Complex Analysis

Instructor: Prof. Balakrishnan

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This is the lecture notes for Prof. Balakrishnan's course on Complex Analysis, given at IIT Madras as an NPTEL module. The course is available on youtube here (Lectures 1-17).

There are a few sections in the notes which are in red. These are statements and calculations that I have concluded on my own and are not stated or performed in the lecture.

If you find any errors in the text, please send an email to me at adithyarao3132001@gmail.com

Contents

1 Analytical Functions

Complex variable — $z = x + iy, \quad x, y \in \mathbb{R}$

We talk about functions which are analytic functions in a specific sense of $x + iy$ on the complex plane. x and y are linearly independent, and $z^* = x - iy$ is linearly independent to z, and by speaking about analytical functions we impose that the function depends on z and not on z^*. We will make this idea much more precise in the future.

Before discussing analytical functions, we need to discuss about stereographic projections. In simple x line, there is only two infinite points, one at $+\infty$ and other at $-\infty$. But in a complex plane, there are infinte number of infinities, one for each direction you can pick. The standard trick is to try and put the points at infinity at the same footing as any finite point. This is done by stereographic projection.

1.1 Stereographic Projection

The idea is to compactify the space by *lifting the plane* and sewing all points at infinity to one single point, which will form a sphere. This will be our model for complex plane.

To make it more concrete, consider a complex plane x, y, and a unit sphere with coordinates $\xi_1 = \sin\theta\cos\phi, \xi_2 = \sin\theta\sin\phi, \xi_3 = \cos\theta$, satisfying the constraint $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$.

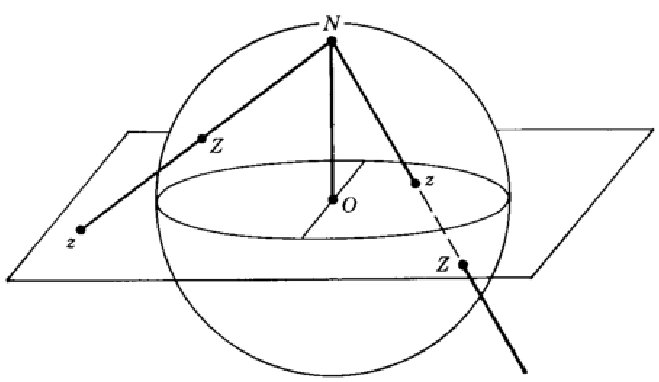


Figure 1: Stereographic projection

What we do is, we pick a point on the complex plane, and draw a line connecting it to the north pole of the sphere. This line will intersect the sphere at some point, and we map these two points as equivalent. We see that all points inside the unit circle on the complex plane get mapped to the southern hemisphere, and all points outside get mapped to the norther hemisphere. The equator of the sphere is the unit circle on the complex plane. All the infinities get mapped to the north pole, while the origin gets mapped to the south pole.

The equations connecting x and y to the ξ's can be found by seeing that, from the similarity of ΔANB

and ΔONP in figure 2, we get

$$\frac{ON}{AN} = \frac{OP}{AB}$$

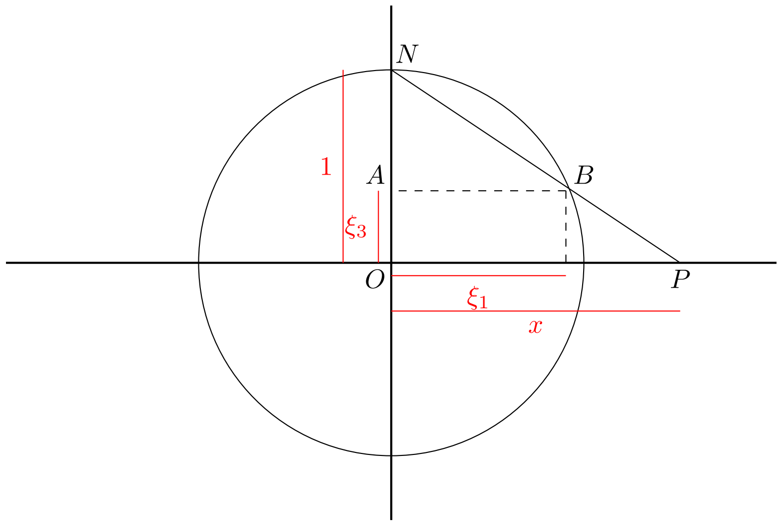


Figure 2: x in terms of ξs

which gives

$$x = \frac{\xi_1}{1-\xi_3} \tag{1.1}$$

$$y = \frac{\xi_2}{1-\xi_3} \tag{1.2}$$

and therefore

$$z = \frac{1}{1-\xi_3}(\xi_1 + i\xi_2) \tag{1.3}$$

Writing in terms of θ and ϕ, we see

$$x = \cot\frac{\theta}{2}\cos\phi \tag{1.4}$$

$$y = \cot\frac{\theta}{2}\sin\phi \tag{1.5}$$

and therefore

$$z = x + iy = \cot\frac{\theta}{2}\mathrm{e}^{i\phi} \tag{1.6}$$

The sphere we constructed above is called the Riemann sphere.

We can find the inverse relations as

$$\xi_1 = \frac{2x}{x^2+y^2+1} = \frac{z+z^*}{|z|^2+1} \tag{1.7}$$

$$\xi_2 = \frac{2y}{x^2+y^2+1} = \frac{z-z^*}{i(|z|^2+1)} \tag{1.8}$$

$$\xi_3 = \frac{x^2+y^2-1}{x^2+y^2+1} = \frac{|z|^2-1}{|z|^2+1} \tag{1.9}$$

Usually we consider complex plane without the infinity, i.e. we define

$$\mathbb{C} := \{z : |z| < \infty\}$$

If we include the point at infinity, we call it the extended complex plane and we denote it as $\hat{\mathbb{C}}$. The Riemann sphere provides a model for the extended complex plane. By doing this compactification, we can now rigorously do calculus on it.

1.2 Notion of distance on the Riemann Sphere

There are many ways one can define a distance on the Riemann sphere, one common way being the geodesic distanec. But we have another distance, which is more convenient called the chordal distance, i.e. given two points z_1 and z_2 on the complex plane, we draw a chord through the hollow sphere connecting the two points when mapped to the sphere, and the length of the chord is the distance. We get that

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(|z_1|^2+1)(|z_2|^2+1)}}$$

This satisfies all the properties we need of a distance function, i.e.

- $d(z_1, z_2) \geq 0$, equality holding only when $z_1 = z_2$
- $d(z_1, z_2) + d(z_2, z_3) \geq d(z_1, z_3)$
- $d(z_2, z_1) = d(z_1, z_2)$

What does it mean when we talk about the distance to infinity?
In the limit $z_2 \to \infty$, we get

$$d(z, \infty) = \frac{2}{\sqrt{|z|^2+1}}$$

which is nothing but the chordal distance between the point z and the north pole.

1.3 Analytic Function in Some Region

A function $f(z) = u(x, y) + iv(x, y)$ is analytic if the Cauchy-Riemann conditions are satisfied, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.10}$$

This is simply a restatement of the requirement that f does not at all depend on z^*, or in other words $\dfrac{\partial f}{\partial z^*} = 0$. That is, f depends only on the combinatin $x + iy$ and not on the combination $x - iy$ at all.

$$\frac{\partial f}{\partial z^*} = 0 \implies \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*} = \frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y} = 0$$

Substituting $f = u + iv$, we get the C-R conditions.

This immediately tells you that for a function to be analytic in a given region, it can not be purely imaginary or purely real, in which case the function would have to depend on z^*. We see that x, y, r, θ etc. are NOT analytic functions, since they all depend on z^*.

One consequence of the C-R conditions is that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \& \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Therefore, the real and imaginary parts of the analytic function should independently satisfy Laplaces equation, and therefore are harmonic functions.

If the function is analytic in the whole of complex plane $\mathbb{C}$ (without ∞), then f is called an entire function.

Euler's theorem — If you have a function that is entire, and at infinity they are not singular and satisfy the C-R conditions, then the function should be a constant function

Therefore, for all normal entire functions like $z, z^2, \mathrm{e}^z, \sin z$ etc, the point at infinity is a singularity and is called an essential singularity.

1.4 Derivative of Complex Function

In the case of complex functions, the derivative,

$$\frac{df(z)}{dz} = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

has an ambiguity in deciding what direction should δz take.
We can write $\delta z = \epsilon \mathrm{e}^{i\alpha}$, where now ϵ is the controlling factor that should go to zero, and α is simply an angle, we get

$$\frac{df(z)}{dz} = \mathrm{e}^{-i\alpha}\left(\lim_{\epsilon \to 0} \frac{u(x + \epsilon \cos\alpha, y + \epsilon \sin\alpha) - u(x, y)}{\epsilon} + i \lim_{\epsilon \to 0} \frac{v(x + \epsilon \cos\alpha, y + \epsilon \sin\alpha) - v(x, y)}{\epsilon}\right)$$

which evaluates to

$$\mathrm{e}^{-i\alpha}\left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha + i\frac{\partial v}{\partial x}\cos\alpha + i\frac{\partial v}{\partial y}\sin\alpha\right)$$

We can rearrange this in the form

$$\frac{df(z)}{dz} = \mathrm{e}^{-i\alpha}\left(\left(\frac{\partial u}{\partial x}\cos\alpha + i\frac{\partial v}{\partial y}\sin\alpha\right) + i\left(\frac{\partial v}{\partial x}\cos\alpha - i\frac{\partial u}{\partial y}\sin\alpha\right)\right)$$

We impose that this shold not have any α dependence, which requires the C-R conditions to be satisfied, in which case the $\mathrm{e}^{i\alpha}$ from inside the bracket cancels the $\mathrm{e}^{-i\alpha}$ from outside.

Why does the condition that there should be no z^* dependence lead to the derivative being direction independent?

