In[ ]:= SetDirectory[NotebookDirectory[]]

```
In[ ]:= (* To find the neighbours of a given site,
given the number of site elements in one dimension and the number of dimensions. *)

(*In each dimension, there are two bounding surfaces. Here we determine the elements
 of the two surfaces for all the dimensions. The boundaries are in the format
 {{x axis boundary points}, {y axis boundary points} ... } *)
getBoundaries[nSites_, nDim_] :=
  Module[{ForwardBoundary, BackwardBoundary, d, n, n1},
   ForwardBoundary = {};
   For[d = 1, d ≤ nDim, d = d + 1,
    AppendTo[ForwardBoundary, {}];
    For[n = 1, n ≤ nSites^(nDim - d), n = n + 1,
     For[n1 = 0, n1 < nSites^(d - 1), n1 = n1 + 1,
      AppendTo[ForwardBoundary[[d]], n * nSites^d - n1 - 1]
     ]
    ]
   ];
   BackwardBoundary = {};
   For[d = 1, d ≤ nDim, d = d + 1,
    AppendTo[BackwardBoundary, {}];
    Do[AppendTo[BackwardBoundary[[d]] , elem - (nSites^(d - 1)) (nSites - 1)],
     {elem, ForwardBoundary[[d]]}
    ]
   ];
   {ForwardBoundary, BackwardBoundary}
  ];

(* Each site element will have 2*nDim neighbours,
i.e. 2 in each dimension. Here we determine the
 2 neighbours of each element in each of the dimensions *)
getNeighbours[nSites_, nDim_] :=
  Module[{boundaries, ForwardBoundary, BackwardBoundary, neighbour, nTot, n, d, nNeigh},
   boundaries = getBoundaries[nSites, nDim];
   ForwardBoundary = boundaries[[1]];
   BackwardBoundary = boundaries[[2]];
   nTot = nSites^nDim;
   neighbour = <||>;
   For[n = 0, n < nTot, n = n + 1,
    nNeigh = {};
    For[d = 0, d < nDim, d = d + 1,
      (* Forward *)
      If[Not[MemberQ[ForwardBoundary[[d + 1]], n]],
       AppendTo[nNeigh, n + nSites^d], AppendTo[nNeigh, n - (nSites^(d)) (nSites - 1)]];
      (* Backward *)
      If[Not[MemberQ[BackwardBoundary[[d + 1]], n]],
```

```
        AppendTo[nNeigh, n - nSites^d],
        AppendTo[nNeigh, n + (nSites^(d)) (nSites - 1)]]
      ] ×
      AppendTo[neighbour, n → nNeigh];
    ];
    neighbour];

treeEqualQ[Tree1_, Tree2_] := (Tree1[[1]] === Tree2[[1]] && Tree1[[3]] === Tree2[[3]]);

getTrees[nSites_, nDim_] := Module[
    {neighbour, Trees, tree, change, count, generatedTrees, newTrees, visited, completed,
     checked, neighbours, unvisitedNeighbours, possibleCombinations, newTree,},

    neighbour = getNeighbours[nSites, nDim];

    Trees = {{{}, {0}, {}, False}}; (*links in the tree ,
    visited sites,checked sites, complete or incomplete*)
change = True;
    SetSharedVariable[change];
While[change,
     change = False; (* while there is atleast one incomplete tree *)
     generatedTrees = WaitAll[
       ParallelTable[(*For each tree*)
        newTrees = {};
        visited = tree[[2]];
        checked = tree[[3]];
        completed = tree[[4]];
        If[completed, AppendTo[newTrees, tree],
         change = True;
         Do[(*For each visited site*)
          If[Not[MemberQ[checked, visitedSite]], (*If site is not already checked *)
           neighbours = neighbour[visitedSite];
           unvisitedNeighbours = Complement[neighbours, visited];
           If[unvisitedNeighbours == {}, , possibleCombinations =
             Subsets[unvisitedNeighbours, {1, Length[unvisitedNeighbours]}]];

            Do[(*For each possible combination of links that can be added*)
             newTree = tree;
             Do[(*Add the links to the tree,
              and the sites to visited*)AppendTo[newTree[[1]], {visitedSite, site}];
              newTree[[1]] = Sort[newTree[[1]]];
              AppendTo[newTree[[2]], site];
              AppendTo[newTree[[3]], visitedSite];
              newTree[[3]] = Sort[newTree[[3]]];
              , {site, combination}];
             (*Check if the newTree created has visited all elements*)
             If[Sort[newTree[[2]]] == Range[0, nSites^nDim - 1], newTree[[4]] = True, ,];
```

```
                (* Append the new tree
                 to the net of newtrees *)AppendTo[newTrees, newTree];
                , {combination, possibleCombinations}
               ]
              ]]
            , {visitedSite, visited}
           ]
         ]; newTrees
         , {tree, Trees}
        ]];
     Trees = DeleteDuplicates[Flatten[generatedTrees, 1], treeEqualQ];
      (* delete duplicates and replace the current set of trees by the new set *)
    ];
    Trees];


(* Generating all possible trees for a 3x3 lattice *)

nSites = 3;
nDim = 2;
neighbour = getNeighbours[nSites, nDim];
Trees = getTrees[nSites, nDim]
```

