Dynamical Symmetries of the Kepler Problem

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PH 311 - Mini Project I
Autumn 2021

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1 Introduction

Symmetries play a pivotal role in physics, especially with the advent of Quantum Field Theory and Standard Model, symmetries have now become central to the entirety of physics. The presence or even absence of symmetries speak volumes about a given physical system. But this view of the universe is a rather recent one. In Newtonian era, the conservation laws were thought to be fundamental rather than the underlying symmetry. In 1915, Emmy Noether laid the founding stones for the symmetry approach to physics. Perhaps one of the greatest theorems in physics - the Noether's theorem states that *"For every differentiable symmetry of a given system, there exists a corresponding conservation law"*. This linked two seemingly unrelated phenomenon in a single cause-effect relation - Symmetries were fundamental and conservation laws arose as an effect of the presence of these symmetries. Quantum mechanics went a step ahead and brought a new symmetry - the exchange of identical particles and the symmetricity and antisymmetricity of wavefunction. This lead to the classification of all elementary particles into bosons and fermions. But the real importance of symmetries came to light in mid 20^{th} century, with the advent of Gell Mann's standard model. The Standard Model explained the elementary particles in terms of the internal symmetries of the underlying fields. Today the standard model is a theory based on the group $SU(3) \times SU(2) \times U(1)$ representing the symmetries of the fields.

The application of symmetries is not limited to the advanced physical theories, in fact, symmetry arguments can lead to beautiful realisations in even the simpler systems. In this project, we study the symmetry of one of the simplest yet non-trivial systems - the Kepler system, and study the action of the symmetry group elements on the system. The Kepler system is a rather peculiar system. Unlike other spherically symmetric systems which have only rotational symmetries, the Kepler system has a higher symmetry group, i.e. it has hidden symmetries.

The work structure is as follows, in section 2 we explore the topic of dynamical symmetries and explain it in the context of classical mechanics, in section 3 we discuss a classical study of the Kepler system, its constants of motion, integrability and other known facts about the system. In section 4 the symmetry group of the system is studied and in section 5 a 4 dimensional description of the system is given based on the symmetry group. In section 6 we discuss the action of group elements on the system and the transformations they bring about. Finally in section 7 we finish with concluding remarks.

2 Dynamical Symmetries

2.1 Canonical transformation

Since Hamiltonian mechanics treats momenta and position variables as independent variables, it gives us a freedom of choice of what coordinate system we want to use. Given a set of coordinates p & q which satisfy the canonical Poisson bracket relations ($\{p_i, p_j\} = \{q_i, q_j\} = 0,\ \{q_i, p_j\} = \delta_{ij}$), a transformation of the form $P = P(p, q, t),\ Q = Q(p, q, t)$ is called a canonical transformation iff the new variables also satisfy the canonical Poisson relations, i.e, iff $\{P_i, P_j\} = \{Q_i, Q_j\} = 0,\ \{Q_i, P_j\} = \delta_{ij}$.

Such transformations can be written in form of matrix multiplication $\eta' = M\eta$, where $\eta = \begin{pmatrix} q_1 \\ \vdots \\ p_1 \\ \vdots \end{pmatrix}$, $\eta' = \begin{pmatrix} Q_1 \\ \vdots \\ P_1 \\ \vdots \end{pmatrix}$ and M is a Jacobian Transformation matrix with elements $M_{ij} = \frac{\partial \eta'_i}{\partial \eta_j}$

The constraint that the canonical Poisson relation should be conserved, reflects on the matrix M as the constraint $M^T J M = J$, where J is a totally antisymmetric $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}$$

The set of such matrices M is called the symplectic group of canonical transformations, $Sp(2n) := \{M \ni M^T J M = J\}$.

2.2 Dynamical Symmetry Group

The dynamical symmetry group of a system is defined as the set of canonical transformations that leave the Hamiltonian form invariant.