Consider the above discussed limit, (with $\delta z = \epsilon \mathrm{e}^{i\alpha} \implies \delta z^* = \epsilon \mathrm{e}^{-i\alpha}$)

$$\lim_{\delta z\to 0}\frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z\to 0}\frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\epsilon\to 0}\mathrm{e}^{-i\alpha}\left(\frac{\frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial z^*}\delta z^*}{\epsilon}\right)$$

which is equal to

$$\mathrm{e}^{-i\alpha}\left(\frac{\partial f}{\partial z}\mathrm{e}^{i\alpha} + \frac{\partial f}{\partial z^*}\mathrm{e}^{-i\alpha}\right) = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial z^*}\mathrm{e}^{-2i\alpha}$$

We see that the direction dependence in the limit comes multiplied with $\dfrac{\partial f}{\partial z^*}$. Therefore, requiring directional independence requires f to not have a dependence on z^* and vice versa.

1.5 Power Series in Complex Variables

Analyticity implies that the derivative itself of an analytic function is analytic, and so on, and therefore all analytic functions on the complex plane are infinitely differentiable. What this means is that we can always represent an analytic function in a neighborhood as a Taylor series, i.e. in a region

$$f(z - z_0) = \sum_n a_n (z - z_0)^n, \quad a_n = \frac{1}{n!}\frac{\partial^n f(z)}{\partial z^n}\bigg|_{z=z_0}$$

For every power series about the point z_0, there is some circle about the point with radius R such that for all points z inside this circle,

$$\sum_n |a_n(z - z_0)^n| < \infty$$

i.e. Absolute convergance. The radius of convergence is given by

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

A better definition is

$$R = \lim_{n\to\infty}\sup |a_n|^{1/n}$$

What happens if R is infinite? Then it means that the function is an entire function.

As examples, consider the function

$$e^z = \sum \frac{1}{n!} z^n$$

For this, the ratio test gives

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{z}{n+1} = 0 \; \forall z < \infty \tag{1.11}$$

Therefore, the series representation of e^z is convergent, and the radius of convergence is ∞.

What about the series

$$\sum z^n$$

This converges for $|z| < 1$, therefore the radius of convergence $R = 1$.
But we know that the function

$$\frac{1}{1-z}$$

represents this infinite series inside the unit circle $|z| < 1$. We are guarenteed that for any z inside the unit circle, the series and the function exactly match. But for $|z| \geq 1$, the series doesn't converge, but the function gives a finite value. The function blows up only at $z = 1$, and is analytic everywhere except at $z = 1$. We call the function the analytic continuation of the series.

<u>Analytic continuation</u> — we have a representation for a series, which matches the series point by point in some region. The series is not convergent in some region, but is not defined at some other, while the representation is defined in a bigger region. Then the representation is called the analytical continuation of the series.

Consider the expansion of $\frac{1}{1-z}$ about some other point, lets say $z = -\frac{1}{2}$.

$$\frac{1}{1-z} = \frac{1}{\frac{3}{2} - \left(z + \frac{1}{2}\right)} = \frac{2}{3} \frac{1}{1 - \frac{2}{3}\left(z + \frac{1}{2}\right)} = \frac{2}{3} \sum \left(\frac{2}{3}\right)^n \left(z + \frac{1}{2}\right)^n$$

This converge in a circle with radius $R = \frac{3}{2}$ centered at $z = -\frac{1}{2}$ This is still undefined at $z = 1$. This series is also valid in the region outside the unit circle. Now we can again write an expansion which is inside this circle, but outside the unit cirlce, and by repeatedly doing so, we can write an infinite number of series for the same function which have overlaps with each other and match the function in different regions, together covering the entire complex plane. Each of the series will have a boundary passing through $z = 1$, and each of the series are analytical continuations of each other, valid in different regions. For each representation, we need to say where it is valid. In general we won't be able to find the master representation function, but we will be able to find the different series representations valid for different regions, and these representations are analytical continuations of each other.

For any power series, there exists a circle of convergence, and the power series is absolutely convergant inside the circle, and divergant outside. Not much can be told about what happens on the circle of convergance, and should be dealt with on case-by-case basis.

As an example consider $z = -1$ on $\sum z^n$. The sum is $1 - 1 + 1 - 1 + 1 - 1 + \cdots$. This doesn't make sense, but we can consider the partial sums. The partial sums would be $1, 0, 1, 0, \cdots$, and the average of the partial sums is $\frac{1}{2}$, and plugging in $z = -1$ in the function also gives $\frac{1}{2}$. Similarly, consider at $z = i$. The series gives $1 + i - 1 - i + \cdots$. The partial sums is $1, 1+i, i, 0, \cdots$ and the average of the partial sums is $\frac{1+i}{2}$. In this case, again the function gives the same value. This sort of thing is called a Cesaro sum. That is, the arithmatic average of the partial sums is guarenteed to match the value of the function.

Theorem — The function which a given power series represents should have atleast one singularity on the circle of convergence.

We might observe that for some series, the power series actually converges at the boundary, in which case on the surface it might look like the above theorem is broken. As an example,

$$f(z) = \sum \frac{z^n}{n^2}$$

is convergent in $|z| \leq 1$. But the function this series represents has a singularity at $z = 1$. This is a subtle type of singularity, to understand which let us consider a function of the form $(1 - z)\log(1 - z)$. At $z = 1$ this has a logarithmic divergance, but the polynomial factor in front of it goes to 0, and therefore in the limit, the entire function goes to zero. This doesn't mean that the function is analytic at $z = 1$. There is still a singularity, and it is this type of singularity that is present on the boundary.

It is also possible that there are infinite number of singularities on the boundary. As an example consider the series

$$f(z) = z + z^2 + z^4 + \cdots \equiv \sum z^{2n}$$

This has radius of convergance $R = 1$, and is convergant in $|z| < 1$.
Now, this series is defninitely singular at $z = 1$. but the function can also be written as

$$f(z) = z + f(z^2)$$

This implies that it is also singular at $z = -1$. We can also write the series as

$$f(z) = z + z^2 + f(z^4)$$

which implies the function is singular at $\pm i$ too, and we can similarly show that for all z on the boundary, the function is singular. This means that there is no way in which we can analytically continue the function to outside the given boundary.

$f(z)$ has a natural boundary $|z| = 1$. This type of series is called a Lacunary series.

If a function is analytical in some region, then within that region, we can also integrate the function. The integration

$$\int_{z_1}^{z_2} f(z)dz$$

is independent of the path chosen, as long as it is a connected path and never leaves the region of analyticity. The immediate consequence is

$$\oint_C f(z)dz = \oint_C f(z)dz = 0$$

This contour can be deformed, while still keeping the value of the integral same.

Why is this true?

This is the consequence of the C-R conditions.

Consider the integral of $f(z)$ over a closed loop which is defined in the region where it is analytic.

$$\oint_C f(z)dz$$

$f(z) = u(x, y) + iv(x, y)$, and $z = x + iy$, meaning

$$f(z)dz = (udx - vdy) + i(vdx + udy)$$

For any contour C, Green's theorem gives that

$$\oint_C udx - vdy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA$$

and

$$\oint_C vdx + udy = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dA$$

Since at all points in the region enclosed by C, the C-R conditions are satisfied, the above contour integrals, and as a result the integral $\oint_C f(z)dz$ is therefore zero.

Now consider two points, z_1 and z_2, and two paths γ_1 and γ_2 connecting them. (we use $-\gamma$ to denote the path γ traced backwards, and the fact that integral over $-\gamma$ is -1 times the integral over γ)

$$\oint_C f(z)dz = 0 \implies \int_{\gamma_1} f(z)dz + \int_{-\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz = 0$$

The above implies that the integral does not depend on the path.

Before discussing integration further, we need to discuss singularities.

1.6 Singularities

There are multiple kinds of singularities, which are as follows

1.6.1 Removable singularities

Example, $\frac{\sin z}{z}$. The limiting value at $z = 0$ is 1, and therefore, we can define the function such that it is $\frac{\sin z}{z}$ at $z \neq 1$ and 1 and $z = 1$, which retains not only the complete function behaviour, but removes the singularity. These type of singularities are trivial, and are not at all to our interest.

1.6.2 Simple Poles

If around $z = a$, a function has the following form

$$f(z) = \frac{c_1}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n$$

That is, it is mostly analytical having a power series representation, with an extra piece $\propto (z-a)^{-1}$, then such a function is said to have a simple pole at a. The residue of the pole is the constant c_{-1}.

For example,

$$f(z) = \frac{\sin z}{z^2}$$

To find the type of singularity, use the fact that sine is an entire function and expand it in power series

$$f(z) = \frac{1}{z} - \frac{z}{3!} + \cdots$$

which has a simple pole at $z = 0$, with residue 1.

In general, if

$$f(z) = \frac{g(z)}{h(z)}$$

with $g(a) \neq 0$, and $h(a)$ has a simple zero, i.e. $h(z)$ is of the form $(z-a)h'(a) + \cdots$, then

$$f(z) = \frac{g(a) + g'(a)(z-a) + \cdots}{h'(a)(z-a) + \cdots}$$

which means that the residue at $z = a$ is $\dfrac{g(a)}{h'(a)}$.

To extract the residue of a function $f(z)$ at $z = a$, we can also do

$$\text{residue} = \lim_{z \to a} (z-a) f(z)$$

This is because when multiplied by $(z-a)$, the singular part becomes simply the constant (residue), while the analytical part goes to zero while taking the limit.

Example,

$$\frac{1}{\sin \pi z}$$

This has singularities at all integers.
At $z = 0$, the leading behaviour of $\sin \pi z$ as $z \to 0$ is $z\pi$, and therefore the residue is $\dfrac{1}{\pi}$. For other integers, we need to take the limit

$$\lim_{z \to n} \frac{z - n}{\sin \pi z} = \frac{(-1)^n}{\pi}$$

1.6.3 Higher Order Poles

What happens if the function is of the form

$$f(z) = \sum c_n (z - a)^n + \frac{c_{-1}}{(z - a)} + \frac{c_{-1}}{(z - a)^2} + \cdots + \frac{c_{-m}}{(z - a)^m}$$

This function is said to have a pole of order m at $z = a$. The residue is still the coefficient of $(z - a)^{-1}$, i.e. c_{-1}. To extract this, we can multiply the series by $(z - a)^m$, and take the derivative with respect to z $m - 1$ times and divide by $(m - 1)!$, and then take the limit $z \to a$.

