \frac{0.8187 \times 0.04}{2}$

$= 0.016374$

Q2 A Book contains 100 misprints distributed randomly throughout its 100 pages. What is the probability that a page observed at a random contains at least two misprints? (Assume Poisson distribution)

Ans $x =$ at least 2 misprint

$(x \geq 2) = 1 - (x < 2)$

$m = \frac{100}{100} = 1$

$x = 0, 1$

$P(x \geq 2) = 1 - \left(\frac{e^{-1} \cdot 1^x}{x!}\right)$

$= 1 - e^{-1}\left(\frac{1^0}{0!} + \frac{1^1}{1!}\right)$

$= 1 - e^{-1}(1+1)$

$= 1 - (e^{-1} \times 2)$

$= 1 - (0.368 \times 2)$

$= 1 - 0.736$

$P(x \geq 2) = 0.264$

Q3 A telephone exchange recieves on an average 4 calls per minute. Find the probability on the basis of poisson distribution $(m = 4)$ of (1) 2 or less calls per minute 2) upto 4 calls per minute and (3) more than 4 calls per minute

Ans $m = 4$

(1) $x = 0, 1, 2$

$P(x=0) + P(x=1) + P(x=2)$

$= \frac{e^{-4} \cdot 4^0}{0!} + \frac{e^{-4} \cdot 4^1}{1!} + \frac{e^{-4} \cdot 4^2}{2!}$

$= e^{-4}\left(1 + \frac{4}{1} + \frac{16}{2}\right)$

$= e^{-4}(1+4+8)$

$= e^{-4}(13)$

$= 0.01831 \times 13$

$= 0.238$

(2) $x = 0, 1, 2, 3, 4$

$P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= \frac{e^{-4} \cdot 4^0}{0!} + \frac{e^{-4} \cdot 4^1}{1!} + \frac{e^{-4} \cdot 4^2}{2!} + \frac{e^{-4} \cdot 4^3}{3!} + \frac{e^{-4} \cdot 4^4}{4!}$

$= e^{-4}\left(1 + 4 + 8 + \frac{64}{6} + \frac{256}{24}\right)$

$= e^{-4}(1 + 4 + 8 + 10.67 + 10.67)$

$= 0.0183 \times 34.34$

$= 0.6284$

(3) $P(x > 4) = 1 - (x \leq 4)$

$P(x > 4) = 1 - P(x \leq 4)$

$= 1 - 0.6284$

$= 0.3716$

Q4 It is found that the number of accidents occurring in a factory follows Poisson distribution with a mean of 2 accidents per week. Find the probability that (i) no accident occurs in a week (ii) number of accidents in a week exceeds 2 (given that $e^{-2} = 0.135$)

Ans mean, $m = 2$

(i) $x = 0$

$P(x=0) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-2} \cdot 2^0}{0!} = e^{-2} \cdot 1$

$P(x=0) = 0.135$

(ii) $x > 2$

ie $P(x > 2) = 1 - (P \leq 2)$

$= 1 - [P(x=0) + P(x=1) + P(x=2)]$

$P(x > 2) = 1 - e^{-2}\left[\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}\right]$

$= 1 - \left[0.135\,[1 + 2 + 2]\right]$

$= 1 - (0.135 \times 5)$

$= 1 - 0.675$

$P(x > 2) = 0.325$

Q5 A manufacturer of pins knows that on an average 5% of his products is defective. He sells pins in boxes of 100 and guarantee that no more than 4 pins will be defective. What is the probability that a box will meet guaranteed quality? ($e^{-5} = 0.0067$)

Ans $m = 5\% = \frac{5}{100} =$

$x = 0, 1, 2, 3, 4$

$P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= e^{-5}\left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!}\right]$

$= 0.0067\left[1 + 5 + 12.5 + 20.8 + 26.04\right]$

$= 0.0067 \times 65.34$

$= 0.4378$

Q6 During 2 hours between 8 and 10 am on an average 1.5 number of phone calls per minute are reported in the switch board of a company. Find the probability that during one particular minute, in the said interval, there will be (1) no phone call at all (2) exactly 3 calls and (3) at least 4 calls (given $e^{-1.5} = 0.2231$)

Ans $m = 1.5$

(1) $x = 0$

$P(x=0) = \frac{e^{-1.5} \times (1.5)^0}{0!}$

$= e^{-1.5} \times 1$

$P(x=0) = 0.2231$

(2) $x = 3$

$P(x=3) = \frac{e^{-1.5} \times (1.5)^3}{3!}$

$= e^{-1.5} \times \frac{3.375}{6}$

$= e^{-1.5} \times 0.5625$

$= 0.2231 \times 0.5625$

$P(x=3) = 0.1255$

(3) $x \geq 4$

$P(x \geq 4) = 1 - P(x < 4)$

$= 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3)]$

$= 1 - e^{-1.5}\left[\frac{(1.5)^0}{0!} + \frac{(1.5)^1}{1!} + \frac{(1.5)^2}{2!} + \frac{(1.5)^3}{3!}\right]$

$= 1 - \left[e^{-1.5}(1 + 1.5 + 1.125 + 0.5625)\right]$

$= 1 - (0.2231 \times 4.1875)$

$P(x \geq 4) = 1 - 0.934$
$= 0.066$ i.e 0.07

Q7 If in the key punching of 80 column cards, the average mistakes per card is 0.3, what percent of cards will have (1) no mistake (2) one mistake (3) three mistakes

Ans $m = 0.3$

(1) $x = 0$

$P(x=0) = \dfrac{e^{-0.3} \times (0.3)^0}{0!}$

$= e^{-0.3} \times 1$

$= 0.7408$

$\% = 0.7408 \times 100$
$= 74.08\%$

(2) $x = 1$

$P(x=1) = \dfrac{e^{-0.3} \times (0.3)^1}{1!}$

$= e^{-0.3} \times 0.3$

$= 0.7408 \times 0.3$

$= 0.22224$

$\% = 0.22224 \times 100$
$= 22.23\%$

(3) $x = 3$

$P(x=3) = \dfrac{e^{-0.3} \times (0.3)^3}{3!}$

$= e^{-0.3} \times \dfrac{0.027}{6}$

$= \dfrac{0.7408 \times 0.027}{6}$

$= \dfrac{0.0200016}{6} = \dfrac{0.0200}{6}$

$= \cancel{0.003333} \quad 0.003333$

$\% = 0.00333333 \times 100$
$= 0.33\%$

Q8 Certain articles, 0.5% of which are damaged are packed in boxes, containing 120 each. What proportion of the boxes are free from the damaged articles, and what proportion contains 2 or more damaged ones?

success - damaged

$p = 0.5\%$

$= \dfrac{0.5}{100} = 0.005$

$n = 120$

$m = np = 0.005 \times 120$

$m = 0.6$

• free from damaged ie $x = 0$

$P(x=0) = \frac{e^{-0.6} \cdot (0.6)^0}{0!} = e^{-0.6}$

$= 0.5488$

Proportion $\Rightarrow 0.5488 \times 100$

$= 54.88\%$

• contains 2 or more damaged

$P(x \geq 2) = 1 - (P(x=0) + P(x=1))$

$= 1 - \left(0.5488 + \frac{e^{-0.6}(0.6)^1}{1!}\right)$

$= 1 - (0.5488 + 0.32928)$

$= 1 - 0.87808$

$= 0.12192$

~~ie 0~~

Proportion $\Rightarrow 0.12192 \times 100$

$= 12.19\%$

Q9 One fifth of per cent of the blades produced by a blade manufacturing factory turn out to be defective. The blades are supplied in packets of 10. Use poisson distribution to calculate the approximate number of packets containing no defective, one defective and two defective blades respectively in a consignment of 1,00,000 packets. (Given $e^{-0.02} = .9802$)

Ans $m = 0.02$

→ no defective − $x = 0$

$P(x=0) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-0.02} \times (0.02)^0}{0!}$

$= e^{-0.02}$

$= 0.9802$

Total no of packets $\Rightarrow 0.9802 \times 100000 = 98020$

→ one defective − $x = 1$

$P(x=1) = \frac{e^{-0.02} (0.02)^1}{1!}$

$= 0.9802 \times 0.02$

$= 0.019604$

no of packets $\Rightarrow 0.019604 \times 100\,000 = 1960$

$\Rightarrow$ two defective $\rightarrow x = 2$

$$P(x=2) = \frac{e^{-0.02} \times (0.02)^2}{2!}$$

$$= \frac{0.9802 \times 0.0004}{2}$$

$$= 0.00019604$$

No. of packets $= 0.00019604 \times 100000$

$= 19.6 \Rightarrow 20$ packets

Q10 Certain automatic screw manufacturing machine produces on the average one slotless screw among every 100 screws. If the screws are packed in boxes of 300, what percentage of these boxes would you expect to have (i) no slotless screw and (ii) atleast one slotless screw.