In[∘]:=
```
DumpSave[StringJoin[{"Trees_", ToString[nSites] , "_" , ToString[nDim], ".mx"}], Trees]
```

Out[∘]=

```
{{{{0, 1}, {1, 2}, {2, 5}, {3, 4}, {4, 7}, {5, 3}, {6, 8}, {7, 6}}, {0, 1, 2, 5, 3, 4, 7, 6, 8},
   {0, 1, 2, 3, 4, 5, 6, 7}, True}, {{{0, 1}, {1, 2}, {2, 5}, {3, 4}, {4, 7}, {5, 3}, {7, 8}, {8, 6}},
   {0, 1, 2, 5, 3, 4, 7, 8, 6}, {0, 1, 2, 3, 4, 5, 7, 8}, True}, … 11 660 …,
  {{{0, 1}, {0, 2}, {0, 3}, {0, 6}, {4, 5}, {6, 7}, {6, 8}, {7, 4}}, {0, 1, 2, 3, 6, 7, 8, 4, 5},
   {0, 0, 0, 0, 4, 6, 6, 7}, True}, {{{0, 1}, {0, 2}, {0, 3}, {0, 6}, {5, 4}, {6, 7}, {6, 8}, {8, 5}},
   {0, 1, 2, 3, 6, 7, 8, 5, 4}, {0, 0, 0, 0, 5, 6, 6, 8}, True}}
```

Full expression not available (original memory size: 16 MB)

In[∘]:=
```
DumpSave[
 StringJoin[{"neighbours_", ToString[nSites] , "_" , ToString[nDim], ".mx"}],  neighbour]
```

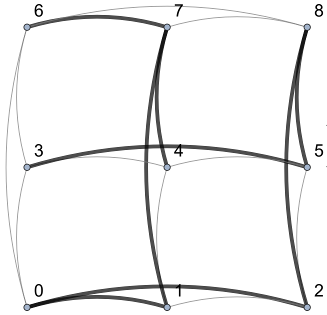
Out[∘]=

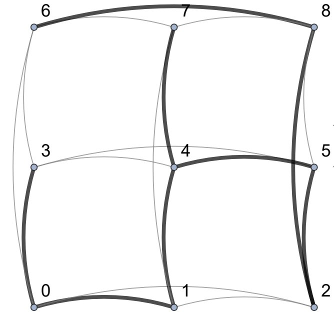
```
{<|0 → {1, 2, 3, 6}, 1 → {2, 0, 4, 7}, 2 → {0, 1, 5, 8}, 3 → {4, 5, 6, 0},
  4 → {5, 3, 7, 1}, 5 → {3, 4, 8, 2}, 6 → {7, 8, 0, 3}, 7 → {8, 6, 1, 4}, 8 → {6, 7, 2, 5}|>}
```

In[∘]:=
```
Get[StringJoin[{"Trees_", ToString[nSites] , "_" , ToString[nDim], ".mx"}]]
(*Get the object "Trees"*)
```