A very important question is where does the singular part of the series converge. The regular part converges inside some region R, and the singular part usually converges outside some region W, which might also be bigger than R. The only way this can make sense is if $W \subset R$, and the series is meaningful in the annular reagion $R - W$. The series of this kind are called Laurent series, and they *typically* converge in some annular region. This is typical behaviour and not necessary behaviour.

2 Calculus of Residues

2.1 The Residue Theorem — Cauchy's Integral Formula

Consider the integral

$$\oint_C \frac{1}{z^{n+1}} dz$$

over a contour enclosing the origin. We can deform this contour to make it a unit circle, on which $z = \mathrm{e}^{i\theta}$, and therefore $dz = i\mathrm{e}^{i\theta} d\theta$. The integral therefore becomes

$$i \int_0^{2\pi} \mathrm{e}^{-(n+1)i\theta} \mathrm{e}^{i\theta} d\theta = i \int_0^{2\pi} \mathrm{e}^{-ni\theta} d\theta = i \int_0^{2\pi} \cos(n\theta) - i\sin(n\theta) d\theta$$

The integral of sine and cosine from 0 to 2π is zero, and therefore, the above integral is 0, unless when $n = 0$, where the integral is simply $i \int d\theta = 2\pi i$.

Therefore the integral

$$\oint_C \frac{1}{z^{n+1}} dz = \begin{cases} 0 & : n \neq 0 \\ 2\pi i & : n = 0 \end{cases}$$

We can shift the contour and the singularity to some point $z = a$ and write the integral on a contour that contains a as

$$\oint_C \frac{1}{(z-a)^{n+1}} dz = \begin{cases} 0 & : n \neq 0 \\ 2\pi i & : n = 0 \end{cases}$$

and similarly for the clockwise contour

$$\oint_C \frac{1}{(z-a)^{n+1}} dz = \begin{cases} 0 & : n \neq 0 \\ -2\pi i & : n = 0 \end{cases}$$

Note that the above integral is also zero if the closed contour does not contain the singularity.

Therefore for a general series,

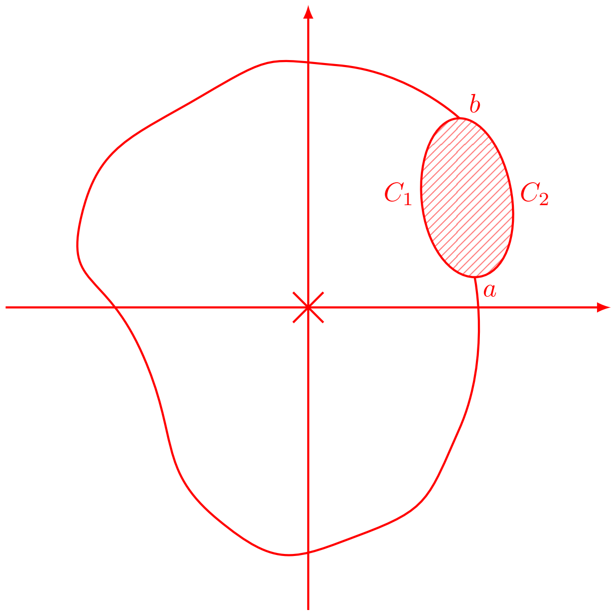
$$f(z) = f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \frac{c_{-1}}{(z-a)} + \frac{c_{-1}}{(z-a)^2} + \cdots$$

the value of

$$\oint f(z) dz = \begin{cases} 2\pi i \; c_{-1} & : \text{ if the contour encloses the point } z = a \\ 0 & : \text{otherwise} \end{cases}$$

since all the other terms in the series integrate out to zero.
This is called the Cauchy's Integral Formula.

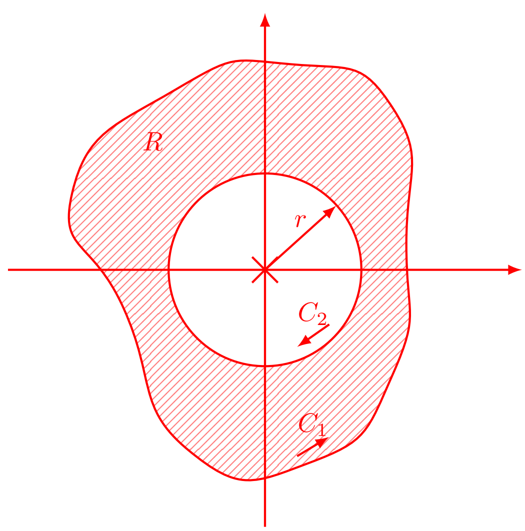
One could question that since the function is not analytic in the entire region inside the contour, how can we claim that the integral is contour independent. To answer this, consider the following case.



Consider taking a section of the path connecting the points a and b — C_1, and deforming it to form another path C_2. Now one can ask if the integral of $f(z)$ over C_1 is equal to the integral over C_2. As one can see, these are indeed equal since the function is analytic in the entire shaded region, making the integral path independent in that region.
Now since we can divide the entire contour into multiple sections, and each of them can be deformed without altering the value of the integral, one can conclude that the entire closed contour integral is also contour independent.

The contour integral changes values, only when while deforming the contour we pass through a singular point, i.e. only if the shaded region in the above figure contained a pole.

Another way to look at this is by considering the following.



Rather than considering the entire region, consider only the annular region R. The analyticity of $f(z)$ in R together with Green's theorem imply that

$$\oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz = 0$$

We can extend the previous integral we performed over the unit circle to be over a circle of any radius r as

$$i\int_0^{2\pi} r^{-n-1}\mathrm{e}^{-(n+1)i\theta} r\mathrm{e}^{i\theta}d\theta = i\int_0^{2\pi} r^{-n}\mathrm{e}^{-ni\theta}d\theta = r^{-n}i\int_0^{2\pi}\cos(n\theta) - i\sin(n\theta)d\theta = \begin{cases} 2\pi i & : n = 0 \\ 0 & : n \neq 0 \end{cases}$$

Which means that

$$\oint_{C_2} f(z)dz = -2\pi i \times \text{residue of } f \text{ at } 0$$

Therefore, one gets

$$\oint_{C_1} f(z)dz = 2\pi i \times \text{residue of } f \text{ at } 0$$

where the integral is now contour independent, as long as the singularity is contained within it. Further, since the integral over C_2 does not depend on r, we can take the limit $r \to 0$ and allow the contour C_1 to be deformed anywhere, as long as it does not pass the singularity.

We will now discuss a few applications of the Cauchy's Integral Formula, with which we can understand it in further depth.

2.2 Application — Recursion Relations

Suppose we are given a recursion relation, typically a two step recursion relation, which is equivalent to a second order differential equation, whose solution can't be written by inspection.

Suppose

$$c_{n+1} = \frac{c_n + c_{n+2}}{2}$$

To solve this uniquely we need to specify initial values, c_0 and c_n, and the question that can be posed is what is the general term c_n.

The first thing we can do is day that

$$c_{n+2} - 2c_{n+1} + c_n = 0$$

Now we define the *generating function*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

Now we can multiply the recursion relation by z^n and sum over n to get

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_{n+2} z^n - \sum_{n=0}^{\infty} 2c_{n+1} z^n + \sum_{n=0}^{\infty} c_n z^n = 0 \\
\implies &\frac{1}{z^2}(f(z) - c_1 z - c_0) - \frac{2}{z}(f(z) - c_0) + f(z) = 0 \\
\implies &f(z) = \frac{c_0 + (c_1 - 2c_0)z}{(z-1)^2}
\end{aligned}$$

If given c_0 and c_1, we get a function. Now since $f(z)$ is a Taylor series, we can write c_n as the n^{th} derivative of $f(z)$ but this is hard.
There is a better way to extract c_n using an integral formula. We can do this by dividing by z^{n+1} and integrating over a contour surrounding the origin.

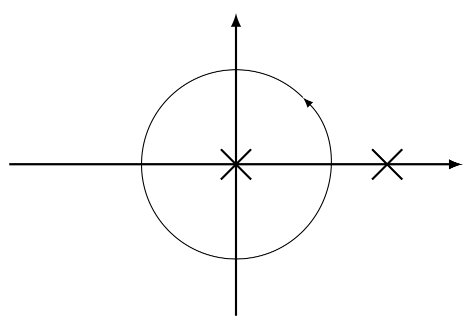
$$c_n = \frac{1}{2\pi i} \oint_C \frac{dz}{z^{n+1}} f(z)$$

where C is a contour that encloses a region containing the origin.

In the example considered, we can do

$$c_n = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} \frac{c_0 + (c_1 - 2c_0)z}{(z-1)^2}$$

We have a pole of order $n+1$ at 0 and a contour surrounding it, and another pole of order 2 at $z = 1$



If we want to do this integral directly, we have to find the residue at $z = 0$ which will involve differentiating $f(z)$ n times. Our life would have been easier if all we had to do was to somehow evaluate the integral at $z = 1$. Well we can do this, by exploiting the path independence. We can deform the contour while keeping track of the orientation without crossing the second pole as given in figure 3 where the dotted line is now at infinity.

What is the contribution of the dotted line to the integral? Since $z = re^{i\theta}$, the relevent terms at infinity and therefore the overall power of r in the above integral are $\dfrac{dz}{z^{n+1}}\dfrac{z}{z^2} \implies \dfrac{r}{r^{n+1}}\dfrac{r}{r^2} = r^{-n}$, where $n > 0$. Therefore the contribution from the dotted part is zero.

