3

Linear Programming Problem —Graphical Solution and Extension

"Graph is the best tool to visualise any concept"

3:1. INTRODUCTION

Linear programming problems involving two decision variables can easily be solved by *graphical method*. The method also provides an insight into the concepts of *Simplex Method*—a powerful technique to solve the linear programming problems involving three or more decision variables.

3:2. GRAPHICAL SOLUTION METHOD

The major steps in the solution of a linear programming problem by graphical method are summarised as follows :

Step 1. Identify the problem — the decision variables, the objective and the restrictions.

Step 2. Set up the mathematical formulation of the problem.

Step 3. Plot a graph representing all the constraints of the problem and identify the feasible region (solution space). The feasible region is the intersection of all the regions represented by the constraints of the problem and is restricted to the first quadrant only.

Step 4. The feasible region obtained in *step 3* may be bounded or unbounded. Compute the coordinates of all the corner points of the feasible region.

Step 5. Find out the value of the objective function at each corner (solution) point determined in *step 4*.

Step 6. Select the corner point that optimizes (maximizes or minimizes) the value of the objective function. It gives the *optimum feasible solution.*

Remarks. *1.* The above method is known as *Search Approach Method.*

2. Another method known as *Iso-Profit* or *Iso-cost approach*, involves the following steps :

(*a*) First *four steps* are same as in the Search Approach. In the *fifth step* we choose a convenient profit (or cost) and draw iso-profit (iso-cost) line so that it falls within the feasible region.

(*b*) Move this iso-profit (or iso-cost) line parallel to itself farther (closer) from (to) the origin.

(*c*) Identify the optimum solution as the coordinates of that point on the feasible region touched by the highest possible iso-profit line (or lower-possible iso-cost line).

(*d*) Compute the *optimum feasible solution.*

SAMPLE PROBLEMS

301. *A company makes two kinds of leather belts. Belt A is a high quality belt, and belt B is of lower quality. The respective profits are Rs. 4.00 and Rs. 3.00 per belt. Each belt of type A requires* *twice as much time as a belt of type B, and if all belts were of type B, the company could make 1000 belts per day. The supply of leather is sufficient for only 800 belts per day (Both A and B combined). Belt A requires a fancy buckle and only 400 buckles per day are available. There are only 700 buckles a day available for belt B. Determine the optimal product mix.* **[Madras M.B.A. 2006; Delhi M.Com. 2005; M.B.A. (Nov.) 2009]**

Solution.

Step 1. The appropriate mathematical formulation of the given linear programming problem is :

Maximize $z = 4x_1 + 3x_2$ subject to the constraints :

$$2x_1 + x_2 \le 1{,}000 \qquad \textit{(Time constraint)}$$
$$x_1 + x_2 \le 800 \qquad \textit{(Availability of Leather)}$$
$$x_1 \le 400 \text{ and } x_2 \le 700 \qquad \textit{(Availability of Buckles)}$$
$$x_1 \ge 0 \text{ and } x_2 \ge 0,$$

where x_1 = number of belts of type A, and x_2 = number of belts of type B.

Step 2. Next we construct the graph by considering the cartesian rectangular axis OX_1X_2 in the plane. As each point has the coordinates of the type (x_1, x_2) ; any point satisfying the conditions $x_1 \ge 0$ and $x_2 \ge 0$ lies in the first quadrant.

Now, the inequalities are graphed taking them as equations, *e.g.*, the first constraint $2x_1 + x_2 \le 1000$ will be graphed as $2x_1 + x_2 = 1000$. The equation is re-written as $\frac{x_1}{500} + \frac{x_2}{1000} = 1$.

This equation indicates that when it is plotted on the graph, it cuts an x_1-intercept of 500 and x_2-intercept of 1000. These two points are then connected by a straight line which is shown in Fig. 3.1(a) as line AB. Any point (representing a combination of x_1 and x_2) that falls on this line or in the area below it, is acceptable in so far as this constraint is concerned. The region OAB formed by two axes and the line representing the equation $2x_1 + x_2 = 1000$ is the region containing acceptable values of x_1 and x_2 in respect of this constraint.

Similarly, the constraint $x_1 + x_2 \le 800$ can be plotted. The line CD in Fig. 3.1(b) represents the equation $x_1 + x_2 = 800$. The region OCD, formed by the two axes and this line represents the area in which any point would satisfy this constraint of leather availability. Further, the constraints $x_1 \le 400$ and $x_2 \le 700$ are also plotted on the graph which represents the area between the two axes and the lines $x_1 = 400$ and $x_2 = 700$ as shown in Fig. 3.1(b).

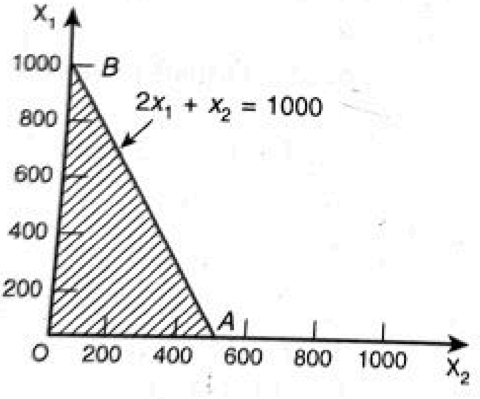

Fig. 3.1(a)

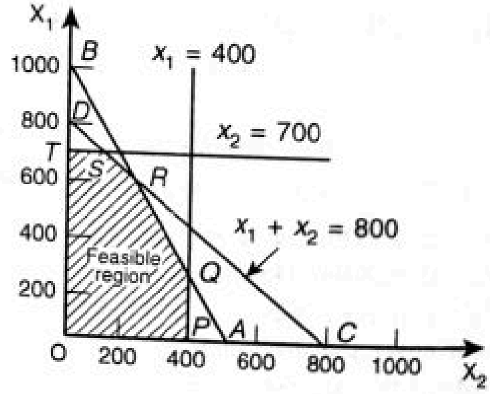

Fig. 3.1(b)

Now all the constraints have been graphed. The area bounded by all these constraints, called feasible *region* or *solution space*, is as shown in Fig. 3.1(b) by the shaded area $OPQRST$.

Step 3. The optimum value of objective function occurs at one of the extreme (corner) points of the feasible region. The coordinates of the extreme points are :

O = (0, 0), P = (400, 0), Q = (400, 200), R = (200, 600), S = (100, 700), and T = (0, 700).

Step 4. We now compute the z-values corresponding to the extreme points :

Extreme point	(x_1, x_2)	$z = 4x_1 + 3x_2$
O	(0, 0)	0
P	(400, 0)	1600
Q	(400, 200)	2200
R	(200, 600)	**2600** ⟵ maximum
S	(100, 700)	2500
T	(0, 700)	2100

Step 5. The optimum solution is that extreme point for which the objective function has the largest value. Thus, the optimum solution occurs at the point R, *i.e.*, $x_1 = 200$ and $x_2 = 600$ with the objective function value of Rs. 2600.

Hence, to maximize profit, the company should produce 200 belts of type A and 600 belts of type B per day.

Alternative Method (Iso-profit approach)

The feasible region (solution space) obtained in *step 2* is as shown in Fig. 3.1(c) by the shaded area *OPQRST*.

Let the profit to the company (arbitrary) is Rs. 1200. The objective function then becomes :

$$4x_1 + 3x_2 = 1200.$$

We draw this equation as a straight line in the feasible region shown in Fig. 3.1(c). This line is known as *iso-profit line*.

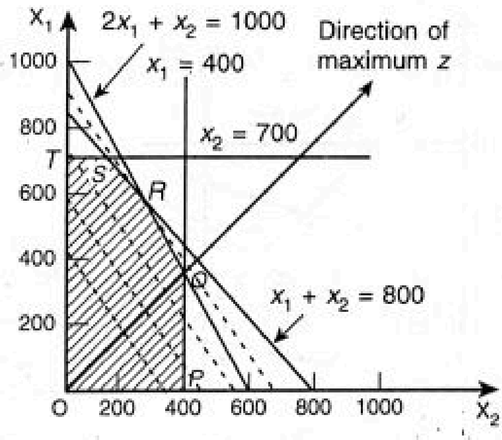

Fig. 3.1(c)

It may be noted that the iso-profit function (objective value function) is a straight line on which every point has the same total profit.

Now, we move the iso-profit line parallel to itself farther from the origin. We observe that one of the iso-profit line touches only point R before leaving the feasible region. This iso-profit line is termed as highest possible iso-profit line and point R gives the extreme point of the solution space.

Hence, the optimum feasible solution is :

$$x_1 = 200 \text{ and } x_2 = 600 \text{ with Maximum } z = \text{Rs. } 2600.$$

302. *Let us assume that you have inherited Rs. 1,00,000 from your father-in-law that can be invested in a combination of only two stock portfolios, with the maximum investment allowed in either portfolio set at Rs. 75,000. The first portfolio has an average rate of return of 10%, whereas the second has 20%. In terms of risk factors associated with these portfolios, the first has a risk rating of 4 (on a scale from 0 to 10), and the second has 9. Since you wish to maximize your return, you will* *not accept an average rate of return below 12% or a risk factor above 6. Hence, you then face the important question. How much should you invest in each portfolio?*

Formulate this as a Linear Programming Problem and solve it by Graphic Method.