$p = \frac{1}{100} = 0.01$ $n = 300$ $m = np = 0.01 \times 300$

$m = 3$

(i) $x = 0$, $P(x=0) = \frac{e^{-3} \cdot (3)^0}{0!}$

$= 0.0498$

$\Rightarrow 0.0498 \times 100$

$= 4.98\%$

(ii) P(atleast one slotless screw)

$= 1 - P(x=0)$

$= 1 - 0.0498$

$= 0.9502$

$= 0.9502 \times 100$

$= 95.02\%$

(i) $\rightarrow 4.98\%$

(ii) $\rightarrow 95.02\%$

Q11 The number of accidents in a year attributed to taxi drivers in a city follows Poisson distribution with mean 3. Out of 1000 taxi drivers find approximately the number of drivers with (1) no accident in a year (2) more than 3 accidents in a year ($e^{-1} = 0.3679$, $e^{-2} = 0.1353$, $e^{-3} = 0.0498$)

$m = 3$

(1) no accidents $P(x=0)$

$$P(x=0) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-3} \cdot 3^0}{0!}$$

$= 0.0498$ ~~×1000~~

no of accidents ⇒ $0.0498 \times 1000 = 49.8$
ie 50

(2) $x > 3$

$P(x>3) = 1 - [P(x=0) + P(x=1) + P(x=2) + P(x=3)]$

$= 1 - \left[e^{-3}\left(\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right)\right]$

$= 1 - [0.0498 \times (1 + 3 + 4.5 + 4.5)]$

$= 1 - (0.0498 \times 13)$

$= 1 - 0.6474$

$= 0.3526$

no of accidents ⇒ 0.3526×1000

$= 352.6$

$= 353$

Q2 A manufacturer who produces medicine bottles, find that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes from the producer of bottles. Using poisson distribution find how many boxes will contain (1) no defective (2) at least 2 defective. (given $e^{-0.5} = .6060$)

$p = 0.1\% = \frac{0.1}{100} = 0.001$

$n = 500$

$m = np = 0.001 \times 500 = 0.5$

(1) $x = 0$

$P(x=0) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-0.5} \cdot (0.5)^0}{0!}$

$= e^{-0.5} = 0.6065$

no of boxes ⇒ 0.6065×100

$= 60.6$

ie 61 boxes

(2) $x \geq 2$

$P(x>2) = 1 - P(x<2)$

$= 1 - [P(x=0) + P(x=1)]$

$= 1 - \left[e^{-0.5}\left(\frac{(0.5)^0}{0!} + \frac{(0.5)^1}{1!}\right)\right]$

$= 1 - (0.6065 \times 1.5)$

$= 1 - 0.9097 = 0.091$

no of boxes ⇒ 0.091×100

$= 9.1$ ie 9 boxes

Q13 A manufacturer finds that the average demands per days for the machines to repair his new product is 1.5, over a period of one year and the demand per day is distributed as a poisson variate. He employes two machines. On how many days in one year (a) both machines would be free (b) some demand is refused.

x - no. of works
~~machines running~~

mean, $m = 1.5$

(a) both machines would be free ~~in one~~ ~~year~~ ie $x = 0$

$$P(x=0) = \frac{e^{-1.5} \cdot (1.5)^0}{0!}$$

$$= e^{-1.5}$$

$$P(x=0) = 0.223$$

total days in 1 year $= 0.223 \times 365$

$= 81.395$

$= 81.4$ days

(b) Some demand is refused ie $x > 2$

$$P(x>2) = 1 - [P(x=0) + P(x=1) + P(x=2)]$$

$$= 1 - \left(0.223 + \frac{e^{-1.5} \cdot (1.5)^1}{1!} + \frac{e^{-1.5} \cdot (1.5)^2}{2!}\right)$$

$$= 1 - (0.223 + 0.3345 + 0.25089)$$

$$= 1 - 0.80839$$

$$= 0.19161$$

total days in a year $= 0.19161 \times 365$

$= 69.93765$

$= 69.9$ days

Q14 If a random variable x has the binomial distribution $B(200, 0.001)$, explain how would you evaluate $P(x>1)$

[Hint: Since n is large and p is small, B.D tends to P.D. So apply poisson approximation. So $m = 200 \times 0.001 = .2$]

Ans $m = 200 \times 0.001 = 0.2$

$$P(x>1) = 1 - P(x \leq 1)$$

$$= 1 - [P(x=0) + P(x=1)]$$

$$= 1 - e^{-0.2}\left[\frac{(0.2)^0}{0!} + \frac{(0.2)^1}{1!}\right]$$

$$= 1 - [0.8187 \times (1 + 0.2)]$$

$$= 1 - [0.8187 \times 1.2]$$

$$= 1 - 0.98244$$

$$= 0.01756$$

Q15 The mean of a poisson distribution is 2.25 find variance and mode of the distribution

Ans mean = m = variance = 2.25

ie mean = 2.25
ie variance = 2.25

mode = 2

Q16 If 10% of certain products are found to be defective, using both Binomial and poisson distribution models find the probability that in a sample of 10 products, exactly two would be defective. Verify that P.D is an approximation of B.D.

Ans $n = 10$ $\quad p = 10\% = \frac{10}{100}$ $\quad p = 0.1$ $\quad \left(m = np = 10 \times 0.1 = \underline{1} \right.$

P.D

$x = 2$

$P(x=2) = \frac{e^{-1} \times (1)^2}{2!}$ $\quad \left(\frac{e^{-m} \cdot m^x}{x!} \right.$

$= \frac{0.367}{2}$

$P(x=2) = \underline{0.183}$

B.D

$p = 10\% = \frac{10}{100} = 0.1$

$n = 10$ $\quad q = 1 - 0.1 = 0.9$

$P(x=2) = {}^{10}C_2 \, (0.1)^2 \, (0.9)^8$

$= \frac{10 \times 9}{2} \times 0.01 \times 0.430$

$= 45 \times 0.01 \times 0.430$

$= 0.193$

$\therefore 0.183 \cong 0.193$

ie P.D $\cong$ B.D

Q17 Out of 1000 houses only one house catches fire in a year. What is the probability that out of 500 houses exactly 4 houses would catch fire?

Ans $m = 1/1000 = 0.001$

$x = 4$

$P(x=4) = \frac{e^{-0.001} \times (0.001)^4}{4!}$

$$= \frac{0.9990 \times 0.00000000000001}{24}$$

$.0045$

Q18 The distribution of typing mistakes committed by a typist given below. Assuming a poisson model find the expected frequencies

Mistake per page	0	1	2	3	4	5
No of pages	142	156	69	27	5	1

x	f	fx
0	142	0
1	156	156
2	69	138
3	27	81
4	5	20
5	1	5
	400	400

$\Sigma f = N = 400$

$$\text{mean, } m = \frac{\Sigma fx}{\Sigma f (\text{or } N)} = \frac{400}{400} = 1$$

$$P(x) = \frac{e^{-1} \cdot 1^x}{x!} \quad \text{where } x = 0, 1, 2, 3, 4, 5$$

x	$P(x)$	Expected frequency $P(x) \times N$
0	$\frac{e^{-1} \cdot 1^0}{0!} = 0.367$	$146.8 = 147$
1	$\frac{e^{-1} \cdot 1^1}{1!} = 0.367$	$146.8 = 147$
2	$\frac{e^{-1} \cdot 1^2}{2!} = 0.183$	$73.2 = 74$
3	$\frac{e^{-1} \cdot 1^3}{3!} = 0.0613 = 0.062$	$24.8 = 25$
4	$\frac{e^{-1} \cdot 1^4}{4!} = 0.015$	6
5	$\frac{e^{-1} \cdot 1^5}{5!} = 0.003$	$1.2 = 1$

Q19 A skilled typist, on routine work, kept a record of mistakes made per day during 300 working days.

Mistakes per day	0	1	2	3	4	5	6
No of days	143	90	42	12	9	3	1

Fit a Poisson distribution and find theoretical frequencies.

x	f	fx
0	143	0
1	90	90
2	42	84
3	12	36
4	9	36
5	3	15
6	1	6
	300	267

mean, $m = \frac{\Sigma fx}{\Sigma f}$

$= \frac{267}{300}$

$= \underline{\underline{0.89}}$

$N = 300$

x	$P(x)$	Expected frequency $P(x) \times N$
0	$\frac{e^{-0.89}(0.89)^0}{0!} = 0.411$	$= 123.3 \approx \underline{\underline{123}}$
1	$\frac{e^{-0.89}(0.89)^1}{1!} = 0.366$	$= 109.8 \approx \underline{\underline{110}}$
2	$\frac{e^{-0.89}(0.89)^2}{2!} = 0.163$	$= 48.9 \approx \underline{\underline{49}}$
3	$\frac{e^{-0.89}(0.89)^3}{3!} = 0.048$	$= 14.4$ $\underline{\underline{14}}$
4	$\frac{e^{-0.89}(0.89)^4}{4!} = 0.010$	$\underline{\underline{3}}$
5	$\frac{e^{-0.89}(0.89)^5}{5!} = 0.0019$	$= 0.57$ $\underline{\underline{1}}$
6	$\frac{e^{-0.89}(0.89)^6}{6!} = 0.0002$	$= 0.06$

Q20 Following mistakes per page were observed in a book

No of mistakes	0	1	2	3	4
No of pages	211	90	19	5	0

Fit a poisson distribution to the above data

x	f	xf
0	211	0
1	90	90
2	19	38
3	5	15
4	0	0
	325	143

Mean, $m = \frac{\Sigma fx}{\Sigma f}$

$= \frac{\cancel{325}}{\cancel{143}} = \frac{143}{325}$

$= 0.44$

x	$P(x)$	$P(x) \times N$
0	$\frac{e^{-0.44} \cdot (0.44)^0}{0!}$	209
1	$\frac{e^{-0.44}(0.44)^1}{1!}$	92
2	$\frac{e^{-0.44}(0.44)^2}{2!}$	20
3	$\frac{e^{-0.44}(0.44)^3}{3!}$	3
4	$\frac{e^{-0.44}(0.44)^4}{4!}$	1

Normal Distribution

Normal distribution . is related to the distribution of errors, made by chance in experimental measurement.

Normal distribution is the most useful theoretical distribution for continuous variables.
Many statistical data concerning business and ecomic problems can be displayed in the form of Normal distribution.

It is now most important probability model in statistical analysis

definition

A continous random variable X is said to follow Normal distribution if its probability function is

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \text{ where}$$

μ and σ are constants (μ being the mean σ being the standard deviation of x)

The variable X varies between $-\infty$ and $+\infty$

Normal distribution as a limiting case of Binomial distribution

Binomial distribution is an important theoretical distribution for discrete variables. Binomial distribution tends to Normal Distribution under the following conditions.

1. Number of trials (n) is very large
2. p and q (ie probability for success in a single trial and the probability for its failure) are almost equal.

Then the Binomial distribution can be approximated to normal.

Probability density function and parameter of the normal distribution

If X is a continous random variable following Normal distribution with mean μ and standard deviation σ then its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-1/2\left[\frac{x-\mu}{\sigma}\right]^2}$$

where π and 'e' are the constants given by $\pi = \frac{22}{7} = 3.14159$ and $e = 2.71828$.

X may vary between $-\infty$ and $+\infty$

Then mean μ and standard deviation σ

are called parameters of the Normal distribution.