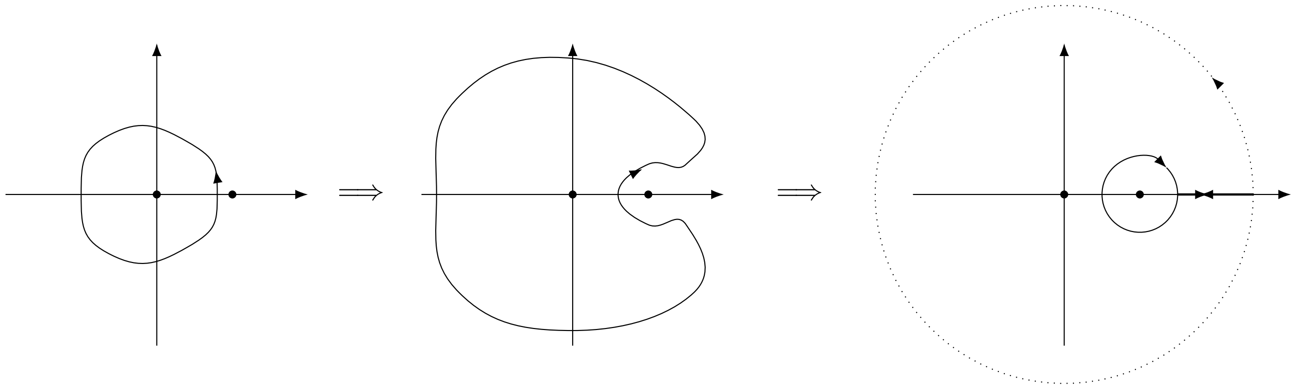


Figure 3: Deforming the contour to wrap around a different pole.

There is a part of the contour on the x axis going to $+\infty$ and another part returning from ∞. There are no other singularities along the x axis, and therefore these two contributions cancel each other, and therefore, the only contribution to the integral is from the, now clockwise, contour going around the pole at $z = 1$. Therefore, the integral is equal to

$$c_n = \frac{1}{2\pi i}(-2\pi i)\frac{1}{1!}\frac{d}{dz}\left(\frac{c_0 + (c_1 - 2c_0)z}{z^{n+1}}\right)\Bigg|_{z=1}$$

which given $c_0 = 1$ and $c_1 = 2$ is equal to $n + 1$.

Excercise — $c_{n+2} = c_{n+1} + c_n$ with $c_0 = 0$ & $c_1 = 1$ Introducing the generating function,

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_{n+2} z^n - \sum_{n=0}^{\infty} c_{n+1} z^n - \sum_{n=0}^{\infty} c_n z^n = 0 \\
\implies &\frac{1}{z^2}(f(z) - c_1 z - c_0) - \frac{1}{z}(f(z) - c_0) - f(z) = 0 \\
\implies &f(z) = \frac{-c_0 - c_1 z + c_0 z}{z^2 + z - 1} = \frac{-z}{z^2 + z - 1}
\end{aligned}$$

The roots of $z^2 + z - 1$ are $z = \frac{-1 \pm \sqrt{5}}{2} = \alpha(\beta)$

$$f(z) = \frac{-z}{(z-\alpha)(z-\beta)}$$

This has two poles at α and β, and we can again deform the counterclockwise contour around 0 to form clockwise contours around α and β. At both, the poles are of order 1, and therefore the formula for c_n is simply

$$c_n = \frac{1}{\alpha^n}\frac{1}{\alpha - \beta} + \frac{1}{\beta^n}\frac{1}{\beta - \alpha} = \frac{1}{\alpha - \beta}\left(\frac{1}{\alpha^n} - \frac{1}{\beta^n}\right)$$

2.3 Application — Evaluation of Infinite Series

As an example, consider the series

$$S(a) = \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$$

where $a > 0$.

First note that the series is even in n, and therefore

$$S(a) = \frac{1}{2} \sum_{n \neq 0, -\infty}^{\infty} \frac{1}{n^2 + a^2}$$

Next we say that it would be very nice if this term appeared as a residue of some function $f(z)$ at $z = n$. That is, we need a function that has simple poles at all $z = n$, and the residue should be 1. We know of such a function

$$g(z) = \pi \frac{\cos \pi z}{\sin \pi z}$$

This is singular at all integer n, and the residue is exactly 1.

Now consider the function

$$f(z) = \pi \frac{\cot \pi z}{z^2 + a^2}$$

This has simple poles at all integers, and the residues is $(n^2 + a^2)^{-1}$.

With this, we can write the infinite series as

$$S(a) = \frac{1}{2} \frac{1}{2\pi i} \sum_{n \neq 0, -\infty}^{\infty} \oint_{C_n} \pi \frac{\cot \pi z}{z^2 + a^2}$$

where C_ns are counterclockwise contours that encloses only n.

Now we can to the reverse trick, we can deform each of the contours in the following fashion

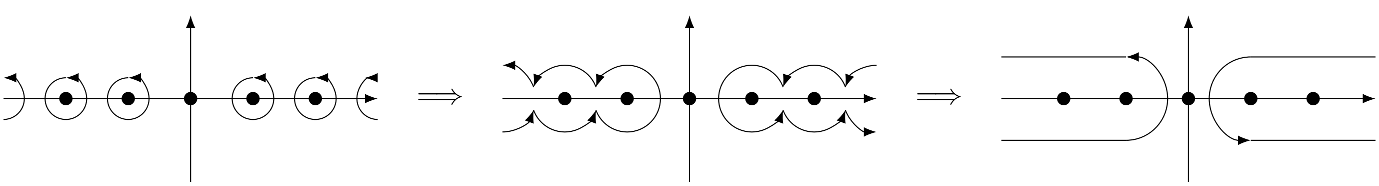


Figure 4: Deforming the contours to merge.

Now since the integrand goes to zero as $r \to \infty$, we can add a carefully chosen zero part, and connect the two contours as in figure 5.

Since now we have a closed contour, we can shrink it back into a clockwise contour that enclose the poles on the imaginary axis (notice that the function also has simple poles at $z = \pm ia$), which can be further shrunken into three contours each around $\pm ia$ & 0. Therefore, we have reduced the infinite sum into a

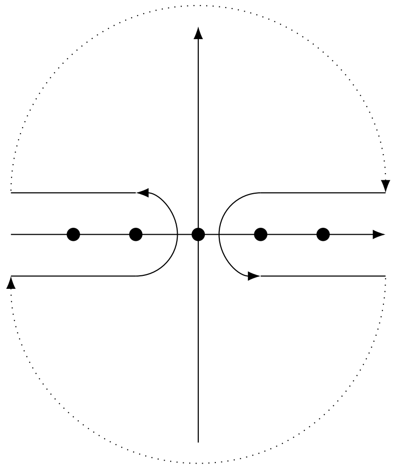


Figure 5: Adding zero

finite sum

$$S(a)=\frac{1}{2}\frac{1}{2\pi i}\sum_{n\in\{0,-ia,ia\}}\oint_{C_n}\pi\frac{\cot\pi z}{(z+ia)(z-ia)}$$

Which can be evaluated as

$$S(a)=-\frac{1}{4\pi i}(-2\pi i)\left(\frac{1}{a^2}+\frac{\pi\cot i\pi a}{2ia}-\frac{\pi\cot -i\pi a}{2ia}\right)$$

since cot is odd function,

$$S(a)=-\frac{1}{2a^2}-\frac{\pi\cot(i\pi a)}{2ia}=-\frac{1}{2a^2}+\frac{\pi}{2a}\frac{e^{\pi a}+\mathrm{e}^{-\pi a}}{e^{\pi a}-\mathrm{e}^{-\pi a}}=\frac{\pi}{2a}\left(\coth\pi a-\frac{1}{\pi a}\right)$$

Now

$$\coth x=\frac{\cosh x}{\sinh x}=\frac{\left(1+\frac{x^2}{2!}+\cdots\right)}{x\left(1+\frac{x^2}{3!}+\cdots\right)}=\frac{\left(1+\frac{x^2}{2!}+\cdots\right)\left(1-\frac{x^2}{3!}+\cdots\right)}{x}=\frac{1}{x}+\frac{x}{2}-\frac{x}{6}\cdots=\frac{1}{x}+\frac{x}{3}+\cdots$$

Therefore,

$$\begin{aligned}S(0)&=\lim_{a\to 0}\frac{\pi}{2a}\left(\coth\pi a-\frac{1}{\pi a}\right)\\&=\lim_{a\to 0}\frac{\pi}{2a}\left(\frac{1}{\pi a}+\frac{\pi a}{3}-\frac{1}{\pi a}\right)\\&=\frac{\pi^2}{6}\end{aligned}$$

2.4 Application — Dirichlet Integral

The integral

$$\int_0^\infty dx\frac{\sin x}{x}$$

This integral is finite. It is not absolutely integrable, i.e. when we try to integrate the absolute value of the integrand, the integral diverges.
We can consider a generalisation of this

$$\int_0^\infty dx \frac{\sin bx}{x} = \frac{\pi}{2}\mathrm{sgn}(b)$$

We try to derive this using contour integrals. To do so, we write this as

$$\int_0^\infty dx \frac{\sin bx}{x} = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin bx}{x}$$

We want to close the contour in the upper and lower half planes and evaluate this integral. But we are in trouble. Since sine is $e^{iz} - \mathrm{e}^{-iz}$, one of these blows up in the upper half and the other blows up in lower half. Therefore we cant *add a 0* to the integral to close the contour.

Therefore, rather we need to consider the integral

$$\frac{1}{2}\mathrm{Im}\int_{-\infty}^\infty dx \frac{\mathrm{e}^{ibx}}{x}$$

but again in this integral, the integrand blows up at $x = 0$. Therefore we need to consider the following contour.