[C.A. Final (May) 1999]

Solution.

Step 1. The appropriate mathematical formulation of the linear programming problem is :

$$\text{Maximize } z = 0.10x_1 + 0.20x_2 \text{ subject to the constraints :}$$

$$x_1 + x_2 \leq 1,00,000, \; x_1 \leq 75,000, \; x_2 \leq 75,000$$

$$0.10x_1 + 0.20x_2 \geq 0.12(x_1 + x_2) \text{ or } -0.02x_1 + 0.08x_2 \geq 0$$

$$4x_1 + 9x_2 \leq 6(x_1 + x_2) \text{ or } -2x_1 + 3x_2 \leq 0$$

$$x_1 \geq 0 \text{ and } x_2 \geq 0$$

where x_1 = amount invested in portfolio *1*, and x_2 = amount invested in portfolio 2.

Step 2. The first constraint $x_1 + x_2 \leq 1,00,000$ can be graphed by plotting the straight line $\frac{x_1}{1,00,000} + \frac{x_2}{1,00,000} = 1$. This cuts a x_1-intercept and x_2-intercept of 1,00,000 each. The area below this line represents the feasible area in respect of this constraint. Similarly, the other constraints are depicted by plotting the straight lines corresponding to the equations $x_1 = 75,000$, $x_2 = 75,000$, $-2x_1 + 3x_2 = 0$, and $-0.02x_1 + 0.08x_2 = 0$. Here, the area below the first three lines and beyond the fourth line gives the feasible region in respect of these four constraints.

Thus, the feasible region in respect of the given problem is as shown in Fig. 3.2.

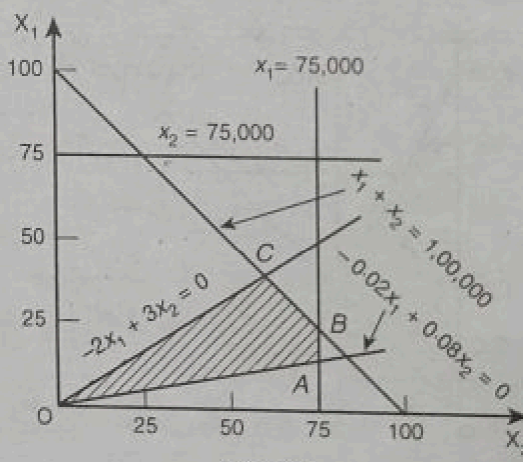

Fig. 3.2

Step 3. The coordinates of the extreme points are : $O = (0, 0)$, $A = (75,000, 18,750)$, $B = (75,000, 25,000)$ and $C = (60,000, 40,000)$.

Step 4. The z-values corresponding to the extreme points are :

Extreme point	(x_1, x_2)	$z = 0.10x_1 + 0.20x_2$
O	(0, 0)	0
A	(75,000, 18,750)	11,250
B	(75,000, 25,000)	12,500
C	(60,000, 40,000)	**14,000** ⟵ maximum

Hence, the optimum solution is :

$$x_1 = 60,000, \; x_2 = 40,000 \text{ and maximum return} = \text{Rs. } 14,000.$$

303. *A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents (call them X, Y and Z), it is necessary* *to buy two additional products, say, A and B. One unit of product A contains 36 units of X, 3 units of Y and 20 units of Z. One unit of product B contains 6 units of X, 12 units of Y and 10 units of Z. The minimum requirement of X, Y and Z is 108 units, 36 units and 100 units respectively. Product A costs Rs. 20 per unit and product B Rs. 40 per unit.*

Formulate the above as a linear programming problem to minimize the total cost, and solve the problem by using graphic method. **[C.A. Final (May) 2002]**

Solution. *Step 1.* The data of the given problem can be summarised as follows :

Nutrient constituents	*Nutrient content in product*		*Minimum amount of nutrient*
	A	*B*	
X	36	06	108
Y	03	12	36
Z	20	10	100
Cost of Product	Rs. 20	Rs. 40	

Making use of above information, the appropriate mathematical formulation of the linear programming problem is :

Minimize $z = 20x_1 + 40x_2$ subject to the constraints :

$$36x_1 + 6x_2 \geq 108,\ 3x_1 + 12x_2 \geq 36,\ 20x_1 + 10x_2 \geq 100, \text{ and } x_1, x_2 \geq 0.$$

where x_1 = number of units of product A, and x_2 = number of units of product B.

Step 2. Consider now a set of cartesian rectangular axis OX_1X_2 in the plane. As each point has the coordinates of the type (x_1, x_2), any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant only.

The constraints of the given problem are plotted as described earlier by treating them as equations :

$$36x_1 + 6x_2 = 108,\ 3x_1 + 12x_2 = 36 \text{ and } 20x_1 + 10x_2 = 100$$

or
$$\frac{x_1}{3} + \frac{x_2}{18} = 1,\ \frac{x_1}{12} + \frac{x_2}{3} \text{ and } \frac{x_1}{5} + \frac{x_2}{10} = 1.$$

The area beyond these lines represents the feasible region in respect of these constraints; any point on the straight lines or in the region above these lines would satisfy the constraints. The feasible region of the problem is as shown in Fig. 3.3.

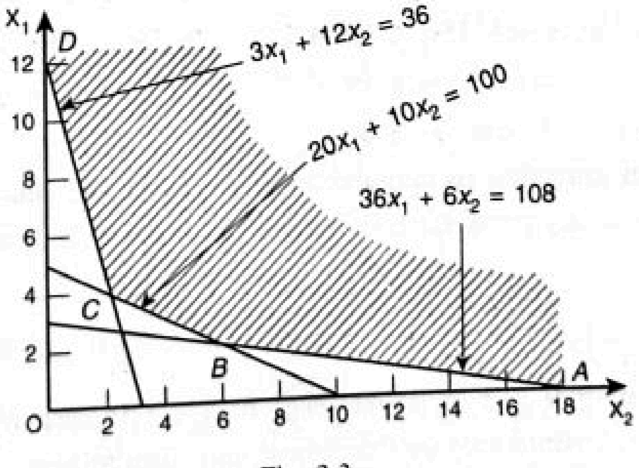

Fig. 3.3

Step 3. The coordinates of the extreme points of the feasible region are :

$$A = (0,\ 18),\ B = (2,\ 6),\ C = (4,\ 2) \text{ and } D = (12,\ 0).$$

Step 4. The value of objective function at each of the extreme points can be evaluated as follows :

Extreme point	(x_1, x_2)	$z = 20x_1 + 40x_2$
A	(0, 18)	720
B	(2, 6)	280
C	(4, 2)	**160** ⟵ minimum
D	(12, 0)	240

Hence, the optimum solution is to purchase 4 units of product *A* and 2 units of product *B* in order to maintain a minimum cost of Rs. 160.

304. *A company has two grades of inspectors 1 and 2 who are to be assigned to a quality inspection work. It is required that at least 1,800 pieces are inspected per 8-hour day. Grade 1 inspectors can check pieces at the rate of 25 per hour with an accuracy of 98%. Grade 2 inspectors can check at the rate of 15 pieces per hour with an accuracy of 95%. The wage rate for grade 1 inspector is Rs. 40 per hour while that of grade 2 is Rs. 30 per hour. Each time an error is caused by the inspector the cost to the company is Rs. 20. The company has eight grade 1 and ten grade 2 inspectors. The company wants to determine the optimal assignment of inspectors to minimise total inspection cost. Formulate it as LPP and solve using graphical method.* ***[Annamalai M.B.A. 2002]***

Solution.

Step 1. The data of the given problem can be summarised as follows :

	Grade 1 inspector	*Grade 2 inspector*
Number of inspectors	8	10
Rate of checking per hour	25 pieces	15 pieces
Inaccuracy in checking	1 − 0.98 = 0.02	1 − 0.95 = 0.05
Cost of inaccuracy in checking	Rs. 20	Rs. 20
Wage rate per hour	Rs. 40	Rs. 30

Let x_1 and x_2 designate the number of Grade 1 and Grade 2 inspectors, respectively.

Hourly costs of each Grade *1* and Grade 2 inspectors are given by :

Grade *1* inspector : Rs. $(40 + 20 \times 0.02 \times 25)$ = Rs. 50.

Grade 2 inspector : Rs. $(30 + 20 \times 0.05 \times 15)$ = Rs. 45.

Using the above information, the appropriate linear programming problem is :

Minimize $z = 8 \times 50x_1 + 8 \times 45x_2 = 400x_1 + 360x_2$ subject to the constraints :

$8 \times 25x_1 + 8 \times 15x_2 \geq 1{,}800$ or $5x_1 + 3x_2 \geq 45$ *(Number of pieces)*

$x_1 \leq 8,\ x_2 \leq 10$ *(Number of inspectors)*

$x_1 \geq 0$ and $x_2 \geq 0$ *(Non-negative restriction)*

Step 2. Consider a set of cartesian rectangular axis OX_1X_2 in the plane. As each point has the coordinates of the type (x_1, x_2), any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant only.

The first constraint $5x_1 + 3x_2 \geq 45$ can be graphed by plotting the straight line $\frac{x_1}{9} + \frac{x_2}{15} = 1$.