A normal distribution is described by its parameters, mean and standard deviation. For various values of mean and standard deviation, we get different normal curves.

Mean of the distribution shows the location of the curve along the X- axis and S.D indicates the spread (or shape) of the curve.

Properties (characteristics) of the Normal Distribution

1. The normal curve is a continous curve.
2. The normal curve is bell shaped
3. Normal curve is symmetric about the mean
4. Mean, Median and Mode are equal for a normal distribution
5. The height of Normal curve is at its maximum at the mean.
6. There is only one maximum point, which occurs at the mean.
7. The ordinate at mean divides the whole area into two equal parts (ie 0.5 on either side)
8. The curve is asymptotic to the base on either side ie the curve approaches nearer and nearer to the base, but it never touches.
9. Coefficient of skewness is 0 ($\beta_1 = 0$)
10. The normal curve is unimodel, ie, it has only one mode.
11. The points of inflexion occur at $\mu \pm \sigma$. At the point of inflexion the curve changes from concavity to convexity.
12. Q_1 and Q_3 are equidistant from median.
13. Mean deviation for Normal distribution is $\frac{4}{5}\sigma$ and QD is $\frac{2}{3}\sigma$
14. All odd moments of the Normal distribution are zero and even moments are interrelated by the following formula:
$$\mu_{2n} = (2n-1)\sigma^2 \mu_{2n-2}$$
15. Normal distribution is mesokurtic. That is measure of kurtosis $= \beta_2 = 3$
16. No portion of the curve lies below the x axis.
17. Theoretically the range of the normal curve is $-\infty$ to $+\infty$. But practically the range is $x - 3\sigma$ to $x + 3\sigma$
18. If x and y are two independent Normal variates, then their sum is also a normal variate. This is called the additive property
19. Area under the normal curve is distributed as follows

$$\left.\begin{array}{l} \mu \pm \sigma \text{ covers } 68.27\% \text{ area} \\ \mu \pm 2\sigma \text{ covers } 95.45\% \text{ area} \\ \mu \pm 3\sigma \text{ covers } 98.73\% \text{ area} \end{array}\right\} \text{area property}$$

Importance (or use) of Normal distribution.

The study of the normal distribution is of central importance in statistical analysis because of the following reasons

1. Most of the discrete probability distributions (eg. Binomial distribution, Poisson distribution etc) tend to normal distribution as 'n' becomes large.

2. Almost all sampling distributions such as student's t-distribution, F-distribution, Z-distribution, χ^2-distribution. etc conform to the normal distribution for large values of n.

3. The various tests of significance like t-test, F-test etc are based on the assumption that the parent population from which the samples have been drawn follows Normal Distribution

4. It is extensively used in large sampling theory to find the estimates of parameters from statistics, confidance limits. etc

5. Normal distribution has the remarkable property stated in the central limit theorem. As per the theorem, when the sample size is increased, the sample means will tend to be normally distributed. Central limit theorem gives the normal distribution its central place in the theory of sampling, since many important problems can be solved by this.

6. In theoritical statistics as well as applied works many problems can be solved only under the assumption of a normal population.

7. The normal distribution has numerous mathematical properties which make it popular and comparatively easy to manupulate. The normal curve is reasonably close to many distributions.

8. It finds applications in statistical quality control and industrial experiments. Many distributions in Social and Economic data are approximately normal.
Eg:- birth, death etc are normally distributed. In psychological and educational data many distributions are of normal type.

Merits and demerits of Normal distribution.

Merits

1. Normal distribution is the mostly used distribution in Inferential Statistics.
2. Most of error of measurements and a large variety of physical observations have approximately Normal distributions.

3. The measurements of linear dimensions of large number of articles produced may show individual variations.
These follow normal distribution

4. The standard normal distribution table shows exhaustively areas for the different intervals of the values of the variable.

5. The Normal distribution has a number of mathematical properties.

6. Most of the distributions in nature are either normal or that can be approximated to normal.

Demerits

1. The variables which are not continous ~~cant~~ cannot be normally distributed. Therefore many distributions in Economics like distribution of number of children per family, cannot be studied under Normal Distribution.

2. The normal distribution cannot be applied to situations where the distribution is highly skewed. For example: distribution on Income is very much skewed. Therefore here Normal distribution will not be appropriate.

Standard Normal Variate (Unit Normal Variate)

If X is a random variable following Normal distribution with mean μ and standard deviation σ then the variable

$$z = \frac{x-\mu}{\sigma}$$ is known as standard normal variate.

This 'z' follows normal distribution with mean '0' and SD = 1.

The distribution of z is known as Standard Normal distribution.

The probability function of z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for } -\infty < z < \infty$$

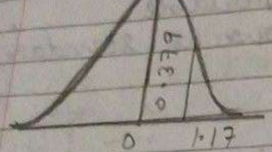
The Standard normal distribution table

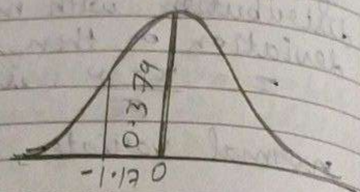
This is a table showing the probability for 'z' taking values between 0 and a given value.

The probability thus obtained is the area of the standard normal curve between the ordinates at $z = 0$ and at the given value.

For example, when $z = 1.17$ table value $= 0.3790$

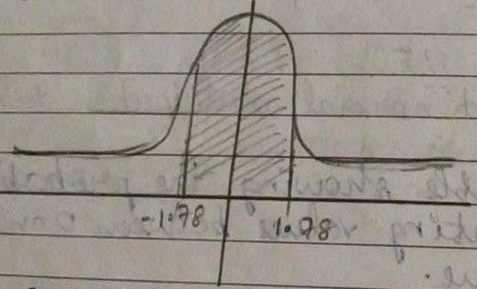
This table value is the area between 0 and 1.17 which can be written as $P(0 < z < 1.17)$
$= 0.3790$





It may be noted that for both negative and positive values of z, area is the same that is, $P(0 < z < 1.17) = P(-1.17 < z < 0)$
$= 0.3790$

eg:- $P(-1.78 < z < 1.78)$

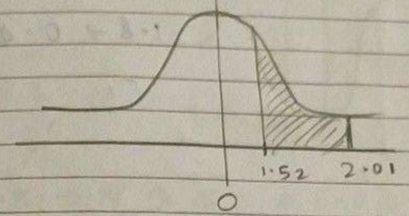


~~=2×~~ $1.78 \rightarrow 0.4625$

$= 2 \times 0.4625$
$= 0.925$

ie 0.925×100
92.5%

eg:- $P(1.52 < z < 2.01)$



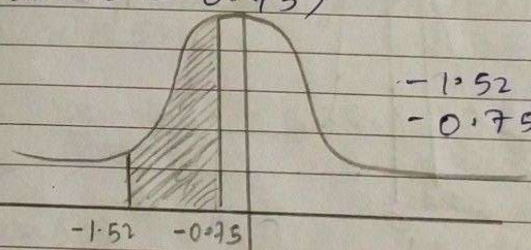
$1.52 \rightarrow 0.4357$
$2.01 \rightarrow 0.4778$

we have to erase value from 0 to 1.52

$\rightarrow 2.01 - 1.52$
$= 0.4778 - 0.4357$
$= 0.0421$

$\Rightarrow 0.0421 \times 100 = 4.21\%$

eg:- $P(-1.52 < z < -0.75)$

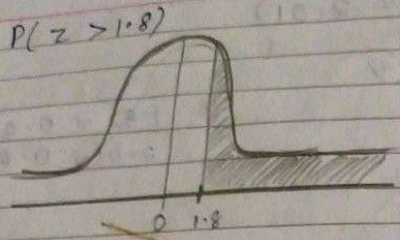


$-1.52 \rightarrow 0.4357$
$-0.75 \rightarrow 0.2734$

$\rightarrow (1.52) - (-0.75)$
$= 0.4357 - 0.2734$
$= 0.1623$

$\Rightarrow 0.1623 \times 100 = 16.23\%$

eg: $P(z > 1.8)$

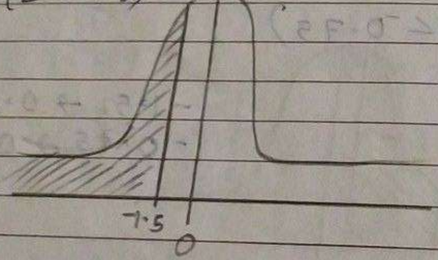


$1.8 \rightarrow 0.4641$

$= 0.5 - P(0 < z < 1.8)$
$= 0.5 - 0.4641$
$= 0.0359$

$\Rightarrow 0.0359 \times 100 = \underline{3.59\%}$

eg $P(z < -1.5)$



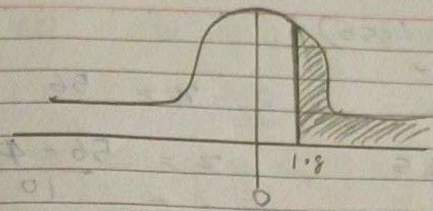
$1.5 \rightarrow 0.4332$

$= 0.5 - P(-1.5 < z < 0)$
$= 0.5 - 0.4332$
$= 0.0668$

$\Rightarrow 0.0668 \times 100 = \underline{6.68\%}$

eg $P(z > 1.8)$

$1.8 \rightarrow 0.4641$



$= 0.5 - P(0 < z < 1.8)$
$= 0.5 - 0.4641$
$= 0.0359$

$\Rightarrow 0.0359 \times 100 = \underline{3.59\%}$

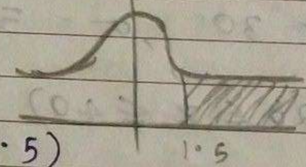
Ques The variable x follows a normal distribution with mean $\mu = 45$ and $\sigma = 10$ (SD.) Find the probability that (1) $x > 60$ (2) $40 < x < 56$

$$z = \frac{x - \mu}{\sigma}$$

(1) $P(x > 60)$

$x = 60$ $\quad z = \frac{60 - 45}{10} = 1.5$

$P(z > 1.5)$



$= 0.5 - P(0 < z < 1.5)$
$= 0.5 - 0.4332$
$= 0.0668$

$\Rightarrow 0.0668 \times 100 = \underline{6.68\%}$

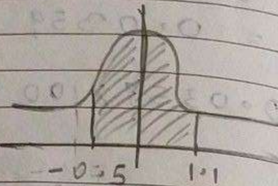
(2) $P(40 < x < 56)$

$x = 40$ $\qquad$ $x = 56$

$z = \frac{40 - 45}{10}$ $\qquad$ $z = \frac{56 - 45}{10}$

$z = -0.5$ $\qquad$ $z = 1.1$

$P(-0.5 < z < 1.1)$



$= P(0.5 < z < 0) +$
$P(0 < z < 1.1)$

$= 0.1915 + 0.3643$
$= 0.5558$

$\Rightarrow 0.5558 \times 100 = \underline{\underline{55.58\%}}$

Ques1 X is a normal variate with mean 30 and SD 5. Find the probabilities that
(1) $26 \leq x \leq 40$ and (2) $x \geq 45$