The infinitesimal transformation of the Hamiltonian under a canonical transformation M can be calculated to be[5] $\partial H = -\epsilon[g, H] - \epsilon\frac{\partial g}{\partial t} = -\epsilon\frac{dg}{dt}$, where g is the generator of the transformation M.

If g is a constant of motion, i.e. if $\frac{dg}{dt} = 0$, then the Hamiltonian remains form invariant under the action of M on the system.

Therefore, we can say that if g is a constant of motion and a generator of canonical transformations, then it is a generator of the dynamical symmetry group.

3 Kepler Problem

The Kepler system is any system that has $1/r$ potential. Two of the important systems in physics, the solar system and the hydrogen atom, are Kepler systems. Due to this reason the Kepler system has been extensively studied, and the all results have been experimentally verified.

If we have an attractive spherically symmetric potential, then for negative energy we get orbits that go round and round, but they are not closed, rather they precess. But in the Kepler system, the orbit is closed and does not precess. Another noteworthy point here is that the hydrogen atom, which is another Kepler system, has a degeneracy more that what is expected from a normal spherically symmetric potential. Both these anomalies can be explained by the fact that the Kepler system has a dynamical symmetry group larger than any regular spherically symmetric potential.

3.1 Constants of Motion of the Kepler system

Liouville-Arnold Condition for Integrability

The equations of motion of a system in matrix form is $\dot{\eta} = J\eta$, where η and J are the same matrices as defined above. This equation for n degrees of freedom forms a set of n coupled differential equations.

The Liouville-Arnold Condition for Integrability states that if there are n Constants of Motion (COMs) $F_1, F_2, ..., F_n$ in *involution* with each other, i.e, $\{F_i, F_j\} = 0 \ \forall \ 1 \leq i, j \leq n$, (where $F_1 = H$) then the system is, in principle, fully integrable.

If there are r COMs where $n < r < 2n-1$ (not necessarily in involution) then the system is superintegrable, and if $r = 2n-1$, then the system is maximally superintegrable.

The Kepler system is a spherically symmetric system. The Lagrangian and Hamiltonian for the system are

$$\mathcal{L} = m\frac{d^2r}{dt^2} + \frac{k}{r}, \quad H = \frac{p^2}{2m} - \frac{k}{r}$$

where k is a positive definite constant.

Since $\mathcal{L}$ does not depend on the coordinates θ and ϕ, the torque on the particle $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = 0$, making the angular momentum vector $\mathbf{L}$ a conserved quantity of the Kepler system. This gives 3 constants of motion $L_x, \ L_y, \ L_z$.

H does not depend explicitly on time, therefore $\dfrac{dH}{dt} = \dfrac{\partial H}{\partial t} = 0$. So H is also a constant of motion.

This gives, together with the components of $\mathbf{L}$, 4 constants of motion, $H, \ L_x, \ L_y, \ L_z$, but these are not in involution with each other. The Poisson bracket relations of the three components of angular momentum reads

$$\{L_i, L_j\} \ = \ \epsilon_{ikj} L_k$$

implying that the three components of angular momentum can not make up for the Liouville-Arnold condition. But we can construct an invariant L^2 quadratic in L, which commutes with all the three components of $\mathbf{L}$ and also H. This gives us we have a set of three constants of motion in involution with each other, $H, \ L_z, \ L^2$, making the system integrable in principle.

The Kepler system, is special as mentioned before. It has, along with the angular momentum vector, another conserved vector called the Laplace-Runge-Lenz (LRL) vector,

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk\mathbf{r}}{r} \tag{1}$$

This lies in the orbital plane, with its direction pointing from the center of force to the perihelion of the orbit. Since this vector is conserved, its direction does not change over

the due course of time, which means that the direction of the major axis of the elliptic orbit remains same. The presence of a conserved LRL vector ensures that the orbits do not precess.
A has two constraints,

$$\mathbf{A}.\mathbf{L} = (\mathbf{p}\times\mathbf{L}).\mathbf{L} - \frac{mk}{r}(\mathbf{r}.\mathbf{L}) = 0 - 0 = 0$$

$$\mathbf{A}.\mathbf{L} = 0$$

and

$$A^2 = \mathbf{A}.\mathbf{A} = (\mathbf{p}\times\mathbf{L}).(\mathbf{p}\times\mathbf{L}) - 2\frac{mk}{r}(\mathbf{p}\times\mathbf{L}).\mathbf{r} + \frac{m^2k^2}{r^2}\mathbf{r}.\mathbf{r}$$