This gives an x_1-intercept of 9 and an x_2-intercept of 15. The area away from the origin, *i.e.*, above this line represents the feasible area of this constraint. Similarly, the other two constraints are depicted by plotting the straight lines corresponding to the equations $x_1 = 8$ and $x_2 = 10$. Here, the area below these lines (towards the origin) gives the feasible area. The common area *ABC* which satisfies all the three constraints is the solution space.

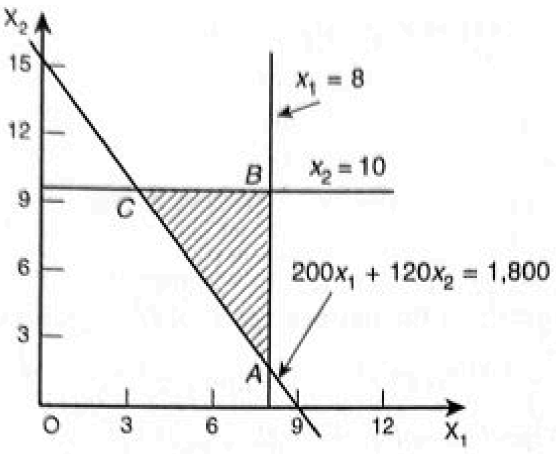

Fig. 3.4

Step 3. The coordinates of the extreme points are : $A = (8, 5/3)$, $B = (8, 10)$, and $C = (3, 10)$.

Step 4. The z-value corresponding to the extreme points are :

Extreme point	(x_1, x_2)	$z = 400x_1 + 360x_2$
A	(8, 5/3)	**3,800** ⟵ minimum
B	(8, 10)	6,800
C	(3, 10)	4,800

Hence, the optimum solution is :

$$x_1 = 8 \text{ and } x_2 = 1.7 \text{ or } 2 \text{ (approximately), minimum } z = 3,800$$

Thus, 8 grade *1* inspectors and 2 grade 2 inspectors should be assigned to have Rs. 3,800 as the total minimum inspection cost.

305. *Use the graphical method to solve the following LPP :*

$$\textit{Minimize} \quad z = -x_1 + 2x_2 \textit{; subject to the constraints :}$$

$$-x_1 + 3x_2 \le 10, \quad x_1 + x_2 \le 6,$$

$$x_1 - x_2 \le 2, \quad \textit{and} \quad x_1 \ge 0, \; x_2 \ge 0.$$

Solution. The constraints are re-written in the intercept form. Thus, we write

$$\frac{x_1}{-10} + \frac{x_2}{10/3} \le 1, \quad \frac{x_1}{6} + \frac{x_2}{6} \le 1, \text{ and } \frac{x_1}{2} + \frac{x_2}{-2} \le 1$$

First, we treat these inequalities as equations and graph them as straight lines. Now, considering the inequalities, shade the feasible region for each constraint. The common region *OABCD* gives the solution space of the LPP. It may be noted that we have considered the feasible region only in the first quadrant of the cartesian axis, as $x_1 \ge 0$ and $x_2 \ge 0$.

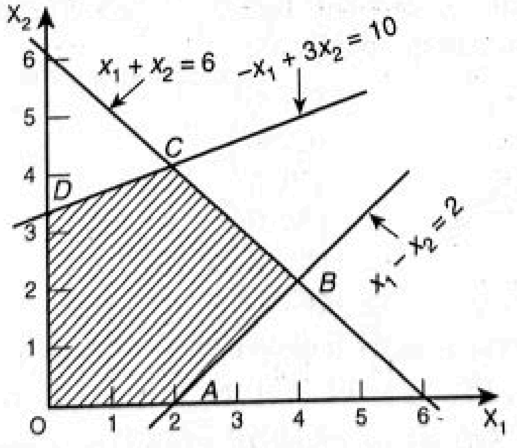

Fig. 3.5

The coordinates of the extreme points are :

$$O = (0, 0), \; A = (2, 0), \; B = (4, 2), \; C = (2, 4) \text{ and } D = \left(0, \frac{10}{3}\right).$$

The z-value corresponding to extreme points are :

Extreme point	(x_1, x_2)	$z = -x_1 + 2x_2$
O	(0, 0)	0
A	(2, 0)	**−2** ⟵ minimum
B	(4, 2)	0
C	(2, 4)	6
D	$(0, \frac{10}{3})$	$\frac{20}{3}$

The minimum value of z occurs at the extreme point A (2, 0). Hence, the optimum solution of the LPP is :

$$x_1 = 2,\ x_2 = 0 \text{ and minimum } z = -2.$$

306. *Use the graphical method to solve the following LPP :*

$$\textit{Maximize } z = 2x_1 + 3x_2; \textit{ subject to the constraints :}$$

$$x_1 + x_2 \le 30,\ x_1 - x_2 \ge 0,\ x_2 \ge 3,$$

$$0 \le x_1 \le 20 \text{ and } 0 \le x_2 \le 12.$$

Solution. To graph the given inequalities, we first treat them as equations $x_1 + x_2 = 30$, $x_1 - x_2 = 0$, $x_2 = 3$, $x_1 = 20$ and $x_2 = 12$ and plot each of these equations as straight lines. We use the inequality condition of each constraint to plot the corresponding feasible region. The common region $ABCDE$ satisfying all the inequalities (feasible region) is shown in shaded Fig. 3.6.

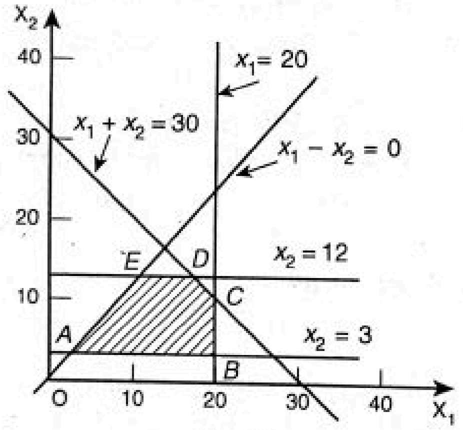

Fig. 3.6

The coordinates of the extreme points of the feasible region are :

$A = (3, 3)$, $B = (20, 3)$, $C = (20, 10)$, $D = (18, 12)$, and $E = (12, 12)$.

The z-values corresponding to extreme points are :

Extreme point	(x_1, x_2)	$z = 2x_1 + 3x_2$
A	(3, 3)	15
B	(20, 3)	49
C	(20, 10)	70
D	(18, 12)	**72** ⟵ maximum
E	(12, 12)	60

The maximum value of z occurs at the extreme point D (18, 12). Hence, the optimum solution is :

$$x_1 = 18,\ x_2 = 12 \text{ and maximum } z = 72.$$

307. *The advertising agency wishes to reach two types of audiences, customers with annual incomes greater than Rs. 40,000 (target audience A) and customers with annual incomes of less than Rs. 40,000 (target audience B). The total advertising budget is Rs. 2,00,000. One programme of TV advertising costs Rs. 50,000; one programme of radio advertising costs Rs. 20,000. For contract* *reasons, at least 3 programmes ought to be on TV and the number of radio programmes must be limited to 5. Surveys indicate that a single TV programme reaches 7,50,000 customers in target audience A and 1,50,000 in target audience B. One radio programme reaches 40,000 in target audience A and 2,60,000 in target audience B.*

Formulate this as a linear programming problem and determine the media mix to maximize the total reach.

[Delhi M.B.A. (Oct.) 2010]

Solution. Let x_1 and x_2 represent the number of programmes to be released on TV and radio, respectively. The mathematical formulation of the LPP is :

$$\begin{aligned} \text{Maximize} \quad z &= \text{TV programme viewers} + \text{Radio programme viewers} \\ &= (7{,}50{,}000 + 1{,}50{,}000)x_1 + (40{,}000 + 2{,}60{,}000)x_2 \\ &= 9{,}00{,}000x_1 + 3{,}00{,}000x_2 \end{aligned}$$

subject to the constraints :

$$50{,}000x_1 + 20{,}000x_2 \le 2{,}00{,}000 \qquad \textit{(Budget constraint)}$$

$$x_1 \ge 3 \text{ and } x_2 \le 5 \qquad \textit{(Programme constraints)}$$

$$x_1 \ge 0 \text{ and } x_2 \ge 0 \qquad \textit{(Non-negative restriction)}$$

For solving this L.P.P. graphically, we first consider the inequalities as equations and draw the straight lines of these equations. Now, consider the inequalities and shade the feasible region satisfying all the inequalities, *viz.*, the region *ABC* consisting of the extreme points *A*, *B* and *C*.

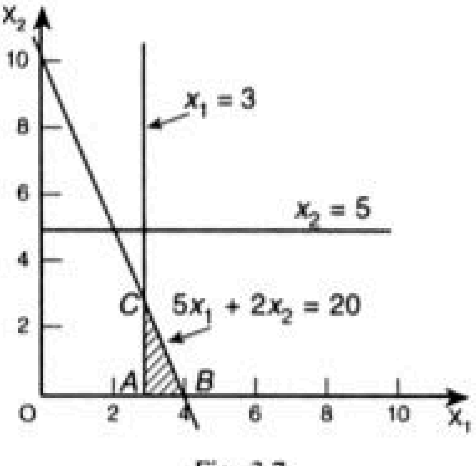

Fig. 3.7

The coordinates of the extreme points of feasible region are :

$$A = (3, 0), \ B = (4, 0) \text{ and } C = (3, 5/2).$$

The z-value corresponding to extreme points are :

Extreme point	(x_1, x_2)	$z = 9{,}00{,}000x_1 + 3{,}00{,}000x_2$	
A	(3, 0)	27,00,000	
B	(4, 0)	**36,00,000**	←—maximum
C	(3, 5/2)	34,50,000	

Since, the maximum value of z occurs at the extreme point $B\,(4, 0)$; the agency must release 4 programmes on TV and no programme on radio to achieve the maximum reach of 36,00,000 audiences.