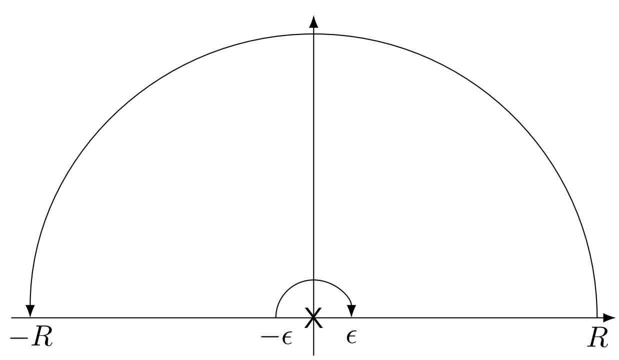


Figure 6: Contour for evaluating the Dirichlet Integral

The contour does not enclose any pole, and therefore, the integral over the contour is zero. Therefore,

$$\int_{-R}^{-\epsilon} dx \frac{\mathrm{e}^{ibx}}{x} + \int_{\epsilon}^{R} dx \frac{\mathrm{e}^{ibx}}{x} + \int_{\pi}^{0} \epsilon \mathrm{e}^{i\theta} i d\theta \frac{\mathrm{e}^{ib\epsilon\mathrm{e}^{i\theta}}}{\epsilon\mathrm{e}^{i\theta}} + \int_0^\pi R\mathrm{e}^{i\theta} i d\theta \frac{\mathrm{e}^{ibR\mathrm{e}^{i\theta}}}{R\mathrm{e}^{i\theta}} = 0$$

The contour integral, where the contour symmetrically avoids the singularity is called the *Cauchy Principle Value Integral*. The first two terms of the above integral added together therefore forms the Cauchy Principle Value Integral, and is denoted by $\mathcal{P}$.

In the limit $\epsilon \to 0$, the exponential in the third term would be simply 1, and therefore the third term would evaluate to $-i\pi$. Since we are in the upper half plane, z has positive imaginary part, and therefore the integrand will be exponentially damped and therefore in the limit $R \to 0$, the last integral contributes 0.

Therefore

$$\mathcal{P}\int_{-\infty}^{\infty} dx \frac{e^{ibx}}{x} - i\pi = 0$$

equating the imaginary part we get

$$\mathcal{P}\int_{-\infty}^{\infty} dx \frac{\sin ibx}{x} = \pi$$

In the Imaginary part, at $x = 0$ there is no singularity, therefore we do not need the principle value, rather this is equal to the integral iteself. Therefore

$$\int_{-\infty}^{\infty} dx \frac{\sin ibx}{x} = \pi \implies \int_{0}^{\infty} dx \frac{\sin ibx}{x} = \frac{\pi}{2}$$

Now if b was negative, then we should have closed in lower half plane, and then we would have picked a negative sign.

We could have taken the path from $-\epsilon$ to ϵ in the lower half plane too, in which case the contour integral would not be 0 but rather $2\pi i$ times the residue. Again we would have the same final answer.

2.5 Mittag-Leffler Representation

A meromorphic function — function who has at best only poles (of any order) in the entire finite part of the complex plane, and do not have other types of singularities (like branch points etc).
For meromorphic functions, we can definitely write, in the neighbourhood of the poles, the function as $c(z-a)^{-1}$ plus some regular part. But the question is can we write the function as the singular parts at all the poles plus some entire function? The answer is yes. If the function has finite number of poles, then it is already in this form, but if it has infinite number of poles, then the question of convergence appears.

Such a representation is called a *Mittag-Leffler representation* and a classic example is the function $\cot \pi z$. We will derive the representation backwards using a known result here. We know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a}\left(\coth \pi a - \frac{1}{\pi a}\right), \quad a \in \mathbb{R}$$

We can consider this as a function of a, and we can analytically continue to make a a complex function. Calling $a = iz$

$$\frac{\pi}{2iz} \coth i\pi z + \frac{1}{2z^2} = \sum_{n=1}^{\infty} \frac{1}{n^2 - z^2}$$

Now since $\coth(ia) = -i\cot(a)$

$$
\begin{aligned}
-\frac{\pi}{2z}\cot\pi z + \frac{1}{2z^2} &= \sum_{n=1}^{\infty}\frac{1}{n^2 - z^2} \\
\implies -\pi\cot\pi z + \frac{1}{z} &= 2\sum_{n=1}^{\infty}\frac{z}{n^2 - z^2} \\
\implies \cot\pi z &= \frac{1}{\pi z} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{z}{z^2 - n^2}
\end{aligned}
$$

For the M-L representation, we need the sum to be explicitly on each pole, and therefore we can write the above as

$$
\cot\pi z = \frac{1}{\pi z} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{z-n} + \frac{1}{z+n}\right)
$$

We still need to separate the two terms in the bracket into two sums. We cant naively do this since separating out means that the individual sums will diverge. That is,

$$
\sum_{n=1}^{\infty}\frac{1}{z-n}
$$

diverges, and only when clubbed together do they converge. To remove this, we can add and subtract the behaviour at infinity, i.e.

$$
\cot\pi z = \frac{1}{\pi z} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\left(\frac{1}{z-n} + \frac{1}{n}\right) + \left(\frac{1}{z+n} - \frac{1}{n}\right)\right]
$$

since the behaviour of $\frac{1}{z-n}$ at large n is $-\frac{1}{n}$ and similarly for the other term. By doing this, we cancel the asymptotially large part of the two series at large n, and therefore we can separate out the sums as

$$
\cot\pi z = \frac{1}{z} + \sum_{n=-\infty}^{\infty}\left(\frac{1}{z-n} + \frac{1}{n}\right)
$$

If we take the differentiation of the above w.r.to z,

$$
\frac{\pi^2}{\sin^2\pi z} = \sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}
$$

where again we see that the LHS has a second order pole at all $z = n$, and the RHS is simply a sum over all double pole terms, and nothing else.

3 Linear Response & Dispersion Relations

We will start with a description of a physical problem — where one disturbs a system and the system responds in some way. Roughly speaking, the greater the stimuli, the greater the response. This is called linear response. Example, Ohm's law — current proportional to voltage, Hook's law — stress proportional to strain, etc.

3.1 Generalised Susceptibility

When we apply a stimulus to a system, the response also depends on how the stimulus depends on time. The most general thing to do is to take an arbitrary stimulus and ask how the system responds. And the easiest way to do this would be to take a stimulus and break it into frequency components and superpose the responses for each component.

Let us called the response of the system $R(t)$, and the force that we apply (stimulus) $F(t)$. The question is how does R depend on F. We know that it should be a linear function, and that causality should be preserved, so the general kind of thing we can write down is

$$R(t) = \int_{-\infty}^{t} dt' \phi(t, t') F(t')$$

ϕ is a measure of response at t per unit force applied at t', and the cutoff of the integral is at t since we can't have the R depend on future forces. Since ϕ doesn't depend on F and therefore the response is linear in F. Another thing we can impose on this is that, since there is time translation symmetry in the system, ϕ should not depend absolutely on t and t', but rather should depend on $t - t'$ (provided the system does not change with time)

$$R(t) = \int_{-\infty}^{t} dt' \phi(t - t') F(t')$$

With this requirement we call ϕ the retarded response, and this is the correct form of response for a causal linear retarded response.

As an example, $F(t)$ can be the electric field vector, with $R(t)$ as the polarization vector, and the most general linear combination of F would require ϕ to be a rank 2 tensor. Let us now stick to real functions.

What we expect the dependency of ϕ on $\tau = t - t'$ is that its value at some non zero τ should not be greater than that at $\tau = 0$, since we do not expect the response due to force applied now to be greater in the future than it is now.

We can Fourier transform R as

$$R(t) = \int_{-\infty}^{\infty} d\omega \mathrm{e}^{-i\omega t}\tilde{R}(\omega) \iff \tilde{R}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \mathrm{e}^{i\omega t}R(t)$$

and $F(t)$ as

$$F(t) = \int_{-\infty}^{\infty} d\omega \mathrm{e}^{-i\omega t}\tilde{F}(\omega) \iff \tilde{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \mathrm{e}^{i\omega t}F(t)$$

Plugging these, we get

$$\int_{-\infty}^{\infty} d\omega \mathrm{e}^{-i\omega t}\tilde{R}(\omega) = \int_{-\infty}^{t} dt' \phi(t-t') \int_{-\infty}^{\infty} d\omega \mathrm{e}^{-i\omega t'}\tilde{F}(\omega)$$

Provided these converge fast and uniformly enough, we can change the order of integration giving

$$\int_{-\infty}^{\infty} d\omega \mathrm{e}^{-i\omega t}\tilde{R}(\omega) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{t} dt' \phi(t-t')\mathrm{e}^{-i\omega t'}\tilde{F}(\omega)$$

Let's change variables from t' to $\tau \implies dt' = -d\tau$ (using this negative sign to change the order of limits)

$$\begin{aligned} \int_{-\infty}^{\infty} d\omega \mathrm{e}^{-i\omega t}\tilde{R}(\omega) &= \int_{-\infty}^{\infty} d\omega \int_{0}^{\infty} d\tau \phi(\tau)\mathrm{e}^{-i\omega(t-\tau)}\tilde{F}(\omega) \\ \implies \int_{-\infty}^{\infty} d\omega \mathrm{e}^{-i\omega t}\tilde{R}(\omega) &= \int_{-\infty}^{\infty} d\omega \mathrm{e}^{i\omega t} \int_{0}^{\infty} d\tau \phi(\tau)\mathrm{e}^{i\omega\tau}\tilde{F}(\omega) \end{aligned}$$

Collecting terms together

$$\int_{-\infty}^{\infty} d\omega \mathrm{e}^{-i\omega t}\left\{\tilde{R}(\omega) - \left(\int_{0}^{\infty} d\tau \phi(\tau)\mathrm{e}^{i\omega\tau}\right)\tilde{F}(\omega)\right\} = 0$$

Since $\mathrm{e}^{-i\omega t}$ forms a complete set of orthonormal functions, the above should be equal to zero coefficient by coefficient, that is,

$$\tilde{R}(\omega) = \left(\int_{0}^{\infty} dt' \phi(\tau)\mathrm{e}^{i\omega\tau}\right)\tilde{F}(\omega) = \chi(\omega)\tilde{F}(\omega) \quad \forall\, \omega \in \mathbb{R}$$

where $\chi(\omega)$ is called the generalised susceptibility. χ gives the response per unit force of frequency ω. It is related to the response function as

$$\chi(\omega) = \int_{0}^{\infty} d\tau \phi(\tau)\mathrm{e}^{i\omega\tau}$$

This is not a Fourier transform, since the integral is not defined at negative τ, this is not even a Laplace transform. So call it Fourier-Laplace transform XD.