Using the identity $(a\times b).(c\times d) = (a.c)(b.d) - (a.d)(b.c)$,

$$A^2 = (\mathbf{p}.\mathbf{p})(\mathbf{L}.\mathbf{L}) - (\mathbf{p}.\mathbf{L})^2 - 2\frac{mk}{r}\mathbf{L}.\mathbf{L} + m^2k^2 = \left(p^2 - 2\frac{mk}{r}\right)L^2 + m^2k^2$$

But $p^2 - 2\dfrac{mk}{r} = 2mH$, giving

$$A^2 = 2mHL^2 + m^2k^2 \tag{2}$$

These two constraints reduce two of the components of **A**, giving only one addition to the set of algebraically independent constants of motions of the Kepler system. The algebraically independent constants of the motion of the Kepler system are $H,\ L_x,\ L_y,\ L_z,\ A_z$. The Kepler system has 5 algebraically independent constants of motion, making the system maximally superintegrable.

4 Symmetry Group of the Kepler system

The angular momentum vector **L** and the Laplace-Runge-Lenz vector **A** generate continuous canonical transformations, and are the elements of the Lie group $sp(2n)$. Since these are also constants of motion, $L_x,\ L_y,\ L_z,\ A_x,\ A_y,\ A_z$ are the generators of the dynamical symmetry group of the Kepler system.

It is already known that the three generators $L_x,\ L_y,\ L_z$ independently form the $so(3)$ group which is the group of rotations in 3 spatial dimensions. These are associated with the geometrical symmetry - the rotational symmetry.
But the symmetry associated with the Runge-Lenz vector is not an associated with any isometry, it is a proper transformation in the phase space, which has no geometric manifestation.

To study the symmetry group the six generators of the Kepler system form, we study the Poisson bracket algebra of the generators.
For the angular momentum components, we have the relations

$$\{L_i, L_j\} = \epsilon_{ijk}L_k$$

The Runge Lenz vector components do not form a group on their own since their Poisson bracket algebra is not closed and it involves angular momentum vectors,

$$\{A_i, A_j\} = -2mH\epsilon_{ijk}L_k$$

The Poisson bracket algebra of Runge-Lenz vector components with angular momentum components reads

$$\{A_i, L_j\} = \epsilon_{ijk}A_k$$

Since H is a constant of motion, we can rescale the Runge-Lenz vector as

$$D_i = \frac{A_i}{\sqrt{2m|H|}}$$

From here on we will restrict ourselves to bound solutions, i.e. $|H| < 0$. With such a rescaling, the Poisson bracket algebra of the Kepler system, taking both the angular momentum and LRL vector together reads,

$$\{L_i, L_j\} = \epsilon_{ijk}L_k,\ \{D_i, L_j\} = \epsilon_{ijk}D_k,\ \{D_i, D_j\} = \epsilon_{ijk}L_k$$

This is isomorphic to the algebra of $so(4)$ - the rotation group in 4 dimensions.

The 4 dimension space-time group of the special relativity is the $SO(3,1)$ group - the Lorentz group. Its elements are rotations in the 4 dimensional Minkowski space, where the metric is $x_0^2 - x_1^2 - x_2^2 - x_3^2$. This describes a hyperbolic geometry where rotations map points to different points lying on same hyperbola.

Unlike the $SO(3,1)$ group, the $SO(4)$ group is a rotation group in 4 dimensional Euclidean space, where the metric is $x_1^2+x_2^2+x_3^2+x_4^2$. This describes a spherical geometry, where rotations map points to different points lying on the same sphere. This group is inherently different from the $SO(3,1)$ group in the very structure of the 4 dimensional space it inhabits.