$\mu = 30$, $\sigma = 5$ $\qquad$ $z = \frac{x - \mu}{\sigma}$

(1) $P(26 \leq x \leq 40)$

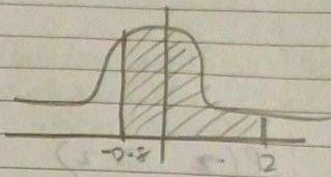
$x = 26$ $\qquad$ $z = \frac{26 - 30}{5} = -\frac{4}{5}$

$z = -0.8$

$x = 40$; $z = \frac{40 - 30}{5} = \frac{10}{5}$

$z = 2$

$P(-0.8 \leq z \leq 2)$



$= P(-0.8 \leq z < 0) +$
$P(0 \leq z < 2)$

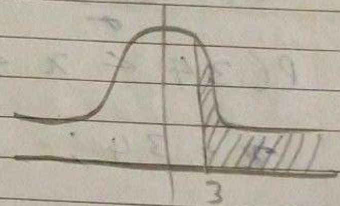
$= 0.2881 + 0.4772$
$= \underline{\underline{0.7653}}$

(2) $P(x \geq 45)$

$x = 45$; $z = \frac{45 - 30}{5} = \frac{15}{5}$

$z = 3$

$P(z \geq 3)$



$= 0.5 - P(0 \leq z \leq 3)$
$= 0.5 - 0.49865$
$= 0.00135$

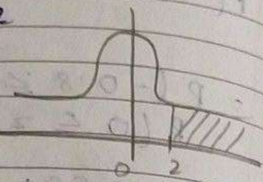
Ques2 If X follows a normal distribution with mean = 12 and variance = 16 find $P(x \geq 20)$

mean, $\mu = 12$
variance, $\sigma^2 = 16$
$\sigma = \sqrt{16} = \underline{\underline{4}}$

$z = \frac{x-\mu}{\sigma}$ $\quad x = 20$

$z = \frac{20-12}{4} = \frac{8}{4}$

$\underline{z = 2}$



$P(z \geq 2)$

$= 0.5 - P(0 \leq z \leq 2)$
$= 0.5 - 0.4772$
$= \underline{0.0228}$

Ques3 Given a normal distribution with $\mu = 50$ and $\sigma = 8$. Find the probability that x assumes a value between 34 and 62.

$z = \frac{x-\mu}{\sigma}$ $\quad \mu = 50 ; \sigma = 8$

$P(34 < x < 62)$

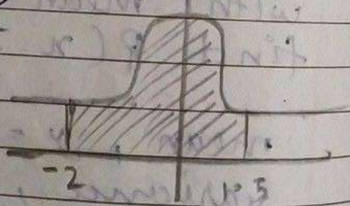
$x = 34 ; \quad z = \frac{34-50}{8} = \frac{-16}{8}$

$\underline{z = -2}$

$x = 62 \quad ; z = \frac{62-50}{8} = \frac{12}{8}$

$\underline{z = 1.5}$

$P(-2 < z < 1.5)$



$= P(-2 < z < 0) +$
$P(0 < z < 1.5)$

$= 0.4772 + 0.4332$
$= \underline{0.9104}$

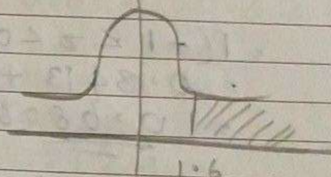
Ques4 Given a normal distribution with $\mu = 200$ and $\sigma = 25$. Find the probability that x assumes a value greater than 240.

$z = \frac{x-\mu}{\sigma}$ $\quad \mu = 200 ; \sigma = 25$

$P(x \geq 240)$

$x = 240 \quad z = \frac{240-200}{25} = \frac{40}{25}$

$\underline{z = 1.6}$



$P(z \geq 1.6)$
$= 0.5 - P(0 \leq z \leq 1.6)$
$= 0.5 - 0.4452$
$= 0.5 - 0.4452$
$= \underline{0.0548}$

Ques5 The scores of students in a test follows Normal distribution with mean = 80 and SD = 15. A sample of 1000 students has been drawn from the population. Find (a) appropriate number of students scoring between 65 and 95. (b) the probability that a randomly chosen student scores greater than 100.

mean, $\mu = 80$; SD, $\sigma = 15$

$z = \frac{x-\mu}{\sigma}$

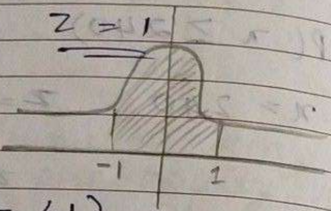
a) $(65 < x < 95)$

when $x = 65$

$z = \frac{65-80}{15} = \frac{-15}{15}$, $z = -1$

when $x = 95$

$z = \frac{95-80}{15} = \frac{15}{15}$, $z = 1$



$P(-1 < z < 1)$

$= P(-1 < z < 0) + P(0 < z < 1)$

$= 0.3413 + 0.3413$ OR (0.3413×2)

$= 0.6826$

Appropriate number of students

$\Rightarrow 0.6826 \times 100 = 682.6$

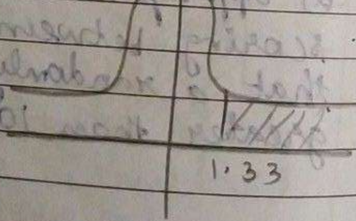
ie 683 students

b) $(x > 100)$ $x = 100$ $z = \frac{100-80}{15}$

$z = \frac{20}{15} = 1.333$

$= 1.33$

$P(z > 1.33)$



$= 0.5 - P(0 < z < 1.33)$

$= 0.5 - 0.4082$

$= 0.0918$

Ques 6 The height of the school children of one Institution is normally distributed with mean of 54 inches and SD of 12 inches. What percentage of students have height between 46 and 56 inches.

mean, $\mu = 54$ SD, $\sigma = 12$

$z = \frac{x-\mu}{\sigma}$

$P(46 < x < 56)$

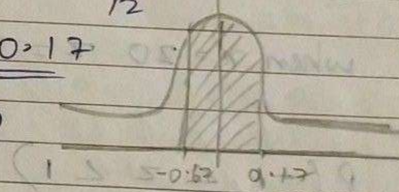
when $x = 46$ $z = \frac{46-54}{12} = \frac{-8}{12}$

$z = -0.666 = -0.67$

when $x = 56$

$z = \frac{56-54}{12} = \frac{2}{12} = 0.166$

$z = 0.17$



$P(-0.67 < z < 0.17)$

$= P(-0.67 < z < 0) + P(0 < z < 0.17)$

$= 0.2486 + 0.0675$

$= 0.3161$

percentage $\Rightarrow 0.3161 \times 100$

$= 31.61\%$

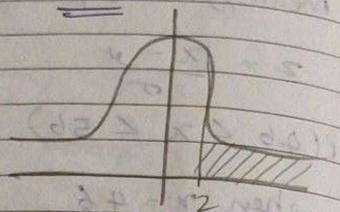
ie 31.61 % of students have height between 46 and 56 inches.

Ques7 Given mean = 15, SD = 5 and the value ~~lies~~ of x is greater than or equal to 25 and also $(10 < x < 20)$

$\mu = 15$; $\sigma = 5$; $(x \geq 25)$ $z = \frac{x-\mu}{\sigma}$

when $x = 25$

$z = \frac{25-15}{5}$ $z = \frac{10}{5}$ $z = 2$



$P(z \geq 2)$

$= 0.5 - P(0 < z < 2)$

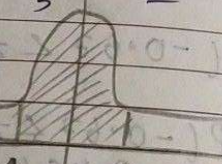
$= 0.5 - 0.4772$

$= 0.0228$

~~when~~ $P(10 < x < 20)$

when $x = 10$ $z = \frac{10-15}{5} = \frac{-5}{5} = -1$

when $x = 20$ $z = \frac{20-15}{5} = \frac{5}{5} = 1$



$P(-1 < z < 1)$

$P(-1 < z < 0) + P(0 < z < 1)$

$= 0.3413 + 0.3413$

$= 0.6826$

Ques8 Find the probability that the number of heads lie in the range 185 and 220 when a fair coin is tossed 400 times.