Assume (very important assumption) that this integral actually exists and is finite. We also assume that the integral exists for 0 frequency mode (the DC mode).

Notice that since ϕ and ω is real, $\mathrm{Re}\chi(-\omega) = \mathrm{Re}\chi(\omega)$ and $\mathrm{Im}\chi(-\omega) = \mathrm{Im}\chi(\omega)$. One of these will correspond to dissipative effect and the other will correspond to reactive part. Which one is which will depend on the system, but in most systems the imaginary part is dissipative. Notice that ALL of response systems are of this form!

3.2 Dispersion Relations

We notice that if the integral exists, then it also exists when there is a damping factor $\mathrm{e}^{-s\tau}$. Therefore we can extend the function to complex ω, with $\mathrm{Im}\omega \geq 0$, and the function $\chi(\omega)$ is analytic in the upper half plane. What happens in the lower half plane, we don't know. W

Our target is to try and show that the analyticity of $\chi(\omega)$ enables us to write the value of $\chi(\omega)$ for any $\omega \in \mathbb{R}$ in terms of an integral over all other frequencies. This is called a dispersion relation.

To do so, let us first define

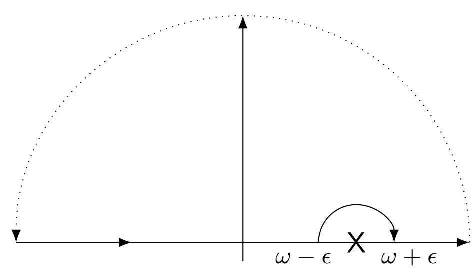
$$f(\omega') = \frac{\chi(\omega')}{\omega' - \omega}$$

where ω' is a complex variable, and ω is a real number.

Since χ is analytic everywhere in upper half plane, f has a pole only at the point ω on the real line. Therefore, over a contour not enclosing ω and lying entirely in the upper half plane ,

$$\oint_C d\omega' \frac{\chi(\omega')}{\omega' - \omega} = 0$$

Now, we do our usual trick of blowing up the contour to form the following



and over this contour too, this integral is zero. From this, we get

$$\mathcal{P}\int_{-R}^{R} \frac{\chi(\omega')}{\omega' - \omega} d\omega' - \int_0^{\pi} i\epsilon \mathrm{e}^{i\theta} d\theta \chi(\omega + \epsilon e^{i\theta}) \frac{1}{\epsilon \mathrm{e}^{i\theta}} + \gamma = 0$$

where as usual

$$\mathcal{P}\int_{-R}^{R} \frac{\chi(\omega')}{\omega' - \omega} d\omega' = \lim_{\epsilon \to 0} \left(\int_{-R}^{\omega-\epsilon} \frac{\chi(\omega')}{\omega' - \omega} d\omega' + \int_{\omega+\epsilon}^{R} \frac{\chi(\omega')}{\omega' - \omega} d\omega' \right)$$

and γ stands for the integral over the semicircle with radius R. As $R \to \infty$, γ will tend to 0 iff $\chi(\omega')$ goes to zero as $|\omega'| \to 0$.

The limit of the integral over the small semicircle in the limit $\epsilon \to 0$ is simply $\pi i \chi(\omega)$, and therefore, we get

$$\chi(\omega) = \frac{1}{i\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\chi(\omega')}{\omega'-\omega}d\omega' \tag{3.1}$$

This integral is over all real frequencies and we do not talk anything about the imaginary frequencies at all.

Now

$$\text{Re}\chi(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\text{Im}\chi(\omega')}{\omega'-\omega}d\omega' \quad \& \quad \text{Im}\chi(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\text{Re}\chi(\omega')}{\omega'-\omega} \tag{3.2}$$

Since the real part and the imaginary part of χ satisfy the above coupled equations, they are called Hilbert transform pair. When we use these to solve for one of the two, we get a double integral, which evaluates a δ function!!

Since in physical systems, negative frequency doesn't make any sense, we can split the above integral into two parts

$$\text{Re}(\chi(\omega)) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{0}\frac{\text{Im}\chi(\omega')}{\omega'-\omega}d\omega' + \frac{1}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\text{Im}\chi(\omega')}{\omega'-\omega}d\omega'$$

and make a change of variables in the first integral and use $\text{Im}\chi(-\omega) = -\text{Im}\chi(\omega)$ to get

$$\text{Re}(\chi(\omega)) = \frac{1}{\pi}\mathcal{P}\int_{0}^{\infty}d\omega'\text{Im}\chi(\omega')\left(\frac{1}{\omega'+\omega}+\frac{1}{\omega'-\omega}\right) = \frac{2}{\pi}\mathcal{P}\int_{0}^{\infty}d\omega'\text{Im}\chi(\omega')\frac{\omega'}{\omega'^2-\omega^2}$$

and similarly for

$$\text{Im}(\chi(\omega)) = \frac{2}{\pi}\mathcal{P}\int_{0}^{\infty}d\omega'\text{Re}\chi(\omega')\frac{\omega}{\omega'^2-\omega^2}$$

Now it makes sense physically. We see that neither of the real and imaginary parts cannot go to zero without the other immediately becoming zero. These relations are called dispersion relations (in physics, they are called Kramers-Kronig relation). Notice that analyticity came from the requirement of causality. If the integral in the definition of χ ran from $-\infty$ to ∞ one could never claim that we could extend the integral to complex ω.

Now let us revisit one assumption we made earlier, about χ going to zero at infinity. It is possible that at infinity, χ goes to some constant value which is non zero. In that case we can not claim the contribution γ to be zero. In this case, there will be an extra contribution $i\pi\chi(\infty)$. This is still manageable. But there is no reason for χ to take on one single constant along all directions, i.e. it might still vary along the semicircle at infinity. In that case, we can perform a trick of subtraction, where we now consider the function

$$f(\omega) = \frac{\chi(\omega')-\chi(\omega_0)}{(\omega'-\omega)(\omega'-\omega_0)}$$

where ω_0 is some frequency for which we know the value of susceptibility. For this function, at infinity, even if χ blows up, as long as it is not growing as fast as ω' itself, the extra ω' in the denominator makes the entire integral go to zero. In doing this, we paid a price, we now need to know the susceptibility at one point explicitly. This is called a subtracted dispersion relation. In case χ blows up faster, we aan subtract further terms to keep it in check. As long as it doesnt blow up as exponential, we can keep it in check.

3.3 Examples

Consider an LCR series circuit,

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int_{-\infty}^{t} dt' I(t') = V$$

where V is now the stimulus and I is the response. To obtain the susceptibility, make the fourier transformation

$$\left(-i\omega L + R - \frac{1}{i\omega C}\right)\tilde{I}(\omega) = \tilde{V}(\omega)$$

which implies,

$$\tilde{I}(\omega) = \frac{i\omega}{L\left(\omega^2 + \frac{R}{L}i\omega - \frac{1}{LC}\right)}\tilde{V}(\omega)$$

which means

$$\chi(\omega) = \frac{i\omega}{L(\omega^2 + i\gamma\omega - \omega_0^2)}$$

where $\omega_0 = \dfrac{1}{\sqrt{LC}}$ and $\gamma = \dfrac{R}{L}$. This has poles. The poles cannot be in upper half plane, and they are in the lower half as we can check by finding their location. We can also show that the real and imaginary part of χ satisfy the previously derived integral relations.

Now for a general physical system, the behaviour of ϕ to the lowest order is an exponential decay

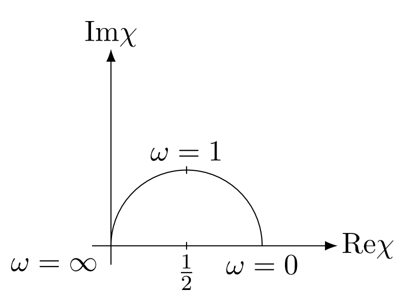
$$\phi(t) = \phi_0 \mathrm{e}^{-\frac{t}{\tau}}$$

where τ is called the relaxation time. In which case we get

$$\chi(\omega) = \int_0^\infty dt\mathrm{e}^{i\omega t}\phi_0\mathrm{e}^{-\frac{t}{\tau}} = \int_0^\infty dt\phi_0\mathrm{e}^{-(1-i\omega\tau)\frac{t}{\tau}} = \frac{\phi_0\tau}{1-i\omega\tau} = \phi_0\tau\left(\frac{1}{1+\omega^2\tau^2} + \frac{i\omega}{1+\omega^2\tau^2}\right)$$

Again as expected, we have even and odd behaviours in the real and imaginary parts, and also a pole which is in the lower half region. Such a model is called Debye relaxation.

What we do next is to plot the real part of χ vs the imaginary part of χ by eliminating the frequency. This is called *Cole-Cole plot*. One can see that for the above χ, it is the following



That is, it is a semicircle centered at $\frac{1}{2}$. When this is experimentally measured and plotted, if the result is exactly this semicircle, we can conclude that the response is an exponential decay. Usually there is a deviation, where the two ends are not perpendicular to the x axis and therefore it doesnt form a semicircle, and then there are more complicated cases, and in these cases the conclusions are that we need more parameters, i.e. more relaxation times to describe the system. More characteristic behaviour of a system is

$$\phi(t) = \int_{\tau_{min}}^{\tau_{max}} d\tau \mathrm{e}^{-\frac{t}{\tau}} \sigma(\tau)$$

where we can have a large possibility of deviations, and we will be able to model a large variety of systems with different behaviours.