5 4-Dimensional Description Of The Kepler Problem

The group $SO(4)$ acts on points in the 4D Euclidean space. From definition, the $SO(4)$ group elements leave distances in 4D space invariant under rotation. This invarient is of the form $l^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, and finding the exact form of this invariant gives us the coordinates of a point in the 4D space in terms of the 3D coordinates (r_1, r_2, r_3) and time (t). The invariant can be found by employing a reparameterisation of the solution curves with a new parameter which we will here call s, subjected to the constraint [4]

$$\frac{ds}{dt} = \sqrt{\frac{-2E}{m}}\frac{1}{r}$$

Defining three constants[3],

$$V=\sqrt{\frac{-2E}{m}},\ R=\frac{-k}{2E},\ T=\frac{R}{V}=-k\sqrt{\frac{-m}{(8E)^3}}$$

the constraint equation reads

$$\frac{ds}{dt}=\frac{V}{r} \tag{3}$$

Under such a parameterisation, the energy equation for the system becomes,

$$\frac{m}{2}\left(\frac{dr}{dt}\right)^2-\frac{k}{r}=E \implies \frac{m}{2}\left(\frac{dr/ds}{dt/ds}\right)^2-\frac{k}{V dt/ds}=E$$

From here, we will be denoting differentiation with respect to s with a prime, i.e. $\frac{d}{ds}(\ )=(\ )'$. The

$$\frac{m}{2}\frac{r'^2}{t'^2}-\frac{k}{Vt'}=E \implies r'^2-\frac{2kVt'}{mV^2}-V^2t'^2=0$$

From the definition, $\frac{k}{mV^2}=R$,

$$r'^2-2(V\ t')(R)+(V\ t')^2=0$$

Completing the square,

$$V^2(t'\ -\ T)^2+r_1'^2+r_2'^2+r_3'^2=R^2 \tag{4}$$

The right hand side of the equation is a constant, and hence this quantity remains invariant under the action of $SO(4)$ group elements, and the coordinates of a point is given, in terms of 3D coordinates, as $x_0=V(t'-T),\ x_1=r_1',\ ,x_2=r_2',\ x_3=r_3'$.

This is the equation of a 4D sphere of radius R. So orbits in 3D, $f(\mathbf{r},t,E)$, are trajectories, $F(\mathbf{r}',t')$ that lie on the hypersphere which is specified by the energy E of the orbit.

5.1 Equation of motion in 4 dimensions

The equation of motion in 3D that reads

$$m\frac{d^2\mathbf{r}}{dt^2}=-k\frac{\mathbf{r}}{r^3}$$

is to be written in the new set of 4D coordinates.

$$\begin{aligned} m\frac{ds}{dt}\frac{d}{ds}\left(\frac{\mathbf{r}'}{t'}\right)=-k\frac{\mathbf{r}}{r^3} &\implies m\frac{V^3}{r}\frac{t'\mathbf{r}''-\mathbf{r}'t''}{r^2}=-k\frac{\mathbf{r}}{r^3} \\ mV^2\frac{r\mathbf{r}''-r'\mathbf{r}'}{r^3}=-k\frac{\mathbf{r}}{r^3} &\implies \mathbf{r}''=-\frac{R\mathbf{r}}{r}+\frac{r'\mathbf{r}'}{r} \end{aligned} \tag{5}$$

The radial equation of motion can be found by calculating the radial component of $\mathbf{r}''$ explicitly. The radial component of $\mathbf{r}''$ is r''

$$r''=(\sqrt{\mathbf{r}.\mathbf{r}})''=\left(\frac{1}{2\sqrt{\mathbf{r}.\mathbf{r}}}(2\mathbf{r}'.\mathbf{r})\right)'=\left(\frac{\mathbf{r}'.\mathbf{r}}{r}\right)'=\frac{r(\mathbf{r}''.\mathbf{r}+\mathbf{r}'.\mathbf{r}')-r'\mathbf{r}'.\mathbf{r}}{r^2}$$