PROBLEMS

308. A firm manufactures two products *A* and *B* on which the profits earned per unit are Rs. 3 and Rs. 4 respectively. Each product is processed on two machines M_1 and M_2. Product *A* requires one minute of processing time on M_1 and two minutes on M_2 while *B* requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hours and 30 minutes, while machine M_2 is available for 10 hours during any working day. Find the number of units of products *A* and *B* to be manufactured to get maximum profit. ***[Madurai M.Com. (Nov.) 2003]***

309. A farmer has a supply of chemical fertilizer of Type I which contains 10% nitrogen and 6% phosphoric acid and Type II fertilizer which contains 5% nitrogen and 10% phosphoric acid. After testing the soil conditions of a field, it is found that at least 14 kg. of nitrogen and 14 kg. of phosphoric acid is required for a good crop. The fertilizer Type I costs Rs. 2 per kg. and the Type II costs Rs. 3 per kg. How many kilograms of each fertilizer should be used to meet the requirement and the cost be minimum? (Use graphic method). ***[Jiwaji M.B.A. March 2010]***

310. The *ABC* Company has been a producer of picture tubes for television sets and certain printed circuits for radios. The company has just expanded into full scale production and marketing of *AM* and *AM-FM* radios. It has built a new plant that can operate 48 hours per week. Production of an *AM* radio in the new plant will require 2 hours and production of an *AM-FM* radio will require 3 hours. Each *AM* radio will contribute Rs. 40 to profits while an *AM-FM* radio will contribute Rs. 80 to profits. The marketing departments, after extensive research, has determined that a maximum of 15 *AM* radios and 10 *AM-FM* radios can be sold each week.

(*i*) Formulate a linear programming model to determine the optimum production mix of *AM-FM* radios that will maximize profits.

(*ii*) Solve the above problem using the graphic model. ***[Delhi M.B.A. (Nov.) 1998]***

311. A firm makes two products *X* and *Y* and has a total production capacity of 9 tonnes per day, *X* and *Y* requiring the same production capacity. The firm has a permanent contract to supply at least 2 tonnes of *X* and at least 3 tonnes of *Y* per day to another company. Each tonne of *X* requires 20 machine hours production time and each tonne of *Y* requires 50 machine hours of production time. The daily maximum possible number of machine hours is 360. All the firm's output can be sold, and the product made is Rs. 80 per tonne of *X* and Rs. 120 per tonne of *Y*. It is required to determine the production schedule for maximum profit and to calculate this profit. ***[Delhi M.B.A. 2009; C.A. Final (Nov.) 2010]***

312. An aeroplane can carry a maximum of 200 passengers. A profit of Rs. 400 is made on each first class ticket and a profit of Rs. 300 is made on each economy class ticket. The airline reserves at least 20 tickets for first class seats. However, at least 4 times as many passengers prefer to travel by economy class as to the first class. Determine how many tickets of each type must be sold in order to maximize the total profit for the airline.

Formulate a linear programming model to determine the optimal mix that will maximize profit. Solve it by graphic method. ***[M.D. Univ. M.B.A. (Nov.) 2010]***

313. A consumer durable manufacturer has potential market for two products P_1 and P_2. In the production shop he finds that two of his machines M_1 and M_2 which are necessary for the products P_1 and P_2 are having limited capacity. The following table gives the machine hours required and machine hours available :

MACHINE-HOURS REQUIRED PER UNIT

Product	M_1	M_2
P_1	3	2
P_2	4	1
Available machine hours	36	16

The costing section has arrived at the contribution of P_1 as Rs. 30 per piece and of P_2 as Rs. 50 per piece. The sales division has indicated that they can sell any amount of P_2 but only 10 units of P_1. Find out the best manufacturing strategy.

314. A small scale manufacturer has production facilities for producing two different products. Each of the products requires three different operations: grinding, assembling and testing. Product *A* requires 15, 20 and 10 minutes to grind, assembly and test respectively, whereas product *B* requires 7.5, 40 and 45 minutes for grinding, assembling and testing. The production run calls for at least 7.5 hours of grinding time, at least 20 hours of assembling time, and at least 15 hours of testing time. If product *A* costs Rs. 60 and product *B* costs Rs. 90 to manufacture, determine the number of units of each product the firm should produce in order to minimize the cost of operations. ***[IGNOU M.B.A. 2008]***

315. An oil refinery can blend three grades of crude oil to produce quality *R* and quality *S* petrol. Two possible blending processes are available. For each production run the older process uses 5 units of crude *A*, 7 units of crude B and 2 units of crude C to produce 9 units of R and 7 units of S. The newer process uses 3 units of crude A, 9 units of crude B and 4 units of crude C to produce 5 units of R and 9 units of S petrol. Because of prior contract commitments, the refinery must produce at least 500 units of R and at least 300 units of S for the next month. It has available 1500 units of crude A, 1900 units of crude B and 1000 units of crude C. For each unit of R the refinery receives Rs. 60, while for each unit of S it receives Rs. 90. Write down the linear programming formulation of the problem so as to maximize the revenue and solve it by graphical method.

316. The manager of an oil refinery must decide on the optimal mix of two possible blending processes of which the inputs and outputs per production run are as follows :

Process	Input (units)		Output (units)	
	Grade A	Grade B	Gasoline X	Gasoline Y
1	5	3	5	8
2	4	5	4	4

The maximum amounts available of crudes A and B are 200 units and 150 units respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profit per production run for process 1 and process 2 are Rs. 300 and Rs. 400 respectively. Solve the LP problem by the graphical method. *[Rajasthan M.B.A. 2008]*

317. A scrap metal dealer has received an order from a customer for at least 2,000 kilograms of scrap metal. The customer requires that at least 1,000 kilograms of the shipment of metal must be high quality copper that can be melted down and used to produce copper tubings. Furthermore, the customer will not accept delivery of the order if it contains more than 175 kilograms of metal that he deems unfit for commercial use, *i.e.*, metal that contains an excessive amount of impurities and cannot be melted down and refined profitably.

The dealer can purchase scrap metal from two different supplies in unlimited quantities with the following percentages (by weight) of high quality copper and unfit scrap.

	Supplier A	Supplier B
Copper	25%	75%
Unfit scrap	5%	10%

The costs per kilogram of metal purchase from supplier A and supplier B are Re. 1 and Rs. 4 respectively. The problem is to determine (using graphic method of LP technique) the optimum quantities of metal for the dealer to purchase from each of the two suppliers. *[Allahabad M.B.A. 2009]*

318. A manufacturer produces two different models, X and Y of the same product. The raw materials r_1 and r_2 are required for production. At least 18 kg. of r_1 and 12 kg. of r_2 must be used daily. Also at most 34 hours of labour are to be utilized. 2 kg. of r_1 are needed for each model X and 1 kg. of r_1 for each model Y. For each model of X and Y, 1 kg. of r_2 is required. It takes 3 hours to manufacture a model X and 2 hours to manufacture a model Y. The profit is Rs. 50 for each model X and Rs. 30 for each model Y. How many units of each model should be produced to maximize the profit. Use graphic model. *[Gujarat M.B.A. 2008]*

319. A company has two grades of inspectors 1 and 2, who are to be assigned for a quality control inspection. It is required at least 2000 pieces be inspected per 8-hour day. Grade 1 inspector can check pieces at the rate of 40 per hour, with an accuracy of 97%. Grade 2 inspector checks at the rate of 30 pieces per hour with an accuracy of 95%.