4 Analytic Continuation & Gamma Function

4.1 The Gamma Function

An instructive example to look at analytic continutation is the Gamma function.

$$\int_0^\infty t^n \mathrm{e}^{-t} = n!$$

Stirling's formula

$$n! = n^n \mathrm{e}^{-n}\sqrt{2\pi n}\left(1 + \frac{1}{12n} + O(n^2)\right)$$

How do we get this formula? We see that t^n is a rapidly increasing function, but e^{-t} decreases faster than t^n, and therefore their product resembles a Gaussian. The approximation is that for the integral the dominant component is only a region surrounding the maxima and the other contributions can be ignore.

That is, the above integral can be rewritten as

$$\int_0^\infty \mathrm{e}^{-(t-n\ln t)} dt$$

and when $f(t) = t - n\ln t$ has extremum, this term contributes, and rest of the time it dies out.

$$f'(t) = 1 - \frac{n}{t}$$

which is 0 at $t = n$, and at $t = n$

$$f''(n) = \frac{1}{n}$$

Now we can expand the function as

$$\int_0^\infty \mathrm{e}^{-(n-n\ln n+\frac{(t-n)^2}{2!n}+\cdots)} dt$$

The leading behaviour of this would be

$$n! = n^n \mathrm{e}^{-n} \int_0^\infty dt \mathrm{e}^{-\frac{(t-n)^2}{2n}}(1+\cdots)$$

Now it can be shown rigorously that the above integral is well damped if extended to $-infty$, in which case it turns into a gaussian integral, whose first term would be $\sqrt{2\pi n}$, and the other correction terms follow.

The gamma function is defined as

$$\int_0^\infty t^{n-1}\mathrm{e}^{-t} = \Gamma(n) = (n-1)!$$

We see that we do not need to restrict ourselves to integers. We can extend it to real numbers $\Gamma(x)$, while being careful that it will blow up for $x \leq 0$. Now the question is, can we extend it to the complex plane and when does such an integral converge.

$$\Gamma(z) = \int_0^\infty dt \mathrm{e}^{-t} t^{z-1}$$

The convergence of the integral depends only on the real part of z, since

$$t^{x+iy-1} = t^{x-1} t^{iy} = t^{x-1} \mathrm{e}^{iy \ln t}$$

and the exponential is of unit magnitude and therefore only oscillates and doesn't blow up, the integral converges for $\mathrm{Re} z > 0$. Just like we said when we define a function by a power series valid in a region, there should be atleast one singularity on the boundary, we suspect that there are one or more singularities on the line $\mathrm{Re} z = 0$ and we want to find out what these singularities are. What we would like to do is try to extend the definition to the left half plane $\mathrm{Re} z < 0$ by analytically continuing it there.

4.2 Analytically continuing the Gamma Function

The part that gave us problem in the definition

$$\Gamma(z) = \int_0^\infty dt \mathrm{e}^{-t} t^{z-1}, \ \mathrm{Re} z > 0$$

is the t^{z-1}. If the power were any higher, then there would not be any problem. We can increase the power by doing integration by parts

$$\Gamma(z) = \mathrm{e}^{-t} \frac{t^z}{z} \Big|_0^\infty - \int_0^\infty -\mathrm{e}^{-t} \frac{t^z}{z}, \ \mathrm{Re} z > 0$$

The first term vanishes in both the limits, provided $z \neq 0$, i.e. for all $z \neq 0$ it vanishes. This is already satisfied in the region $\mathrm{Re} z > 0$.

$$\Gamma(z) = \frac{1}{z} \int_0^\infty \mathrm{e}^{-t} t^z$$

We therefore now have a representation that point by point matches the original function in the region $\mathrm{Re} z > 0$, has a pole explicitly at $z = 0$, and converges in the region $\mathrm{Re} z > -1$. Therefore, starting with a representation that converged only when $\mathrm{Re} z > 0$, we have now found a representation that matches this exactly in $\mathrm{Re} z > 0$, has a pole at $z = 0$, and converges in the region $\mathrm{Re} z > -1$. This is valid in a bigger region and also exposes the singularity at $z = 0$.

Now we can play the same trick one more time to get

$$\Gamma(z) = \frac{1}{z(z+1)} \int_0^\infty t^{z+1} e^{-t}$$

which now converges in $\mathrm{Re} z > -2$ and has poles at $z = 0, -1$

We can similarly extend this to Re$z > -n$ region in the left half of the complex plane, and therefore $\Gamma(z)$ is a meromorphic function with simple poles at all $z = -n$ where $n = 0, 1, 2, \cdots$, with residue $\dfrac{(-1)^n}{n!}$.

What we have done, is analytically continue the function $\Gamma(z)$ step by step by using integration by parts to extend it to the a larger region in the complex plane, but we have not given one single representation for the entire complex plane. The above representations are still convergent only in a finite reason. There is a single representation that is valid for the whole complex plane but this would need a discussion of branch points, and therefore we postpone our discussion.

Another trick would be to use the fact that the Γ function satisfies

$$\Gamma(z+1) = z\Gamma(z) \implies \Gamma(z) = \frac{1}{z}\Gamma(z+1)$$

to analytically continue.

4.3 Log Derivative of Gamma Function

What happens if we took the derivative of the Γ function?
Define

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{1}{\Gamma(z)}\frac{d}{dz}\Gamma(z)$$

The functional equation satisfied by this $\psi(z)$ can be found by seeing that

$$\ln\Gamma(z+1) = \ln z + \ln\Gamma(z) \implies \psi(z+1) = \frac{1}{z} + \psi(z)$$

What is the singularity spectrum of $\psi(z)$?
Near $z = -n$, since the Γ function has a pole there, $\Gamma(z)$ is guarenteed to be of the form

$$\Gamma(z) = \frac{(-1)^n}{n!(z+n)} + \text{Regular part}$$

To find what happens to ψ, we need to differentiate this w.r.to z and divide by $\Gamma(z)$, which will imply that near $z = -n$, the ψ will be proportional to

$$\frac{-1}{z+n}$$

Therefore, the $\psi(z)$ is also a meromorphic function with simple poles at all $z = -n$ with residue -1.

Now the question is what is the values of ψ at different z like $1, 2$, etc.

See that

$$\Gamma(z) \underset{(z\approx 0)}{=} \frac{1}{z} + \sum_{n=0}^{\infty} c_n z^n$$

The constants can be found by starting with a representation of the Gamma function and working out the answer. Turns out the term c_0 is a fundamental constant called the Euler-Mascheroni constant defined as

$$\gamma = \lim_{N\to\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n} - \ln(N)\right) \approx 0.5772$$

It is a conjecture that this is an irrational number, but it is not definitely known if it is indeed or not.

Therefore

$$\Gamma(z) \underset{(z\approx 0)}{=} \frac{1}{z} + \gamma + \sum_{n=1}^{\infty} c_n z^n$$

with this,

$$\psi(z) \underset{(z\approx 0)}{=} -\frac{1}{z} - \gamma + \cdots$$

Substituting this in the functional equation we wrote above and letting z go to zero gives

$$\psi(1) = -\gamma$$

This immediately tells that $\psi(2) = 1 - \gamma$, $\psi(3) = 1 + \frac{1}{2} - \gamma$ and

$$\psi(n+1) = -\gamma + \sum_{i=1}^{n}\frac{1}{i}$$

4.3.1 Gaussian Integrals

A general Gaussian integral is of the form

$$\int_0^\infty du \mathrm{e}^{-au^2} u^r,\ \mathrm{Re}a > 0,\ r > -1$$

A way to do this integral is to substitute

$$au^2 = t \implies u = \frac{\sqrt{t}}{\sqrt{a}} \implies du = \frac{dt}{2\sqrt{at}}$$

which leads to the integral

$$\int_0^\infty \frac{dt}{2\sqrt{at}}\mathrm{e}^{-t}\left(\frac{\sqrt{t}}{\sqrt{a}}\right)^r = \frac{1}{2a^{\frac{r+1}{2}}}\int_0^\infty dt e^{-t} t^{\frac{r}{2}} = \frac{\Gamma\left(\frac{r+1}{2}\right)}{2a^{\frac{r+1}{2}}}$$

Since

$$\int_0^\infty du \mathrm{e}^{-au^2} = \sqrt{\frac{\pi}{a}}$$

this immediately implies $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ using which we can now write the Gamma functions for all half odd integrers.

4.4 Mittag-Leffler Representation for Gamma Function

We know the locations of all the poles of the Gamma function, and therefore we can ask if it is possible to find the Mittag-Leffler expansion for the function. The Γ function is of the form

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}$$

The singularity in the Gamma function comes from t^{z-1} term as $t \to 0$. Therefore, we split the integral as

$$\Gamma(z) = \int_0^1 dt\ t^{z-1}e^{-t} + \int_1^\infty dt\ t^{z-1}e^{-t}$$

The second term is regular, and the first term is singular, and therefore we need to find the series representation for the first part only. We can perform the integral explicitly

$$\int_0^1 dt\ t^{z-1}e^{-t} = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\int_0^1 dt\ t^{n+z-1} = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{1}{z+n}$$

and therefore, we have the Mittag-Leffler representation

$$\Gamma(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{1}{z+n} + \int_1^\infty dt\ t^{z-1}e^{-t}$$

We did not choose a random splitting at a because in such a splitting we would not get the residues in the series expansion, but there would be factors of a in the series, and therefore it would not be a Mittag-Leffler representation.

4.5 The reciprocal of Γ

We saw that $\Gamma(z)$ is a meromorphic function, that is it has only poles. It turns out that $\dfrac{1}{\Gamma(z)}$ is an entire function. The condition for the reciprocal of a meromorphic function to be entire is that the function should have no zeroes, since if there are no zeroes, there are no poles.