Substituting $\mathbf{r}''$ from equation (5), the radial component now reads,

$$r'' = \frac{(-R\mathbf{r} + r'\mathbf{r}').\mathbf{r} + r|\mathbf{r}'|^2}{r^2} - \frac{r'\mathbf{r}'.\mathbf{r}}{r^2}r'' = -\frac{R\mathbf{r}.\mathbf{r}}{r^2} + \frac{|\mathbf{r}'|^2}{r} = -R + \frac{|\mathbf{r}'|^2}{r} \tag{6}$$

From equation (4),

$$|\mathbf{r}'|^2 = R^2 - V^2(t' - T)^2 = R^2 - V^2(\frac{r}{V} - T)^2 = R^2 - (r - R)^2 = -r^2 + 2Rr$$

Inserting this in equation (6), we get

$$\begin{aligned} r'' &= -R - \frac{-r^2 + 2Rr}{r} = -R - r + 2R \\ r'' &= -(r - R) \end{aligned} \tag{7}$$

This, using the parameterisation constraint, this can be written as

$$t''' = -(t' - T) \tag{8}$$

Differentiating (5) with respect to s and making use of (7) and (5),

$$\mathbf{r}''' = -R\frac{r\mathbf{r}' - r'\mathbf{r}}{r^2} + \frac{r(r'\mathbf{r}'' + r''\mathbf{r}') - \mathbf{r}'(r')^2}{r^2}$$
$$\mathbf{r}''' = -R\frac{\mathbf{r}'}{r} + R\frac{r'\mathbf{r}}{r^2} + \frac{-R\mathbf{r}r'}{r^2} + \frac{(r')^2\mathbf{r}'}{r^2} + R\frac{\mathbf{r}'}{r} - \mathbf{r}' - \frac{\mathbf{r}'(r')^2}{r^2}$$

which upon simplification gives the equation of motion for $\mathbf{r}$

$$\mathbf{r}''' = -\mathbf{r}' \tag{9}$$

If we define a 4D vector $\mathbf{v}$ as

$$\mathbf{v} = (t' - T)\mathbf{e_t} + \mathbf{r}'$$

then the equation of motion can be expressed as

$$\mathbf{v}'' = -\mathbf{v} \tag{10}$$

This equation, along with the constant energy constraint, $||\mathbf{v}|| = R$, specifies the trajectory of a particle in the 4D space.

Equation (10) is the equation of simple harmonic motion about the origin, and equation (4) is an equation of sphere. These two equations together imply that the trajectory of a particle in the 4D space is a great circle that lies on the sphere of constant energy.

The projection of the great circle onto the spatial subspace is an ellipse with the force center at the origin. The vector $\mathbf{r}'$ describes an ellipse with center at origin. Since at every point of the orbit, $\mathbf{r}''$ is perpendicular to $\mathbf{r}'$, the vector $\mathbf{r}''$ also describes the same ellipse as $\mathbf{r}'$.

The LRL vector can be written in terms of s derivatives as

$$\mathbf{A} = m^2\dot{\mathbf{r}} \times (\mathbf{r} \times \dot{\mathbf{r}}) - \frac{mk\mathbf{r}}{r}$$
$$= m^2\frac{V^2}{r^2}\mathbf{r}' \times (\mathbf{r} \times \mathbf{r}') - \frac{mk\mathbf{r}}{r}$$
$$= \frac{m^2V^2}{r^2}((\mathbf{r}'.\mathbf{r}')\mathbf{r} - (\mathbf{r}'.\mathbf{r})\mathbf{r}')) - \frac{mk\mathbf{r}}{r} = \frac{m^2V^2}{r^2}((\mathbf{r}'.\mathbf{r}')\mathbf{r} - (\mathbf{r}'.\mathbf{r})\mathbf{r}')) - \frac{m^2V^2R\mathbf{r}}{r}$$
$$= -m^2V^2\mathbf{r} + 2\frac{m^2V^2R}{r}\mathbf{r} - \frac{m^2V^2}{r^2}(\mathbf{r}'.\mathbf{r})\mathbf{r}' - \frac{m^2V^2R}{r}\mathbf{r}$$
$$\mathbf{A} = \frac{m^2V^2}{r}((-r+R)\mathbf{r} - r'\mathbf{r}')$$