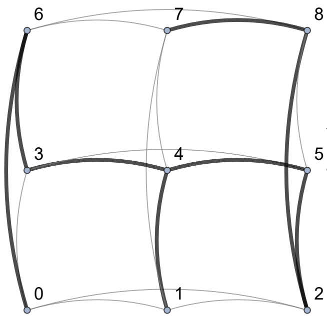
```
In[]:= getVertexCoords[nSites_, nDim_] := Module[ (*Only for 2D lattices*)
         {nTot, n, x, y},
         vertexPos = {};
         nTot = nSites^nDim;
         x = 0;
         y = 0;
         For[n = 0, n < nTot, n = n + 1,
          AppendTo[vertexPos, n → {x, y}];
          If[Mod[n, nSites] == nSites - 1, x = 0; y = y + 1, x = x + 1];
         ];
         vertexPos
        ]
       showTree[nSites_, nDim_, tree_] := Module[
         {links, neighbour, n, nTot, treeLinks},
         links = {};
         nTot = nSites^nDim;
         neighbour = getNeighbours[nSites, nDim];
         For[n = 0, n < nTot, n = n + 1,
          Do[
            AppendTo[links,
             UndirectedEdge[Sort[{n, i}][[1]], Sort[{n, i}][[2]]]], {i, neighbour[n]}
          ]
         ];
         treeLinks = Table[UndirectedEdge[link[[1]], link[[2]]], {link, tree}];
         HighlightGraph[ Graph[DeleteDuplicates[links], VertexLabels → "Name",
           VertexCoordinates → getVertexCoords[nSites, nDim], EdgeShapeFunction → "CurvedEdge",
           EdgeStyle → {Gray}], Style[treeLinks, {Black, Thick}], ImageSize → Small]
        ]

In[]:= (* Showing 8 random trees from the generated
        trees. The thin gray lines are the links of the lattice,
       and the thick black lines are the links set to I via spanning tree*)
       Table[index = RandomChoice[Range@Length@Trees];
        {StringJoin["Index: ", ToString@index] → showTree[nSites, nDim, Trees[[ index, 1]]]},
        {i, Range[1, 8]}]
```

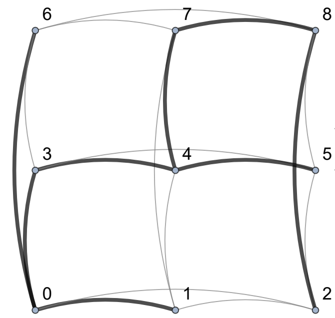
Out[ ]=

{{Index: 4974 → 

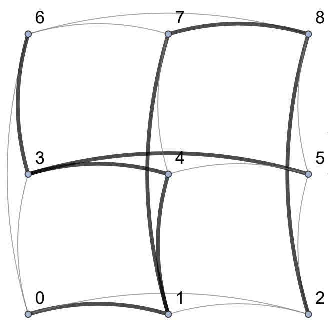
}, {Index: 5664 → 

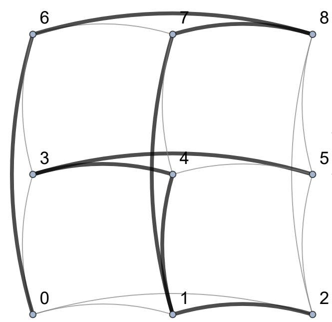
},

{Index: 3572 → 

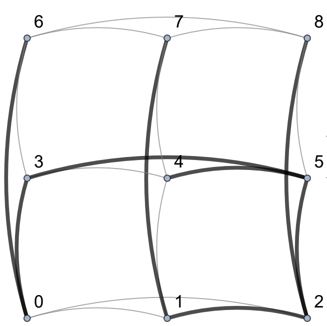
}, {Index: 10960 → 

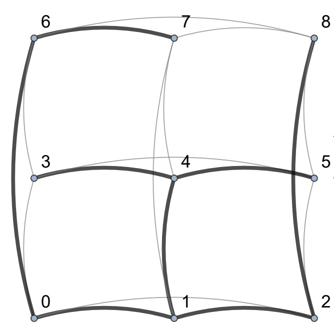
},

{Index: 981 → 

}, {Index: 3987 → 

},

{Index: 9421 → 

}, {Index: 6761 → 

}}