Just like there is a Mittag-Leffler representation for meromorphic functions in terms of its poles, there is a representation for an entire function in terms of its zeroes. That is, we ask that if α, β, etc to be the locations of zeroes, then for a representation of the form $(z-\alpha)(z-\beta)\cdots$. Such a product is called Weierstrass product, and for the reciprocal of the Gamma function it is

$$\frac{1}{\Gamma(z)} = z\mathrm{e}^{\gamma z}\prod_{n=1}^\infty (1+\frac{z}{n})\mathrm{e}^{-\frac{z}{n}}$$

Note that at $z=-n$, $\Gamma(z)$ had poles, so its reciprocal will have zeroes at these points.
Such a representation will typically have portions that are zeroes, and exponential of an entire function.

4.6 Euler's Beta Function

Defined as

$$\beta(m,n) = \int_0^1 dt\ t^{m-1}(1-t)^{n-1}$$

with $m,n > 0$. At $m = 0$ it blows up due to the lower limit in the integration, and at $n = 0$ it blows up due to the upper limit of the integration.

We can define this also as a function of two complex variables

$$\beta(z,w) = \int_0^1 dt\ t^{z-1}(1-t)^{w-1}$$

This converges as long as $\mathrm{Re}z, \mathrm{Re}w > 0$.

We can again do integration by parts. If we try to extend to larger region in z, we need to integrate the first term, while differentiating the second, and therefore we now move to a smaller region in the w plane. If we tried it the other way, then the opposite would have happened. And therefore, it is pretty clear that by using this *trick*, we cannot analytically continue in both the variables. There are other ways of doing this, which we will discuss later.

There is yet another way, which is to show that the β functio is related to the Γ function and to use its properties to analytically continue. We will take this up later after we discuss branch points. This diversion was to explicitly show that the trick of integration by parts doesn't already works.

There is no general prescription for analytic continuation. It varies case by case, and we need to come up with such *tricks* to continue based on the analytic properties of the functions.

What is the connection between beta functions and gamma functions.
Consider the double integral

$$I(z,w) = \int_0^\infty du \int_0^\infty dv\ \mathrm{e}^{-(u^2+v^2)} u^{2z-1} v^{2w-1}$$

We can factor this integral into u and v integrals, the individual integrals being

$$\int_0^\infty du\ \mathrm{e}^{-u^2} u^{2z-1}$$

which with the substitution $u^2 = t$ reduces to

$$\frac{1}{2}\int_0^\infty \mathrm{e}^{-t} t^{z-1} = \frac{1}{2}\Gamma(z)$$

and therfore we would get

$$I(z,w) = \frac{1}{4}\Gamma(z)\Gamma(w)$$

We can also do the integral $I(z,w)$ in polar coordinates

$$\int_0^\infty rdr \int_0^{\frac{\pi}{2}} d\theta\ \mathrm{e}^{-r^2}(r^2)^{z+w-1}(\cos\theta)^{2z-1}(\sin\theta)^{2w-1}$$

changing the variables $s = r^2$

$$I(z,w) = \frac{1}{2}\int_0^\infty ds \mathrm{e}^{-s}(s)^{z+w-1}\int_0^{\frac{\pi}{2}} d\theta\ (\cos\theta)^{2z-1}(\sin\theta)^{2w-1} = \frac{1}{2}\Gamma(z+w)\int_0^{\frac{\pi}{2}} d\theta\ (\cos\theta)^{2z-1}(\sin\theta)^{2w-1}$$

Now if we put $(\cos\theta)^2 = \xi \implies d\xi = -2\cos\theta\sin\theta d\theta$, then we get

$$\int_0^{\frac{\pi}{2}} d\theta\ (\cos\theta)^{2z-1}(\sin\theta)^{2w-1} = -\frac{1}{2}\int_1^0 d\xi\ (\xi)^{z-1}(1-\xi)^{w-1} = \frac{1}{2}\beta(z,w)$$

Therefore, we get

$$I(z,w) = \frac{1}{4}\Gamma(z+w)\beta(z,w) = \frac{1}{4}\Gamma(z)\Gamma(w)$$

from which we get

$$\beta(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

One of the consequences of this is that if we put $w = 1-z$, we get

$$\beta(z,1-z) = \Gamma(z)\Gamma(1-z)$$

and one can show that this is equal to

$$\beta(z,1-z) = \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

This formula is called the reflection formula for $\Gamma(z)$

We can also take log on both sides and differentiate to get

$$\psi(1-z) - \psi(z) = \pi\cot\pi z$$

The Γ function also has the following *Doubling* property

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

Taking the logarithm, we get

$$\ln\Gamma(2z) = (2z-1)\ln 2 + \ln\Gamma(z) + \ln\Gamma\left(z+\frac{1}{2}\right) - \frac{1}{2}\ln\pi$$

Differentiating this,

$$2\psi(2z) = 2\ln 2 + \psi(z) + \psi\left(z + \frac{1}{2}\right)$$

Using this, we can write ψ for half integers.

$$2\psi(1) = 2\ln 2 + \psi\left(\frac{1}{2}\right) + \psi(1) \implies \psi\left(\frac{1}{2}\right) = -2\ln 2 + \psi(1) = -2\ln 2 - \gamma$$

We can also find ψ for other half integers too.

5 Möbius Transformations

We can regard every analytic function as a map of the complex plane to the complex plane. For analytic functions, the map has very specific properties.
Suppose an analytic function

$$w = f(z) = u + iv$$

This defines a transformation $z \to w$. and this transformation preserves angles between curves.

To see how, suppose two curves $y = s(x),\ y = t(x)$ in the z plane intersecting at $z_0 = x_0 + iy_0$ making an angle θ. At z_0, the tangent vectors (observe that working on the complex plane is exactly the same as vector analysis 2D plane, with Rez being the x component of the vector and Imz being the y component) on to the two curves would be

$$s \to \delta x + i\frac{ds}{dx}\delta x, \quad t \to \delta x + i\frac{dt}{dx}\delta x$$

The curves intersect at an angle θ means that the angle between the tangents is θ, i.e.

$$\cos\theta = \frac{\left(1 + \frac{ds}{dx}\frac{dt}{ds}\right)}{\sqrt{\left(1 + \left(\frac{ds}{dx}\right)^2\right)\left(1 + \left(\frac{dt}{dx}\right)^2\right)}}$$

Now consider the map

$$w = f(z) = u(z) + iv(z)$$

under this, the points $x_0 + s/t(x_0),\ x_0 + \delta x + i\left(s/t(x_0) + \frac{d}{dx}s/t(x_0)\delta x\right)$transform as

$$u(x_0, s/t(x_0)) + iv(x_0, s/t(x_0))$$
$$u\left(x_0 + \delta x, \left(s/t(x_0) + \frac{d}{dx}s/t(x_0)\delta x\right)\right) + iv\left(x_0 + \delta x, \left(s/t(x_0) + \frac{d}{dx}s/t(x_0)\delta x\right)\right)$$

which can be expanded in first order as

$$u(x_0, s/t(x_0)) + iv(x_0, s/t(x_0))$$
$$u(x_0, s/t(x_0)) + u_x\delta x + u_y\left(\frac{d}{dx}s/t(x_0)\delta x\right) + iv(x_0, s/t(x_0)) + iv_x\delta x + iv_y\left(\frac{d}{dx}s/t(x_0)\delta x\right)$$

where the subscripts indicate partial derivative w.r.to the variables.
Therefore, the tangent vectors to the curves in w space would be

$$\left(u_x\delta x + u_y\left(\frac{d}{dx}s/t(x_0)\delta x\right)\right) + i\left(v_x\delta x + v_y\left(\frac{d}{dx}s/t(x_0)\delta x\right)\right)$$

and therefore the angle between them will be $\cos\theta' =$

$$\frac{\left(\left(u_x + u_y\frac{ds}{dx}\right)\left(u_x + u_y\frac{dt}{dx}\right) + \left(v_x + v_y\frac{ds}{dx}\right)\left(v_x + v_y\frac{dt}{dx}\right)\right)}{\sqrt{\left(\left(u_x + u_y\frac{ds}{dx}\right)\left(u_x + u_y\frac{ds}{dx}\right) + \left(v_x + v_y\frac{ds}{dx}\right)\left(v_x + v_y\frac{ds}{dx}\right)\right)\left(\left(u_x + u_y\frac{dt}{dx}\right)\left(u_x + u_y\frac{dt}{dx}\right) + \left(v_x + v_y\frac{dt}{dx}\right)\left(v_x + v_y\frac{dt}{dx}\right)\right)}}$$

Now using the analyticity condition, $u_x = v_y,\ u_y = -v_x$

$$\frac{\left(\left(u_x - v_x\frac{ds}{dx}\right)\left(u_x - v_x\frac{dt}{dx}\right) + \left(v_x + u_x\frac{ds}{dx}\right)\left(v_x + u_x\frac{dt}{dx}\right)\right)}{\sqrt{\left(\left(u_x - v_x\frac{ds}{dx}\right)\left(u_x - v_x\frac{ds}{dx}\right) + \left(v_x + u_x\frac{ds}{dx}\right)\left(v_x + u_x\frac{ds}{dx}\right)\right)\left(\left(u_x - v_x\frac{dt}{dx}\right)\left(u_x - v_x\frac{dt}{dx}\right) + \left(v_x + u_x\frac{dt}{dx}\right)\left(v_x + u_x\frac{dt}{dx}\right)\right)}}$$

which can be expanded as

$$\frac{\left(u_x^2 + v_x^2\right)\left(1 + \frac{ds}{dx}\frac{dt}{dx}\right)}{\sqrt{\left(\left(u_x^2 + v_x^2\right)\left(1 + \left(\frac{ds}{dx}\right)^2\right)\right)\left(\left(u_x^2 + v_x^2\right)\left(1 + \left(\frac{dt}{dx}\right)^2\right)\right)}}$$

which is exactly equal to

$$\frac{\left(1 + \dfrac{ds}{dx}\dfrac{dt}{ds}\right)}{\sqrt{\left(1 + \left(\dfrac{ds}{dx}\right)^2\right)\left(1 + \left(\dfrac{dt}{dx}\right)^2\right)}} = \cos\theta$$

Therefore $\theta' = \theta$, and the mapping $w = f(z)$ preserves angles, i.e. it is a conformal transformation.