The wage rate of a Grade 1 inspector is Rs. 5 per hour, while that of a Grade 2 inspector is Rs. 4 per hour. An error made by an inspector costs Rs. 3 to the company. There are only nine Grade 1 inspectors and eleven Grade 2 inspectors available in the company. The company wishes to assign work to the available inspectors so as to minimize the total cost of the inspection. Formulate this problem as a linear programming model and solve it by graphical method. *[Delhi M.B.A. 2004; Panjab B.Com. 2008]*

320. A manufacturer employs three inputs: man-hours, machine-hours and cloth material to manufacture two types of dresses. Type A dress fetches him a profit of Rs. 160 per piece, while type B, that of Rs. 180 per piece. The manufacturer has enough man-hours to manufacture 50 pieces of type A or 20 pieces of type B dresses per day, while the machine-hours he possesses suffice only for 36 pieces of type A or for 24 pieces for type B dresses. Cloth material available per day is limited but sufficient enough for 30 pieces of either type of dresses. Formulate the linear programming model and solve it graphically. *[Delhi M.Com. 2005]*

$\left[\textbf{Hint: } \text{Constraints are } \frac{x}{50} + \frac{y}{20} \le 1,\ \frac{x}{36} + \frac{y}{24} \le 1,\ \frac{x}{30} + \frac{y}{30} \le 1.\right]$

321. The standard weight of a special purpose brick is 5 kg. and it contains two basic ingredients B_1 and B_2. B_1 costs Rs. 5 per kg. and B_2 costs Rs. 8 per kg. Strength considerations dictate that the brick contains not more than 4 kg. of B_1 and minimum of 2 kg. of B_2. Since the demand for the product is likely to be related to the price of the brick, find out graphically minimum cost of the brick satisfying the above conditions. *[Bangalore M.B.A. 1997]*

322. Find the maximum value of $z = 50x_1 + 60x_2$ subject to the constraints :

$$2x_1 + 3x_2 \le 1500,\ 3x_1 + 2x_2 \le 1500,\ 0 \le x_1 \le 400,\ 0 \le x_2 \le 400.$$

323. Maximize $z = 1.75x_1 + 1.50x_2$ subject to the constraints :

$$8x_1 + 5x_2 \le 320,\ 4x_1 + 5x_2 \le 200,\ 0 \le x_1 \ge 15,\ 0 \le x_2 \ge 10.$$

324. Find the maximum value of $z = 7x_1 + 3x_2$ subject to the constraints :

$$x_1 + 2x_2 \ge 3,\ x_1 + x_2 \le 4,\ 0 \le x_2 \le 5/2,\ 0 \le x_2 \le 3/2.$$

325. Show graphically that the maximum or minimum values of the objective functions for the following problem are same :

Maximize (or Minimize) $z = 5x_1 + 3x_2$ subject to the constraints :

$$x_1 + x_2 \le 6,\ 2x_1 + 3x_2 \ge 3,\ 0 \le x_1 \ge 3,\ 0 \le x_2 \ge 3.$$ *[Purvanchal M.C.A. 1996]*

326. Find the maximum value of $z = 5x_1 + 3x_2$ subject to the constraints :

$$x_1 + x_2 \le 6,\ 2x_1 + 3x_2 \ge 6,\ 0 \le x_1 \le 4,\ 0 \le x_2 \le 3.$$ *[Madurai M.B.A. (DPL) 2009]*

327. Solve graphically the following L.P.P. :

Maximize $z = 3x_1 + 2x_2$ subject to the constraints :

$$-2x_1 + x_2 = 1,\ x_1 \le 2,\ x_1 + x_2 \le 3,\ x_1, x_2 \ge 0.$$

328. Find the minimum or/and maximum value of $z = 3x_1 + 5x_2$ subject to the constraints :

(*a*) $-3x_1 + 4x_2 \le 12,\ 2x_1 - x_2 \ge -2$
$2x_1 + 3x_2 \ge 12,\ 4 \ge x_2 \ge 0,\ 2 \ge x_2 \ge 0.$

(*b*) $x_1 \le 4,\ 2x_2 \le 6,\ 3x_1 + 2x_2 \le 18$
$x_1 + x_2 \le 9,\ x_1, x_2 \ge 0.$

3:3. SOME EXCEPTIONAL CASES

In the preceding sections, we discussed some linear programming problems which may be called 'well-behaved' problems. In each case, a solution was obtained, in some cases it took less effort while in some others it took less effort while in some others it took a little more. But a solution was finally obtained. It should not be taken as a rule. There may be an L.P.P. for which no solution exists or for which the only solution obtained is an unbounded one. Though such problems seldom occur in real situations, it will be an omission, if at this stage, the reader is not exposed to such exceptional cases.

This section considers the following three special cases that arise in the application of the graphical method :

(*i*) Alternative optima, (*ii*) Unbounded solution, (*iii*) Infeasible (or non-existing) solution.

1. **Alternative optima.** When the objective function is parallel to a binding constraint (*i.e.*, a constraint that is staisfied as an equation by the optimal solution), the objective function will assume the same optimum value at more than one solution point. For this reason they are called alternative optima. The problem given below shows that there is an infinity number of such solutions.

SAMPLE PROBLEMS

329. *Use graphical method to solve the L.P.P. :*

Maximize $z = 2x_1 + 4x_2$ *subject to the constraints :*

$$x_1 + 2x_2 \le 5,\ x_1 + x_2 \le 4;\ \text{and}\ x_1, x_2 \ge 0.$$

Solution. The problem is depicted graphically in Fig. 3.8. The extreme points of the feasible region are O, A, B and C.

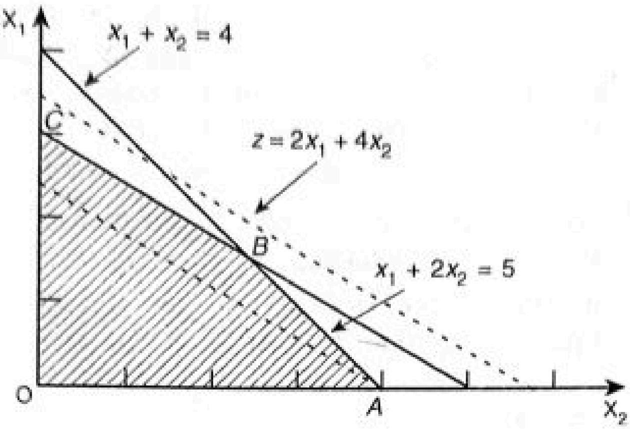

Fig. 3.8

We observe that our objective function (iso-profit line) is parallel to the line BC (or the first constraint), which forms the boundary of the feasible region. Thus as we move the iso-profit line away from the origin, it coincides with the portion BC of the constraint line which forms the boundary of the feasible region. This implies that any point including extreme points B and C on the same line between B and C is an optimal solution. Therefore, in fact an infinite number of values of x_1, x_2 give the same value of objective function.

Now, the value of objective function at each of the extreme points is evaluated as follows :

Extreme point	(x_1, x_2)	$z = 2x_1 + 4x_2$
O	(0, 0)	0
A	(4, 0)	8
B	(3, 1)	**10** ⟵ maximum
C	(0, 2.5)	**10** ⟵ maximum

Since, any point on the line segment BC gives the maximum value ($z = 10$) of the objective function, there exists an alternative optima.

2. **Unbounded solution.** When the values of the decision variables may be increased indefinitely without violating any of the constraints, the solution space (feasible region) is unbounded. The value of objective function, in such cases, may increase (for maximization) or decrease (for minimization) indefinitely. Thus, both the solution space and the objective function value are unbounded.

330. *Use graphical method to solve the following L.P.P. :*

Maximize $z = 6x_1 + x_2$ *subject to the constraints :*

$$2x_1 + x_2 \geq 3,\ x_2 - x_1 \geq 0 \text{ and } x_1, x_2 \geq 0.$$

Solution. The problem is depicted graphically in Fig. 3.9. The two extreme points of the feasible region are A and B.

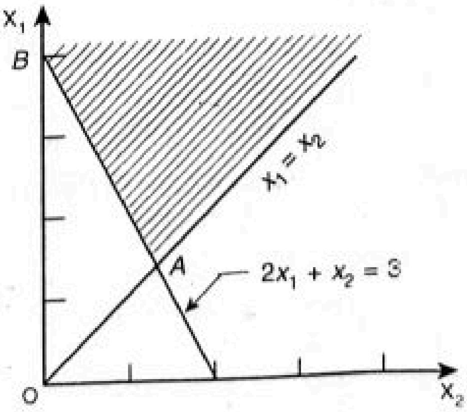

Fig. 3.9

We observe, that the feasible region (solution space) is unbounded. The value of the objective function at the extreme points $A(1, 1)$ and $B(0, 3)$ are 7 and 3 respectively.

But there exist number of points in the feasible region for which the value of the objective function is more than 7. For example, the point (3, 6) lies in the feasible region and the objective function value at this point is 24 which is more than 7. Thus both the variables x_1 and x_2 can be made arbitrarily large and the value of z also increased. Hence, the problem has an unbounded solution.

Remark. An unbounded solution means that there exist an infinite number of solutions to the given problem and the optimal value of z lies at infinity.

3. **Infeasible solution.** When the constraints are not satisfied simultaneously, the linear programming problem has no feasible solution. This situation can never occur if all the constraints are of the '≤' type.

331. *Solve the following L.P.P. :*

Maximixe $z = x_1 + x_2$ *subject to the constraints :*

$$x_1 + x_2 \le 1, \; -3x_1 + x_2 \ge 3$$

$$x_1 \ge 0, \; x_2 \ge 0.$$

[Guru Nanak Dev Univ. B.Com. 2006]

Solution. The problem is depicted graphically in Fig. 3.10. As shown in the figure, there is no point (x_1, x_2) which can lie in both the regions (satisfy both the constraints), there exists no solution to the given problem. Hence, there is infeasible solution.