Since $m^2V^2 = \dfrac{mk}{R}$,

$$\mathbf{A} = \frac{mk}{Rr}((-r+R)\mathbf{r} - r'\mathbf{r}') \tag{11}$$

Defining a new vector $\mathbf{e} = \dfrac{\mathbf{A}R}{mK}$, this vector lies along the major axis, pointing from center in the direction of one of the focii, and the magnitude of $\mathbf{E}$ is

$$|\mathbf{e}| = \frac{|\mathbf{A}|R}{mk}$$

Since R is the radius of the sphere which is the same as the length of major axis of the ellipse, and $\dfrac{|\mathbf{A}|}{mk} = e$, the eccentricity of the orbit, $|\mathbf{e}|$ is equal to the distance of the focus from the origin.

Using

$$\mathbf{e} = \frac{1}{r}((R-r)\mathbf{r} - r'\mathbf{r}')$$

in equation (5),

$$\mathbf{r}'' = \frac{1}{r}(-R\mathbf{r} + r'\mathbf{r}') = -\mathbf{e} - \mathbf{r}$$

This equation implies that the orbit in 3D space is the same ellipse as $\mathbf{r}''$, with major axis R , but with origin shifted by the vector $\mathbf{e}$. i.e. the origin of the force is now at one of the focii.

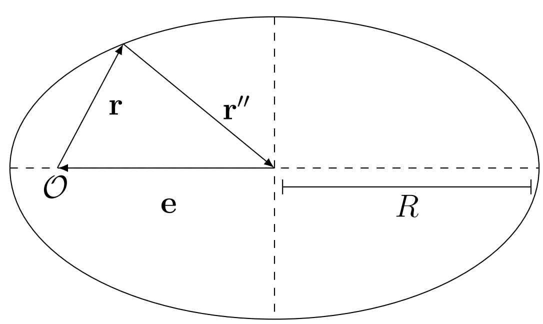

Figure 1: Relation between the $\mathbf{r}''$ trajectory and the $\mathbf{r}$ trajectory

6 Group Action

In the 4D space, the $SO(4)$ group elements rotate the great circles without changing their radius, i.e. they rotate the trajectories while keeping their energy constant. Thus the action of the group elements is to take a given orbit o_1 to another orbit o_2 such that the energy of both o_1 and o_2 is the same. Application of $SO(4)$ group elements generate a family of orbits all having same energy. To study the effect of the 4D rotations on the 3D projections, it is fruitful to consider a Kepler system in 2D space.

In this case, the trajectories are great circles that lie on a 3D sphere. In 2D, the angular momentum is not a vector, rather it is a bivector having one component, and the LRL vector has two components. For bound orbit solutions, the three together constitute the group $SO(3)$, the group of rotations in 3D Euclidean space. The projection of the circle onto the 2D plane - $r'_1 r'_2$ is again a centered ellipse while $\mathbf{r}$ traces the same ellipse but with origin at one of the focii.

The $SO(3)$ rotations can be broadly classified into two categories

1. rotations that do not alter the t' component of the trajectory (rotation of the $r'_1 r'_2$ plane) and

2. rotations that alter the t' component (rotation of the $t' r'_1$ and $t' r'_2$ planes)

When the $r'_1 r'_2$ plane itself is rotated, the orbit as a whole rotates without any change in the eccentricity or other properties. These rotations only change the direction of the LRL vector but not its magnitude. In 2D space, L independently forms a group $SO(1)$ of spatial rotations that rotate the orbit without changing its shape. Thus, rotations that leave t' component unchanged correspond to elements generated by angular momentum L.