$P(185 < x < 220)$

$n = 400$ $p = 1/2$ $q = 1/2$

$\mu = np$ $\mu = 400 \times \frac{1}{2}$

$\sigma^2 = npq$ $\mu = 200$

$\sigma = \sqrt{npq}$

$\sigma^2 = 400 \times \frac{1}{4}$

$\sigma^2 = 100$

$\sigma = \sqrt{100} = 10$

$z = \frac{x-\mu}{\sigma}$

when $x = 185$ $z = \frac{185-200}{10} = \frac{-15}{10}$

$z = -1.5$

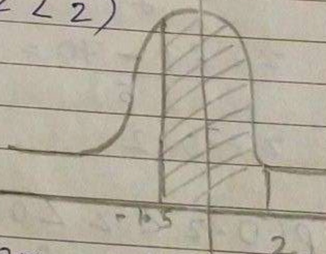
when $x = 220$ $z = \frac{220-200}{10} = \frac{20}{10} = 2$

$P(-1.5 < z < 2)$

$= P(-1.5 < z < 0) + P(0 < z < 2)$

$= 0.4332 + 0.4772$

$= 0.9104$



Ques9 The weekly wages of 1000 workmen are normally distributed around a mean of Rs 70 and with a SD of 5. Estimate the number of workers whose weekly wages

will be (a) between Rs 70 and Rs 72 (b) between Rs 69 and 72 (c) more than Rs 75 (d) less than Rs 63.

mean, $\mu = 70$; SD, $\sigma = 5$

(a) b/w Rs 70 and Rs 72

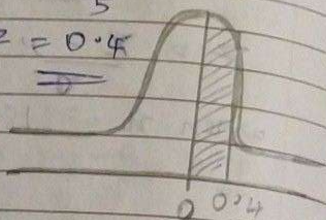
$P(70 < x < 72)$

when $x = 70$ when $x = 72$

$z = \frac{x - \mu}{\sigma}$ $z = \frac{x-\mu}{\sigma}$

$z = \frac{70-70}{5} = \frac{0}{5}$ $z = \frac{72-70}{5}$

$z = 0$ $z = 0.4$



$P(0 < z < 0.4)$

$= 0.1554$

(b) b/w Rs 69 and 72

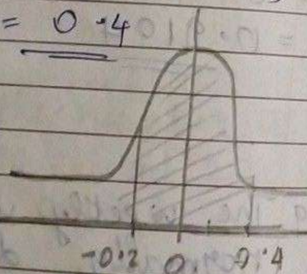
$P(69 < x < 72)$

when $x = 69$ when $x = 72$

$z = \frac{x-\mu}{\sigma}$ $z = \frac{x-\mu}{\sigma}$

$z = \frac{69-70}{5} = \frac{-1}{5}$ $z = \frac{72-70}{5} = \frac{2}{5}$

$z = -0.2$ $z = 0.4$



$P(-0.2 < z < 0.4)$

$P(-0.2 < z < 0) + P(0 < z < 0.4)$

$= 0.0793 + 0.1554$

$= 0.2347$

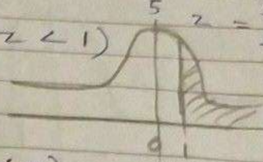
(c) more than Rs 75 $P(x > 75)$

when $x = 75$; $z = \frac{x-\mu}{\sigma} = \frac{75-70}{5} = \frac{5}{5}$ $z = 1$

$P(z > 1) = 0.5 - P(0 < z < 1)$

$= 0.5 - 0.3413$

$= 0.1587$

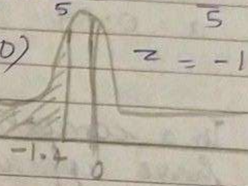


(d) less than Rs 63 $P(x < 63)$

when $x = 63$; $z = \frac{x-\mu}{\sigma} = \frac{63-70}{5} = \frac{-7}{5}$ $z = -1.4$

$P(z < -1.4) = P(-1.4 < z < 0)$ $0.5 -$

$= 0.5 - 0.4192 = 0.0808$



Number of workers

(a) $0.1554 \times 1000 \Rightarrow 155.4$

(b) $0.2347 \times 1000 \Rightarrow 234.7$

(c) $0.1587 \times 1000 \Rightarrow 158.7$

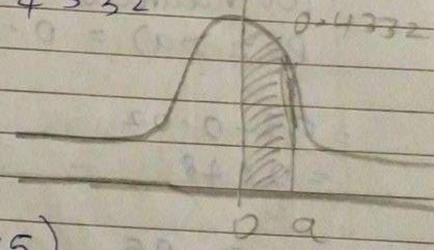
(d) $0.0808 \times 1000 \Rightarrow 80.8$

Problems based on giving the area and finding the value of z.

eg. If $(0 < z < a) = 0.4332$ find the value of a

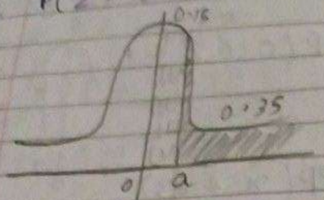
area $\rightarrow 0.4332$

$a = 1.5$



$(\because 0.4332 \rightarrow 1.5)$

eg $P(z > a) = 0.35$. Find the value of a



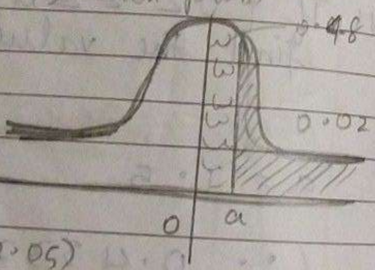
$= 0.5 - 0.35$
$= \underline{0.15}$

$a = 0.39$ $(0.1517 \to 0.39)$

Ques1 In a competitive examination 5000 students have appeared for a paper in statistics. Their average marks was 62 and standard deviation was 12. If there are only 100 vacancies, find the minimum marks that one should score inorder to get selected.

% of students who can get $= \frac{100}{5000} \times 100 = 2\%$
$= \underline{0.02}$

100 vacancies $\Rightarrow 0.02$ (selection prob)
$P(z > a) = 0.02$



$= 0.5 - 0.02$
$= 0.48$

$a = 2.05$ $(0.4798 \to 2.05)$
ie $\underline{z = 2.05}$

$z = \frac{x - \mu}{\sigma}$ $\quad z = 2.05;\ x = ?$
$\mu = 62 \quad \sigma = 12$

$2.05 = \frac{x - 62}{12}$
$24.6 = x - 62$

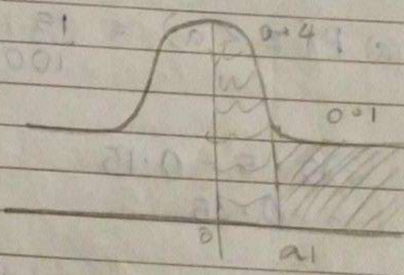
$x = 24.6 + 62$
$x = \underline{86.6}$

marks we get $\Rightarrow$ 86.6 (the minimum marks)

Q The weekly wages of 1000 workmen are normally distributed around a mean of Rs.70 and with a SD of 5. Estimate

proportion of highly paid workers $= \frac{100}{1000} \times 100 = 10\% = \underline{0.1}$

$P(z > a) = 0.1$



$= 0.5 - 0.1$
$= 0.4$

$a = 1.28$
ie $z = \underline{1.28}$ $(\because 0.3997 \to 1.28)$

$z = \frac{x-\mu}{\sigma}$ $\quad z = 1.28 \quad \mu = 70 \quad x = ? \quad \sigma = 5$

$1.28 = \frac{x - 70}{5}$

$x - 70 = 6.4$

$x = 6.4 + 70 = \underline{\underline{76.4}}$

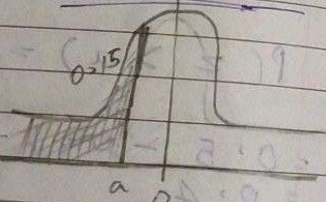
Rupees 76.4

Ques 3 Given a normal distribution with mean 40 and SD 10. Find the value of x that has (1) 15% of its area to its left (2) 20% of area to its right.

(1) $P(z<a)$ (2) $P(z>a)$

mean, $\mu = 40$; SD, $\sigma = 10$

(a) $P(z<a) = \frac{15}{100}$ ie $\underline{\underline{P(z<a) = 0.15}}$



$= 0.5 - 0.15$
$= 0.35$

$\underline{\underline{a = -1.04}}$ $(\because 0.3508 \rightarrow 1.04)$

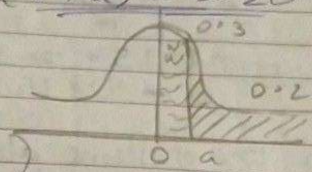
ie $\underline{\underline{z = -1.04}}$

$z = \frac{x-\mu}{\sigma}$

$-1.04 = \frac{x-40}{10}$

$x - 40 = -10.4$
$x = -10.4 + 40$
$x = \underline{\underline{29.6}}$

(b) $P(z>a) = \frac{20}{100}$ ie $\underline{\underline{P(z>a) = 0.20}}$



$= 0.5 - 0.20$
$= 0.3$

$\underline{\underline{a = 0.84}}$ $(\because 0.29955 \rightarrow 0.84)$

ie $\underline{\underline{z = 0.84}}$

$z = \frac{x-\mu}{\sigma}$

$0.84 = \frac{x-40}{10}$

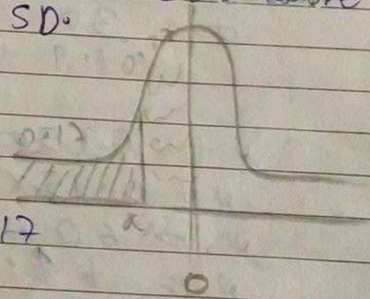
$x - 40 = 8.4$
$x = 8.4 + 40$
$x = \underline{\underline{48.4}}$

Ques 4 In a normal distribution 17% of the items are below 30 and 17% of the items are above 60. Find the mean and SD.

(a) 17% of items below 30 $(x < 30)$

$= \frac{17}{100} = 0.17$

$P(z<a) = 0.17$



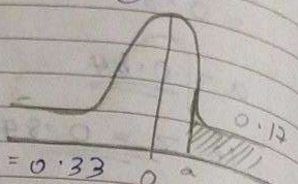
$P(x<30)$
$= 0.5 - 0.17 = 0.33$
$a = -0.95$ $(\because 0.3289 \rightarrow 0.95)$
ie $z = -0.95$

$z = \frac{x-\mu}{\sigma}$; $-0.95 = \frac{30-\mu}{\sigma}$

$-0.95\sigma = 30 - \mu$.