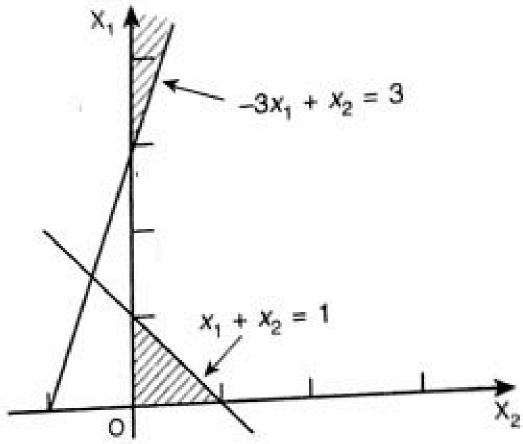

Fig. 3.10

PROBLEMS

332. Find the maximum value of $z = 10x_1 + 6x_2$ subject to the constraints :

$$5x_1 + 3x_2 \le 30, \; x_1 + 2x_2 \le 18 \text{ and } x_1, x_2 \ge 0.$$

333. Find the maximum value of $z = x_1 + 2x_2$ subject to the constraints :

$$x_1 - x_2 \le 1, \; x_1 + x_2 \ge 3 \text{ and } x_1 \ge 0, \; x_2 \ge 0.$$

334. Find the maximum value of $z = 2x_1 + x_2$ subject to the constraints :

$$x_1 - x_2 \le 10, \; x_1 \le 20 \text{ and } x_1 \ge 0, \; x_2 \ge 0.$$

335. Find the maximum value of $z = 3x_1 + 2x_2$ subject to the constraints :

$$2x_1 + x_2 \le 2, \; 3x_1 + 4x_2 \ge 12; \; x_1, x_2 \ge 0.$$

336. Find the maximum value of $z = 3x_1 + 2x_2$ subject to the constraints :

$$-2x_1 + 3x_2 \le 9, \; x_1 - 5x_2 \ge -20 \text{ and } x_1, x_2 \ge 0.$$

[Panjab Tech. Univ. M.B.A. (Dec.) 2009]

337. Consider the following LPP :

Maximize $z = 3x_1 + 7x_2$ subject to the constraints :

$$4x_1 + 5x_2 \le 20, \; 2x_1 + x_2 \le 6, \; 2x_1 \ge 7, \; 2x_2 \le 7; \; x_1 \ge 0 \text{ and } x_2 \ge 0.$$

Demonstrate whether it is possible to get a solution to the above LPP. ***[Delhi M.B.A. (HCA) 2015]***

338. A company buying scrap metal has two types of scrap available to them. The first type of scrap metal has 20% of metal A, 10% of impurity and 20% of metal B by weight. The second type of scrap has 30% of metal A, 10% of impurity and 15% of metal B by weight. The company requires at least 120 kg. of metal A, at most 40 kg. of impurity and at least 90 kg. of metal B. The prices for the two scraps are Rs. 200 and Rs. 300 per kg. respectively. Determine the optimum quantities of the two scraps to be purchased so that the requirements of the two metals and the restriction on impurity are satisfied at minimum cost. *[Nagpur M.B.A. 1997]*

339. A manufacturing firm has a long history of production troubles. It produces two products A and B, which are equally profitable. Recently, the company has entered into contract to supply 40 units of A and 20 units of B per week to another company. The technology of the chemical process implies that production of A must always be at least as large as of B. There are two raw material constraints to be satisfied:

$$5A + 8B \leq 400 \quad \text{and} \quad 55A + 50B \leq 2{,}750.$$

Solve the problem graphically and comment on the solution obtained.

3:4. GENERAL LINEAR PROGRAMMING PROBLEM

We shall now consider the L.P.P. in the general context, that is, when the number of variables is more than two.

***Definition 1* (General Linear Programming Problem).** *Let z be a linear function on R^n defined by*

(*a*) $$z = c_1x_2 + c_2x_2 + \dots + c_nx_n$$

where c_j's are constants. Let (a_{ij}) be an $m \times n$ real matrix and $\{b_1, b_2, \dots, b_m\}$ be a set of constants such that

(*b*) $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq \text{ or } \leq \text{ or } = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq \text{ or } \leq \text{ or } = b_2 \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq \text{ or } \leq \text{ or } = b_m \end{cases}$$

and finally let

(*c*) $$x_j \geq 0, \quad j = 1, 2, \dots, n.$$

The problem of determining an n-tuple $(x_1, x_2, \dots, x_n)$ which makes z a minimum (or maximum) and satisfies (b) and (c) is called the ***general linear programming problem.***

Objective function. *The linear function*

$$z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

which is to be minimized (or maximized) is called the ***objective function*** *of the General L.P.P.*

Constraints. *The inequations (b) are called the* ***constraints*** *of the General L.P.P.*

Non-negative restrictions. *The set of inequations (c) is usually known as the set of* ***non-negative restrictions*** *of the General L.P.P.*

Definition 2* (Solution).** *An n-tuple $(x_1, x_2, \dots, x_n)$ of real numbers which satisfies the constraints of a General L.P.P. is called a* ***solution *to the General L.P.P.*

Definition 3* (Feasible solution).** *Any solution to a General L.P.P. which also satisfies the non-negative restrictions of the problem, is called a* ***feasible solution *to the General L.P.P.*

Definition 4* (Optimum solution).** *Any feasible solution which optimizes (minimizes or maximizes) the bjective function of a General L.P.P. is called an* ***optimum solution *to the General L.P.P.*

Note. The term *optimal solution* is also used for optimum solution.

***Example of a general L.P.P.* (Diet problem).** *Given the nutrient contents of a number of different foodstuffs and the daily minimum requirement of each nutrient for a diet, determine the balanced diet which staisfied the minimum daily requirements and at the same time has the minimum cost.*

Mathematical Formulation

Let there be n different types of foodstuffs available and m different types of nutrients required. Let a_{ij} denote the number of units of nutrient i in one unit of foodstuff j, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Let x_j be the number of units of food j in the desired diet. Then, the total number of units of nutrient i in the desired diet is

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n$$

Let b_i be the number of units of the minimum daily requirement of nutrient i. Then, we must have

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i \qquad i = 1, 2, \ldots, m.$$

Also, each x_j must be either positive or zero. Thus, we also have

$$x_j \geq 0, \qquad j = 1, 2, \ldots, n.$$

Finally, consider the cost. Let c_j be the cost per unit of food j. Thus, the total cost of the diet is given by

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Thus, the problem of selecting the best diet reduces to the following mathematical form :

Find an n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers, such that

(a) $$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i \qquad i = 1, 2, \ldots, n$$

(b) $$x_j \geq 0 \qquad j = 1, 2, \ldots, n$$

and for which the expression (objective function)

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

may be a minimum (least).

This is a General L.P.P. It is general in the sense that the data b_i, a_{ij} and c_j are parameters, which for different sets of values will give rise to different problems.

Slack and Surplus Variables

Definition 1 **(Slack variables).** *Let the constraints of a General L.P.P. be*

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad i = 1, 2, \ldots, k$$

Then, the non-negative variables x_{n+i} *which satisfy*

$$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i \qquad i = 1, 2, \ldots, k$$

are called ***slack variables.***

Definition 2 **(Surplus variables).** *Let the constraints of a General L.P.P. be*

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \qquad i = k+1, k+2, \ldots, l$$

Then, the non-negative variables x_{n+i} *which satisfy*

$$\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i \qquad i = k+1, k+2, \ldots, l$$

are called ***surplus (or negative slack) variables.***

3:5. CANONICAL AND STANDARD FORMS OF L.P.P.

After the formulation of linear programming problem (L.P.P.), the next step is to obtain its solution. But for the solution of any linear programming problem, the problem must be available in a particular form. Two forms are dealt with here, the *canonical form* and the *standard form*.

The Canonical Form

The general formulation of linear programming problem discussed in the previous section can always be put in the following form :

Maximize $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ subject to the constraints :

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \le b_i; \qquad i = 1, 2, \dots, m$$

$$x_1, x_2, \dots, x_n \ge 0$$

by making use of some elementary transformations. This form of L.P.P. *is called the* **canonical form** of L.P.P. The characteristics of this form are :

(*i*) *The objective function is of the maximization type.*

The minimization of a function, $f(x)$, is equivalent to the maximization of the negative expression of this function, $-f(x)$, *i.e.*,

$$\text{Minimize } f(x) = -\text{Maximize}\{-f(x)\}$$

For example, the linear objective function

$$\text{Minimize } z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

is equivalent to

$$-\text{Maximize } h = -c_1x_1 - c_2x_2 - \dots - c_nx_n$$

with $z = -h$.

(*ii*) *All the constraints are of the "$\le$" type, except for the non-negative restrictions.*

An inequality of "$\ge$" type can be changed to an inequality of the "$\le$" type by multiplying both sides of the inequality by -1.

For example, the linear constraint

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \ge b_i$$

is equivalent to

$$-a_{i1}x_1 - a_{i1}x_2 - \dots - a_{in}x_n \le -b_i$$

An equation may be replaced by two weak inequalities in opposite directions. For example,

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i$$

is equivalent to

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \le b_i$$

and

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \ge b_i.$$

(*iii*) *All the variables are non-negative.*

A variable which is *unrestricted* in sign (*i.e.*, positive, negative or zero) is equivalent to the difference between two non-negative variables. Thus, if x_j is unrestricted in sign, it can be replaced by $(x_j' - x_j'')$, where x_j' and x_j'' are both non-negative, *i.e.*,

$$x_j = x_j' - x_j'', \qquad \text{where } x_j' \ge 0 \text{ and } x_j'' \ge 0.$$

The Standard Form

The general linear programming problem in the form

Maximize or Minimize $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ subject to the constraints :

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i, \qquad i = 1, 2, \dots, n$$

$$x_1, x_2, \dots, x_n \ge 0$$

is known as in **standard form**. The characteristics of this form are :

(*i*) All the constraints are expressed in the form of equations, except for the non-negative restrictions.