On the other hand, rotations of planes $t' r'_1$ and $t' r'_2$ tilt the orbit at an angle with the t' axis, changing the r'_2 and r'_1 component of the projection respectively. In other words, these rotations change the length of minor axis without changing the length of major axis of the projections, i.e. they change the eccentricity, and hence $|\mathbf{A}|$ of the orbit. This change is accompanied by a change in L such that equation (2) is satisfied keeping the energy constant. Thus LRL vectors generate elements that change the eccentricity and angular momentum of the orbit, leaving its energy unchanged.

The inferences made from the 2D Kepler system can be extrapolated to the 3D Kepler system. The 4D description has 4 axes t', r'_1, r'_2, r'_3 giving us 6 planes to rotate. The rotations that do not change t' component - rotations of planes $r'_1 r'_2$, $r'_1 r'_2, r'_2 r'_3$ change the orientations of orbit in 3D space without changing its shape or energy. These are elements generated by the components of the angular momentum vector.

The rotations that change the t' components - rotations of planes $t' r'_1$, $t' r'_2, t' r'_3$ change the eccentricity of the orbit compensating for it with a change in $|\mathbf{L}|$ leaving energy unchanged. These take the orbit from a perfect circle to ellipses of higher eccentricity ending up in a straight line where the particle approaches the origin head on and then recedes. These are elements generated by the components of the Laplace-Runge-Lenz vector.

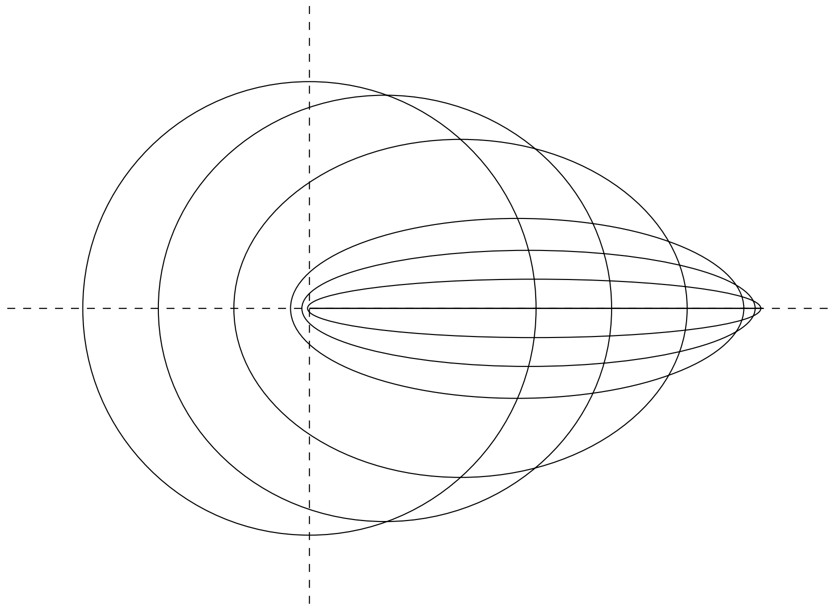

Figure 2: Few members of the family of infinite orbits having same energy but different angular momentum, related to each other by rotations generated by LRL vector components.

The existence of conserved LRL vector provides us a new degree of freedom to transform an existing orbit into another orbit of different angular momentum but same energy. In the language of quantum mechanics, what this means is that for different values of the angular momentum quantum number l, we get stationary states of same energy E. The hydrogen atom, which is a Kepler system, has additional degeneracy which is not expected from a trivial spherically symmetric system. The hydrogen atom has an additional degeneracy where the energy of a state, which trivially should have depended on the angular momentum quantum number l, depends only on the principle quantum number n. This is exactly what follows from the freedom LRL vector provides us - the accidental degeneracy of the Hydrogen atom is a consequence of the conserved LRL vector in a Kepler system.