$\mu - 0.95\sigma = 30$ ——— ①

(b) P% of items above 60 ($x > 60$)



$P(z > a) = 0.17$

$P(x > 60) = 0.5 - 0.17 = 0.33$

$\therefore a = 0.95$ ie $z = 0.95$

$z = \frac{x-\mu}{\sigma}$; $0.95 = \frac{60-\mu}{\sigma}$

$0.95\sigma = 60 - \mu$

$\mu + 0.95\sigma = 60$ ——— ②

② − ①

$$\begin{aligned} \mu + 0.95\sigma &= 60 \\ -\ \mu - 0.95\sigma &= 30 \\ \hline 0 + 1.9\sigma &= 30 \end{aligned}$$

$\sigma = \frac{30}{1.9} = 15.8$

$\mu + 0.95\sigma = 60$

$\mu + 0.95 \times 15.8 = 60$

$\mu = 60 - 15.01$

$\mu = 44.99$

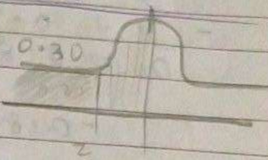
$\mu = 45$

ie $\sigma = 15.8$. $\mu = 45$

Ques5 The distribution is exactly normal. Find the mean and SD.

x	Frequency
less than 40	30
40 or more or less than 50	33
50 and more	37

(a) less than 40 ; $x < 40$



area $= 30/100 = 0.30$

$= 0.5 - 0.3$

$= 0.2$

ie $z = -0.52$ ($\because 0.1985 \to 0.52$)

$z = \frac{x-\mu}{\sigma}$

$-0.52 = \frac{40-\mu}{\sigma}$

$-0.52\sigma = 40 - \mu$

$\mu - 0.52\sigma = 40$ ——— ①

(b) 50 and more, $x > 50$

area $= 37/100 = 0.37$

$= 0.5 - 0.37$

$=$

ie $z = 0.33$ ($\because 0.1293 \to 0.33$)

$z = \frac{x-\mu}{\sigma}$

$0.33 = \frac{50-\mu}{\sigma}$

$0.33\sigma = 50 - \mu$

$\mu + 0.33\sigma = 50$ ——— (2)

(1) − (2)

$$\begin{aligned} \mu - 0.52\sigma &= 40 \\ -\ \mu + 0.33\sigma &= 50 \\ \hline 0 - 0.85\sigma &= -10 \end{aligned}$$

$-0.85\sigma = -10$

$\sigma = \dfrac{-10}{-0.85}$

$\sigma = 11.765$

$\sigma = \underline{\underline{11.8}}$

$\mu - 0.52\sigma = 40$

$\mu - 0.52 \times 11.8 = 40$

$\mu - 6.136 = 40$

$\mu = 40 + 6.136$

$\mu = 46.136$

OR

$\mu + 0.33\sigma = 50$

$\mu + 0.33 \times 11.8 = 50$

$\mu + 3.894 = 50$

$\mu = 50 - 3.894$

$\mu = 46.106$

ie $\sigma = 11.8$; $\mu = 46.1$

Ques 6 As a result of tests on 20000 electric bulbs manufactured by a company it was found that the life time of the bulb was normally distributed with an average life of 2040 hours and S.D of 60 hours. Estimate the number of bulbs that are expected to burn for (a) more than 2150 hours (b) less than 1960 hours

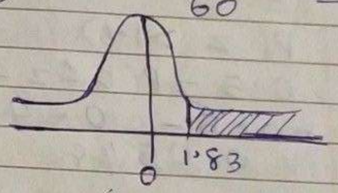
mean, $\mu = 2040$; SD, $\sigma = 60$

(a) more than 2150 hours

$P(x > 2150)$

$z = \dfrac{x - \mu}{\sigma} = \dfrac{2150 - 2040}{60} = \dfrac{110}{60} = \underline{\underline{1.83}}$

$P(z > 1.83)$



$= 0.5 - P(0 < z < 1.83)$

$= 0.5 - 0.4664$

$= \underline{\underline{0.0336}}$

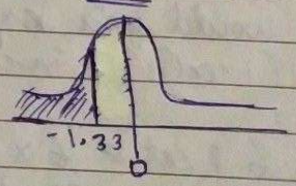
no of bulbs $\rightarrow 0.0336 \times 20{,}000$

$= \underline{\underline{672}}$

(b) less than 1960 hours $P(x < 1960)$

$z = \dfrac{1960 - 2040}{60} = \dfrac{-80}{60} = \underline{\underline{-1.33}}$

$P(z < \bar{1}.33)$



$= 0.5 - P(-1.33 < z < 0)$

$= 0.5 - 0.4082$

$= 0.0918$

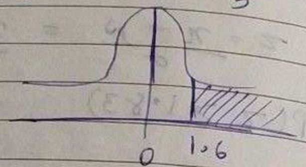
no. of bulbs → $0.0918 \times 20{,}000$
$= \underline{1836}$

Ques7 (27) If the heights of 1000 soldiers in a regiment are distributed normally with a mean of 172 cms and a standard deviation of 5cms, how many soldiers have heights greater than 180 cms

mean, $\mu = 172$, SD, $\sigma = 5$

$P(x > 180)$, $z = \frac{x-\mu}{\sigma} = \frac{180-172}{5} = \frac{8}{5}$

$\underline{z = 1.6}$



$P(z > 1.6)$
$= 0.5 - P(0 < z < 1.6)$
$= 0.5 - 0.4452$
$= 0.0548$

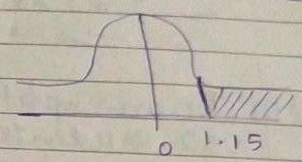
no of soldiers → 0.0548×1000
$= 54.8$
ie 55 soldiers

Ques8 (24) The mean height of the soldiers is 68.22 inches with a variance of $10.8\ (in)^2$. How many soldiers in a regiment of 1000 would you expect to be over 6 feet tall? (1 feet = 12 inches)

6 feet = 6×12
= 72 inches

mean $\mu = 68.22$ SD, $\sigma = 3.29$
variance of $\sigma^2 = 10.8$

$P(x > 72)$ $z = \frac{x-\mu}{\sigma} = \frac{72 - 68.22}{3.29}$
$= \frac{3.78}{3.29} = \underline{1.1489}$ $z = 1.149 = \underline{1.15}$



$P(z > 1.15)$
$= 0.5 - P(0 < z < 1.15)$
$= 0.5 - 0.3749$
$= \underline{0.1251}$

no of soldiers → 0.1251×1000
$= \underline{125.1}$
ie 125 soldiers

2.28

Ques9 (26) Packets of a certain washing powder are filled with an automatic machine with an average weight of 5 kg and SD of 50gm. If the weight of packets are normally distributed, find the percentage of packets having weight above 5.10 kg

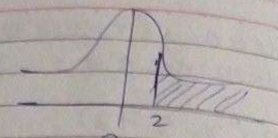
mean, $\mu = 5$ kg SD, $\sigma = 50$ gm $= 0.05$

$P(x > 5.10)$ $z = \frac{x-\mu}{\sigma} = \frac{5.10 - 5.0}{0.05} = \frac{0.10}{0.05}$

$\underline{z = 2}$

$P(z > 2) = 0.5 - P(0 < z < 2)$

$= 0.5 - 0.4772$
$= 0.0228$



percentage of pulets
0.0228×100
$= 2.28\%$

Ques 10 (20) In an aptitude test administered to 1000 students the mean score is 60 and the standard deviation being 20. Find (i) the number of students whose scores are between 35 and 75 (ii) whose scores exceeds 70 and (iii) whose scores are below 45

mean $\mu = 60$, SD, $\sigma = 20$ $\quad z = \frac{x-\mu}{\sigma}$

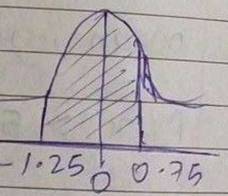
(i) $P(35 < x < 75)$

when $x = 35$ $\quad$ when $x = 75$

$z = \frac{35-60}{20} = \frac{-25}{20}$ $\quad z = \frac{75-60}{20} = \frac{15}{20}$

$z = -1.25$ $\quad z = 0.75$

$P(-1.25 < z < 0.75)$



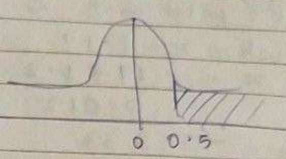
$= P(-1.25 < z < 0) + P(0 < z < 0.75)$
$= 0.3944 + 0.2734$
$= 0.6678$

no of students $\rightarrow 0.6678 \times 1000 = 667.8$
$= 668$

(ii) $P(x > 70)$ $\quad z = \frac{70-60}{20} = \frac{10}{20}$

$z = 0.5$

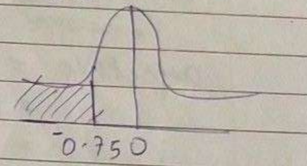
$P(z > 0.5)$
$= 0.5 - P(0 < z < 0.5)$
$= 0.5 - 0.1915$
$= 0.3085$



no of students $\rightarrow 0.3085 \times 1000 = 308.5$
$= 309$ stu

(iii) $P(x < 45)$ $\quad z = \frac{45-60}{20} = \frac{-15}{20} = -0.75$

$P(z < -0.75)$
$= 0.5 - P(-0.75 < z < 0)$
$= 0.5 - 0.2734$
$= 0.2266$



no of students $\rightarrow 0.2266 \times 1000 = 226.6$
$= 227$

Ques 11 (20) The mean of the inner diameter (in cm) of a sample of 200 tubes produced by a machine is 1.2 cm and SD is 0.012. The purpose for which these are intended allows a maximum tolerance in the diameter of 1.19 to 1.21 (ie otherwise the tubes are considered defective). What percentage of the tubes produced by the machine is defective if the diameters are found to be normally distributed?

mean, $\mu = 1.2$, SD, $\sigma = 0.012$, $z = \frac{x-\mu}{\sigma}$

$P(1.19 < x < 1.21)$

when $x = 1.19$

$z = \frac{1.19 - 1.2}{0.012} = \frac{-0.01}{0.012}$

$z = -0.833$

$z = \underline{-0.83}$

when $x = 1.21$

$z = \frac{1.21 - 1.20}{0.012}$

$z = \frac{0.01}{0.012} = \underline{0.83}$

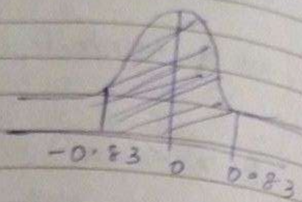
$P(-0.83 < z < 0.83)$

$= P(-0.83 < z < 0) +$

$P(0 < z < 0.83)$

$= 0.2967 + 0.2967$

$= \underline{0.5934}$



Defective = 1 − nondefective

$= 1 - 0.5934$

$= 0.4066$

percentage ⇒ 0.4066×100

$\underline{40.66\,\%}$

Ques 12 (32) Among 10000 random digits, in how many cases do we expect that the digit 3 appears at the most 960 times.