(*ii*) The right hand side of each constraint equation is non-negative.

The inequality constraint can be changed into equation by introducing a non-negative variable on the left hand side of such constraint. It is to be added (slack variable) if the constraint is of "$\leq$" type and subtracted (surplus variable) if the constraint is of "$\geq$" type.

In matrix notation the standard form of L.P.P. can be expressed as :

Maximize or Minimize $z = \mathbf{cx}$ *(objective function)*

subject to the constraints :

$$\mathbf{Ax} = \mathbf{b}, \qquad \textit{(constraints)}$$

$$\mathbf{x} \geq \mathbf{0} \qquad \textit{(non-negative restrictions)}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{c} = (c_1, c_2, \ldots, c_n)$, $\mathbf{b}^T = (b_1, b_2, \ldots, b_m)$ and $\mathbf{A} = (a_{ij})$; $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Remarks 1. The coefficients of slack and/or surplus variables in the objective function are always assumed to be zero, so that the conversion of the constraints to a system of simultaneous linear equations does not change the objective function under consideration.

2. The linear programming form :

Maximize $z = \mathbf{cx}$ subject to the constraints : $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$

is known as the **canonical form** of the L.P.P.

Theorem 2-1. *The set of feasible solutions to an L.P.P. is a convex set.*

Proof. Let the L.P.P. be to determine $\mathbf{x}$ so as to maximize the linear function $z = \mathbf{c}^T\mathbf{x}$ subject to the constraints : $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.

Let $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$ be two feasible solutions to this problem, so that

$$\mathbf{Ax}^{(1)} = \mathbf{b}\,;\ \ \mathbf{Ax}^{(2)} = \mathbf{b}\,;\ \ \mathbf{x}^{(1)} \geq \mathbf{0} \ \text{ and } \ \mathbf{x}^{(2)} \geq \mathbf{0}.$$

Now, consider convex combination of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, namely,

$$\mathbf{x} = \lambda \mathbf{x}^{(1)} + (1 - \lambda)\,\mathbf{x}^{(2)}, \quad 0 \leq \lambda \leq 1.$$

Clearly

$$\begin{aligned} \mathbf{Ax} &= \mathbf{A}\,[\lambda \mathbf{x}^{(1)} + (1 - \lambda)\,\mathbf{x}^{(2)}] = \lambda\,\mathbf{Ax}^{(1)} + (1 - \lambda)\,\mathbf{Ax}^{(2)} \\ &= \lambda \mathbf{b} + (1 - \lambda)\,\mathbf{b} = \mathbf{b} \end{aligned}$$

Again, since $\mathbf{x}^{(1)} \geq \mathbf{0}$, $\mathbf{x}^{(2)} \geq \mathbf{0}$, and λ, $1 - \lambda \geq 0$, $\therefore\ \mathbf{x} \geq \mathbf{0}$.

Hence, $\mathbf{x}$ is also feasible solution to the problem. Thus, the set

$$S = \{\mathbf{x} \mid \mathbf{x} \text{ is a feasible solution to an L.P.P.}\}$$

is a convex set.

Remark. In general a convex set S is either empty or unbounded or closed.

The *empty set* occurs when the constraints are not satisfied simultaneously. In this case the system yields no solution. An unbounded set implies that the region of feasible solution is not constrained in at least one direction.

A *closed set* implies that the region of feasible solution is a convex polyhedron, since it is defined by the intersections of a finite number of linear constraints.

SAMPLE PROBLEMS

340. *Rewrite in standard form the following linear programming problem :*

Minimize $z = 2x_1 + x_2 + 4x_3$ *subject to the constraints :*

$$-2x_1 + 4x_2 \leq 4,\ \ x_1 + 2x_2 + x_3 \geq 5,\ \ 2x_1 + 3x_3 \leq 2,$$

$$x_1,\ x_2 \geq 0 \textit{ and } x_3 \textit{ unrestricted in sign.}$$

Solution. As the constraints are in the form of inequalities, we introduce slack variables $x_4 \geq 0$ and $x_5 \geq 0$ in the first and third inequalities and a surplus variable $x_6 \geq 0$ in the second inequality of the constraints.

Further since x_3 is unrestricted in sign, we write $x_3 = x_3' - x_3''$, where $x_3' \geq 0$ and $x_3'' \geq 0$. Thus the constraints of the problem become :

$$-2x_1 + 4x_2 + x_4 = 4$$
$$x_1 + 2x_2 + (x_3' - x_3'') - x_6 = 5$$
$$2x_1 + 3(x_3' - x_3'') + x_5 = 2$$

The non-negative restrictions are :

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3' \geq 0,\ x_3'' \geq 0,\ x_4 \geq 0,\ x_5 \geq 0 \text{ and } x_6 \geq 0.$$

The objective function of the problem is converted into that of maximization by multiplying it by (–1). Also we assign a cost zero to each slack/surplus variable. Thus, the revised objective function is :

$$\text{Maximize } z^* = -2x_2 - x_2 - 4(x_3' - x_3'') + 0.x_4 + 0.x_5 + 0.x_6$$

where $z^* = -z$.

Thus, the given LPP in standard form is :

Maximize $z^* = -2x_1 - x_2 - 4x_3' + 4x_3'' + 0.x_4 + 0.x_5 + 0.x_6$ subject to the constraints :

$$-2x_1 + 4x_2 + x_4 = 4,\ x_1 + 2x_2 + x_3' - x_3'' - x_6 = 5$$
$$2x_1 + 3x_3' - 3x_3'' + x_5 = 2,\ x_j \geq 0;\ j = 1, 2, 3', 3'', 4, 5, 6.$$

341. *Reduce the following LPP to its standard form :*

Maximize $z^* = 3x_1 + 4x_2 + 6x_3$ *subject to the constraints :*

$$2x_1 + x_2 + 2x_3 \geq 6,\ 3x_1 + 2x_2 = 8,$$
$$7x_1 - 3x_2 + 5x_3 \geq 9;\ x_1 \geq 0,\ x_2 \geq 0 \text{ and } x_3 \text{ unrestricted in sign.}$$

Solution. Since, the first and third constraints are inequalities with $\geq$ sign, we introduce surplus variables $x_4 \geq 0$ and $x_5 \geq 0$ in the respective inequalities.

Further since x_3 is unrestricted in sign, we write $x_3 = x_3' - x_3''$, where $x_3' \geq 0$ and $x_3'' \geq 0$.

Thus, the constraints of the given LPP are :

$$2x_1 + x_2 + 2(x_3' - x_3'') + x_4 = 6,$$
$$3x_1 + 2x_2 = 8,$$
$$7x_1 - 3x_2 + 5(x_3' - x_3'') + x_5 = 9$$

The objective function will be :

$$\text{Maximize } z = 3x_1 + 4x_2 + 6(x_3' - x_3'').$$

$\therefore$ The given LPP in its standard form is :

Maximize $z = 3x_1 + 4x_2 + 6x_3' - 6x_3'' + 0 \cdot x_4 + 0 \cdot x_5$ subject to the constraints :

$$2x_1 + x_2 + 2x_3' - 2x_3'' + x_4 = 6,\ 3x_1 + 2x_2 = 8$$
$$7x_1 - 3x_2 + 5x_3' - 5x_3'' + x_5 = 9$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3' \geq 0,\ x_3'' \geq 0,\ x_4 \geq 0 \text{ and } x_5 \geq 0.$$

PROBLEMS

342. Reduce the following L.P.P. to its standard form :

Maximize $z = x_1 - 3x_2$ subject to the constraints :

$$-x_1 + 2x_2 \leq 15,\ x_1 + 3x_2 = 10$$

x_1 and x_2 unrestricted in sign.

343. Write down the following LPP in standard form :

Maximize $z = 3x_1 + 2x_2 + 5x_3$ subject to the constraints :

$$2x_1 - 3x_2 \leq 3,\ x_1 + 2x_2 + 3x_3 \geq 5,\ 3x_1 + 2x_3 \leq 2$$
$$x_1 \geq 0,\ x_2 \geq 0 \text{ and } x_3 \geq 0.$$

344. Transform the following LPP into the form where all the constraints are of equality type :

$$\text{Maximize} \quad z = 2x_1 + x_2 - 6x_3 - 4x_4 \quad \text{subject to the constraints :}$$
$$3x_1 + x_4 \le 25, \; x_1 + x_2 + x_3 + x_4 = 20$$
$$4x_1 + 6x_3 \ge 5, \; 2 \le x_1 + 3x_3 + 2x_4 \le 30 ; \; x_j \ge 0, \; (j = 1, 2, 3, 4).$$

345. Consider the Linear Programme

$$\text{Minimize} \quad z = -3x_1 + x_2 + x_3 \quad \text{subject to the constraints :}$$
$$x_1 - 2x_2 + x_3 \le 11$$
$$-4x_1 + x_2 + 2x_3 \ge 3, \; 2x_1 - x_3 = -1$$
$$x_1, x_2 \ge 0, \; x_3 \ge \text{ or } < 0.$$

Cast it in standard form, adding variables and/or constraints if necessary. How many decision variables and constraints are present in this problem?