7 Conclusion

We have discussed the classical definition of the dynamical symmetry group of a system. The Kepler system has two conserved vectors - **L** and **A** which makes it a maximally superintegrable system. The 6 components of the vectors together generate an $SO(4)$ algebra, which is a higher symmetry group than $SO(3)$ group which is expected from a spherically symmetric system. We have also discussed a detailed 4 dimensional description of the Kepler problem wherein trajectories in 4D which are great circles get projected as ellipses in spatial 3 dimensions. The group action of the $SO(4)$ on the 4D space is rotation of the great circles without changing its energy. The effect of rotations generated by components of **L** is to spatially rotate an orbit without changing its shape - a geometric symmetry. The rotations generated by components of **A** change eccentricity and angular momentum of the orbit while essentially keeping the energy constant. We also discussed how the existence of conserved LRL vector explain the accidental degeneracy of the hydrogen atom.

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A Group Theory

The formal mathematical way to talk about symmetries is the group theory. If a system remains invariant under a transformation G_i, then G_i is a member of the symmetry group G of the system. Mathematically a group is a set $G = \{G_i | \ \forall \ i\}$, along with an operation $\circ$ which follows these postulates.

- Closure - if $G_i, G_j \in G$, $G_i \circ G_j = G_k \in G$
- Associativity - $(G_i \circ G_j) \circ G_k = G_i \circ (G_j \circ G_k) \ \forall \ G_i, G_j, G_k \in G$
- Existence of identity - $\exists \ e \in G$ s.t. $G_i \circ e = e \circ G_i = G_i \ \forall \ G_i \in G$
- Existence of inverse - $\exists \ G_i^{-1} \in G$ s.t. $G_i \circ G_i^{-1} = G_i^{-1} \circ G_i = e \ \forall \ G_i \in G$

Symmetries can be broadly categorized in two categories - Discrete symmetries and continuous symmetries. If we restrict ourselves to the symmetries which are generated by a function of a parameter which is analytic over its entire domain, we will be talking about Lie groups

A.1 Lie Groups

Lie groups, being continuous, have infinite elements. Each element in the lie group can be written in the form $G_i = exp(g)$. This element g is called a generator of the lie group. g forms a vector space, spanned by the basis generators g_i. If α is the continuous parameter, then g can be written as a linear combination of the basis generators as $g = \sum \alpha_i g_i$. The group element can be, hence, written as $G_i = exp(\sum \alpha_k g_k)$. As an example, if a system is invariant under rotations, then the basis generators of the symmetry group are the components of angular momentum vector L_i. If θ is the angle of rotation, then the rotation element can be written as $exp(\sum \theta_i L_i)$

The group elements can be, in general, mapped to matrices such that all properties of the elements are respected by the matrices and the group operation can be mapped to matrix multiplication. In the case of rotational symmetry, the group is called SO(3), and the group elements can be expressed as 3×3 orthogonal matrices having determinant $+1$. Here S represents 'special', implying determinant $+1$, and O refers to 'orthogonal', and *3* indicates that the lowest non trivial representation will be the 3×3 matrices.

The generators of a group follow the commutator algebra, and this algebra is specific for a given group. Formally Lie algebra is defined as a vector space g which follows the following postulates.

- Alternativity - $[X, X] = 0 \ \forall \ X \in \mathrm{g}$
- Anti-commutativity - $[X, Y] = -[Y, X] \ \forall \ X, Y \in \mathrm{g}$
- Bilinearity - $[aX + bY, Z] = a[X, Z] + b[Y, Z] \ \forall \ X, Y, Z \in \mathrm{g}$
- Jacobi Identity - $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \ \forall \ X, Y, Z \in \mathrm{g}$

The SO(3) rotations group follows the algebra $[L_i, L_j] \equiv \{L_i, L_j\} = \epsilon_{ijk} L_k$.

Some other well known symmetry groups in physics are

- Lorentz group: Rotational invariance + boost invariance.
  Number of generators = 6
- Poincare group: Lorentz group + space-time translation invariance.
  Number of generators = 10
- SU(2): 2×2 traceless hermitian matrices. The algebra is similar to that of SO(3)
  Number of generators = 3
- SU(3): 3×3 traceless hermitian matrices.
  Number of generators = 8