Ques 13 (34) If the heights of 1000 college men closely follow a normal distribution with a mean of 69.0 inches and a standard deviation of 2.5 inches (a) How many of these men would you expect to be at least 6 feet in height? (b) what range of heights would you expect include the middle 75% of the men in this group?

mean, $\mu = 69.0$ SD, $\sigma = 2.5$

$Z = \frac{x - \mu}{\sigma}$

(1 feet − 12 inches)

6 feet − 12 × 6 − 72

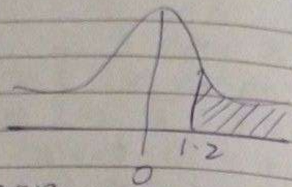
(a) $P(x > 72)$ $\quad z = \frac{72-69}{2.5} = \frac{3}{2.5} = 1.2$

$P(z > 1.2)$

$= 0.5 - P(0 < z < 1.2)$

$= 0.5 - 0.3849$

$= 0.1151$



no of men $\rightarrow 0.1151 \times 1000$

$= 115.1$

ie $\underline{\underline{115}}$

66.13 – 71.88

(b)

61.18 $\qquad$ 51.0, 204, 75.6

Ques 14 (a) In an university examination of a particular year 60% of the students ~~fal~~ failed when mean of the marks was 50%. and SD 5%. University decided to relax the conditions of passing by lowering the pass mark to show 70%. Find minimum marks for a student to pass, supposing the marks to be normally distributed and no change in the performance of students take place.

mean, $\mu = 50\% = \frac{50}{100} = \underline{\underline{0.5}}$ $\quad$ SD, $\sigma = 5\% = \underline{\underline{0.05}}$

$x = 70\% = \frac{70}{100} = 0.7$ $\qquad z = \frac{0.6 - 0.5}{0.05} = \underline{0.2}$

$P(-z < 0.7)$ $\qquad 0.4772$

$0.5 - 0.2580$

Ques 15 (38) In an aptitude test administered to 900 college students, the mean score is 50 and SD = 20. Find the number of students securing scores (i) between 30 and 70 (ii) exceeding 65. Find the value of the score exceeded by the top 90 students.

mean, $\mu = 50$; SD, $\sigma = 20$ $z = \frac{x-\mu}{\sigma}$

(i) $P(30 < x < 70)$

when $x = 30$

$z = \frac{30-50}{20} = \frac{-20}{20}$

$z = -1$

when $x = 70$

$z = \frac{70-50}{20} = \frac{20}{20}$

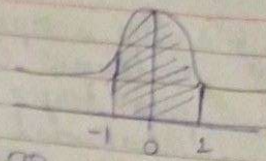
$z = 1$

$P(-1 < z < 1)$

$= P(-1 < z < 0) + P(0 < z < 1)$

$= 0.3413 + 0.3413$

$= 0.6826$



no of students $= 0.6826 \times 900$

$= 614.34$

614 students

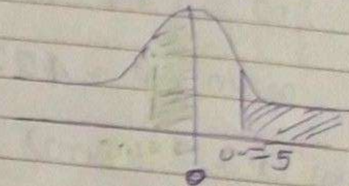
(ii) $P(x > 65)$ $z = \frac{65-50}{20} = \frac{15}{20} = +0.75$

$P(z > +0.75)$

$= 0.5 - P(0 < z < 0.75)$

$= 0.5 - 0.2734$

$= 0.2266$



no of students $= 0.2266 \times 900$

$= 203.94$

204 students

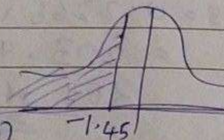
Ques16 A city corporation has installed 1000
(40) sodium lamps throughout the city on the same day. The life of bulbs is normally distributed with a mean 4000 hours and SD = 550 hours. Find the number of lamps that would be expected to fail
a) in the first 3200 hours of burning
b) between 3000 and 5200 hours (c) after what period of time would you expect only 10% of the lamps to be still burning.

mean, $\mu = 4000$, SD, $\sigma = 550$ $\quad z = \frac{x-\mu}{\sigma}$

(a) $P(x < 3200)$ $z = \frac{3200-4000}{550} = \frac{-800}{550}$

$z = -1.45$

$P(z < -1.45)$



$= 0.5 - P(-1.45 < z < 0)$
$= 0.5 - 0.4265$
$= 0.0735$

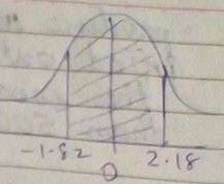
no of lamps → $0.0735 \times 1000 = 73.5$ ie $\underline{\underline{74}}$

(b) $P(3000 < x < 5200)$

when $x = 3000$	when $x = 5200$
$z = \frac{3000-4000}{550} = \frac{-1000}{550}$	$z = \frac{5200-4000}{550}$
$z = -1.82$	$= \frac{1200}{550} = 2.18$

$P(-1.82 < z < 2.18)$

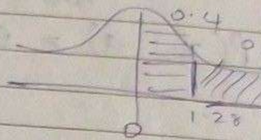
$= P(-1.82 < z < 0) + P(0 < z < 2.18)$
$= 0.4656 + 0.4854$
$= 0.951$



no of lamps → 0.951×1000
$= \underline{\underline{951}}$

(c) 10% of lamps $\frac{10}{100} = 0.1$ (area)

$P(z \geq a) = 0.1$
$= 0.5 - 0.1$
$= 0.4$



ie $a = 1.28$
$z = 1.28$ ($\because 0.3997 \Rightarrow 1.28$)

$z = \frac{x-\mu}{\sigma}$, $1.28 = \frac{x-4000}{550}$

$1.28 \times 550 = x - 4000$
$704 = x - 4000$
$x = 704 + 4000$
$x = \underline{\underline{4704}}$ lamps

Ques17 A baker from experience has found that the
(42) daily demand for his bread is approximately normally distributed with a mean of 7200 loaves and a standard deviation of 300 loaves. Based upon cost consideration the baker has decided that it is best to produce a sufficient number of loaves so that it will fully meet the demand on 94% of all the days. (i) determine how many loaves of bread the baker should

2·68, 45·627.

produce ~~as~~ based on production as above and (b) find out the percentage of days on which the company will be left with more than 500 loaves of unsold breads.

mean, $\mu = 7200$, SD, $\sigma = 30$ $\quad z = \frac{x-\mu}{\sigma}$

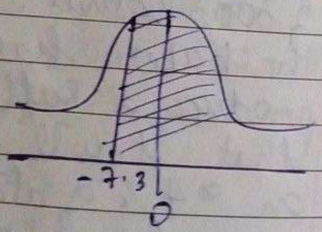
(i) 94% $\quad \frac{94}{100} = \underline{\underline{0·94}}$ (area)

$P(z \quad a) = 0·94$

(ii) $P(x > 500)$ $\quad z = \frac{500 - 7200}{30} = -\frac{[illegible]}{30}$

$z = [illegible]$

$P(z > [illegible])$

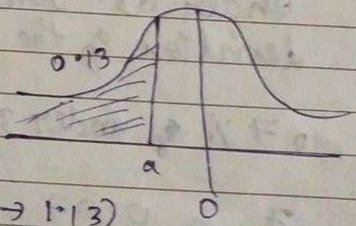


Ques18 (44) Give a normal distribution with mean = 50 and SD = 10. find the value of x that has (a) 13% of the area to its left and (b) 14% of the area to its right

mean, $\mu = 50$; SD, $\sigma = 10$

(a) $P(z < a) = \frac{13}{100}$ = ie $P(z < a) = 0·13$

$= 0·5 - 0·13$
$= 0·37$

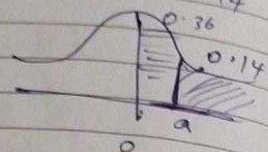


$a = -1·13$ ($\because 0·3708 \to 1·13$)
ie $z = -1·13$

$z = \frac{x-\mu}{\sigma}$; $-1·13 = \frac{x-50}{10}$

$-11.3 = x - 50$
$x = -11.3 + 50$
$x = \underline{\underline{38.7}}$

(b) $P(z > a) = \frac{14}{100}$ ie $P(z > a) = 0.14$



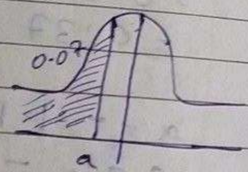
$= 0.5 - 0.14$
$= 0.36$
$a = 1.08$ ($\because 0.3599 - 1.08$)
$z = 1.08$

$z = \frac{x - \mu}{\sigma}$; $1.08 = \frac{x - 50}{10}$

$10.8 = x - 50$
$x = 10.8 + 50$
$x = \underline{\underline{60.8}}$

Ques19 (40) In a distribution exactly normal, 7% of the items are under 35 and 89% are under 63. What are mean and standard deviation of the distribution?