346. Transform the following LP problem into the form where all constraints are of equality type :

$$\text{Maximize} \quad z = 2x_1 + x_2 - 5x_3 + 3x_4 \quad \text{subject to the constraints :}$$
$$3x_1 + 2x_2 \le 15, \; 4x_1 + 5x_2 \ge 20, \; x_1 + x_2 - x_3 + 2x_4 = 10$$
$$2 \le 2x_1 + 4x_2 - x_3 \le 30, \; x_i \ge 0 ; \; i = 1, 2, 3, 4$$

Identify the variables introduced.

3:6. INSIGHTS INTO THE SIMPLEX METHOD

While solving an LPP graphically, we noticed the following important characteristics :

(*a*) Whenever feasible solutions existed, the region of feasible solutions was *convex*, bounded by *lines* or *planes* (more than two variable case). For each such convex region, there were *corners* (or vertices) on the boundary and *edges* joining these corners.

(*b*) For each value of z, the objective function could be represented by a line or plane and whenever the maximum or minimum value of z was finite, the optimal solution occurred at some corner (or vertex) of the convex region of feasible solutions. If the optimal solution were not unique, there were points (on an edge) that were optimal, but in every event at least one vertex was optimal. In the case of unbounded optimal solution; however, no corner point was optimal.

Interestingly, these observations, derived from simple graphical examples, hold true for the general LPP also, if we think of a geometrical representation in n dimensional space. The region of the feasible solutions is a convex region or a convex set, has corners or the vertices. The linear constraints of the problem are represented by half spaces determined by the hyperplanes. The optimal solution, if it exists, corresponds to some vertex of the convex region.

The problem essentially is that of determining which particular vertex of the convex region corresponds to the optimal solution. No method is so far known that will locate this optimal vertex in a single step and one has to adopt an iterative procedure for locating the same.

The best known and the most widely used procedure for locating the optimal vertex is the "simplex method" that determines the optimal vertex in a finite number of steps. The *simplex method* is a procedure which consists in moving step by step from a given vertex to an optimal vertex. At each step, it is possible to move only to an adjacent vertex. The method consists of moving along an 'edge' of the convex region of feasible solutions from one vertex to an adjacent one. Of all the adjacent vertices the one yielding an *improved* value of the objective function over that of the preceding vertex, is chosen. At each vertex point, the method tells us whether that extreme point is optimal, and if not, the procedure of jumping from one extreme point to another is repeated. Since the number of vertices is finite, simplex method leads to an optimal vertex in a finite number of steps. If at any stage, the procedure leads us to a vertex which has an edge leading to infinity and if the objective function value can be further improved by moving along that edge, the simplex method tells us that there is an unbounded solution.

MULTIPLE CHOICE QUESTIONS

1. Which of the following is not correct ?
   (a) The graphic approach to an LPP is most suitable when there are only two decision variables.
   (b) A possible solution on the graph corresponds to every point (x, y).
   (c) The graphic approach to an LPP is applicable when the number of decision variables are more than the number of constraints.
   (d) The common region that satisfies all the contraints is called the feasible (convex) region.
2. A feasible solution to an LPP
   (a) must satisfy all of the problem's constraints simultaneously.
   (b) must be a corner point of the feasible region.
   (c) need not satisfy all of the constraints, only some of them.
   (d) must optimize the value of the objective function.
3. Which of the following is not correct ?
   (a) The graphic approach to the solution of LPP's cannot handle problems with more than three variables.
   (b) A feasible solution to an LPP is one that satisfies at least one of the constraints of the problem.
   (c) An optimum solution to an LPP is a feasible solution which optimizes the objective function.
   (d) The feasible region is also termed as the solution space.
4. Which of the following is not correct ?
   (a) Feasible solution of an LPP is independent of the objective function.
   (b) The feasible region of an LPP must be a convex set.
   (c) The feasible region for a constraint is restricted if its '≥' or '≤' sign is replaced by a '=' sign.
   (d) It is not possible to obtain feasible solution of an LPP by graphical method.
5. An iso-profit line represents
   (a) an infinite number of solutions all of which yield the same profit.
   (b) an infinite number of optimum solutions.
   (c) an infinite number of solutions all of which yield the same cost.
   (d) a boundary of the feasible region.
6. If an iso-profit line yielding the optimum solution coincides with a constraint line, then
   (a) the solution is unbounded.
   (b) the solution is infeasible.
   (c) the constraint which coincides is redundant.
   (d) none of the above.
7. If two constraints do not intersect in the positive quadrant of the graph, then
   (a) one of the constraint is redundant. (b) the solution is infeasible.
   (c) the solution is unbounded. (d) none of these.
8. Which of the following is not correct ?
   (a) An infeasible solution exists, when there is no feasible solution.
   (b) Iso-profit lines on a graph of an LPP would always be parallel to each other.
   (c) An iso-cost line cannot be parallel to the line of any constraint.
   (d) Every LPP has a unique optimum solution.
9. Using graphic method, the optimum solution of the LPP of maximizing $z = 10x + 15y$ subject to $2x + y \leq 26$, $x + 2y \leq 28$, $y - x \leq 5$ and $x \geq 0$, $y \geq 0$ is obtained as
   (a) $x = 8$ and $y = 10$. (b) $x = 6$ and $y = 1$.
   (c) $x = 6$ and $y = 10$. (d) $x = 8$ and $y = 8$.
10. Given an LPP to maximize $z = -5x_2$ subject to $x_1 + x_2 \leq 1$, $0.5x_1 + 5x_2 \geq 0$, and $x_1 \geq 0$, $x_2 \geq 0$. Using graphical method, we have
    (a) no feasible solution. (b) unbounded solution.
    (c) unique optimum solution. (d) multiple optimum solutions.

11. Which of the following is not correct ?
    (*a*) An LPP with an unbounded feasible region would obviously have unbounded solution.
    (*b*) For an LPP, the feasible region may change if non-binding constraints are deleted.
    (*c*) A redundant constraint represents an abundant resource.
    (*d*) It is possible for the objective function value of an LPP to be the same at two distinct extreme points.
12. The general linear programming problem is in standard form, if
    (*a*) the constraints are strict equations.
    (*b*) the constraints are inequalities of '≤' type.
    (*c*) the constraints are inequalities of '≥' type.
    (*d*) the decision variables are unrestricted in sign.
13. Which of the following is not correct ?
    (*a*) Graphical method of linear programming is not useful when there are only two decision variables.
    (*b*) Solution of a maximization LPP when permitted to be infinitely large is called unbounded.
    (*c*) Optimum solution to an LPP always lies at least on the two vertices of the feasible region.
    (*d*) It is possible for the objective function value of an LPP to be the same at two distinct extreme points.
14. Which of the following statements is wrong ?
    (*a*) Slack variables are used to convert the inequalities of the type '≤' into equations.
    (*b*) Surplus variables are used to convert the inequalities of the type '≥' into equations.
    (*c*) An LPP with all its constraints are of the type '≥' is said to be in *standard form*.
    (*d*) An LPP with all its constraints are of the type '≤' is said to be in *canonical form*.

ANSWERS

1. (*c*)	**2.** (*a*)	**3.** (*b*)	**4.** (*d*)	**5.** (*a*)
6. (*d*)	**7.** (*b*)	**8.** (*c*)	**9.** (*a*)	**10.** (*a*)
11. (*b*)	**12.** (*a*)	**13.** (*c*)	**14.** (*c*).	

REVIEW QUESTIONS

1. Write down the standard form of a LPP. ***[Annamalai M.B.A. 2010]***
2. Explain the graphical method of solving a LPP involving two variables. ***[Madurai M.Com. (Nov.) 2002]***
3. What is feasible region ? Is it necessary that it should always be a convex set ? ***[Madurai M.Com. 2003]***
4. Define iso-profit and iso-cost lines. How do these help in obtaining an optimum solution to an LPP ?
5. Discuss the special cases of infeasible and unbounded solution by giving examples in LPP. ***[Panjab Tech. Univ. M.B.A. (Dec.) 2011]***
6. Graphical solution is not possible for LPP with more than two constraints. True or False ? Justify your answer. ***[Annamalai M.B.A. (Nov.) 2002]***
7. What is meant by the term feasible solution space? What determines the region? ***[Panjab Tech. Univ. M.B.A. (June) 2011]***
8. Illustrate graphically the following special cases of linear programming problems :
   (*i*) Multiple optimal solutions, (*ii*) No feasible solution, (*iii*) Unbounded solution.
9. Define and explain the following terms :
   (*i*) Optimum solution, (*ii*) Feasible solution, (*iii*) Degenerate basic feasible solution. ***[Madras M.B.A. 2004]***
10. Define slack and surplus variables in LPP. ***[Delhi B.Sc. (Stat.) 2000; Madras B.E. 1999]***
11. State the general linear programming problem in : (*a*) Standard form, (*b*) Canonical form. ***[Delhi B.Sc. (Stat,) 2000]***
12. With the help of suitable sketches, define convex, non-convex, and infeasible region in relation to the graphical solution of a linear programming problem. ***[Delhi M.Com. 2008]***