(i) 7% under 35 $(x < 35)$



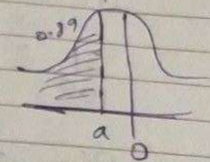
$\frac{7}{100} = 0.07$
$P(z < a) = 0.07$

$P(x < 35)$
$= 0.5 - 0.07 = 0.43$
$a = -1.48$ ie $z = -1.48$
($\because 1.48 \to 0.4306$)

$z = \frac{x - \mu}{\sigma}$; $-1.48 = \frac{35 - \mu}{\sigma}$

$-1.48\sigma = 35 - \mu$
$\mu - 1.48\sigma = 35$ —— (1)

(ii) 89% under 63 $(x < 63)$



$\frac{89}{100} = 0.89$
$P(z < a) = 0.89$

$P(x < 63)$
$= 0.5 - 0.89$
$= -0.39$
$a = -1.23$ ($\because 0.3907 \to 1.23$)
ie $z = -1.23$

$z = \frac{x - \mu}{\sigma}$; $-1.23 = \frac{63 - \mu}{\sigma}$

$-1.23\sigma = 63 - \mu$
$\mu - 1.23\sigma = 63$ —— (2)

(1) − (2)

$\mu - 1.48\sigma = 35$
$\mu - 1.23\sigma = 63$
$0 - 0.25\sigma = -28$

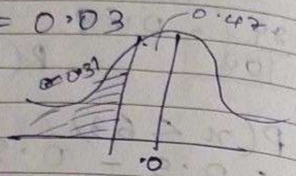
$\sigma = \frac{-28}{-0.25}$

Ques 20 (48) In a normal distribution 31% of the items are under 45 and 8% are over 64. Find the mean and SD

(i) 31% under 45 $(x < 45)$

$\frac{31}{100} = 0.31$

$P(z < a) = 0.03$



$P(x < 45)$
$= 0.5 - 0.31$
$= 0.19$
$a = -0.5 \; (\because 0.5 \to 0.1915)$
$z = -0.5$

$z = \frac{x-\mu}{\sigma}$; $-0.5 = \frac{45-\mu}{\sigma}$

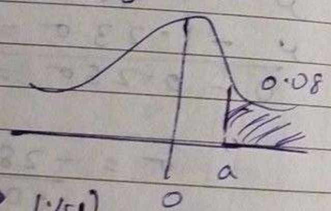
$-0.5\sigma = 45 - \mu$
$\mu - 0.5\sigma = 45$ — (1)

(ii) 8% over 64 $(x > 64)$

$\frac{8}{100} = 0.08$

$P(z > a) = 0.08$

$P(x > 64)$
$= 0.5 - 0.08$
$= 0.42$
$a = 1.41 \; (\because 0.4207 \to 1.41)$
$z = 1.41$



$z = \frac{x-\mu}{\sigma}$; $1.41 = \frac{64-\mu}{\sigma}$

$1.41\sigma = 64 - \mu$
$\mu + 1.41\sigma = 64$ — (2)

(1) − (2)

$$\begin{array}{r} \mu - 0.5\sigma = 45 \\ - \; \mu + 1.41\sigma = 64 \\ \hline 0 - 1.91\sigma = -19 \end{array}$$

$\sigma = \frac{-19}{-1.91} = \frac{19}{1.91}$

$\sigma = 9.947$
$\sigma = 10$

$\mu - 0.5 \times 10 = 45$
$\mu - 5 = 45$
$\mu = 45 + 5$
$\mu = 50$

$\sigma = 10$ and $\mu = 50$

Ques 21 (50) The wage distribution of the workers in a factory is normal with mean Rs 400 and SD = 50. If the wage of 40 workers be less than Rs 350, what is the total number of workers in the factory?

mean, $\mu = 400$, SD, $\sigma = 50$ $z = \frac{x-\mu}{\sigma}$

Ques22 (52) Fit a normal curve to the following data

Class :	60-62	63-65	66-68	69-71	72-74
f :	5	18	42	27	8

Class	f	(x) Mid	d' $\left(\frac{x-a}{c}\right)$	fd'	fd'^{2}
59.5 - 62.5	5	61	-2	-10	20
62.5 - 65.5	18	64	-1	-18	18
65.5 - 68.5	42	67	0	0	0
68.5 - 71.5	27	70	1	27	27
71.5 - 74.5	8	73	2	16	32
	100 Σf			15 $\Sigma fd'$	97 $\Sigma fd'^2$

$$\text{Mean} = a + \frac{\Sigma fd'}{N} \times c \qquad (a \to 67,\ N \to 100,\ c \to 3,\ \Sigma fd' \to 15)$$

$$= 67 + \frac{15}{100} \times 3$$

$$\text{Mean} = 67 + 0.45 = \underline{67.45}$$

$$SD = \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} \times c$$

$$SD = \sqrt{\frac{97}{100} - \left(\frac{15}{100}\right)^2} \times \cancel{3} = \sqrt{0.97 - 0.0225} \times \cancel{3}$$

$$SD = \sqrt{0.9475} \times \cancel{3} = 0.973396 \times 3$$

$$= \cancel{2.92}$$

$$= 0.9734 \times 3$$

$$SD = 2.920\cancel{2}$$

$$\mu = \underline{\underline{67.45}} \qquad \sigma = \underline{\underline{2.92}}$$

$z = \frac{x - 67.05}{2.92}$

class limit	$z = \frac{x-\mu}{\sigma}$	Area from table	Area for classes	area × N (area × 100)	rounding off
$-\infty$	$-\infty$	0.5	—	—	
62.5	−1.69	0.4545	0.0455	4.55	5
65.5	−0.66	0.2454	0.2091	20.91	21
68.5	0.35	0.1368	0.3822	38.22	38
71.5	1.38	0.4162	0.2794	27.94	28
∞	∞	0.5	0.0838	8.38	8

Ques23 Fit a normal curve to the following data.
(54)

height	59.5-62.5	62.5-65.5	65.5-68.5	68.5-71.5	71.5-74.5
No	5	18	42	27	8

(SAME AS LAST QUESTION)

Ques24 In an examination it is laid down that a
(56) student passes if he ~~scores~~ secures 30% or more marks. He is placed in the first, second or third division according as he secures 60% or more marks, between 45% and 60% marks and marks between 30% and 45% respectively.

He gets a distinction in case he secures 80% or more marks. It is noticed from the results that 10% of the students failed in the examination whereas 5% of them obtained distinction. Calculate the % of students placed in the second division (Assume normal distribution)

34%

Fitting Normal Curve

Main problem in fitting the normal distribution is estimating the parameters of the distribution. Parameters of the distribution are μ and σ. So find mean and standard deviation of the given frequency distribution,

Take them as the estimates of the parameters μ and σ. Then we get the density function

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$$

By substituting values of μ and σ in this equation we get the equation of the best fitting normal distribution.

To get the theoretical frequencies, following steps are applied.

1. Calculate mean and S.D. of the given distribution
2. Calculate z value for all class limits, using the formula $z = \frac{x-\mu}{\sigma}$
3. Find the area of each z value from the standard normal table.
4. Find the area of each class (by subtracting or adding, as the case maybe, with the areas obtained for 'z' values.)
5. Multiply the area of each class by the total frequency to get the class frequency

The new frequency distribution with theoretical frequencies, will be a normal approximation to the given ~~distribution~~ frequency distribution.

Ques) Fit a normal distribution to the following data:

Marks:	10-20	20-30	30-40	40-50	50-60	60-70	70-80
No. of students:	4	22	48	66	40	16	4

To find mean and SD $(x-a)$

Class	f	Mid x	d' $\left(\frac{x-a}{c}\right)$	fd'	d'^2	fd'^2
10-20	4	15				
20-30	22	25				
30-40	48	35				
40-50	66	45				
50-60	40	55				
60-70	16	65				
70-80	4	75				

$$\text{Mean} = a + \frac{\Sigma fd'}{N} \times c$$

$=$

$$SD = \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} \times c$$

The density function $f(x)$ is

(1) Class limit of x	(2) $z = \frac{x-\mu}{\sigma}$	(3) Area from table	(4) Area for interval	(5) = (4) × N Theoretical freq.	Round off

Various steps

1. Find first mean and SD of the given distribution i.e. μ and σ.
2. Then for each 'x' value (the class limit), get z value using the formula, $z = \frac{x-\mu}{\sigma}$. These z values form column 3
3. Column 4 is the difference between two adjacent values of column 3 except two values where z changes from negative to positive.
4. Column 5 is obtained by multiplying all the values in column 4 with total frequency 100.

NOTE: (a) First and last 'z' values are taken as $-\infty$ and $+\infty$

(b) In column 4, third value is obtained by adding two values of the previous column instead of taking the difference (when z values changes from negative to positive we add

Q. Fit a normal curve to the following data.

class interval:	5-9	10-14	15-19	20-24	25-29
frequency:	10	22	40	21	7

To find mean and SD $(x-17)/5$ $c = 5$

class	f	x value (mid)	$d' = \frac{x-a}{c}$	fd'	d'^2	fd'^2
4.5-9.5	10	7	-2	-20	4	40
9.5-14.5	22	12	-1	-22	1	22
14.5-19.5	40	17 a	0	0	0	0
19.5-24.5	21	22	1	21	1	21
24.5-29.5	7	27	2	14	4	28
	$\Sigma f = 100$			$\Sigma fd' = -7$		$\Sigma f d'^2 = 111$

N=100

$$\text{Mean} = a + \frac{\Sigma fd'}{N} \times c = 17 + \frac{-7}{100} \times 5$$

$$\text{Mean} = 17 - 0.35 = 16.65$$

$$SD = \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} \times c = \sqrt{\frac{111}{100} - \left(\frac{7}{100}\right)^2} \times 5$$

$$= \sqrt{\frac{111}{100} - \frac{49}{1000}} \times 5 = \sqrt{\frac{11051}{1000}} \times 5 = \sqrt{1.1051} \times 5$$

$$= 1.0512 \times 5$$

$$SD = 5.256 = 5.25$$

$(x-16.65)/5.25$

Class limit	$z = \frac{x-\mu}{\sigma}$	area from table	Area for classes	frequency area × N	Round of values
$-\infty$	$-\infty$	0.5	–	–	–
9.5	-1.36	0.4131	0.0869	8.69	9
14.5	-0.40	0.1554	0.2577	25.77	26
19.5	0.54	0.2054	0.3608	36.08	36
24.5	1.49	0.4319	+0.2265	22.65	23
∞	∞	0.5	+0.0681	6.81	7

101

$$f(x) = \frac{1}{\sqrt{2\pi} \times 5.25} e^{-\frac{1}{2}\left(\frac{x-16.65}{5.25}\right)^2}$$